FACULTY WORKING PAPER 93-0103

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January 1993

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December 1992

In this paper we propose simple diagnostic tests, based on OLS residuals, for spatial residual autocorrelation (or spatially lagged dependent variable) in the presence of spatially lagged dependent variable (or spatial residual autocorrelation), applying the modified LM test developed in Bera and Yoon (1990). Our new tests may be viewed as computationally simple and robust alternatives to some existing procedures in spatial econometrics.

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1. Introduction

In spatial data analysis, model specification issues have recently become an integrated part of spatial econometric modelling [see, for example, Anselin (1988b, 1989) and Blommestein (1983)]. In a recent paper by Anselin (1988a) several diagnostics for spatial econometric models have been proposed based on the Lagrange multiplier (LM) principle. In particular the focus was on detecting model misspecification due to spatial dependence (in the form of an omitted spatially lagged dependent variable and spatial residual autocorrelation) as well as spatial heterogeneity (in the form of heteroskedasticity). In deriving a joint test for spatial dependence and spatial heterogeneity, Anselin (1988a) has observed that the inverse of the information matrix for the joint LM test is block diagonal between the spatially dependent and the heteroskedastic components, and hence the joint test statistic is the sum of the two corresponding component statistics where the test for the heteroskedastic part is identical to the Breusch and Pagan (1979) statistic. However, the spatially dependent part cannot be decomposed further into two one-directional test statistics corresponding to spatially lagged dependent variable and spatial residual autocorrelation respectively. As emphasized by the author, this is due to the structural relationship between spatial autoregressive processes in the dependent variable and the disturbance term resulting in the non block diagonality of the information matrix between the corresponding elements [see Anselin (1988a p. 8)].

Noting this, Anselin (1988a) poposes an LM test for spatial residual autocorrelation in the presence of a spatially lagged dependent variable. The implementation of the suggested test, however, requires nonlinear optimization or some numerical search techniques. In this paper we propose simple diagnostic tests, based on ordinary least squares (OLS) residuals, for spatial dependence, applying the modified LM test developed in Bera and Yoon (1990) to spatial models.

In section 2, we briefly summarize the main results on the distribution of standard LM test when the alternative hypothesis is misspecified, and present the modified LM test which is robust under local misspecification. Section 3 develops new diagnostic tests for spatial residual autocorrelation (or spatially lagged dependent variable) in the presence of spatially lagged dependent variable (or spatial residual autocorrelation). Final section 4 contains some concluding remarks.

2. A General Approach to Testing in the Presence of a Nuisance Parameter

Consider a general statistical model represented by the log-likelihood $L(\gamma, \psi, \phi)$ where γ is a parameter vector, and for simplicity ψ and ϕ are assumed to be scalars. Suppose an investigator sets $\phi = 0$ and tests $H_0 : \psi = 0$ using the likelihood function $L_1(\gamma, \psi) = L(\gamma, \psi, 0)$. The LM statistic for testing H_0 in $L_1(\gamma, \psi)$ will be denoted by LM_ψ. Let us also denote $\theta = (\gamma', \psi, \phi)'$ and $\widetilde{\theta} = (\widetilde{\gamma}', 0, 0)'$ where $\widetilde{\gamma}$ is the maximum likelihood estimator (MLE) of γ when $\psi = 0$ and $\phi = 0$. The score vector and the information matrix are defined, respectively as

$$d_a(\theta) = \frac{\partial L(\theta)}{\partial a} \quad \text{for} \quad a = \gamma, \psi, \phi,$$

and

$$J(\theta) = -\text{plim}\frac{1}{N}\frac{\partial^2 L(\theta)}{\partial\theta\partial\theta'} = \begin{bmatrix} J_\gamma & J_{\gamma\psi} & J_{\gamma\phi} \\ J_{\psi\gamma} & J_\psi & J_{\psi\phi} \\ J_{\phi\gamma} & J_{\phi\psi} & J_\phi \end{bmatrix}.$$

If $L_1(\gamma, \psi)$ were the true model, then it is well known that under $H_0 : \psi = 0$,

$$LM_\psi = \frac{1}{N} d_\psi(\widetilde{\theta})' J_{\psi\cdot\gamma}^{-1}(\widetilde{\theta}) d_\psi(\widetilde{\theta}) \xrightarrow{D} \chi_1^2(0)$$

where $J_{\psi\cdot\gamma}(\theta) = J_\psi(\theta) - J_{\psi\gamma}(\theta)J_\gamma^{-1}(\theta)J_{\gamma\psi}(\theta)$. We use $\xrightarrow{D}$ to denote convergence in distribution. Under this set-up, asymptotically the test will have correct size and will be locally optimal. Now suppose that the true log-likelihood function is $L_2(\gamma, \phi)$ so that the alternative $L_1(\gamma, \psi)$ becomes misspecified. Using a sequence of local values $\phi = \delta/\sqrt{N}$, Davidson and McKinnon (1987) and Saikkonen (1989) obtained the asymptotic distribution of LM_ψ under $L_2(\gamma, \phi)$ as

$$LM_\psi \xrightarrow{D} \chi_1^2(\lambda) \tag{2.1}$$

where the non-centrality parameter λ is given by $\lambda = \delta' J_{\phi\psi\cdot\gamma} J_{\psi\cdot\gamma}^{-1} J_{\psi\phi\cdot\gamma}\delta$ with $J_{\psi\phi\cdot\gamma} = J_{\psi\phi} - J_{\psi\gamma}J_\gamma^{-1}J_{\gamma\phi}$. Due to this non-centrality parameter, LM_ψ will have power in the model $L(\gamma, \psi, \phi)$ even when $\psi = 0$, and therefore, the test will have incorrect size. Notice that the crucial quantity is $J_{\psi\phi\cdot\gamma}$ which can be interpreted as the partial covariance between d_ψ and d_ϕ after eliminating the effect of d_γ on d_ψ and d_ϕ. If $J_{\psi\phi\cdot\gamma} = 0$, then the local presence of the parameter ϕ has no effect on LM_ψ.

Using the result (2.1), Bera and Yoon (1990) suggested a modification to LM_ψ so that the resulting test is robust to the presence of ϕ. The modified statistic is given by

$$LM_\psi^* = \frac{1}{N}\left[d_\psi(\tilde{\theta}) - J_{\psi\phi\cdot\gamma}(\tilde{\theta})J_{\phi\cdot\gamma}^{-1}(\tilde{\theta})d_\phi(\tilde{\theta})\right]' \tag{2.2}$$

$$\left[J_{\psi\cdot\gamma}(\tilde{\theta}) - J_{\psi\phi\cdot\gamma}(\tilde{\theta})J_{\phi\cdot\gamma}^{-1}(\tilde{\theta})J_{\phi\psi\cdot\gamma}(\tilde{\theta})\right]^{-1}$$

$$\left[d_\psi(\tilde{\theta}) - J_{\psi\phi\cdot\gamma}(\tilde{\theta})J_{\phi\cdot\gamma}^{-1}(\tilde{\theta})d_\phi(\tilde{\theta})\right]$$

This new test essentially adjusts the mean and variance of the standard LM_ψ. Bera and Yoon (1990) further showed that under $\psi = 0$ and $\phi = \delta/\sqrt{N}$ LM_ψ^* has a central χ_1^2 distribution. Thus LM_ψ^* has the same asymptotic null distribution as the LM_ψ based on the correct specification, thereby producing an asymptotically correct size test under locally misspecified model. Two things regarding LM_ψ^* are worth noting. First, LM_ψ^* requires estimation only under the joint null, namely $\psi = 0$ and $\phi = 0$. Given the full specification of the model $L(\gamma, \psi, \phi)$ it is of course possible to derive an LM test for $\psi = 0$ in the presence of ϕ. However, that requires MLE of ϕ which could be difficult to obtain in some cases. Second, when $J_{\psi\phi\cdot\gamma} = 0$, $LM_\psi^* = LM_\psi$. This is a very simple condition to check in practice. As mentioned before, if this condition is true, LM_ψ is an asymptotically valid test in the local presence of ϕ.

3. Tests for Spatial Dependence

We consider the mixed regressive-spatial autoregressive model with a spatial autoregressive disturbance

$$y = \phi W_1 y + X\gamma + u$$

$$u = \psi W_2 u + \epsilon$$

$$\epsilon \sim \mathcal{N}(0, I\sigma^2) \tag{3.1}$$

In this model y is a $(N \times 1)$ vector of observations on a dependent variable recorded at each of N locations, X is an $(N \times k)$ matrix of exogenous variables, and γ is a $(k \times 1)$ vector of parameters. ϕ and ψ are scalar parameters. W_1 and W_2 are $(N \times N)$ spatial observable weight matrices associated with the spatially lagged dependent variable and the spatial autoregressive disturbance respectively. These spatial weight matrices represent the 'degree of possible interaction' between neighboring locations and are scaled such that the sum of the row elements in each matrix is equal to one [see Ord (1975) and Upton and Fingleton (1985) for discussions of W matrix]. Note that it is the inclusion of these spatial weight matrices that renders the spatial models to depart from the standard linear model limiting the applicability of the standard econometric procedures based on OLS method.

We are interested in the problem of testing $H_0 : \psi = 0$ in the presence of the nuisance parameter ϕ. As before, let $\theta = (\gamma', \psi, \phi)'$. Since the information matrix is block diagonal between the θ and σ^2 parameters, we need only to consider the scores and the information matrix evaluated at

$\theta_0 = (\gamma', 0, 0)'$, i.e.,

$$d_\gamma = \frac{1}{\sigma^2} X'u,$$

$$d_\psi = \frac{1}{\sigma^2} u'W_2u,$$

$$d_\phi = \frac{1}{\sigma^2} u'W_1y,$$

and

$$J = \frac{1}{N\sigma^2} \begin{bmatrix} X'X & 0 & X'(W_1X\gamma) \\ 0 & T_{22}\sigma^2 & T_{21}\sigma^2 \\ (W_1X\gamma)'X & T_{12}\sigma^2 & (W_1X\gamma)'(W_1X\gamma) + T_{11}\sigma^2 \end{bmatrix} \tag{3.2}$$

where, as in Anselin (1988a), we use the notation $T_{ij} = tr\{W_iW_j + W_i'W_j\}$, $i, j = 1, 2$, with tr denoting trace of a matrix. From (3.2) it follows that

$$J_{\psi\phi\cdot\gamma} = \frac{1}{N} T_{21},$$

$$J_{\psi\cdot\gamma} = \frac{1}{N} T_{22},$$

and

$$J_{\phi\cdot\gamma} = \frac{1}{N\sigma^2} \left[(W_1X\gamma)'M(W_1X\gamma) + T_{11}\sigma^2\right]$$

where $M = I - X(X'X)^{-1}X'$. Note that $J_{\psi\phi\cdot\gamma} \neq 0$. This indicates the asymptotic correlation between the scores corresponding to the two spatial autoregressive parameters in the dependent variable and the disturbance term. Our modified LM test can be easily obtained as

$$LM_{\psi}^{*} = \frac{\left[\widetilde{u}'W_2\widetilde{u}/\widetilde{\sigma}^2 - T_{21}(N\widetilde{J}_{\phi\cdot\gamma})^{-1}\widetilde{u}'W_1y/\widetilde{\sigma}^2\right]^2}{T_{22} - (T_{21})^2(N\widetilde{J}_{\phi\cdot\gamma})^{-1}} \tag{3.3}$$

where $\widetilde{u} = y - X\widetilde{\gamma}$ are the OLS residuals with $\widetilde{\sigma}^2 = \widetilde{u}'\widetilde{u}/N$ and $(N\widetilde{J}_{\phi\cdot\gamma})^{-1} = \widetilde{\sigma}^2[(W_1X\widetilde{\gamma})'M(W_1X\widetilde{\gamma}) + T_{11}\widetilde{\sigma}^2]^{-1}$. One can interpret $(W_1X\widetilde{\gamma})$ as the spatially lagged OLS predicted values. If we assume that the spatial weight matrices W_1 and W_2 are the same, i.e., $T_{11} = T_{21} = T_{22} = T = tr\{(W'+W)W\}$, the LM_{ψ}^{*} in (3.3) can be simplified further to give

$$LM_{\psi}^{*} = \frac{\left[\widetilde{u}'W_2\widetilde{u}/\widetilde{\sigma}^2 - T(N\widetilde{J}_{\phi\cdot\gamma})^{-1}\widetilde{u}'W_1y/\widetilde{\sigma}^2\right]^2}{T\left[1 - T(N\widetilde{J}_{\phi\cdot\gamma})^{-1}\right]} \tag{3.4}$$

The conventional one-directional test LM_{ψ} given in Burridge (1980) is obtained by setting $\phi = 0$ to yield

$$LM_{\psi} = \frac{\left[\widetilde{u}'W_2\widetilde{u}/\widetilde{\sigma}^2\right]^2}{T} \tag{3.5}$$

Comparison of (3.4) with (3.5) clearly reveals that the LM_{ψ}^{*} modifies the standard LM_ψ by correcting the mean and variance of the score for the asymptotic correlation between d_ψ and d_ϕ.

Let us now consider the LM test for $H_0 : \psi = 0$ in the presence of the ϕ parameter derived in Anselin (1988a). We can denote this statistic by LM_ψ^A:

$$LM_\psi^A = \frac{\left[\widehat{u}' W_2 \widehat{u} / \widehat{\sigma}^2\right]^2}{T_{22} - (T_{21A})^2 \widehat{\mathrm{var}}(\widehat{\phi})} \tag{3.6}$$

where $\widehat{u}$ are the maximum likelihood residuals under the null model $y = \phi W_1 y + X\gamma + u$ obtained by nonlinear optimization or some search techniques. T_{21A} denotes $tr\{W_2 W_1 A^{-1} + W_2' W_1 A^{-1}\}$ with $A = I - \widehat{\rho} W_1$. Comparing the LM_ψ^A with the LM_ψ^* in (3.3), it is readily seen that the LM_ψ^A does not have the mean correction factor in LM_ψ^*. This is because LM_ψ^A uses the restricted MLE of ϕ for which $d_\phi = 0$. We may view LM_ψ^A as the spatial version of the Durbin h statistic which can also be derived from the general LM principle. Unlike Durbin's h, however, LM_ψ^A cannot be computed using the OLS residuals while LM_ψ^* can be, since here the model is nonlinear even under $H_0 : \psi = 0$.

We can also obtain LM_ϕ^* easily to test $H_0 : \phi = 0$ in the presence of ψ yielding

$$LM_\phi^* = \frac{\left[\widetilde{u}' W_1 y / \widetilde{\sigma}^2 - T_{12} T_{22}^{-1} \widetilde{u}' W_2 \widetilde{u} / \widetilde{\sigma}^2\right]^2}{N \widetilde{J}_{\phi\cdot\gamma} - (T_{21})^2 T_{22}^{-1}} \tag{3.7}$$

Assuming $W_1 = W_2$, this simplifies to

$$LM_\phi^* = \frac{\left[\widetilde{u}'W_1 y/\widetilde{\sigma}^2 - \widetilde{u}'W_2\widetilde{u}/\widetilde{\sigma}^2\right]^2}{N\widetilde{J}_{\phi\cdot\gamma} - T} \tag{3.8}$$

It is straightforward to see that the standard one-directional test LM_ϕ given $\psi = 0$ is obtained as

$$LM_\phi = \frac{\left[\widetilde{u}'W_1 y/\widetilde{\sigma}^2\right]^2}{N\widetilde{J}_{\phi\cdot\gamma}} \tag{3.9}$$

Note that this statistic is identical to the one shown in equation (32) in Anselin (1988a).

Anselin (1988a) also derives an LM test for spatial residual autocorrelation in the presence of heteroskedasticity assuming no spatially lagged dependent variable. The statistic is given by

$$\frac{\left[\widehat{u}'\widehat{\Omega}^{-1}W_2\widehat{u}\right]^2}{T} \tag{3.10}$$

where Ω denotes the diagonal error covariance matrix incorporating heteroskedasticity. Using the information matrix given in Anselin (1988a) it is easy to check that $J_{\psi\phi\cdot\gamma} = 0$ in this model. This implies that our modified LM^* would revert to the conventional LM test given in (3.5). In other words, the simple standard LM statistic in (3.5) would give asymptotically same inference as (3.10) in the presence of <u>local</u> heteroskedasticity without the computational difficulties associated with (3.10).

4. Concluding Remarks

In this paper we have proposed simple diagnostic tests for spatial dependence. The proposed tests can be implemented using OLS residuals and are robust to local presence of a nuisance parameter. Anselin (1990) reviews some robust approaches to specification testing in the context of spatial econometric models, focusing on techniques that are robust to the presence of heteroskedasticity of an unknown form. For example, following Davidson and MacKinnon (1985), Anselin (1990) considers heteroskedasticity-robust tests for spatial error autocorrelation as well as spatial lag. Essentially, these may be viewed as tests for conditional mean specification robust against misspecification of the conditional variance. It is worth pointing out, however, that the information matrix between the parameters of the conditional mean function and those of the conditional variance will be block diagonal when the unknown heteroskedasticity is parametrized as, for example, in Breush and Pagan (1979). Davidson and MacKinnon's approach is not applicable when the information matrix is not block diagonal. Therefore, our proposed tests may be viewed as computationally simple and robust alternatives to some available procedures in spatial econometrics.

References

Anselin, L., 1988a, Lagrange multiplier test diagnostics for spatial dependence and spatial heterogeneity, Geographical Analysis 20, 1–17.

Anselin, L., 1988b, Model validation in a spatial econometrics: A review and evaluation of alternative approaches, International Regional Science Review 11, 279–316.

Anselin, L., 1989, Spatial dependence and spatial heterogeneity: Model specification issues in the spatial expansion paradigm, in: J.P. Jones and E. Casetti, eds., Applications of the expansion method (Croom Helm, London).

Anselin, L., 1990, Some robust approaches to testing and estimation in spatial econometrics, Regional Science and Urban Economics 20, 141–163.

Bera, A. and M. Yoon, 1990, Specification testing with misspecified alternatives, Paper presented at the Winter Meeting of the Econometric Society, Washington, D.C., U.S.A.

Blommestein, H., 1983, Specification and estimation of spatial econometric models: A discussion of alternative strategies for spatial economic modelling, Regional Science and Urban Economics 13, 251–270.

Breusch, T. and A. Pagan, 1979, A simple test for heteroscedasticity and random coefficient variation, Econometrica 47, 1287–1294.

Burridge, P., 1980, On the Cliff-Ord test for spatial correlation, Journal of the Royal Statistical Society B 42, 107–108.

Davidson, R. and J. MacKinnon, 1985, Heteroskedasticity-robust tests in regression directions, Annales De L'INSEE 59/60, 183–217.

Davidson, R. and J. MacKinnon, 1987, Implicit alternatives and the local power of test statistics, Econometrica 55, 1305–1329.

Ord, J., 1975, Estimation methods for models of spatial interaction, Journal of the American Statistical Association 70, 120–126.

Saikkonen, P., 1989, Asymptotic relative efficiency of the classical test statistics under misspecification, Journal of Econometrics 42, 351–369.

Upton, G. and B. Fingleton 1985, Spatial data analysis by example, (Wiley, New Nork).