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STEADY-STATE BIFURCATION WITH EUCLIDEAN SYMMETRY

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Abstract. We consider systems of partial differential equations equivariant under the Euclidean group $\mathbf{E}(n)$ and undergoing steady-state bifurcation (with nonzero critical wavenumber) from a fully symmetric equilibrium. A rigorous reduction procedure is presented that leads locally to an optimally small system of equations. In particular, when $n = 1$ and $n = 2$ and for reaction-diffusion equations with general n, reduction leads to a single equation. (Our results are valid generically, with perturbations consisting of relatively bounded partial differential operators.)

In analogy with equivariant bifurcation theory for compact groups, we give a classification of the different types of reduced systems in terms of the absolutely irreducible unitary representations of $\mathbf{E}(n)$. The representation theory of $\mathbf{E}(n)$ is driven by the irreducible representations of $\mathbf{O}(n-1)$. For $n = 1$, this constitutes a mathematical statement of the 'universality' of the Ginzburg-Landau equation on the line. (In recent work, we addressed the validity of this equation using related techniques.)

When $n = 2$, there are precisely two significantly different types of reduced equation: *scalar* and *pseudoscalar*, corresponding to the trivial and nontrivial one-dimensional representations of $\mathbf{O}(1)$. There are infinitely many possibilities for each $n \geq 3$.

1. Introduction

Certain systems of partial differential equations (PDEs) such as the Navier-Stokes equations, the Boussinesq equations (modeling the planar Bénard problem), the Kuramoto-Sivashinsky equation and reaction-diffusion equations have Euclidean symmetry when posed on an unbounded domain such as the whole of $\mathbb{R}^n$. For an overview, see [4]. One approach to such systems of PDEs is to restrict to solutions with a prescribed spatial periodicity. It is then possible to derive a finite-dimensional ordinary differential equation (ODE) or 'Landau equation'.

Of course, solutions need not be spatially periodic, and consequently these techniques are somewhat limited. In addition, when $n \geq 2$ there are many ways to prescribe the spatial periodicity, and these cannot be captured simultaneously by a single ODE. (In general, we consider PDEs posed on domains of the form $\mathbb{R}^n \times \Omega$ where Ω is a bounded subset of $\mathbb{R}^d$, $d \geq 0$. Hence, n refers throughout to the number of unbounded spatial variables.)

In order to include solutions that are not spatially periodic, it is customary to consider infinite-dimensional modulation equations such as the Ginzburg-Landau

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equation ($n = 1$) and the Newell-Whitehead-Segel equation ($n = 2$) [24, 30]. The underlying ansatz is that there is some 'basic' or 'preferred' spatially periodic state bifurcating at criticality. The Ginzburg-Landau and Newell-Whitehead-Segel equations are 'universal' modulation equations around this basic state.

From the mathematical point of view, there are serious difficulties in substantiating both the validity and the universality (or model-independence) of the modulation equations. This is in contrast with the Landau equations, where there are completely satisfactory interpretations both of their validity (in terms of Liapunov-Schmidt or center manifold reduction) and of their universality (in terms of the absolutely irreducible representations of the compact Lie group of symmetries present in these problems). These issues make up a large part of the subject known as equivariant bifurcation theory [8, 26, 27, 28, 32]. Of course, the discussion of absolutely irreducible representations is at the heart of the purely phenomenological Landau theory [17, 21]. (The problem of determining existence and stability of branches of solutions is also a significant issue in Landau theory and equivariant bifurcation theory.)

Recently, there has been a great deal of progress on the validity of the Ginzburg-Landau equations when $n = 1$. In this paper, we consider problems with Euclidean symmetry quite generally (for all $n \geq 1$). In particular, we give a complete answer to the question of universality, as well as making progress on validity. To put the results in context it is worthwhile to review the methods and results of Landau theory and equivariant bifurcation theory when there is a compact group of symmetries.

Steady-state bifurcation with a compact symmetry group. Suppose that Γ is a compact Lie group and that a Γ-equivariant system of PDEs undergoes a steady-state bifurcation: a fully symmetric 'trivial solution' loses stability as an eigenvalue passes through zero. The Landau equations can be derived in several ways:

1. phenomenologically,
2. asymptotic expansion,
3. Liapunov-Schmidt/center manifold reduction.

All three approaches lead to an ODE that is equivariant under the group Γ. The equations are 'universal' in the sense that the symmetry comes in, generically, in only countably many ways (finitely many if Γ is finite), and these can be enumerated as the absolutely irreducible representations of Γ, see Golubitsky, Stewart and Schaeffer [8]. Once the representation is known, the precise details of the original problem enter only in the Taylor coefficients of the reduced equation. Thus, for some purposes it is not even necessary to have a PDE model in the first place, and this brings us back to the original phenomenological approach of Landau [17] in the theory of second order phase transitions.

When there is an underlying PDE, Liapunov-Schmidt/center manifold reduction makes the asymptotic expansion method completely rigorous. However, there is an additional step involved in obtaining the Landau equations where the reduced equations are truncated at low order. The truncation step is not rigorous in general (in many cases it can be shown not to be valid) but can sometimes be justified via a scaling argument, at least for certain classes of solutions (Sattinger [27]).

Mathematically, the derivation and universality of the Landau equations can be summarized as follows.

(a) **Enumeration of universality classes** of reduced ODEs in terms of the absolutely irreducible representations of the compact symmetry group Γ. When there is an underlying PDE, these representations correspond to the action (which is generically absolutely irreducible) of Γ on the kernel of the linearized PDE.

(b) **Rigorous justification, generically, of reduction of a PDE to one of the universality classes** via Liapunov-Schmidt reduction or center manifold reduction. Locally (that is, for small amplitude solutions near criticality) solutions to the reduced ODE are in one-to-one correspondence with solutions to the original PDE.

(c) **Enumeration of the Landau equations as truncations** (via scalings) of the ODEs in each universality class. 'Nondegenerate' or hyperbolic solutions for the Landau equations extend to branches of solutions to the full PDE (by the implicit function theorem).

Steady-state bifurcation with Euclidean symmetry. We are now in a position to discuss the situation for the noncompact group of Euclidean symmetries $\mathbf{E}(n)$. There are two different kinds of steady-state bifurcation, depending on whether the so-called critical wavenumber is zero or nonzero. We concentrate throughout on the more interesting case where the critical wavenumber is nonzero (Type I_s in the physics nomenclature [4]). This assumption is a crucial factor in the formulation of the Ginzburg-Landau and Newell-Whitehead-Segel equations.

First, we state our result on universality, which extends step (a) above to the $\mathbf{E}(n)$-equivariant context.

Theorem 1.1. *Suppose an $\mathbf{E}(n)$-equivariant system of PDEs undergoes steady-state bifurcation (with nonzero critical wavenumber) from a fully-symmetric equilibrium. Then, generically, the kernel of the linearized PDE is absolutely irreducible under $\mathbf{E}(n)$ and corresponds to an irreducible representation of $\mathbf{O}(n-1)$.*

In fact, the different universality classes corresponding to steady-state bifurcation with $\mathbf{E}(n)$-symmetry are in one-to-one correspondence with the irreducible representations of $\mathbf{O}(n-1)$. There is one such representation (the trivial one) when $n = 1$, and hence one universality class, which we call the *scalar class* for reasons explained below after Theorem 1.3. When $n = 2$, we have $\mathbf{O}(1) \cong \mathbb{Z}_2$ and there are two universality classes, the scalar class and the *pseudoscalar class*, corresponding to the trivial and nontrivial irreducible representations of $\mathbb{Z}_2$. Once $n \geq 3$, there are a countable infinity of universality classes.

Remark 1.2. (a) The significance of the group $\mathbf{O}(n-1)$ in Theorem 1.1 can be explained in terms of Mackey's classification of the irreducible unitary representations of $\mathbf{E}(n)$. (See also Ito [13].) Let $\mathbf{O}(n)$ act on $\mathbb{R}^n$ in the standard way. For each $a \geq 0$, choose $x_a \in \mathbb{R}^n$ at a distance a from the origin. Now define $H_a \subset \mathbf{O}(n)$ to be the isotropy subgroup of x_a. It follows from Mackey [19, Theorem 14.1] that the irreducible representations of $\mathbf{E}(n)$ are in one-to-one correspondence with pairs consisting of a number $a \geq 0$ and an irreducible representation of H_a. It turns out that a can be identified with the critical wavenumber, so the case $a = 0$ (with $H_a = \mathbf{O}(n)$) is not relevant here. When the critical wavenumber is nonzero we have $H_a = \mathbf{O}(n-1)$ as required.
(b) The precise definition of what we mean by a generic property is rather technical and is deferred until later in this paper, see Section 4.1. The main points are that we work within the class of PDEs (even though the reduced equations are not PDEs, see Remark 1.4(c)) and we do not allow singular perturbations.

Next, we turn to the reduction step (b) described above and give a reasonably precise statement of our results for $n = 1$ and $n = 2$. (The analogous result for general n is stated in Section 2.)

Theorem 1.3. *Let $n = 1$ or $n = 2$, and suppose that an $\mathbf{E}(n)$-equivariant system of PDEs posed on $\mathbb{R}^n \times \Omega$ undergoes steady-state bifurcation (with nonzero critical wavenumber) from a fully-symmetric equilibrium. Then, generically, there is a reduction from the original PDE to a single reduced equation posed on $\mathbb{R}^n$. This reduction preserves essential solutions bifurcating from the trivial solution near criticality. The reduced equation is scalar or pseudoscalar, depending on the representation of $\mathbf{O}(n-1)$ associated with the kernel of the linearized PDE.*

The terminology scalar and pseudoscalar is introduced in [2] and refers to the way functions $u : \mathbb{R}^n \to \mathbb{R}$ transform under the group $\mathbf{E}(n)$. The scalar action of $\mathbf{E}(n)$ is given by $u(x) \mapsto u(\gamma^{-1}x)$, where $\gamma \in \mathbf{E}(n)$ acts in the standard way on the unbounded spatial variables $x \in \mathbb{R}^n$. If we write $\gamma = (A, t)$ where A is an orthogonal transformation and t is a translation, then $u(x) \mapsto (\det A)u(\gamma^{-1}x)$ defines the pseudoscalar action of $\mathbf{E}(n)$. A single $\mathbf{E}(n)$-equivariant PDE posed on $\mathbb{R}^n$ is said to be scalar or pseudoscalar, depending on whether the action of $\mathbf{E}(n)$ is scalar or pseudoscalar.

The fact that the kernel of the linearized PDE need not transform under the scalar action of $\mathbf{E}(n)$ seems to have first been observed by Sattinger [27].

Remark 1.4. (a) Our reduction simultaneously removes the bounded variables and reduces from a system to a single equation. Mielke [23] and Haragus [9] have previously presented an alternative approach which removes the bounded variables but does not reduce to a single equation.
(b) The reduction does not preserve all the local dynamics but only the so-called *essential solutions* [1]. These are solutions that are bounded and small over the whole of space and time. The idea of using Liapunov-Schmidt reduction (as in this paper) to preserve essential solutions and not just time-independent solutions is already present in the above-mentioned work of [23, 9].
(c) The (nontruncated) reduced equation is a pseudodifferential equation in common with the reduced equations of [12, 23, 9].

In [20], attention was concentrated on the case $n = 1$. Using techniques similar to those in this paper, Theorem 1.3 was proved for specific examples. Moreover, in the special case $n = 1$ it was possible to achieve three goals simultaneously: (i) removal of the bounded spatial variables so that the reduced equation is posed on $\mathbb{R}$, (ii) reduction from a system to a single equation, and (iii) extraction of modulation equations (in a complex amplitude function A related to the underlying solution u via the ansatz $u = Ae^{iax} + \bar{A}e^{-iax}$, where $a > 0$ is the critical wavenumber). At present, the third goal is not possible when $n \geq 2$. Combining the results in this paper with those in [20], we have the following result.

Corollary 1.5 (Universal validity of the Ginzburg-Landau equation)**.** *Suppose an $\mathbf{E}(1)$-equivariant system of PDEs posed on $\mathbb{R} \times \Omega$ undergoes steady-state bifurcation (with nonzero critical wavenumber). Generically, there is a reduction (that preserves essential solutions) to a single scalar modulation equation posed on $\mathbb{R}$. When truncated, this equation is precisely the Ginzburg-Landau equation.*

Remark 1.6. In stating this corollary, we have adopted the point of view that justification of the Ginzburg-Landau equation means finding a rigorous reduction to an equation with terms (and derivatives) of all orders that yields the cubic order Ginzburg-Landau equation when truncated with respect to the standard weighting (or scaling). In this introduction, we have described the historical precedent for taking this viewpoint. More recently, the work of Iooss, Mielke and Demay [12] (who consider the steady Ginzburg-Landau equation) and the previously mentioned work of Mielke [23] and Haragus [9] fits into this framework.

There is a completely different point of view where the truncated Ginzburg-Landau equation (with $n = 1$) is justified in the sense that solutions of this equation and the underlying PDE are approximately the same over long but finite timescales; see [29] and the references therein. Such an approximation clearly does not preserve significant qualitative features (such as quasiperiodicity) of the solutions and does not address the convergence of the asymptotic expansion underlying the formal derivation of the Ginzburg-Landau equation and its solutions.

Solutions. Up to this point we have not addressed the issue of how our reduced equations might be useful in determining solutions to the underlying PDEs. In general, the rigorous determination of branches of solutions to problems with Euclidean symmetry remains an important and challenging question. There is, however, an immediate application of our results which we now describe. Dionne and Golubitsky [5] classify a certain class of spatially periodic solutions known as *axial planforms* that bifurcate simultaneously for scalar equations when $n = 2$. These include the well-known planforms such as rolls and simple hexagons, and also more exotic planforms such as anti-squares and super hexagons. Bosch-Vivancos *et al.* [2] classify the axial planforms that bifurcate simultaneously in the pseudoscalar case together with their branching type. (For example, rolls and simple hexagons are replaced by new planforms called anti-rolls and oriented hexagons. Whereas simple hexagons bifurcate transcritically, oriented hexagons undergo a pitchfork bifurcation.) It now follows immediately from Theorem 1.3 that the corresponding classification for any $\mathbf{E}(2)$-equivariant system of PDEs on $\mathbb{R}^2 \times \Omega$ is given by the classification in either the scalar or pseudoscalar case.

As far as spatially aperiodic solutions go, the only completely satisfactory approach is that of Kirchgässner [16] and Mielke [22]. Center manifold reduction in a spatially unbounded variable leads to an ODE for steady-state solutions that are small and bounded in space. In particular, many equilibrium solutions that are not spatially periodic can be derived in this way [12, 11]. Unfortunately, this elegant method is restricted to the case $n = 1$ and yields only solutions that are stationary or time-periodic.

The remainder of this paper is organized as follows. In Section 2, we describe the class of ‘physical’ actions of $\mathbf{E}(n)$ that we work with in this paper. In addition, we state the generalization of Theorem 1.3 for general n, whereby any steady-state bifurcation can be reduced to a ‘minimal’ representation of $\mathbf{E}(n)$.

The functional-analytic framework for the results in this paper is the subject of Section 3. Section 4 is concerned with the proof of Theorem 1.1. In particular, we define a space $\mathcal{S}$ of $\mathbf{E}(n)$-equivariant partial differential operators endowed with a ‘relative boundedness’ topology and study the generic properties of steady-state bifurcations within this topology. In Section 5, we prove Theorem 1.3, together with its generalization Theorem 2.2.

2. Actions of the Euclidean group

We consider systems of PDEs posed on $\mathbb{R}^n \times \Omega$, where $\Omega \subset \mathbb{R}^d$ is bounded. The PDEs are supposed to be equivariant with respect to an action of the Euclidean group on functions $u : \mathbb{R}^n \times \Omega \to \mathbb{R}^s$, where s is the size of the system of PDEs. We make the standing assumptions that

(i) The symmetries act on the domain variables $\mathbb{R}^n \times \Omega$ by acting in the standard way on $\mathbb{R}^n$ and trivially on Ω.
(ii) Translations act trivially on the range variables $\mathbb{R}^s$.

These assumptions are made precise in Definition 2.1 below.

The Euclidean group $\mathbf{E}(n)$ consists of rigid transformations or isometries in $\mathbb{R}^n$. If $\gamma \in \mathbf{E}(n)$ is an isometry, then there is an orthogonal matrix $A \in \mathbf{O}(n)$ and a translation $t \in \mathbf{T}(n) \cong \mathbb{R}^n$ such that $\gamma x = Ax + t$ for all $x \in \mathbb{R}^n$. Multiplication in $\mathbf{E}(n)$ is defined as follows: if $\gamma_i = (A_i, t_i)$, $i = 1, 2$, then $\gamma_2\gamma_1 = (A_2A_1, A_2t_1 + t_2)$. Then $\mathbf{T}(n)$ is a normal subgroup and $\mathbf{E}(n)$ is the semi-direct product $\mathbf{E}(n) = \mathbf{O}(n)\dot{+}\mathbf{T}(n)$.

2.1. Physical actions of $\mathbf{E}(n)$.

There are various ways that $\mathbf{E}(n)$ can act on functions $u : \mathbb{R}^n \times \Omega \to \mathbb{R}^s$ depending on the value of s. We restrict to a class of representations for which assumptions (i) and (ii) above are satisfied. This class includes the representations that are typically encountered in applications. More precisely, suppose that $\rho : \mathbf{O}(n) \to \mathrm{GL}(\mathbb{R}^s)$ is a representation of $\mathbf{O}(n)$ on $\mathbb{R}^s$ and let ρ_A denote the image of $A \in \mathbf{O}(n)$ under ρ. We denote the unbounded domain variables by x and the bounded domain variables by z, so that $(x, z) \in \mathbb{R}^n \times \Omega$.

Definition 2.1. A *physical action* of $\mathbf{E}(n)$ on functions $u : \mathbb{R}^n \times \Omega \to \mathbb{R}^s$ is an action that takes the form

$$(\gamma \cdot u)(x, z) = \rho_A \cdot u(\gamma^{-1}(x), z), \quad \text{for all} \quad \gamma = (A, t) \in \mathbf{E}(n),\ (x, z) \in \mathbb{R}^n \times \Omega.$$

The most commonly encountered actions ρ on the range $\mathbb{R}^s$ are as follows:

$\rho_A = I$ in reaction diffusion equations.
$s = n$ and $\rho_A = A$ in vector field PDEs such as the Navier-Stokes equations.

When $s = 1$ the only physical actions of $\mathbf{E}(n)$ are the *scalar* action $\rho_A = I$ and the *pseudoscalar* action $\rho_A = \det A$.

2.2. Minimal actions of $\mathbf{E}(n)$.

Let $\mathbf{O}(n)$ act on $\mathbb{R}^n$ in the standard way, and choose $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$. The isotropy subgroup of x_0 is a copy of $\mathbf{O}(n-1)$. Different choices of x_0 lead to conjugate copies of $\mathbf{O}(n-1)$ in $\mathbf{O}(n)$. According to the main results described in the introduction, steady-state bifurcation with $\mathbf{E}(n)$ symmetry is organized to a large extent by the irreducible representations of $\mathbf{O}(n-1)$. In this subsection, we describe the irreducible representations of $\mathbf{O}(n-1)$ supported by a physical representation of $\mathbf{E}(n)$. Conversely, it is useful to define the 'minimal' physical representation of $\mathbf{E}(n)$ that supports a given irreducible representation of $\mathbf{O}(n-1)$.

Suppose that we are given a physical action of $\mathbf{E}(n)$ on $\mathbb{R}^s$ determined by the representation ρ of $\mathbf{O}(n)$ on $\mathbb{R}^s$. The action ρ restricts to an action of $\mathbf{O}(n-1)$ on $\mathbb{R}^s$. Write $\mathbb{R}^s = V_1 \oplus \cdots \oplus V_\ell$, where $V_1, \ldots, V_\ell$ are $\mathbf{O}(n-1)$-irreducible subspaces. We say that $V_1, \ldots, V_\ell$ are the $\mathbf{O}(n-1)$-irreducible representations *supported* by the physical action of $\mathbf{E}(n)$. It turns out that the irreducible representation of $\mathbf{O}(n-1)$ mentioned in Theorem 1.1 is one of the V_j, $j = 1, \ldots, \ell$.

Conversely, given an $\mathbf{O}(n-1)$-irreducible representation V, it is possible to choose $s' \geq 1$ and an action ρ of $\mathbf{O}(n)$ on $\mathbb{R}^{s'}$ that restricts to the $\mathbf{O}(n-1)$-irreducible subspace V [3, Chapter III, Theorem 4.5]. Hence, there is a physical action of $\mathbf{E}(n)$ that supports V. The physical representation is *minimal with respect to* V if s' is as small as possible.

For example, when V is one-dimensional it is clear that the corresponding minimal representations of $\mathbf{E}(n)$ are precisely the scalar and pseudoscalar representations, the latter occurring only when $n \geq 2$. We have the following generalization of Theorem 1.3.

Theorem 2.2. *Let $n \geq 1$ and suppose that an $\mathbf{E}(n)$-equivariant system of PDEs posed on $\mathbb{R}^n \times \Omega$ undergoes steady-state bifurcation (with nonzero critical wavenumber) from a fully-symmetric equilibrium. Then, generically, there is a reduction (preserving essential solutions) to a system of equations posed on $\mathbb{R}^n$. The reduced system is equivariant under an action of $\mathbf{E}(n)$ that is minimal with respect to the representation of $\mathbf{O}(n-1)$ associated with the kernel of the linearized PDE.*

Remark 2.3. The size of the reduced system is given by the value of s' in the definition of minimal representation, irrespective of the size s of the underlying system.

When $n = 3$, there is (in addition to the scalar and pseudoscalar actions of $\mathbf{E}(3)$) a minimal action of $\mathbf{E}(3)$ for each two-dimensional irreducible representation of $\mathbf{O}(2)$. The standard representation of $\mathbf{O}(2)$ is contained in the standard action of $\mathbf{O}(3)$ ($s' = 3$), but in general we require $s' = 2\ell + 1$ in order to account for the ℓ-fold action of $\mathbf{O}(2)$ on $\mathbb{R}^2$. In particular, we have a countable infinity of minimal physical actions of $\mathbf{E}(3)$ with s' arbitrarily large. The situation for $n = 3$ is indicative of the general case $n \geq 3$.

Given a particular physical action of $\mathbf{E}(n)$ defined by the homomorphism $\rho : \mathbf{O}(n) \to \mathrm{GL}(\mathbb{R}^s)$, it is clear that s' is bounded by s and moreover that s' is the dimension of an $\mathbf{O}(n)$-irreducible subspace of $\mathbb{R}^s$. For example, if $\rho_A = \pm I_s$ for each A then generically $s' = 1$ and we reduce to the scalar and pseudoscalar case. In particular, reaction-diffusion equations reduce generically to scalar equations.

3. The functional-analytic framework

In this section, we lay out the functional-analytic framework used in this paper. The framework is somewhat technical: our basic function space consists of the Fourier transforms of bounded vector-valued Borel measures on $\mathbb{R}^n$, the measures taking values in some Banach space Z^s of functions $f : \Omega \to \mathbb{C}^s$ (subject to suitable reality conditions). We recall that $\Omega \subset \mathbb{R}^d$ represents the bounded variables in the problem. The technical (and notational) difficulties are alleviated to some extent by restricting to the case of no bounded variables. This is done in Subsection 3.1. In Subsection 3.2, we consider the linear operators (especially the partial differential operators) that commute with the action of $\mathbf{E}(n)$. In Subsection 3.3, we reintroduce the bounded variables Ω into the general framework.

Properties of the function space. As motivation, we describe briefly the desired properties of the function spaces considered in this paper.

The crucial property is the closed splitting property described in Proposition 3.2(b) below. As in Melbourne [20] we must overcome the well-known obstruction to reduction of Euclidean-symmetric problems presented by the continuity of the spectra of certain linear operators. Proposition 3.2(b) guarantees the existence of closed splittings even in the absence of spectral splittings, thus making possible the reduction in Corollary 5.8 and Subsection 5.2.

A second requirement is that pointwise multiplication of functions is a smooth operation—see Proposition 3.2(a) and Remark 5.1(a).

Two final (but somewhat contradictory) requirements are that the function space contains large enough classes of functions, Remark 3.3, yet is amenable to harmonic analysis so that linear operators commuting with translations are multiplication operators (see Subsection 3.2).

3.1. **Function space, no bounded variables.** Consider the Banach space $\mathcal{M}^1 = \mathcal{M}^1(\mathbb{R}^n)$ of complex-valued Borel measures on $\mathbb{R}^n$ (see for example [25]). Associated to each measure $\mu \in \mathcal{M}^1$ is the 'total variation' measure $|\mu|$ defined by $|\mu|(B) = \sup\sum_{j=1}^\infty |\mu(E_j)|$, where the supremum is taken over all countable partitions of the Borel set B. The positive measure $|\mu|$ is finite, and the norm of the complex measure μ is defined to be $\|\mu\| = |\mu|(\mathbb{R}^n)$. The space $\mathcal{M}^1$ is a Banach algebra under convolution of measures,

$$\|\mu \star \nu\| \le \|\mu\|\|\nu\|. \tag{3.1}$$

(The convolution $\mu \star \nu \in \mathcal{M}^1$ is defined by $(\mu \star \nu)(E) = \int \mu(E-k)d\nu(k)$.)

More generally, we consider the space $\mathcal{M}^s$ of $\mathbb{C}^s$-valued measures μ with components $\mu_1, \dots, \mu_s \in \mathcal{M}^1$. Define $\|\mu\| = \left(\|\mu_1\|^2 + \cdots + \|\mu_s\|^2\right)^{1/2}$. Then $\mathcal{M}^s$ is a Banach module over $\mathcal{M}^1$.

If $B \subset \mathbb{R}^n$ is a Borel set, then the subspace $\mathcal{M}^s(B)$ consists of those measures in $\mathcal{M}^s$ that are supported on B. We have the closed splitting

$$\mathcal{M}^s = \mathcal{M}^s(B) \oplus \mathcal{M}^s(\mathbb{R}^n - B). \tag{3.2}$$

Let $\mathcal{M}^s_c$ consist of the measures $\mu \in \mathcal{M}^s$ with compact support.

Proposition 3.1. *The subspace $\mathcal{M}^s_c$ is dense in $\mathcal{M}^s$.*

Proof. Let $\mu \in \mathcal{M}^s$ and define $\mu_m \in \mathcal{M}^s_c$, $\mu_m(E) = \mu|_{D_m}$, where D_m is the disk of radius m in $\mathbb{R}^n$. Suppose that $\mathbb{R}^n = \bigcup_{j=1}^\infty E_j$, where the E_j are disjoint Borel subsets. Then

$$\begin{aligned}\sum_{j=1}^\infty |(\mu - \mu_m)(E_j)| &= \sum_{j=1}^\infty |\mu(E_j) - \mu(E_j \cap D_m)| \\ &= \sum_{j=1}^\infty |\mu(E_j - D_m)| \le |\mu|(\mathbb{R}^n - D_m).\end{aligned}$$

It follows from the finiteness of $|\mu|$ that $|\mu|(\mathbb{R}^n - D_m) \to 0$ as $m \to \infty$. Hence, $\|\mu - \mu_m\| \to 0$ as required. □

For each $\mu \in \mathcal{M}^s(B)$, we define the Fourier-Stieltjes transform $\mathcal{F}\mu : \mathbb{R}^n \to \mathbb{C}^s$,

$$\mathcal{F}\mu(x) = \int_B e^{-ik\cdot x} d\mu(k);$$

see for example [15]. Define $\mathcal{X}^s(B)$ to be the *real*-valued functions $u : \mathbb{R}^n \to \mathbb{R}^s$ obtained in this way:

$$\mathcal{X}^s(B) = \{u : \mathbb{R}^n \to \mathbb{R}^s,\ u = \mathcal{F}\mu \text{ for some } \mu \in \mathcal{M}^s(B)\}$$

and write $\mathcal{X}^s = \mathcal{X}^s(\mathbb{R}^n)$. The Fourier transform converts convolution of measures into pointwise multiplication of functions, and so $\mathcal{X}^1$ is a (proper) subalgebra of the Banach algebra $C_{\text{unif}}(\mathbb{R}^n)$.

Since the Fourier transform is invertible, $\mathcal{M}^s$ and $\mathcal{X}^s$ are isomorphic as vector spaces, and we define a norm on $\mathcal{X}^s$ so as to obtain an isometric isomorphism. In other words, if $u \in \mathcal{X}^s$, there is a unique $\mu \in \mathcal{M}^s$ such that $\mathcal{F}\mu = u$. Set $\|u\| = \|\mu\|$. Via the isometric isomorphism, properties (3.1) and (3.2) become:

Proposition 3.2. (a) *$\mathcal{X}^s$ is a Banach module (under pointwise multiplication) over the Banach algebra $\mathcal{X}^1$: if $u \in \mathcal{X}^1$ and $v \in \mathcal{X}^s$, then $uv \in \mathcal{X}^s$ and $\|uv\| \leq \|u\|\|v\|$.*
(b) *If $B \subset \mathbb{R}^n$ is a Borel set, then $\mathcal{X}^s = \mathcal{X}^s(B) \oplus \mathcal{X}^s(\mathbb{R}^n - B)$.*

Remark 3.3. (a) The absolutely continuous measures (with respect to Lebesgue measure) in $\mathcal{M}^1$ can be identified with the L^1 functions. Hence $\mathcal{X}^1$ contains the real-valued functions in $\mathcal{F}L^1$ and is a proper but uniformly dense subspace of $C_0(\mathbb{R}^n)$.
(b) The closed subspace generated by the Dirac measures is isomorphic to $\ell^1(\mathbb{R}^n)$, and the corresponding subspace of $\mathcal{X}^1$ consists of spatially-quasiperiodic functions of the form $u(x) = \sum_{j=1}^{\infty} a_j e^{-ik_j \cdot x}$ for which $\sum_{j=1}^{\infty} |a_j| < \infty$.

Now, suppose that we are given a physical action of $\mathbf{E}(n)$ on the space of functions $u : \mathbb{R}^n \to \mathbb{R}^s$ as in Subsection 2.1. The action restricts to an action of $\mathbf{E}(n)$ on $\mathcal{X}^s$ and leads to an action (via the Fourier transform) on $\mathcal{M}^s$. We can assume that the action ρ of $\mathbf{O}(n)$ on $\mathbb{R}^s$ is orthogonal. By definition of the norm on $\mathcal{M}^s$, $\mathbf{O}(n)$ acts isometrically on $\mathcal{M}^s$.

Proposition 3.4. *Any physical action of $\mathbf{E}(n)$ defines an isometric action on $\mathcal{X}^s$. The corresponding action on $\mathcal{M}^s$ is given by $\mu \mapsto \gamma\mu$, where, for $\gamma = (A, t) \in \mathbf{E}(n)$,*

$$\gamma\mu(E) = \rho_A \int_E e^{ik\cdot t} d\mu(A^{-1}k).$$

Proof. First, we compute the action on measures. Write $u(x) = \int e^{-ik\cdot x} d\mu(k)$. Then

$$\begin{aligned}(\gamma u)(x) &= \rho_A u(\gamma^{-1}x) = \rho_A u(A^{-1}(x-t)) = \rho_A \int e^{-ik\cdot A^{-1}(x-t)} d\mu(k) \\ &= \int e^{-iAk\cdot(x-t)} d(\rho_A \mu(k)) = \int e^{-ik\cdot x} e^{ik\cdot t} d(\rho_A \mu(A^{-1}k)) = \int e^{-ik\cdot x} d(\gamma\mu)(k),\end{aligned}$$

where $\gamma\mu$ is the transformed measure

$$\gamma\mu(E) = \int_E e^{ik\cdot t} d(\rho_A \mu(A^{-1}k)) = \rho_A \int_E e^{ik\cdot t} d(\mu(A^{-1}k)).$$

In particular, $(\gamma\mu)_j(E) = \int_E e^{ik\cdot t} d(\rho_A\mu)_j(A^{-1}k)$, so that

$$|(\gamma\mu)_j(E)| \leq |(\rho_A\mu)_j|(A^{-1}E).$$

Hence, $\|(\gamma\mu)_j\| \leq \|(\rho_A\mu)_j\|$, and it follows that $\|\gamma\mu\| \leq \|\rho_A\mu\| = \|\mu\|$. We have proved that $\|\gamma\mu\| \leq \|\mu\|$ for all $\gamma \in \mathbf{E}(n)$, and hence $\|\gamma\mu\| = \|\mu\|$. Thus $\mathbf{E}(n)$ acts isometrically on $\mathcal{M}^s$ and hence on $\mathcal{X}^s$. ☐

Remark 3.5. The action of $\mathbf{E}(n)$ on $\mathcal{X}^s$ is not continuous, indeed most $\mathbf{SO}(n)$-orbits are discrete. (For example, let $u(x) = e^{ik_0\cdot x} + e^{-ik_0\cdot x}$ for some fixed $k_0 \in \mathbb{R}^n$. Then the relative topology on $\mathbf{SO}(n) \cdot u \subset \mathcal{X}^s$ is the discrete topology.) This fact will be of no consequence in the sequel.

A subspace of the form $\mathcal{X}^s(B)$ is $\mathbf{E}(n)$-invariant if and only if B is invariant under the action of $\mathbf{O}(n)$ on $\mathbb{R}^n$. If $J \subset [0,\infty)$, then $\mathcal{X}^s(J)$ is defined (with a slight ambiguity of notation) to be the invariant subspace $\mathcal{X}^s(B)$, where $B = \{k \in \mathbb{R}^n,\ |k| \in J\}$. In particular, $\mathcal{X}^s(1)$ consists of the Fourier transforms of the Borel measures supported on the unit sphere in $\mathbb{R}^n$.

3.2. **$\mathbf{E}(n)$-equivariant linear operators.** In this subsection, we investigate the structure of Euclidean-equivariant systems of linear operators on $\mathcal{X}^s$. Throughout the subsection, we write $\mathcal{X}$ instead of $\mathcal{X}^s$.

Translation equivariance. It is well-known that, on 'reasonable' spaces, linear operators that commute with translations are multiplication operators. Hence, an element $u(x) = \int e^{-ik\cdot x} d\mu(k)$ should be transformed into $Lu(x) = \int e^{-ik\cdot x} d\nu(k)$, where $d\nu = q\, d\mu$ for some fixed $q : \mathbb{R}^n \to L(\mathbb{C}^s)$; here $L(\mathbb{C}^s)$ is the space of $s \times s$ matrices. The entries of q lie in a suitable function space (which may be difficult to characterize, see [31, pp. 28–30]). Unfortunately, by this criterion $\mathcal{X}$ fails to be a reasonable space — there are linear operators on $\mathcal{X}$ that commute with translations yet are not multiplication operators.

Example 3.6. We have the Lebesgue decomposition $\mathcal{X} = \mathcal{X}_{\mathrm{ac}} \oplus \mathcal{X}_{\mathrm{sing}}$ (absolutely continuous measures and singular measures). Consider the bounded linear operator $L : \mathcal{X} \to \mathcal{X}$ that restricts to the identity on $\mathcal{X}_{\mathrm{ac}}$ and twice the identity on $\mathcal{X}_{\mathrm{sing}}$, that is, $Lu = u_{\mathrm{ac}} + 2u_{\mathrm{sing}}$ where $u = u_{\mathrm{ac}} + u_{\mathrm{sing}}$. This operator commutes with translations (indeed, with all elements of $\mathbf{E}(n)$) but is not a multiplication operator. A less trivial example is the unbounded operator $Lu = \Delta u_{\mathrm{ac}} + (1+\Delta)^2 u_{\mathrm{sing}}$.

Although it is not the case that any bounded linear operator on $\mathcal{X}$ that commutes with translations is a multiplication operator, we show in the appendix that this property becomes valid on restriction to certain subspaces:

(i) The subspace $\mathcal{X}_{\mathrm{ac}}$ consisting of Fourier transforms of absolutely continuous measures (L^1 functions).
(ii) The subspace $\mathcal{X}_{\mathrm{Dirac}}$ generated by the Fourier transforms of the Dirac measures.

Motivated by these considerations, we make the following definition.

Definition 3.7. An (unbounded) linear operator $L : \mathcal{X} \to \mathcal{X}$ is an $\mathbf{E}(n)$-*equivariant linear operator* if there is a measurable map $q : \mathbb{R}^n \to L(\mathbb{C}^s)$ (the *multiplier*) such that

1. $u(x) = \int e^{-ik\cdot x} d\mu(k)$ transforms under L to $Lu(x) = \int e^{-ik\cdot x} q_k d\mu(k)$.
2. L commutes with the action of $\mathbf{O}(n) \subset \mathbf{E}(n)$.

Remark 3.8. (a) In the theory of linear operators that commute with a finite-dimensional action of a compact Lie group, the terms 'commuting' and 'equivariant' are used interchangeably. Example 3.6 shows that the $\mathbf{E}(n)$-equivariant linear operators are a proper subset of the commuting linear operators.
(b) In applications, the entries of q are usually polynomials, in which case we say that L is an $\mathbf{E}(n)$-equivariant linear *partial differential* operator. However, C^∞ multipliers arise in our reduction procedure later in the paper.
(c) An $\mathbf{E}(n)$-equivariant linear operator $L : \mathcal{X} \to \mathcal{X}$ is bounded if and only if the entries of q are bounded. For each $k \in \mathbb{R}^n$, q_k is an element of $L(\mathbb{C}^s)$ and we can define the operator norm $|q_k|$. Then L is bounded if and only if $\sup_k |q_k| < \infty$, in which case $\|L\| = \sup_k |q_k|$.

We say that $q : \mathbb{R}^n \to L(\mathbb{C}^s)$ is *locally bounded* if q is bounded on each bounded subset of $\mathbb{R}^n$.

Proposition 3.9. *Suppose that L is an $\mathbf{E}(n)$-equivariant linear operator with measurable, locally bounded multiplier q. Then L is densely defined and closable.*

Proof. The domain of L contains $\mathcal{FM}_c$ (Fourier transforms of compactly supported measures), which is dense by Proposition 3.1. Next, we prove that the linear operator $\hat{L}$ induced on $\mathcal{M}$ is closable. Suppose that $\{\mu_n\}$ is a sequence in the domain of $\hat{L}$ and $\mu_n \to 0$, $\hat{L}\mu_n \to \nu \in \mathcal{M}$. It is sufficient to prove that $(\hat{L}\mu_n)(E) \to 0$ for any bounded subset $E \subset \mathbb{R}^n$. But $|\hat{L}\mu_n(E)| = |\int_E q\, d\mu_n| \le \sup_{k\in E} |q_k| \|\mu_n\| \to 0$. □

Under the hypotheses of the proposition, L extends to a closed operator (which we also denote by L) with domain $\mathcal{X}[L]$ which is a Banach space in the graph norm $\|u\|_L = \|u\| + \|Lu\|$. With respect to this norm, $L : \mathcal{X}[L] \to \mathcal{X}$ is a bounded linear operator.

$\mathbf{O}(n)$*-equivariance.*

Proposition 3.10. *Suppose that $q : \mathbb{R}^n \to L(\mathbb{C}^s)$ is measurable. Then q is the multiplier for an $\mathbf{E}(n)$-equivariant linear operator $L : \mathcal{X} \to \mathcal{X}$ (defined as in condition 1 of Definition 3.7) if and only if*

$$q_{Ak} = \rho_A q_k \rho_A^{-1}, \quad A \in \mathbf{O}(n), \qquad\qquad q_{-k} = \overline{q}_k. \tag{3.3}$$

Proof. Acting first by L and then by $A \in \mathbf{O}(n)$ on u yields

$$\begin{aligned}(\rho_A Lu)(A^{-1}x) = \int e^{-ik\cdot A^{-1}x}(\rho_A q_k)d\mu(k) &= \int e^{-iAk\cdot x}(\rho_A q_k)d\mu(k)\\ &= \int e^{-ik\cdot x}(\rho_A q_{A^{-1}k})d\mu(A^{-1}k).\end{aligned}$$

On the other hand, acting first by A yields

$$\int e^{-ik\cdot A^{-1}x}\rho_A d\mu(k) = \int e^{-ik\cdot x}\rho_A d\mu(A^{-1}k),$$

and applying L yields

$$\int e^{-ik\cdot x}(q_k\rho_A)d\mu(A^{-1}k).$$

This establishes the first condition in (3.3), and the second condition is the reality condition. □

Since $\mathbf{O}(n)$ acts transitively on vectors in $\mathbb{R}^n$ of the same norm, the multiplier $q : \mathbb{R}^n \to L(\mathbb{C}^s)$ is determined by its values on vectors of the form $k = (a, 0, \dots, 0)$, $a \ge 0$. In other words, q is determined by the matrices $Q_a = q_{(a,0,\dots,0)}$, $a \in [0, \infty)$. The isotropy subgroup of vectors $(a, 0 \dots, 0) \in \mathbb{R}^n$ is a copy of $\mathbf{O}(n-1)$, and conditions (3.3) reduce to the conditions

$$\begin{gathered} Q_0 \text{ is } \mathbf{O}(n)\text{-equivariant}, \qquad Q_a \text{ is } \mathbf{O}(n-1)\text{-equivariant},\\ Q_{-a} = \rho_{-I} Q_a \rho_{-I} = \overline{Q}_a. \end{gathered} \tag{3.4}$$

Hence, there is a one-to-one-correspondence between equivariant linear operators $L : X \to X$ and *symbols* $Q : \mathbb{R} \to L(\mathbb{C}^s)$ satisfying conditions (3.4).

Remark 3.11. (a) When $s = 1$, the symbol is an even function $Q : \mathbb{R} \to \mathbb{R}$ and we can write $Q_a = P(a^2)$. It is easily seen that $L = P(-\Delta)$, where $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ is the Laplacian.
(b) If $\rho_{-I} = \pm I_s$, then the matrices Q_a have real entries. This is the case, for example, for the Navier-Stokes equations. In general, we can choose coordinates on $\mathbb{R}^s$ so that ρ_{-I} is diagonal (with diagonal entries ± 1). For the Boussinesq equations, $\rho_{-I} = \mathrm{diag}(-1, -1, -1, 1) = (-I_3) \oplus I_1$ and Q_a need not have real entries.

In general, write $\rho_{-I} = I_{s_1} \oplus (-I_{s_2})$ where $s_1 + s_2 = s$. Then Q_a has a corresponding 2×2 block structure (independent of a) where the diagonal blocks have real even entries and the off-diagonal blocks have purely imaginary odd entries.

Proposition 3.12. *Let $Q : \mathbb{R} \to L(\mathbb{C}^s)$ and define $\widetilde{Q}_a = B^{-1} Q_a B$, where $B = I_{s_1} \oplus (iI_{s_2})$. Then, $\widetilde{Q}_a$ has real entries and Q satisfies conditions* (3.4) *if and only if*

$$\widetilde{Q}_0 \textit{ is } \mathbf{O}(n)\textit{-equivariant}, \qquad \widetilde{Q}_a \textit{ is } \mathbf{O}(n-1)\textit{-equivariant},$$
$$\widetilde{Q}_{-a} = \rho_{-I} \widetilde{Q}_a \rho_{-I}.$$

Hence, there is a one-to-one correspondence between maps $\widetilde{Q} : \mathbb{R} \to L(\mathbb{R}^s)$ satisfying these conditions and symbols $Q : \mathbb{R} \to L(\mathbb{C}^s)$ satisfying conditions (3.4).

Proof. It is an easy calculation to verify that $\widetilde{Q}_a$ has real entries. Since the action of $\mathbf{O}(n)$ commutes with ρ_{-I}, the matrix B is $\mathbf{O}(n)$-equivariant, and it follows that $\widetilde{Q}_a$ is $\mathbf{O}(n-1)$ or $\mathbf{O}(n)$-equivariant if and only if Q_a is. Finally, $\overline{B} = \rho_{-I} B$, and a simple calculation shows that the last condition in the proposition is equivalent to the last condition in (3.4). ☐

The point of this reformulation is that for each fixed a, $\widetilde{Q}_a$ is a *general* real linear map equivariant under the action of the compact Lie group $\mathbf{O}(n)$ or $\mathbf{O}(n-1)$. Hence, we can apply the methods of equivariant bifurcation theory [8] to these maps. This is important in Section 4 (particularly, Theorem 4.10).

3.3. **Function space in the presence of bounded variables.** In this subsection we generalize to $\mathbf{E}(n)$-equivariant operators on domains of the form $\mathbb{R}^n \times \Omega$ where Ω is a bounded subset of $\mathbb{R}^d$.

The functional-analytic prerequisites required for this generalization are not completely standard. An elementary treatment of the integration of vector-valued functions with respect to a positive measure (Bochner integral) can be found in Lang [18]. The generalization to integration of operator-valued functions with respect to a (bounded) vector-valued measure is straightforward. Details on convolutions of vector-valued measures can be found, for example, in Dinculeanu [6, §24]. All measures that we consider are bounded (finite).

We assume that the functional analysis has been worked out for systems of equations posed on Ω alone. Let Z be a suitable Banach space of functions from Ω to $\mathbb{C}$. We suppose that Z is closed under pointwise multiplication and complex conjugation: if $f, g \in Z$, then $fg, \bar{f} \in Z$, where $(fg)(z) = f(z)g(z)$ and $\bar{f}(z) = \overline{f(z)}$. For simplicity, we assume that Z is a Banach algebra under pointwise multiplication.

Let μ be a vector-valued Borel measure with values in Z ($\mu : \mathcal{B} \to Z$ is countably additive and $\mu(\emptyset) = 0$). As for complex measures, there is an associated positive 'total variation' measure $|\mu|$ defined by $|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)|$, where the supremum is taken over all partitions of E. It is no longer the case that $|\mu|$ is automatically finite. We restrict attention to the space $\mathcal{M}_Z$ of *bounded* measures μ for which $\|\mu\| = |\mu|(\mathbb{R}^n) < \infty$. Then $\mathcal{M}_Z$ is a Banach space [18] and is moreover a Banach algebra under convolution of measures [6]. Similarly, we define the Banach module $\mathcal{M}_Z^s$.

Let $\mathcal{X}_Z^s$ consist of the Fourier transforms $u(x) = \int e^{-ik\cdot x} d\mu(k)$ of measures in $\mathcal{M}_Z^s$ subject to the usual reality condition and with norm derived from the norm on $\mathcal{M}_Z^s$. The basic properties of $\mathcal{X}^s$, Proposition 3.2 and so on, generalize immediately to $\mathcal{X}_Z^s$.

Remark 3.13. The definition of the norm on $\mathcal{X}_Z^s$ relies on the fact that the Fourier transform operator on $\mathcal{M}_Z^s$ is one-to-one. This is immediate from the injectivity for complex-valued measures together with the fact that bounded linear functionals separate points of Z^s.

Remark 3.14. Given $\mu \in \mathcal{M}_Z^s$ and $z \in \Omega$, we define $\mu_z \in \mathcal{M}^s$, $\mu_z(E) = \mu(E, z)$. At the level of functions, $u \in \mathcal{X}_Z^s$ naturally determines a function $\tilde{u} : \mathbb{R}^n \times \Omega \to \mathbb{R}^s$, $\tilde{u}(x, z) = \int e^{-ik\cdot x} d\mu_z(k)$.

Symmetry. Next, suppose that $\rho : \mathbf{O}(n) \to \mathrm{GL}(\mathbb{R}^s)$ defines a physical action of $\mathbf{E}(n)$ on functions $\tilde{u} : \mathbb{R}^n \times \Omega \to \mathbb{R}^s$, $(\gamma \cdot \tilde{u})(x, z) = \rho_A \tilde{u}(\gamma^{-1}x, z)$. As usual, we can assume that the action of $\mathbf{O}(n)$ on $\mathbb{R}^s$ is orthogonal. A computation as in the proof of Proposition 3.4 shows that the action of $\mathbf{E}(n)$ on $\mathcal{X}_Z^s$ is isometric. Moreover, if $u(x) = \int e^{-ik\cdot x} d\mu(k)$ and $\gamma = (A, t) \in \mathbf{E}(n)$, then

$$(\gamma \cdot u)(x) = \rho_A \int e^{ik\cdot t} d\mu(A^{-1}k).$$

Bounded linear operators. During the remainder of this subsection, we write $\mathcal{X} = \mathcal{X}_Z^s$ and $\mathcal{M} = \mathcal{M}_Z^s$. Let μ be a fixed measure in $\mathcal{M}$ and let $B(Z^s)$ denote the space of bounded linear operators on Z^s. Suppose that $q : \mathbb{R}^n \to B(Z^s)$ is simple: $q = \sum_{i=1}^r g_i \chi_{A_i}$, where $g_i \in B(Z^s)$ and $A_i \in \mathcal{B}$. We define the integral $\int q\, d\mu \in Z^s$ by the formula $\int q\, d\mu = \sum_{i=1}^r g_i(\mu(A_i))$.

Recall that $q : \mathbb{R}^n \to B(Z^s)$ is strongly measurable if q is the pointwise limit μ-almost everywhere of simple functions ϕ_n. The ϕ_n can be modified so that $|\phi_n| \le 2|q|$ almost everywhere. The function q is said to be integrable if $\int |q|\, d|\mu| < \infty$. In this case, the integral $\int q\, d\mu = \lim \int \phi_n d\mu \in Z^s$ is well-defined and $|\int q\, d\mu| \le \int |q|\, d|\mu|$.

We say that a map $q : \mathbb{R}^n \to B(Z^s)$ is *completely measurable* if q is strongly measurable with respect to every measure $\mu \in \mathcal{M}$. Suppose in addition that q is bounded, $\|q\|_\infty = \sup_k |q_k| < \infty$, and that q satisfies conditions (3.3). Then q induces a bounded linear operator $\hat{L} : \mathcal{M} \to \mathcal{M}$, where $\hat{L}\mu(E) = \int_E q\, d\mu$, and a bounded $\mathbf{E}(n)$-*equivariant* linear operator $L : \mathcal{X} \to \mathcal{X}$ with $\|L\| = \|\hat{L}\| = \|q\|_\infty$. If $u(x) = \int e^{-ik\cdot x} d\mu(k)$ is an element of $\mathcal{X}$, we write $Lu(x) = \int e^{-ik\cdot x} q_k d\mu(k)$.

Remark 3.15. (a) Consider the closed subspace $\mathcal{X}_{\mathrm{Dirac}}$ generated by the Dirac measures. A typical element of $\mathcal{X}_{\mathrm{Dirac}}$ has the form $u(x) = \sum e^{-ik\cdot x} f_k$ (countable sum), where $f_k \in Z^s$, $f_{-k} = \bar{f}_k$ and $\|u\| = \sum |f_k| < \infty$. In this case, $Lu(x) = \sum e^{-ik\cdot x} q_k(f_k)$.
(b) Next, we let $\mathcal{X}_{L^1}$ denote the closed subspace of $\mathcal{X}$ consisting of functions $u(x) = \int e^{-ik\cdot x} f_k dk$, where $f \in L^1(\mathbb{R}^n, Z^s)$ (f is strongly measurable and norm integrable with respect to Lebesgue measure). When Z is finite-dimensional, $\mathcal{X}_{L^1}$

coincides with the subspace $\mathcal{X}_{\rm ac}$ of absolutely continuous measures, but in general $\mathcal{X}_{L^1}$ is a proper subspace of $\mathcal{X}_{\rm ac}$. We have $Lu(x) = \int e^{-ik\cdot x} q_k(f_k)dk$.

Unbounded linear operators. Let D be a fixed subspace of Z^s. Suppose that the values of q are (unbounded) operators on Z^s and that the domain of q_k contains D for each $k \in \mathbb{R}^n$. Suppose moreover that there is a norm $|\,|_D$ on D such that $(D, |\,|_D)$ is a Banach space and $q_k : D \to Z^s$ is a bounded linear operator for each k. As before, we require that the map $q : \mathbb{R}^n \to B(D, Z^s)$ is completely measurable.

Let $\mathcal{M}_D \subset \mathcal{M}$ denote the Banach space of Borel measures with values in D. If q is bounded, then q induces a bounded linear operator $\hat{L} : \mathcal{M}_D \to \mathcal{M}$, $\hat{L}\mu(E) = \int_E q\, d\mu$. More generally, if q is locally bounded, then q induces an unbounded operator $\hat{L} : \mathcal{M}_D \to \mathcal{M}$, whose domain includes the dense subspace $\mathcal{M}_{D,c}$ of compactly supported measures. We can also regard $\hat{L}$ as an unbounded operator $\hat{L} : \mathcal{M} \to \mathcal{M}$. The corresponding unbounded linear operator $L : \mathcal{X} \to \mathcal{X}$ is by definition $\mathbf{E}(n)$-equivariant. More precisely, $L : \mathcal{X} \to \mathcal{X}$ is an $\mathbf{E}(n)$-*equivariant linear operator* if the multiplier $q : \mathbb{R}^n \to L(Z^s)$ can be viewed as a completely measurable, locally bounded map $q : \mathbb{R}^n \to B(D, Z^s)$ (for some subspace $D \subset Z^s$) satisfying conditions (3.3).

Let $Q : \mathbb{R} \to L(Z^s)$ be the corresponding symbol. We say that L is an equivariant *partial differential* linear operator if Q is polynomial, that is, $Q_a = a^d M_d + \cdots + aM_1 + M_0$ where $M_0, \ldots, M_d \in L(Z^s)$. (Since the operators M_j are not bounded, polynomial does not imply analytic.)

In this definition, the operators M_j need not be partial differential operators on Z^s. However, nothing in the sequel is changed if we insist that the M_j are partial differential operators.

Remark 3.16. A necessary condition for $L : \mathcal{X} \to \mathcal{X}$ to be densely-defined and closable is that each operator $q_k \in L(Z^s)$ is densely-defined and closable. Conversely, it seems reasonable to conjecture that if the subspace $D \subset Z^s$ is dense and each operator q_k is closable, then $L : \mathcal{X} \to \mathcal{X}$ is densely-defined and closable. Four special cases of this conjecture are easily verified. It is clear that L is densely-defined and closable on restriction to $\mathcal{X}_{\rm Dirac}$. The same is true on restriction to $\mathcal{X}_{L^1}$ (since L^1-convergence implies that there is a subsequence that converges almost everywhere). The closability of L holds on the whole of $\mathcal{X}$ if q is a simple function. Finally, if $Z = \mathbb{C}$ we have Proposition 3.9.

4. Steady-state bifurcation with nonzero critical wavenumber

A basic result of equivariant bifurcation theory (Golubitsky *et al.* [8, Proposition XIII, 3.2]) catalogues steady-state bifurcation with a compact symmetry group Γ in terms of the absolutely irreducible representations of Γ. Suppose that Γ is a compact Lie group acting on a finite-dimensional vector space V and that $L : V \to V$ is a linear map commuting with the action of Γ. If L has a zero eigenvalue, then generically $\ker L$ is an absolutely irreducible representation of Γ (that is, the only commuting linear maps are real scalar multiples of the identity). Moreover, it is generically the case that $\ker L$ is the entire center subspace of L. Ruelle [26, Theorem 1.2 and p. 140] proves an infinite-dimensional version of this result under the technical assumption that 0 is an isolated eigenvalue of finite multiplicity.

When Γ is not compact, zero eigenvalues are typically neither isolated nor of finite multiplicity. In particular, the results of [8, 26] do not apply to problems with (noncompact) Euclidean symmetry. The aim of this section is to obtain the required generalization. In particular, we prove Theorem 1.1. Throughout, we assume a physical action of $\mathbf{E}(n)$ on $\mathcal{X} = \mathcal{X}_Z^s$. In view of Example 3.6, we modify the definition of absolute irreducibility. We recall that the $\mathbf{E}(n)$-equivariant linear operators are the subclass of commuting linear operators that are defined by a completely measurable, locally bounded multiplier $q : \mathbb{R}^n \to B(D, Z^s)$.

Definition 4.1. An $\mathbf{E}(n)$-invariant subspace Y of $\mathcal{X}$ is *absolutely irreducible* if every bounded $\mathbf{E}(n)$-equivariant linear operator on $\mathcal{X}$ leaving Y invariant restricts to a real multiple of the identity on Y.

On the subspace $\mathcal{X}_{\mathrm{Dirac}}$, the definitions of absolute irreducibility in terms of commuting linear operators and equivariant linear operators coincide:

Proposition 4.2. *Suppose that $\mathcal{X}_{\mathrm{Dirac}} = Y \oplus \widetilde{Y}$, where Y, $\widetilde{Y}$ are closed $\mathbf{E}(n)$-invariant subspaces. Then Y is absolutely irreducible if and only if every bounded commuting linear operator on Y is a real scalar multiple of the identity.*

This proposition is not required in the sequel, and the proof is deferred to the appendix.

In Subsection 4.1, we define a suitable notion of genericity for $\mathbf{E}(n)$-equivariant linear partial differential operators. In Subsection 4.2, we analyze the structure of the spectrum of an $\mathbf{E}(n)$-equivariant linear operator, leading to a classification of the four types of local bifurcation (depending on whether there is a steady-state or Hopf bifurcation with zero or nonzero critical wavenumber). The specific case of steady-state bifurcation with nonzero wavenumber is considered in detail in Subsections 4.3 and 4.4. In particular, in Subsection 4.4, we prove that the kernel is generically absolutely irreducible.

4.1. **Relatively bounded perturbations and genericity.** The results in this paper rely on various genericity assumptions on the linearizations of $\mathbf{E}(n)$-equivariant systems equations. We require genericity within the class of linear partial differential operators. Hence, we must consider unbounded perturbations while avoiding *singular* perturbations. For example, suppose that $Z^s = \mathbb{C}$, so Remark 3.11(a) implies that L is a polynomial function of the Laplacian, $L = b_d\Delta^d + \cdots + b_1\Delta + b_0$, where d is a positive integer and $b_d \neq 0$. Then $L + \epsilon\Delta^p$ is an allowable 'small' perturbation if and only if $p \leq d$. This is made precise through the notion of relative boundedness.

Suppose that L, M are (unbounded) operators defined on $\mathcal{X}$. Following Kato [14, p.190], we say that the operator M is *relatively bounded with respect to* L if the domain of M includes the domain of L and there exist constants $a, b \geq 0$ such that

$$\|Mu\| \leq a\|u\| + b\|Lu\|, \text{ for all } u \text{ in the domain of } L.$$

When L is bounded, there are no relatively bounded perturbations other than the bounded ones. This degenerate situation is excluded by the following definition. We say that an $\mathbf{E}(n)$-equivariant linear operator M is *second order* if the symbol Q takes the form $Q_a = a^2 H$ where $H : Z^s \to Z^s$ is a bounded linear operator.

Definition 4.3. An equivariant linear operator $L : \mathcal{X} \to \mathcal{X}$ is *nondegenerate* if every second order equivariant linear operator is relatively bounded with respect to L.

A closed, densely defined linear operator $L : \mathcal{X} \to \mathcal{X}$ is *sectorial* if there is an open sector S of the complex plane with vertex $\beta_0 \in \mathbb{R}$ such that S is symmetric with respect to the real axis, S contains the half-line $(-\infty, \beta_0)$ and the angle opening of the sector is less than π radians, see Figure 1(a), and moreover, $\|(L - \sigma I)^{-1}\| = O(1/|\sigma - \beta_0|)$ as $|\sigma| \to \infty$ for $\sigma \in \mathbb{C} - S$. (This is equivalent to the definition in Henry [10] with L replaced by $-L$. The sector can be chosen so that $\operatorname{spec} L \subset S$.) The sectorial operators are precisely those operators that generate analytic semigroups (semiflows) on $\mathcal{X}$ [10]. Define $\beta^*(L) = \sup \Re(\operatorname{spec} L) = \sup\{\Re(\sigma),\ \sigma \in \operatorname{spec} L\}$. (By convention, $\sup(\emptyset) = -\infty$.) Then $\beta^*(L) < \infty$ for L sectorial, and this supremum is attained provided the spectrum is nonempty.

Proposition 4.4. *Suppose that $L : \mathcal{X} \to \mathcal{X}$ is sectorial. Then the sectoriality of L is preserved under small relatively bounded perturbations. That is, if M is relatively bounded and ϵ is small enough, then $L + \epsilon M$ is sectorial. Moreover, $\beta^*(L + \epsilon M)$ is upper-semicontinuous at $\epsilon = 0$.*

Proof. We give the proof, which is standard, for completeness. Since L is sectorial, there are constants $K, R > 0$ such that $\|(L-\sigma I)^{-1}\| \le K/|\sigma-\beta_0|$ for all $\sigma \in \mathbb{C}-S$, $|\sigma - \beta_0| \ge R$. Let M be relatively bounded. We can suppose without loss that the constants a and b in the definition of relatively bounded satisfy $a, b \le 1$.

It is immediate that $L + \epsilon M$ is densely defined for all ϵ. Moreover $L + \epsilon M$ is closed if $|\epsilon| < 1$ [14, Theorem IV,1.1]. Let $u \in D_L \subset D_M \subset \mathcal{X}$. We compute that

$$\begin{aligned}\|M(L - \sigma I)^{-1}u\| &\le \|(L - \sigma I)^{-1}u\| + \|L(L - \sigma I)^{-1}u\| \\ &\le \|(L - \sigma I)^{-1}\|\|u\| + \|u\| + |\sigma|\|(L - \sigma I)^{-1}\|\|u\|.\end{aligned}$$

Hence

$$\|M(L - \sigma I)^{-1}\| \le 1 + K/|\sigma - \beta_0| + |\sigma|K/|\sigma - \beta_0| \le K',$$

where $K' = 1 + K/R + K + |\beta_0|K/R$. In particular, $I + \epsilon M(L - \sigma I)$ is invertible if $|\epsilon| < 1/K'$, and

$$\begin{aligned}\|(L + \epsilon M - \sigma I)^{-1}\| &\le \|(L - \sigma I)^{-1}\|\|(I + \epsilon M(L - \sigma I)^{-1})^{-1}\| \\ &\le K(1 - \epsilon K')^{-1}/|\sigma - \beta_0|.\end{aligned}$$

This shows that $L + \epsilon M$ is sectorial. Moreover, if S is the sector for L, then we have shown that the spectrum of $L + \epsilon M$ is contained in the union of S and the ball of radius R, center β_0. We have also reproved the well-known statement that any compact subset Γ of the resolvent of L lies also in the resolvent of $L + \epsilon M$ for ϵ small ('upper-semicontinuity' of the spectrum [14]): replace K' by $\sup_{\sigma\in\Gamma} \{1 + (1 + |\sigma|)\|(L - \sigma I)^{-1}\|\}$.

Now choose $R > |\beta^*(L) - \beta_0|$ and for each $\delta > 0$ consider the compact set

$$\Gamma_\delta = \{\sigma \in \mathbb{C},\ |\sigma - \beta_0| \le R;\ \Re(\sigma) \ge \beta^*(L) + \delta \text{ or } \sigma \notin S\};$$

see Figure 1(b). For ϵ small, we have $\operatorname{spec}(L+\epsilon M) \subset S-\Gamma$ and hence $\beta^*(L+\epsilon M) < \beta^*(L) + \delta$. ∎

Remark 4.5. Suppose that $L : \mathcal{X} \to \mathcal{X}$ is an $\mathbf{E}(n)$-equivariant linear operator on $\mathcal{X}$ with symbol $Q : \mathbb{R} \to L(Z^s)$. If L is sectorial on $\mathcal{X}$, then each operator Q_a is sectorial on Z^s. In particular, the conclusions of Proposition 4.4 are valid for Q_a.

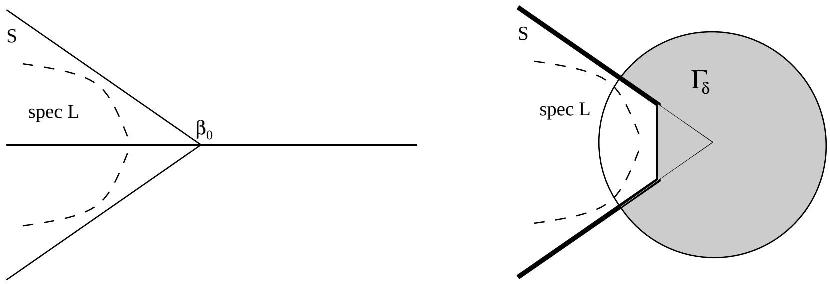

FIGURE 1. (a) A sector S with vertex β_0 containing the spectrum of the sectorial operator L. (b) Under a relatively bounded perturbation, the spectrum of L is constrained to lie in $S - \Gamma_\delta$.

Let $\mathcal{S}$ denote the set of sectorial, nondegenerate $\mathbf{E}(n)$-equivariant linear partial differential operators on $\mathcal{X}$. We define a topology on $\mathcal{S}$ in terms of the relative bounded perturbations. For each $L \in \mathcal{S}$ and for each $r > 0$, define $B_r(L)$ to consist of all operators in $\mathcal{S}$ of the form $L + M$, where $D_M \supset D_L$ and there is an $r' < r$ such that $\|Mu\| \le r'(\|u\| + \|Lu\|)$ for all $u \in D_L$.

Proposition 4.6. *The family $\{B_r(L)\}$ is a basis for a topology on $\mathcal{S}$.*

Proof. Let $L_0 \in B_{r_1}(L_1)$. It is sufficient to prove that $B_r(L_0) \subset B_{r_1}(L_1)$ for some $r > 0$. By definition of $B_{r_1}(L_1)$, there is an $r_1' < r_1$ such that $\|(L_0 - L_1)x\| \le r_1'(\|x\| + \|L_1 x\|)$ for all $x \in D_{L_1}$. Choose $r \in (0, (r_1 - r_1')/(1 + r_1'))$. If $L \in B_r(L_0)$, we compute that

$$\begin{aligned}\|(L - L_1)x\| &\le \|(L - L_0)x\| + \|(L_0 - L_1)x\| \le (r + r_1')\|x\| + r\|L_0 x\| + r_1'\|L_1 x\| \\ &\le (r + r_1')\|x\| + r(\|(L_0 - L_1)x\| + \|L_1 x\|) + r_1'\|L_1 x\| \\ &\le (r + r_1' + r r_1')(\|x\| + \|L_1 x\|),\end{aligned}$$

where $r + r_1' + r r_1' < r_1$. Hence $L \in B_{r_1}(L_1)$ and so $B_r(L_0) \subset B_{r_1}(L_1)$, as required. □

From now on, we assume that $\mathcal{S}$ is endowed with the topology in Proposition 4.6. It follows from Proposition 4.4 that $\beta^*(L) = \sup \Re(\operatorname{spec} L)$ is an uppersemicontinuous function on $\mathcal{S}$. A *generic property* is (for our purposes) one that holds on a open and dense subset of $\mathcal{S}$.

4.2. **Classification of bifurcations.** Recall that $\beta^* = \sup \Re(\operatorname{spec} L) < \infty$. Let $\beta(a) = \sup \Re(\operatorname{spec} Q_a) \le \beta^*$. Condition (3.4) implies that Q_{-a} is similar to Q_a and has the same spectrum. Hence β is even.

Proposition 4.7. *If $L \in \mathcal{S}$, then $\beta(a) \to -\infty$ as $a \to \pm\infty$.*

Proof. Let $s = \limsup_{a\to\infty} \beta(a) \le \beta^* < \infty$. We show that $s = -\infty$. Suppose for contradiction that s is finite. Since L is nondegenerate, the perturbation with symbol $\epsilon a^2 I_{Z^s}$ is relatively bounded. Hence for ϵ small enough, the perturbed operator is sectorial with $\beta_\epsilon(a) = \beta(a) + \epsilon a^2$. Hence $\beta_\epsilon^* \ge \limsup \beta_\epsilon(a) = +\infty$, which is a contradiction. □

Remark 4.8. It is clear that $\bigcup_a \operatorname{spec} Q_a \subset \operatorname{spec} L$. The reverse inclusion fails in certain pathological cases (even if the closure is taken on the left-hand side). For example, suppose that there are no bounded variables ($Z = \mathbb{C}$) and let $s = 2$, $n \geq 1$. Consider the nondegenerate linear partial differential operator $L(u, v) = (-\Delta^2 u - 2u - \Delta v, -\Delta u - 2v)$, which is Euclidean equivariant when the action of $\mathbf{O}(n)$ on $\mathbb{R}^2$ is trivial. Then $Q_a = \begin{pmatrix} -a^4 - 2 & a^2 \\ a^2 & -2 \end{pmatrix}$. A calculation shows that L is sectorial and that the spectrum of L lies inside the real axis. However,

$$\bigcup_a \operatorname{spec} Q_a = (-\infty, -2], \qquad \operatorname{spec} L = (-\infty, -2] \cup \{-1\}.$$

We note that $\det(Q_a - \sigma) = (\sigma + 1)a^4 + \sigma^2 + 4\sigma + 4$. It follows that the matrix family $(Q_a - \sigma)^{-1}$ is uniformly bounded eventually in a if and only if $\sigma \neq -1$. This accounts for the fact that $-1 \in \operatorname{spec} L$ (see Proposition A.3).

On the other hand, it is clear that this situation is nongeneric: after applying the relatively bounded perturbation $\epsilon(0, \Delta v)$ say, the determinant of $Q_a - \sigma$ is a polynomial of degree six in a, so that $(Q_a - \sigma)^{-1} \to 0$ as $a \to \infty$ for each $\sigma \in \mathbb{C}$. In particular, $(Q_a - \sigma I)^{-1}$ is always eventually uniformly bounded, and it follows that $\operatorname{spec} L = \bigcup_a \operatorname{spec} Q_a$ for the perturbed operator. Further details can be found in the appendix, where it is shown in particular that generically $\operatorname{spec} L = \bigcup_a \operatorname{spec} Q_a$ when there are no bounded variables.

In view of Remark 4.8 and the associated results in the appendix, the following assumption is justified.

(H1) $\operatorname{spec} L = \bigcup \operatorname{spec} Q_a$.

The origin in $\mathcal{X}$ is a sink under L if $\beta^* < 0$ and is unstable if $\beta^* > 0$. Now we restrict to the critical case $\beta^* = 0$. It follows from assumption (H1), the evenness of β and Proposition 4.7 that $\beta(a) = 0$ for some $a \geq 0$. Define the *critical wavenumber* $a_0 = \inf\{a \geq 0,\ \beta(a) = 0\}$.

(H2) The map $a \mapsto Q_a$ is analytic at a_0.

Here, we mean analyticity in the sense of Kato [14, p. 375], 'holomorphic of type (A)'. That is, there is a neighborhood I of a_0 such that the Q_a, $a \in I$, are closed on a common domain $D \subset Z^s$ and, for each fixed $f \in D$, the map $a \to Q_a f$ is analytic on I.

Since Q is polynomial in a, assumption (H2) is automatic when $Z = \mathbb{C}$ and for infinite dimensional Z if the operators Q_a are bounded. In general, write $Q_a = M_0 + M_1(a - a_0) + \cdots + M_p(a - a_0)^p$. If each M_j is relatively bounded with respect to M_0, then assumption (H2) is valid.

Assumption (H2) implies that the operators Q_a are mutually relatively bounded for a near a_0. Hence, β is upper-semicontinuous at a_0 and it follows that $\beta(a_0) = 0$.

(H3) 0 is isolated in $\Re(\operatorname{spec} Q_{a_0})$.

Hence, there is a spectral splitting $Z^s = E_c \oplus E_s$ where E_c, E_s are closed subspaces invariant under Q_{a_0} (the center subspace and stable subspace respectively) such that $\Re(\operatorname{spec} Q_{a_0}|_{E_c}) = 0$ and $\Re(\operatorname{spec} Q_{a_0}|_{E_s}) < 0$. By conditions (3.4), E_c and E_s are $\mathbf{O}(n)$-invariant (resp. $\mathbf{O}(n-1)$-invariant) when $a_0 = 0$ (resp. $a_0 > 0$).

(H4) $\dim E_c < \infty$.

There are four quite distinct situations (local bifurcations) to consider. After rescalings, these are as follows:

(i) $a_0 = 0$, $0 \in \operatorname{spec} Q_0$, (steady-state bifurcation with zero wavenumber).
(ii) $a_0 = 1$, $0 \in \operatorname{spec} Q_1$, (steady-state bifurcation with nonzero wavenumber).

(iii) $a_0 = 0$, $\pm i \in \operatorname{spec} Q_0$, (Hopf bifurcation with zero wavenumber).
(iv) $a_0 = 1$, $\pm i \in \operatorname{spec} Q_1$, (Hopf bifurcation with nonzero wavenumber).

We focus on case (ii) in this paper, but it should be fairly clear how to proceed with the other cases. Case (i) occurs in the nonlinear heat equation, and case (ii) is relevant for the Boussinesq equations and for Ginzburg-Landau theory in general.

4.3. Steady-state bifurcation problems with nonzero critical wavenumber.

Definition 4.9. Let $L : \mathcal{X} \to \mathcal{X}$ be a sectorial nondegenerate $\mathbf{E}(n)$-equivariant partial differential operator ($L \in \mathcal{S}$) and assume that L satisfies hypotheses (H1–H4). In case (ii) above ($a_0 = 1$, $0 \in \operatorname{spec} Q_1$) we say that L is a *steady-state bifurcation problem with nonzero wavenumber*.

Theorem 4.10. *Let L be a steady-state bifurcation problem with nonzero wavenumber. Generically,*

(a) $\ker Q_1$ *is an* $\mathbf{O}(n-1)$*-irreducible subspace of* Z^s *and* $E_c = \ker Q_1$.
(b) $\beta(a)$ *is an isolated eigenvalue of* Q_a *for* a *close to* 1*, and the corresponding eigenspace is* $\mathbf{O}(n-1)$*-irreducible and isomorphic to* $\ker Q_1$.
(c) $\beta(a) < 0$ *for* $a \neq \pm 1$.
(d) β *is analytic at* 1*, and* 1 *is a nondegenerate critical point.*

Proof. We begin by establishing part (a). Write $E_c = E_0 \oplus F$, where E_0 is the zero generalized eigenspace of the finite-dimensional matrix $Q_1|_{E_c}$ and F is the sum of the remaining generalized eigenspaces in E_c. Since Q_1 commutes with the action of $\mathbf{O}(n-1)$ on Z^s, the subspaces $\ker Q_1$, E_0, E_c and so on are $\mathbf{O}(n-1)$-invariant. Perturbing by second order (hence relatively bounded) partial differential operators as necessary, we may arrange that $Q_1|_{E_0}$ is semisimple. Hence, without loss of generality, we may suppose that $E_0 = \ker Q_1$.

Choose an $\mathbf{O}(n-1)$-invariant inner product on the finite-dimensional subspace $\ker Q_1$. Write $\ker Q_1 = V \oplus V^\perp$, where V is $\mathbf{O}(n-1)$-irreducible and $V^\perp$ is an invariant complement. Let $W = V^\perp \oplus F \oplus E_s$, so that $Z^s = V \oplus W$ is a closed $\mathbf{O}(n-1)$-invariant splitting.

Now consider second order perturbations of the form $L + \epsilon M$, $\epsilon > 0$, where M has symbol R defined by $R_a|_V = 0$, $R_a|_W = -a^2 I_W$. (Note that R_0 is required to be $\mathbf{O}(n)$-equivariant.) Using this perturbation, we can arrange that $E_c = \ker Q_1 = V$. In fact, the irreducibility of $\ker Q_1 = E_c$ is generic, openness following from upper-semicontinuity of the function $\max \Re \operatorname{spec} Q_1|_{E_s}$ (see Remark 4.5).

Suppose then that $V = \ker Q_1 = E_c$ is $\mathbf{O}(n-1)$-irreducible (and $Z^s = \ker Q_1 \oplus E_s$). The zero eigenvalues of Q_1 constitute a finite system of eigenvalues in the terminology of [14]. Since 0 is isolated in the spectrum of Q_1, we can apply [14, Chapter VII, Theorem 1.7]: for a near 1, there is an analytic family of operators T_a similar to the operators Q_a (the similarity transformation also is analytic in a), such that

(i) T_a preserves V and E_s, and
(ii) $\operatorname{spec} T_a|_{E_s}$ is uniformly bounded into the left-half-plane.

It follows that $\beta(a) = \max \Re(\operatorname{spec} Q_a|_V)$ for a close to 1.

The symmetry properties of Q_a are inherited by T_a, and by Proposition 3.12 we can regard $T_a|_V$ as a *real* matrix equivariant under the action of $\mathbf{O}(n-1)$. The irreducible representations of $\mathbf{O}(n-1)$ are absolutely irreducible and we can write

$T_a|_V = \sigma(a)I_V$, where $\sigma(a) \in \mathbb{R}$. It is immediate from the above considerations that $\beta = \sigma$ is analytic at $a = 1$. In addition, part (b) is proved.

Next we prove part (c). It is convenient to momentarily drop the normalizing assumptions $\beta^* = 0$ and $a_0 = 1$. Let $a_0 \geq 0$ be least such that $\beta(a_0) = \beta^* < \infty$, and suppose that $a_0 > 0$. Consider the relatively bounded perturbation with symbol $R_a = -\epsilon a^2 I_{Z^s}$, where $\epsilon > 0$. Then β is transformed under perturbation to $\beta^\epsilon(a) = \beta(a) - \epsilon a^2$.

It follows from the definition that a_0 is an isolated critical point. (Otherwise the analytic map β is constant on a neighborhood of a_0, contradicting the minimality of a_0.) There is a neighborhood $(a_0 - \delta, a_0 + \delta)$ on which β is analytic and such that a_0 is the unique critical point in this neighborhood. After perturbation, generically a_0 becomes a nondegenerate critical point a_0^ϵ (completing the proof of part (d)).

By construction, $\beta(a)$ is bounded away from β^* for $0 \leq a \leq a_0 - \delta$. This property is clearly preserved for β^ϵ provided $\epsilon > 0$ is small enough. Since $-\epsilon a^2$ is strictly decreasing, we have $\beta^\epsilon(a) < \beta^\epsilon(a_0^\epsilon)$ for all $a > a_0 + \delta$. This completes the proof of part (c). □

4.4. **Absolute irreducibility of** $\ker L$**.** Recall from Subsection 3.1 that $\mathcal{X}(1)$ is the subspace of $\mathcal{X}$ arising from the Borel measures supported on the unit sphere S^{n-1} in $\mathbb{R}^n$. We show that there is a natural correspondence between $\mathbf{E}(n)$-invariant subspaces U of $\mathcal{X}(1)$ and $\mathbf{O}(n-1)$-invariant subspaces V (complexified) of Z^s.

This correspondence is most easily seen in the context of measures defined by L^1 functions on S^{n-1}. Since $\mathbf{O}(n)$ acts transitively on S^{n-1}, given $k \in S^{n-1}$, we can choose $A_k \in \mathbf{O}(n)$ such that $A_k(1, 0, \ldots, 0) = k$. Let V be an $\mathbf{O}(n-1)$-invariant subspace of Z^s and define $V^k = \rho_{A_k} V$ (the $\mathbf{O}(n-1)$-invariance of V guarantees that V_k is independent of the choice of A_k). Let U consist of those functions $u \in \mathcal{X}(1)$ of the form $u(x) = \int_{S^{n-1}} e^{-ik\cdot x} f(k)dk$, where $f : S^{n-1} \to Z^s$ is in L^1 and satisfies $f(k) \in V^k$ almost everywhere. Then it is readily checked that U is $\mathbf{E}(n)$-invariant and is absolutely irreducible if and only if V is irreducible under the complexified action of $\mathbf{O}(n-1)$.

For transforms of general Borel measures, we choose A_k as before but ensuring that there is a piecewise smooth (measurable and bounded suffices) dependence on $k \in S^{n-1}$. Define $B(k) = \rho_{A_k}^{-1}$ for $k \in S^{n-1}$. Let $U \subset \mathcal{X}(1)$ consist of the Fourier transforms of measures $\mu \in \mathcal{M}$ supported on S^{n-1} that satisfy

$$\int_E B(k) d\mu(k) \in V \quad \text{for all Borel sets } E \subset S^{n-1}.$$

Proposition 4.11. *Suppose that V is a closed $\mathbf{O}(n-1)$-invariant subspace of Z^s and that there is a closed splitting $Z^s = V \oplus \widetilde{V}$. Then the corresponding subspace $U \subset \mathcal{X}$ is well-defined (independent of the choice of A_k) and is $\mathbf{E}(n)$-invariant. Moreover, U is absolutely irreducible if and only if V is $\mathbf{O}(n-1)$-irreducible.*

Proof. We define a symbol $Q : \mathbb{R} \to L(Z^s)$ as follows: $Q_a = I_{\widetilde{V}}$ for $a = \pm 1$ and $Q_a = I_{Z^s}$ elsewhere. Let $L : \mathcal{X} \to \mathcal{X}$, $\hat{L} : \mathcal{M} \to \mathcal{M}$ be the corresponding bounded $\mathbf{E}(n)$-equivariant linear operators. We claim that $\ker L = U$. It follows immediately from the claim that U is well-defined and $\mathbf{E}(n)$-invariant.

We now verify the claim. It is clear that $\ker L \subset \mathcal{X}(1)$, so we restrict attention to measures $\mu \in \mathcal{M}$ supported in S^{n-1}. Let $q : \mathbb{R}^n \to L(Z^s)$ be the multiplier associated to Q. Then $\mu \in \ker \hat{L}$ iff $\int_E q_k d\mu(k) = 0$ for all Borel sets $E \subset S^{n-1}$.

But

$$q_k = q_{A_k(1,0,\dots,0)} = \rho_{A_k} q_{(1,0,\dots,0)} \rho_{A_k}^{-1} = \rho_{A_k} Q_1 B(k).$$

Since ρ_{A_k} is invertible for all k, we deduce that $\mu \in \ker \hat{L}$ iff $Q_1 \int_E B(k) d\mu(k) = 0$ for all E. In other words, $\int_E B(k) d\mu(k) \in \ker Q_1 = V$. It follows by definition of U that $\ker L = U$.

It remains to prove the statement about the absolute irreducibility of U. Suppose that $L : \mathcal{X} \to \mathcal{X}$ is an $\mathbf{E}(n)$-equivariant linear operator leaving U invariant. Let Q be the symbol of L with corresponding multiplier q. If $u \in U$ then u is the transform of a measure μ supported in S^{n-1} satisfying $\int_E B(k) d\mu(k) \in V$ for all Borel sets $E \in S^{n-1}$. The fact that L leaves U invariant translates into the condition that $\int_E B(k) q_k d\mu(k) \in V$ for all E. Using the definition of B and conditions (3.3), we have $Q_1 \int_E B(k) d\mu(k) \in V$. Now the vectors $\int_E B(k) q_k d\mu(k)$ span V (consider $\mu = B_{k_0}^{-1} v \delta_{k_0}$, where δ_{k_0} is a Dirac measure supported at some fixed k_0 and v is any element of V). It follows that $Q_1(V) \subset V$, and that the action of L on U determines and is determined by the action of Q_1 on V. But $Q_1|_V$ is forced to be a real multiple of the identity if and only if V is $\mathbf{O}(n-1)$-irreducible. In this case, $L|_U$ is the same real multiple of the identity and U is absolutely irreducible. □

Theorem 1.1 now follows easily. More precisely, we have

Corollary 4.12. *Suppose that* $L : \mathcal{X} \to \mathcal{X}$ *is a steady-state bifurcation problem with nonzero wavenumber. Generically,* $\ker L \subset \mathcal{X}(1)$*, in which case* $\ker L$ *is the* $\mathbf{E}(n)$*-invariant subspace of* $\mathcal{X}(1)$ *corresponding to the* $\mathbf{O}(n-1)$*-invariant subspace* $\ker Q_1 \subset Z^s$*. Moreover, generically* $\ker L$ *is* $\mathbf{E}(n)$*-absolutely irreducible. Furthermore, the center subspace of* $L|_{\mathcal{X}(1)}$ *is well-defined and coincides with* $\ker L$*. Finally,* $\Re(\operatorname{spec} L|_{\mathcal{X}(a)}) < 0$ *for* $a \neq \pm 1$*.*

Proof. It follows from Theorem 4.10(c) that $\ker L \subset \mathcal{X}(1)$. The argument used in the proof of Proposition 4.11 shows that $\ker L$ is the $\mathbf{E}(n)$-invariant subspace corresponding to $\ker Q_1$. Absolute irreducibility follows from Theorem 4.10(a) and Proposition 4.11. The remaining statements are immediate from Theorem 4.10. □

Since the spectrum of L is continuous, there is no splitting into center and stable subspaces. Corollary 4.12 includes the statement that, in the best possible sense, the 'center subspace' of L generically coincides with the kernel of L.

5. Reduction of nonlinear $\mathbf{E}(n)$-equivariant PDEs

In this section, we consider $\mathbf{E}(n)$-equivariant systems of nonlinear PDEs undergoing steady-state bifurcation (with nonzero wavenumber) from a trivial solution.

Let $\mathcal{X}_Z^s$ denote the Banach space of functions $u : \mathbb{R}^n \times \Omega \to \mathbb{R}^s$ introduced in Section 3. As usual, we suppose that we are given a physical action of $\mathbf{E}(n)$, $u(x,z) \mapsto \rho_A u(\gamma^{-1}x, z)$, where $\gamma = (A,t) \in \mathbf{E}(n)$ and ρ is an orthogonal action of $\mathbf{O}(n)$ on $\mathbb{R}^s$ (Subsection 2.1). Let $L : \mathcal{X}_Z^s \to \mathcal{X}_Z^s$ be a sectorial nondegenerate $\mathbf{E}(n)$-equivariant partial differential operator ($L \in \mathcal{S}$, Subsection 4.1) and suppose moreover that L is a steady-state bifurcation problem with nonzero wavenumber $a_0 = 1$ (Definition 4.9).

Consider the system of nonlinear PDEs

$$du/dt = \Phi(u,\lambda) = Lu + N(u,\lambda), \tag{5.1}$$

where $\lambda \in \mathbb{R}$ is a bifurcation parameter and $N : \mathcal{X}_Z^s \times \mathbb{R} \to \mathcal{X}_Z^s$ is a nonlinear partial differential operator. We assume that $N(\gamma \cdot u, \lambda) = \gamma \cdot N(u, \lambda)$ for all $\gamma \in \mathbf{E}(n)$ and that $N(0, \lambda) \equiv 0$. Thus the PDE (5.1) is $\mathbf{E}(n)$-equivariant and possesses a trivial solution $u \equiv 0$. When the domain $\mathcal{X}_Z^s[L]$ of L is endowed with the graph norm, $L : \mathcal{X}_Z^s[L] \to \mathcal{X}_Z^s$ is bounded. We suppose that $N : \mathcal{X}_Z^s[L] \times \mathbb{R} \to \mathcal{X}_Z^s$ is analytic and $(dN)_{0,0} = 0$. In particular, $\Phi : \mathcal{X}_Z^s[L] \times \mathbb{R} \to \mathcal{X}_Z^s$ is analytic and $(d\Phi)_{0,0} = L$.

Provided L and N satisfy the conditions described above, we say that $\Phi : \mathcal{X}_Z^s \times \mathbb{R} \to \mathcal{X}_Z^s$ is a *nonlinear steady-state bifurcation problem with nonzero wavenumber.*

Remark 5.1. (a) It follows from Proposition 3.2(a) that the assumption of analyticity of Φ is satisfied by semilinear operators, where N consists of nonlinear terms in u and λ possessing derivatives of lower (or equal) order than the highest order derivatives in L.
(b) Our assumptions do not guarantee that equation (5.1) defines a local dynamical system on $\mathcal{X}$. The required technical hypothesis (analyticity of N on the domain of a fractional power of L) can be found in Henry [10] but is not required for any of our results.

Set $\beta^*(\lambda) = \sup \Re(\operatorname{spec}(d\Phi)_{0,\lambda})$. It is easy to see that generically, $d\beta^*/d\lambda \neq 0$. To fix ideas let us suppose that $d\beta^*/d\lambda > 0$. Under the additional technical hypothesis, the principle of linear stability [10, Theorem 5.1.1, Corollary 5.1.6] states that the trivial solution $u = 0$ is asymptotically stable if $\lambda < 0$ and unstable if $\lambda > 0$.

By Corollary 4.12, we have generically that $\ker L$ is absolutely irreducible. Corresponding to $\ker L$ is the $\mathbf{O}(n-1)$-irreducible subspace $\ker Q_1 \subset Z^s$. Choose a minimal physical representation of $\mathbf{E}(n)$ on $\mathcal{X}^{s'}$ corresponding to the irreducible representation of $\mathbf{O}(n-1)$. (In other words, $\mathbb{C}^{s'}$ contains an $\mathbf{O}(n-1)$-irreducible subspace isomorphic to $\ker Q_1$, and s' is as small as possible.)

Now, we state precisely an 'equilibrium' version of Theorems 1.3 and 2.2.

Theorem 5.2. *Suppose that $\Phi : \mathcal{X}_Z^s \times \mathbb{R} \to \mathcal{X}_Z^s$ is a nonlinear steady-state bifurcation problem with nonzero wavenumber. Generically, equilibrium solutions of the partial differential equation $du/dt = \Phi(u, \lambda) = Lu + N(u, \lambda)$ on $\mathcal{X}_Z^s$ are locally (near $(u, \lambda) = (0, 0)$) in one-to-one correspondence with equilibrium solutions of a reduced pseudodifferential equation $dv/dt = \Phi'(v, \lambda) = L'v + N'(v, \lambda)$ on $\mathcal{X}^{s'}$ (near $(v, \lambda) = (0, 0)$).*

Remark 5.3. (a) Except for the fact that the reduced nonlinear operator Φ' is not a partial differential operator, Φ' enjoys all the properties of the original bifurcation problem Φ. In particular, the linear and nonlinear operators $L', N' : \mathcal{X}^{s'} \times \mathbb{R} \to \mathcal{X}^{s'}$ are analytic when viewed as operators $L', N' : \mathcal{X}^{s'}[L'] \times \mathbb{R} \to \mathcal{X}^{s'}$.
(b) The linear pseudodifferential operator L' has the same structure as an $\mathbf{E}(n)$-equivariant partial differential operator on $\mathcal{X}_Z^{s'}$ except that the symbol is smooth (C^∞) rather than polynomial; see Proposition 5.9 and Remark 5.10 below. Analogous statements apply to the nonlinear operator N'. Such considerations are not necessary for the proof of Theorem 5.2, and we refer to [20] for more details.

To consider nonequilibrium solutions, we replace the space $\mathcal{X}_Z^s$ by the space $\mathcal{X}_{Z,t}^s$ consisting of functions $u : \mathbb{R}^{n+1} \times \Omega \to \mathbb{R}^s$ where the additional unbounded domain variable is time. Define $L_t = -d/dt + L$ and $\Phi_t = -d/dt + \Phi = L_t + N$. If we define $\mathcal{X}_{Z,t}^s[L_t]$ using the graph norm, then the operators $L_t, N : \mathcal{X}_{Z,t}^s[L_t] \to \mathcal{X}_{Z,t}^s$ are analytic.

Theorem 5.4. *Suppose that $\Phi : \mathcal{X}_Z^s \times \mathbb{R} \to \mathcal{X}_Z^s$ is a nonlinear steady-state bifurcation problem with nonzero wavenumber. Generically, zeroes of the partial differential operator $\Phi_t(u,\lambda) = -du/dt + Lu + N(u,\lambda)$ on $\mathcal{X}_{Z,t}^s$ are locally (near $(u,\lambda) = (0,0)$) in one-to-one correspondence with zeroes of a reduced pseudodifferential operator $\Phi_t'(v,\lambda) = -dv/dt + L'v + N_t'(v,\lambda)$ on $\mathcal{X}_t^{s'}$ (near $(v,\lambda) = (0,0)$).*

Remark 5.5. (a) The zeroes of Φ_t in Theorem 5.4 are of course solutions to the original partial differential equation $du/dt = \Phi(u,\lambda)$. However, the local nature of the reduction means that only 'small' zeroes/solutions are preserved. (Roughly speaking, small means of small norm in Z^s for all $(x,t) \in \mathbb{R}^{n+1}$. This interpretation would be more precise if we were working with a 'sup norm' rather than the norm inherited from Fourier space.) Such solutions are called *essential solutions* [1].
(b) The reduced linear terms L' in Theorems 5.2 and 5.4 are identical. The new nonlinearity N_t' is analytic on $\mathcal{X}_t^{s'}[L_t']$ but contains time derivatives as well as spatial derivatives.

The remainder of this section is concerned with the proof of Theorems 5.2 and 5.4. In Subsection 5.1 we reduce the linear operator. The full nonlinear operator is reduced in Subsection 5.2.

5.1. **Reduction of the linear operator.** We continue to suppose that we are given a physical action of $\mathbf{E}(n)$ on $\mathcal{X}_Z^s$. As usual $L : \mathcal{X}_Z^s \to \mathcal{X}_Z^s$ is a steady-state bifurcation problem with nonzero wavenumber and polynomial symbol $Q : \mathbb{R} \to L(Z^s)$. We have the following consequence of Theorem 4.10.

Proposition 5.6. *Generically, there is an open interval $J = (\delta_1, \delta_2) \subset (0,\infty)$, $0 < \delta_1 < 1 < \delta_2$, such that*

(i) $\beta(\delta_1) = \beta(\delta_2)$.
(ii) *β is analytic on J with unique critical point $a = 1$.*
(iii) *$\ker(Q_a - \beta(a)I_s)$ is an analytic family of isomorphic $\mathbf{O}(n-1)$-irreducible subspaces, $a \in J$.*
(iv) *$\beta(a)$ is an isolated eigenvalue of Q_a, $a \in J$.*

(See Figure 2.)

Let $V_a = \ker(Q_a - \beta(a)I_s)$ for $a \in J$. Since $\beta(a)$ is an isolated eigenvalue of Q_a and V_a is the corresponding generalized eigenspace, there is a closed $\mathbf{O}(n-1)$-invariant splitting $Z^s = V_a \oplus \widetilde{V}_a$. As in the proof of Theorem 4.10, there is an analytic family of $\mathbf{O}(n-1)$-equivariant isomorphisms on Z^s that transform the splitting $Z^s = V_a \oplus \widetilde{V}_a$ into the constant splitting $Z^s = V_1 \oplus \widetilde{V}_1$ for $a \in J$. There is an obvious decomposition of $\mathcal{X}_Z^s(J)$ corresponding to the constant splitting, and we use the family of isomorphisms to obtain a closed, L-invariant, $\mathbf{E}(n)$-equivariant splitting $\mathcal{X}_Z^s(J) = \mathcal{U} \oplus \widetilde{\mathcal{U}}$, where $\operatorname{spec} L|_{\mathcal{U}}$ consists of the real eigenvalues $\beta(a)$, $a \in J$. Then we have the closed, L-invariant, $\mathbf{E}(n)$-invariant splitting for the full space $\mathcal{X}_Z^s$:

$$\mathcal{X}_Z^s = \mathcal{U} \oplus \mathcal{Y}, \quad \text{where} \quad \mathcal{Y} = \widetilde{\mathcal{U}} \oplus \mathcal{X}_Z^s([0,\infty) - J).$$

Proposition 5.7. *Generically, we can choose $J \subset (0,\infty)$ so that properties (i)–(iv) in Proposition 5.6 are satisfied and in addition*

(v) $\operatorname{spec}(L|_{\mathcal{U}}) = [r, 0]$ *and* $\Re(\operatorname{spec} L|_{\mathcal{Y}}) \le r$,

where $r < 0$ is the common value of $\beta(\delta_j)$, $j = 1,2$ (see Figure 2).

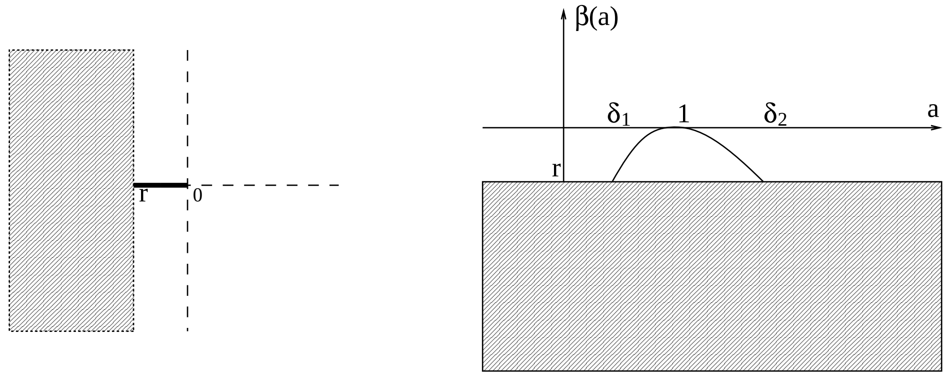

FIGURE 2. (a) The part of the spectrum of L with real part greater than r and (b) the part of the graph of β with values greater than r in a generic steady-state bifurcation with nonzero wavenumber

The splitting $\mathcal{X}_Z^s = \mathcal{U} \oplus \mathcal{Y}$ induces a splitting $\mathcal{X}_Z^s[L] = \mathcal{U}[L] \oplus \mathcal{Y}[L]$, and the bounded linear operator $L : \mathcal{X}_Z^s[L] \to \mathcal{X}_Z^s$ respects the splittings.

Corollary 5.8. *Under the hypotheses and conclusions of Proposition 5.7, the linear operator $L|_{\mathcal{Y}[L]} : \mathcal{Y}[L] \to \mathcal{Y}$ is an isomorphism.*

Proof. By Proposition 5.7, the spectrum of this bounded linear operator does not contain zero, so there is a bounded inverse defined on the whole of $\mathcal{Y}$. □

We can interpret Proposition 5.7 as saying that $\mathcal{U}$ corresponds to the critical eigenspaces of L. Now, $\mathcal{U}$ can be identified with a subspace of a minimal physical $\mathbf{E}(n)$-representation $\mathcal{X}^{s'}$ as defined in Subsection 2.2. Recall that s' is chosen as small as possible so that there is an irreducible representation $\rho' : \mathbf{O}(n) \to \mathrm{GL}(\mathbb{R}^{s'})$ of $\mathbf{O}(n)$ that contains the $\mathbf{O}(n-1)$-irreducible representation V_1. We can then identify the subspaces V_a of Z^s with the (constant) $\mathbf{O}(n-1)$-irreducible subspace $V' = V_1$ of $\mathbb{C}^{s'}$. We have the corresponding closed $\mathbf{E}(n)$-invariant splitting $\mathcal{X}^{s'}(J) = \mathcal{U}' \oplus \widetilde{\mathcal{U}}'$.

Proposition 5.9. *Suppose $L : \mathcal{X}_Z^s \to \mathcal{X}_Z^s$ satisfies the generic consequences (i)–(v) in Propositions 5.6 and 5.7. Then there is an $\mathbf{E}(n)$-equivariant linear operator $L' : \mathcal{X}^{s'} \to \mathcal{X}^{s'}$ with smooth symbol such that (i)–(v) are also satisfied by L', and, in addition, the restricted operators $L|_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$ and $L'|_{\mathcal{U}'} : \mathcal{U}' \to \mathcal{U}'$ are similar under an $\mathbf{E}(n)$-equivariant change of coordinates (with analytic symbol).*

Proof. Write $\mathbb{C}^s = V' \oplus \widetilde{V}'$ and define $Q' : J \to L(\mathbb{C}^s)$, $Q'_a = (\beta(a) I_{V'}) \oplus (-I_{\widetilde{V}'})$. Then Q' is analytic and can be extended to a smooth symbol $Q' : \mathbb{R} \to L(\mathbb{C}^s)$ by choosing a smooth extension $\beta' : \mathbb{R} \to \mathbb{R}$ of β. Provided we choose β' even with $\beta'(0) = -1$, the symbol Q' satisfies conditions (3.4). The corresponding $\mathbf{E}(n)$-equivariant linear operator $L' : \mathcal{X}^{s'} \to \mathcal{X}^{s'}$ is related to L as required. For a suitable choice of β' (it suffices that β' has critical points only at 0, ± 1) conditions (i)–(v) are valid. □

Remark 5.10. There is some flexibility in the choice of the reduced linear operator L', especially with respect to the boundedness of L'. In particular we can always choose L' to be bounded. As defined in the proof of Proposition 5.9, L' is degenerate. By modifying the definition of Q'_a on $\widetilde{V}'$, and modifying the extension β' of β, we can arrange that L' is nondegenerate and sectorial (with $\operatorname{spec} L' = (-\infty, 0]$), so that L' satisfies all of the properties of an operator in $\mathcal{S}$ with the exception that L' is not a partial differential operator.

5.2. **Reduction of the nonlinear operator.** In this subsection, we prove Theorem 5.4, thereby proving also Theorem 5.2.

The spectrum of L_t is the Cartesian product $\Re(\operatorname{spec} L) \times i\mathbb{R}$. It follows from Theorem 4.10(a) that generically $\ker L_t = \ker L$. Indeed, $\ker(L_t - \sigma I) = \ker(L - \sigma I)$ for all $\sigma \in \mathbb{C}$ with $\Re\sigma \geq r$, where $r < 0$ is as defined in Proposition 5.7. Hence, there is generically a closed L_t-invariant $\mathbf{E}(n)$-invariant splitting $\mathcal{X}^s_{Z,t} = \mathcal{U} \oplus \mathcal{Y}$, where $\mathcal{U}$ consists of the critical eigenspaces ($\Re\sigma$ close to 0) such that the conclusions of Corollary 5.8 and Proposition 5.9 are valid now for L_t. Since $\mathbf{E}(n)$ acts trivially on the time variable, the modified subspace $\mathcal{U}$ remains $\mathbf{E}(n)$-invariant.

Define the corresponding splitting $\mathcal{X}^s_{Z,t}[L_t] = \mathcal{U}[L_t] \oplus \mathcal{Y}[L_t]$ and the complementary projections $I - E : \mathcal{X}^s_{Z,t} \to \mathcal{U}$, $E : \mathcal{X}^s_{Z,t} \to \mathcal{Y}$. Now we proceed as in the standard Liapunov-Schmidt reduction (see for example [7]). By the implicit function theorem and Corollary 5.8, the equation $E\Phi_t(v + w, \lambda) = 0$ allows us to solve locally for $w = W(v, \lambda)$, where $W : \mathcal{U}[L_t] \times \mathbb{R} \to \mathcal{Y}[L_t]$ satisfies $W(0, 0) = 0$. Substituting into $(I - E)\Phi_t(v + w, \lambda) = 0$ yields the reduced operator $\phi : \mathcal{U}[L_t] \times \mathbb{R} \to \mathcal{U}$,

$$\phi(v, \lambda) = (I - E)\Phi_t(v + W(v, \lambda), \lambda).$$

Locally, zeroes of ϕ are in one-to-one correspondence with zeroes of Φ_t. Moreover, $(d\phi)_{0,0} = L_t|_{\mathcal{U}[L_t]}$.

The next step is to lift ϕ back to an operator on $\mathcal{X}^{s'}_t$ while preserving the local correspondence of zeroes. As in Subsection 5.1, we can embed $\mathcal{U}$ inside $\mathcal{X}^{s'}_t$. Then a crude but sufficient approach is to extend ϕ to the nonlinear operator $\Phi'_t(v+w, \lambda) = \phi(v, \lambda) \oplus L'_t|_{\mathcal{Y}[L_t]} w$, where L' is as in Proposition 5.9 and $L'_t = -d/dt + L'$. The zeroes of Φ'_t are identical to the zeroes of ϕ and hence are locally in one-to-one correspondence with the zeroes of Φ_t. Moreover, $(d\Phi'_t)_{0,0} = -d/dt + L'$, and the theorem is proved.

With extra effort, we can perform a 'reverse Liapunov-Schmidt reduction' as in Melbourne [20] to obtain a more natural reduced operator Φ'_t. Such refinements are not required for the results in this paper.

Appendix

First, we prove some auxiliary results concerning the structure of commuting linear operators (as opposed to equivariant linear operators) as promised in Subsection 3.2. The issue is to what extent a bounded linear operator commuting with translations is a multiplication operator. Throughout the appendix, we suppose that we are in the situation of no bounded variables, $Z = \mathbb{C}$. Write $\mathcal{X} = \mathcal{X}^s$.

Lemma A.1. *Suppose that $L : \mathcal{X} \to \mathcal{X}$ is a bounded linear operator that commutes with translations, and let $\hat{L}$ be the operator induced on $\mathcal{M}$. If $\mu \in \mathcal{M}$, then $\hat{L}\mu$ is supported on the support of μ.*

Proof. If $h \in L^\infty(\mu)$, then the measure $h\mu \in \mathcal{M}$ is defined by $h\mu(E) = \int_E h d\mu$ and satisfies $\|h\mu\| \leq \|h\|_\infty \|\mu\|$. In this notation, the condition that $\hat{L} : \mathcal{M} \to \mathcal{M}$ commutes with translations implies that $\hat{L}(e^{ik\cdot t}\mu) = e^{ik\cdot t}\hat{L}\mu$ for all $t \in \mathbb{R}^n$.

Suppose that $h : \mathbb{R}^n \to \mathbb{R}$ is C^∞ and periodic (in the sense that h can be regarded as an element of $C^\infty(T^n)$ for a suitable choice of torus T^n). In particular, the Fourier series of h converges uniformly on T^n. Hence, we can find sequences $a_j \in \mathbb{R}$, $t_j \in \mathbb{R}^n$ such that $\|h - \sum_{j=1}^N a_j e^{ik\cdot t_j}\|_\infty \to 0$ as $N \to \infty$. We compute that

$$\begin{aligned}\|\hat{L}(h\mu) - h\hat{L}(\mu)\| &\le \|\hat{L}(h\mu) - \sum_{j=1}^N a_j \hat{L}(e^{ik\cdot t_j}\mu)\| + \|\sum_{j=1}^N a_j e^{ik\cdot t_j}\hat{L}\mu - h\hat{L}\mu\| \\ &\le \|\hat{L}\|\|(h - \sum a_j e^{ik\cdot t_j})\mu\| + \|(\sum a_j e^{ik\cdot t_j} - h)\hat{L}\mu\| \\ &\le 2\|\hat{L}\|\|\mu\|\|h - \sum a_j e^{ik\cdot t_j}\|_\infty.\end{aligned}$$

Hence $\hat{L}(h\mu) = h\hat{L}\mu$.

Now suppose that $\mu \in \mathcal{M}_c$ with compact support E. Let F be a compact set containing E, and choose h smooth and periodic such that supp $h|_F = F - E$. Then $h\mu = 0$, and we have $h\hat{L}(\mu) = \hat{L}(h\mu) = 0$. It follows that the measure $\hat{L}(\mu)$ vanishes when restricted to $F - E$. Since F is arbitrary, we deduce that supp $\hat{L}(\mu) \subset$ supp μ.

We can approximate $\mu \in \mathcal{M}$ by $\mu_m \in \mathcal{M}_c$ as in the proof of Proposition 3.1. In particular, we can arrange that supp $\mu_m \subset$ supp μ. It follows that supp $\hat{L}\mu_m \subset$ supp μ. Since L is bounded, $\hat{L}\mu_m \to \hat{L}\mu$, and the result follows. □

Corollary A.2. *Let $L : \mathcal{X} \to \mathcal{X}$ be a bounded linear operator that commutes with translations. Then L restricts to a multiplication operator on each of the following closed $\mathbf{E}(n)$-invariant subalgebras of $\mathcal{X}$:*

(i) *The subspace $\mathcal{X}_{\rm ac}$ consisting of Fourier transforms of absolutely continuous measures (L^1 functions).*
(ii) *The subspace $\mathcal{X}_{\rm Dirac}$ generated by the Fourier transforms of the Dirac measures.*

Proof. It is immediate from Lemma A.1 that L restricts to a commuting linear operator on $\mathcal{X}_{\rm Dirac}$ and $\mathcal{X}_{\rm ac}$. The corollary then follows from well-known results. Case (ii) is particularly straightforward, and case (i) is contained in [31]. There is an elementary proof of case (i) using Lemma A.1 that we now sketch.

We identify $\mathcal{X}_{\rm ac}$ with $L^1(\mathbb{R}^n, \mathbb{C}^s)$. Consider the L^1 function $b\chi_E$, where $b \in \mathbb{C}^s$ and $E \in \mathcal{B}$. By Lemma A.1, $\hat{L}(b\chi_E) = p_E(b)\chi_E$, where $p_E \in L^1(E, L(\mathbb{C}^s))$. Moreover, it follows from Lemma A.1 that p_E and p_F coincide on $E \cap F$: consider $\hat{L}(b\chi_E - b\chi_F)$. Hence, there is a locally integrable map $q : \mathbb{R}^n \to L(\mathbb{C}^s)$ such that $\hat{L}(b\chi_E) = q(b)\chi_E$ for all Borel sets E. By linearity, $\hat{L}\phi = q\phi$ for $\phi : \mathbb{R}^n \to \mathbb{C}^s$ simple. Since $\hat{L}$ is bounded, it follows that $q \in L^\infty(\mathbb{R}^n, \mathbb{C}^s)$. Simple functions are dense in L^1, and hence $\hat{L}$ has the required form on $L^1(\mathbb{R}^n, \mathbb{C}^s)$. □

The corollary provides us with a proof of the characterization of absolutely irreducible subspaces of $\mathcal{X}_{\rm Dirac}$ stated in Section 4.

Proof of Proposition 4.2. First suppose that every bounded commuting linear operator on Y is a scalar multiple of the identity. Let $L : \mathcal{X} \to \mathcal{X}$ be a bounded $\mathbf{E}(n)$-equivariant linear operator such that $L(Y) \subset Y$. Then $L|_Y$ is a bounded commuting linear operator and is a scalar multiple of the identity; hence Y is absolutely irreducible.

Conversely, if Y is absolutely irreducible, then $Y \subset \mathcal{X}(a_0)$ for some $a_0 \ge 0$ (otherwise the operator with symbol $Q_a = f(a^2)I_{Z^s}$ is not a scalar multiple of the identity on Y, where $f : [0, \infty) \to \mathbb{R}$ is any injective bounded completely measurable function). Any commuting linear operator $L_0 : Y \to Y$ extends to a commuting linear operator $L' : \mathcal{X}_{\rm Dirac} \to \mathcal{X}_{\rm Dirac}$ (set $L'|_{\tilde{Y}} = 0$, say). By Corollary A.2, L' is a multiplication operator and hence is determined on $\mathcal{X}_{\rm Dirac}(a_0)$ by a multiplier $q : S^{n-1} \to B(Z^s)$, where S^{n-1} is the sphere in $\mathbb{R}^n$ of radius a_0. We can extend q to a completely measurable bounded multiplier on $\mathbb{R}^n$: if $a_0 = 0$, take q constant; otherwise take q to be constant on half lines in $\mathbb{R}^n - \{0\}$. Since q is completely measurable, we obtain a bounded $\mathbf{E}(n)$-equivariant linear operator L defined on the whole of $\mathcal{X}$. Moreover, L agrees with L' on $\mathcal{X}_{\rm Dirac}(a_0)$ and hence on Y. Since $Y \subset \mathcal{X}_{\rm Dirac}(a_0)$ is absolutely irreducible, it follows that $L_0|_Y = L'|_Y = L|_Y$ is a scalar multiple of the identity. □

Finally, we prove some auxiliary results on the spectrum of an $\mathbf{E}(n)$-equivariant linear operator $L : \mathcal{X} \to \mathcal{X}$ (see Remark 4.8). We continue to suppose that $\mathcal{X} = \mathcal{X}^s$ (no bounded variables).

Proposition A.3. *Suppose that L is an $\mathbf{E}(n)$-equivariant linear operator on $\mathcal{X}$ with measurable, locally bounded symbol Q. Let $\sigma \in \mathbb{C}$ and suppose that $\sigma \notin \operatorname{spec} Q_a$ for all $a \in \mathbb{R}$. Then $\sigma \in \operatorname{spec} L$ if and only if the family $|(Q_a - \sigma I)^{-1}|$ is not uniformly bounded (in a).*

Proof. First, suppose that the family is uniformly bounded: $|(q_k - \sigma I)^{-1}| \le c$ for all k. By Remark 3.8(c), $\|(L - \sigma I)^{-1}\| \le c$, so that $L - \sigma I$ has a bounded inverse. Since L is closed, the domain of $(L - \sigma I)^{-1}$ is closed. Moreover, the domain of $(L - \sigma I)^{-1}$ contains the dense subspace $\mathcal{X}_c = \mathcal{F}\mathcal{M}_c$ and hence is the whole of $\mathcal{X}$. It follows that $\sigma \notin \operatorname{spec} L$.

Conversely, suppose that the family is not uniformly bounded. Then we can choose $v_j \in \mathbb{C}^s$ with $|v_j| = 1/2$ and $k_j \in \mathbb{R}^n$ such that $|(q_{k_j} - \sigma I)^{-1} v_j| \ge j$. Define $u_j \in \mathcal{X}$ by setting $u_j(x) = v_j e^{i k_j x_1} + \text{c.c.}$ Then $\|u_j\| = 1$ but $\|(L - \sigma I)^{-1} u_j\| = 2|(q_{k_j} - \sigma I)^{-1} v_j| \ge 2j$. Hence $(L - \sigma I)^{-1}$ is not bounded, so $\sigma \in \operatorname{spec} L$. □

Proposition A.4. *Let L be a nondegenerate equivariant linear partial differential operator with symbol Q. Generically, for all $\sigma \in \mathbb{C}$, the family $(Q_a - \sigma I)^{-1}$ is eventually defined and converges to the zero matrix as $a \to \infty$.*

Proof. The matrix family Q_a commutes with the action of $\mathbf{O}(n-1)$ on $\mathbb{C}^s$ and block-diagonalizes according to the isotypic decomposition of $\mathbb{C}^s$ under $\mathbf{O}(n-1)$. Consider an isotypic component consisting of c copies of an $\mathbf{O}(n-1)$-irreducible representation, of dimension d, say. The corresponding $cd \times cd$ diagonal block of Q_a can be identified with a $c \times c$ matrix.

Without loss of generality, we may restrict attention to this single block of Q_a, which can be identified, by Proposition 3.12, with a single almost arbitrary $c \times c$ matrix with real polynomial entries. (There may be additional restrictions on the constant terms, since Q_0 commutes with the whole of $\mathbf{O}(n)$. We consider only perturbations that have no constant terms.)

By Cramer's rule, each entry of $(Q_a - \sigma I)^{-1}$ is a rational function whose denominator is the determinant of $Q_a - \sigma I$ and whose numerator is the determinant of an $(s-1) \times (s-1)$ minor. Let d be the maximum degree of the determinants of the $(s-1) \times (s-1)$ minors. It is sufficient to show that generically $\det Q_a$ has degree greater than d. This condition is clearly open, and is dense since one can make arbitrarily small perturbations to the linear and quadratic terms of the entries of Q_a. (Such perturbations are relatively bounded since L is assumed to be nondegenerate.) □

Corollary A.5. *If L is a nondegenerate equivariant linear partial differential operator on $\mathcal{X}$, then generically,* $\operatorname{spec} L = \bigcup_a \operatorname{spec} Q_a$.

Proof. One inclusion is automatic. To prove the other inclusion, suppose that $\sigma \notin \operatorname{spec} Q_a$ for all a. Then the continuous function $f(a) = |(Q_a - \sigma I)^{-1}|$ is defined for all a and generically converges to 0 as $a \to \infty$ by Proposition A.4. Hence $f(a)$ is bounded, and it follows from Proposition A.3 that $\sigma \notin \operatorname{spec} L$. □

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References

[1] C. Baesens and R. S. Mackay. *Uniformly traveling water waves from a dynamical systems viewpoint: some insight into bifurcations from the Stokes family*, J. Fluid Mech. **241** (1992), 333-347. MR **93h:**76007

[2] I. Bosch Vivancos, P. Chossat and I. Melbourne. *New planforms in systems of partial differential equations with Euclidean symmetry*, Arch. Rat. Mech. Anal. **131** (1995), 199-224. MR **96g:**58037

[3] T. Bröcker and T. tom Dieck. *Representations of Compact Lie Groups*, Grad. Texts in Math. **98**, Springer, New York, 1985. MR **86i:**22023

[4] M. C. Cross and P. C. Hohenberg. *Pattern formation outside of equilibrium*, Rev. of Mod. Phys. **65** (1993), 851-1112.

[5] B. Dionne and M. Golubitsky. *Planforms in two and three dimensions*, ZAMP **43** (1992), 36-62. MR **93e:**58026

[6] N. Dinculeanu. *Integration on Locally Compact Spaces*, Noordhoff International Pub., Leyden, The Netherlands, 1974. MR **50:**13428

[7] M. Golubitsky and D. G. Schaeffer. *Singularities and Groups in Bifurcation Theory*, Vol. 1. Appl. Math. Sci. **51**, Springer, New York, 1984. MR **86e:**58014

[8] M. Golubitsky, I. N. Stewart and D. G. Schaeffer. *Singularities and Groups in Bifurcation Theory*, Vol. 2. Appl. Math. Sci. **69**, Springer, New York, 1988. MR **89m:**58038

[9] M. Haragus. *Reduction of PDEs on unbounded domains. Application: unsteady water waves problem*, Preprint, Institut Non Linéaire de Nice, 1994.

[10] D. Henry. *Geometrical Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. **840**, Springer, 1981. MR **83j:**35084

[11] G. Iooss and J. Los. *Bifurcation of spatially quasiperiodic solutions in hydrodynamic stability problems*, *Nonlinearity* **3** (1990), 851-871. MR **92b:**35026

[12] G. Iooss, A. Mielke and Y. Demay. *Theory of steady Ginzburg-Landau equation in hydrodynamic stability problems*, Eur. J. Mech. B Fluids **8** (1989), 229-268. MR **90g:**76057

[13] S. Ito. *Unitary representations of some linear groups II*, Nagoya Math. J. **5** (1953), 79-96. MR **14:**533d

[14] T. Kato. *Perturbation Theory for Linear Operators*, Grundlehren **132**, Springer, New York, 1976. MR **53:**11389

[15] Y. Katznelson. *An Introduction to Harmonic Analysis*, Dover, New York, 1976. MR **54:**10976

[16] K. Kirchgässner. *Wave solutions of reversible systems and applications*, J. Diff. Eq. **45** (1982), 113-127. MR **83j:**35063

[17] L. D. Landau and E. M. Lifshitz. *Statistical Physics*, Part 1, Pergamon, New York, 1958. MR **24:**B2416

[18] S. Lang. *Real and Functional Analysis*, Grad. Texts in Math. **142**, Springer, New York, 1993. MR **94b:**00005

[19] G. W. Mackey. *Induced representations of locally compact groups I*, Annals of Math. **55** (1952), 101-139. MR **13:**434a

[20] I. Melbourne. *Derivation of the time-dependent Ginzburg-Landau equation on the line.* J. Nonlin. Sci. **8** (1998), 1–15. CMP 98:07

[21] L. Michel. *Symmetry defects and broken symmetry. Configurations. Hidden symmetry*, Rev. of Mod. Phys. **52** (1980), 617-651. MR **81k:**82006

[22] A. Mielke. *Reduction of quasilinear elliptic equations in cylindrical domains with applications*, Math. Meth. Appl. Sci. **10** (1988), 51-66. MR **89d:**35063

[23] ——— *Reduction of PDEs on domains with several unbounded directions: A first step towards modulation equations*, ZAMP **43** (1992), 449-470. MR **93f:**35023

[24] A. C. Newell and J. A. Whitehead. *Finite bandwidth, finite amplitude convection*, J. Fluid Mech. **38** (1969), 279-303.

[25] W. Rudin. *Real and Complex Analysis*, McGraw-Hill, New York, 1966. MR **35:**1420

[26] D. Ruelle. *Bifurcations in the presence of a symmetry group*, Arch. Rat. Mech. Anal. **51** (1973), 136-152. MR **50:**1291

[27] D. Sattinger. *Group representation theory, bifurcation theory and pattern formation*, J. Funct. Anal. **28** (1978), 58-101. MR **82f:**58034

[28] ——— *Group Theoretic Methods in Bifurcation Theory*, Lecture Notes in Math. **762** Springer, Berlin, 1979. MR **81e:**58022

[29] G. Schneider. *Error estimates for the Ginzburg-Landau approximation*, ZAMP **45** (1994), 433-457. MR **96m:**35035

[30] L. A. Segel. *Distant side-walls cause slow amplitude modulation of cellular convection*, J. Fluid Mech. **38** (1969), 203-224.

[31] E. M. Stein and G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, New Jersey, 1971. MR **46:**4102

[32] A. Vanderbauwhede. *Local Bifurcation and Symmetry*, Pitman Research Notes in Math. **75**, Boston, 1982. MR **85f:**58026

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