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A C^∞ diffeomorphism with infinitely many intermingled basins

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Abstract. Let M be the four-dimensional compact manifold $M = T^2 \times S^2$ and let $k \geq 2$. We construct a C^∞ diffeomorphism $F : M \to M$ with precisely k intermingled minimal attractors $A_1, \ldots, A_k$. Moreover the union of the basins is a set of full Lebesgue measure. This means that Lebesgue almost every point in M lies in the basin of attraction of A_j for some j, but every non-empty open set in M has a positive measure intersection with each basin. We also construct $F : M \to M$ with a countable infinity of intermingled minimal attractors.

1. *Introduction*

Let M be a topological space with a Borel probability measure m. Let $k \geq 2$. Measurable sets $B_1, \ldots, B_k \subset M$ are *intermingled* if they are measure-theoretically dense in each other. In other words, if one of the B_j meets an open set U in a set of positive measure, then U meets each of the B_j in a set of positive measure.

An *attractor* A is a compact invariant set such that the *basin of attraction* $b(A) = \{x : \omega(x) \subset A\}$ has positive Lebesgue measure and such that there is no strictly smaller compact invariant set A' such that $b(A)\backslash b(A')$ has zero Lebesgue measure [**6**]. An invariant set A is *minimal* if $\omega(x) = A$ for all $x \in A$.

We say that attractors $A_1, \ldots, A_k$ for a dynamical system are *intermingled* if the basins of attraction are intermingled. Similarly, we can speak of countably many intermingled sets/attractors.

Numerical evidence for the existence of intermingled attractors was first presented in Alexander *et al* [**2**] for a certain class of non-invertible maps of the plane. A proof is presented in [**1**]. They did not verify that the basins occupy a set of full measure, but did show that the regular parts of the basin (those characterized by typical Lyapunov exponents) are intermingled for a set of parameters with positive measure.

van Strien [**7**, Lemma 2.2] gave an example of a transitive polynomial interval map with two intermingled attractors. In the invertible context, Kan [**5**] announced the existence of an open set of C^k diffeomorphisms on the three-dimensional manifold with boundary $M = T^2 \times [0, 1]$ with two intermingled attractors (unfortunately the details do not appear in print).

Recently, Fayad [**4**] gave a new simpler construction of a C^∞ diffeomorphism $F : T^3 \to T^3$ that has two intermingled attractors. This was based on the following result of Windsor [**8**] (although related results implicit in Anosov and Katok [**3**] are sufficient for these purposes.) We use a variation on Fayad's idea for our construction.

THEOREM 1.1. *For each $k \geq 2$ there exists a minimal C^∞ diffeomorphism $f : T^2 \to T^2$ preserving Haar measure that has exactly k ergodic measures each of which is absolutely continuous. Similarly, there exists a minimal C^∞ diffeomorphism $f : T^2 \to T^2$ preserving Haar measure that has countably many absolutely continuous ergodic measures the union of whose basins has full measure.*

In this paper, we construct examples of diffeomorphisms on the four-dimensional compact manifold $M = T^2 \times S^2$ with arbitrarily many (even countably infinitely many) intermingled attractors.

THEOREM 1.2. *Let $k \geq 2$. There exists a C^∞ diffeomorphism $F : T^2 \times S^2 \to T^2 \times S^2$ with precisely k intermingled minimal attractors $A_1, \dots, A_k$ with* $\mathrm{Leb}\big(\bigcup b(A_j)\big) = 1$. *Moreover, $\omega(q) = A_j$ for some $j = 1, \dots, k$, for almost every $q \in T^2 \times S^2$.*

THEOREM 1.3. *There exists a C^∞ diffeomorphism $F : T^2 \times S^2 \to T^2 \times S^2$ with a countable infinity of intermingled minimal attractors $A_1, A_2, \dots$ with* $\mathrm{Leb}\big(\bigcup b(A_j)\big) = 1$. *Moreover, $\omega(q) = A_j$ for some $j \geq 1$ for almost every $q \in T^2 \times S^2$.*

Remark 1.4. When $k = 2$, the construction in this paper can clearly be made to work on T^3 (giving an alternative to [**4, 5**]). It is an interesting open problem to construct three or more intermingled attractors for a three-dimensional diffeomorphism.

The proofs of Theorems 1.2 and 1.3 are given in §2, except that a technical detail regarding smoothness is postponed to Appendix A. For completeness, the proof of Theorem 1.1 is outlined in Appendix B.

2. *The construction of $F : T^2 \times S^2 \to T^2 \times S^2$*

First, we describe the construction for finite k. Let $f : T^2 \to T^2$ be as in Theorem 1.1. Denote the k absolutely continuous ergodic measures by $\mu_1, \dots, \mu_k$. Let $\phi : T^2 \to \mathbb{R}^2$ be a C^∞ map and define $v_j = \int \phi \, d\mu_j \in \mathbb{R}^2$. Then $(1/N) \sum_{j=0}^{N-1} \phi \circ f^j$ converges almost everywhere with each pointwise limit lying in the set $\{v_1, \dots, v_k\}$. We choose ϕ so that $v_j \neq 0$ for each j, and such that the k unit vectors $w_j = v_j/|v_j|$ are distinct. (An open and dense set of C^∞ maps ϕ satisfies these properties.)

Let $D = \{z \in \mathbb{R}^2 : |z| < 1\}$ and $\overline{D} = \{z \in \mathbb{R}^2 : |z| \leq 1\}$. Choose $p : \mathbb{R}^2 \to D$ to be a direction-preserving diffeomorphism (that is, $\arg p(z) = \arg z$). Define $F : T^2 \times D \to T^2 \times D$ by

$$F(x, z) = (fx, p[\, p^{-1}(z) + \phi(x)]).$$

Extend F to a homeomorphism on $T^2 \times \overline{D}$ by setting $F(x, z) = (fx, z)$ for all $(x, z) \in X \times \partial D$.

LEMMA 2.1. *The homeomorphism $F : T^2 \times \overline{D} \to T^2 \times \overline{D}$ has precisely k intermingled minimal attractors $A_j = T^2 \times \{w_j\}$, $j = 1, \ldots, k$. Moreover, $\omega(p) = A_j$ for some $j = 1, \ldots, k$, for almost every $p \in T^2 \times \overline{D}$.*

Proof. Since $f : T^2 \to T^2$ is minimal, it is immediate from the definitions that A_j is a minimal set for each j. Let $(x, z) \in T^2 \times D$ such that $(1/N) \sum_{j=0}^{N-1} \phi(f^j x) \to v_j$. We show that $\omega(x, z) = A_j$. Note that

$$F^N(x, z) = \bigg(f^N x, p\bigg[p^{-1}(z) + \sum_{j=0}^{N-1} \phi(f^j x)\bigg]\bigg) = (f^N x, p[Nv_j + o(N)]).$$

Now $|Nv_j + o(N)| \to \infty$ and $\arg(Nv_j + o(N)) \to \arg v_j$. Since $p : \mathbb{R}^2 \to D$ is a direction-preserving diffeomorphism, it follows that $p[Nv_j + o(N)] \to w_j$. Also, f is minimal, so $\omega_f(x) = T^2$. Hence, $\omega(x, z) = A_j$ as required. □

LEMMA 2.2. *The diffeomorphism $p : \mathbb{R}^2 \to D$ can be chosen (independent of f and ϕ) in such a way that $F - \mathrm{Id}$ is C^∞ flat at the boundary ∂D.*

Proof. The key here is to choose p decaying sufficiently *slowly* at infinity. It turns out that polynomial decay is too fast, but logarithmic decay suffices. The calculations are given in Appendix A. □

Proof of Theorem 1.2. Glue two copies of $F : T^2 \times \overline{D} \to T^2 \times \overline{D}$ together at the equator, to obtain a homeomorphism $F : T^2 \times S^2 \to T^2 \times S^2$. The dynamical properties required of F follow from Lemma 2.1. Clearly, F is a C^∞ diffeomorphism away from the equator. By Lemma 2.2, $F - \mathrm{Id}$ is C^∞ flat at the equator, so F is C^∞ everywhere. Since

$$F^{-1}(x, z) = (f^{-1}x, p[p^{-1}(z) - \phi(f^{-1}x)])$$

has the same structure as F, it follows that F is a C^∞ diffeomorphism. □

Proof of Theorem 1.3. The proof is identical, except that we start with a minimal diffeomorphism $f : T^2 \to T^2$ with a countable infinity of absolutely continuous ergodic components (again using [**8**]), and we choose ϕ so that each of the time-averages v_j is non-zero and such that the unit vectors $w_j = v_j/|v_j|$, $j \geq 1$, are distinct. Moreover, the construction of f is such that for Haar almost every point in T^2, the time average is v_j for some j. □

A. *Appendix. The diffeomorphism $p : \mathbb{R}^2 \to D$*

In this appendix, we prove Lemma 2.2, constructing a direction-preserving diffeomorphism $p : \mathbb{R}^2 \to D$ with the desired properties at infinity. To illustrate the issues involved, we begin with the one-dimensional analogue, taking $\phi(x) \equiv \beta$ constant.

A.1. *One dimension.* Let $p_1 : \mathbb{R} \to (-1,1)$ be an odd orientation-preserving diffeomorphism. For s near $+\infty$ we take $p_1(s) = 1 - 1/\ln s$.

PROPOSITION A.1. *Let $\beta \in \mathbb{R}$. Define $G_1 : (-1,1) \to (-1,1)$ by $G_1(r) = p_1(p_1^{-1}(r) + \beta)$. Then $G_1(r) - r$ is C^∞ flat at ± 1.*

Proof. Near $r = 1$, we have $p_1^{-1}(r) = e^{1/(1-r)}$. Let $\widetilde{G}_1(r) = G_1(r) - r$, so $\widetilde{G}_1(r) = (1-r) - 1/\ln[e^{1/(1-r)} + \beta]$. Define $H(y) = \widetilde{G}_1(1-y)$. We show that H is C^∞ flat at $y = 0$ as $y \to 0^+$. A calculation yields

$$H(y) = \frac{y^2 \ln[1 + \beta e^{-1/y}]}{1 + y \ln[1 + \beta e^{-1/y}]}.$$

Now $e^{-1/y}$ is flat, and $\ln(1+g)$ is flat whenever g is flat. Also, flatness is preserved after multiplication by a smooth function (or dividing by a non-vanishing smooth function). Hence, H is flat as required. □

Remark. It is important that $p_1(s) \to 1$ sufficiently slowly as $s \to \infty$. If the decay is polynomial ($p_1(s) = 1 - 1/s^\alpha$ say), then $G_1(r) - r$ is only C^k flat where k is finite. Indeed, for C^k-flatness we require that $\alpha < 1/(k-1)$.

A.2. *Two dimensions.* Suppose that $p_1 : \mathbb{R} \to (-1,1)$ is as above, and additionally that $p_1(s) \equiv s$ for s close to 0. We use (r,θ) for polar coordinates on D, and (s,θ) for polar coordinates on $\mathbb{R}^2$. Define $p : \mathbb{R}^2 \to D$ by setting $p(s,\theta) = (p_1(s),\theta)$. Let $(s,\theta) \mapsto t_{\phi(x)}(s,\theta)$ be the transformation corresponding to translation by $\phi(x) = (\phi_1(x), \phi_2(x)) \in \mathbb{R}^2$.

PROPOSITION A.2. *Define $G : X \times D \to D$ by $G = p \circ t_\phi \circ p^{-1}$. Then $G(x,r,\theta) - (r,\theta)$ is C^∞ flat at ∂D.*

Proof. We have $G(x,r,\theta) = (p_1(\widehat{s}), \widehat{\theta})$, where

$$\widehat{s}^2 = [p_1^{-1}(r)]^2 + 2p_1^{-1}(r)(\phi_1(x)\cos\theta + \phi_2(x)\sin\theta) + \phi_1(x)^2 + \phi_2(x)^2,$$
$$\widehat{\theta} = \arctan\left(\frac{p_1^{-1}(r)\sin\theta + \phi_2(x)}{p_1^{-1}(r)\cos\theta + \phi_1(x)}\right).$$

Define $\widetilde{G}(x,r,\theta) = G(x,r,\theta) - (r,\theta)$ and $H(x,y,\theta) = \widetilde{G}(x,1-y,\theta)$. By rotation symmetry, it suffices to show that H is flat at $y = 0$ (as $y \to 0^+$).

Write $H = (H_1, H_2)$ and compute that

$$H_1(x,y,\theta) = \frac{y^2 \ln[1 + g(x,y,\theta)]}{2 + y \ln[1 + g(x,y,\theta)]},$$

$$H_2(x,y,\theta) = \arctan\left(\frac{\sin\theta + \phi_2(x)e^{-1/y}}{\cos\theta + \phi_1(x)e^{-1/y}}\right) - \theta,$$

where $g(x,y,\theta) = 2e^{-1/y}(\phi_1(x)\cos\theta + \phi_2(x)\sin\theta) + e^{-2/y}(\phi_1(x)^2 + \phi_2(x)^2)$.

As in the one-dimensional case, we argue that flatness of H_1 follows from flatness of g. Since the arctangent of a flat function is flat, it suffices to verify flatness of

$$\tan(H_2(x, y, \theta)) = e^{-1/y}\Big(\frac{\phi_2(x)\cos\theta - \phi_1(x)\sin\theta}{1 + \phi_1(x)e^{-1/y}\cos\theta + \phi_2(x)e^{-1/y}\sin\theta}\Big).$$

This is a product of the flat function $e^{-1/y}$ and a smooth function, and hence is flat. □

B. *Appendix. Intermingled ergodic components*

For the sake of completeness, in this appendix we sketch the proof of Theorem 1.1. (The results in [**8**] are formulated for any compact manifold that admits a free circle action. By specializing to T^2, we bypass many of the technicalities in [**8**].) The argument could be made marginally simpler by dropping the requirement that the diffeomorphism is area preserving (which is not required for our main results) but the simplification does not seem worthwhile.

Let T^2 denote the 2-torus with normalized Haar measure μ and metric d. For a measurable set $E \subset T^2$ with $\mu(E) > 0$, let $\mu\big|_E(A) := \mu(A \cap E)/\mu(E)$ denote the normalized restriction of the measure μ to E.

The required diffeomorphism is constructed using a variant of the fast approximation-conjugation method pioneered by Anosov and Katok [**3**]. For $t > 0$, let $S_t : T^2 \to T^2$ be the translation defined by

$$S_t(x, y) := (x, y + t \mod 1).$$

Let k denote the number of absolutely continuous ergodic measures desired. We divide T^2 into k vertical strips $M_i = [(i-1)/k, i/k) \times [0, 1)$, $1 \le i \le k$, with associated S_t-invariant probability measures $\mu^{(i)} := \mu\big|_{M_i}$.

The required diffeomorphism $f : T^2 \to T^2$ is the limit of a sequence of periodic diffeomorphisms f_n given by

$$f_n := H_n^{-1} S_{\omega_n} H_n$$

where $\omega_n = p_n/q_n$ with $(p_n, q_n) = 1$, and $H_n : T^2 \to T^2$ is an area-preserving diffeomorphism. (We construct ω_n and H_n in Appendix B.2.) Clearly f_n preserves the measures

$$\mu_n^{(i)} := H_n^* \mu^{(i)} = \mu\big|_{H_n^{-1} M_i}.$$

The required ergodic measures appear as the limits $\mu_\infty^{(i)} = \lim_{n\to\infty} \mu_n^{(i)}$ in the total variation norm.

B.1. *Convergence.* Let ϵ_n be a summable sequence of positive real numbers and let $E_n = \sum_{m=n}^{\infty} \epsilon_m$. Let $\{\varphi_i\}_{i=1}^{\infty}$ be a countable dense set of continuous real-valued functions on T^2. Let ρ_n denote the standard metric on C^n diffeomorphisms of T^2, $\rho_n(f, g) = \widetilde{\rho}_n(f, g) + \widetilde{\rho}_n(f^{-1}, g^{-1})$, where $\widetilde{\rho}_n(f, g) = \max_{j=0,1,\ldots,n} \sup_{x\in T^2} d(f^{(j)}(x), g^{(j)}(x))$.

We construct the maps H_n such that the following properties are obtained.

1. $\rho_n(f_n, f_{n+1}) < \epsilon_n$.
2. $\sup_x \max_{1\le i\le q_n} d(f_n^i x, f_{n+1}^i x) < \epsilon_n$.

3. f_n is ϵ_n-minimal, every orbit meets every ϵ_n-ball.
4. For $\varphi \in \{\varphi_1, \dots, \varphi_n\}$ and for every $x \in T^2$, there exists ν^x_{n-1} in the simplex generated by the measures $\mu^{(1)}_{n-1}, \dots, \mu^{(k)}_{n-1}$ such that

$$\left| \frac{1}{q_n} \sum_{i=0}^{q_n-1} \varphi(f_n^i x) - \int_{T^2} \varphi \, d\nu^x_{n-1} \right| < \epsilon_n.$$

5. $\mu(H_n^{-1} M_i \triangle H_{n+1}^{-1} M_i) < \epsilon_n$ for $1 \le i \le k$.

In the remainder of this appendix, we show how Theorem 1.1 follows from the above conditions.

Condition 1 guarantees the convergence of the sequence f_n to a C^∞ area-preserving diffeomorphism f. Minimality of f is ensured by conditions 2 and 3 as follows. Given $\epsilon > 0$, consider n such that $E_n < \epsilon/2$. The periodic diffeomorphism f_n is $\epsilon/2$ dense and every point on the f_n orbit of x can be approximated within $\epsilon/2$ by a point on the f orbit of x. Hence, the f orbit of x meets every ϵ-ball. Since ϵ was arbitrary, f is minimal.

Condition 5 guarantees that for each $1 \le i \le k$ the sequence $\mu^{(i)}_n$ converges in the variation norm to an invariant probability measure $\mu^{(i)}_\infty$. Indeed $\mu(\cdot \triangle \cdot)$ makes the measure algebra into a complete metric space. The sequence of sets $H_n^{-1} M_i$ is a Cauchy sequence in this metric and hence converges to a (unique modulo null sets) measurable set. The limiting measure is the normalized restriction of μ to this set. Since the M_i are mutually disjoint the limiting measures $\mu^{(i)}_\infty$ are mutually singular.

If ν is an ergodic measure for f then there is a point $x_0 \in T^2$ such that for every continuous function φ,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x_0) = \int_{T^2} \varphi \, d\nu.$$

However, by conditions 2 and 4,

$$\lim_{n\to\infty} \frac{1}{q_n} \sum_{i=0}^{q_n-1} \varphi(f^i x_0) = \lim_{n\to\infty} \frac{1}{q_n} \sum_{i=0}^{q_n-1} \varphi(f_n^i x_0) = \lim_{n\to\infty} \int_{T^2} \varphi \, d\nu^{x_0}_{n-1},$$

and so ν is the weak limit of the sequence $\nu^{x_0}_n$. This means ν must be in the simplex generated by $\mu^{(1)}_\infty, \dots, \mu^{(k)}_\infty$. Hence, any ergodic measure must be one of $\mu^{(1)}_\infty, \dots, \mu^{(k)}_\infty$. Since the $\mu^{(i)}_\infty$ are mutually singular they must all be ergodic.

B.2. *Construction of the H_n.* In this appendix, we complete the proof of Theorem 1.1 by constructing the conjugacies H_n so that conditions 1–5 in B.1 are satisfied. Recall that $f_n = H_n^{-1} S_{\omega_n} H_n$. The conjugating maps H_n are constructed inductively

$$H_n := h_n \circ \cdots \circ h_1$$

where each h_n is a C^∞ area-preserving diffeomorphism on T^2.

We require that h_{n+1} commutes with S_{ω_n}. Then we can write

$$f_{n+1} = H_n^{-1} S_{\omega_n} h_{n+1}^{-1} S_{\omega_{n+1}-\omega_n} h_{n+1} H_n.$$

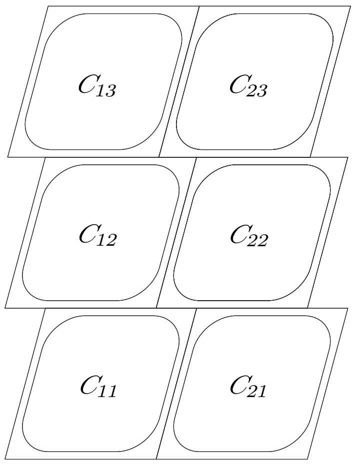

FIGURE 1. Parallelograms and cores C_{ij}.

Once the diffeomorphism h_{n+1} is fixed we may always choose ω_{n+1} sufficiently close to ω_n to ensure that conditions 1 and 2 hold.

In the following, ℓ, m, N denote positive integers that will be chosen later sufficiently large. We construct h_{n+1} on the horizontal strip $\Delta := [0, 1) \times \left[0, 1/\ell q_n\right)$ and extend it by requiring that h_{n+1} commute with $S_{1/\ell q_n}$. This naturally ensures that h_{n+1} commutes with S_{ω_n}.

We partition Δ into equally sized parallelograms $P_{i,j}$, $1 \leq i \leq kN$, $1 \leq j \leq kmN$ with sides horizontal and at 45 degrees, starting from $(0, 0)$, see Figure 1. The parallelograms have base $1/kN$ and height $1/\ell q_n kmN$. Let $P_{i,j}$ denote the parallelogram in the ith column and jth row.

Let $\widetilde{M}_i$ denote the approximation to M_i by parallelograms, and $\widetilde{\mu}^{(i)}$ the associated measures. Let $\Delta_0 = \bigcup_{1 \leq j \leq kN} P_{i,j}$ and $\Delta_1 = \bigcup_{kN < j \leq kmN} P_{i,j}$ denote the lower and upper portions of Δ. Choosing m and N large enough, we can ensure that Δ_1 is arbitrarily close to full measure in Δ and that $\widetilde{M}_i$ is arbitrarily close to M_i, so that condition 5 is satisfied.

For each i, j, we choose a *core* $C_{i,j} \subset \operatorname{Int} P_{i,j}$ diffeomorphic to a closed disk.

Let $\mathfrak{C} = \bigcup_{i,j} C_{i,j}$. Choose h_{n+1} to be an area-preserving C^∞ diffeomorphism such that

$$h_{n+1} C_{i,j} = \begin{cases} C_{\alpha^j(i),j} & j \leq kN \\ C_{\beta^j(i),j} & \text{otherwise} \end{cases}$$

where α and β are the permutations given by

$$\alpha = (1 \cdots kN), \quad \beta = (1 \cdots N)(N+1 \cdots 2N) \cdots ((k-1)N + 1 \cdots kN).$$

Note that α acts on Δ_0 and β acts on each $\widetilde{M}_i \cap \Delta_1$. This permutation is constructed by exhibiting a transposition of adjacent cores and then using the fact that any permutation can be written as a product of transpositions.

Consider the partition of T^2 given by the columns of parallelograms $K_{a,b,c}$, $1 \le a \le kN$, $1 \le b \le m$, $0 \le c \le \ell q_n - 1$, where

$$K_{a,b,c} = S_{c/\ell q_n} \bigcup_{j=(b-1)kN+1}^{bkN} P_{a,j},$$

with dimensions $1/kN \times 1/\ell q_n m$.

We choose the cores $C_{i,j}$ large enough so that every vertical line in T^2 intersects every row of cores and, moreover, there exists a column i_0 such that the vertical line intersects $C_{i_0,j}$ for each $j \ge 1$. For every x, the orbit $\{h_{n+1}^{-1} S_t x : t > 0\}$ intersects every column $K_{a,1,c}$ and is uniformly distributed amongst $\{K_{a,b,c} \cap \mathfrak{C} : K_{a,b,c} \subset \widetilde{M}_i \cap \Delta_1\}$ for each i. Hence, for q_{n+1} large enough, the orbit $\{h_{n+1}^{-1} S_{\omega_{n+1}}^j x : j \ge 1\}$ intersects every column $K_{a,1,c}$ and is almost uniformly distributed amongst $\{K_{a,b,c} \cap \mathfrak{C} : K_{a,b,c} \subset \widetilde{M}_i \cap \Delta_1\}$ for each i.

Next, we prove ϵ_{n+1}-minimality. For ℓ large enough, it suffices to prove that each f_{n+1} orbit intersects every ϵ_{n+1}-ball. Choose ℓ and N large enough that for every ϵ_{n+1}-ball B, there exists a, c such that $K_{a,1,c} \subset H_n B$. Hence, $\{H_{n+1}^{-1} S_{\omega_{n+1}}^j x\ ;\ j \ge 1\}$ intersects every ϵ_{n+1}-ball. Since $f_{n+1} = H_{n+1}^{-1} S_{\omega_{n+1}} H_{n+1}$, this gives condition 3.

Finally, we choose ℓ and N large enough that for all $\varphi \in \{\varphi_1, \dots, \varphi_{n+1}\}$ we have

$$\max_{x \in K_{a,b,c}} \varphi \circ H_n^{-1}(x) - \min_{x \in K_{a,b,c}} \varphi \circ H_n^{-1}(x) < \frac{\epsilon_{n+1}}{6}. \tag{B.1}$$

Let $\Pi_x^{(i)} := \{j \in \{1, \dots, q_{n+1}\} : h_{n+1}^{-1} S_{\omega_{n+1}}^j x \in \mathfrak{C} \cap \widetilde{M}_i \cap \Delta_1\}$. If $\Pi_x^{(i)} \ne \varnothing$, then

$$\left| \frac{1}{\#\Pi_x^{(i)}} \sum_{j \in \Pi_x^{(i)}} \varphi H_n^{-1} h_{n+1}^{-1} S_{\omega_{n+1}}^j x - \int \varphi H_n^{-1} d\widetilde{\mu}^{(i)}|_{\mathfrak{C} \cap \Delta_1} \right| < \frac{\epsilon_{n+1}}{3},$$

by (B.1) and (almost) uniform distribution. If we let $\Pi_x := \bigcup_i \Pi_x^{(i)}$, then

$$\left| \frac{1}{\#\Pi_x} \sum_{j \in \Pi_x} \varphi H_n^{-1} h_{n+1}^{-1} S_{\omega_{n+1}}^j x - \int \varphi H_n^{-1} d\widetilde{\nu}_x|_{\mathfrak{C} \cap \Delta_1} \right| < \frac{\epsilon_{n+1}}{3},$$

where $\widetilde{\nu}_x$ is in the simplex of measures $\widetilde{\mu}^{(1)}, \dots, \widetilde{\mu}^{(k)}$. Since we can make the cores $\mathfrak{C}$ capture almost all of every orbit and since Δ_1 has almost full measure in Δ we obtain

$$\left| \frac{1}{q_{n+1}} \sum_{j=1}^{q_{n+1}-1} \varphi H_n^{-1} h_{n+1}^{-1} S_{\omega_{n+1}}^j x - \int \varphi H_n^{-1}\, d\nu_x \right| < \epsilon_{n+1},$$

which is condition 4.

Remark. A modification [**8**] to the above arguments yields minimality and countably many absolutely continuous ergodic measures. Moreover, the union of the 'supports' of the absolutely continuous measures is of full measure. (By support, we mean the set of generic points for a given invariant measure.) The argument to show there are no more ergodic measures now shows only that there are no more absolutely continuous ergodic measures. Indeed there must be at least one singular ergodic measure by weak-* compactness. By carefully choosing the approximation by parallelograms it is possible to ensure that there is precisely one singular ergodic measure, but this is not necessary for Theorem 1.3.

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