A Novel Framework of Fuzzy Complex Numbers and its Application to Compositional Modelling

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Abstract— **Dealing with various inexact pieces of information has become an intrinsically important issue in knowledge-based reasoning, because many problem domains involve imprecise, incomplete and uncertain information. Indeed, different approaches exist for reasoning with inexact knowledge and data. However, the common strategy they adopt is to integrate various types of inexact information into a global measure. This may destroy the underlying semantics associated with different information components. This paper presents an innovative notion of fuzzy complex numbers (FCNs), which extends real complex numbers to representing two-dimensional uncertainties conjunctively without necessarily integrating them. This new framework is applied to supporting Compositional Modelling (CM). In particular, calculus of FCNs over arithmetic and propositional relations is developed to entail scenario model synthesis from model fragments, and modulus of FCNs is introduced to constrain the scenario descriptions. The utility and usefulness of this work are illustrated by means of an example for constructing possible scenario descriptions from given evidence in the crime investigation domain.**

I. INTRODUCTION

Solving many real-world problems requires the use of inexact information captured in a variety of forms. Conceptually, such information may be classified into the following four types, which are of particular interest to the present work: 1) *Vagueness*: It occurs when the boundary of a piece of information can not be determined precisely (e.g. Bob is *tall*). 2) *Uncertainty*: It depicts the reliability or confidential weight of a given piece of information, usually represented by a numerical value (e.g. The suspect overpowers the victim, with certainty degree 0.7). 3) *Both vagueness and uncertainty*: That is, types 1 and 2 coexist (e.g. The amount of collected fiber is *a_lot*, with certainty degree 0.7). 4) *Both vagueness and uncertainty with the uncertainty also expressed in vague terms:* Instead of using numerical values, the uncertainty is described by a fuzzy number or linguistic term (e.g. The amount of collected fiber is *a_lot*, with certainty degree *very_likely*).

To address such information, much work has been developed, especially to support reasoning with inexact knowledge and data. Although the application domains and problem-solving approaches may be rather different, they all aim to integrate the underlying distinct bits of inexact information into a global measure. Whilst different kinds of information may be effectively integrated in these approaches, the underlying semantics associated with different information components may be destroyed. Thus, it is of great interest and potentially beneficial to establish a new mechanism which will maintain the associated semantics when perform reasoning without necessarily using just one global scale.

Inspired by this observation, this paper proposes a novel framework of *fuzzy complex numbers* (FCNs) that will entail effective and efficient representation of aforementioned types of inexact information conjunctively and explicitly. Note that the term FCNs is not new; the concept of conventional complex numbers has been extended with fuzzy set theory for the purpose of providing richer representations. In particular, a *fuzzy complex number* has been defined in [1] as a type-1 fuzzy set, mapping from the complex plane to $[0, 1]$. The basic arithmetic operations including addition, substraction, multiplication and division have been developed for such *fuzzy complex numbers*, using the extension principle. Further, the differentiation and integration of this type of *fuzzy complex number* is proposed in [2], [3], with more advanced follow-on work given in [14], [15]. Also, another notion that connects real complex numbers to fuzzy sets is introduced in [11], where a new type of set, named *complex fuzzy sets*, is suggested to allow the membership value of a standard fuzzy set to be represented using a classical complex number. Basic set theoretic operations are defined on *complex fuzzy sets*, including complement, union and intersection. However, as indicated in [11], it can be a difficult task to acquire intuition for the concept of complex-valued membership. This limits significantly the potential application of such fuzzy sets. Nevertheless, work has continued along this theme of research. This is evident in that a framework of fuzzy reasoning systems that incorporate *complex fuzzy sets* in propositional logic has been proposed in [4].

Briefly, existing work on so-called *fuzzy complex numbers* or *complex fuzzy sets* is framed by either giving conventional complex numbers a real-valued membership or assigning a fuzzy set element with a complex number as its membership value. These are rather different from what is proposed in this paper, where both the real and imaginary values of an FCN are in general, themselves fuzzy numbers, with each having an embedded semantic meaning. In particular, the paper introduces for the first time the mathematical definition and operations of such FCNs. This work is applied to support Compositional Modelling (CM) in an effort to deal with inexact knowledge and data. CM has been developed to synthesize and store plausible scenarios in many problem domains with promising results [5], [9], [10], [12]. In this paper, the calculus of FCNs over propositional and arithmetic relations is introduced, thereby entailing scenario model generation from model fragments. Also, it defines the modulus of FCNs, which is applied to constrain the scenario descriptions.

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978-1-4244-3597-5/09/$25.00 ©2009 IEEE

The rest of this paper is organized as follows. Section II introduces the background that sets the scene for the present work. Section III proposes the innovative notion of FCNs, which extends real-valued complex numbers to representing two-dimensional uncertainties. The proposed new type of FCNs is integrated in the process of model composition, with a propagation algorithm provided, in Section IV. This is followed by an illustrative example in Section V, showing the utility and usefulness of the proposed approach to support equivocal death investigation. Finally, Section VI concludes the paper and points out future research.

II. Outline of Compositional Modelling

One of the most significant advantages of using CM is its ability to automatically construct many variations of a given type of scenario from a relatively small knowledge base. In existing work, it is traditionally assumed that the generic and reusable model fragments within the knowledge base can all be expressed by precise and crisp information. However, significant challenges arise for problems where the degree of inexactness of available knowledge and data can vary greatly. This work extends the existing CM work to allow the generation of scenarios which is capable of representing, storing and supporting inference about inexact information, by the use of the propopsed notion of FCNs. For completeness, the basic concepts of CM are briefly introduced below.

The knowledge base (KB) in a compositional modeller consists of a number of generic and reusable model fragments, each of which represents a set of relationships between domain objects and their states for a certain type of partial scenario. A model fragment (MF) has two parts that encode domain knowledge: 1) The relations between domain elements, often represented in a form that is similar to the style of conventional production rules, but involving much more general contents; and 2) a set of rule instances that represent how certainly it is that the corresponding relationships hold. More formally, an MF is a tuple $\langle \mathbf{X}, \mathbf{Y}, \mathbf{H} \rangle$, as generally described below:

If $\mathbf{X}$
Assuming $\mathbf{H}$
Relations $\{\mathbf{X},\ \mathbf{H} \rightarrow \mathbf{Y}\}$
Distribution $\mathbf{Y}$
$\{A^1_{j_1}, \ldots, A^n_{j_n}, C^1_{i_1}, \ldots, C^q_{i_q} \rightarrow B^1_{j_{n+1}}, \ldots, B^m_{j_{n+m}} : p\}$

In this definition, $\mathbf{X} = \{x_1, \ldots, x_n\}$ is a set of antecedent predicates. The **If** statement describes the required conditions for a partial scenario to become applicable; $\mathbf{H} = \{h_1, \ldots, h_q\}$ is a (possibly empty) set of assumptions, referring to those pieces of information which are unknown or cannot be inferred from others, but they may be presumed to hold for the sake of performing hypothetical reasoning; $\mathbf{Y} = \{y_1, \ldots, y_m\}$ is a set of consequent predicates, which describe the consequences when the conditions and assumptions hold, including pieces of new knowledge or relations which are derived from the hypothetical reasoning; p indicates the certainty degree of its corresponding outcome if the MF is instantiated; and finally within the **Distribution** statement, the left hand side of the "implication" sign in each propositional instance is a combination of value pairs, involving the values of antecedent and assumption variables, and the right hand side indicates the corresponding possible outcome if the MF is instantiated.

Note that, in CM, it is not required that every possible combination of antecedent and assumption values has to be assigned a certainty degree. This is because the number of combinations will increase exponentially with the number of variables. In acquiring domain knowledge, it is unlikely to obtain all such details but the most significant components. In the present work, the default certainty degree for those unassigned combinations is set to 0.

For simplicity, in this work, the assumptions are treated as part of the antecedent due to their logical equivalence. Hence, a rule instance in an MF is of the form:

$$\text{IF } x_1 \text{ is } A^1_{j_1} \oplus \cdots \oplus x_n \text{ is } A^n_{j_n}, \text{ THEN } y_1 \text{ is } B^1_{j_{n+1}} \otimes \cdots \otimes y_m \text{ is } B^m_{j_{n+m}}, \tag{1}$$

where the operator $\otimes$ in the consequence is restricted to logical conjunction in this paper, but $\oplus$ in the antecedence denotes either a conjunctive or disjunctive operator. Further, this rule can be decomposed to multiple simpler rules each involving a single consequence, and can equivalently be written as:

$$\text{IF } x_1 \text{ is } A^1_{j_1} \oplus \cdots \oplus x_n \text{ is } A^n_{j_n}, \text{ THEN } y_1 \text{ is } B^1_{j_{n+1}},$$
$$\cdots$$
$$\text{IF } x_1 \text{ is } A^1_{j_1} \oplus \cdots \oplus x_n \text{ is } A^n_{j_n}, \text{ THEN } y_m \text{ is } B^m_{j_{n+m}}.$$

Therefore, in the reminder of this work, only the rules with single consequence will be considered.

In order to depict vague information in MFs, fuzzy representation has been introduced to CM in [6]. In particular, the values of fuzzy variables can be represented by fuzzy sets and certainty degrees can be represented in linguistic terms or fuzzy numbers. Interested readers can refer to [6] for further details.

III. Fuzzy Complex Numbers

A. Definition of FCNs

Inherit from the real complex numbers, an FCN, $\tilde{z}$, is defined in the form of:

$$\tilde{z} = \tilde{a} + i\tilde{b}, \tag{2}$$

where both $\tilde{a}$ and $\tilde{b}$ are fuzzy numbers with membership functions $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$, regarding a given domain variable x. $\tilde{a}$ is the real part of $\tilde{z}$ while $\tilde{b}$ represents the imaginary part, i.e. $Re(\tilde{z}) = \tilde{a}$ and $Im(\tilde{z}) = \tilde{b}$.

An FCN can be visually shown as in Fig. 1. Importantly, in general, for a given $\tilde{z}$, both $Re(\tilde{z})$ and $Im(\tilde{z})$ are fuzzy. If $\tilde{b}$ does not exist, $\tilde{z}$ degenerates to a fuzzy number. Further, if $\tilde{b}$ does not exist and $\tilde{a}$ iteself degenerates to a real number, then $\tilde{z}$ degenerates to a real number.

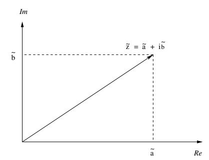

Fig. 1. A fuzzy complex number

B. Operations on FCNs

The basic operations on FCNs form a straightforward extension of those on real complex numbers. Let $\widetilde{z}_1 = \tilde{a} + i\tilde{b}$ and $\widetilde{z}_2 = \tilde{c} + i\tilde{d}$ be two FCNs, where $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$ are fuzzy numbers with membership functions $\mu_{\tilde{a}}(x)$, $\mu_{\tilde{b}}(x)$, $\mu_{\tilde{c}}(x)$ and $\mu_{\tilde{d}}(x)$, respectively. The basic arithmetic operations on $\widetilde{z}_1$ and $\widetilde{z}_2$ are defined as follows:

Addition

$$\widetilde{z}_1 + \widetilde{z}_2 = (\tilde{a} + \tilde{c}) + i(\tilde{b} + \tilde{d}), \tag{3}$$

where $\tilde{a} + \tilde{c}$ and $\tilde{b} + \tilde{d}$ are newly derived fuzzy numbers with the following membership functions:

$$\begin{aligned} \mu_{\tilde{a}+\tilde{c}}(y) &= \bigvee_{y=x_1+x_2} (\mu_{\tilde{a}}(x_1) \wedge \mu_{\tilde{c}}(x_2)), \\ \mu_{\tilde{b}+\tilde{d}}(y) &= \bigvee_{y=x_1+x_2} (\mu_{\tilde{b}}(x_1) \wedge \mu_{\tilde{d}}(x_2)). \end{aligned} \tag{4}$$

Subtraction

$$\widetilde{z}_1 - \widetilde{z}_2 = (\tilde{a} - \tilde{c}) + i(\tilde{b} - \tilde{d}), \tag{5}$$

where $\tilde{a} - \tilde{c}$ and $\tilde{b} - \tilde{d}$ are newly derived fuzzy numbers with the following membership functions:

$$\begin{aligned} \mu_{\tilde{a}-\tilde{c}}(y) &= \bigvee_{y=x_1-x_2} (\mu_{\tilde{a}}(x_1) \wedge \mu_{\tilde{c}}(x_2)), \\ \mu_{\tilde{b}-\tilde{d}}(y) &= \bigvee_{y=x_1-x_2} (\mu_{\tilde{b}}(x_1) \wedge \mu_{\tilde{d}}(x_2)). \end{aligned} \tag{6}$$

Multiplication

$$\widetilde{z}_1 \times \widetilde{z}_2 = (\tilde{a}\tilde{c} - \tilde{b}\tilde{d}) + i(\tilde{b}\tilde{c} + \tilde{a}\tilde{d}), \tag{7}$$

where $\tilde{a}\tilde{c} - \tilde{b}\tilde{d}$ and $\tilde{b}\tilde{c} + \tilde{a}\tilde{d}$ are newly derived fuzzy numbers with the following membership functions:

$$\begin{aligned} \mu_{\tilde{a}\tilde{c}-\tilde{b}\tilde{d}}(y) &= \bigvee_{y=x_1x_2-x_3x_4} (\mu_{\tilde{a}}(x_1) \wedge \mu_{\tilde{c}}(x_2) \wedge \mu_{\tilde{b}}(x_3) \wedge \mu_{\tilde{d}}(x_4)), \\ \mu_{\tilde{b}\tilde{c}+\tilde{a}\tilde{d}}(y) &= \bigvee_{y=x_1x_2+x_3x_4} (\mu_{\tilde{b}}(x_1) \wedge \mu_{\tilde{c}}(x_2) \wedge \mu_{\tilde{a}}(x_3) \wedge \mu_{\tilde{d}}(x_4)). \end{aligned} \tag{8}$$

Division

$$\frac{\widetilde{z}_1}{\widetilde{z}_2} = \left(\frac{\tilde{a}\tilde{c} + \tilde{b}\tilde{d}}{\tilde{c}^2 + \tilde{d}^2}\right) + i\left(\frac{\tilde{b}\tilde{c} - \tilde{a}\tilde{d}}{\tilde{c}^2 + \tilde{d}^2}\right). \tag{9}$$

For notational simplicity, let $\tilde{t}_1 = \frac{\tilde{a}\tilde{c}+\tilde{b}\tilde{d}}{\tilde{c}^2+\tilde{d}^2}$ and $\tilde{t}_2 = \frac{\tilde{b}\tilde{c}-\tilde{a}\tilde{d}}{\tilde{c}^2+\tilde{d}^2}$, where $\tilde{t}_1$ and $\tilde{t}_2$ are newly derived fuzzy numbers with the following membership functions:

$$\begin{aligned} \mu_{\tilde{t}_1}(y) &= \bigvee_{y=\frac{x_1x_3+x_2x_4}{x_3^2+x_4^2},\, x_3^2+x_4^2\neq 0} (\mu_{\tilde{a}}(x_1) \wedge \mu_{\tilde{b}}(x_2) \wedge \mu_{\tilde{c}}(x_3) \wedge \mu_{\tilde{d}}(x_4)), \\ \mu_{\tilde{t}_2}(y) &= \bigvee_{y=\frac{x_2x_3-x_1x_4}{x_3^2+x_4^2},\, x_3^2+x_4^2\neq 0} (\mu_{\tilde{a}}(x_1) \wedge \mu_{\tilde{b}}(x_2) \wedge \mu_{\tilde{c}}(x_3) \wedge \mu_{\tilde{d}}(x_4)). \end{aligned} \tag{10}$$

Modulus Given $\tilde{z}_1 = \tilde{a} + i\tilde{b}$, the modulus of $\tilde{z}_1$ is defined:

$$|\tilde{z}_1| = \sqrt{\tilde{a}^2 + \tilde{b}^2}. \tag{11}$$

It is obvious that $|\tilde{z}_1|$ is a newly derived fuzzy number with the following membership function:

$$\mu_{|\tilde{z}_1|}(y) = \bigvee_{y=\sqrt{x_1^2+x_2^2}} (\mu_{\tilde{a}}(x_1) \wedge \mu_{\tilde{b}}(x_2)). \tag{12}$$

IV. Propagation of FCNs during Model Composition

When dynamically composing instantiated MFs into plausible scenarios, the vague and uncertain information (which can be concisely represented in terms of FCNs) needs to be propagated from individual fragments to their related ones. This section describes how inexact information captured in FCNs may be combined and propagated through relations. Note that whilst CM is herein used to illustrate the combinational propagation of FCNs, the underlying mechanism is general and can be readily adapted to suit other problems.

A. Propagation

In CM, each proposition which reflects a node in the space of plausible scenario descriptions (called emerging sceanrio space hereafter) has an FCN attached. Also, each rule instance in any generic MF has an FCN attached, denoted by $\tilde{z}_r$, indicating the certainty degree of the corresponding causal proposition. The propagation of FCNs during model composition is a process of combining the FCNs given to the variables (antecedents or consequent) with the FCN attached to an instantiated rule instance ($\tilde{z}_r$).

For simplicity, the following rule instance in an MF:

IF x_1 is $A^1_{j_1} \oplus \cdots \oplus x_n$ is $A^n_{j_n}$, THEN y is $B_{j_{n+1}}$ ($\tilde{z}_r$)

is rewritten as:

IF $p_1 \oplus \cdots \oplus p_n$, THEN c ($\tilde{z}_r$),

where $p_1, \ldots, p_n$ are the antecedent propositions, c is the consequent proposition and $\tilde{z}_r$ is the FCN attached to the rule. Again, $\oplus$ in the antecedence may be interpreted as either a conjunctive or disjunctive operator.

Proposition 1: If $\oplus$ is a conjunctive operator, the aggregated FCN of the antecedent is:

$$\begin{aligned} \tilde{z}_{antecedent} &= \min(Re(\tilde{z}_{p_1}), \ldots, Re(\tilde{z}_{p_n})) \\ &\quad + i\min(Im(\tilde{z}_{p_1}), \ldots, Im(\tilde{z}_{p_n})). \end{aligned} \tag{13}$$

Proposition 2: If $\oplus$ is a disjunctive operator, the aggregated FCN of the antecedent is:

$$\begin{aligned} \tilde{z}_{antecedent} &= \max(Re(\tilde{z}_{p_1}), \ldots, Re(\tilde{z}_{p_n})) \\ &\quad + i\max(Im(\tilde{z}_{p_1}), \ldots, Im(\tilde{z}_{p_n})). \end{aligned} \tag{14}$$

Since the rule instance in an MF only has the certainty degree attached, i.e. $\tilde{z}_r = Re(\tilde{z}_r)$, the newly derived FCN attached to the consequence is computed as:

$$\begin{aligned} \tilde{z}_{new} &= \tilde{z}_{antecedent} \times \tilde{z}_r \\ &= Re(\tilde{z}_{antecedent}) \times \tilde{z}_r + iIm(\tilde{z}_{antecedent}) \times \tilde{z}_r, \end{aligned} \tag{15}$$

where $Re(\tilde{z}_{antecedent})$, $Im(\tilde{z}_{antecedent})$ and $\tilde{z}_r$ are represented by different fuzzy numbers.

During model composition, given a set of collected evidence and a pre-defined KB, plausible scenarios can be generated by joint use of two conventional inference techniques, namely *backward chaining* and *forward chaining*. In forward-chaining, the values of the antecedent variables are known, the value of the consequent variable is derived by using the compositional rule of inference. However, due to the lack of inverse operators over fuzzy sets, given the value of the consequent variable, no general method exists to derive the exact values of antecedent variables by direct computation. This is especially the case when there are more than one antecedent variable involved. To address this problem, in this work, the fuzzy rule instances contained within MFs are not used to derive the unknown values of the antecedent variables. Instead, such causal constraints over the antecedent and consequent variables are employed to check for consistency amongst their respective values. Both forward and backward chaining processes are explained in detail below. Due to the nature of evidence-driven scenario generation, backward chaining is described first.

1) Backward chaining: This involves the abduction of domain variables and their states which might have led to the available evidence. Plausible causes can be identified by instantiating the conditions and assumptions of those MFs whose consequent matches the given evidence. However, for efficiency, only those possible causes whose consequences matching with the given evidence have resulted in the greatest FCN modulus are created in the emerging scenario space. As indicated earlier, during backward chaining, where $\tilde{z}_c$ and $\tilde{z}_r$ are given, no general method exists to derive the exact value of $\tilde{z}_{antecedent}$ in a closed form. Hence, specific fuzzy quantity spaces are built to approximately represent the possible values of the real and imagery parts of $\tilde{z}_{antecedent}$, assuming that they can only take values from such predefined quantity spaces. For computational simplicity, in this work, it is further assumed that only fuzzy values from a common quantity space, $Q_{FN} = \{V_{FN_1}, \ldots, V_{FN_j}, \ldots, V_{FN_n}\}$, are possibly taken by $\tilde{z}_c$ and $\tilde{z}_r$.

Algorithm 1 is constructed by ensuring the generality that each element in Q_{FN} may be the possible value of $\tilde{z}_{antecedent}$. Thus, all such values are checked using (15) as the constraint. The value that best satisfies this constraint when given $\tilde{z}_c$, is selected to represent $\tilde{z}_{antecedent}$. Note that the $Re(\tilde{z}_c)$ and $Im(\tilde{z}_c)$ are checked against the constraint separately.

Algorithm 1 BackwardPropagation($\tilde{z}_c, \tilde{z}_r$)

```
Re(z̃_antecedent) ← ∅
Im(z̃_antecedent) ← ∅
S_Re = 0
S_Im = 0
for j = 1 : n do
    if S(Re(z̃_c), V_FN_j × z̃_r) > S_Re then
        S_Re = S(Re(z̃_c), V_FN_j × z̃_r)
        Re(z̃_antecedent) = V_FN_j
    end if
    if S(Im(z̃_c), V_FN_j × z̃_r) > S_Im then
        S_Im = S(Im(z̃_c), V_FN_j × z̃_r)
        Im(z̃_antecedent) = V_FN_j
    end if
end for
return z̃_antecedent
```

Once $\tilde{z}_{antecedent}$ is obtained, the next step is to derive the individual FCNs of each antecedent variable. Take the conjunctive operator for example, the following constraints are introduced:

$$\begin{aligned} &\min(Re(\tilde{z}_{p_1}), \ldots, Re(\tilde{z}_{p_n})) = Re(\tilde{z}_{antecedent}), \\ &\min(Im(\tilde{z}_{p_1}), \ldots, Im(\tilde{z}_{p_n})) = Im(\tilde{z}_{antecedent}). \end{aligned} \tag{16}$$

These constraints simply state that, for any $(V_{FN_m}, \ldots, V_{FN_k}) \in Q_{FN} \times \cdots \times Q_{FN}$, where Q_{FN} is the fuzzy quantity space [13] for both real and imaginary parts of FCN. If $\min(V_{FN_m}, \ldots, V_{FN_k}) = Re(\tilde{z}_{antecedent})$, then $V_{FN_m}, \ldots, V_{FN_k}$ are possible solutions for each real part of $\tilde{z}_{p_1}, \ldots, \tilde{z}_{p_n}$, respectively. Similarly, the possible solutions of the imaginary parts can be obtained. Note that, if $\oplus$ is interpreted as a disjunctive operator, all is needed is to change the *min* operator in (16) to *max*.

2) Forward chaining: This procedure applies logical deduction to all such the MFs in the KB whose conditions match the existing nodes in the emerging scenario space. Therefore, the corresponding FCN propagation algorithm is straightforward, as described in Algorithm 2. In particular, $\tilde{z}_{antecedent}$ and $\tilde{z}_r$ are known, the $\tilde{z}_{new}$ is computed by applying (15), where $\tilde{z}_{new}$ is the calculated FCN of the consequent variable. Given the fixed Q_{FN} in this work, once $\tilde{z}_{new}$ is computed, it is used to match against the elements of Q_{FN}, the element receives the highest fuzzy matching degree with $\tilde{z}_{new}$ is returned to represent $\tilde{z}_c$.

Algorithm 2 ForwardPropagation($\tilde{z}_{p_1}, \ldots, \tilde{z}_{p_n}, \tilde{z}_r$)

```
Calculate z̃_antecedent
z̃_new = z̃_antecedent × z̃_r
S_Re = 0
S_Im = 0
for j = 1 : n do
    if S(V_FN_j, Re(z̃_new)) > S_Re then
        S_Re = S(V_FN_j, Re(z̃_new))
        Re(z̃_c) = V_FN_j
    end if
    if S(V_FN_j, Im(z̃_new)) > S_Im then
        S_Im = S(V_FN_j, Im(z̃_new))
        Im(z̃_c) = V_FN_j
    end if
end for
return z̃_c
```

B. FCN updating

In the emerging scenario space, if an existing node resulted from instantiating another MF such that two instantiated MFs share a common variable value, then the existing FCNs associated with this node needs to be updated for consistency. Suppose that for a certain variable to take value A, and that the following two FCNs are obtained through different inference procedures:

$$\tilde{z}_A = V_{FN_i} + iV_{FN_j},$$
$$\tilde{z}'_A = V_{FN_p} + iV_{FN_q},$$

where $V_{FN_i}, V_{FN_j}, V_{FN_p}$ and V_{FN_q} denote different elements in the pre-defined Q_{FN} respectively. The work here uses the modulus of an FCN to evaluate the overall information content of its associated variable value. If $|\tilde{z}_A|$ and $|\tilde{z}'_A|$ are the same (whilst $\tilde{z}_A \neq \tilde{z}'_A$), then both $\tilde{z}_A$ and $\tilde{z}'_A$ will be kept as possible FCNs. However, if $|\tilde{z}_A| \neq |\tilde{z}'_A|$, then the newly updated FCN of this variable taking value A is obtained by:

$$\tilde{z}''_A = \min(V_{FN_i}, V_{FN_p}) + i\min(V_{FN_j}, V_{FN_q}). \tag{17}$$

This has an intuitive appeal because if (17) holds, then both $\tilde{z}_A$ and $\tilde{z}'_A$ will also hold.

V. Application to Crime Investigation

A. The crime case

This illustrative example is extracted and adapted from a realistic case described in [7]. Consider the following case description: *The victim was discovered lying on this back across his bed with a semiautomatic Remington .308 model 742 rifle between his legs. There were two bullet holes in* *the roof of the victim's trailer, suggesting that two shots had been fired upward towards the ceiling. Also, the window near the bed has been shattered. The preliminary observations suggested that the victim had shot himself in the head with the weapon and this case was classified as a suicide at the beginning. However, according to the expertise, it is very unlikely for a weapon to discharge twice during the suicide process due to a reflex action, it has never been reported in any forensic journal. Hence, another explanation comes up, this is possibly a homicidal case.*

B. Knowledge representation

In order to model the above case, four fuzzy variables are extracted from the description as defined in Table II. Suppose that the KB contains the following two MFs:

If $\{x_1\}$
Relations $\{x_1 \rightarrow x_2\}$
Distribution {
r_1: low_powered $\rightarrow$ low ($Very\_high$)
r_2: low_powered $\rightarrow$ medium ($Medium$)
r_3: low_powered $\rightarrow$ aloud ($Very\_small$)
r_4: medium $\rightarrow$ low ($Small$)
r_5: medium $\rightarrow$ medium ($Very\_high$)
r_6: medium $\rightarrow$ aloud ($Small$)
r_7: high_powered $\rightarrow$ low ($Very\_small$)
r_8: high_powered $\rightarrow$ medium ($Medium$)
r_9: high_powered $\rightarrow$ aloud ($Very\_high$) } (MF_1)

If $\{x_2, x_3\}$
Relations $\{x_2, x_3 \rightarrow x_4\}$
Distribution {
r_1: aloud, near $\rightarrow$ high ($Very\_high$)
r_2: aloud, medium $\rightarrow$ high($High$)
r_3: aloud, far $\rightarrow$ high ($Small$)
r_4: medium, near $\rightarrow$ high ($High$)
r_5: medium, medium $\rightarrow$ high ($Medium$)
r_6: medium, far $\rightarrow$ high ($Small$)
r_7: low, near $\rightarrow$ high ($Medium$)
r_8: low, medium $\rightarrow$ high ($Small$)
r_9: low, far $\rightarrow$ high ($Very\_small$) } (MF_2)

In this work for simplicity, the fuzzy numbers used to form FCNs (namely, both certainty and fuzzy matching degrees) are defined in the following fuzzy quantity space, as illustrated in Fig. 2:

$$Q_{FN} = \{Very\_small, Small, Medium, High, Very\_high\}.$$

C. Generation of plausible scenario space

Suppose that after examination of the death scene, two pieces of evidence are collected:

e_1: The degree of window shattered is *quite_high*, with certainty degree *Very_high*.

e_2: The Remington .308 model 742 rifle was found, with certainty degree *Very_high*.

Initial creation of an emerging scenario space is done by matching a given piece of evidence, say e_1, against the KB. Assume that the matching degree of e_1 and "x_4 is *high*" in MF_2 is *High*. Hence, the first node, "x_4 is *high*", is added to the emerging scenario space with the FCN: $Very\_high + iHigh$. Given this and the instantiated MF_2, backward chaining is performed from x_4. This leads to x_2 and x_3 being added to the emerging scenario space. By applying Algorithm 1 to MF_2, the results of Table I are obtained.

In order to avoid generating unnecessary explanations, the modeller produces only the current best or the most plausible explanations in the first instance. That is, the approach will not create alternative but less plausible explanation unless the current best has to be discarded. With regard to the use of the modulus of a derived FCN for plausibility evaluation, r_2 and r_4 are established to outperform the rest. This means that either "the sound volume is *aloud* and the distance is *medium*" or "the sound volume is *medium* and the distance is *near*" is the most plausible situation to have caused the *quite_high* degree of window shattered.

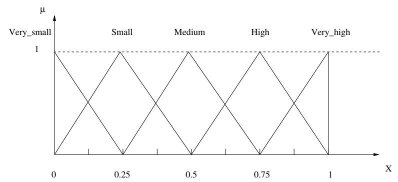

Fig. 2. Fuzzy quantity space for both certainty and fuzzy matching degrees

TABLE I

RESULTS OF BACKWARD CHAINING DUE TO e_1

Rule index	$Re(\tilde{z}_{antecedent})$	$Im(\tilde{z}_{antecedent})$
r_1	Very_high	High
r_2	Very_high	Very_high
r_3	N/A	N/A
r_4	Very_high	Very_high
r_5	N/A	Very_high
r_6	N/A	N/A
r_7	N/A	Very_high
r_8	N/A	N/A
r_9	N/A	N/A

TABLE II

FUZZY VARIABLES

Var	Interpretation	Domain
x_1	Power level of the weapon	{low_powered, medium, high_powered}
x_2	Sound volume of the weapon	{low, medium, aloud}
x_3	Distance between the discharge position and the window	{near, medium, far}
x_4	Degree of window shattered	{very_low, low, medium, quite_high, high}

According to Table I, $\tilde{z}_{antecedent_{r_2}} = Very\_high + iVery\_high$, it follows from (16) that

$$Re(\tilde{z}_{antecedent_{r_2}}) = \min(Re(\tilde{z}_{x_2:aloud}), Re(\tilde{z}_{x_3:medium})),$$
$$Im(\tilde{z}_{antecedent_{r_2}}) = \min(Im(\tilde{z}_{x_2:aloud}), Im(\tilde{z}_{x_3:medium})).$$

It is obvious that

$$Re(\tilde{z}_{antecedent_{r_2}}) = Very\_high \Longleftrightarrow \begin{cases} Re(\tilde{z}_{x_2:aloud}) = Very\_high \\ Re(\tilde{z}_{x_3:medium}) = Very\_high \end{cases}$$
$$Im(\tilde{z}_{antecedent_{r_2}}) = Very\_high \Longleftrightarrow \begin{cases} Im(\tilde{z}_{x_2:aloud}) = Very\_high \\ Im(\tilde{z}_{x_3:medium}) = Very\_high \end{cases}$$

Therefore, the FCNs associated with $x_2 : aloud$ and $x_3 : medium$ can be respectively written as:

$$\tilde{z}_{x_2:aloud} = Very\_high + iVery\_high,$$
$$\tilde{z}_{x_3:medium} = Very\_high + iVery\_high.$$

Similarly, the FCNs attached to r_4 can be derived. After that, the newly created instances of plausible causes are recursively used in the same manner as if each of them was the original piece of evidence. As a result, MF_1 is then instantiated, x_1 is added and the FCNs associated with the states of x_1 can be obtained in the same way as above. This phase of backward chaining (with given e_1) leads to what is shown in Fig. 3.

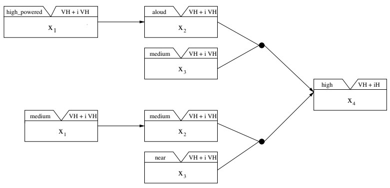

Fig. 3. Emerging scenario space after backward chaining initiated by e_1

Given e_2 and the same KB, no backward chaining is needed as the information Remington .308 model 742 rifle only matches the antecedent of MF_1 as a *high_powered* weapon with the $FCN : Very\_high + iHigh$. This piece of evidence is then used to revise the emerging scenario space of Fig. 3 by forward chaining. In particular, the FCN associated with the node "x_1 is high_powered" is updated by using constraint (17). This leads to the result that the FCN associated with x_1 being *high_powered* becomes $Very\_high + iHigh$. As the variable value for x_1 is *high_powered*, the strategy of best-first generation of the scenario space will ensure the removal of the lower branch of Fig. 3. The forward chaining procedure is much easier to complete as it involves straightforward calculations only. The results are listed in Table III.

TABLE III

RESULTS OF FORWARD CHAINING PROPAGATION DUE TO e_2

Rule index	$Re(\tilde{z}_c)$	$Im(\tilde{z}_c)$
r_7	Very_small	Very_small
r_8	Medium	Medium
r_9	Very_high	High

According to the modulus of the FCN obtained for each matched propositional instance, the one associated with r_9 in MF_1 outperforms those associated with the rest. The consequence of r_9 in MF_1, "x_2 is *aloud*", is therefore firstly instantiated. Because "x_2 is *aloud*" already exists in the emerging space, it is only needed to update the FCN associated with it, which is done in the same manner as above. The revised scenario space is depicted in Fig. 4. For this simple example, no further instantiations and propagations of variable states are possible. Hence, Fig. 4 is the final outcome of the entire CM process.

In summary, the resulting scenario space offers the best explanation possible for the collected pieces of evidence. According to the initial case description, the bed is *near* the window, hence, the victim lying on the bed did not shot himself, because the distance between the discharge position and the window is establish to be *medium* in the resultant scenario description. Therefore, this case is suggested being a homicide rather than a suicide.

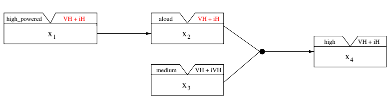

Fig. 4. Emerging scenario space after forward chaining

VI. CONCLUSIONS

This paper has presented a novel framework of fuzzy complex numbers, which is capable of representing and propagating different kinds of inexact knowledge and data in a unified manner. This ability is demonstrated by exploiting the framework to support the task of compositional modelling. Additionally, from the CM point of view, the proposed method not only provides a more concise and flexible knowledge representation formalism, but also enhances the capability of CM to handle a wide range of inexact information. Finally, the utility and usefulness of the proposed framework are illustrated by means of an application to the construction of plausible descriptions from given evidence in the crime investigation domain. Note that whilst crime investigation is herein used to demonstrate the applicability of FCNs, the proposed framework is general and can be readily adapted to suit different problems such as representing knowledge for multifinered robot hand manipulation in [8], [10].

Whilst the results are promising, the present notion of FCNs is only capable of jointly representing two kinds of inexactness. It remains as an important piece of further research to extend this framework to arbitrary n-types of inexact information. Also, a more general constraint satisfaction mechanism suitable for constraining variables modified by FCNs would help to improve the generality of this work. In particular, this may avoid the need to prefix just one common quantity space which the real and imaginary parts of any FCN may take values from.

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