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INDUCED MAPS OF HYPERBOLIC BERNOULLI SYSTEMS

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Abstract. Let (f, T^n, μ) be a linear hyperbolic automorphism of the n-torus. We show that if $A \subset T^n$ has a boundary which is a finite union of C^1 submanifolds which have no tangents in the stable (E^s) or unstable (E^u) direction then the induced map on A, (f_A, A, μ_A) is also Bernoulli. We show that Poincáre maps for uniformly transverse C^1 Poincáre sections in smooth Bernoulli Anosov flows preserving a volume measure are Bernoulli if they are also transverse to the strongly stable and strongly unstable foliation.

1. **Introduction.** Let (f, X, μ) be a dynamical system (that is, a measure-preserving transformation of a measure space (X, μ)). If $A \subset X$ is a set of positive measure then we define the induced map f_A by $f_A(x) = f^{n(x)}(x)$ where $n(x) = \min\{k > 0 : f^k(x) \in A\}$. f_A is a transformation $f_A : A \to A$ which preserves the induced measure $\mu_A(.) := \mu(.)/\mu(A)$.

Induced maps arising from a dynamical system (f, X, μ) may be given a topology from the symmetric-difference metric on sets of positive measure by defining $D(f_A, f_{A'}) = \mu(A \Delta A')$. By a residual set we mean a countable intersection of open, dense sets (a G_δ set).

If X is assumed to be a smooth n-dimensional Riemannian manifold we consider the following metric ρ on a class of sets A in X with nonempty interior, whose boundaries consist of a finite collection of $n-1$ dimensional C^1 submanifolds ie such that $\partial A = \cup_{i=1}^m \phi_i(I)$ where $\phi_i : I^{n-1} \to X$ is a C^1 mapping of the unit $n-1$ cube into X.

We define $\rho(\partial A, \partial A') := d_H(\partial A, \partial A') + |H^{n-1}(\partial A) - H^{n-1}(\partial A')|$ where d_H is the Hausdorff metric and H^{n-1} is $n-1$ dimensional Hausdorff measure. It is easy to check that ρ is a metric.

It is well known that if T is ergodic then the dynamical system (T_A, X, μ_A) is also ergodic. A natural question is which stronger mixing properties are inherited by induced maps? In this paper we are concerned, in particular, with the question of when the Bernoulli property is inherited by an induced map.

We say two dynamical systems (f, X, μ) and (g, Y, ν) are Kakutani equivalent if there exist sets $A \subset X$ and $B \subset Y$ such that (f_A, A, μ_A) is isomorphic to (g_B, B, ν_B). The study of Kakutani equivalence led Feldman to develop the notion of the $\bar{f}$ metric and the theory of Loosely Bernoulli (LB) processes—a theory which in many ways parallels the Bernoulli theory of Ornstein and Weiss which rests upon the $\bar{d}$ metric. We refer the reader to [7] for details. A LB process is

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one whch satisfies the VWB condition with the $\bar{f}$ metric in place of the $\bar{d}$ metric. B-processes and processes arising from translations of compact abelian groups are examples of LB processes. The product of a LB process and a B process is LB. One of the striking results of this theory is that if (f, X, μ) is LB of positive entropy then the set of A for which (f_A, A, μ_A) is Bernoulli is dense in the symmetric difference metric and the set of A for which (f_A, A, μ_A) is not Bernoulli is also dense.

Hans-Otto Georgii [2] has given an interesting example which illustrates that induced maps of Bernoulli systems need only inherit ergodicity. Consider the stationary Markov chain with state space $\{0, 1, 2\}$ and transition probabilities $p_{00} = p_{01} = 1/2$ and $p_{12} = p_{20} = 1$. This is Bernoulli yet the induced map on the set A of sequences $\{x_i\}$ (where $x_i \in \{0, 1, 2\}$) defined by $x_0 = 1$ or $x_0 = 2$ is not even weak mixing.

Rudolph and del Junco [3] using a shift construction recently proved that if T is ergodic, loosely Bernoulli of positive entropy then T_A is Bernoulli for a residual set of A. They also showed that if T is ergodic of positive entropy then T_A is a K-automorphism for a residual set of A.

It is natural to ask what is the case in a smooth setting- is it possible to give sufficient conditions on a class of sets in a class of smooth dynamical systems to ensure that the induced map is Bernoulli ?

In Section 3 we consider the class of linear hyperbolic toral automorphisms and show that if $A \subset T^n$ has a boundary which is a finite union of C^1 submanifolds such that no tangent to the boundary is contained in the stable or unstable direction then the induced map on A is Bernoulli. It is fairly easy to show that the induced map on such sets is either Bernoulli or Bernoulli times a rotation- in the latter case the transformation is not mixing.

In Section 4 we show that if a transverse Poincáre section in a uniformly hyperbolic Bernoulli flow is C^1 then it's associated Poincáre map is Bernoulli if the section is also transverse to the strongly stable and strongly unstable foliation. Ornstein and Weiss [9, Theorem 6.1, page 453] have shown that the associated Poincáre map for transverse smooth sections is Bernoulli or Bernoulli times a rotation. We give conditions which rule out the latter (non-mixing) case.

2. **Notation and Background.** To introduce some notation recall that linear hyperbolic toral automorphisms are smooth uniformly hyperbolic maps $f : X \mapsto X$ (where $X = T^n$) i.e. there exists a (continuous) Df invariant splitting $T(X) = E^s \oplus E^u$, $C > 0$ and $0 < \lambda < 1$ such that

$$||Df^n(v)|| \leq C\lambda^n ||v||, \text{ for } v \in E^s \tag{2.1}$$

$$||Df^{-n}(v)|| \leq C\lambda^n ||v||, \text{ for } v \in E^u \tag{2.2}$$

where $n \geq 0$. For smooth Anosov flows $\phi_t : M \mapsto M$ there is a central flow direction E^c and a continuous invariant splitting $E^s \oplus E^c \oplus E^u$ such that the spaces E^s, E^u satisfy the hyperbolic properties of equations (2.1),(2.2) with ϕ_1, the time-one map of ϕ_t, in place of f.

3. **Statement and proof of results.** In this section we apply standard hyperbolic techniques ([6], [10], [1], [5], [4]) to prove our main result:

Theorem 3.1. *Let (f, T^n, μ) be a linear hyperbolic toral automorphism of the n-dimensional torus, where μ is Haar measure. Let $A \subset T^n$ be a set with nonempty interior (not necessarily connected) which is bounded by a finite number of C^1*

submanifolds $\{C_i\}$ such that for each point p in each submanifold C_i the tangent space to the submanifold at p is not contained in either $E^u(p)$ or $E^s(p)$. Then (f_A, A, μ_A) is Bernoulli.

Proof of Theorem 3.1

To simplify the exposition we present the argument in the case that F is taken to be the map

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

of the two-dimensional torus T^2. Although some of the details of the proof make reference to two-dimensions (we talk of 1-dimensional stable and unstable manifolds) this is not an essential feature of the proof and the argument easily generalises to an n-dimensional hyperbolic toral automorphism.

Lemma 3.2. *Let*

$$F := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Suppose $A \subset T^2$ is a set with nonempty interior (not necessarily connected) which is bounded by a finite number of C^1 curves $\{C_i\}$ such that for each i, for each point $x \in C_i$ neither the right nor left hand derivative to C_i at x (both are defined except at endpoints) is contained in $E^u(x)$ or $E^s(x)$. Then (F_A, A, μ_A) is Bernoulli.

Proof. First we consider the boundary of A to be a singular set for the mapping F. Let $S = \partial A$ denote the singularity set for F_A. We will construct using techniques of [6] a foliation into local stable manifolds (lsm) and local unstable manifolds (lum) from which it will be possible to prove Bernoullicity of F_A.

The set S satisfies the *regularity* conditions for singularity sets as given in [6]. That is to say for each n, $S_n := \cup_{i=0}^{n} F^n S$ consists of a finite number of 1-dimensional submanifolds which intersect at a finite number of points.

Using standard techniques we may construct C^1 local stable manifolds (lsm) $W^s_{\epsilon(x)}(x)$ for μ a.e. point $x \in T^2$. The notation $\epsilon(x)$ indicates that the stable leaves have a length $\epsilon(x)$ which depends upon x. These lsm have the property that $F^k W^s_{\epsilon(x)}(x) \cap S = \emptyset$ for all $k \geq 0$ and $W^s_{\epsilon(x)}(x)$ has a length $\epsilon(x) \geq \delta > 0$ (δ independent of x) or endpoints on $\cup_{k=0}^{\infty} F^{-k}(S)$. The leaves $W^s_{\epsilon(x)}(x)$ will be straight line segments in the direction $E^s(x)$.

We may construct C^1 local unstable manifolds $\{W^u_{\epsilon(x)}(x)\}$ with analogous properties. The leaves $W^u_{\epsilon(x)}(x)$ will be straight line segments in the direction $E^u(x)$.

These foliations are characterised by $y \in W^u_\epsilon(x)$ if

$$d(F^{-n}x, F^{-n}y) \leq \epsilon \text{ for all } n \geq 0$$

and $y \in W^s_\epsilon(x)$ if

$$d(F^n x, F^n y) \leq \epsilon \text{ for all } n \geq 0$$

These foliations persist for any k'th power of F_A, since $(F_A^k)^j W^s_{\epsilon(x)}(x) \cap S = \emptyset$ for $j \geq 0$ and $(F_A^k)^j W^u_{\epsilon(x)}(x) \cap S = \emptyset$ for $j \leq 0$.

The foliation into lsm and lum is absolutely continuous (the holonomy maps are C^∞ since the leaves are straight line segments) and we may cover T^2 (and hence A) up to full measure with a countable number of hyperbolic blocks ([8],[9, Section 4]) (also called rectangles or quadrilaterals) formed by the local product structure of the lum and lsm. Thus A is covered up to full μ_A measure with a countable number of hyperbolic blocks formed by the local product structure of the lum and lsm for F_A. Standard arguments ([8],[9, Section 4]) show that (F_A, A, μ_A) is Bernoulli or

Bernoulli times a rotation. If we show that for any integer $k > 0$, (F_A^k, A, μ_A) is ergodic then we may rule out the possibility of a rotation factor and show that (F_A, A, μ_A) is Bernoulli.

To do this we will show that the foliation we have constructed has enough leaves of a suitable length to allow a Hopf chain argument to prove ergodicity for F_A (of course we already know that F_A is ergodic by other considerations but the key point is that it is the foliation that enables us to prove ergodicity- this foliation persists for any power of F_A). Since this foliation is also a foliation for F_A^k for any k the same Hopf chain argument shows that any power of F_A^k is ergodic and hence F_A is Bernoulli.

We now show that under the conditions we have imposed on the singularity set ∂A the proof of [6, Main Lemma] is readily modified to show that F_A is ergodic *by a Hopf chain argument.*

We first show the foliation has enough long leaves to prove local ergodicity of F. Let $0 < c < 1$ be a number to be specified later. Define the set

$$\mathcal{N}(n, c) = \{c/n(m, k) \in U | m, k \in \mathbb{Z}\}$$

and let $G_n(c)$ be the collection of squares with centers at points from $\mathcal{N}(n, c)$ and having parallel sides of length $1/n$ contained in the stable E^s and unstable E^u directions respectively.

We call two squares in $G_n(c)$ neighbours if the distance between their centers is c/n. We can naturally define a column of squares and a row of squares. We say that a lum $W^u_{\epsilon(x)}(x)$ (lsm $W^s_{\epsilon(x)}(x)$) is long in a square R if it intersects both sides of R which lie in the stable (unstable) direction.

Let $\alpha > 0$. We say that a square $R \in G_n(c)$ is α-connecting if the measure of the set of points x in R such that $W^s_{\epsilon(x)}(x)$ and $W^u_{\epsilon(x)}(x)$ is long in R is at least a fraction α of the total measure of R. We say a square is good if it is α-connecting. We say a square is bad if it is not α-connecting. In the case of F we may take α to be $3/4$ and $c < \alpha$. In the case of an n-dimensional hyperbolic toral automorphism, or more generally a uniformly hyperbolic system with singularities satisfying the conditions of [6], α may need to be taken very small in which case we must take c small enough to ensure that neighboring squares intersect on a set of measure greater than $1 - \alpha$. Note that for a positive value of c each square can overlap with only finitely many of it's neighbours.

For each x let $C_1(x)$ be a sector in R^2 based at the point x containing the line along the unstable direction $E^s(x)$ which is sufficiently narrow that it's intersection with a square from $G_n(c)$ centered at x does not exceed $1/8$ of the area of the square. Similarly for each x let $C_2(x)$ be a sector with vertex at x containing the line along the stable direction through x which is sufficiently narrow that it's intersection with a square from $G_n(c)$ centered at x does not exceed $1/8$ of the area of the square. The quantity $1/8$ is to ensure that the intersection of $C_1(x) \cup C_2(x)$ with such a square does not exceed $1/4$ of the measure of the square.

Take N large enough so that for any $i \geq N$ all of the segments in $F^i S$ are contained in the cone family $\{C_1(x)\}$ and all of the segments in $F^{-i}S$ are contained in the cone family $\{C_2(x)\}$. This is possible because all line segments based at y not parallel to the stable direction eventually are contained in $C_1(F^n y)$ under F (uniformly in y) and similarly all line segments not parallel to the unstable direction based at y are eventually contained in $C_2(F^{-n}y)$ under F^{-1}.

We define $U(x)$ as the largest open square (with sides parallel to the stable $E^s(x)$ and unstable $E^u(x)$ directions at x) which does not intersect $\cup_{i=0}^N F^i S \cup \cup_{i=0}^N F^{-i} S$.

The proof of Proposition 2.3 [6] goes through with no essential modification to show that a.e. lsm and lum for F in $U(x)$ has the same ergodic average and hence $U(x)$ is contained in one ergodic component of F. We will repeat this argument for the sake of exposition.

The main part of the proof consists in establishing that the union of squares in $G_n(c)$ which are not α-connecting has measure $o(n)$ (Sinai's Theorem). This enables a Hopf chain argument to show that the forward time average of any continuous function is constant on $U(x)$ and equal to the backwards time average on $U(x)$. This implies local ergodicity.

(In the general case of a dynamical system satisfying the conditions given in [6] we need to take $\alpha, c > 0$ sufficiently small to ensure this. The value of α we need to take depends upon the geometry of the cone foliation $\{C_i(x)\}$ $(i = 1, 2)$. For T^2, $\alpha = 3/4$ and any $c < \alpha$ will do.).

The proof of Sinai's Theorem has two main ingredients. The first is that there exists an integer N such that all the segments in $\cup_{i=N}^{\infty} F^i S$ are contained in a cone family $\{C_1(x)\}$ which is centered along the unstable direction and all the segments in $\cup_{i=N}^{\infty} F^{-i} S$ are contained in a cone family $\{C_2(x)\}$ which is centered along the stable direction.

The second is that the singularity set S satisfies a regularity condition which requires that for any $M \geq N$ the union of $M - N$ iterates under F of the singularity set S, $\cup_{i=N}^{M} F^i S$, is a finite union of codimension-1 submanifolds which intersect in codimension-2 submanifolds (see [6] for a precise statement of the regularity condition in the general case). Since $\cup_{i=N}^{M} F^i S$ consists of a finite collection of closed segments with a finite number of intersection points the regularity condition is clearly satisfied.

The first ingredient is used in the following way. If a square $G \in G_n(c)$ intersects only one segment in $\cup_{i=N}^{M} F^i S$ then since the segment is aligned to lie in the cone family $C_1(x)$ the measure of points in G whose lum intersect the segment is smaller than 1/8 of the measure of square. Thus we need only look at those squares which contain the intersection points of two or more segments. We first analyse the measure of those points whose lum is cut by the singularity set—the analysis in the case of lsm is analogous. We call a square $G \in G_n(c)$ M-bad if the set of $y \in G$ such that $W^u_{\epsilon(y)}(y) \cap \cup_{i=N}^{M} F^i S \neq \emptyset$ has measure greater than a fraction 1/8 of the measure of the square. Thus an M-bad square has to intersect at least two segments in $\cup_{i=N}^{M} F^i S$. The regularity condition ensures that the set of squares in $G_n(c)$ which do so is of order $o(n)$. In fact there are a finite number of intersection points and hence the area of the union of squares within a distance $1/n$ of the set of intersection points is of the form $K(1/n^2)$, where K is a constant.

The measure of the set of squares which are bad but not M-bad is of the form $C(1/n) \sum_{i=M+1}^{\infty} \lambda^i$ where $\lambda < 1$ and C is a constant. To see this suppose a square G is bad but not M-bad. This means that at least a fraction α of the measure of the square lies in the set $G \cap (\cup_{i=M+1}^{\infty} F^i S \cup \cup_{i=M+1}^{\infty} F^{-i} S)$.

We first consider the set $G \cap (\cup_{i=M+1}^{\infty} F^i S)$. If we define $S_t = \{x | d(x, S) \leq t\}$ then $F^{-i}(W^u_{\epsilon(x)}(x) \cap B) \subset S_{\frac{\lambda^i}{n}}$. Similarly $F^i(W^s_{\epsilon(x)}(x) \cap B) \subset S_{\frac{\lambda^i}{n}}$. Thus each square which is bad but not M-bad has at least a fraction α of it's area covered by $\cup_{i=M+1}^{\infty} F^i S_{\frac{\lambda^i}{n}} \cup \cup_{i=M+1}^{\infty} F^{-i} S_{\frac{\lambda^i}{n}}$. Since F^i preserves measure the measure of those squares which are bad but not M-bad takes the form of the product of a constant $C(c, \alpha)$ and the tail-end of a converging series. This establishes that the set of bad squares in $G_n(c) \cap U(x)$ is $o(n)$.

Now a "Hopf chain" argument to prove local ergodicity proceeds in the following way. Let ϕ be a continuous function on the torus and denote it's forward time average by ϕ^+ and it's backwards time average by ϕ^-. We call a point y typical if $\phi^-(y) = \phi^+(y)$. We call a lsm $W^s_{\epsilon(y)}(y)$ ϕ typical if $\phi^+(y)$ and $\phi^-(y)$ are defined and equal except for a set of points $z \in W^s_\epsilon(y)$ of zero arc-length measure on $W^s_{\epsilon(y)}(y)$ (and similarly for $W^u_{\epsilon(y)}(y)$).

Since $0 < c < \alpha$ any two α connecting squares which are neighbours contain in their intersection a set of connecting lsm and lum of positive measure. The absolute continuity of the foliation shows that if ϕ is a continuous function then ϕ^+ and ϕ^- are the same for typical connecting lsm and lum and moreover neighbouring squares have the same common value of ϕ^+ and ϕ^-. Let x and y be two typical points in a connected component of A.

We construct a Hopf chain of lsm and lum linking x to y to show that $\phi^+(x) = \phi^-(y)$. The Sinai Theorem allows us to claim that for sufficiently large n, $W^u_{\epsilon(x)}(x)$ is connecting in a column of connecting squares in $G_n(c)$ (that is, a column of squares in which all the squares are connecting and hence have a common value of ϕ^+ and ϕ^- for typical lsms and lums). Similarly $W^s_{\epsilon(y)}(y)$ is connecting in a row of connecting squares for sufficiently large n. To see this suppose that $W^u_{\epsilon(x)}(x)$ is not connecting in a column of connecting squares for large n- then every column of squares in $G_n(c)$ contains at least one non-connecting square. The number of columns in which $W^u_{\epsilon(x)}(x)$ is connecting grows linearly in n and the measure of one square shrinks as $1/n^2$. This implies that the number of non-connecting squares is $O(n)$- a contradiction. Once we have that $W^u_{\epsilon(x)}(x)$ is connecting in a column of connecting squares in $G_n(c)$ and $W^s_{\epsilon(y)}(y)$ is connecting in a row of connecting squares then it is easy to see that $\phi^+(x) = \phi^-(x) = \phi^+(y) = \phi^-(y)$ (perhaps owing to the geometry of the connected component of A that x, y lie in we need a few more zigzags of connecting rows and columns in our chain, but the number of such is uniformly bounded and determined by the geometry of this connected component of A). Since ϕ was arbitrary this implies local ergodicity.

Hence *by considering the lsm and lum* we may conclude that the partition of T^2 into ergodic components (for F, F_A and any power of F_A) is coarser than the partition of T^2 into the connected components $T^2 \setminus (\cup_{i=0}^N T^i S \cup \cup_{i=0}^N T^{-i} S)$. Note that S consists of the boundary of A and hence the boundaries of the connected components of A. Thus there are at most a finite number of ergodic components. Now fix $k > 0$. The proof of local ergodicity of the k'th power of F_A, F_A^k, follows exactly as above, using the same foliation. Recall that if $x \in A$ then it's lsm (lum) will also be lsm (lum) for F_A^k for any $k > 0$. In fact these foliations persist for any power of F_A, since $F^j W^s_{\epsilon(x)}(x) \cap S = \emptyset$ for $j \geq 0$ and $F^j W^u_{\epsilon(x)}(x) \cap S = \emptyset$ for $j \leq 0$ imply that $(F_A^k)^j W^s_{\epsilon(x)}(x) \cap S = \emptyset$ for $j \geq 0$ and $(F_A^k)^j W^u_{\epsilon(x)}(x) \cap S = \emptyset$ for $j \leq 0$. This is a trivial but key point.

We will now show that F_A^k is ergodic for any $k > 0$. This will rule out a rotation factor and show that (F_A, A, μ_A) is Bernoulli. Fix an integer $k > 0$. Define the set $G_n(c)$ as before. Let $U(x) \subset A$ be the largest connected neighbourhood of x contained in A which does not intersect $\cup_0^N F^i S \cup \cup_0^{-N} F^{-i} S$, where N is defined as above. In exactly the same way we may show that a.e. lsm and lum for F_A^k in $U(x)$ has the same ergodic average and hence $U(x)$ is contained in one ergodic component of F_A^k.

Thus the partition of A into ergodic components for F_A^k is coarser than the partition of A into the connected components $A \setminus (\cup_{i=0}^N F^i S \cup \cup_{i=0}^N F^{-i} S)$. We will now show that there is only one ergodic component. First we use a special feature of our system to make the following observation. Note that for each segment I of $\cup_{i=0}^N F^i \cup \cup_{i=0}^N F^{-i}S$ there exists a j_I such that $(F)^{j_I}(I)$ consists of segments which intersect $\cup_{i=0}^N F^i S \cup \cup_{i=0}^N F^{-i}$ transversally and hence the set $\cup_{i=0}^N F^i S \cup \cup_{i=0}^N F^{-i}S$ is not invariant under F. This is because I is not tangent to the stable or unstable foliation.

This implies that for each segment I of $\cup_{i=0}^N F^i S \cup \cup_{i=0}^N F^{-i}S$ there exists an integer j_I such that $(F_A^k)^{j_I}(I)$ is a collection of segments which intersect $\cup_{i=0}^N F^i S_- \cup \cup_{i=0}^N F^{-i}S$ transversally.

Hence the set $\cup_{i=0}^N F^i S \cup \cup_{i=0}^N F^{-i}S$ is not invariant under F_A^k. Thus $A \backslash (\cup_{i=0}^N F^i S \cup \cup_{i=0}^N F^{-i}S)$ cannot partition A into ergodic components for F_A^k. Thus F_A^k is ergodic. Since k was arbitrary (F_A, A, μ_A) is Bernoulli.

□

4. **Poincáre Sections.** We do not attempt to give results in their greatest generality but only illustrate an application of these methods in a simple context. Suppose (ϕ_t, M, μ) is a C^2 Bernoulli flow on a smooth, compact manifold M preserving a volume measure μ. Let N be a codimension one embedded C^1 submanifold uniformly transverse to the flow direction. We refer to N as a uniformly transverse C^1 cross-section. The measure μ induces a measure with density $\frac{d\mu}{\nabla \phi_t}$ which is preserved by the Poincáre map $\phi_N : N \to N$. We will call this measure μ_N. The following theorem gives a condition under which [9, Theorem 6.1, p 453] implies the Bernoulli property rather than Bernoulli times a rotation.

Theorem 4.1. *The Poincáre map (ϕ_N, N, μ_N) of a uniformly transverse C^1 cross-section is Bernoulli if the section is also transverse to the strongly stable and strongly unstable direction.*

Proof. We will use a similar line of proof to that we gave for induced maps of hyperbolic toral automorphisms and [9, Section 6].

First suppose that N is a C^1 cross-section transverse to the flow direction and the tangent space to N at each point is transverse to the strongly stable and strongly unstable foliation of ϕ_t. Since the flow direction is a neutral direction we also have the weakly-stable $\{W_\epsilon^{sc}(p)\}$ and weakly-unstable $\{W_\epsilon^{uc}(p)\}$ foliations. We will use the notation lum and lsm to refer to strongly unstable and strongly stable manifolds.

Let $p \in N$ and let $W_\epsilon^s(p)$ and $W_\epsilon^u(p)$ be the lsm and lum of p for ϕ_t.

We may project a piece of $W_\epsilon^s(p)$ onto N by forming $W_\epsilon^{sc}(p) \cap N$. This gives a set $\tilde{W}_\epsilon^s(p)$ in N. Similarly we may define $\tilde{W}_\epsilon^u(p) := W_\epsilon^{uc}(p) \cap N$. If $y \in \tilde{W}_\epsilon^s(p)$ then the flow lines $\phi_t x, \phi_t y$ approach each other at a uniform exponential rate. This means that the iterates $\phi_N^i x$ and $\phi_N^i y$ will approach each other at a uniform exponential rate until a time i at which $\phi_t \tilde{W}_\epsilon^s(p) \cap \partial N \neq \emptyset$ for $0 < t \leq i$. The fact that N has a C^1 boundary implies that the standard Borel-Cantelli argument shows that there exists $\epsilon(x) > 0$ such that $\tilde{W}_{\epsilon(x)}^s(p)$ is a strong stable manifold for ϕ_N through x where $\epsilon(x)$ refers to the length of the strongly stable manifold. An analogous discussion applies to the projection of the weakly-unstable foliation $\{W_\epsilon^{uc}(p)\}$ to produce the strongly unstable foliation $\tilde{W}_{\epsilon(x)}^u(p)$ for $\phi_N : N \mapsto N$. These foliations form a hyperbolic block structure on N and this implies that $\phi_N : N \mapsto N$ is either Bernoulli or Bernoulli times a rotation [9, Theorem 6.1, p 453].

We may now adapt the proof of Section 3 to show that there are sufficiently long leaves in the foliations $\{\tilde{W}_{\epsilon(x)}^s(p)\}$, $\{\tilde{W}_{\epsilon(x)}^u(p)\}$ to enable a Hopf chain to be constructed between arbitrary points $x, y \in N$. Note that $\phi_t \tilde{W}_\epsilon^s(p) \cap \partial N \neq \emptyset$ for

$t > 0$ only if $\tilde{W}^s_\epsilon(p) \cap \phi_{-t}\partial N \neq \emptyset$ and similarly $\phi_{-t}\tilde{W}^u_\epsilon(p) \cap \partial N \neq \emptyset$ for $t > 0$ only if $\tilde{W}^u_\epsilon(p) \cap \phi_t \partial N \neq \emptyset$.

Consider for $t > 0$ the singularity sets $\cup \phi_{-t}\partial N \cap N$ and $\cup \phi_t \partial N \cap N$. Since the flow is transverse to N (hence return times to N are bounded away from zero as N is uniformly transverse to the flow direction) and the C^1 boundary of N is not aligned in the direction of the strongly stable or strongly unstable direction for the flow there exists $\tau > 0$ such that $\cup_{|t|>\tau}\phi_t \partial N \cap N$ consists of a finite number of submanifolds with tangents contained in a family of sectors $C_1(p)$ oriented along the strongly unstable direction in N and $\cup_{-t>\tau}\phi_t\partial N \cap N$ consists of submanifolds with tangents contained in a family of sectors $C_2(p)$ oriented along the strongly stable direction in N. Furthermore for any $M > \tau$ the codimension one (in the cross-section N) submanifolds in $\cup_{M>|t|>\tau}\phi_t\partial N \cap N$ intersect in a finite number of codimension two (in the cross-section N) submanifolds. These conditions are all that is needed to use the same argument as that of Section 3 to establish a Hopf chain for ϕ_N between arbitrary points in N. This Hopf chain persists for any power of ϕ_N as $\phi_t\tilde{W}^s_\epsilon(p) \cap \partial N = \emptyset$ and $\phi_{-t}\tilde{W}^u_\epsilon(p) \cap \partial N = \emptyset$ for all t. Thus all powers of ϕ_N are ergodic and hence (ϕ_n, N, μ_N) is Bernoulli.

□

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