FINITE AND INFINITE DIMENSIONAL ATTRACTORS FOR POROUS MEDIA EQUATIONS

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Abstract. The fractal dimension of the global attractors of porous media equations in bounded domains is studied. The conditions which guarantee this attractor to be finite dimensional are found and the examples of infinite-dimensional attractors beyond of that conditions are constructed. The upper and lower bounds for the Kolmogorov's ε-entropy of infinite-dimensional attractors are also obtained.

Introduction.

It is well-known that the long-time behaviour of many dissipative systems generated by evolution PDEs of mathematical physics can be described in terms of the so-called attractors. By definition, a global attractor is a compact invariant set in the phase space which attracts the images of all bounded subsets under the temporal evolution. Thus, on the one hand, the global attractor (if it exists) contains all of the nontrivial dynamics and, on the other hand, it is usually essentially smaller than the initial phase space.

In particular, in the case of dissipative PDEs in bounded domains, this attractor usually has finite Hausdorff and fractal dimension, see [2], [18], [23] and references therein. Consequently, in spite of the infinite-dimensionality of the initial phase space, the reduced dynamics on the attractor is (in a sense) finite-dimensional and can be studied by the methods of the classical (finite-dimensional) theory of dynamical systems.

In contrast to that, infinite-dimensional global/uniform attractors are typical for dissipative PDEs in *unbounded* domains or/and for the *nonautonomous* equations. In order to study such attractors one usually uses the concept of Kolmogorov's ε-entropy, see [5], [7], [11], [14], [24-27] for the details.

We however note that the above results have been obtained mainly for evolution PDEs with more or less regular structure (e.g., uniformly parabolic or uniformly hyperbolic). In contrast to this, very little is known about the equations with singularities or degeneration (even in the relatively simple case of scalar second order equations, like porous media equations, elliptic-parabolic equations etc.) which also play a significant role in modern mathematical physics, see [3], [22] and references therein. Indeed, although the attractors for such equations have been considered in a number of papers (see [1], [9-10], [16]), to the best of our knowledge, the questions related to the finite or infinite-dimensionality of these attractors have been not studied yet (as an exception, we mention the recent paper [21], where the finite dimensionality of the attractor of a 3D Cahn-Hilliard problem with logarithmic nonlinearity is proved).

The main aim of the present paper is to give a detailed study of the fractal dimension and Kolmogorov's entropy of attractors of the following degenerate porous media equation:

$$\partial_t u = \Delta_x(f(u)) - g(u) + h \tag{0.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ (equipped with Dirichlet boundary conditions). Here $f(u) \sim u|u|^{p-1}$ has a degeneracy at $u = 0$ $(p > 1)$, the function g satisfies the standard dissipativity assumptions and $h = h(x)$ is a given external force (see

Section 1 for the rigorous conditions).

The paper is organized as follows. In Section 1, we briefly recall some basic results on the existence, uniqueness and regularity of solutions of equation (0.1).

A natural class of equations of the form (0.1) whose global attractors are finite-dimensional is considered in Section 2. The finite-dimensionality is proved under the additional assumption

$$g'(0) > 0 \tag{0.2}$$

and strongly based on the global Hölder continuity of solutions of equation (0.1), see Theorem 2.1 of Section 2.

A finite-dimensional exponential attractor (in the sense of [15]) for problem (0.1) under assumption (0.2) is constructed in Section 3.

Finally, in Section 4, we show that the global attractor is usually infinite-dimensional if condition (0.2) is violated and, thus, the sign of $g'(0)$ appears to be crucial for finite or infinite-dimensionality of the global attractor. Namely, we consider here the particular case of (0.1) of the following form:

$$\partial_t u = \Delta_x(u|u|^{p-1}) + u - g(u) \tag{0.3}$$

with $p > 1$ and $g(u)$ vanishing near $u = 0$. Under these assumptions, we prove (see Theorem 4.1) that the associated global attractor is infinite-dimensional.

Moreover, we study also the Kolmogorov's ε-entropy $\mathbb{H}_\varepsilon(\mathcal{A})$ of this attractor (which, by definition, is a logarithm from the minimal number of ε-balls which cover the compact set $\mathcal{A}$). To be more precise, we establish the following lower bounds for that quantity:

$$\mathbb{H}_\varepsilon(\mathcal{A}) \geq C\left(\frac{1}{\varepsilon}\right)^{n(p-1)/2} \tag{0.4}$$

where C is some positive number independent of $\varepsilon > 0$.

Thus, porous media equations of the form (0.3) give natural examples of dissipative equations of mathematical physics in *bounded* domains with infinite-dimensional attractors. It is also worth noting that, although the infinite-dimensional global attractors are typical for regular equations in unbounded domains, even in that case the asymptotics of their Kolmogorov's ε-entropy were always logarithmical (like $(\log_2 \frac{1}{\varepsilon})^{n+1}$, see [24-26]). To the best of our knowledge, it is a first example of a global attractor whose ε-entropy has polynomial (with respect to ε^{-1}) asymptotics.

We also note that equation (0.3) with the nonlinearity g *vanishing* in the neighbourhood of zero looks rather artificial. That is the reason why, we extend (in Section 4) the above result on the following equation:

$$\partial_t u = \Delta_x(u^3) + u - u^3, \quad u\big|_{\partial\Omega} = 0 \tag{0.5}$$

which has analytic nonlinearities and can be considered as a natural degenerate analogue of the Chafee-Infante equation.

We finally note that the method of the study of the dimension of global attractors of degenerate parabolic equations developed in this paper seems to have a general nature and can be applied for other classes of degenerate equations (e.g., for elliptic-parabolic equations). We will return to these questions in forthcoming papers.

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§1 A priori estimates and regularity of solutions.

In this section, we briefly recall the known results on the regularity of solutions of porous media equations which will be systematically used in the next sections, see e.g. [4], [6], [8] and [17] fore more details.

In a bounded domain $\Omega \subset\subset \mathbb{R}^n$ with a sufficiently smooth boundary we consider the porous media equation in the following form:

$$\begin{cases} \partial_t u = \Delta_x f(u) - g(u) + h, \\ u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0. \end{cases} \tag{1.1}$$

where $u = u(t,x)$ is an unknown function, Δ_x is the Laplacian with respect to the variable $x = (x_1, \cdots, x_n)$, f and g are given functions and $h = h(x)$ is a given external force.

We assume that the function $f \in C^2(\mathbb{R})$ has a polynomial degeneracy at $u = 0$ and is nondegenerate for $u \neq 0$. To be more precise, we assume that

$$C_1|u|^{p-1} \le f'(u) \le C_2|u|^{p-1}, \quad f(0) = 0, \tag{1.2}$$

for some positive constants C_i and $p > 1$. It is also assumed that the function g satisfies the following dissipativity condition

$$g'(u) \ge -C + \kappa|u|^{q-1}, \tag{1.3}$$

for some $q > 1$, $\kappa > 0$ and the external force h belongs to $L^\infty(\Omega)$.

As usual, in order to prove the existence of a solution of problem (1.1), one considers the nondegenerate analogue of (1.1)

$$\begin{cases} \partial_t u = \Delta_x f(u) + \varepsilon\Delta_x u - g(u) + h, \\ u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0 \end{cases} \tag{1.4}$$

which obviously has a unique solution for every $\varepsilon > 0$ and sufficiently smooth u_0, see e.g. [20] and then passes to the limit $\varepsilon \to 0$. Following this scheme, we first derive uniform with respect to ε estimates for equation (1.4). We start from the standard $L^1 - L^\infty$-estimates.

Lemma 1.1. *Let the above assumptions hold and let u be a solution of equation (1.4). Then the following estimates hold:*

$$\begin{cases} 1) \ \|u(t)\|_{L^1(\Omega)} \le \|u(0)\|_{L^1(\Omega)} e^{-\alpha t} + C(1 + \|h\|_{L^1(\Omega)}), \\ 2) \ \|u(t)\|_{L^\infty(\Omega)} \le \|u(0)\|_{L^\infty(\Omega)} e^{-\alpha t} + C(1 + \|h\|_{L^\infty(\Omega)}) \end{cases} \tag{1.5}$$

where the positive constants α and C depend only on the function g and are independent of ε, t, u_0 and u. Moreover, the following L^1-L^∞-smoothing property holds:

$$\|u(t)\|_{L^\infty(\Omega)} \le Q(t^{-1} + \|h\|_{L^\infty(\Omega)}), \quad t > 0 \tag{1.6}$$

where the monotonic function Q is independent of ε, t and u.

Proof. Indeed, multiplying equation (1.4) scalarly in $L^2(\Omega)$ by the function $\operatorname{sgn} u = \operatorname{sgn}(f(u) + \varepsilon u)$ and using the Kato inequality $(\Delta_x v, \operatorname{sgn} v) \le 0$ and the dissipativity assumption (1.3), we deduce that

$$\partial_t \|u(t)\|_{L^1(\Omega)} + \kappa\|u(t)\|^q_{L^q(\Omega)} \le C + \|h\|_{L^1(\Omega)}. \tag{1.7}$$

Since $\kappa\|u\|^q_{L^q(\Omega)} \ge \|u(t)\|_{L^1(\Omega)} - C$, the the Gronwall inequality applied to (1.7) implies the first estimate of (1.5). In order to deduce the second estimate of (1.5), we use the comparison principle for second order parabolic equations and deduce that

$$y_-(t) \le u(t,x) \le y_+(t) \tag{1.8}$$

where $y_\pm(t)$ solve the following ODEs

$$y'_\pm(t) + g(y_\pm(t)) = \pm\|h\|_{L^\infty(\Omega)}, \quad y_\pm(0) = \pm\|u_0\|_{L^\infty(\Omega)}. \tag{1.9}$$

It remains to note that, due to the dissipativity assumption (1.3), the solutions $y_\pm(t)$ satisfy the analogue of estimate (1.5)(2) which together with (1.8) finishes the proof of estimate (1.5)(2). Finally, in order to verify (1.6), it remains to recall that, due to our assumptions, g has a superlinear growth rate as $u \to \infty$. Consequently, the solutions $y_\pm(t)$ satisfy estimate (1.6) (see e.g., [21]) which together with estimate (1.8) imply estimate (1.6) for the solution u and finishes the proof of the lemma.

The next Lemma gives some kind of energy estimates for equation (1.4).

Lemma 1.2. *Let the above assumptions hold and let u be a solution of* (1.4). *Then, for every $\delta > 0$, the following estimate holds:*

$$\begin{aligned} \|u\|_{W^{1/p-\delta,2p}([t,t+1]\times\Omega)} + \|\partial_t f(u)\|_{L^2([t,t+1]\times\Omega)} + \\ + \|\nabla_x f(u)\|_{L^\infty([t,t+1],L^2(\Omega))} \le Q(t^{-1} + \|h\|_{L^\infty(\Omega)}) \end{aligned} \tag{1.10}$$

where the monotonic function Q is independent of ε, $t > 0$ and u (here and below, $W^{s,p}$ denotes the Sobolev space of distributions whose derivatives up to order s belong to L^p, see e.g. [20]*).*

Proof. Without loss of generality we can assume that $t \le 1$. Then, multiplying equation (1.4) scalarly in $L^2(\Omega)$ by $f_\varepsilon(u) := f(u) + \varepsilon u$, and integrating over $[\delta, 2]$, $\delta > 0$, we get

$$\begin{aligned} (F_\varepsilon(u(2)),1) - (F_\varepsilon(u(\delta)),1) + \int_\delta^2 \|\nabla_x f_\varepsilon(u(t))\|^2_{L^2(\Omega)}\,dt + \\ + \int_\delta^2 (f_\varepsilon(u(t), g(u(t))) - (h, f_\varepsilon(u(t)))\,dt = 0 \end{aligned} \tag{1.11}$$

(here and below we denote by $(\cdot,\cdot)$ the standard inner product in $L^2(\Omega)$ and $F_\varepsilon(u) = \int_0^v f_\varepsilon(v)\,dv$). Together with L^∞-estimate (1.6) this estimates give

$$\int_\delta^2 \|\nabla_x f_\varepsilon(u)\|^2_{L^2(\Omega)}\,dt \le Q(\delta^{-1} + \|h\|_{L^\infty(\Omega)}). \tag{1.12}$$

Let us now multiply equation (1.4) by $(t-\delta)\partial_t f_\varepsilon(u)$ and integrate over $[\delta, T]\times\Omega$, $\delta \le T \le 2$. Then, we have

$$\begin{aligned} &\int_\delta^2 (t-\varepsilon) f'_\varepsilon(u(t))|\partial_t u(t)|^2\,dt + \\ &+ (T-\delta)(1/2\|\nabla_x f_\varepsilon(u(T))\|^2_{L^2(\Omega)} + (F_{\varepsilon,g}(u(T)),1) - (f_\varepsilon(u(T)),h)) = \\ &= \int_\delta^T 1/2\|\nabla_x f_\varepsilon(u(t))\|^2_{L^2(\Omega)} + (F_{\varepsilon,g}(u(t)),1) - (f_\varepsilon(u(t)),h)\,dt \end{aligned} \tag{1.13}$$

where $F_{\varepsilon,g}(u) := \int_0^u f'_\varepsilon(v)g(v)\,dv$. This estimate, together with L^∞-estimate (1.6) and estimate (1.22) implies that

$$\int_{2\delta}^2 f'(u(t))|\partial_t u(t)|^2\,dt + \|\nabla_x f(u)\|^2_{L^\infty([2\delta,2],L^2(\Omega))} \le Q(\delta^{-1} + \|h\|_{L^\infty(\Omega)}) \tag{1.14}$$

for the appropriate monotonic function Q. This estimate, together with the L^∞-estimate implies, in turn, that

$$\|\partial_t f(u)\|_{L^2([t,t+1]\times\Omega)} + \|\nabla_x f(u)\|_{L^\infty([t,t+1],L^2(\Omega))} \le Q(t^{-1} + \|h\|_{L^\infty(\Omega)}). \tag{1.15}$$

Thus, it only remains to estimate the first term in the left-hand side of (1.10). To this end we note that, according to (1.15), we have $v = f(u) \in W^{1,2}([t,t+1]\times\Omega) \subset W^{1-\delta,2}([t,t+1]\times\Omega)$. Then, due to Proposition A.1, see Appendix below, we have

$$\|u\|_{W^{1/p-\delta,2p}([t,t+1]\times\Omega)} \le C_{f,\delta}\|v\|^{1/p}_{W^{1,2}([t,t+1]\times\Omega)}$$

where the constant $C_{f,\delta}$ depends only on f and $\delta > 0$. Lemma 1.2 is proved.

The next lemma gives the uniform Lipschitz continuity of solutions in $L^1(\Omega)$.

Lemma 1.3. *Let the above assumptions hold and let $u_1(t)$ and $u_2(t)$ be two solutions of equation (1.4). Then, the following estimate holds:*

$$\|u_1(t) - u_2(t)\|_{L^1(\Omega)} \le e^{Kt}\|u_1(0) - u_2(0)\|_{L^1(\Omega)} \tag{1.16}$$

where $K := \max_{v\in\mathbb{R}}\{-g'(v)\}$.

Proof. Indeed, let $v(t) := u_1(t) - u_2(t)$. Then, this function satisfies the following linear equation:

$$\partial_t v = \Delta_x(l_1(t)v) + \varepsilon\Delta_x v - l_2(t)v, \quad v\big|_{\partial\Omega} = 0, \quad v\big|_{t=0} = u_1(0) - u_2(0) \tag{1.17}$$

where $l_1(t) := \int_0^1 f'(su_1(t) + (1-s)u_2(t))\,ds \geq 0$, $l_2(t) := \int_0^1 g'(su_1(t) + (1-s)u_2(t))\,ds \geq -K$. Multiplying now equation (1.17) by $\operatorname{sgn} v = \operatorname{sgn}((l_1(t)+\varepsilon)v)$ and using again the Kato inequality, we arrive at

$$\partial_t\|v(t)\|_{L^1(\Omega)} - K\|v(t)\|_{L^1(\Omega)} \leq 0. \tag{1.18}$$

Applying the Gronwall inequality to this relation, we finish the proof of Lemma 1.3.

We are now ready to verify the existence and uniqueness of a solution for the initial degenerate problem (1.1). To this end, we first formulate the definition of a weak solution of that problem.

Definition 1.1. We say that a function u is a weak solution of (1.1) if $u \in C([0,T], L^1(\Omega))$, $u \in L^\infty([t,T]\times\Omega)$ and $f(u) \in L^2([t,T], W_0^{1,2}(\Omega))$, for every $t > 0$ and it satisfies (1.1) in the sense of distributions.

The following theorem can be considered as the main result of the section

Theorem 1.1. *Let the above assumptions hold. Then, for every* $u_0 \in L^1(\Omega)$*, there exists a unique weak solution of problem* (1.1) *and this solution satisfies all of the estimates, formulated in Lemmata 1.1–1.3.*

Proof. We first establish the existence of a solution and assume additionally that u_0 is smooth enough. Let us consider a sequence $u_{\varepsilon_n}(t)$ of solutions of the auxiliary problem (1.4) with $\varepsilon_n \to 0$. Then, this sequence satisfies estimates (1.5) and (1.10) uniformly with respect to n. Moreover, since u_0 is smooth, then estimate (1.10) holds for $t = 0$ as well. In particular,

$$\|u_{\varepsilon_n}\|_{W^{1/p-\delta,2p}([t,t+1]\times\Omega)} \leq C \tag{1.19}$$

uniformly with respect to t and n. Thus, without loss of generality, we can assume that $u_{\varepsilon_n} \to u$ strongly in $C([0,T],L^1(\Omega))$ (due to the compactness of the embedding $W^{1/p-\delta,2p}([0,T]\times\Omega) \subset C([0,T],L^1(\Omega))$ if δ is small enough). Passing now in a standard way (see e.g., [2]) to the limit $n\to\infty$ in equations (1.4), we verify that u_0 satisfies the initial equation (1.1) (in the sense of distributions) and passing to the limit $n\to\infty$ in the uniform estimates of Lemmata 1.1–1.3, we verify that the solution thus constructed satisfies estimates (1.5), (1.6), (1.10) and (1.16). In particular, these estimates show that u is a weak solution in the sense of Definition 1.1. Thus, for smooth initial data u_0 the existence of a solution is verified. In order to relax the smoothness assumption, it remains to recall that the solutions constructed satisfy (1.16) with the constants which are independent of the initial data, consequently, approximating in $L^1(\Omega)$ the nonsmooth initial data $u_0\in L^1(\Omega)$ by a sequence of the smooth ones u_0^n, constructing the associated solutions $u^n(t)$ and passing to the limit $n\to\infty$, we obtain a weak solution $u(t)$ for every $u_0\in L^1(\Omega)$. Obviously, this solution will also satisfy all of the estimates of Lemma 1.1.–1.3. Thus, the existence is verified.

Let us now prove the uniqueness. Indeed, let $u_1(t)$ and $u_2(t)$ be two weak solutions of equation (1.1) and let $v(t) := u_1(t) - u_2(t)$. Then, this function satisfies the equation

$$\partial_t v = \Delta_x(l_1(t)v) - l_2(t)v \tag{1.20}$$

where $l_i(t)$ are the same as in (1.17). It would be natural (analogously to the proof of Lemma 1.3) to multiply equation (1.20) by $\operatorname{sgn}(v)$ and use the Kato inequality which would immediately give estimate (1.16) and finish the proof of the uniqueness, but, unfortunately, in contrast to the situation in Lemma 1.3, we do not have now enough regularity for the expression $(\Delta_x(l(t)v),\operatorname{sgn}(v))$ to have sense. Thus, we need to proceed in a little more precise way. To this end, we assume, in addition, that $u_i\in L^\infty([0,T]\times\Omega)$ and introduce the following "regularized" version of the conjugate equation for (1.20):

$$-\partial_t w = l_1(t)\Delta_x w + \varepsilon\Delta_x w,\quad w\big|_{t=T} = w_T,\quad w\big|_{\partial\Omega} = 0. \tag{1.21}$$

which we will consider in the space $W^{(1,2),2}([0,T]\times\Omega)$ (here and below, we denote by $W^{(1,2),q}$ the anisotropic Sobolev space consisting of distributions whose t-derivatives up to order one and x-derivatives up to order two belong to L^q, see [20]).

The next Lemma gives the solvability result for that equation.

Lemma 1.4. *Let the above assumptions hold. Then, for every $w_T \in W^{1,2}_0(\Omega)$ and every $\varepsilon > 0$, equation (1.21) possesses a unique solution $w \in W^{(1,2),2}([0,T]\times\Omega)$ and the following estimate holds:*

$$\|\nabla_x w(t)\|^2_{L^2(\Omega)} + 2\varepsilon \int_0^T \|\Delta_x w(t)\|^2_{L^2(\Omega)} \le \|\nabla_x w(T)\|^2_{L^2(\Omega)}, \quad t \in [0,T]. \tag{1.22}$$

Moreover, if in addition, $C_1 \le w_T(x) \le C_2$, then

$$C_1 \le w(t,x) \le C_2, \quad t \in [0,T]. \tag{1.23}$$

Proof. Indeed, according to our assumption, $l_i \in L^\infty([0,T]\times\Omega)$ and, moreover, due to (1.2), $l_1(t) \ge 0$. Therefore, equation (1.21) is non-degenerate. A priori estimate (1.22) can be obtained by multiplying (1.21) by $\Delta_x w$ and integrating over $[0,T]\times\Omega$ and the L^2-estimate for the derivative follows then from (1.22) and equation (1.21). Thus, the a priori estimate in $W^{(1,2),2}([0,T]\times\Omega)$ is obtained. The existence of a solution can be easily verified by e.g. the Galerkin method, see [2]. Finally, estimate (1.23) is just a maximum principle for the linear second order parabolic equation (1.21) (Being pedants, we cannot apply the classical maximum principle directly to equation (1.21) since the function $l_1(t,x)$ is only from L^∞ (and not smooth), but approximating it by the smooth ones, say, in $L^2([0,T]\times\Omega)$, we may apply the maximum principle for the solutions of the associated smooth equations and then pass to the limit in a standard way.) Lemma 1.4 is proved.

We are now ready to finish the proof of the uniqueness for weak solutions of (1.1). To this end, we multiply equation (1.20) by the solution $w(t)$ of the "conjugate" equation (1.21) (with some w_T) and integrate over $[\delta,T]\times\Omega$. Then, after the integration by parts, we have

$$(1.24)\quad (v(T),w(T))-(v(0),w(0))+\\ +\varepsilon\int_0^T(\Delta_x w(t),v(t))\,dt+\int_0^T(l_2(t)v(t),w(t))\,dt=0.$$

We now approximate the function $w_T^0:=\operatorname{sgn}(v(T))$ in the $L^2(\Omega)$ metric by $w_T^n\in W_0^{1,2}(\Omega)$ in such way that $-1\le w_T^n\le 1$ and construct the appropriate solutions $w^n(t)$ of equation (1.21). Then, due to (1.23), $-1\le w^n(t,x)\le 1$ and, consequently, (1.24) reads

$$(1.25)\quad (v(T),w_T^n)+\varepsilon\int_0^T(\Delta_x w^n(t),v(t))\,dt\le\|v(0)\|_{L^1(\Omega)}+L_2\int_0^T\|v(t)\|_{L^1(\Omega)}\,dt$$

where $L_2=\|l_2(t,x)\|_{L^\infty([0,T]\times\Omega)}$. We are now pass to the limit $\varepsilon\to 0$ (with a fixed n) in the inequality (1.25) using (1.22) and

$$\varepsilon\int_0^T(\Delta_x w^n(t),v(t))\,dt\le\varepsilon^{1/4}(\varepsilon^{1/2}\|\Delta_x w^n\|^2_{L^2([0,T]\times\Omega)}+\|v\|^2_{L^2([0,T]\times\Omega)}).$$

Then, we have

$$(1.26)\qquad (v(T),w_T^n)\le\|v(0)\|_{L^1(\Omega)}+L_2\int_0^T\|v(t)\|_{L^1(\Omega)}\,dt.$$

Finally, passing to the limit $n\to\infty$ in (1.26), we get

$$(1.27)\qquad \|v(T)\|_{L^1(\Omega)}\le\|v(0)\|_{L^1(\Omega)}+L_2\int_0^T\|v(t)\|_{L^1(\Omega)}\,dt.$$

Since $T>0$ is arbitrary, then the Gronwall inequality, applied to (1.27) implies that

$$\|v(t)\|_{L^1(\Omega)}\le e^{L_2t}\|v(0)\|_{L^1(\Omega)}.$$

Thus, we have proved that every weak solution $u(t)$ of (1.1) is unique under the additional assumption $u\in L^\infty([0,T]\times\Omega)$. Therefore, every such solution coincides with the solution obtained by passing to the limit $\varepsilon\to 0$ in the nondegenerate equations (1.4). This, implies, in turns, that all such solutions should satisfy estimate (1.16).

Let us now consider the general case of two weak solutions u_1 and u_2 which do not belong to $L^\infty([0,T]\times\Omega)$. Then, due to the definition of a weak solution, $u_i \in L^\infty([\delta,T]\times\Omega)$ for every $\delta>0$ and, consequently, due to (1.16), we have

$$\|u_1(t)-u_2(t)\|_{L^1(\Omega)} \le e^{K(t-\delta)}\|u_1(\delta)-u_2(\delta)\|_{L^1(\Omega)}. \tag{1.28}$$

Passing now to the limit $\delta \to 0$ in (1.28) and taking into account that $u_i \in C([0,T],L^1(\Omega))$, we obtain estimate (1.16) for any two weak solutions of (1.1). Theorem 1.1 is proved.

Remark 1.1. In Theorem 1.1, we have proved, in particular, that every weak solution of (1.1) can be approximated by smooth solutions of the nondegenerate problem (1.4). This allows us in the sequel to use the Kato inequality for deriving more delicate estimates without taking care about the regularity. Indeed, all that estimates can be easily justified by this approximating procedure.

We also note that the rather strong dissipativity condition (which guarantees, in particular, the superlinear growth rate of the nonlinearity g) has been posed only in order to avoid the technicalities in proving the L^1-L^∞ smoothing property for the solutions of (1.1) and can be relaxed to the standard dissipativity condition:

$$\limsup_{|u|\to\infty}\frac{g(u)}{u}>0.$$

We conclude this Section by formulating the result on the Hölder continuity of solutions of degenerate parabolic equations which is crucial for our study of the dimension of the attractor.

Theorem 1.2. *Let the above assumptions hold and let u be a weak solution of* (1.1)*. Then, there exists a positive constant α such that*

$$\|u\|_{C^\alpha([t,t+1]\times\Omega)} \le Q(t^{-1}+\|h\|_{L^\infty(\Omega)}) \tag{1.29}$$

where $t>0$ and Q is some monotonic function.

In the multidimensional case $n\ge 2$ the Hölder continuity (1.29) is a rather delicate fact and its proof is based on the proper modification of the De Giorgi technique, see [6], [8] and [17]. By contrast, in the one-dimensional case, it can be easily derived from standard energy estimates. For the convenience of the reader, we give the proof for the 1D case.

Proof: 1D case. Indeed, according to Lemma 1.2 and Theorem 1.1, any weak solution u satisfies

$$\|f(u)\|_{L^\infty([t,t+1],W^{1,2}(\Omega))\cap W^{1,2}([t,t+1],L^2(\Omega))} \le Q(t^{-1}+\|h\|_{L^\infty(\Omega)}). \tag{1.30}$$

Moreover, by interpolation, see [20], we have

$$\|v\|_{C^\alpha([t,t+1],W^{1-2\alpha,2}(\Omega))} \le C_\alpha \|v\|_{L^\infty([t,t+1],W^{1,2}(\Omega))\cap W^{1,2}([t,t+1],L^2(\Omega))} \tag{1.31}$$

for $0 \le \alpha < 1/2$. In $1D$-case, we have the embedding $W^{1-2\alpha,2}(\Omega) \subset C^{1/2-2\alpha}(\Omega)$. Taking $\alpha = 1/6$, we finally derive

$$\|f(u)\|_{C^{1/6}([t,t+1]\times\Omega)} \le Q(t^{-1}+\|h\|_{L^\infty(\Omega)}) \tag{1.32}$$

for some monotonic function Q. Proposition A.1 together with (1.32) imply (1.29) with $\alpha = 1/(6p)$. Theorem 1.2 for 1D is proved.

§2 The finite-dimensional case: global attractors.

In the previous section we have proved that equation (1.1) generates a uniformly Lipschitz continuous semigroup $S(t)$ on the phase space $\Phi = L^1(\Omega)$ via

$$S(t)u_0 = u(t), \quad u_0 \in L^1(\Omega), \quad t > 0 \tag{2.1}$$

where $u(t)$ is a unique weak solution of (1.1) (see Theorem 1.1). The present section is devoted to study of the long-time behaviour of the trajectories of that semigroup in terms of finite-dimensional global attractors. The case where the limit dynamics is infinite-dimensional will be considered in Section 4.

We first recall that, by definition, the set $\mathcal{A} \subset \Phi$ is a global attractor of the semigroup $S(t)$ if the following conditions are satisfied:

1) the set $\mathcal{A}$ is a compact subset of the phase space $\Phi = L^1(\Omega)$;

2) it is strictly invariant, i.e. $S(t)\mathcal{A} = \mathcal{A}$, for all $t \geq 0$;

3) it attracts the images of all bounded subsets as time tends to infinity, i.e., for every bounded subset $B \subset \Phi$ and every neighbourhood $\mathcal{O}(\mathcal{A})$ there exists time $T = T(B, \mathcal{O})$, such that

$$S(t)B \subset \mathcal{O}(\mathcal{A}), \quad \text{for all} \quad t \geq T. \tag{2.2}$$

This assumption can be reformulated in the following equivalent form: for every bounded set B

$$\operatorname{dist}(S(t)B, \mathcal{A}) \to 0 \quad \text{as } t \to \infty \tag{2.3}$$

where $\operatorname{dist}(\cdot, \cdot)$ is a non-symmetric Hausdorff distance between sets in Φ:

$$\operatorname{dist}(X, Y) = \sup_{x \in X} \inf_{y \in Y} \|x - y\|_{\Phi}. \tag{2.4}$$

The next lemma states the existence of such an attractor.

Lemma 2.1. *Let the assumptions of Section 1 hold. Then, the semigroup $S(t)$ associated with equation* (1.1) *possesses a global attractor $\mathcal{A}$ in the phase space $L^1(\Omega)$ which is globally bounded in $C^\alpha(\Omega)$ (for some sufficiently small α) and has the following structure:*

$$\mathcal{A} = \mathcal{K}\big|_{t=0} \tag{2.5}$$

where $\mathcal{K}$ is a set of all bounded solutions of (1.1) *defined for all t. Moreover, this set satisfies*

$$\|\mathcal{K}\|_{C^\alpha(\mathbb{R}\times\Omega)} \leq Q(\|h\|_{L^\infty(\Omega)}). \tag{2.6}$$

for some monotone function Q.

Proof. As usual, in order to verify the existence of a global attractor, one needs to verify two properties:

1) the maps $S(t) : \Phi \to \Phi$ are continuous for every fixed t;

2) the semigroup $S(t)$ possesses a (pre)compact absorbing set in Φ, see [2], [18].

In our case, the first property is obvious, since, due to Lemma 1.3, the semigroup $S(t)$ is even globally Lipschitz continuous in Φ. Moreover, the existence of an absorbing set, bounded in $C^\alpha(\Omega)$, is an immediate corollary of Theorem 1.2. Thus, due to the abstract theorem on the attractor's existence, this semigroup possesses a global attractor $\mathcal{A}$, bounded in $C^\alpha(\Omega)$. Formula (2.5) is also a corollary of that theorem and (2.6) follows from Theorem 1.2. Lemma 2.1 is proved.

For the further investigation of the constructed global attractor we recall the definition of the so-called Kolmogorov ε-entropy, see [19] for the details.

Definition 2.1. Let K be a (pre)compact set in a metric space M. Then, for every $\varepsilon > 0$, K can be covered by the finite number of ε-balls in M. Let $N_\varepsilon(K, M)$ be the minimal number of such balls. Then, by definition, the Kolmogorov ε-entropy of K is a binary logarithm of that number:

$$\mathbb{H}_\varepsilon(K, M) := \log_2 N_\varepsilon(K, M). \tag{2.7}$$

The fractal dimension $\dim_f(K)$ of the set K can be expressed in terms of this entropy via

$$\dim_f(K, M) := \frac{\mathbb{H}_\varepsilon(K, M)}{\log_2 1/\varepsilon}. \tag{2.8}$$

We also recall that the Kolmogorov entropy is finite for every compact set K and every $\varepsilon > 0$ and the fractal dimension can be infinite (if the space M is infinite-dimensional).

The next theorem which establishes the finite-dimensionality of the global attractor under the additional assumption that equation (1.1) is asymptotically stable near $u = 0$ can be considered as the main result of the section.

Theorem 2.1. *Let the assumptions of Section 1 hold and let, in addition,*

$$g'(0) > 0. \tag{2.9}$$

Then the fractal dimension of $\mathcal{A}$ in $C(\Omega)$ is finite:

$$\dim_f(\mathcal{A}, C(\Omega)) < \infty. \tag{2.10}$$

Proof. As usual, see [5],[12-13],[23] in order to prove the finite-dimensionality of the attractor, we need to consider an arbitrary finite ε-net V_ε in $\mathcal{A}$ in the metric of $L^1(\Omega)$ (with a sufficiently small positive ε) and to construct, using this net, a $\kappa\varepsilon$-net $V_{\kappa\varepsilon}$ (with $\kappa<1$) in $\mathcal{A}$ satisfying

$$\#V_{\kappa\varepsilon}\le L\#V_\varepsilon \tag{2.11}$$

where the constants κ and L are independent of ε and of the initial covering V_ε (here and below $\#S$ means the number of elements of the finite set S). Then, iterating this procedure we can prove the finite dimensionality of the attractor.

Let $V_\varepsilon=\{u_0^i\}_{i=1}^{N_\varepsilon}$, $V_\varepsilon\subset\mathcal{A}$ be an arbitrary ε-net in $\mathcal{A}$ (with $N_\varepsilon=\#V_\varepsilon$). Then, in order to construct the required $\kappa\varepsilon$-net, it is sufficient to construct, for every $u_0\in\mathcal{A}$, the $\kappa\varepsilon$-net $V_{\kappa\varepsilon}(u_0)$ in the the image $S(T)(B(\varepsilon,u_0,L^1)\cap\mathcal{A}))$ (for some positive T) of the ε-ball centered at u_0 intersected with the attractor (here and below we denote by $B(R,x,X)$ an R-ball in the space X centered at $x\in X$) satisfying

$$\#V_\varepsilon(u_0)\le L. \tag{2.12}$$

Then, obviously, the set $V_{\kappa\varepsilon}:=\cup_{u_0\in V_\varepsilon}V_{\kappa\varepsilon}(u_0)$ gives a $\kappa\varepsilon$-net in $S(T)\mathcal{A}$ satisfying (2.11). Finally, since $S(T)\mathcal{A}=\mathcal{A}$, the required $\kappa\varepsilon$-net in $\mathcal{A}$ would be constructed.

Thus, we only need to construct the $\kappa\varepsilon$-net in the set $S(T)(B(\varepsilon,u_0,L^1(\Omega))\cap\mathcal{A})$ for all sufficiently small ε, $u_0\in\mathcal{A}$ and some $T>0$ satisfying (2.12) with the constant L independent on ε and u_0. So, let $u_0\in\mathcal{A}$ and $\varepsilon\ll1$ be fixed.

Let us introduce, for every $\theta>0$, the following sets:

$$\begin{aligned}L(\theta)=L(\theta,u_0)&:=\{x\in\Omega,\quad |u_0(x)|>\theta\},\\ S(\theta)=S(\theta,u_0)&:=\{x\in\Omega,\quad |u_0(x)|<\theta\}.\end{aligned} \tag{2.13}$$

Then, obviously, $S(\theta_1)\subset S(\theta_2)$ and $L(\theta_2)\subset L(\theta_1)$ if $\theta_1\le\theta_2$. Moreover, since $u_0\in C^\alpha(\Omega)$ and $\|u_0\|_{C^\alpha}\le M$, then these sets are open,

$$\partial S(\theta)=\partial L(\theta)=\{x\in\Omega,\ u_0(x)=\theta\},\quad \Omega=S(\theta)\cup L(\theta)\cup\partial L(\theta)$$

and, for every $\delta>0$,

$$d[\partial S(\theta+\delta),\partial S(\theta)]\ge C_\delta \tag{2.14}$$

where the constant C_δ depends only on δ, α and M and is independent on θ and on the concrete choice of $u_0 \in \mathcal{A}$. Here and below we denote by $d(X,Y)$ the standard metric distance between sets in $\mathbb{R}^n$:

$$d[X,Y] := \inf_{x \in X} \inf_{y \in Y} \|x - y\|.$$

Let us fix now $\theta > 0$ and $\beta > 0$ in such way that

$$g'(u) > 3\beta > 0, \quad \forall |u| < 5\theta \tag{2.15}$$

(this is possible by assumption (2.9)) and the cut-off function $\phi \in C^\infty(\mathbb{R}^n)$, $\phi \geq 0$ such that:

$$\phi(x) = \begin{cases} 1, & x \in S(4\theta), \\ 0, & x \in L(5\theta). \end{cases} \tag{2.16}$$

Due to (2.14), and Proposition A.2 this cut-off function ϕ can be chosen in such a way that

$$\|\phi\|_{C^k(\Omega)} \leq C_k \tag{2.17}$$

where the constants C_k depend only on M, α and k and are independent of u_0.

We recall that the trajectory $u(t) := S_t u_0$ belongs to C^α with respect to t and x, consequently, there exists time $T > 0$ (also depending only on M, β and θ) such that

$$\begin{aligned} g'(u(t,x)) > 2\beta, \quad x \in S(5\theta),\ t \in [0,T], \\ |u(t,x)| > \theta/2, \quad x \in L(\theta),\ t \in [0,T]. \end{aligned} \tag{2.18}$$

On the other hand, due to the interpolation inequality

$$\|w\|_{C(\Omega)} \leq C\|w\|^{\gamma}_{L^1(\Omega)}\|w\|^{1-\gamma}_{C^\alpha(\Omega)} \tag{2.19}$$

(for some $0 < \gamma < 1$) and the Hölder continuity, we obtain that

$$|v(t,x) - u(t,x)| \leq C_1\varepsilon^\gamma, \quad x \in \Omega,\ t \in [0,T] \tag{2.20}$$

for every solution $v(t)$ such that $v(0) \in \mathcal{A} \cap B(\varepsilon, u_0, L^1)$. Thus, assuming that ε is small enough ($\varepsilon \le \varepsilon_0 \ll 1$ where $\varepsilon_0 > 0$ is independent of $u_0 \in \mathcal{A}$), we may improve (2.18) in the following way:

$$\begin{aligned} &g'(v(t,x)) > \beta, \quad x \in S(5\theta),\ t \in [0,T],\\ &|v(t,x)| > \theta/4, \quad x \in L(\theta),\ t \in [0,T] \end{aligned} \tag{2.21}$$

uniformly with respect to $v_0 \in \mathcal{A} \cap B(\varepsilon, u_0, L^1)$.

In order to construct the required $\kappa\varepsilon$-net in $S(T)(\mathcal{A} \cap B(\varepsilon, u_0, L^1))$, we need to derive some smoothing property for differences of solutions. To this end, we consider the difference $w(t) := u_1(t) - u_2(t)$ of two solutions satisfying $u_i(0) \in B(\varepsilon, u_0, L^1)$. Then the function $w(t)$ solves the following equation:

$$\partial_t w(t) = \Delta_x(l_1(t)w) - l_2(t)w, \quad w\big|_{t=0} = u_1(0) - u_2(0), \quad t \in [0,T] \tag{2.22}$$

where $l_i(t)$ are the same as in (1.17).

Let us first consider the case of domains $L(\theta)$ where the equation (1.1) is, in a sense, nondegenerate. To this end, we need the following lemma which is similar to the classical interior regularity estimates for the linear parabolic equation (2.22).

Lemma 2.2. *Let $u_0 \in \mathcal{A}$ be arbitrary, the sets $L(\theta, u_0)$ be defined via* (2.13). *Assume also that $u_1(t)$ and $u_2(t)$ are two solutions of* (1.1) *such that $u_i(0) \in \mathcal{A} \cap B(\varepsilon, u_0, L^1(\Omega))$. Then, the following estimate holds for every $t_0 \in (0,T)$:*

$$\|u_1 - u_2\|_{C^\alpha([t_0,T]\times L(3\theta))} \le C_{t_0}\|u_1(0) - u_2(0)\|_{L^1(\Omega)} \tag{2.23}$$

where the constant C_{t_0} depends on t_0 and is independent of ε, u_0, u_1 and u_2.

Proof. We first prove that, for every $r > 2$, the functions u_1 and u_2 satisfy the following estimate:

$$\|u_i\|_{W^{(1,2),r}([t_0/2,T]\times L(2\theta))} \le C_r, \quad i = 1,2 \tag{2.24}$$

where the constant C_r depends on r, but is independent of the concrete choice of u_0, ε and of the trajectories u_1 and u_2 (starting from $\mathcal{A} \cap B(\varepsilon, u_0, L^1)$). Indeed, let us verify it for $u = u_1$ (for $u = u_2$ it can be verified analogously). To this end, we introduce a new dependent variable $v(t,x) := f(u(t,x))$. Then, since $f(u)$ is nondegenerate if $|u| > \theta > 0$, one can easily verify that the function v solves the following equation:

$$\partial_t v = a\Delta_x v + h_u, \quad (t,x) \in [0,T] \times L(\theta) \tag{2.25}$$

where $a(t,x) := f'(u(t,x))$ and $h_u(t,x) := f'(u(t,x))[h(x) - g(u(t,x))]$. Moreover, due to (2.6), the coefficient a is uniformly (with respect to $u \in \mathcal{K}$) Hölder continuous and the function h_u is uniformly bounded in L^∞. Furthermore, due to the second inequality of (2.21) and assumption (1.2), we have

$$a(t,x) \geq C_1, \quad (t,x) \in [0,T] \times L(\theta)$$

where the constant C_1 is also independent of the choice of u_0 and u. Thus, we can apply the standard L^r-interior regularity estimate for the solution of the linear nondegenerate equation (2.25), see Proposition A.4 and Corollary A.1. Due to (2.14) with $\delta = \theta$, this estimate implies

$$\|v\|_{W^{(1,2),r}([t_0/2,T]\times L(2\theta))} \leq C_r(\|h\|_{L^r([0,T]\times L(\theta))} + \|v\|_{L^1([0,T]\times L(\theta))}) \leq C'_r.$$

Returning back to the variable $u = f^{-1}(v)$ and using that $f \in C^2$ (and nondegenerate outside of zero), we deduce estimate (2.24).

We now return to equation (2.22) which will be now considered in the domain $[t_0/2, T] \times L(2\theta)$. To this end, we first need to study the regularity of the coefficient $l_1(t)$. Indeed, since $f \in C^2$ and estimate (2.6) holds, then

$$\|l_1\|_{C^\alpha([0,T]\times\Omega)} \leq C \tag{2.26}$$

where the constant C is independent of u_1 and u_2. Moreover, due to (2.24), we have

$$\|\partial_t l_1\|_{L^r([t_0/2,T]\times L(2\theta))} \leq C\sum_{i=1}^{2}\|\partial_t u_i\|_{L^r([t_0/2,T]\times L(2\theta))} \leq C''_r \tag{2.27}$$

and, finally, due to the second inequality of (2.21), we also have

$$l_1(t,x) \ge \kappa > 0, \quad (t,x) \in [t_0/2, T] \times L(2\theta) \tag{2.28}$$

where the constants C, C_r'' and κ are independent of the concrete choice of u_0, u_1 and u_2.

Let us introduce a new dependent variable $Z(t) := l_1(t)w(t)$. Then, this function solves

$$\partial_t Z = a(t,x)\Delta_x Z + l(t,x)Z, \quad (t,x) \in [t_0/2, T] \times L(2\theta) \tag{2.29}$$

where $a(t,x) := l_1(t,x)$ and $l(t,x) := l_2(t,x) - \frac{\partial_t l_1(t,x)}{l_1(t,x)}$. Furthermore, estimates (2.26)–(2.28) (together with the obvious fact that l_2 is uniformly bounded in the L^∞-norm) allows us to apply the L^q-interior regularity estimate for equation (2.29) which gives, see Proposition A.4 and Corollary A.1,

$$\|Z\|_{W^{(1,2),q}([t_0,T]\times L(3\theta))} \le C_q \|Z\|_{L^1([t_0/2,T]\times L(2\theta)} \le C_q' \|w\|_{L^1([0,T]\times\Omega)}. \tag{2.30}$$

Fixing now q large enough to have the embedding $W^{(1,2),q} \subset C^\alpha$, returning to the initial variable w and using (2.26), we have

$$\|w\|_{C^\alpha([t_0,T]\times L(3\theta))} \le C\|w\|_{L^1([0,T]\times\Omega)}. \tag{2.31}$$

Estimating the right-hand side of (2.31) using (1.16), we deduce (2.23) and finish the proof of Lemma 2.2.

Let us consider now equation (2.22) on the set $S(4\theta)$ where, due to the first condition of (2.21), we have, in a sense, the contraction property for the differences of solutions. Indeed, let us multiply equation (2.22) by

$$\phi(x)\operatorname{sgn}(w(t,x)) = \phi(x)\operatorname{sgn}(\phi(x)l_1(t,x)w(t,x)) \tag{2.32}$$

(where ϕ is defined by (2.16)) and use the equation

$$\phi\Delta_x[l_1(t)w] = \Delta_x(\phi(x)l_1(t)w) - 2\nabla_x\phi.\nabla_x(l_1(t)w) - \Delta_x\phi l_1(t)w. \tag{2.33}$$

Integrating then over $x \in \Omega$ and using the Kato inequality, we derive that

$$\partial_t(\phi, |w|) \le (\Delta_x \phi, l_1(t)|w|) - (g(u_1) - g(u_2), \phi \operatorname{sgn}(u_1 - u_2)). \tag{2.34}$$

Taking into account the first inequality of (2.21) and the fact that $\Delta_x \phi(x) = 0$ for $x \in S(4\theta)$, we deduce from (2.34) that

$$\partial_t(\phi, |w(t)|) + \beta(\phi, |w(t)|) \le C\|w(t)\|_{L^1(L(4\theta))} \tag{2.35}$$

and consequently, due to the Gronwall inequality and estimate (1.16), we infer

$$\begin{aligned} \|u_1(T) - u_2(T)\|_{L^1(S(4\theta))} \le e^{Kt_0 - \beta(T-t_0)} \|u_1(0) - u_2(0)\|_{L^1(\Omega)} + \\ + C_{t_0}\|u_1 - u_2\|_{L^1([t_0,T]\times L(4\theta))} \end{aligned} \tag{2.36}$$

where t_0 is an arbitrary time in the interval $(0, T)$.

Let us now fix t_0 in such way that

$$e^{Kt_0 - \beta(T-t_0)} < 1 - \delta < 1.$$

In this case (2.29) really gives a contraction in $S(4\theta)$. Moreover, using that

$$\|w\|_{L^1(\Omega)} \le \|w\|_{L^1(S(4\theta))} + \|w\|_{L^1(L(7\theta/2))}$$

and that $\|w(T)\|_{L^1(L(7\theta/2))} \le C\|w\|_{C([t_0,T]\times L(7\theta/2))}$, we derive from (2.23) and (2.36) the following basic inequalities:

$$\begin{cases} \|u_1 - u_2\|_{C^\alpha([t_0,T]\times L(3\theta))} \le P\|u_1(0) - u_2(0)\|_{L^1(\Omega)}, \\ \|u_1(T) - u_2(T)\|_{L^1(\Omega)} \le (1-\delta)\|u_1(0) - u_2(0)\|_{L^1(\Omega)} + \\ +P\|u_1 - u_2\|_{C([t_0,T]\times L(7\theta/2))} \end{cases} \tag{2.37}$$

which is valid for all solutions u_i such that $u_i(0) \in B(\varepsilon, u_0, L^1) \cap \mathcal{A}$ where the constants $T > 0$, $\delta > 0$ and P are independent of the concrete choice of $\varepsilon \le \varepsilon_0$ and $u_0 \in \mathcal{A}$.

Our next observation is the fact that the embedding $C^\alpha([t_0, T] \times L(3\theta, u_0)) \subset C([t_0, T] \times L(7\theta/2, u_0))$ is compact. Moreover, since $L(7\theta/2, u_0) \subset L(3\theta, u_0)$ and

$d[\partial L(3\theta, u_0), L(7\theta/2, u_0)] \ge C_\theta$ with the constant C_θ independent of $u_0 \in \mathcal{A}$, then this embedding is uniformly (with respect to $u_0 \in \mathcal{A}$) compact. This means that there exists a monotone decreasing function $\mathbb{M}(\delta)$ such that

$$\mathbb{H}_\delta\left(B(1, 0, C^\alpha([t_0, T] \times L(3\theta, u_0))), C([t_0, T] \times L(7\theta/2, u_0))\right) \le \mathbb{M}(\delta) \tag{2.38}$$

uniformly with respect to $u_0 \in \mathcal{A}$ and $\delta > 0$, see Proposition A.5.

We are now ready to construct the required $\kappa\varepsilon$-net in the set $S(T)(B(\varepsilon, u_0, L^1) \cap \mathcal{A})$. To this end, we fix a minimal $\delta\varepsilon/(4P)$-net V in the ball $B(P, u, C^\alpha([t_0, T] \times L(3\theta)))$, where $u(t) := S(t)u_0$, endowed with the metric of $C([t_0, T] \times L(7\theta/2))$. Then, due to (2.38), the number of points in that net can be estimated via

$$\begin{aligned} \#V = N_{\varepsilon\delta/(4P)}(B(P\varepsilon, u, C^\alpha([t_0, T] \times L(3\theta)), C([t_0, T] \times L(7\theta/2))) = \\ = N_{\delta/(4P^2)}(B(1, 0, C^\alpha([t_0, T] \times L(3\theta)), C([t_0, T] \times L(7\theta/2))) \le e^{\mathbb{M}(\delta/(4P^2))} := L \end{aligned} \tag{2.39}$$

where L is independent of u_0. Moreover, since we only need to control the trajectories $v(t)$ starting from $\mathcal{A} \cap B(\varepsilon, u_0, L^1)$ (all these trajectories are contained in the ball $B(P\varepsilon, u, C^\alpha([t_0, T] \times L(3\theta))$ due to the first estimate of (2.37)), then increasing the radii of the balls by a factor of two, we may construct the $\delta\varepsilon/(2P)$-net $\bar{V} = \{u^1, \cdots, u^N\}$ in the set of these trajectories (in the metric of $C([t_0, T] \times L(7\theta/2))$) such that the functions $\{u^1, \cdots, u^L\}$ are also the trajectories of (1.1) started from $\mathcal{A} \cap B(\varepsilon, u_0, L^1)$ and $\#\bar{V} \le L$. We claim that the set

$$V_{\kappa\varepsilon}(u_0) := \bar{V}\big|_{t=T} \tag{2.40}$$

is the required $\kappa\varepsilon$-net in $S(T)(B(\varepsilon, u_0, L^1) \cap \mathcal{A})$ with $\kappa = 1 - \delta/2 < 1$. Indeed, let $v(t)$ be an arbitrary trajectory starting from the $B(\varepsilon, u_0, L^1) \cap \mathcal{A}$. Then, due to our construction of the net $\bar{V}$, there exists a solution $u^i \in \bar{V}$ satisfying

$$\|u^i - v\|_{C([t_0, T] \times L(7\theta/2)))} \le \delta\varepsilon/(2P). \tag{2.41}$$

Inserting this estimate into second estimate of (2.37) and using that $\|u^i(0) - v(0)\|_{L^1(\Omega)} \le \varepsilon$, we infer

$$\|u^i(T) - v(T)\|_{L^1(\Omega)} \le (1 - \delta)\varepsilon + \delta\varepsilon/2 = (1 - \delta/2)\varepsilon.$$

Thus, (2.40) is indeed the required $\kappa\varepsilon$-net in $S(T)(B(\varepsilon, u_0, L^1)\cap\mathcal{A})$. Since an ε-ball of the attractor has been chosen arbitrarily, then the reccurrent formula (2.11) is verified for $\varepsilon \le \varepsilon_0$.

We are now ready to finish the proof of the theorem. Indeed, since the attractor $\mathcal{A}$ is compact in $L^1(\Omega)$, then

$$\mathbb{H}_{\varepsilon_0}(\mathcal{A}, L^1(\Omega)) \le C_{\varepsilon_0} < \infty. \tag{2.42}$$

Moreover, starting from that ε_0-net and using the reccurrent procedure described above, we prove that

$$\mathbb{H}_{\kappa^m\varepsilon_0}(\mathcal{A}, L^1(\Omega)) \le C_{\varepsilon_0} + m\log_2 L \tag{2.43}$$

for all $m \in \mathbb{N}$. Together with (2.8) this estimate gives

$$\dim_f(\mathcal{A}, L^1(\Omega)) \le \frac{\log_2 L}{\log_2 1/\kappa} < \infty. \tag{2.44}$$

The finite-dimensionality in $C(\Omega)$ is now an immediate corollary of Hölder continuity (2.6) and the interpolation inequality (2.19). Theorem 2.1 is proved.

§3 The finite dimensional case: exponential attractors.

In the previous section, we have proved the existence of a finite dimensional global attractor $\mathcal{A}$ for problem (1.1). However, according to the definition of $\mathcal{A}$, we know only that $\operatorname{dist}(S(T)B, \mathcal{A})$ tends to zero as $t \to \infty$ (for every bounded subset B) and do not have any information on the *rate* of convergence in (2.3). Moreover, this rate of convergence can be arbitrarily slow and, to the best of our knowledge, there is no way to control this rate of convergence in a more or less general situation (e.g., to express it in terms of physical parameters of the system considered). This leads to essential difficulties in numerical simulations of global attractors and even makes them, in a sense, unobservable.

In order to overcome this difficulty, the concept of the so-called *exponential* attractor has been suggested in [15]. By definition, a set $\mathcal{M} \subset \Phi$ is an exponential attractor of the semigroup $S(t)$ if the following conditions are satisfied:

1) the set $\mathcal{M}$ is compact in $\Phi = L^1(\Omega)$;

2) it is semi-invariant, i.e. $S(t)\mathcal{M} \subset \mathcal{M}$;

3) it attracts *exponentially* the images of all bounded sets, i.e., for every $B \subset \Phi$ bounded,

$$\operatorname{dist}(S(t)B, \mathcal{M}) \le Q(\|B\|_\Phi)e^{-\alpha t} \tag{3.1}$$

where the positive constant α and the monotonic function Q are independent of B;

4) it has finite fractal dimension in Φ:

$$\dim_f(\mathcal{M}, \Phi) \le C < \infty. \tag{3.2}$$

We recall that in contrast to global attractors, an exponential attractor is not unique and, consequently, the particular choice of the exponential attractor is, in a sense, artificial (of course, it is natural to find "the simplest" construction of an exponential attractor). An essential advantage of exponential attractors (in comparison with global ones) is, however, the fact that the function Q and the constant α can be usually explicitly found in terms of physical parameters of the equation considered. Moreover, the exponential attractors are much more robust with respect to perturbations, in particular, upper and lower semicontinuos and even Hölder continuous in the symmetric Hausdorff distance, see [12-15], [21] and the references therein.

In the present section, we construct the exponential attractor for the porous media equation (1.1). The main result of the section is formulated in the following theorem.

Theorem 3.1. *Let the assumptions of Theorem 2.1 hold. Then, the semigroup* (2.1) *generated in* $\Phi = L^1(\Omega)$ *by equation* (1.1) *possesses an exponential attractor* $\mathcal{M}$ *in the sense of the above definition. Moreover, this attractor is bounded in* $C^\alpha(\Omega)$*, for some* $\alpha > 0$.

Proof. Let us introduce the set

$$\mathcal{C} := \left[\cup_{t\ge 1} S(t)\Phi\right]_{C^\alpha(\Omega)} \tag{3.3}$$

where $[\cdot]_V$ is a closure in the space V. Then, due to Theorem 1.2, we have

$$\|\mathcal{C}\|_{C^\alpha(\Omega)} \le M \tag{3.4}$$

and, due to the construction of $\mathcal{C}$, we have also

$$S(t)\mathcal{C} \subset \mathcal{C}, \quad t \ge 0. \tag{3.5}$$

Thus, instead of constructing an exponential attractor for $S(t)$ on the whole phase space Φ, it is sufficient to construct it only for the restriction of that semigroup on a compact invariant subset $\mathcal{C}$. To this end, we will use the algorithm of constructing ε-nets, developed in the proof of Theorem 2.1. To be more precise, let $V_{\varepsilon_0} \subset \mathcal{C}$ be an ε_0-net in the set $\mathcal{C}$ with sufficiently small ε_0. Then, arguing exactly as in the proof of Theorem 2.1, we can find positive numbers T, L and $\kappa < 1$ such that, for every $u_0 \in \mathcal{C}$, the set $S(T)(B(\varepsilon_0, u_0, L^1) \cap \mathcal{C})$ possesses a $\kappa\varepsilon_0$-net with L-points. Thus, starting from the ε_0-net V_{ε_0} of $\mathcal{C}$, we construct the $\kappa\varepsilon_0$-net $V_{\kappa\varepsilon_0} \subset S(T)\mathcal{C}$ of the set $S(T)\mathcal{C}$ such that

$$\#V_{\kappa\varepsilon_0} \le L\#V_{\varepsilon_0}.$$

Iterating this procedure, we construct then, for every $n \in \mathbb{N}$, $\kappa^n\varepsilon_0$-nets $V_{\kappa^n\varepsilon_0} \subset S(nT)\mathcal{C}$ in the set $S(nT)\mathcal{C}$ which satisfy

$$\#V_{\kappa^n\varepsilon_0} \le L^n\#V_{\varepsilon_0}. \tag{3.6}$$

These $\kappa^n\varepsilon_0$-nets in $S(nT)\mathcal{C}$ allow us to construct in a standard way the exponential attractor $\mathcal{M}_d$ for the *discrete* dynamical system, generated by the map $S = S(T) : \mathcal{C} \to \mathcal{C}$. This exponential attractor can be defined via the following expression:

$$\mathcal{M}_d := \left[\cup_{n=0}^{\infty} \cup_{m=0}^{\infty} S(m)V_{\kappa^n\varepsilon_0}\right]_{L^1(\Omega)}. \tag{3.7}$$

Indeed, the semi-invariantness and exponential attraction property are obvious since

$$\operatorname{dist}(S(nT)\mathcal{C}, V_{\kappa^n\varepsilon_0}) \le \kappa^n\varepsilon_0, \quad n \in \mathbb{N}, \quad \kappa < 1. \tag{3.8}$$

The finitness of the fractal dimension of $\mathcal{M}_d$ can be easily verified using (3.6), (3.8) and the fact that $V_{\kappa^n\varepsilon_0}\subset S(nT)\mathcal{C}$, see [12] for details. Thus, since $\mathcal{M}_d$ is closed, it is indeed an exponential attractor for the map $S=S(T):\mathcal{C}\to\mathcal{C}$. As usual, the required exponential attractor $\mathcal{M}$ for the semigroup $S(t)$ with continuous time can be defined via

$$\mathcal{M}=\cup_{t\in[T,2T]}S(t)\mathcal{M}_d. \tag{3.9}$$

Indeed, the semi-invariantness and exponential attraction property follow immediately from the analogous properties of the discrete attractor $\mathcal{M}_d$ and the finiteness of a fractal dimension in $L^1(\Omega)$ can be easily verified using that the dimension of $\mathcal{M}_d$ is finite and that the map $S(t)$ is uniformly Hölder continuous on $\mathcal{M}_d$, see [12-15] for the details. Thus, $\mathcal{M}$ is indeed the required exponential attractor and Theorem 3.1 is proved.

Remark 3.1. There exists a rather important exceptional class of dynamical systems whose global attractors are simultaneously the exponential ones. These are the so-called *regular* attractors which appear in *smooth* dynamical systems with the global Lyapunov function under the additional assumption that all of the equilibria are hyperbolic, see [2]. In our case of the porous media equation (1.1), we obviously have the global Lyapunov function. Indeed, arguing as in Lemma 1.2, we can easily verify that the functional

$$\mathcal{G}(u):=\int_{x\in\Omega}\frac{1}{2}|\nabla_x f(u(x))|^2+F_{0,g}(u(x))-h(x)F_0(u(x))\,dx \tag{3.10}$$

where $F_{0,g}$ and F_0 are the same as in Lemma 1.2, satisfies

$$\mathcal{G}(u(t))-\mathcal{G}(u(0))=-\int_0^T\int_{x\in\Omega}f'(u(t,x))|\partial_t u(t,x)|^2\,dx\,dt \tag{3.11}$$

and, consequently, gives a global Lyapunov function for (1.1).

Nevertheless, the regular attractor's theory seems to be not applicable here, since equation (1.1) is *degenerate* and we cannot obtain the differentiability of semigroup $S(t)$ with respect to the initial data and the hyperbolicity of the equilibria.

Remark 3.2. As we have already mentioned, the appropriate smoothing properties for differences of solutions play a crucial role in the modern theory of exponential attractors, see [12-15], [21]. The simplest abstract version (which gives existence of an exponential attractor for the map S) of such a smoothing property is the following one:

$$\|Su_1 - Su_2\|_{H_1} \le K\|u_1 - u_2\|_H \tag{3.12}$$

where the constant K is independent of u_i belonging to a bounded invariant subset and H_1 and H are two *Banach* spaces such that H_1 is compactly embedded in H, see [13].

Our proof of the existence of an exponential attractor can also be embedded in an abstract scheme, but, in contrast to (3.12), in our situation, the spaces H_1 and H should *depend* on u_1 and u_0.

To be more precise, let S be an abstract map acting on some Banach space X and let $\mathcal{C}$ be a compact subset of X such that

$$S\mathcal{C} \subset \mathcal{C}. \tag{3.13}$$

Let us assume also that, for every $u_0 \in \mathcal{C}$ and for every $\varepsilon \le \varepsilon_0$, there exist a pair of Banach spaces $H_1(u_0, \varepsilon)$ and $H(u_0, \varepsilon)$ such that H_1 is compactly embedded in H and this embedding is *uniformly* (with respect to ε and u_0) compact in the sense of Kolmogorov's ε-entropy, compare with (2.38) and a map $\mathcal{T}_{u_0,\varepsilon} : B(\varepsilon, u_0, X) \cap \mathcal{C} \to H_1(u_0, \varepsilon)$ such that, for every $u_1, u_2 \in B(\varepsilon, u_0, X) \cap \mathcal{C}$

$$\begin{cases} \|\mathcal{T}_{u_0,\varepsilon}u_1 - \mathcal{T}_{u_0,\varepsilon}u_2\|_{H_1} \le P\|u_1 - u_2\|_X, \\ \|Su_1 - Su_2\|_X \le (1-\delta)\|u_1 - u_2\|_X + P\|\mathcal{T}_{u_0,\varepsilon}u_1 - \mathcal{T}_{u_0,\varepsilon}u_2\|_H, \end{cases} \tag{3.14}$$

compare with (2.37). Then, arguing exactly as in the proof of Theorems 2.1 and 3.1, we can verify the existence of an exponential attractor for the abstract map S.

Remark 3.3. It would be very interesting to develop the perturbation theory for the exponential attractor $\mathcal{M}$ of degenerate porous media equation (1.1). In particular, it would be interesting to construct exponential attractors $\mathcal{M}_\varepsilon$ for the non-degenerate approximations (1.4) in such way that

$$\operatorname{dist}^{symm}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \le C\varepsilon^\kappa, \tag{3.15}$$

for some positive constants C and κ. We shall return to that problem elsewhere.

§4 The global attractor: the case of infinite dimension.

We now show that the attractor $\mathcal{A}$ can be infinite-dimensional if condition (2.9) is violated. To be more precise, we consider the following equation of the form (1.1):

$$\partial_t u = \Delta_x(u|u|^{p-1}) + u - g(u), \quad u\big|_{\partial\Omega} = 0 \tag{4.1}$$

where $p > 1$ and the function g vanishes near zero and satisfies assumption (1.3) at infinity. As we will show below the associated attractor has infinite dimension. That is why we will study below its Kolmogorov ε-entropy. The following theorem which gives a natural lower bound for the entropy of the attractor can be considered to be the main result of this section.

Theorem 4.1. *Let the above assumptions hold. Then the global attractor $\mathcal{A}$ associated with equation* (4.1) *is infinite-dimensional and its ε-entropy possesses the following estimate:*

$$\mathbb{H}_\varepsilon(\mathcal{A}, L^\infty) \ge C\left(\frac{1}{\varepsilon}\right)^{n(p-1)/2}, \tag{4.2}$$

for some positive constant C independent of ε.

Proof. In order to prove the theorem, we will study as usual the so-called unstable set $M_+(0)$ of the equilibrium $u \equiv 0$ of equation (4.1). By definition,

$$M_+(0) = \{u_0 \in L^\infty(\Omega), \ \ \exists u \in \mathcal{K}, \ \ \lim_{t\to-\infty} \|u(t)\|_{L^\infty} = 0, \ \ u(0) = u_0\}. \tag{4.3}$$

Obviously $M_+(0) \subset \mathcal{A}$. On the other hand, since the nonlinearity g vanishes at the origin, it is sufficient to consider only the backward solutions of the following "linearized" problem:

$$\partial_t u = \Delta_x(u|u|^{p-1}) + u, \quad u(0) = u_0, \quad t \le 0 \tag{4.4}$$

tending to zero as $t \to -\infty$ (all such solutions belonging to the sufficiently small ball in L^∞ will satisfy also equation (4.1)). In order to solve equation (4.4), we change to the unknown $v(t) := e^{-t}u(t)$. Then we arrive at

$$\partial_t v = e^{(p-1)t}\Delta_x(v|v|^{p-1}), \quad v(0) = u_0, \quad t \in (-\infty, 0). \tag{4.5}$$

Finally, making one more variable change $s := e^{(p-1)t}$, we obtain

$$\partial_s \tilde{v} = (p-1)\Delta_x(\tilde{v}|\tilde{v}|^{p-1}), \quad \tilde{v}(1) = u_0, \quad s \in (0, 1]. \tag{4.6}$$

Let $\mathcal{S}_t : L^\infty(\Omega) \to L^\infty(\Omega)$ be the solution operator of the following problem:

$$\partial_t w = (p-1)\Delta_x(w|w|^{p-1}), \quad w\big|_{t=0} = w_0, \quad t \geq 0. \tag{4.7}$$

Then, we have shown that the unstable set $M_+(0)$ contains the image of a sufficiently small ball $B(r_0) := B(r_0, 0, L^\infty)$:

$$\mathcal{S}_1 B(r_0) \subset M_+(0) \subset \mathcal{A}. \tag{4.8}$$

Thus, it is sufficient to estimate the ε-entropy of the set $\mathcal{S}_1 B(r_0)$. To this end, we recall that in contrast to the nondegenerate case, equation (4.7) possesses spatially localized solutions, i.e. there exists a nonzero solution $W(t,x) \geq 0$ of equation (4.7) such that $W(0) \in B(r_0)$ and

$$\operatorname{supp} W(s, \cdot) \subset K \subset\subset \Omega, \tag{4.9}$$

for all $s \in [0,1]$. For simplicity, we assume that $\|W(1)\|_{L^\infty} = 1$. On the one hand, if $W(s,x)$ solves (4.7) then the scaled function

$$W_\varepsilon(s,x) := \varepsilon W(s, \varepsilon^{(1-p)/2}x) \tag{4.10}$$

also solves (4.7) for every $\varepsilon \neq 0$ and

$$\operatorname{supp} W_\varepsilon(s,x) \subset K_\varepsilon := \varepsilon^{(p-1)/2}K. \tag{4.11}$$

Therefore, for every sufficiently small ε, there exists a finite set $R_\varepsilon := \{x_i\} \subset \Omega$ such that

$$
\begin{aligned}
&1)\ \ (x+K_\varepsilon)\cap(y+K_\varepsilon)=\varnothing, \quad \forall x,y\in R_\varepsilon, \quad x\neq y,\\
&2)\ \ \#R_\varepsilon \geq C\left(\frac{1}{\varepsilon}\right)^{n(p-1)/2},\\
&3)\ \ x+K_\varepsilon \subset\subset \Omega, \quad \forall x\in R_\varepsilon.
\end{aligned}
\tag{4.12}
$$

Consequently, for every $m\in\{0,1\}^{R_\varepsilon}$ the function

$$
W_{m,\varepsilon}(s,x) := \sum_{i=1}^{\#R_\varepsilon} m_i W_\varepsilon(s,x-x_i) \tag{4.13}
$$

solves (4.7) in Ω. On the other hand, obviously we have

$$
\|W_{m^1,\varepsilon}(1,\cdot)-W_{m^2,\varepsilon}(1,\cdot)\|_{L^\infty}\geq\varepsilon \tag{4.14}
$$

for $m^1\neq m^2$. Since we have $2^{\#R_\varepsilon}$ different functions of that form, then

$$
\mathbb{H}_\varepsilon(\mathcal{A},L^\infty)\geq\mathbb{H}_\varepsilon(\mathcal{S}_1B(r_0),L^\infty)\geq\#R_\varepsilon\geq C\left(\frac{1}{\varepsilon}\right)^{n(p-1)/2}. \tag{4.15}
$$

Theorem 4.1 is proved

Remark 4.1. It is worth recalling the usual method of obtaining lower bounds for the attractor dimension based on unstable manifolds theory. Namely, if we are able to find a (hyperbolic) equilibrium with large/infinite instability index then, due to this theory, the attractor contains a manifold of large/infinite dimension (which is equal to the instability index, see [2]). But this method is not applicable for degenerate equations since the associated semigroups are usually *not differentiable*. Indeed, under the assumptions of Theorem 4.1 the formal linearization near the zero equilibrium reads

$$
\partial_t w = w
$$

which, of course, has infinite instability index. But, in contrast to the nondegenerate case the backward solutions of that equation are not associated with the backward solutions of the whole nonlinear equation (due to the lack of regularity) and, consequently, do not give the infinite-dimensionality of the associated unstable set. That is the reason why we needed to develop above the alternative method based on the existence of a localized solution and scaling technique which is closely related with the degenerate nature of the problem considered.

Remark 4.2. It is also worth noting that, for nondegenerate parabolic equations, the asymptotics for the image of a ball under the evolution operator is usually logarithmic:

$$C^{-1}\left(\log_2\frac{1}{\varepsilon}\right)^{1+n/2}\le \mathbb{H}_\varepsilon(\mathcal{S}_1B(r_0),\Phi)\le C\left(\log_2\frac{1}{\varepsilon}\right)^{1+n/2}$$

where n is the space dimension, see [28]. The proof of Theorem 4.1 shows that the degeneracy changes drastically type of these asymptotics.

The next corollary gives the lower bounds for the ε-entropy in the initial phase space $L^1(\Omega)$.

Corollary 4.1. *Let the assumptions of Theorem 4.1 hold. Then, the Kolmogorov ε-entropy of the attractor $\mathcal{A}$ in $L^1(\Omega)$ possesses the following estimate:*

$$\mathbb{H}_\varepsilon(\mathcal{A},L^1(\Omega))\ge C\left(\frac{1}{\varepsilon}\right)^{\frac{n(p-1)}{2+n(p-1)}} \tag{4.16}$$

where the constant C is independent of ε.

Proof. Indeed, according to (4.10),

$$\|W_\varepsilon(1,x)\|_{L^1(\Omega)}=C\varepsilon^{1+n(p-1)/2} \tag{4.17}$$

and, consequently, instead of (4.14), we now have

$$\|W_{m^1,\varepsilon}(1,\cdot)-W_{m^2,\varepsilon}(1,\cdot)\|_{L^1}\ge C\varepsilon^{1+n(p-1)/2}. \tag{4.18}$$

Therefore, the distance between any two functions of the form (4.13) is not less than $C\varepsilon^{1+n(p-1)/2}$. Since we have $2^{\#R_\varepsilon}$ of such functions, estimate (4.16) is verified and Corollary 4.1 is proved.

We note that, in contrast to the lower bounds for the entropy in L^∞-metric given in Theorem 4.1, estimate (4.16) seems to be very rough (in particular, the exponent in the right-hand side of it remains bounded as $p \to \infty$ or $n \to \infty$). Nevertheless, it allows us to establish the infinite-dimensionality of global attractors for an essentially more general class of porous media equations. We illustrate this on the following example of the degenerate Chafee-Infante equation:

$$\partial_t u = \Delta_x(u^3) + u - u^3, \quad u\big|_{\partial\Omega} = 0. \tag{4.19}$$

Corollary 4.2. *Let $\mathcal{A}$ be the attractor of equation* (4.19). *Then, its Kolmogorov ε-entropy satisfies:*

$$\mathbb{H}_\varepsilon(\mathcal{A}, L^1(\Omega)) \geq C \begin{cases} \varepsilon^{-1/2}, & n = 1, \\ \varepsilon^{-2/(n+1)}, & n \geq 2, \end{cases} \tag{4.20}$$

for some $C > 0$ independent of ε.

Proof. Indeed, analogously to the proof of Theorem 4.1, replacing the dependent variable $u(t) = e^t v(t)$ in equation (4.19) and scaling time $s = e^{2t}$, we arrive at

$$\partial_s v = 2\Delta_x(v^3) - v^3, \quad s \in [0, 1]. \tag{4.21}$$

Let now $W_\varepsilon(s, x) := \varepsilon W(s, \varepsilon^{-1}x)$, $\varepsilon \ll 1$ be the solutions of equation

$$\partial_s w = 2\Delta_x(w^3) \tag{4.22}$$

constructed in the proof of Theorem 4.1 and define, for every $m \in \{0, 1\}^{R_\varepsilon}$, the functions $W_{m,\varepsilon}(s, x)$ via (4.13). We also recall that the L^1-norm of every solution of (4.22) with compact support is preserved, consequently,

$$\|W_{m,\varepsilon}(s, \cdot)\|_{L^1(\Omega)} = C\varepsilon^{1+n}|m| \tag{4.23}$$

where $|m| = \sum m_i$. Let us now define the associated solutions $\bar{W}_{m,\varepsilon}(s, x)$ of (4.21) with $\bar{W}_{m,\varepsilon}(0, x) = W_{m,\varepsilon}(0, x)$. Then, the difference $Z(s) = \bar{W}_{m,\varepsilon}(s) - W_{m,\varepsilon}(s)$ satisfies

$$\partial_s Z = 2\Delta_x(\bar{W}^3_{m,\varepsilon} - W^3_{m,\varepsilon}) - (\bar{W}^3_{m,\varepsilon} - W^3_{m,\varepsilon}) - W^3_{m,\varepsilon}. \tag{4.24}$$

Multiplying (4.24) by $\operatorname{sgn} Z$ integrating and using the Kato inequality together with (4.23), we obtain

$$(4.25)\qquad \|\bar W_{m,\varepsilon}(1,\cdot)-W_{m,\varepsilon}(1,\cdot)\|_{L^1(\Omega)}\le\int_0^1 (W_{m,\varepsilon}(s))^3\,ds\le C\varepsilon^{3+n}|m|.$$

Thus, due to (4.18) with $p=3$,

$$(4.26)\qquad \|\bar W_{m^1,\varepsilon}(1)-\bar W_{m^2,\varepsilon}(1)\|_{L^1(\Omega)}\ge\|W_{m^1,\varepsilon}(1)-W_{m^2,\varepsilon}(1)\|_{L^1(\Omega)}-\\ -C\varepsilon^{3+n}(|m^1|+|m^2|)\ge C\varepsilon^{1+n}(1-\varepsilon^2(|m^1|+|m^2|))$$

and, consequently, the functions $\bar W_{m^i,\varepsilon}$ are $\varepsilon^{1+n}/(2C)$-separated if

$$(4.27)\qquad |m^i|\le 1/4\varepsilon^{-2}.$$

We recall that $\#R_\varepsilon\sim\varepsilon^{-n}$. Then, for $n=1$, (4.27) is automatically satisfied for small ε and so, the number N of $1/2C\varepsilon^2$ separated functions is equal to $2^{\#R_\varepsilon}\sim 2^{C\varepsilon^{-1}}$. In the case of $n\ge 2$ this number N, obviously satisfies $N\ge 2^{1/4\varepsilon^{-2}}$. These estimates immediately imply (4.20). This finishes the proof of Corollary 4.2.

To conclude, we discuss also the upper bounds for the Kolmogorov's ε-entropy of the attractors of porous media equations of the form of (4.1). To this end, we recall that the polynomial asymptotics of the Kolmogorov entropy (like ε^{-k}) are typical for the embeddings of Sobolev spaces, and, consequently, the upper bounds of the entropy in the same form can be obtained by studying the maximal smoothness of the attractor. In particular, Theorem 1.2 together with the standard asymptotics for the Kolmogorov entropy of the embedding $C^\alpha\subset C$, see [19], gives

$$(4.28)\qquad \mathbb{H}_\varepsilon(\mathcal{A},L^\infty)\le C\left(\frac{1}{\varepsilon}\right)^{n/\alpha}.$$

In particular, for $n=1$ under the assumptions of Theorem 4.1, we have

$$(4.29)\qquad C^{-1}\left(\frac{1}{\varepsilon}\right)^{(p-1)/2}\le\mathbb{H}_\varepsilon(\mathcal{A},L^\infty)\le C\left(\frac{1}{\varepsilon}\right)^{6p}.$$

In turns, estimate (4.2) (and the scaling method, introduced in Theorem 4.1) give the natural upper bounds for the smoothness of the attractor.

Corollary 4.3. *The Hölder constant α in* (1.29) *satisfies $\alpha \le 2/(p-1)$. Moreover, if the inequality*

$$\|u_0\|_{W^{1,1}(\Omega)} \le C \tag{4.30}$$

holds uniformly with respect u_0 belonging to the attractor $\mathcal{A}$ of (4.1) *then, necessarily, $p \le 3$.*

Proof. Indeed, analogously to the proof of Theorem 4.1, all functions $W_\varepsilon(1,x) = \varepsilon W(1, \varepsilon^{(1-p)/2}x)$ belong to the attractor. On the other hand,

$$\|W_\varepsilon(1)\|_{C^\alpha(\Omega)} = \varepsilon^{1-\alpha(p-1)/2}\|W(1)\|_{C^\alpha(\Omega)}. \tag{4.31}$$

Since the left-hand side of (4.31) should be bounded as $\varepsilon \to 0$, then, necessarily, $\alpha \le 2/(p-1)$.

Analogously,

$$\|W_\varepsilon(1)\|_{W^{1,1}(\Omega)} = \varepsilon^{(3-p)/2}\varepsilon^{n(p-1)/2}\|W(1)\|_{W^{1,1}(\Omega)}. \tag{4.32}$$

Let us now consider the function $W_{\vec{1},\varepsilon}(1,x)$ associated with (4.13) with all $m_i = 1$. Then, since $\#R_\varepsilon \sim \varepsilon^{n(1-p)/2}$, (4.32) implies that

$$\|W_{\vec{1},\varepsilon}(1)\|_{W^{1,1}(\Omega)} = C\varepsilon^{(3-p)/2}\|W(1)\|_{W^{1,1}(\Omega)}.$$

Thus, (4.30) implies indeed that $p \le 3$ and Corollary 4.3 is proved.

Appendix. Some technicalities.

In this concluding section, we give, for the convenience of the reader, a more detailed exposition of several known technical issues used above. We start with the smoothness relations between u and $f(u)$.

Proposition A.1. *Let the function $f \in C^2(\mathbb{R}, \mathbb{R})$ satisfy* (1.2). *Then, for every $s \in (0,1)$ and $1 < q \le \infty$, we have*

$$\|u\|_{W^{s/p,pq}(\Omega)} \le C_p \|f(u)\|^{1/p}_{W^{s,q}(\Omega)} \tag{A.1}$$

where the constant C_p is independent of u.

Proof. Indeed, let f^{-1} be the inverse function to f. Then, due to conditions (1.2), the function $G(v) := \operatorname{sgn} v|f^{-1}(v)|^p$ is nondegenerate and satisfies

$$C_2 \le G'(v) \le C_1, \tag{A.2}$$

for some positive constants C_1 and C_2. Therefore, we have

$$|f^{-1}(v_1) - f^{-1}(v_2)|^p \le C_p|G(v_1) - G(v_2)| \le C_p'|v_1 - v_2|, \tag{A.3}$$

for all $v_1, v_2 \in \mathbb{R}$. Finally, according to the definition of the fractional Sobolev spaces (see e.g. [20]),

$$\|f^{-1}(v)\|^{pq}_{W^{s/p,qp}(\Omega)} := \|f^{-1}(v)\|^{pq}_{L^{pq}(\Omega)} + \int_\Omega\int_\Omega \frac{|f^{-1}(v(x)) - f^{-1}(v(y)|^{pq}}{|x-y|^{n+sq}}\,dx\,dy \le$$
$$\le C\|v\|^q_{L^q(\Omega)} + C_p'\int_\Omega\int_\Omega \frac{|v(x) - v(y)|^q}{|x-y|^{n+sq}}\,dx\,dy = C_p''\|v\|^q_{W^{s,q}(\Omega)},$$

where we have implicitly used that $f^{-1}(v) \sim \operatorname{sgn} v|v|^{1/p}$. Proposition A.1 is proved.

We are now going to discuss the interior regularity estimates for linear parabolic equations. To this end, we first construct special cut–off functions.

Proposition A.2. *Let $V \subset B(R, 0, \mathbb{R}^l)$ be a bounded set in $\mathbb{R}^l$ and let $V_\delta := \mathcal{O}_\delta(V)$ be its δ-neighbourhood. Then, there exists a cut-off function $\phi \in C^\infty(\mathbb{R})$, $\phi(x) \in [0,1]$, such that, for every $\beta \in (0,1)$ and every $k \in \mathbb{Z}_+$,*

$$\begin{cases} 1.\ \ \phi(x) = 1, \quad \text{for } x \in V \text{ and } \phi(x) = 0 \ \text{ for } x \notin V_\delta, \\ 2.\ \ |D^k_x\phi(x)| \le C_{k,\beta}[\phi(x)]^{1-\beta},\ x \in \mathbb{R}^l, \end{cases} \tag{A.4}$$

where the constant $C_{k,\beta} = C(k,\beta,\delta,R)$ is independent of x and the concrete choice of V and D^k_x means the collection of all x-derivatives of order k.

Proof. Indeed, let us introduce the standard bump function in $\mathbb{R}^l$:

$$\psi_r(x) := \begin{cases} e^{-\frac{1}{r^2-|x|^2}}, & |x| < r, \\ 0, & |x| \ge r. \end{cases} \tag{A.5}$$

Then, this function obviously satisfies estimate (A.4)(2).

Let us fix now a covering of the $\mathbb{R}^l$ by the balls of radius $\delta/2$ and let $W_\delta := \delta/2\mathbb{Z}^l$ be centers of that covering. Let us now construct also partition of unity associated with that covering and (A.5) via

$$\phi_q(x) := \frac{\psi_{\delta/2}(x-q)}{\sum_{p\in W_\delta}\psi_{\delta/2}(x-p)}, \quad q \in W_\delta. \tag{A.6}$$

Obviously, $\{\phi_q(x)\}_{q\in W_\delta}$ is a partition of unity associated with the above covering and, moreover, these functions satisfy (A.4)(2) *uniformly* with respect to $q \in W_\delta$.

Let us define now the required cut-off function $\phi(x) = \phi_V(x)$ by the following expression:

$$\phi_V(x) := \sum_{q\in W_{\delta/2}\cap V_{\delta/2}} \phi_q(x). \tag{A.7}$$

Indeed, since $\operatorname{supp}\phi_q \subset B(\delta/2, q, \mathbb{R}^l)$ and the sum of all such functions equals one identically, the function ϕ_V thus defined satisfies (A.4)(1). Moreover, since the number of points

$$\#(W_{\delta/2}\cap V_{\delta/2}) \le \#(W_{\delta/2}\cap B(R+\delta, 0, \mathbb{R}^l)) \le N(\delta, R)$$

is finite and uniformly bounded with respect to $V \subset B(R, 0, \mathbb{R}^l)$ and the functions $\phi_q(x)$ satisfy (A.4)(2) uniformly with respect to $q \in W_{\delta/2}$, then the function $\phi_V(x)$ also satisfies this inequality uniformly with respect to $V \subset B(R, 0, \mathbb{R}^l)$. Proposition A.2 is proved.

We now recall the classical L^q-regularity estimate for second order parabolic equations on the following model example:

$$\begin{cases} \partial_t w = a(t,x)\Delta_x w + b(t,x)w + h, \\ w\big|_{\partial\Omega} = 0, \quad w\big|_{t=0} = 0. \end{cases} \tag{A.8}$$

Proposition A.3. *Let Ω be a smooth domain and let $a \in C^\alpha(\Omega_T)$ (with $\alpha > 0$ and $\Omega_T := [0,T]\times\Omega$) satisfy*

$$0 < C_1 \le a(t,x) \le C_2, \quad (t,x) \in \Omega_T, \tag{A.9}$$

for some positive C_i. Let also $h \in L^q(\Omega_T)$ for some $1 < q < \infty$, $q \neq 3/2$. Assume finally that

$$b \in L^r(\Omega_T) \tag{A.10}$$

for a sufficiently large r depending on q ($r > \max\{q, \frac{n+2}{2}\}$). Then, problem (A.8) *possesses a unique solution $w \in W^{(1,2),q}(\Omega_T)$ and the following estimate holds:*

$$\|w\|_{W^{(1,2),q}(\Omega_T)} \le C\|h\|_{L^q(\Omega_T)}, \tag{A.11}$$

where the constant C depends on q, Ω, $\|a\|_{C^\alpha}$, $\|b\|_{L^r}$ and on the constants C_i from (A.9)*, but is independent of the concrete choice of a, b and h.*

The proof of this proposition (in more general setting) can be found in [20], see Chapter IV, §9 Th. 9.1. In particular, the assertion of the proposition is proved there without the assumption on Hölder continuity of a and the constant C in (A.11) depends on the modulus of continuity of the function a. However, for our purposes it is more convenient to control this modulus of continuity by the Hölder norm.

We are now able to verify the L^q-interior regularity estimate for equation (A.8) (which is analogous to estimate (10.12) of [20], see Chapter IV, §10, page 355).

Proposition A.4. *Let the above assumptions hold and let V be an arbitrary open set in Ω. Then, for every $0 < t_0 < T$, $\delta > 0$ and $q > 2$, the solution w satisfies*

$$\|w\|_{W^{(1,2),q}([t_0,T]\times V)} \le C(\|h\|_{L^q([0,T]\times V_\delta)} + \|w\|_{L^1([0,T]\times V_\delta)}), \tag{A.12}$$

where $V_\delta := \mathcal{O}_\delta(V) \cap \Omega$ and the constant C is independent of w and of the concrete choice of a, b and h.

Proof. According to Proposition A.2 there exists a cut-off function $\phi \in C^\infty(\mathbb{R}^{n+1})$ such that

$$\begin{cases} 1. \ \ \phi(t,x) \equiv 1, \quad \text{for } (t,x) \in [t_0,T] \times V, \\ 2. \ \ \phi(t,x) \equiv 0, \quad \text{for } (t,x) \notin [3t_0/4,T] \times V_{\delta/2}, \\ 3. \ \ |D^k_{(t,x)}\phi(t,x)| \le C_{k,\delta,\beta}[\phi(t,x)]^{1-\beta}, \end{cases} \tag{A.13}$$

where $\beta > 0$ is arbitrary and the constant $C_{k,\delta,\beta}$ is independent of V. Let us now introduce a function $w_\phi(t,x) := w(t,x)\phi(t,x)$ which obviously satisfies the following equation:

$$\partial_t w_\phi = a\Delta_x w_\phi + bw_\phi + h_\phi, \quad w_\phi\big|_{t=0} = 0, \quad w_\phi\big|_{\partial\Omega} = 0 \tag{A.14}$$

where

$$h_\phi := h\phi + w\partial_t\phi - 2\nabla_x\phi\nabla_x w - w\Delta_x\phi. \tag{A.15}$$

Applying now the L^q-regularity estimate (see Proposition A.3) to equation (A.14) and using (A.13), we infer

$$\begin{aligned} \|w_\phi\|^q_{W^{(1,2),q}(\Omega_T)} \le C\|h_\phi\|^q_{L^q(\Omega_T} \le C_1(\|h\|^q_{L^q([0,T]\times V_\delta)} + \\ + \int_{\Omega_T} [\phi(t,x)]^{q(1-\beta)}(|w(t,x)|^q + |\nabla_x w(t,x)|^q)\,dx\,dt). \end{aligned} \tag{A.16}$$

Let us assume for the moment that we have proved the following interpolation inequality:

$$\begin{aligned} \int_{\Omega_T} \phi^{q(1-\beta)}(|w|^q + |\nabla_x w|^q)\,dx\,dt \le \\ \le \varepsilon\|w_\phi\|^q_{W^{(1,2),q}(\Omega_T)} + C_\varepsilon\|w\|^q_{L^1([0,T]\times V_\delta)} \end{aligned} \tag{A.17}$$

which holds for every $\varepsilon > 0$. Then, inserting it into the right-hand side of (A.16) and fixing ε to be small enough, we have

$$\|w_\phi\|^q_{W^{(1,2),q}(\Omega_T)} \le C(\|h\|^q_{L^q([0,T]\times V_\delta)} + \|w\|^q_{L^1([0,T]\times V_\delta)}) \tag{A.18}$$

which together with (A.13)(1) implies (A.12) and finishes the proof of the proposition.

Thus, we only need to verify inequality (A.17). Indeed, due to Hölder's inequality,

$$\begin{aligned} \int_{\Omega_T} \phi^{q(1-\beta)}|w|^q\,dx\,dt = \int_{\Omega_T} (\chi_{V_\delta}(x)|w|)^{\beta q}|w_\phi|^{(1-\beta)q}\,dx\,dt \\ \le C\|w\|^{\beta q}_{L^1([0,T]\times V_\delta)}\|w_\phi\|^{(1-\beta)q}_{L^s(\Omega_T)} \le \varepsilon\|w_\phi\|^q_{L^s(\Omega_T)} + C_\varepsilon\|w\|^q_{L^1([0,T]\times V_\delta)} \end{aligned} \tag{A.19}$$

where $\chi_{V_\delta}(x)$ is the characteristic function of the set V_δ and $s = s(\beta) := \frac{q(1-\beta)}{1-q\beta}$. Fixing now β so small that the Sobolev embedding $W^{(1,2),q}(\Omega_T) \subset L^s(\Omega_T)$ holds, we verify inequality (A.17) for the term $\phi^{q(1-\beta)}|w|^q$.

Thus, it now remains to verify (A.17) for the term containing $\nabla_x w$. To this end, we transform this term as follows,

$$\begin{aligned}\text{(A.20)}\quad \int_{\Omega_T} \phi^{q(1-\beta)}|\nabla_x w|^q\,dx\,dt \le C\int_{\Omega_T} \phi^{1-q\beta}\nabla_x w.\nabla_x w_\phi|\nabla_x w_\phi|^{q-2}\,dx\,dt \\ + C\int_{\Omega_T}(\phi^{1-\beta}|\nabla_x w|)(\phi^{1-2\beta}|w|)^{q-1}\,dx\,dt.\end{aligned}$$

The last integral on the right-hand side can be, in turn, estimated via Hölder's inequality

$$\text{(A.21)}\qquad I_1 \le \varepsilon\int_{\Omega_T}\phi^{q(1-\beta)}|\nabla_x w|^q\,dx\,dt + C_\varepsilon\int_{\Omega_T}\phi^{q(1-2\beta)}|w|^q\,dx\,dt.$$

The last term on the right-hand of (1.21) can be estimated exactly as (A.19) and the first one coincides with the left-hand side of (A.20), but with arbitrarily small coefficient. This implies

$$\begin{aligned}\text{(A.22)}\quad \int_{\Omega_T}\phi^{q(1-\beta)}|\nabla_x w|^q\,dx \le C\int_{\Omega_T}\phi^{1-q\beta}\nabla_x w.\nabla_x w_\phi|\nabla_x w_\phi|^{q-2}\,dx\,dt \\ + \varepsilon\|w_\phi\|^q_{W^{(1,2),q}(\Omega_T)} + C_\varepsilon\|w\|^q_{L^1([0,T]\times V_\delta)}.\end{aligned}$$

So, one only needs to estimate the first term on the right-hand side of (A.22). Integrating by parts in that term and using again (A.13)(3), we infer

$$\begin{aligned}\text{(A.23)}\quad I_2 \le C\int_{\Omega_T}\phi^{1-q\beta}|w|\cdot|\Delta_x w_\phi|\cdot|\nabla_x w_\phi|^{q-2}\,dx\,dt+ \\ + C\int_{\Omega_T}\phi^{1-(q+1)\beta}|w|\cdot|\nabla_x w_\phi|^{q-1}\,dx\,dt\end{aligned}$$

(here we have implicitly used that $w\big|_{\partial\Omega} = 0$ and that $q > 2$). Applying now once more the Hölder inequality to both integrals in the right-hand side of (A.23), we finally arrive at

$$\begin{aligned}\text{(A.24)}\quad I_2 \le \varepsilon\int_{\Omega_T}|\Delta_x(w_\phi)|^q + |\nabla_x w_\phi|^q\,dx\,dt + C_\varepsilon\int_{\Omega_T}\phi^{q(1-(q+1)\beta)}|w|^q\,dx\,dt \le \\ \le C\varepsilon\|w_\phi\|^q_{W^{(1,2),q}(\Omega_T)} + C_\varepsilon\int_{\Omega_T}\phi^{q(1-(q+1)\beta}|w|^q\,dx\,dt.\end{aligned}$$

Estimating now the last term on the right-hand side of (1.24) analogously to (A.19), we deduce the analogue of estimate (A.17) for the term I_2. Inserting then this estimate to (A.20)–(A.22) and using (A.19), we finish the proof of estimate (A.17). Thus, Proposition A.4 is proved.

Corollary A.1. *Let the solution $w(t,x)$ of equation* (A.8) *be defined only for $(t,x) \in [t_0/2,T] \times V_\delta$ and the coefficients a, b and the external force h be also defined only in $[t_0/2,T] \times V_\delta$ and satisfy the assumptions of Proposition A.4 in this domain. Then, the solution w satisfy the interior regularity estimate* (A.12) *with the constant C independent of the concrete choice of V, a, b, h and w.*

Proof. Indeed, the function $w_\phi(t,x) := w(t,x)\phi(t,x)$ introduced in the proof of Proposition A.4 equals zero identically for (t,x) outside of $[3t_0/4,T] \times V_{\delta/2}$. Therefore, we can construct an extension $\tilde a$ and $\tilde b$ of the coefficients a and b from the initial domain of definition $[t_0/2,T] \times V_\delta$ to the whole domain $[0,T] \times \Omega$ in such a way that

$$\text{(A.25)} \qquad \begin{cases} 1.\ \ \tilde a(t,x) = a(t,x), \quad \tilde b(t,x) = b(t,x), \quad (t,x) \in [t_0/2,T] \times V_\delta \\ 2.\ \ \|\tilde a\|_{C^\alpha(\Omega_T)} \le C\|a\|_{C^\alpha([t_0/2,T]\times V_\delta)}, \ \ \|\tilde b\|_{L^r(\Omega_T)} \le C\|b\|_{L^r([t_0/2,T]\times V_\delta)} \\ 3.\ \ C_1 \le \tilde a \le C_2, \end{cases}$$

where the constant C is independent of a, b and V and the constants C_i are the same as in (A.9). Such an extension can be constructed e.g. via

$$\tilde a(t,x) := C_1(1-\psi(t,x)) + \psi(t,x)a(t,x), \quad \tilde b(t,x) := \psi(t,x)b(t,x) \tag{A.26}$$

where the cut-off function ψ equals one for $(t,x) \in [3t_0/4,T] \times V_{\delta/2}$ and zero for (t,x) outside of $[t_0/2,T] \times V_\delta$ (this cut-off function exists due to Proposition A.2).

Thus, due to (A.25)(1), the function w_ϕ satisfies the equation

$$\partial_t w_\phi = \tilde a \Delta_x w_\phi + \tilde b w_\phi + h_\phi \tag{A.27}$$

in the whole domain $[0,T]\times\Omega$ and, due to (A.25)(2) and (A.25)(3), the L^q-regularity estimate is applicable to (A.27) in Ω_T and gives (A.16). The rest of the proof of

Corollary A.1 repeats word by word the proof of Proposition A.4. Corollary A.1 is proved.

We conclude by verifying the uniform compactness of the embedding $C^\alpha(V_\delta) \subset C(V)$ which is crucial for our proof of the finite-dimensionality given in Section 2.

Proposition A.5. *Let $V \subset \Omega$ be an open bounded set and $\delta > 0$ some positive number. Let us also consider a unit ball $\mathbb{B}_\alpha := B(0,1,C^\alpha([t_0,T]\times V_\delta))$ and its restriction $\Pi_V\mathbb{B}_\alpha$ to the domain $[t_0,T]\times V$ for some $t_0 < T$ and positive α. Then, the embedding $\Pi_V\mathbb{B}_\alpha \subset C([t_0,T]\times V)$ is uniformly compact with respect to $V \subset \Omega$ in the following sense: there exists a monotone decreasing function $\varepsilon \to \mathbb{M}(\varepsilon)$ (which depends on α, t_0, T and δ, but is independent of $V \subset \Omega$) such that*

$$\mathbb{H}_\varepsilon(\Pi_V\mathbb{B}_\alpha, C([t_0,T]\times V)) \le \mathbb{M}(\varepsilon) \tag{A.28}$$

holds for every $\varepsilon > 0$.

Proof. Let us fix a cut-off function $\phi(x)$ such that $\phi(x) = 1$ for $x \in V$ and $\phi(x) = 0$ for $x \notin V_{\delta/2}$ (see Proposition A.2). Then, since the norms of derivatives of ϕ are uniformly bounded (with respect to V), we have the following embedding:

$$\phi\mathbb{B}_\alpha \subset B(0,r,C^\alpha([t_0,T]\times\Omega)) \tag{A.29}$$

where the radius r depends on α and δ, but is independent of V.

Let us now fix an arbitrary $\varepsilon > 0$ and find a finite ε-net W_ε of $B(0,r,C^\alpha([t_0,T]\times\Omega))$ relative to the metric of $C([t_0,T]\times\Omega)$ (such net exists since the embedding $C^\alpha \subset C$ is compact). Then, embedding (A.29) guarantees that the finite set $\Pi_V W_\varepsilon$ will be the required ε-net in the set $\Pi_V\mathbb{B}_\alpha$. As usual, increasing the radiii of the balls by the factor of two, we can construct a 2ε-net with the centers belonging to $\Pi_V\mathbb{B}_\alpha$. Proposition A.5 is proved.

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