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Essays in Mechanism Design

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ABSTRACT

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This thesis addresses problems in the area of mechanism design. In many settings in which collective decisions are made, individuals' actual preferences are not publicly observable. As a result, individuals should be relied on to reveal this information. We are interested in an important application of mechanism design, which is the construction of desirable procedures for deciding upon resource allocation or task assignment.

We make two main contributions. First, we propose a new mechanism for allocating a divisible commodity between a number of buyers efficiently and fairly. Buyers are assumed to behave as price-anticipators rather than as price-takers. The proposed mechanism is as parsimonious as possible, in the sense that it requires participants to report a one-dimensional message (scalar strategy) instead of an entire utility function, as required by Vickrey-Clarke-Groves (VCG) mechanisms. We show that this

mechanism yields efficient allocations in Nash equilibria and moreover, that these equilibria are envy-free. Additionally, we present distinct results that this mechanism is the only simple scalar strategy mechanism that both implements efficient Nash equilibria and satisfies the no envy axiom of fairness. The mechanism's Nash equilibria are proven to satisfy the fairness properties of both Ranking and Voluntary Participation.

Our second contribution is to develop optimal VCG mechanisms in order to assign identical economic "bads" (for example, costly tasks) to agents. An optimal VCG mechanism minimizes the largest ratio of budget imbalance to efficient surplus over all cost profiles. The optimal non-deficit VCG mechanism achieves asymptotic budget balance, yet the non-deficit requirement is incompatible with reasonable welfare bounds. If we omit the non-deficit requirement, individual rationality greatly changes the behavior of surplus loss and deficit loss. Allowing a slight deficit, the optimal individually rational VCG mechanism becomes asymptotically budget balanced. Such a phenomenon cannot be found in the case of assigning economic "goods."

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Chapter 1

Introduction

1.1 Vickrey-Clarke-Groves Mechanisms

This thesis deals with an important application of mechanism design, which is the construction of procedures for deciding upon resource allocation or task assignment. We denote by $N = \{1, \cdots, n\}$ the set of agents, and assume that there are at least two agents. We focus on the special class of environments in which agents have quasi-linear preferences. Each agent $i \in N$ has a private monetary valuation on consumption x_i. There is a monetary transfer $t_i \in \mathcal{R}$ between each agent i and the mechanism. Agent i may need to pay some money to the mechanism or the mechanism may subsidize the agent. The net utility of each agent $i \in N$ has the following quasi-linear preference:

$$V_i = u_i(x_i) - t_i. \tag{1.1}$$

The efficient allocation (of resources) is defined to be an allocation x such that:

$$x \in \underset{x \in \mathcal{X}}{\operatorname{argmax}} \sum_{i \in N} u_i(x_i) \tag{1.2}$$

where $\mathcal{X}$ is the set of feasible allocations. A direct revelation mechanism[1] is strategy-proof (dominant strategy incentive compatible) if for every agent, truth telling is a dominant strategy equilibrium of the mechanism.

In social choice environments with quasi-linear preferences and private valuations, a group of mechanisms derived from the seminal work of Vickrey (1961), Clarke (1971) and Groves (1973) consists of mechanisms whose allocation rules select the efficient and strategy-proof outcome. Green and Laffont (1977, 1979) discovered that any direct revelation mechanism satisfying those two properties is a member of Vickrey-Clarke-Groves mechanisms (VCG mechanisms). Holmstrom (1979) proved that VCG mechanisms are unique on restricted domains which are smoothly connected, in particular convex domains (Suijs (1996) and Carbajal-Ponce (2007) investigated further into the uniqueness of VCG mechanisms).

After collecting reported valuations $\hat{u}$ from agents, a **VCG Mechanism** selects resource allocation x such that:

$$x(\hat{u}) \in \underset{x \in \mathcal{X}}{\operatorname{argmax}} \sum_{i \in N} \hat{u}_i(x_i) \tag{1.3}$$

[1] A mechanism is a direct revelation mechanism if the strategy each agent reports is his valuations on all possible allocations

and implements the following payment scheme for each $i \in N$:

$$t_i(\hat{u}) = -\sum_{j \neq i} \hat{u}_j(x_j(\hat{u})) + h_i(\hat{u}_{-i}) \tag{1.4}$$

where $h_i(\cdot)$ is an arbitrary function of $\hat{u}_{-i}$.

Therefore, agent i's net utility for VCG mechanisms is:

$$u_i(x_i(\hat{u}_i, \hat{u}_{-i})) + \sum_{j \neq i} \hat{u}_j(x_j(\hat{u}_i, \hat{u}_{-i})) - h_i(\hat{u}_{-i}). \tag{1.5}$$

Since h_i depends only on the messages $\hat{u}_{-i}$ sent by other agents, agent i tries to maximize $u_i(x_i(\hat{u}_i, \hat{u}_{-i})) + \sum_{j \neq i} \hat{u}_j(x_j(\hat{u}_i, \hat{u}_{-i}))$ by choosing $\hat{u}_i$. This expression has a maximum, $\max_{x \in \mathcal{X}} u_i(x_i) + \sum_{j \neq i} \hat{u}_j(x_j)$. Agent i can achieve this maximum by reporting his true utility function u_i according to expression (1.3). Truth telling is a dominant strategy for every agent, and therefore, VCG mechanisms select efficient allocation $x(u)$.

Despite satisfying strong incentive compatibility and efficiency, VCG mechanisms can be too complicated to be of use in some cases (Nisan and Ronen, 2007), and are not guaranteed to be budget balanced (Green and Laffont (1979)). These difficulties of VCG mechanisms have recently led economists and computer scientists to search alternative mechanisms (Johari and Tsitsiklis (2007), Maheswaran and Basar (2006), Yang and Hajek (2006a, 2006b)), or to identify particular VCG mechanisms minimizing budget imbalance (Bailey (1997), Deb, Gosh and Seo (2002), Green et al.(1976), Green and Laffont (1979), Guo and Conitzer (2009), Zhou (2007), Moulin (1986), Deb and Seo (1998)).

As we explain in detail in Chapter 2, when a resource is divisible (for example, electricity in power grid), agents' reporting utility functions to a VCG mechanism implies that each agent should submit an infinite number of valuations to describe his utility function. Therefore, the communication between every agent and the mechanism requires exponential growth of effort which probably causes an extra cost to the agent (Rothkopf (2007)). Additionally, computation of the efficient allocation and payment is almost intractable (NP-hard) in VCG mechanisms when a resource is divisible (Nisan and Ronen, (2007), Rothkopf (2007)).

On the other hand, when objects to be allocated are identical and indivisible, implementing VCG mechanisms would not cause informational burden on agents. However, we still face the fact that every VCG mechanism cannot be budget balanced at all profiles. If a VCG mechanism generates a budget surplus, then it needs to be given away to a passive residual claimant in order to preserve the incentive compatibility of a VCG mechanism. In case of budget deficit, the residual claimant must finance the mechanism using outside monetary source. Interpreting the budget imbalance of a mechanism as its implementation cost, Chapter 3 focuses on minimizing budget imbalance in the original VCG mechanisms.

1.2 Divisible Commodity Allocation and Scalar Strategy Mechanisms

For the problem of allocating a divisible commodity where the total amount of the resource is $R > 0$, the monetary value of agent i's resource share $x_i \in [0, R]$ is represented by a utility function, u_i, that is strictly increasing, concave, and smooth.

For the case of a divisible resource, the amount of information each agent should report to a VCG mechanism is infinitely dimensional. As a class of alternative mechanisms to complicated VCG mechanisms, scalar strategy mechanisms have received intense attention from computer scientists, interested in designing network capacity allocation mechanisms. Kelly (1997) and Kelly et al. (1998) have proposed network bandwidth allocation algorithms, where participants submit scalar bids, and then the algorithms achieve efficiency under price taking behavior.

Since every agent is required to submit only a one-dimensional message in scalar strategy mechanisms, strategy-proofness is no longer achievable incentive compatibility. Instead, we are interested in Nash incentive compatibility of scalar strategy mechanisms, that is, the scalar strategy mechanism always induces a Nash equilibrium in which the resource is allocated efficiently for a given preference profile. Adopting Nash incentive compatibility, for the uniform price model in Kelly (1997), the recent literature has focused on its efficiency loss under price anticipating behavior (Maheswaran and Basar (2005), Johari and Tsitsiklis (2004) and Hajek and Yang (2004)).

For multi-price models, Kelly et al. (1998) provided the original idea of a mechanism that maximizes total surrogate utilities, and this has inspired the following multi-price scalar strategy mechanisms: VCG-like mechanisms by Johari and Tsitsiklis (2007); g-mechanisms by Maheswaran and Basar (2006) and Yang and Hajek (2006a); and VCG-Kelly mechanisms by Yang and Hajek (2006b). These mechanisms have been proven to implement efficient Nash equilibria. Among multi-price scalar strategy mechanisms, VCG-like mechanisms in Johari and Tsitsiklis (2007) have the most general form and thus, we will study VCG-like mechanisms in Chapter 2.

1.2.1 VCG-like Mechanisms

VCG-like mechanisms use one-dimensional message (scalar strategy) spaces and differentiated unit prices. They are similar in both spirit and form to the VCG mechanisms except that each individual reports a one-dimensional message, rather than his entire utility function.

A **VCG-like mechanism** first requires each agent i to report a nonnegative real number θ_i which selects a surrogate utility function $\bar{u}(\cdot, \theta_i)$ from a given single parameter family of functions. The set of surrogate utility functions is the same for all agents. After the profile of messages θ is collected, VCG-like mechanisms choose the resource allocation x such that:

$$x(\theta) \in \operatorname*{argmax}_{x \in \mathcal{X}} \sum_{i \in N} \bar{u}(x_i, \theta_i) \tag{1.6}$$

and the payment scheme t designed similarly to VCG mechanisms such that:

$$t_i(\theta) = -\sum_{j \neq i} \bar{u}(x_j(\theta), \theta_j) + h_i(\theta_{-i}) \quad \text{for all } i \in N \tag{1.7}$$

where $h_i(\cdot)$ is an arbitrary function of θ_{-i}. Thus, the net utility of agent i is written as:

$$u_i(x(\theta)) + \sum_{j \neq i} \bar{u}(x_j(\theta), \theta_j) - h_i(\theta_{-i}). \tag{1.8}$$

According to expression (1.6), VCG-like mechanisms choose the resource allocation to maximize the sum of surrogate functions, while VCG mechanisms try to maximize the sum of reported utility functions as written in expression (1.3). By comparing equation (1.4) and equation (1.7), we notice that both payment schemes for VCG mechanisms and VCG-like mechanisms are similarly designed, except that VCG mechanisms determine one's payment based on others' reported utility functions, whereas VCG-like mechanisms decide one's payment based on surrogate functions (or scalar strategies) chosen by others.

As VCG-like mechanisms use the similar idea to that of VCG mechanisms, we can anticipate that VCG-like mechanisms would achieve some type of incentive compatibility and efficiency, but cannot be budget balanced at all profiles. In VCG-like mechanisms, agents select their surrogate functions equating their true marginal utilities with marginal price in such a way that the VCG-like mechanisms achieve efficient allocations in Nash equilibria. VCG-like mechanisms are indeed shown to be the only scalar strategy mechanisms which achieve efficient allocations for a given utility profile under regularity assumptions, but we also show that it is not possible for them to be budget balanced at all profiles in Chapter 2.

1.2.2 Scalar Strategy Mechanisms and No Envy Axiom

Our investigation into VCG-like mechanisms will reach further to consider fairness properties of VCG-like mechanisms. For multi-price mechanisms such as VCG-like mechanisms, buyers may dislike price discrimination and thus become more sensitive to fairness issues. No envy (envy-freeness) axiom is a persuasive standard of fairness since each participant is maximally satisfied with his resource share and payment compared to what others receive and pay (Foley (1967), Thomson (2007)). A mechanism is *envy-free* (or a mechanism satisfies *no envy*) if:

$$u_i(x_i) - t_i \geq u_i(x_j) - t_j \tag{1.9}$$

holds for all pairs in N at equilibrium allocations x and payment t, given a vector of utility functions u.

For the problem of allocating heterogeneous indivisible objects, Papai (2003) identifies a class of envy-free VCG mechanisms when utilities are superadditive.[2] For the same problem, Yengin (2008) characterizes a class of VCG mechanisms satisfying envy-free and egalitarian-equivalence axioms on restricted domains.[3]

However, for the problem of allocating a divisible resource, finding the complete set of efficient and envy-free mechanisms in quasi-linear environments poses a problem. Moulin (2008) discusses that efficient cost sharing demand mechanisms for divisible commodities cannot reach no envy.

Maskin (1999) and Fleurbaey and Maniquet (1997) show that, for preferences satisfying *monotonic closedness*[4], the no envy axiom is satisfied if an allocation rule is Nash implementable in addition to satisfying equal treatment of equals. Unfortunately, quasi-linear preferences are not monotonically closed, rendering Maskin's, and Fluerbaey and Maniquet's promising results inapplicable. Likewise, Zhang (2005) and Feldman et al (2005) have studied a modified version of the no envy axiom, *c*-

[2]Utilities are *superadditive* if the utility of a set of objects is at least the sum of the utilities of any combination of bundles of objects that it contains.

[3]Egalitarian equivalence requires a mechanism to choose those allocations such that each agent is indifferent between a common reference and his assigned resource share with payment.

[4]Let X denote an agent's consumption set with typical elements $a, b, ...$, and $\tilde{\mathcal{R}}$ denote the domain of admissible preferences over X. We define *Monotonic Closedness* as follows: $\forall \tilde{R}, \tilde{R}' \in \tilde{\mathcal{R}}$, $\forall a, b \in X$ such that $a\tilde{P}b$, $\exists \tilde{R}_i'' \in \tilde{\mathcal{R}}$, $\forall c \in X$, (i) $a\tilde{R}'c \Rightarrow a\tilde{R}''c$, $(ii) b\tilde{R}c \Rightarrow b\tilde{R}''c$, and $(iii) \sim (a\tilde{I}''b)$.

approximate envy-freeness[5], but their results are only applicable to cases of multiple resource allocation. Finding a closed form solution to describe the general structure of efficient and envy-free mechanisms is a challenging task for the problem of divisible resource allocation.

1.2.3 Simple Envy-Free Mechanism

Because there is no literature that studies fairness implication of scalar strategy mechanisms, the main contribution of Chapter 2 is bringing the focus on fairness to this problem arena, and presenting the Simple-Envy-Free mechanism (SEF mechanism) and its properties.

The SEF mechanism is constructed in the following way. Resource allocation is determined to be proportional to strategies: $x_i = \frac{\theta_i}{\theta_N}R$, and the payment scheme t_i assigned to each i is linear in agent i's strategy θ_i: $t_i = \theta_i\theta_{N\setminus i} - S_{-i}$ where $\theta_N = \sum_{i\in N}\theta_i > 0$, $\theta_{N\setminus i} = \theta_N - \theta_i$, and $S_{-i} = \sum_{j\neq i}\theta_j^2$. Therefore, in **Simple Envy-Free Mechanism (SEF mechanism)**, agent i's net utility from submitting θ_i is:

$$V_i(\theta_i, \theta_{-i}) = u_i(\frac{\theta_i}{\theta_N}R) - \theta_i\theta_{N\setminus i} + S_{-i}.$$

If we set a surrogate function of a VCG-like mechanism to be $\bar{u}(x_i, \theta_i) = -\frac{\theta_i^2}{x_i}R$ and set residual payment scheme properly, we can see that this VCG-like mechanism is the SEF mechanism.

To maximize the net utility, every agent i equates his marginal utility to marginal

[5] c-approximately envy-free is defined as follows: let $\rho(x) = \min_{i\neq j}\frac{u_i(x_i)}{u_i(x_j)}$. When $\rho(x) \geq 1$, the allocation x is known as an envy-free allocation. We call a mechanism c-approximately envy-free if for any x, $\rho(x) \geq c$.

price. This is written in the first order condition $u_i'(x_i) = \frac{(\theta_N)^2}{R}$ for every agent i and we can see easily that an equilibrium allocation is efficient. The SEF mechanism not only allocates the commodity efficiently, but also fairly in the sense of no-envy. We first plug into inequality (1.9) the forms of allocation rule $x_i = \frac{\theta_i}{\theta_N} R$ and payment scheme $t_i = \theta_i \theta_{N\setminus i} - S_{-i}$ of SEF mechanism. Then, by writing $\theta_i = \frac{x_i \theta_N}{R}$ and using equilibrium condition $u_i'(x_i) = \frac{\theta_N^2}{R}$, inequality (1.9) is written as:

$$u_i(x_i) - u_i(x_j) \geq (x_i - x_j) u_i'(x_i).$$

This holds true because of the concavity of utility functions. The SEF mechanism also satisfies other desirable fairness axioms such as ranking and voluntary participation.

1.3 Multiple Tasks Assignment and Asymptotically Budget Balanced VCG Mechanisms

There are mainly two ways to approach the problem of minimizing the budget imbalance of VCG mechanisms. First, we weaken the incentive criterion from dominant strategy and use Bayesian assumptions for the distribution of utility functions. Then, we can calculate the expected budget imbalance (Bailey (1997)).[6] For the problem of provisioning public goods, Deb, Gosh and Seo (2002), Green et al.(1976) and Green and Laffont (1979) give the asymptotic behavior of the expected budget imbalance under the pivotal mechanism (Vickrey Auction), and Zhou (2007) provides the same for the problem of private good exchange.

[6] In both the public good provision problem and the bilateral trading problem, there exists no budget balanced mechanism that is Bayesian-incentive compatible, efficient, and individually rational (Laffont and Maskin (1979) and Myerson and Satterthwaite (1983)).

Second, we can maintain dominant strategy incentive compatibility, so that we assume no prior and approach the problem using the worst case analysis. The Operations Research and Computer Science literatures commonly use the worst case analysis, and it is often referred to as competitive analysis (e.g., Tennenholtz (2001)). The worst case analysis plays a central role in the algorithmic approach to mechanism design. Anshelevich et al. (2004), Koutsoupias and Papadimitriou (1999), Porter et al. (2004), Roughgarden and Tardos (2002) use the worst case analysis to evaluate the competitiveness of Nash equilibrium behavior in congestion problems on a network. Chen and Zhang (2005), Johari and Tsitsiklis (2004, 2007), Sanghavi and Hajek (2004), Yang and Hajek (2005) adopt the worst case analysis in one-dimensional cost sharing problems. Moulin and Shenker (2001) as well as Roughgarden and Sundararajan (2006a, 2006b)) use it to discuss the tradeoff between budget balance and allocative efficiency for (group) strategy-proof cost sharing mechanisms. Goldberg et al.(2001, 2006) and Aggarwal et al.(2005) as well as Hartline and McGrew (2005) design worst case profit maximizing mechanisms.

As precedents in the economic literature that use the worst case analysis on VCG mechanisms, Moulin (1986) as well as Deb and Seo (1998) investigate the pivotal mechanism in the worst scenario for a public good provision problem, and Moulin and Shenker (2001) do the same for a cost sharing problem. Guo and Conitzer (2009) and Moulin (2009, 2010) construct asymptotically budget balanced VCG mechanisms in the worst case scenario.

1.3.1 Indivisible Economic "Bads" and VCG Mechanisms

In the case of assigning identical economic "bads" (for example, performing a costly task and locating waste disposal facilities) we will see in Chapter 3, each agent is required to take at most an object. m of the n agents should perform m identical tasks together where $1 \leq m \leq n-1$. Each agent i, $i \in N$ can perform a task with cost c_i, which is private information. We denote by c^{*k} the kth lowest cost among $c_1, \cdots, c_n$.

Given a cost profile c, *efficient cost* for performing m tasks is the minimal cost $\tau_m(c) = \sum_{k=1}^{m} c^{*k}$. VCG mechanisms assign tasks to a subset of m agents whose total cost to perform m tasks together is minimal. Agent i's net disutility V_i in a VCG mechanism is written as:

$$V_i(c) = \tau_m(c) + h_i(c_{-i})$$

where h_i is an arbitrary function.

A typical example of VCG mechanisms is the pivotal mechanism. We denote by $V_i^p(c)$ the net utility of agent i under the pivotal mechanism. Agent i's net utility in the pivotal mechanism is simplified as $V_i^p(c) = c_i$ if $c_i \leq c^{*m}$ or $V_i^p(c) = c^{*m}$ if $c_i \geq c^{*(m+1)}$. If agent i's cost consists of efficient cost, he will pay only his own cost. If agent i's cost is greater than mth smallest cost, he will pay the mth smallest cost.

If we rewrite the function $h_i(c_{-i})$ as $h_i(c_{-i}) = -\tau_{m-1}(c_{-i}) - r(i; c_{-i})$, where $r(i; c_{-i})$ is a *redistribution scheme* for agent i, the general form of VCG mechanisms is given as:

$$V_i(c) = \tau_m(c) - \tau_{m-1}(c_{-i}) - r(i; c_{-i}) = V_i^p(c) - r(i; c_{-i}).$$

Our VCG mechanisms ask the residual claimant to first run the pivotal mechanism. Then, the residual claimant distributes a suitable rebate to each agent if there is a budget surplus, or charges agents of additional tax if there is a deficit. With this interpretation, we write the budget imbalance of a VCG mechanism with a redistribution scheme r as:

$$\Delta(c, r) = ps(c) - \sum_{i=1}^{n} r(i; c_{-i}) = (n - m)c^{*m} - \sum_{i=1}^{n} r(i; c_{-i})$$

where $ps(c)$ is the budget surplus of the pivotal mechanism at cost profile c.

1.3.2 Efficient Surplus and Optimality

As we mentioned earlier, the budget imbalance is considered as implementation cost. On the other hand, drawing on the concept of opportunity cost, we notice that implementing a VCG mechanism actually saves costs when performing tasks. To perform tasks, a VCG mechanism will spend the efficient cost while a random assignment, as the primitive benchmark, will spend average cost. The saved cost garnered by the VCG mechanism is the difference between the average cost and the efficient cost. Thus, we define efficient surplus (es) as follows:

$$es(c) = \frac{m}{n} \sum_{i \in N} c_i - \tau_m(c).$$

The performance of VCG mechanisms can be fairly compared when we compute for each mechanism the implementation cost relative to generated efficient surplus as implementation gain. We adopt the worst case analysis to measure the performance of a VCG mechanism. Unlike the previous literature, however, we differentiate budget surplus and budget deficit, by considering their different natures such that the former is wasted money and the latter is borrowed money from the outside of the mechanism. Thus, we define the worst case budget surplus as the largest ratio of budget surplus to efficient surplus, and the worst case budget deficit is defined accordingly. With these definitions, we can identify the set of feasible pairs of worst case budget surplus and worst case budget deficit.

The ratio of budget surplus to efficient surplus is bounded by λ such as:

$$\lambda = \sup_{c \in \mathcal{R}_+^N} \frac{\Delta(c)}{es(c)}$$

and the absolute ratio of budget deficit to efficient surplus is bounded by μ such as:

$$\mu = \sup_{c \in \mathcal{R}_+^N} -\frac{\Delta(c)}{es(c)}.$$

Since we desire to decrease worst case ratios, we can order pairs of a worst case surplus and a worst case deficit by a relation of dominance in a two-dimensional space, and will eventually find the frontier of the feasible set. These minimal undominated pairs on the frontier are called **optimal pairs**, and a VCG mechanism generating an optimal pair is said to be an optimal mechanism. This definition of optimality from optimal frontier is more general than the optimality from efficiency loss (the largest ratio of absolute budget imbalance to efficient surplus), thus we provide a broad framework to analyze VCG mechanisms.

1.3.3 Optimal Tradeoffs between Surplus and Deficit

The main point of Chapter 3 is that once we differentiate budget surplus and budget deficit, and impose individual rationality as a natural fairness requirement, optimal mechanisms for "goods" and the optimal mechanisms for "bads" behave very differently. The striking asymmetry resides in the asymptotic behavior of the optimal pairs of surplus and deficit under individual rationality for the case of "bads".

For the problem of assigning economic "goods," whether or not we impose individual rationality does not change the relationship between budget surplus and budget deficit on the optimal frontier. For the case of economic "goods," a mechanism satisfies individual rationality if no agent suffers a net loss as a result of participating, i.e., $V_i \geq 0$ for all i. We can easily check that for the case of economic "goods," unit worst case surplus can only be replaced with unit worst case deficit, regardless of individual rationality.

In the case of economic "bads," individual rationality is defined differently. A mechanism satisfies individual rationality if participation in the mechanism brings each agent a smaller net loss than the loss he would experience in an anarchistic state where everyone performs one task on his own, i.e., $V_i \leq c_i$ for all $i \in N$. The different interpretations of individual rationality for the case of economic "goods" and the case of "bads" turn out to affect the behavior of optimal tradeoffs between budget surplus and budget deficit.

The case of assigning a single "bad" has a unique feature that does not exist for the case of multiple "bads." When we compute the optimal pairs of surplus and deficit of any individually rational VCG mechanism and the corresponding optimal mechanisms, there exist only two optimal individually rational mechanisms. One is the pivotal mechanism whose worst case (relative) budget surplus is infinite ($\lambda = \infty$) but generates no budget deficit. For the other, its worst case (relative) budget deficit is 1 ($\mu = 1$) with no budget surplus, and its linear redistribution scheme is $r^*(c_{-i}) = \frac{n-1}{n}(c_{-i})^{*1}$ for all $i \in N$. On the contrary, for the case of multiple "bads," we can find an infinite number of optimal individually rational mechanisms. This result differs from the outcome of allocating economic "goods" in that there are always infinitely many optimal pairs for any number of economic "goods."

For the case of multiple "bads," $2 \leq m \leq n-1$, we find that the optimal frontier of any anonymous and individually rational VCG mechanism is given as:

$$\frac{\lambda^*_{n,m}}{A(n,m)} + \frac{\mu^*_{n,m}}{B(n,m)} = 1$$

where

$$A(n,m) = \frac{\binom{n-1}{m-1}}{\sum_{k=0}^{m-2}\binom{n-2}{k}};$$
$$B(n,m) = \frac{\binom{n-1}{m-1}}{\sum_{k=0}^{m-3}\binom{n-2}{k} + \frac{m}{n-m}\sum_{k=m-1}^{n-2}\binom{n-2}{k}}.$$

We observe that $\lim_{n\to\infty} A(n,m)/\frac{n}{m-1} = 1$ and $\lim_{n\to\infty} P(n,m)/G(n,m) = 1$ where $P(n,m) = \frac{B(n,m)}{A(n,m)}$ and $G(n,m) = \frac{n^{m-1}}{m(m-2)!2^{n-2}}$. If we do not allow any deficit, the optimal mechanism converges to the pivotal mechanism (the worst case surplus of the pivotal mechanism is $\frac{n}{m-1}$) and its surplus loss diverges in n. On the other hand, we can see that for a fixed m, the function $P(n,m)$ is strictly decreasing in n and converges exponentially fast to zero in n. This implies that as more agents participate, a very minute amount of deficit can replace unit surplus. By allowing a slight deficit, we can almost achieve budget balanced VCG mechanisms. This result of extremely asymmetric tradeoffs between optimal surplus and deficit stands in stark contrast to the outcome of assigning economic "goods."

Chapter 2

Envy-Free and Incentive Compatible Division of a Commodity

We will investigate the problem of allocating a perfectly divisible object between a finite number of buyers. Examples of divisible commodity allocation can be found in auctions of Treasury notes (Back and Zender (1993), Keloharju, Nyborg and Rydqvist (2005)), the sale of communication network capacity (Kelly et al. (1998)), the design of electricity markets (Green and Newbery (1992), Ausubel (2006)) and auctions for spectrum licenses (Levin (1966)). Auctioning pollution permits (Cramton and Kerr (2002)) can be another interesting case where we can study the problem of allocating a divisible commodity.

We assume that each buyer has quasi-linear preferences and participates in a game defined by a mechanism. Each participant submits a one-dimensional bid (also known as a message or a signal) to the mechanism. Once all bids have been collected, the mechanism determines both the allocation of a resource and the payment scheme for each participant. This mechanism is called a *scalar strategy mechanism.* Nash equilibrium points are considered to be predictors of the behavior of agents.

Among scalar strategy mechanisms, we are particularly interested in multi-price mechanisms, *VCG-like mechanisms.* The VCG-like mechanisms use one dimensional message spaces and differentiated unit prices. They are similar in both spirit and form to the Vickrey-Clarke-Groves mechanisms (VCG mechanisms, Green and Laffont (1979)) except that each individual reports a one-dimensional message, rather than his entire utility function. Therefore, the VCG-like mechanisms have an advantage: the informational burden is lower in VCG-like mechanisms compared to the size of information in VCG mechanisms, since the latter requires agents to report infinite dimensional vectors in divisible commodity allocation.

The basic idea behind VCG-like mechanisms is that each agent selects a surrogate utility function from a set of scalar parametrized functions. The mechanisms determine resource shares to maximize the sum of surrogate utilities with payment rules designed similarly to VCG mechanisms. Agents select their surrogate functions equating their true marginal utilities with marginal price in such a way that the VCG-like mechanisms achieve efficient allocations in Nash equilibria.

Our goal is to investigate the fairness and budget balance properties of the VCG-like mechanisms, and to eventually design a scalar strategy mechanism that will achieve allocative efficiency, Nash incentive-compatibility, and *no envy* fairness.[1] For multi-price mechanisms such as VCG-like mechanisms, buyers may dislike price dis-

[1]The *no envy* axiom is a central standard of fairness in mechanism design theory (Foley (1967), Thomson (2007)).

crimination and thus become more sensitive to fairness issues.[2] An important issue is whether a mechanism's implemented allocation is fair enough to meet every individual's need for justice. *No envy (envy-freeness)* axiom is a persuasive standard of fairness since each participant is maximally satisfied with his resource share and payment compared to what others receive and pay. In addition, for mechanisms that are not concerned with maximizing revenue, it is often desirable to keep as small a budget imbalance as possible, so that the side payment collected or subsidized by a mechanism is perceived as a cost of implementation.

First, we will provide a characterization of VCG-like mechanisms such that they are the only scalar strategy mechanisms which achieve efficient allocations for a given utility profile (Theorem 1). This is in contrast to the result of inefficiency of scalar strategy mechanisms with a uniform price (Johari (2004), Johari and Tsitsiklis (2004), Yang and Hajek (2004)). Uniform price scalar strategy mechanisms fail to implement efficient allocations for some utility profiles and therefore, they do not satisfy the no envy property.[3] In addition, we will discuss both no envy and budget balance of VCG-like mechanisms. Example 1 demonstrates that many VCG-like mechanisms fail no envy property. Proposition 3 shows that VCG-like mechanisms are never budget-balanced. It is well-known that no VCG mechanism results in balanced budget (Green and Laffont (1979)), and we will show that VCG-like mechanisms inherit this property.[4]

In Section 2.3, we will construct a VCG-like mechanism that not only implements

[2]In 2000, Amazon engaged in price discrimination but stopped its pricing variations since the company received complaints from DVDTalk members (Perloff (2004)).

[3]The Appendix (Proposition 10) shows that no uniform pricing scalar strategy mechanism with proportional allocations is efficient or envy-free.

[4]VCG mechanisms that are almost budget-balanced have just started to be designed. For example, see Moulin (2008, 2009).

efficient Nash equilibria, but also satisfies the no envy axiom. Furthermore, this mechanism involves simply-formed payment rules and satisfies the *Ranking* and *Voluntary Participation* properties. We call this mechanism *the Simple Envy-Free* mechanism (SEF mechanism) (Theorem 2, Theorem 3). The SEF mechanism is a VCG-like mechanism in which each agent's resource share is proportional to his signal, and the payment to each agent is linear in his signal. Proposition 2 shows that no envy property is stronger than efficiency in the environment of quasilinear utilities. Using this result, we can identify the SEF mechanism without considering efficiency. Therefore, we characterize the SEF mechanism as a scalar strategy mechanism with proportional shares, no envy, and symmetric marginal price (Proposition 5, Proposition 6).

Every VCG-like mechanism has at least one efficient Nash equilibrium for every utility profile, but it may also have multiple equilibria with inefficient equilibria for some utility profiles. These properties are discussed in detail within the concrete context of the SEF mechanism (Example 3, Example 4). The SEF mechanism may have inefficient equilibria only when every agent except one submits a zero strategy. We can eliminate these inefficient equilibria, assuming the Inada condition such that there are at least two agents whose marginal utilities at zero shares are infinite.[5] In Section 2.3.4, we will discuss what happens to the SEF mechanism if we drop the Inada condition. Proposition 7 computes the worst case of relative efficiency of the SEF mechanism when agents are required to submit only positive strategies. Proposition 8 shows that engaging two virtual players in the game guarantees approximate efficiency without the Inada condition.[6]

The remainder of the paper is organized in the following manner. We will de-

[5]This statement holds for all VCG-like mechanisms.
[6]This idea is suggested by Yang and Hajek (2006).

scribe the model and VCG mechanisms in Section 2.1. In Section 2.2, we will introduce VCG-like mechanisms. In Section 2.2.1, we will provide a characterization of VCG-like mechanisms as efficient scalar strategy mechanisms. In Section 2.2.2, we will discuss the properties of VCG-like mechanisms in terms of both fairness and budget balance. We will show that many of them fail the no envy test and all of them fail to be budget-balanced. In Section 2.3, we will construct the SEF mechanism and discuss its incentive compatibility, fairness properties, and the size of budget imbalance in great detail. The SEF mechanism satisfies other axioms such as Ranking and Voluntary Participation and is both efficient and envy-free. Two different characterizations of the SEF mechanism are illustrated in Section 2.3.3. In Section 2.3.4, we drop the Inada condition which enables VCG-like mechanisms to have only efficient equilibria. We suggest two methods that would improve efficiency of the SEF mechanism without the condition. In the final section, we qualify the need for future research to identify a general class of envy-free VCG-like mechanisms, and to construct VCG-like mechanisms with the smallest budget imbalances. All proofs are gathered in Appendix 2.5.

2.1 Model

We are interested in allocating a fixed amount of a divisible resource to a finite number of agents. There is a center that possesses a resource and the total amount of the resource is $R > 0$. Let $n \geq 2$ be the number of agents, and let the set of agents be denoted as $N = \{1, \cdots, n\}$.

Let x_i be the resource share of agent i and $x = (x_1, \cdots, x_n)$. A resource allocation

x is *feasible* if it belongs to the set $\mathcal{X} = \{x : \sum_{i \in N} x_i \leq R, \ x_i \geq 0 \ \text{ for all } i \in N\}$.

When agent i receives his resource share, the monetary value of the share is represented by a utility function, u_i, that is continuous, strictly increasing, concave, and continuously differentiable on $[0, +\infty)$. Let $u_i(0) = 0$ for each $i \in N$. Denoted by $\mathcal{U}$, the set of utility functions satisfies the aforementioned properties. Let $u = (u_1, \cdots, u_n)$ and $u \in \mathcal{U}^n$.

The center tries to maximize the sum of agents' utilities (economic surplus) through the allocation of a resource. When a resource allocation determined by the center maximizes the economic surplus, the allocation is *efficient.*

Efficiency: If given $u \in \mathcal{U}^n$, a resource allocation x is chosen to be

$$x \in \underset{x \in \mathcal{X}}{\operatorname{argmax}} \sum_{i \in N} u_i(x_i),$$

then the resource allocation x is said to be *efficient.*

In addition to an allocatable resource, there can be a *money transfer (side payment)* between agents and the center. A transfer of money from agent i to the center is denoted by $t_i \in \mathcal{R}$. $t_i > 0$ means that agent i pays $|t_i|$ amount of money to the center. Likewise, $t_i < 0$ means that the center subsidizes agent i by granting $|t_i|$ amount of money to him. A vector of transfers is denoted by $t = (t_1, \cdots, t_n)$.

We will focus on the special environments in which agents have quasilinear preferences. Agent i's *net utility* function p_i takes the quasilinear form

$$p_i(x_i, t_i) = u_i(x_i) - t_i.$$

For all $i \in N$, agent i's resource share, x_i, and money transfer, t_i, are decided by the center. Once the center knows the utility functions for all agents, it tries to achieve its main goal, "efficiency" for given $u \in \mathcal{U}^n$.

However, utility functions $u \in \mathcal{U}^n$ of agents are mostly unknown to the center. To achieve desired outcomes, the center has to set up a message process (*a mechanism*) through which relevant information is collected. Let Θ_i denote the set of messages that agent i can send to the center. Each agent $i \in N$ selects a *m-dimensional message* θ_i from $\Theta_i = \{\theta_i | \ \theta_i \in \mathcal{R}^m_+\}$. Let θ be a vector $(\theta_1, \cdots, \theta_n) \in \Theta$ where $\Theta = \times_{i \in N} \Theta_i$. Let $\Theta_{-i} = \times_{j \neq i} \Theta_j$ and $\theta_{-i} = (\theta_1, \cdots, \theta_{i-1}, \theta_{i+1}, \cdots, \theta_n)$.

A *mechanism* F assigns to each message θ a solution $(x, t) = F(\theta)$ such that x is a vector of feasible allocations and t is a vector of money transfers. If a mechanism requires agents to submit m-dimensional messages, the mechanism is called *m-dimensional strategy mechanism.* In addition, if each agent submits his entire utility function, i.e., $\Theta_i = \mathcal{U}$ for all $i \in N$, a mechanism F is a *direct (revelation) mechanism.*

Among direct mechanisms, the *Vickrey-Clarke-Groves mechanisms* (VCG mechanisms) are proven to be the only mechanisms in which agents report their utility functions truthfully in dominant strategy equilibrium[7] and allocations are efficient.

VCG Mechanisms: Given a vector of reported utility functions $\hat{u} \in \mathcal{U}^n$, *VCG*

[7]The strategy profile $\theta^* = (\theta_1^*, \cdots, \theta_n^*)$ is a *dominant strategy equilibrium* of mechanism F if, for all $i \in N$ and all $u_i \in \mathcal{U}$,

$$p_i(F(\theta_i^*, \theta_{-i})) \geq p_i(F(\theta_i', \theta_{-i}))$$

for all $\theta_i' \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$.

mechanisms select resource allocation x such that

$$x(\hat{u}) \in \underset{x \in \mathcal{X}}{\operatorname{argmax}} \sum_{i \in N} \hat{u}_i(x_i) \tag{2.1}$$

and choose payment scheme for each $i \in N$ such that

$$t_i(\hat{u}) = -\sum_{j \neq i} \hat{u}_j(x_j(\hat{u})) + h_i(\hat{u}_{-i})$$

where $h_i(\cdot)$ is an arbitrary function of $\hat{u}_{-i}$.

Therefore, agent i's net utility for VCG mechanisms is

$$u_i(x_i(\hat{u}_i, \hat{u}_{-i})) + \sum_{j \neq i} \hat{u}_j(x_j(\hat{u}_i, \hat{u}_{-i})) - h_i(\hat{u}_{-i}).$$

VCG mechanisms are strategy-proof and efficient.[8] Since h_i depends only on the messages $\hat{u}_{-i}$ sent by other agents, agent i tries to maximize $u_i(x_i(\hat{u}_i, \hat{u}_{-i})) + \sum_{j \neq i} \hat{u}_j(x_j(\hat{u}_i, \hat{u}_{-i}))$ by choosing $\hat{u}_i$. This expression has a maximum, $\max_{x \in \mathcal{X}} u_i(x_i) + \sum_{j \neq i} \hat{u}_j(x_j)$. Agent i can achieve this maximum by reporting his true utility function u_i according to expression (2.1). Truth telling is a dominant strategy for every agent. Therefore, VCG mechanisms select efficient allocation $x(u)$.

A problem with VCG mechanisms is that when a resource is perfectly divisible, each individual should report a function which is in infinite dimensional space. In

[8]A direct revelation mechanism F is *strategy-proof* (or *dominant strategy incentive compatible*) if for all $i \in N$ and all $u_i \in \mathcal{U}$, truth telling is a dominant strategy equilibrium of the mechanism. That is, for all $i \in N$, all $u \in \mathcal{U}^n$, and all $u_i' \in \mathcal{U}$,

$$p_i(F(u_i, u_{-i})) \geq p_i(F(u_i', u_{-i})).$$

this case, the informational demand is too high, so the VCG mechanism is very difficult to use. Instead, we can consider a mechanism whose informational request is quite low, while it still maintains the spirit of the VCG mechanism. In the following section, we will introduce *scalar strategy mechanisms* in which each agent reports a one-dimensional message (scalar strategy).

2.2 VCG-like Scalar Strategy Mechanisms

A *scalar strategy* (one-dimensional strategy) mechanism requires each agent i to submit a one-dimensional bid θ_i such that $\theta_i \in [0, +\infty)$. It collects these bids, $\theta = (\theta_1, \cdots, \theta_n)$ and decides the resource allocation as well as the payment scheme for each participant. Therefore, a scalar strategy mechanism consists of a triple (Θ, x, t) where Θ is the set of allowable strategies of the form $\theta = (\theta_1, \cdots, \theta_n)$ with $\theta_i \in \mathcal{R}_+$, $i \in N$, x is the allocation vector, and t is the payment scheme. Each agent i's net utility is written as $u_i(x_i(\theta)) - t_i(\theta)$. Since agent i's net utility is determined by θ_i and θ_{-i}, agent i's net utility is denoted by $p_i(\theta_i, \theta_{-i})$ where θ_i is his message and θ_{-i} is a vector of messages submitted by others.

Within the class of all scalar strategy mechanisms, this paper discusses *VCG-like scalar strategy mechanisms* (VCG-like mechanisms).

2.2.1 Basic Idea and Characterization of VCG-like Mechanisms

A VCG-like mechanism imitates VCG mechanisms as follows: first the mechanism requires each agent i to report a one-dimensional signal $\theta_i \in [0, \infty)$ which selects a ***surrogate utility*** function $\bar{u}(\cdot, \theta_i)$ from a given *single parameter* family of functions, $\bar{\mathcal{U}} = \{\bar{u}(\cdot, \theta_i) | \ \theta_i \in [0, \infty)\}$. The set of surrogate utility functions $\bar{\mathcal{U}}$ is the same for all agents. If $\theta_i = 0$, then $\bar{u}(x_i, \theta_i) = 0$.

We assume that for all $i \in N$, given a positive real number θ_i, $\bar{u}(x_i, \theta_i)$ is strictly concave, strictly increasing, continuous and continuously differentiable for $x_i > 0$. In addition, for every $\gamma \in (0, \infty)$ and $x_i > 0$, there exists a $\theta_i > 0$ such that $\bar{u}'(x_i, \theta_i) = \gamma$.

The last assumption about $\bar{u}$ implies that all the functions in $\bar{\mathcal{U}}$ can cover the space $\mathcal{R}^2_{++}$ whose single element is (x_i, γ). Because of this property of $\bar{u}$, each agent i can express his marginal utility at any amount of resource by selecting θ_i.

Once the mechanism collects θ, i.e., $\bar{u} = (\bar{u}(\cdot, \theta_1), \cdots, \bar{u}(\cdot, \theta_n))$, it chooses the resource allocation $x \in \mathcal{X}$ that maximizes the sum of surrogate utilities, $\sum_{i \in N} \bar{u}(x_i, \theta_i)$ for the given θ. The VCG-like mechanism sets its payment scheme analogously to the payment scheme of VCG mechanisms, such that each agent's payment depends on both the sum of surrogate utilities of other agents (except his surrogate utility) and an arbitrary function of strategies submitted by other agents.

VCG-like Mechanisms: For θ collected, *VCG-like mechanisms* choose the resource allocation x such that

$$x(\theta) \in \operatorname*{argmax}_{x \in \mathcal{X}} \sum_{i \in N} \bar{u}(x_i, \theta_i) \tag{2.2}$$

and the payment scheme t such that

$$t_i(\theta) = -\sum_{j \neq i} \bar{u}(x_j(\theta), \theta_j) + h_i(\theta_{-i}) \quad \text{for all } i \in N$$

where $h_i(\cdot)$ is an arbitrary function of θ_{-i}.

Therefore, agent i's net utility in VCG-like mechanisms is

$$u_i(x_i(\theta)) + \sum_{j \neq i} \bar{u}(x_j(\theta), \theta_j) - h_i(\theta_{-i})$$

where $h_i(\cdot)$ is an arbitrary function of θ_{-i}.

As expressed in (2.2), a VCG-like mechanism determines the vector of resource shares x for given θ, and thus, agent i's resource share x_i, $i \in N$ is a function of θ. Since $x = x(\theta)$ maximizes $\sum_{i \in N} \bar{u}(x_i, \theta_i)$, the first order conditions give $\bar{u}'(x_i, \theta_i) = \bar{u}'(x_j, \theta_j)$ for $i \neq j$, $i, j \in N$, and $\theta_i, \theta_j > 0$. Plugging $x_i = x_i(\theta)$ for all $i \in N$ into the previous expression, we can write agent i's marginal surrogate function as $\bar{u}'(x_i(\theta), \theta_i) = g(\theta)$ for all $i \in N$. Therefore, when a VCG-like mechanism provides a set of surrogate utility functions $\bar{\mathcal{U}}$, the mechanism specifies the function $g(\theta)$.

In order to predict the behavior of agents, we will use Nash equilibrium to express incentive compatibility. We denote agent i's net utility by $p_i(\theta)$ to emphasize that his resource share x_i and payment scheme t_i depend on his report θ_i and reports by others θ_{-i}. He tries to maximize his net utility by selecting θ_i based on unilateral decision making. We define a *Nash equilibrium* as follows.

Nash Equilibrium: θ is a *Nash equilibrium* if and only if, for every $i \in N$, $p_i(\theta_i, \theta_{-i}) \geq p_i(\theta_i', \theta_{-i})$ for every $\theta_i' \in \mathcal{R}_+$.

A Nash equilibrium θ is an *efficient equilibrium* if the resource allocation $x(\theta)$ is efficient. A mechanism is an *efficient mechanism* if for each $u \in \mathcal{U}^n$, every Nash equilibrium is efficient (unfortunately, there is no efficient scalar strategy mechanism which implements dominant strategy equilibrium.[9])

Now, we will characterize the VCG-like mechanisms in Theorem 1. We show that among scalar strategy mechanisms that determine the resource share x according to (2.2), VCG-like mechanisms are the only mechanisms in which there exists an efficient Nash equilibrium.[10]

Theorem 1. *Let a scalar strategy mechanism determine an allocation vector x according to (2.2). Assume that for every $i \in N$, the net utility function $p_i(\theta)$ is concave in θ_i. Then, the scalar strategy mechanism has an efficient Nash equilibrium for each $u \in \mathcal{U}^n$ if and only if it is a VCG-like mechanism.*

As Theorem 1 states, VCG-like mechanisms achieve efficient Nash equilibria. We can explain the reason in the following way.

Agent i, $i \in N$ chooses his strategy θ_i to maximize his net utility $u_i(x_i(\theta)) - t_i(\theta)$. His optimal strategy θ_i given θ_{-i} is determined by the first order condition: $u_i'(x_i(\theta)) \cdot \frac{\partial x_i(\theta)}{\partial \theta_i} = \frac{\partial t_i(\theta)}{\partial \theta_i}$. That is, agent i chooses his strategy θ_i such that his utility increases from a change in his strategy θ_i to equal his payment increased from a change in his strategy θ_i. This first order condition is rewritten as $u_i'(x_i) = \frac{\partial t_i(\theta)}{\partial \theta_i} / \frac{\partial x_i(\theta)}{\partial \theta_i}$. The expression in the right hand side is the ratio of the additional amount of money agent i has to pay for the additional units of the good agent i receives when he increases his strategy θ_i. The term $\frac{\partial t_i(\theta)}{\partial \theta_i} / \frac{\partial x_i(\theta)}{\partial \theta_i}$ can be interpreted as ***marginal price*** for the

[9]This is shown in the Appendix (Proposition 9).

[10]Moulin (2008) characterizes cost sharing mechanisms which guarantee the existence of surplus maximizing Nash equilibrium demands.

divisible good agent i faces at equilibrium.

VCG-like mechanisms set the same marginal price for all agents such that $\frac{\partial t_i(\theta)}{\partial \theta_i} / \frac{\partial x_i(\theta)}{\partial \theta_i} = g(\theta)$ where $g(\theta) = \bar{u}'(x_i, \theta_i)$ for all $i \in N$. Therefore, agent i selects an equilibrium strategy θ_i satisfying $u_i'(x_i) = \bar{u}'(x_i, \theta_i)$. Notice that efficient allocations are essentially determined by marginal utilities in first order conditions such that $u_i'(x_i) = \lambda$ for $x_i > 0$ for all $i \in N$. For the efficient allocation x^*, each agent i chooses θ_i^* satisfying $u_i'(x_i^*) = \bar{u}'(x_i^*, \theta_i^*)$. Using surrogate utility functions and properly designed payment schemes, VCG-like mechanisms extract the information of true marginal utilities at an efficient allocation and therefore achieve efficient equilibria.

The main goals of a mechanism are typically achieving efficiency, incentive compatibility, and fairness. When there are side payments from agents to a mechanism, the size of the budget imbalance could present a concern for the center. We will discuss the fairness and budget imbalance of VCG-like mechanisms in the following subsections.

2.2.2 Fairness and Budget Balance of VCG-like Mechanisms

We will mainly consider *no envy (envy-freeness)* as a central fairness concept in this subsection. A mechanism is *envy-free*, or satisfies *no envy*, if no agent envies others in every equilibrium. Agent i doesn't envy agent j if his own equilibrium allocation of resource and payment gives net utility at least as high as his net utility from the case in which he receives agent j's share and payment allocation instead. The *envy-free* state of agent i compared against agent j is written as $u_i(x_i) - t_i \geq u_i(x_j) - t_j$ at equilibrium allocations.

Envy-Freeness (No Envy): If $u_i(x_i) - t_i \geq u_i(x_j) - t_j$ holds for all pairs in N at equilibrium allocations x and payment t, given a vector of utility functions $u \in \mathcal{U}^n$, the mechanism is *envy-free* (or the mechanism satisfies *no envy*).

Proposition 1 provides a necessary and sufficient condition for mechanisms to be envy-free in quasilinear environments.

Proposition 1. *An allocation mechanism with side payment is envy-free if and only if it satisfies the following* **no envy condition***: Given $u \in \mathcal{U}^n$, at every equilibrium allocation and payment (x,t), for every $i \in N$ and all $j \neq i$, if $x_j \neq x_i$,*

$$\begin{aligned} u_i'(x_i) &= \frac{t_i - t_j}{x_i - x_j} \quad \text{for } x_i > 0 \\ &\leq \frac{t_i - t_j}{x_i - x_j} \quad \text{for } x_i = 0, \end{aligned}$$

and if $x_j = x_i$, $t_i = t_j$.

The no envy condition conveys efficiency. That is, if a mechanism is envy-free, then it is efficient.[11]

Proposition 2. *An envy-free allocation mechanism with side payment is efficient.*

Remark. The reverse statement is not true. Unless $\lambda = \frac{t_i - t_j}{x_i - x_j}$ for all $i, j \in N$ and $i \neq j$ where λ is the market clearing price for price taking buyers, an efficient allocation is not necessarily envy-free.

No envy is a stronger property than efficiency, so it follows that many VCG-like

[11]Svensson (1983) provided a statement analogous to our Proposition 2. He studied the problem of allocating indivisible commodities with sidepayment when each agent gets at most one indivisible good. He noted that envy-freeness implies efficiency when the number of agents is equal to the number of indivisible objects. Alkan et al. (1991) extended this observation, allowing any number of people and objects.

mechanisms fail the no envy test as in the following example.

Example 1. Envious VCG-like mechanisms: select a VCG-like mechanism with $\bar{u}(x_i, \theta_i) = \theta_i \ln x_i$ for all $i \in N$. Then, $x_i = \frac{\theta_i}{\theta_N} R$ and $\frac{\partial p_i}{\partial \theta_i} = \frac{R\theta_{N\setminus i}}{\theta_N^2}[u_i'(x_i) - \frac{\theta_N}{R}]$. Suppose $\theta_{N\setminus i} \neq 0$. The first order condition of the equilibrium is $u_i'(x_i) = \frac{\theta_N}{R}$ if $x_i > 0$ and $u_i'(x_i) \leq \frac{\theta_N}{R}$ if $x_i = 0$. Applying the no envy condition, we have $\frac{t_i - t_j}{x_i - x_j} = \frac{\theta_N}{R}$ for $x_i > 0$. Thus, no envy holds if and only if

$$h_i(\theta_{-i}) - h_j(\theta_{-j}) = (\theta_i - \theta_j) - \theta_i \ln(\theta_i R) + \theta_j \ln(\theta_j R) + (\theta_i - \theta_j) \ln \theta_N.$$

The left hand side of the equation is additively separable w.r.t. θ_i and θ_j, so that its cross derivative, $\frac{\partial^2 (h_i(\theta_{-i}) - h_j(\theta_{-j}))}{\partial \theta_i \partial \theta_j}$ should be zero. However, the right hand side's cross derivative w.r.t. θ_i and θ_j is $(\theta_j - \theta_i)/\theta_N^2 \neq 0$. Therefore, with the surrogate utility function $\bar{u}(x_i, \theta_i) = \theta_i \ln x_i$, the VCG-like mechanism is not envy-free. Likewise, we can show that with $\bar{u}(x_i, \theta_i) = \theta_i \sqrt{x_i}$, the VCG-like mechanism generates envy among agents.

When the sum of payments made by agents at an equilibrium is positive, the mechanism produces revenue. We do not assume that the center has an objective to achieve when spending the residual money. To eliminate possible manipulations by the center or by the participants, the revenue cannot be a desirable income of the center or be paid back to agents. It should be burnt or wasted by the benevolent center. In the other case, the mechanism needs financial inflow from an outside source if the sum of payments is negative. It is a burden for the center to acquire money inflow to subsidize agents. Therefore, it is good to have no revenue or no financial inflow. We will discuss what happens in VCG-like mechanisms in terms of money waste or money inflow.

Given $u \in \mathcal{U}^n$, The *budget imbalance* of a mechanism is denoted by $\Delta(u)$. When a mechanism charges each agent i of t_i, the budget imbalance is

$$\Delta(u) = \sum_{i \in N} t_i(\theta)$$

where θ is a vector of equilibrium strategies. If $\Delta(u) > 0$ for every u, the mechanism has a *budget surplus*; if $\Delta(u) = 0$ for every u, it is *budget balanced*; for the case of $\Delta(u) < 0$ for every u, it has a *budget deficit.*

As VCG mechanisms fail to be budget balanced,[12] we will show that VCG-like mechanisms cannot achieve budget balance.

Proposition 3. *Every VCG-like mechanism fails to be budget balanced.*

The mechanism we propose in the next section is a VCG-like scalar strategy mechanism. Since its payment scheme has a very simple form and the mechanism is envy-free, we call the mechanism *the Simple Envy-Free (SEF) mechanism.* The efficiency of the SEF mechanism will be discussed in great detail. Its fairness and budget imbalance will be given concrete descriptions.

2.3 The SEF Mechanism

In this section, we will introduce an envy-free VCG-like mechanism, the SEF mechanism, and discuss its properties. The SEF mechanism is constructed in the following way. Resource allocation is determined to be proportional to strategies, so that the allocation to individual i is $x_i = \frac{\theta_i}{\theta_N} R$ where $\theta_N = \sum_{i \in N} \theta_i > 0$. If $\theta_N = 0$, then

[12] Refer to Green and Laffont (1979).

$x_i = 0$ for all $i \in N$. The payment scheme t_i assigned to each i is $t_i = \theta_i\theta_{N\backslash i} - S_{-i}$ where $\theta_{N\backslash i} = \theta_N - \theta_i, S = \sum_{i \in N} \theta_i^2$, and $S_{-i} = S - \theta_i^2$. Additionally, let $\theta_N^2 = (\theta_N)^2$. Payment made by some agents can be negative, which means that they are subsidized by the mechanism.

Simple Envy-Free Mechanism: In *the Simple Envy-Free mechanism* (SEF mechanism), agent i's net utility from submitting θ_i is

$$p_i(\theta_i, \theta_{-i}) = u_i(\frac{\theta_i}{\theta_N}R) - \theta_i\theta_{N\backslash i} + S_{-i}.$$

For all $i \in N$, if we set a surrogate function of a VCG-like mechanism to be $\bar{u}(x_i, \theta_i) = -\frac{\theta_i^2}{x_i}R$ for $x_i, \theta_i \in \mathcal{R}_{++}$ and residual payment scheme to be $h(\theta_{-i}) = -\theta_{N\backslash i}^2 - S_{-i}$ where $\theta_{N\backslash i}^2 = (\theta_{N\backslash i})^2$, we can see that this VCG-like mechanism is the SEF mechanism.

There is a caveat when we use the SEF mechanism. The identities of participants should be known to the mechanism. Otherwise, as we can see in the following example, some agents can benefit from submitting shill bids.

Example 2. If two agents i and j can shield their identities pretending to be one agent, they merge their bids to be $\tilde{\theta} = \theta_i + \theta_j$ and then submit it to the mechanism. Under this bid, they receive $\tilde{x} = \frac{\tilde{\theta}}{\tilde{\theta} + \theta_{N\backslash i,j}}R$ and pay $\tilde{t} = \tilde{\theta} \cdot \theta_{N\backslash i,j} - S_{-i-j}$ jointly. Here $\theta_{N\backslash i,j} = \theta_N - \theta_i - \theta_j$ and $S_{-i-j} = \sum_{l \neq i,j} \theta_l^2$.

Since $\tilde{x} = x_i + x_j$, the resource share that the agents i and j receive jointly by submitting the merged bid is equal to the sum of their original shares. The total payment originally made by two agents is

$$
\begin{aligned}
t_i(\theta_i, \theta_{-i}) + t_j(\theta_j, \theta_{-j}) &= \theta_i(\theta_j + \theta_{N\setminus i,j}) - \theta_j^2 - S_{-i-j} + \theta_j(\theta_i + \theta_{N\setminus i,j}) - \theta_i^2 - S_{-i-j} \\
&= (\theta_i + \theta_j)\theta_{N\setminus i,j} + 2\theta_i\theta_j - \theta_i^2 - \theta_j^2 - 2S_{-i-j} \\
&= \tilde{t} - (\theta_i - \theta_j)^2 - S_{-i-j}
\end{aligned}
$$

and this gives $\tilde{t} - t_i(\theta_i, \theta_{-i}) - t_j(\theta_j, \theta_{-j}) = (\theta_i - \theta_j)^2 + S_{-i-j} \geq 0$. This implies that by merging their bids, agents i and j may jointly pay more than the sum of their original payments although they receive the same shares. Thus, merging bids is not profitable. However, by the same logic, splitting bids is profitable.

Therefore, the mechanism should prevent identity shielding. For this reason, we assume that the number of agents and their identities are known to the mechanism.

2.3.1 Incentive Compatibility

The efficiency property and examples discussed in this section hold for all VCG-like mechanisms. We will use a concrete form of the SEF mechanism in order to make it easier to discuss these aspects.

To see if a Nash equilibrium exists for this mechanism, we first consider a case where every agent submits a bid of 0 to the mechanism. If agent 1 changes his strategy from 0 to ϵ such that $\epsilon > 0$, then his net utility becomes $u_1(R)$, which is positive, while his net utility is 0 when staying with $\theta_1 = 0$. Thus, $\theta = (0, \cdots, 0)$ is not a Nash equilibrium.

There could be an equilibrium in which only one agent submits a positive strategy, while the other agents return zeros, but typically this equilibrium is inefficient as we will see in the following example.

Example 3. Inefficient equilibria: let $n = 2$ and $R = 1$. Suppose $u_1(x_1) = ax_1$ and $u_2(x_2) = bx_2$ for $0 < a < b$. At an efficient allocation, agent 1 receives nothing and agent 2 should receive 1. As a result, the efficient surplus is b. However, there are multiple equilibria where agent 1 receives everything and agent 2 gets nothing. Assume that agent 1 reports $\epsilon > 0$ and agent 2 reports 0. Agent 1 does not have any incentive to change his strategy since he receives all of the resource but pays nothing. Agent 2 does not have any incentive to change his strategy if his net utility decreases by submitting a positive number. This is the case when

$$\frac{\partial p_2(\epsilon, 0)}{\partial \theta_2} \leq 0,$$

that is, $b \leq \epsilon^2$. Therefore, $(\epsilon, 0)$ is a Nash equilibrium if $\epsilon \geq \sqrt{b}$, however, the allocation is inefficient.

Using the same logic as in the previous example, we can identify all inefficient equilibria for the SEF mechanism. Note $p_j'(0, \theta_{-j}) = \frac{R}{\epsilon}\big(u_j'(0) - \frac{\epsilon^2}{R}\big)$ where θ_{-j} is $(n-1)$ dimensional vector with $\epsilon > 0$ for i'th coordinate and zero for others. $p_j'(0, \theta_{-j}) \leq 0$ if and only if $\sqrt{Ru_j'(0)} \leq \epsilon$. Therefore, $(0, \cdots, 0, \cdots, \epsilon, 0, \cdots, 0)$ is a Nash equilibrium if and only if $\epsilon \geq \max_{j \neq i} \sqrt{Ru_j'(0)}$. This type of Nash equilibrium, where one agent i receives all of the resource, is inefficient unless his utility function has the property such as $u_i'(R) \geq u_j'(0)$ for all $j \neq i$.

Looking at the structure of inefficient equilibria, we make the following interpretation. As long as every other agent $j \in N$, $j \neq i$ has finite $u_j'(0)$, agent i has an opportunity to take the entirety of the resource, resulting in an inefficient equilibrium.

Thus, to prevent an inefficient equilibrium, for each agent $i \in N$, there should be at least one other agent $j \neq i$ with $u_j'(0) = +\infty$. From this example, we can make the following assumption which ensures that there are at least two agents whose strategies are positive. This assumption is commonly used in macroeconomics for production functions and is called the Inada condition. However, we do not assume that the limits of the derivatives of utility functions towards positive infinity are 0.

Inada Condition: $u_i'(0) = \infty$ for at least two agents.

The Inada condition excludes all inefficient equilibria in which one agent receives all of the resource. When there are at least two agents receiving positive shares, any Nash equilibria in the SEF mechanism are efficient. We prove below that the SEF mechanism has Nash equilibria and that all of its Nash equilibria are efficient.

Theorem 2. *Every Nash equilibrium of the SEF mechanism is efficient.*

Since we only assume concavity of utility functions, the SEF mechanism can have multiple efficient equilibria in the following example. When utility functions are strictly concave, there is a unique efficient equilibrium.

Example 4. Multiple efficient equilibria: consider a case with two agents. If each agent's utility function has a constant slope over a part of the domain, there can be multiple equilibria. Let u_1 and u_2 have the same constant slope over $[x_1, \frac{R}{2}]$ and $[\frac{R}{2}, x_2]$, respectively, where $x_1 + x_2 = R$. Then, $u_1'(x_1) = u_2'(x_2)$, and $x = (x_1, x_2)$ is a Nash equilibrium allocation. Therefore, there are a pair of equilibrium strategies θ_1, θ_2 which satisfy $x_1 = \frac{\theta_1}{\theta_1+\theta_2}R$ and $x_2 = \frac{\theta_2}{\theta_1+\theta_2}R$. Likewise, if $Q_1 \in [x_1, \frac{R}{2}]$ and $Q_2 \in [\frac{R}{2}, x_2]$ with $Q_1 + Q_2 = R$, we again have $u_1'(Q_1) = u_2'(Q_2)$, and there is a pair of equilibrium strategies θ_1' and θ_2' which satisfies $Q_1 = \frac{\theta_1'}{\theta_1'+\theta_2'}R$ and $Q_2 = \frac{\theta_2'}{\theta_1'+\theta_2'}R$. We can find infinitely many equilibria in this example.

2.3.2 Fairness and Budget Balance

We will examine whether the SEF mechanism satisfies desirable fairness properties and will measure the size of budget imbalance. In addition to the no envy concept, we will introduce two additional fairness concepts, *Ranking* and *Voluntary Participation.*

If an agent receives a bigger share of the resource than the other agents, he has to pay a greater amount than the others. This primitive notion of fairness is represented as *Ranking* (RK).

Ranking: A mechanism satisfies *Ranking* if $x_i \leq x_j$ implies $t_i \leq t_j$ for any $i, j \in N$, $i \neq j$ at every equilibrium.

Individuals are not forced to participate in the mechanism if they would be made worse off by participating. There is neither punishment nor discrimination between participants and non-participants, so that agents are free to choose whether or not they will participate in the game. If equilibrium allocations satisfy this property, the mechanism is said to satisfy *Voluntary Participation* (VP). We assume that $x_i = t_i = 0$ if agent i doesn't participate, that is, he doesn't submit any bid. Then, *Voluntary Participation* is expressed as follows.

Voluntary Participation: A mechanism satisfies *Voluntary Participation* if each agent $i \in N$ has net utility $p_i(\theta)$ which is nonnegative at equilibrium θ.

The following result shows that the SEF mechanism is not only envy-free but also satisfies the two aforementioned fairness properties.

Theorem 3. *The SEF mechanism (i) satisfies Ranking, (ii) achieves Voluntary Participation, and (iii) guarantees no envy.*

Remark. Voluntary Participation and Ranking hold without the Inada condition, but no envy holds only with the Inada condition. As in Example 3, let $(\epsilon, 0)$ be a Nash equilibrium for $u_1(x) = ax$, $u_2(x) = bx$ and $0 < a < b$. At this equilibrium, $x_1 = 1, x_2 = 0$ and $t_1 = 0, t_2 = -\epsilon^2$. Plugging in the no envy condition, we get $u_1'(x_1) = \epsilon^2$ and $u_2'(x_2) \leq \epsilon^2$. This implies $a \geq b$ which contradicts $a < b$. Therefore, this equilibrium allocation is not envy-free. Consequently, we maintain the Inada condition in this section.

Though the SEF mechanism satisfies useful fairness properties as well as efficiency, it can generate a budget deficit and the center may need financial inflow to subsidize some agents.

Proposition 4. *The SEF mechanism yields a budget deficit which can range from 0 to $R\lambda(n-1)$, where λ is the market clearing price for price taking buyers. When every agent submits the same strategy, $\theta = (\alpha, \cdots, \alpha)$, the mechanism's budget is balanced.*

2.3.3 Characterizations of the SEF Mechanism

We will show that the SEF mechanism is characterized by the combination of allocations, determined in proportion to the agents' strategies, as well as no envy fairness under the Inada condition. We provide two characterizations according to different properties of the payment scheme.

Characterization A

The first characterization assumes that an agent's payment is linear in his own strategy.

Property A1. There are at least two agents. The set of strategies Θ equals $\mathcal{R}_{+}^{n}$, and the allocation is proportional to the submitted strategies: $x_i = \frac{\theta_i}{\theta_N} R$ if $\theta \neq 0$ and $x_i = 0$ if $\theta = 0$.

Property A2. The symmetric payment by agent i is the sum of a variable price in θ_i and a fixed price independent of θ_i: $t_i(\theta)$ is linear in θ_i, i.e., $t_i(\theta) = \alpha(\theta_{-i})\cdot\theta_i + \beta(\theta_{-i})$.

Property A3. For any utility profiles $u_1, \cdots, u_n$ such that each u_i is strictly increasing, concave, continuous and continuously differentiable for all $i \in N$, the mechanism is envy-free.

Proposition 5. *The SEF mechanism is the only scalar strategy mechanism satisfying Properties A1-A3 up to affine transformations.*

Property *A1* makes computation much easier. Almost all scalar strategy mechanisms that have been developed so far use this proportional form for resource allocation. For the case of uniform price scalar strategy mechanisms, this form of proportional resource shares can be derived by assuming concavity of net utilities (Johari and Tsitsiklis (2007)).

Now we drop the Property *A2* of linear payment scheme and provide another characterization of the SEF mechanism.

Characterization B

Recall that VCG-like mechanisms set a marginal price function $g(\theta)$ where $g(\theta) = \bar{u}'(x_i, \theta_i)$ for all $i \in N$ and g is a positive and continuous function. Instead of linear payment schemes, we consider symmetric marginal price functions.

Property B1. There are at least two agents. The set of strategies Θ equals $\mathcal{R}_{+}^{n}$, and the allocation is proportional to the submitted strategies: $x_i = \frac{\theta_i}{\theta_N} R$ if $\theta \neq 0$ and $x_i = 0$ if $\theta = 0$.

Property B2. Marginal price, $g(\theta)$, is a function of the sum of strategies, therefore $g(\theta) = g(\theta_N)$.

Property B3. For any utility profiles $u_1, \cdots, u_n$ such that each u_i is strictly increasing, concave, continuous and continuously differentiable for all $i \in N$, the mechanism is envy-free.

Proposition 6. *The SEF mechanism is the only scalar strategy mechanism satisfying Properties B1-B3 up to affine transformations.*

Remark. If we assume that $g(\theta) = \sum_{i \in N} f_i(\theta_i)$ or $g(\theta) = f(\theta^N)$ where $\theta^N = \prod_{i \in N} \theta_i$, the no envy with proportional allocation results in $g(\theta) = c_1 \log \theta^N$. However, we cannot easily adopt $g(\theta) = c_1 \log \theta^N$ since $g(\cdot)$ may not be positive.

2.3.4 Efficiency without the Inada Condition.

Without the Inada condition, we observed that the SEF mechanism could yield inefficient equilibria. Recall that the Inada condition has to have at least two agents whose equilibrium strategies are positive. We now introduce two methods to replace the Inada condition. Since any alternative assumptions other than the Inada condition may result in an inefficient equilibrium, we introduce an efficiency index to gauge the efficiency loss.

The ***worst-case relative surplus*** of a mechanism is a real number such that

$$\inf_{\substack{u \in \mathcal{U}^n \\ x \in E(u)}} \frac{\sum_{i \in N} u_i(x_i)}{\sum_{i \in N} u_i(x_i^*)}$$

where $E(u)$ is the set of equilibrium allocations and x^* is an efficient allocation, given $u \in \mathcal{U}^n$.

The worst-case analysis is commonly used in computer science and operation research. There are also precedents in economic literature. For example, Moulin (1986) uses it to discuss the pivotal mechanism in the public good provision problem. Moulin and Shenker (2001) is an example of worst-case analysis in a cost sharing problem.

Without the Inada condition, it is easy to see that the worst-case relative surplus is 0. For instance, let $u_1(x) = ax$ and $u_2(x) = bx$ for $0 < a < b$. An inefficient equilibrium has a surplus of a and a relative surplus is a/b. As a gets closer to 0, the worst-case relative surplus converges to 0.

Recall that inefficient equilibria can occur when all agents except one submit zero bids. In order to prevent the extreme case mentioned above, the mechanism may restrict strategies to be strictly positive (Method A) or introduce two virtual players whose utilities satisfy the Inada condition (Method B).

Method A. Restricted Strategies

Method A imposes a positive lower bound on strategies. Each agent can select θ from $[\epsilon, \infty)$ for a positive real number ϵ. We will show that using Method A, the worst-case relative surplus increases from 0 to $1/n$ for the case of n participants. First we need to determine the shape of the equilibria under the restriction.

Lemma 1. *Let $n = 2$ and let the efficient equilibrium strategy be denoted by (θ_1^*, θ_2^*). If $\theta_1^* \leq \theta_2^*$, the adjusted equilibrium (θ_1, θ_2) with a lower bound ϵ satisfies $\theta_1 \leq \theta_2$. If $\theta_1^* \leq \epsilon \leq \theta_2^*$ or $\theta_1^* \leq \theta_2^* \leq \epsilon$, the adjusted equilibrium has the form of (ϵ, θ_2) for $\theta_2 \geq \epsilon$.*

Applying the same logic that we used for the proof of Lemma 1, we can find the adjusted equilibrium for the case of n agents.

By Lemma 4 of Johari and Tsitsiklis (2004), the worst-case relative surplus occurs with linear utility functions. Therefore, computing the worst-case relative surplus for linear utility functions is enough to calculate the worst-case relative surplus of the SEF mechanism.

Proposition 7. *The worst-case relative surplus of the SEF mechanism improves from 0 to 1/n when we impose a positive lower bound on strategy sets.*

Under Method A, the SEF mechanism may have inefficient equilibria, so it cannot be envy-free. However, Voluntary Participation and Ranking still hold.

Method B. Virtual Players

The mechanism can ensure that the equilibria are efficient by introducing two virtual players. These virtual players have infinite marginal utilities at zero shares, so the Inada condition is satisfied. This idea is suggested by Yang and Hajek (2006). They showed approximate efficiency for a group of VCG-like mechanisms where every agent has a strictly concave utility function. We apply their idea to the case of concave utilities and use different utility functions for virtual players.

Let us describe this idea in detail. A VCG-like mechanism introduces two virtual players whose utility functions are $u_{n+1}(x_{n+1}) = w\epsilon\sqrt{x_{n+1}}$ and $u_{n+2}(x_{n+2}) = (1-w)\epsilon\sqrt{x_{n+2}}$, respectively for $\epsilon > 0$ and $0 < w < 1$. These virtual players choose their strategies $\theta_{n+1} \geq 0$ and $\theta_{n+2} \geq 0$ to maximize their net utilities. Let $\theta_\epsilon = (\theta_1, \cdots, \theta_n, \theta_{n+1}, \theta_{n+2})$ be the extension of θ including the virtual players' strategies and let $x_\epsilon = (x_1, \cdots, x_n, x_{n+1}, x_{n+2})$. We can prove that in this ϵ-extended game, equilibrium strategies for the first n players are converging to efficient equilibrium strategies in the game without virtual players, as virtual players have negligible utility functions.

Proposition 8. (Yang and Hajek (2006)) *Let $\tilde{\theta} = \lim_{\epsilon \to 0} \theta_\epsilon$ and let the vector of the first n elements of $\tilde{\theta}$ be denoted by θ. Then the limit, $\tilde{\theta}$ exists and θ is the efficient equilibrium of the original game without virtual players.*

By introducing negligible virtual players, the mechanism achieves approximately efficient and envy-free equilibria.

2.4 Conclusion.

We showed that VCG-like mechanisms are the only scalar strategy mechanisms which achieve efficient Nash equilibria for the problem of allocating a divisible commodity. Furthermore, we identified the SEF mechanism as a VCG-like mechanism that is envy-free and uses a linear payment scheme. In the future, properties of VCG-like mechanisms need to be studied in-depth. We could also consider finding other envy-free mechanisms among VCG-like mechanisms. Designing an efficient scalar strategy mechanism with the smallest budget imbalance should be our most pressing concern.

2.5 Appendix

Proof of Theorem 1. We first prove "only if" part. Fix $u \in \mathcal{U}^n$ and let $K = \{i \in N | \, \theta_i = 0\}$ for a vector of strategies, θ for given u. For $i \in N \backslash K$, the scalar mechanism returns a unique vector of allocations x at given θ such that $\bar{u}'(x_i, \theta_i) = \lambda(\theta)$. Let $f(x_i, \theta) = \bar{u}'(x_i, \theta_i) - \lambda(\theta)$. Assuming $\lambda(\theta)$ and $\bar{u}'(x_i, \theta_i)$ are continuously differentiable in θ, f is continuously differentiable and $\frac{\partial f(x_i, \theta)}{\partial x_i} = \bar{u}''(x_i, \theta_i) < 0$ for all $x_i, \theta_i > 0$. By the implicit function theorem, there exists a unique continuously differentiable function $x_i(\theta)$. Furthermore, the feasibility condition of $\sum_{i \in N} x_i(\theta) = R$ assures that x_i is not a constant function of θ_{-i}.

A vector of efficient allocations for u is denoted by x^u. Due to the assumptions about $\bar{u}$, for every $i \in N \setminus K$, there exists $\theta_i^u > 0$ such that $u_i(x_i^u) = \bar{u}'(x_i^u, \theta_i^u)$ where $x_i^u > 0$ and for $i \in K$, $\theta_i^u = 0$ where $x_i^u = 0$. The vector of strategies θ yielding efficient allocations should be θ^u.

Recall that a vector of strategies θ is a Nash equilibrium if and only if

$$u_i'(x_i) \cdot \frac{\partial x_i(\theta)}{\partial \theta_i} = \frac{\partial t_i(\theta)}{\partial \theta_i} \text{ for } \theta_i > 0$$
$$u_i'(x_i) \cdot \frac{\partial x_i(\theta)}{\partial \theta_i} \leq \frac{\partial t_i(\theta)}{\partial \theta_i} \text{ for } \theta_i = 0.$$

Given θ, the mechanism allocates the entirety of the resource, so $\sum_{i \in N \backslash K} x_i(\theta) = R$ should hold. Thus, for $i \in N \backslash K$, $\frac{\partial x_i(\theta)}{\partial \theta_i} = \frac{\partial(-\sum_{j \neq i, j \in N \backslash K} x_j(\theta))}{\partial \theta_i} = -\sum_{j \neq i, j \in N \backslash K} \frac{\partial x_j(\theta)}{\partial \theta_i}$. At the efficient allocation x^u (which satisfy $u_i'(x_i^u) = u_j'(x_j^u)$) and the strategy θ^u

corresponding to x^u, we have

$$u_i'(x_i^u) \cdot \frac{\partial x_i(\theta^u)}{\partial \theta_i} = - \sum_{j \neq i, j \in N \setminus K} u_j'(x_j^u) \cdot \frac{\partial x_j(\theta^u)}{\partial \theta_i} = - \sum_{j \neq i, j \in N \setminus K} \bar{u}'(x_j^u, \theta_j^u) \cdot \frac{\partial x_j(\theta^u)}{\partial \theta_i}.$$

When θ^u is an Nash equilibrium, we should have

$$- \sum_{j \neq i, j \in N \setminus K} \bar{u}'(x_j^u, \theta_j^u) \cdot \frac{\partial x_j(\theta^u)}{\partial \theta_i} = \frac{\partial t_i(\theta^u)}{\partial \theta_i}.$$

This relation should hold for an arbitrary $u \in \mathcal{U}^n$ and then we have

$$- \sum_{j \neq i, j \in N \setminus K} \bar{u}'(x_j, \theta_j) \cdot \frac{\partial x_j(\theta)}{\partial \theta_i} = \frac{\partial t_i(\theta)}{\partial \theta_i}.$$

Therefore, the payment scheme for $i \in N \setminus K$ is

$$t_i(\theta) = - \sum_{j \neq i, j \in N \setminus K} \bar{u}(x_j, \theta_j) + h_i(\theta_{-i}) = - \sum_{j \neq i} \bar{u}(x_j, \theta_j) + h_i(\theta_{-i}).$$

The last equality holds since $\bar{u}(x_i, \theta_i) = 0$ for $\theta_i = 0$. For $i \in K$, $x_i = 0$ and the mechanism determines $x(\theta)$ to maximize $\sum_{i \in N \setminus K} \bar{u}(x_i, \theta_i)$ where $\sum_{i \in N \setminus K} x_i = R$. Thus, for $j \neq i$, $x_j(\theta) = x_j(\theta_{-i})$ and $t_i(\theta) = t_i(0, \theta_{-i})$. Thus, we can write $t_i(\theta) = - \sum_{j \neq i} \bar{u}(x_j, \theta_j) + h_i(\theta_{-i})$. We conclude that the mechanism is a VCG-like mechanism.

Now we prove "if" part. We can use the argument of Lemma 1 in Johari and Tsitsiklis (2007). First we show that θ is a Nash equilibrium if and only if for all $i \in N$,

$$x(\theta) \in \operatorname*{argmax}_{x \in \mathcal{X}} \left[u_i(x_i) + \sum_{j \neq i} \bar{u}(x_j, \theta_j) \right]. \tag{2.3}$$

The optimal value of (2.3) is an upper bound to agent i's net utility without $h_i(\theta_{-i})$. Given θ, if (2.3) holds for all agents, then their net utilities are maximized so that θ is a Nash equilibrium. For the sake of contradiction, assume that given a Nash equilibrium θ, (2.3) is not satisfied for some agent i. The problem (2.3) has an optimal solution, x^* since $\mathcal{X}$ is compact and $x^* \neq x(\theta)$. Then, x^* satisfies the first order conditions such that $u_i'(x_i^*) = \lambda$ for $x_i^* > 0$, $u_i'(x_i^*) \leq \lambda$ for $x_i^* = 0$ and $\bar{u}'(x_j^*, \theta_j) = \lambda$ for $j \in N \setminus K$. For $i \in N \setminus K$, let agent i choose $\theta_i' > 0$ such that $u_i'(x_i^*) = \bar{u}'(x_i^*, \theta_i')$. Then, x^* is also a solution (2.2) when a strategy vector is (θ_i', θ_{-i}). Since the solution of (2.2) is unique for given (θ_i', θ_{-i}), we have $x^* = x(\theta_i', \theta_{-i})$. Then, we have

$$\begin{aligned} u_i(x_i(\theta)) + \sum_{j \neq i} \bar{u}(x_j(\theta), \theta_j) - h_i(\theta_{-i}) &< u_i(x_i^*) + \sum_{j \neq i} \bar{u}(x_j^*, \theta_j) - h_i(\theta_{-i}) \\ &= u_i(x_i(\theta_i', \theta_{-i})) + \sum_{j \neq i} \bar{u}(x_j(\theta_i', \theta_{-i}), \theta_j) - h_i(\theta_{-i}) \end{aligned}$$

which contradicts that θ is a Nash equilibrium.

Finally we prove that the VCG-like mechanism has an efficient equilibrium. For a vector of efficient allocations x^*, each agent i, $i \in N$ chooses $\theta_i > 0$ such that $u_i'(x_i^*) = \bar{u}'(x_i^*, \theta_i)$ for $x_i^* > 0$ or selects $\theta_i = 0$ for $x_i^* = 0$. Since (2.2) has a unique solution for θ, we have $x^* = x(\theta)$. By the same logic, $x^*(= x(\theta))$ is also a solution of (2.3). Therefore, we conclude that θ is a Nash equilibrium. ■

Proof of Proposition 1. Given $u \in \mathcal{U}^n$, let the mechanism have equilibrium allocation and payment (x, t). If $x_i < x_j$, by definition, no envy holds if and only if $u_i(x_i) - u_i(x_j) \geq t_i - t_j$ and $u_j(x_j) - u_j(x_i) \geq t_j - t_i$. This is equivalent to

$$\frac{u_i(x_i) - u_i(x_j)}{x_i - x_j} \leq \frac{t_i - t_j}{x_i - x_j} \text{ and } \frac{u_j(x_j) - u_j(x_i)}{x_j - x_i} \geq \frac{t_j - t_i}{x_j - x_i}.$$

By the concavity of $u_i \in \mathcal{U}$, for $x_i < x_j$ we have $\frac{u_i(x_i)-u_i(x_j)}{x_i-x_j} \leq u_i'(x_i)$. Considering a case in which $\frac{u_i(x_i)-u_i(x_j)}{x_i-x_j} = u_i'(x_i)$, agent i is envy-free if and only if $u_i'(x_i) \leq \frac{t_i-t_j}{x_i-x_j}$. Likewise, for another equilibrium allocation where $x_i >' x_j$, agent i is envy-free if and only if $u_i'(x_i) \geq \frac{t_i-t_j}{x_i-x_j}$. If $x_i = x_j$, no envy holds if and only if $t_i = t_j$. Thus, agent i is envy-free if and only if $u_i'(x_i) = \frac{t_i-t_j}{x_i-x_j}$ for $x_i > 0$ and $u_i'(x_i) \leq \frac{t_i-t_j}{x_i-x_j}$ for $x_i = 0$. ■

Proof of Proposition 2. Let a mechanism satisfy the no envy condition. Given $u \in \mathcal{U}^n$, let (x, t) be an equilibrium allocation and payment. For any pair of $i, j \in N$, $i \neq j$, we have $\frac{t_i-t_j}{x_i-x_j} = \frac{t_j-t_i}{x_j-x_i}$ so that at allocations $x_i, x_j > 0$, we have $u_i'(x_i) = u_j'(x_j)$. If $x_j = 0$, then we have $u_j'(x_j) \leq u_i'(x_i)$. It is easy to see that this is the first order condition for efficient allocations. ■

Proof of Proposition 3. An agent i's net utility in a VCG-like mechanism is

$$p_i(\theta) = u_i(x_i) + \sum_{j \neq i} \bar{u}(\theta_j, x_j) - h_i(\theta_{-i}),$$

so his payment is $t_i(\theta) = -\sum_{j\neq i} \bar{u}(\theta_j, x_j) + h_i(\theta_{-i})$ at an equilibrium θ and corresponding allocation x. The mechanism's budget is

$$\sum_{i \in N} t_i = -\sum_{i \in N} \sum_{j \neq i} \bar{u}(\theta_j, x_j) + \sum_{i \in N} h_i(\theta_{-i}).$$

Budget balance means that $\sum_{i\in N} h_i(\theta_{-i}) = \sum_{i\in N} \sum_{j\neq i} \bar{u}(\theta_j, x_j)$ holds for any pair of equilibrium strategies θ.

Suppose that there exist a $\bar{u}$ which yields budget balance. Let $n = 2$. Without loss of generality, we can assume that $x_1 > 0$, $x_2 > 0$ at equilibrium allocations. Then, budget balance implies that $h_1(\theta_2) + h_2(\theta_1) = \bar{u}(\theta_1, x_1) + \bar{u}(\theta_2, x_2)$ for any equilibrium strategies θ and corresponding allocations x. Equilibrium strategies θ

vary as utility profiles u vary. Thus, the right hand side of this equation should be additively separable in θ_1 and θ_2. We assume that for $i = 1, 2$, $\bar{u}(\theta_i, x_i)$ is twice differentiable in θ_i. Therefore, budget balance implies

$$\frac{\partial^2[\bar{u}(\theta_1, x_1) + \bar{u}(\theta_2, x_2)]}{\partial\theta_1\partial\theta_2} = 0.$$

The partial derivative of $\bar{u}$ in x_i, $\bar{u}'(\theta_i, x_i)$ is denoted by $\bar{u}_{(0,1)}(\theta_i, x_i)$ for $i = 1, 2$. Recall that the equilibrium condition is written as $\bar{u}_{(0,1)}(\theta_1, x_1) = \bar{u}_{(0,1)}(\theta_2, x_2) = g(\theta)$. Now we have

$$\begin{aligned}\frac{\partial(\bar{u}(\theta_1, x_1) + \bar{u}(\theta_2, x_2))}{\partial\theta_1} &= \bar{u}_{(1,0)}(\theta_1, x_1) + \bar{u}_{(0,1)}(\theta_1, x_1)\frac{\partial x_1}{\partial\theta_1} + \bar{u}_{(0,1)}(\theta_2, x_2)\frac{\partial x_2}{\partial\theta_1} \\ &= \bar{u}_{(1,0)}(\theta_1, x_1) + g(\theta)(\frac{\partial x_1}{\partial\theta_1} + \frac{\partial x_2}{\partial\theta_1}) = \bar{u}_{(1,0)}(\theta_1, x_1).\end{aligned}$$

The last equality holds since $x_1(\theta) + x_2(\theta) = R$. Likewise, we have $\frac{\partial}{\partial\theta_2}(\bar{u}(\theta_1, x_1) + \bar{u}(\theta_2, x_2)) = \bar{u}_{(1,0)}(\theta_2, x_2)$. Because $\frac{\partial^2}{\partial\theta_1\partial\theta_2} = \frac{\partial^2}{\partial\theta_2\partial\theta_1}$, budget balance holds if and only if

$$\frac{\partial\bar{u}_{(1,0)}(\theta_1, x_1)}{\partial\theta_2} = \frac{\partial\bar{u}_{(1,0)}(\theta_2, x_2)}{\partial\theta_1} = 0.$$

This is equivalent to $\bar{u}_{(1,1)}(\theta_1, x_1)\frac{\partial x_1}{\partial\theta_2} = \bar{u}_{(1,1)}(\theta_2, x_2)\frac{\partial x_2}{\partial\theta_1} = 0$. We proved that x_i is a differentiable function of θ, so $\frac{\partial x_2}{\partial\theta_1}$ and $\frac{\partial x_1}{\partial\theta_2}$ cannot be zero. Thus, the budget balance requests $\bar{u}_{(1,1)}(\theta_1, x_1) = \bar{u}_{(1,1)}(\theta_2, x_2) = 0$. This implies that for $i = 1, 2$, $\bar{u}(x_i, \theta_i)$ is additively separable in x_i and θ_i, that is, we should have $\bar{u}(\theta_i, x_i) = f(\theta_i) + k(x_i)$ for some functions f and k. Then, for $i = 1, 2$, $\bar{u}'(\theta_i, x_i) = \frac{\partial k(x_i)}{\partial x_i}$. This violates an assumption about $\bar{u}$ of VCG-like mechanisms such that for every $\gamma \in (0, \infty)$ and $x_i > 0$, there exists $\theta_i > 0$ s.t. $\bar{u}'(x_i, \theta_i) = \gamma$ for $i = 1, 2$. Therefore, there is no $\bar{u}$ that satisfies the budget balance. ■

Proof of Theorem 2. The net utility of agent i with strategy θ_i when others submit θ_{-i} is

$$p_i(\theta_i, \theta_{-i}) = u_i(\frac{\theta_i}{\theta_N} R) - \theta_i \theta_{N \setminus i} + S_{-i}.$$

Note that for each agent i, $\theta_{N\setminus i} \neq 0$. Agent i tries to maximize $p_i(\theta_i, \theta_{-i})$ for a given θ_{-i} where p_i is continuous and concave in θ_i. Therefore, the first order conditions (FOC) are the sufficient and necessary condition to find Nash equilibria. The conditions are

$$\begin{aligned} u_i'(\frac{\theta_i}{\theta_N} R) \frac{\theta_{N\setminus i}}{\theta_N^2} R - \theta_{N\setminus i} &= 0 \quad \text{if } \theta_i > 0 \\ u_i'(\frac{\theta_i}{\theta_N} R) \frac{\theta_{N\setminus i}}{\theta_N^2} R - \theta_{N\setminus i} &\leq 0 \quad \text{if } \theta_i = 0. \end{aligned}$$

Since $\theta_{N\setminus i} > 0$, these conditions equal

$$\begin{aligned} u_i'(\frac{\theta_i}{\theta_N} R) &= \frac{\theta_N^2}{R} \quad \text{if } \theta_i > 0 \\ u_i'(\frac{\theta_i}{\theta_N} R) &\leq \frac{\theta_N^2}{R} \quad \text{if } \theta_i = 0. \end{aligned}$$

Let $\mu = \frac{\theta_N^2}{R}$ and $x_i = \frac{\theta_i}{\theta_N} R$ for $\forall i \in N$. Then the FOC can be rewritten as

$$\begin{aligned} u_i'(x_i) &= \mu \quad \text{if } x_i > 0 \\ u_i'(x_i) &\leq \mu \quad \text{if } x_i = 0. \end{aligned}$$

Thus, θ is a Nash equilibrium if and only if for all $i \in N$, we have

$$u_i'(x_i) = \mu \quad \text{if } x_i > 0$$
$$u_i'(x_i) \leq \mu \quad \text{if } x_i = 0$$

where $\mu = \frac{\theta_N^2}{R}$, $x_i = \frac{\theta_i}{\theta_N}R$, and $\sum_{i\in N} x_i \leq R$. We know that an allocation, x^*, is efficient if and only if it satisfies $x^* \in \text{argmax}_{x\in\mathcal{X}} \sum_{i\in N} u_i(x_i)$. Since $\sum_{i\in N} u_i(x_i)$ is continuous in x and $\mathcal{X}$ is compact, efficient allocations exist. Also, $\sum_{i\in N} u_i(x_i)$ is concave, so the necessary and sufficient first order conditions are

$$\begin{aligned} u_i'(x_i^*) &= \lambda \quad \text{if } x_i^* > 0 \\ u_i'(x_i^*) &\leq \lambda \quad \text{if } x_i^* = 0 \end{aligned}$$

where $\lambda > 0$. We can show that $\mu = \lambda$.

For the sake of contradiction, suppose that $\mu > \lambda$. We denote an equilibrium allocation by x and an efficient allocation by x^*. Choose i such that $x_i > 0$. Then, $u_i'(x_i) = \mu > \lambda \geq u_i'(x_i^*)$. This implies $x_i < x_i^*$, so $R - x_i = \sum_{j\neq i} x_j > R - x_i^* = \sum_{j\neq i} x_j^*$. If this is the case, there should be $j \neq i$ such that $x_j > x_j^*$ and we have $u_j'(x_j) \leq u_j'(x_j^*)$. Since $x_j^* \geq 0$, we have $x_j > 0$ and $\mu = u_j'(x_j) \leq u_j'(x_j^*) \leq \lambda$. Hence, $\mu \leq \lambda$ and this contradicts the previous assumption. Therefore, $\mu = \lambda$. We conclude the two FOC's are indeed the same, so that θ is Nash if and only if $x = x(\theta)$ is an efficient allocation. The existence of efficient allocations also guarantees the existence of Nash equilibria. Therefore, Nash equilibria exist and they are efficient, as desired. ■

Proof of Theorem 3. (i) Suppose that $x_i \leq x_j$, which is equivalent to $\theta_i \leq \theta_j$.

Remember that $t_i = \theta_i\theta_{N\setminus i} - S_{-i}$. Then,

$$t_i - t_j = \theta_i\theta_{N\setminus i} - S_{-i} - \theta_j\theta_{N\setminus j} + S_{-j} = \theta_N(\theta_i - \theta_j).$$

Thus $t_i \le t_j$ if and only if $\theta_i \le \theta_j (x_i \le x_j)$.

(ii) Since $p_i(\theta_i, \theta_{-i})$ is concave in θ_i, it is sufficient to check if $p_i(0, \theta_{-i}) \ge 0$. We see $p_i(0, \theta_{-i}) = S_{-i} > 0$ and so VP holds.

(iii) By the no envy condition from Proposition 1, no envy holds if and only if $\frac{t_i - t_j}{x_i - x_j} = u_i'(x_i)$ at equilibrium allocations. For the SEF mechanism, $u_i'(x_i) = \frac{\theta_N^2}{R}$ and it is easy to check that the no envy condition holds. ■

Proof of Proposition 4. The mechanism collects $\sum_{i \in N} t_i$ and we have

$$\begin{aligned} \sum_{i \in N} t_i &= \sum_{i \in N} [\theta_i\theta_{N\setminus i} - S_{-i}] = \sum_{i \in N} \theta_i\theta_N - nS \\ &= (\sum_{i \in N} \theta_i)^2 - n\sum_{i \in N} \theta_i^2 \le 0. \end{aligned}$$

The second to last inequality holds due to the Cauchy-Schwartz inequality. Thus, the SEF mechanism yields a budget deficit.

Budget Deficit (BD) $= n\sum_{i \in N} \theta_i^2 - (\sum_{i \in N} \theta_i)^2$. Substituting in $\theta_i = x_i\frac{\theta_N}{R}$,

$$BD = n(\sum_{i \in N} x_i^2)\frac{\theta_N^2}{R} - \frac{\theta_N^2}{R}(\sum_{i \in N} x_i)^2 = n(\sum_{i \in N} x_i^2)\frac{\theta_N^2}{R^2} - \theta_N^2.$$

There is a $i \in N$ with $x_i > 0$ and so $u_i'(x_i) = \frac{\theta_N^2}{R}$.

$$BD = \frac{n}{R}u_i'(x_i)\sum_{i \in N} x_i^2 - Ru_i'(x_i) = \frac{n}{R}\lambda[\sum_{i \in N} x_i^2 - \frac{R^2}{n}].$$

Observations:

(a) If $u_i = u_j$ for all $i \neq j$ and $i, j \in N$, then $\theta_i = \theta_j$ for all $i \neq j$ and $x_i = \frac{R}{n}$ for all $i \in N$. Then it is easy to check that the mechanism has a balanced budget.

(b) Note that the supremum of $\sum_{i \in N} x_i^2$ for $x \in \mathcal{X}$ is achieved at the extreme points of $x \in \mathcal{X}$. Then, we have

$$BD = \frac{n}{R}\lambda[\sum_{i \in N} x_i^2 - \frac{R^2}{n}] \leq \frac{n}{R}\lambda(R^2 - \frac{R^2}{n}) = \lambda R(n-1). \blacksquare$$

Proof of Proposition 5. With these assumptions, agent i's net utility is $p_i(\theta_i, \theta_{-i}) = u_i(\frac{\theta_i}{\theta_N}R) - g_i(\theta) - h(\theta_{-i})$. p_i is concave in θ_i since g_i is linear in θ_i. θ is an Nash equilibrium if and only if we have

$$\begin{aligned} u_i'(\frac{\theta_i}{\theta_N}R)\frac{\theta_{N\backslash i}}{\theta_N^2}R &= g_i'(\theta) \quad \text{if } \theta_i > 0 \\ u_i'(\frac{\theta_i}{\theta_N}R)\frac{\theta_{N\backslash i}}{\theta_N^2}R &\leq g_i'(\theta) \quad \text{if } \theta_i = 0 \end{aligned}$$

where $g_i'(\theta) = \frac{\partial g_i(\theta)}{\partial \theta_i}$. With $x_i = \frac{\theta_i}{\theta_N}R$, these FOC conditions equal

$$\begin{aligned} u_i'(x_i) &= g_i'(\theta)\frac{\theta_N^2}{\theta_{N\backslash i}R} \quad \text{if } \theta_i > 0 \\ u_i'(x_i) &\leq g_i'(\theta)\frac{\theta_N^2}{\theta_{N\backslash i}R} \quad \text{if } \theta_i = 0. \end{aligned}$$

The mechanism is envy-free, so every Nash equilibrium is efficient. We should have

$$\lambda = g_i'(\theta)\frac{\theta_N^2}{\theta_{N\backslash i}R}$$

where λ is the market clearing price for price taking buyers and $\lambda > 0$. Since $g_i(\theta) =$

$\theta_i\alpha(\theta_{-i}) + \beta(\theta_{-i})$ for $\alpha(\theta_{-i}) > 0$, $g_i'(\theta) = \alpha(\theta_{-i})$. Again, $\lambda = \alpha(\theta_{-i})\frac{\theta_N^2}{\theta_{N\setminus i}C}$ for all $i \in N$ implies $\alpha(\theta_{-i}) = k\theta_{N\setminus i}$ where k is a positive constant. Thus, $t_i = k\theta_{N\setminus i}\theta_i + \beta(\theta_{-i}) + h(\theta_{-i})$.

The envy-free condition holds if and only if $\frac{t_i - t_j}{x_i - x_j} = u_i'(x_i)$. That is to say, no envy holds if and only if

$$\begin{aligned}
\frac{k\theta_N^2}{R} &= k\frac{(\theta_{N\setminus i}\theta_i - \theta_{N\setminus j}\theta_j)}{\theta_i - \theta_j}\frac{\theta_N}{R} + \frac{[h(\theta_{-i}) - h(\theta_{-j}) + \beta(\theta_{-i}) - \beta(\theta_{-j})]}{\theta_i - \theta_j}\frac{\theta_N}{R} \\
&= k\frac{\theta_N}{R}\frac{\theta_N(\theta_i - \theta_j) - (\theta_i^2 - \theta_j^2)}{\theta_i - \theta_j} + \frac{[h(\theta_{-i}) - h(\theta_{-j}) + \beta(\theta_{-i}) - \beta(\theta_{-j})]}{\theta_i - \theta_j}\frac{\theta_N}{R} \\
&= k\frac{\theta_N}{R}[\sum_{l \neq i,j}\theta_l] + \frac{[h(\theta_{-i}) - h(\theta_{-j}) + \beta(\theta_{-i}) - \beta(\theta_{-j})]}{\theta_i - \theta_j}\frac{\theta_N}{R}
\end{aligned}$$

for any $i \neq j$ such that $i, j \in N$. This is equivalent to

$$k\theta_N = k\sum_{l \neq i,j}\theta_l + \frac{[h(\theta_{-i}) - h(\theta_{-j}) + \beta(\theta_{-i}) - \beta(\theta_{-j})]}{\theta_i - \theta_j}$$

and again, in the same way, to

$$[h(\theta_{-i}) + \beta(\theta_{-i})] - [h(\theta_{-j}) + \beta(\theta_{-j})] = k(\theta_i^2 - \theta_j^2).$$

Then, $h(\theta_{-i}) + \beta(\theta_{-i}) = -kS_{-i} + \gamma$ where γ is an arbitrary constant. Therefore, we have $t_i = k\theta_i\theta_{N\setminus i} - kS_{-i} + \gamma$. ■

Proof of Proposition 6. Recall $u_i'(x_i) = \frac{\partial t_i}{\partial \theta_i}/\frac{\partial x_i}{\partial \theta_i} = g(\theta)$ for every $i \in N$. If agent i's resource share x_i is determined proportionally to his strategy θ_i, then

$\frac{\partial x_i}{\partial \theta_i} = \frac{\theta_{N\backslash i} R}{\theta_N^2}$ and his payment is

$$t_i(\theta) = R\theta_{N\backslash i} \int_0^{\theta_i} \frac{g(t, \theta_{-i})}{(t + \theta_{N\backslash i})^2} dt + \beta(\theta_{-i}).$$

When the equilibrium allocation is envy-free, we have $u_i'(x_i) = \frac{t_i - t_j}{x_i - x_j}$ for $\forall i, j \in N$, $i \neq j$. This equation is the same as $g(\theta) = \frac{t_i - t_j}{\theta_i - \theta_j} \cdot \frac{\theta_N}{R}$. Thus, envy-free Nash implementation holds if and only if we have

$$g(\theta) = \frac{\theta_N}{\theta_i - \theta_j} \Big[\theta_{N\backslash i} \int_0^{\theta_i} \frac{g(t, \theta_{-i})}{(t + \theta_{N\backslash i})^2} dt - \theta_{N\backslash j} \int_0^{\theta_j} \frac{g(t, \theta_{-j})}{(t + \theta_{N\backslash j})^2} dt + \frac{\beta(\theta_{-i}) - \beta(\theta_{-j})}{R} \Big]$$

for every $i \neq j \in N$.

Assume $g(\theta) = g(\theta_N)$. We want to find a function $g(\theta)$ and $\beta(\theta_{-i})$ satisfying the following equation:

$$g(\theta_N) = \frac{\theta_N}{\theta_i - \theta_j} \Big[\theta_{N\backslash i} \int_0^{\theta_i} \frac{g(t + \theta_{N\backslash i})}{(t + \theta_{N\backslash i})^2} dt - \theta_{N\backslash j} \int_0^{\theta_j} \frac{g(t + \theta_{N\backslash j})}{(t + \theta_{N\backslash j})^2} dt + \frac{\beta(\theta_{-i}) - \beta(\theta_{-j})}{R} \Big]. \tag{2.4}$$

Multiplying both sides by $(\theta_i - \theta_j)/\theta_N$ and fixing $\theta_{N\backslash i,j}$ as $\tilde{c}$, the equation is written as

$$0 = \frac{(\theta_i - \theta_j) \cdot g(\theta_i + \theta_j + \tilde{c})}{\theta_i + \theta_j + \tilde{c}} - (\theta_j + \tilde{c}) \int_0^{\theta_i} \frac{g(t + \theta_j + \tilde{c})}{(t + \theta_j + \tilde{c})^2} dt + (\theta_i + \tilde{c}) \int_0^{\theta_j} \frac{g(t + \theta_i + \tilde{c})}{(t + \theta_i + \tilde{c})^2} dt$$
$$- \frac{\beta(\theta_{-i}) - \beta(\theta_{-j})}{R}.$$

Assume that marginal price $g(\theta)$ is twice-differentiable. Differentiating the previous equation with respect to θ_i and θ_j, we have

$$\Big(- g'(\theta_i+\theta_j+\tilde{c}) + (\theta_i+\theta_j+\tilde{c})g''(\theta_i+\theta_j+\tilde{c})\Big)\frac{\theta_i-\theta_j}{(\theta_i+\theta_j+\tilde{c})^2} = 0.$$

Since we do not consider the case of $\theta_i = \theta_j$, this is equivalent to

$$-g'(\theta_N) + (\theta_N)g''(\theta_N) = 0.$$

The solution of this equation should be $g(x) = a_1x^2 + a_2$ for constants $a_1 > 0$ and a_2. Without loss of generality, set $a_2 = 0$. Inserting $g(x) = a_1x^2$ into the equation (2.4), we have $Ra_1(\theta_i^2 - \theta_j^2) = \beta(\theta_{-i}) - \beta(\theta_{-j})$ and $\beta(\theta_{-i}) = -Ra_1S_{-i} + a_0$ for a constant a_0. ■

Proof of Lemma 1. Note that for the efficient equilibrium (θ_1^*, θ_2^*) such that $\theta_1^* \leq \theta_2^*$, we have $u_1'(1/2) \leq u_1'(\frac{\theta_1^*}{\theta_1^*+\theta_2^*}) = (\theta_1^*+\theta_2^*)^2 = u_2'(\frac{\theta_2^*}{\theta_1^*+\theta_2^*}) \leq u_2'(1/2)$. If $\theta_1^* \geq \epsilon$ and $\theta_2^* \geq \epsilon$, the efficient strategy will be the adjusted equilibrium and $\theta_1 \leq \theta_2$. When $\theta_1^* \leq \epsilon \leq \theta_2^*$ or $\theta_1^* \leq \theta_2^* \leq \epsilon$, we can prove that the adjusted equilibrium has the form of (ϵ, θ_2) for $\theta_2 \geq \epsilon$.

Let $\theta_1 = \epsilon$. Agent 2 responds with θ_2 such that $u_2'(\frac{\theta_2}{\epsilon+\theta_2}) = (\epsilon+\theta_2)^2$. If this $\theta_2 \leq \epsilon$, i.e., $u_2'(1/2) \leq 4\epsilon^2$, agent 2 likes to play ϵ. If $u_1'(1/2) \leq 4\epsilon^2$, agent 1 doesn't have an incentive to change his strategy from ϵ. Thus, if $u_1'(1/2) \leq 4\epsilon^2$ and $u_2'(1/2) \leq 4\epsilon^2$, then (ϵ, ϵ) is a Nash equilibrium. If agent 2 responds to agent 1's strategy ϵ with $\theta_2 \geq \epsilon$, i.e., $u_2'(1/2) \geq 4\epsilon^2$, agent 1 still plays ϵ as $u_1'(\frac{\epsilon}{\epsilon+\theta_2}) \leq (\epsilon+\theta_2)^2$. Thus, (ϵ, θ_2) is a Nash equilibrium if

$$u_1'(\frac{\epsilon}{\epsilon+\theta_2}) \leq (\epsilon+\theta_2)^2 = u_2'(\frac{\theta_2}{\epsilon+\theta_2}). \tag{2.5}$$

Note that $u_1'(1/2) \leq u_1'(\frac{\epsilon}{\epsilon+\theta_2}) \leq u_2'(\frac{\theta_2}{\epsilon+\theta_2}) \leq u_2'(1/2)$ and the solution θ_2 exists for the equation (2.5). However, if we let $\theta_2 = \epsilon$ and $\theta_1 > \epsilon$, applying the same logic as before leads to $u_1'(1/2) > u_2'(1/2)$, which is contradictory to the condition from the efficient equilibrium. Therefore, the adjusted equilibrium is (ϵ, θ_2) for $\theta_2 \geq \epsilon$. ■

Proof of Proposition 7. Suppose that every agent has a linear utility function. Let $n = 2$. Let $u_1 = ax$ and $u_2 = bx$ for $0 < a < b$. Agent 1's net utility is $p_1 = a\frac{\theta_1}{\theta_1+\theta_2} - \theta_1\theta_2 + \theta_2^2$ and agent 2's net utility is $p_2 = b\frac{\theta_2}{\theta_1+\theta_2} - \theta_1\theta_2 + \theta_1^2$. The first order condition for an interior solution is $\theta_1+\theta_2 = \sqrt{a}$ for agent 1 and $\theta_1+\theta_2 = \sqrt{b}$ for agent 2, respectively. The equilibrium strategy (θ_1^*, θ_2^*) without lower bound cannot satisfy both first order conditions, so we cannot have $\theta_1^* \geq \epsilon$ and $\theta_2^* \geq \epsilon$. Suppose that $\theta_1^* \geq \epsilon$ and $\theta_2^* \leq \epsilon$. Since the net utility function is concave, agent 2 will play ϵ and agent 1 will play $\theta_1 = \sqrt{a} - \epsilon$ for $\sqrt{a} > \epsilon$. Then, agent 2 will adjust his strategy according to $\theta_2' = \sqrt{b} - \theta_1 = \sqrt{b} - (\sqrt{a} - \epsilon)$. We want ϵ to be an equilibrium strategy for agent 2, so that $\theta_2' \leq \epsilon$ holds. This is equivalent to $\sqrt{b} \leq \sqrt{a}$ and contradicts $a < b$. Therefore, $\theta_1^* \geq \epsilon, \theta_2^* \leq \epsilon$ cannot happen for the case where $a < b$ to have an adjusted equilibrium.

We can instead think of the case where $\theta_1^* \leq \epsilon$ and $\theta_2^* \geq \epsilon$. Since the net utility function is concave, agent 1 will play ϵ when lower bound ϵ is imposed on his strategy set. Then agent 2 will change his strategy to be $\theta_2 = \sqrt{b} - \epsilon$. Note that from $\theta_2^* \geq \epsilon$, we have $b \geq 4\epsilon^2$. Thus, if $b \geq 4\epsilon^2$, $(\epsilon, \sqrt{b} - \epsilon)$ is a unique Nash equilibrium; and so is (ϵ, ϵ) otherwise. The worst-case relative surplus is $\inf_{a,b} \frac{a\epsilon+b(\sqrt{b}-\epsilon)}{b\sqrt{b}} = 0.5$ for $b \geq 4\epsilon^2$ and $\inf_{a,b} \frac{a+b}{2b} = 0.5$ otherwise. Therefore, the worst-case relative surplus is 0.5 for $n = 2$.

Applying the same logic as $n = 2$ case, we can prove that for utility functions,

$u_1(x) = a_1 x, \cdots, u_n(x) = a_n x$ such that $a_1 < \cdots < a_n$, there is an equilibrium $(\epsilon, \cdots, \epsilon, \sqrt{a_n} - (n-1)\epsilon)$ where $\sqrt{a_n} - (n-1)\epsilon \geq \epsilon$. The relative economic surplus (res) is

$$\frac{\epsilon(a_1 + \cdots + a_{n-1}) + a_n(\sqrt{a_n} - (n-1)\epsilon)}{a_n\sqrt{a_n}}.$$

We have $\inf_{a_1,\cdots,a_{n-1}} res = 1 - \frac{(n-1)\epsilon}{\sqrt{a_n}}$. From $\sqrt{a_n} \geq n\epsilon$, $1 - \frac{(n-1)\epsilon}{\sqrt{a_n}} \geq \frac{1}{n}$ and the worst-case relative surplus is $\frac{1}{n}$. ■

Proof of Proposition 8. Note that $u'_{n+1}(0) = u'_{n+2}(0) = \infty$. In the ϵ-extended game, $\theta_{n+1} > 0, \theta_{n+2} > 0$ and the first order conditions of the equilibria are for each $i \in N$,

$$u'_i(x_i(\epsilon)) = \frac{w\epsilon}{2\sqrt{x_{n+1}(\epsilon)}} = \frac{(1-w)\epsilon}{2\sqrt{x_{n+2}(\epsilon)}} = \lambda(\epsilon) \text{ if } x_i > 0,$$

$$u'_i(0) \leq \lambda(\epsilon) \text{ if } x_i = 0.$$

Suppose $\lambda(\epsilon)$ is not strictly increasing in ϵ, i.e., for $0 < \epsilon_1 < \epsilon_2$, $\lambda(\epsilon_1) \geq \lambda(\epsilon_2)$. Then, it is easy to check that $x_i(\epsilon_1) \leq x_i(\epsilon_2)$ for all $i \in N$, $x_{n+1}(\epsilon_1) < x_{n+1}(\epsilon_2)$, and $x_{n+2}(\epsilon_1) < x_{n+2}(\epsilon_2)$. This contradicts the fact that $\sum_{i \in N} x_i + x_{n+1} + x_{n+2} = R$ for any $\epsilon > 0$. Thus, $\lambda(\epsilon)$ is strictly increasing in ϵ. In addition, notice that $\lambda \geq \min_{i \in N} u'_i(R) > 0$. Since $\lambda(\epsilon)$ is strictly monotone and bounded from below, we have $\lim_{\epsilon \to 0} \lambda(\epsilon) = \lambda > 0$. In addition, we get $\lim_{\epsilon \to 0} x_{n+1}(\epsilon) = 0$ and $\lim_{\epsilon \to 0} x_{n+2}(\epsilon) = 0$. Since $u'_i(x_i)$ is decreasing in x_i for all $i \in N$, $x_i(\epsilon)$ is strictly increasing as ϵ approaches 0 and it is bounded above by $R > 0$. Thus, $x_i(\epsilon)$ converges to x_i. Therefore, the solution $(x(\epsilon), \lambda(\epsilon))$ of the above FOC converges to the solution (x, λ) of the efficient allocation's FOC. ■

Proposition 9. *There is no efficient scalar strategy mechanism which implements a dominant strategy equilibrium.*

Proof. We will show the nonexistence for the simplest case of $n = 2$. Remember that a scalar mechanism implements efficient equilibrium if and only if, for a given utility profile $u \in U$, each equilibrium θ with an equilibrium allocation $x(\theta)$ satisfies the first order condition for efficient equilibria,

$$u_i'(x_i(\theta)) = g(\theta) \text{ for all } i \in N$$

where g is a continuous and nonnegative function. The function g determines properties of the scalar strategy mechanism.

Suppose that a scalar strategy mechanism $M(g)$ implements efficient dominant strategy equilibrium. Given the mechanism $M(g)$, for every pair of utility profiles (u_1, u_2), there is a corresponding pair of dominant strategy equilibrium $(\tilde{\theta_1}, \tilde{\theta_2})$. $\tilde{\theta_1}$ should be a best response of agent 1 to every $\theta_2 \in \mathcal{R}_+$, that is, we have

$$u_1'(x_1(\tilde{\theta_1}, \theta_2)) = g(\tilde{\theta_1}, \theta_2) \text{ for every } \theta_2 \in \mathcal{R}_+ \qquad (1).$$

Likewise, $\tilde{\theta_2}$ should be a best response of agent 2 to every $\theta_1 \in \Theta$, that is, we have

$$u_2'(x_2(\theta_1, \tilde{\theta_2})) = g(\theta_1, \tilde{\theta_2}) \text{ for every } \theta_1 \in \mathcal{R}_+ \qquad (2).$$

To use simple notations, we will denote u_1' by f and u_2' by h. Then, f and h are functions of θ_1 and θ_2, i.e., $f = f(\theta_1, \theta_2)$ and $h = h(\theta_1, \theta_2)$. As a pair of utility profiles $u \in \mathcal{U}^2$ can be chosen arbitrarily, we can say that there are a set F and a set H such that

$$\begin{aligned} F &= \{f(x,y): \ f \text{ is continuous and nonnegative}\} \\ H &= \{h(x,y): \ h \text{ is continuous and nonnegative}\}. \end{aligned}$$

Denoting $x = \theta_1$, $y = \theta_2$, $x^* = \tilde{\theta_1}$ and $y^* = \tilde{\theta_2}$, equations (1) and (2) are rewritten as follows:

$$\begin{aligned} f(x^*, y) &= g(x^*, y) \text{ for every } y \in Y \qquad (3) \\ h(x, y^*) &= g(x, y^*) \text{ for every } x \in X \qquad (4) \end{aligned}$$

where $X = Y = [0, +\infty)$ and $X \times Y$ are the domains of functions f and h. In addition, we should have

$$f(x^*, y^*) = h(x^*, y^*) \qquad (5).$$

The equations (3)-(5) should hold for any pair of f and h from F and H, respectively. Note that x^* and y^* depend on the choice of f and h, but g is fixed by the mechanism.

Let us choose three pairs of (f_1, h_1), (f_2, h_2) and (f_3, h_3) from F and H. There are corresponding dominant strategy equilibrium pairs: (x_1, y_1), (x_2, y_2) and (x_3, y_3), respectively. According to equation (5), we have

$$\begin{aligned} f_1(x_1, y_1) = h_1(x_1, y_1) = g(x_1, y_1), \\ f_2(x_2, y_2) = h_2(x_2, y_2) = g(x_2, y_2), \\ f_3(x_3, y_3) = h_3(x_3, y_3) = g(x_3, y_3). \end{aligned}$$

Notice that the function $g(x, y)$ should have same values at each point (x_1, y_2), (x_1, y_3), (x_2, y_1), (x_2, y_3), (x_3, y_1), and (x_3, y_2), so we have

$$f_1(x_1, y_2) = h_2(x_1, y_2) = g(x_1, y_2),$$
$$h_1(x_2, y_1) = f_2(x_2, y_1) = g(x_2, y_1),$$
$$f_1(x_1, y_3) = h_3(x_1, y_3) = g(x_1, y_3),$$
$$f_2(x_2, y_3) = h_3(x_2, y_3) = g(x_2, y_3),$$
$$h_1(x_3, y_1) = f_3(x_3, y_1) = g(x_3, y_1),$$
$$f_3(x_3, y_2) = h_2(x_3, y_2) = g(x_3, y_2).$$

Taking x_1, x_2, x_3, y_1, y_2, and y_3 as unknown variables to solve, we have 6 unknown variables with 9 equations. Considering that f and h are arbitrarily selected, a function g cannot exist in this situation. ■

Proposition 10. *Neither uniform pricing scalar strategy mechanism with proportional allocations is efficient or envy-free.*

Proof. Under a uniform pricing scalar strategy mechanism, agent i's net utility is $u_i(\frac{\theta_i}{\theta_N}R) - \tilde{p}(\theta)\frac{\theta_i}{\theta_N}R$ where $\tilde{p}(\theta)$ is the uniform price. The first order equilibrium condition is $u_i'(x_i) = \tilde{p}(\theta) + \frac{\partial \tilde{p}(\theta)}{\partial \theta_i}\frac{\theta_N \theta_i}{\theta_{N\setminus i}}$ for every $i \in N$. For an equilibrium θ to be efficient, $\frac{\partial \tilde{p}(\theta)}{\partial \theta_i}\frac{\theta_N \theta_i}{\theta_{N\setminus i}} = g(\theta)$ for every $i \in N$ and a continuous function g. Then, we have

$$\tilde{p}(\theta) = \theta_{N\setminus i} \int_0^{\theta_i} \frac{g(t, \theta_{-i})}{t(t + \theta_{N\setminus i})} dt + \beta(\theta_{-i})$$

for every $i \in N$ where β is a continuous function.

We show that this equation cannot hold for $n = 2$. For the sake of contradiction, suppose that it holds for the case of two agents. We have

$$\tilde{p}(\theta) = \theta_2 \int_0^{\theta_1} \frac{g(t,\theta_2)}{t(t+\theta_2)} dt + \beta(\theta_2) = \theta_1 \int_0^{\theta_2} \frac{g(\theta_1,t)}{t(t+\theta_1)} dt + \beta(\theta_1).$$

From the second and third equations, we have

$$\beta(\theta_2) - \beta(\theta_1) = \theta_1 \int_0^{\theta_2} \frac{g(\theta_1,t)}{t(t+\theta_1)} dt - \theta_2 \int_0^{\theta_1} \frac{g(t,\theta_2)}{t(t+\theta_2)} dt.$$

This equation should hold for any $\theta_1, \theta_2 \in [0, \infty)$ and the left hand side is additively separable in θ_1 and θ_2. The cross derivative of the right hand side is

$$\begin{aligned}
&\frac{\partial^2}{\partial\theta_1 \partial\theta_2} (\theta_1 \int_0^{\theta_2} \frac{g(\theta_1,t)}{t(t+\theta_1)} dt - \theta_2 \int_0^{\theta_1} \frac{g(t,\theta_2)}{t(t+\theta_2)} dt) \\
&= \frac{\theta_2 g(\theta_1,\theta_2)}{\theta_1(\theta_1+\theta_2)^2} - \frac{g(\theta_1,\theta_2)}{\theta_1(\theta_1+\theta_2)} + \frac{g(\theta_1,\theta_2)}{\theta_2(\theta_1+\theta_2)} - \frac{\theta_2 g_2(\theta_1,\theta_2)}{\theta_1(\theta_1+\theta_2)} \\
&+ \theta_1(-\frac{g(\theta_1,\theta_2)}{\theta_2(\theta_1+\theta_2)^2} + \frac{g_1(\theta_1,\theta_2)}{\theta_2(\theta_1+\theta_2)})
\end{aligned}$$

and it should be zero where $g_1(\theta_1,\theta_2) = \frac{\partial g(\theta_1,\theta_2)}{\partial\theta_1}$ and $g_2(\theta_1,\theta_2) = \frac{\partial g(\theta_1,\theta_2)}{\partial\theta_2}$. Solving the equation such that the cross derivative of the right hand side equals zero, we have $g(\theta) = k\frac{\theta_1+\theta_2}{\theta_1\theta_2}$ for a constant $k > 0$. However, with this function g, $\theta_1 \int_0^{\theta_2} \frac{g(\theta_1,t)}{t(t+\theta_1)} dt$ and $\theta_2 \int_0^{\theta_1} \frac{g(t,\theta_2)}{t(t+\theta_2)} dt$ do not converge. Therefore, there does not exist a g function that allows the uniform pricing mechanism to be efficient. Finally, due to the no envy condition, we can conclude that if a mechanism is not efficient, then it fails to be envy-free. ■

Chapter 3

Optimal VCG Mechanisms to Assign Multiple Tasks

There exist m undesirable objects (or "bads") which are identical and which need to be allocated to n, $n > m$ strategic agents. Cries of NIMBY greeting waste disposal facilities represents a problem of allocating economic bads (Kunreuther and Easterling (1996)). Each agent is required to take at most an object. For the problem of assigning economic bads, the seminal Vickrey-Clarke-Groves (VCG) mechanisms (Generalized Vickrey Auction) achieve both allocative efficiency and incentives by way of money transfer. They are uniquely characterized by strategy-proofness[1] and allocative efficiency (Green and Laffont (1977, 1979), Suijs (1996), Holmstrom (1979)). However, it is not possible for VCG mechanisms to be budget balanced at all valuation profiles (Green and Laffont (1979)). If there is a budget surplus, then it needs to be discarded by a benevolent residual claimant in order to preserve the incentive compatibility of

[1] A mechanism is said to be *strategy-proof* if truth telling is a dominant strategy for every agent.

a VCG mechanism. In case of budget deficit, the residual claimant must finance the mechanism. Interpreting any budget imbalance as a mechanism implementation cost, our aim is to design VCG mechanisms that minimize the budget imbalance.[2]

If we weaken the incentive criterion from dominant strategy, we can use Bayesian assumptions for the distribution of utility functions, and therefore calculate the expected budget imbalance (Bailey (1997)).[3] For the problem of provisioning public goods, Deb, Gosh and Seo (2002), Green et al.(1976) and Green and Laffont (1979) give the asymptotic behavior of the expected budget imbalance under the pivotal mechanism (Vickrey Auction), and Zhou (2007) provides the same for the problem of private good exchange. If we wish to maintain dominant strategy incentive compatibility, we assume no prior and approach the problem using the worst case analysis. Moulin (1986) as well as Deb and Seo (1998) investigate the pivotal mechanism in the worst scenario for a public good provision problem, and Moulin and Shenker (2001) do the same for a cost sharing problem. Goldberg et al.(2001, 2006) and Aggarwal et al.(2005) as well as Hartline and McGrew (2005) design worst case profit maximizing mechanisms.

Favoring the prior-free above Bayesian approach, we will adopt the worst case analysis. We will measure the performance of a VCG mechanism with the worst ratio of budget imbalance to efficient surplus over all utility profiles. This index is called *efficiency loss* of the VCG mechanism. Efficiency loss is interpreted as the worst implementation cost relative to the created benefit in the mechanism. When a

[2]Parkes et al. (2001) and Faltings (2005) construct budget balanced mechanisms forsaking efficiency or strategy-proofness.

[3]In both the public good provision problem and the bilateral trading problem, there exists no budget balanced mechanism that is Bayesian-incentive compatible, efficient, and individually rational (Laffont and Maskin (1979) and Myerson and Satterthwaite (1983)).

mechanism has a minimal efficiency loss among a group of mechanisms, it is said to be *optimal*, and its efficiency loss is called *optimal efficiency loss.*[4] We will develop optimal VCG mechanisms in the problem of allocating bads.[5]

The main results are presented in Section 3.2. We not only compute optimal VCG mechanisms, but also conduct basic tests of fairness as well. For the basic fairness tests, we will adopt unanimity upper bound and individual rationality. If a mechanism guarantees each participant a net loss smaller than the loss he would experience under random assignment, the mechanism satisfies *unanimity upper bound.* A mechanism is said to be *individually rational* if participation in the mechanism brings each agent a smaller net loss than the loss he would experience in an anarchistic state where everyone performs one task on his own. Our intention is to show that the optimal mechanisms for "goods" and the optimal mechanisms for "bads" behave very differently when we require individual rationality.

The most relevant articles to our problem have been written by Moulin (2009) and Guo and Conitzer (2009). They investigate the problem of assigning multiple "goods" and develop optimal VCG mechanisms using the worst case analysis. The resulting optimal VCG mechanisms significantly improve upon the previous in Cavallo (2006).[6]

[4]Apt et al.(2008) and Guo and Conitzer (2008a) use a different concept of optimality. Their optimal mechanisms are defined to be *undominated.* A VCG mechanism dominates another if it always charges less payment against each agent.

[5]As an application of VCG mechanisms to the assignment problem of identical economic bads, Porter, Shoham and Tennenholtz (2004) provide an equity test called k-fairness and develop a 3-Fair mechanism. Moulin (2010) discusses tradeoffs between efficiency and k-fairness. He constructs a VCG mechanism which guarantees each participant a fair share of the q^{th} highest valuation and minimizes the efficiency loss in the allocation problem of a single object.

[6]Cavallo (2006) constructs a VCG mechanism to redistribute some of the payment back to the agents in a way that will not affect incentives. For the instance of a single object auction, Cavallo's mechanism redistributes to agent i $\frac{1}{n}$ times the second highest bid among bids other than his own bid.

For the problem of assigning economic goods, Moulin (2009) makes two interesting points. The first being that the optimal loss of any non-deficit VCG mechanism is strictly smaller than the optimal loss of any individually rational and non-deficit VCG mechanism. Thus individual rationality plays a role when $m \geq 2$. Both indices converge exponentially fast to zero in n if the scarcity ratio $\frac{m}{n}$ is less than $\frac{1}{2}$, and as $\frac{1}{\sqrt{n}}$ if $\frac{m}{n} \simeq \frac{1}{2}$. Their behavior, however, is quite different if $\frac{m}{n} > \frac{1}{2}$. The optimal loss, excluding individual rationality, still converges fast to zero in n, while the optimal loss under individual rationality does not converge to zero in n.

Secondly, Moulin (2009) points that whether or not deficit is allowed does not make an essential difference in total optimal loss. The optimal loss of any VCG mechanism (allowing deficit) is about one-half (saying exactly, between $\frac{1}{2+\frac{1}{n-1}}$ and $\frac{1}{2}$) of the optimal loss of any non-deficit VCG mechanism. He conjectures that this property still holds true even if individual rationality is imposed.

On the other hand, Guo and Conitzer (2009) use the worst ratio of budget imbalance to the budget surplus of the pivotal mechanism to measure performance. Although their design goal is different from the goal in Moulin (2009), their non-deficit optimal mechanism is the same as the non-deficit and individually rational optimal mechanism in Moulin (2009). Individual rationality is irrelevant in Guo and Conitzer (2009), since their non-deficit optimal mechanism remains the same even if we impose individual rationality.

In addition, the optimal loss of any non-deficit VCG mechanism in Guo and Conitzer (2009) equals the optimal loss of any non-deficit and individually rational VCG mechanism in Moulin (2009). This demonstrates that the non-deficit optimal VCG mechanism in the former fails asymptotic budget balance altogether when the scarcity ratio is greater than one-half. In addition, according to Guo and Conitzer (2009), when $m = n-1$, the pivotal mechanism will always be optimal among all VCG mechanisms. This is undesirable since the efficiency loss of the pivotal mechanism is always greater than 1. In addition, similarly to Moulin's findings, allowing deficit does not essentially change the optimal loss for Guo and Conitzer (2009) either.[7]

In the problem of assigning "bads", we show that the performance measurement suggested by Guo and Conitzer (2009) fails to be in use for all m, $m < n$. If we measure the performance of a mechanism and find the optimal mechanism according to the standards in Guo and Conitzer (2009), it rarely redistributes the surplus of the pivotal mechanism for every m, $m \geq 2$. For $m = 1$, the pivotal mechanism is optimal, therefore there is no redistribution. Thus, the optimal mechanism is far from achieving Guo and Conitzer's original objective of redistributing the surplus of the pivotal mechanism. We can predict that this optimal mechanism will have a large efficiency loss since the pivotal mechanism generates the largest efficiency loss among all non-deficit and individually rational VCG mechanisms. This point is shown in detail in Appendix 3.4.1.

In Section 3.2.1, we compute the optimal efficiency loss $\lambda^*_{n,m}$ of any non-deficit VCG mechanism and its corresponding optimal mechanism for all m and n. For both $m = 1$ and $m = n-1$, the worst case surplus in the optimal mechanism never exceeds $\frac{n}{2^{n-3}}$ of efficient surplus (Theorem 1.1 and Theorem 1.3). For m, $2 \leq m \leq n-2$, the optimal efficiency loss of any non-deficit VCG mechanism vanishes fast at exponential speed in n: $\lambda^*_{n,m} \simeq \frac{n^m}{m!2^{n-2}}$ (Theorem 1.2). This tells that efficiency loss works well as a performance index for the problem of assigning "bads". In addition, similarly

[7]The optimal loss with no deficit λ_G and the optimal loss allowing deficit μ_G in Guo and Conitzer (2009) relate as follows: for $m \leq n-2$, $\frac{\mu_G}{\lambda_G} = \frac{1}{2-\lambda_G}$ and $\frac{\mu_G}{\lambda_G}$ converges to $\frac{1}{2}$ in n given m.

to the problem of assigning economic goods, Proposition 1 shows that whether we require non-deficit property or not, has no bearing on the total optimal efficiency loss in the problem of assigning economic bads as well.

Section 3.2.2, however, shows that the non-deficit property is incompatible with preliminary tests of welfare bounds. Proposition 2 shows that the unanimity upper bound test fails under the non-deficit constraint. The non-deficit constraint also makes the pivotal mechanism the uniquely optimal individually rational VCG mechanism (Corollary 1).

Interestingly, if the non-deficit constraint is abandoned, individual rationality becomes greatly significant to our problem. We compute the optimal pairs of surplus loss $\lambda^*_{m,n}$ and deficit loss $\mu^*_{m,n}$ of any individually rational VCG mechanism and the corresponding optimal mechanisms for all n and m.

Theorem 2.1 shows that when assigning a single bad, there exist only two optimal individually rational mechanisms. This result differs from the case of multiple bads in which we can find an infinite number of optimal individually rational mechanisms. For the case of a single bad, the pivotal mechanism is optimal and non-deficit, but generates infinite efficiency loss. In contrast, another optimal VCG mechanism does not generate any surplus and its efficiency loss due to deficit is 1. The optimal surplus loss is infinite times the optimal deficit loss.

Theorem 2.3 shows that to assign multiple bads, $m \geq 3$, we can find an infinite number of optimal pairs of surplus and deficit loss of any individually rational VCG mechanisms. The optimal pairs of surplus loss $\lambda^*_{n,m}$ and deficit loss $\mu^*_{n,m}$ consist of a frontier such that $\lambda^*_{n,m}/A(n,m) + \mu^*_{n,m}/B(n,m) = 1$ where $A(n,m) > B(n,m)$ for all n and m. The asymptotic behavior of the ratio $B(n,m)/A(n,m)$ such that

$B(n,m)/A(n,m) \simeq \frac{n^{m-1}}{(m-1)m!2^{n-2}}$ implies that as more agents participate, a very minute amount of deficit loss can replace unit surplus loss. The deficit becomes much more inexpensive than surplus as the number of agents increases. By allowing a slight deficit, we can almost achieve budget balanced VCG mechanisms. This result stands in stark contrast to the outcome of assigning economic goods. For the case of economic goods, regardless of individual rationality, unit surplus loss can only be replaced with unit deficit loss. Theorem 2.2 also provides similar results for $m = 2$.

All proofs are gathered in Appendix 3.4.2.

3.1 The Model

Let $N = \{1, \cdots, n\}$ be the set of agents. m of the n agents should perform m identical tasks together. The tasks are undesirable, and thus, they are economic "bads" which are costly to agents. Every agent is equally responsible and is liable for at most one task. It is assumed that $1 \leq m \leq n-1$ (if $n = m$, everyone performs a task) and that a monetary transfer occurs.

Each agent i, $i \in N$ can perform a task with cost c_i, which is private information. Performing a task causes agent i disutility c_i. Let $c = (c_1, c_2, \cdots, c_n)$. Given a cost profile $c \in \mathcal{R}^N_+$, the vector $c^* \in \mathcal{R}^N_+$ is the permutation of c whose coordinates are arranged increasingly:

$$c^{*1} \leq c^{*2} \leq \cdots \leq c^{*n}.$$

Let $c_{-i} = (c_1, \cdots, c_{i-1}, c_{i+1}, \cdots, c_n)$. We denote by $(c_{-i})^{*k}$ the kth lowest cost among

$c_1, \cdots, c_{i-1}, c_{i+1}, \cdots, c_n$. Given a cost profile $c \in \mathcal{R}_+^N$, ***efficient cost*** for performing m tasks is the minimal cost $\tau_m(c) = \sum_{k=1}^{m} c^{*k}$.

VCG mechanisms assign tasks to a subset of m agents whose total cost to perform m tasks together is minimal. And each VCG mechanism is defined by n arbitrary real-valued functions t_i on $\mathcal{R}_+^{N\setminus\{i\}}$. The function $t_i(c_{-i})$ represents a monetary transfer from agent i to the mechanism given a cost profile c. Agent i's net disutility V_i in a VCG mechanism is written as:

$$V_i(c) = \tau_m(c) + t_i(c_{-i}) \quad \text{for all } c \in \mathcal{R}_+^N.$$

Every VCG mechanism is *efficient* since an allocation determined by the mechanism always minimizes the total cost to perform m tasks.[8] It is *strategy-proof* since every agent is always better off when he reveals his private information truthfully. Holmstrom (1979) proves that VCG mechanisms are the only strategy-proof and allocatively efficient mechanisms in our model.

We use Δ to denote the budget imbalance of a VCG mechanism as follows:

$$\Delta(c) = \sum_{i \in N} V_i(c) - \tau_m(c) = (n-1)\tau_m(c) + \sum_{i \in N} t_i(c_{-i}).$$

Given a cost profile $c \in \mathcal{R}_+^N$, if $\Delta(c) = 0$, then we have a *balanced budget*, if $\Delta(c) > 0$, a *budget surplus* exists, and if $\Delta(c) < 0$, a *budget deficit* is indicated.

Among VCG mechanisms, the pivotal mechanism (the Vickrey auction) is a benchmark mechanism (Green and Laffont (1979)). In the pivotal mechanism, each

[8] The objects go to the agents with the highest valuations in the case of (desirable) "goods", the lowest disutilities in the case of bads.

agent i's net disutility equals "efficient cost to perform m tasks - efficient cost to perform $(m-1)$ tasks with agent i ignored." If agent i is ignored, other agents force agent i to perform one task and allocate residual $(m-1)$ tasks efficiently among themselves. This implies $t_i(c_{-i}) = -\tau_{m-1}(c_{-i})$. Thus, the net disutility under the pivotal mechanism is written as:

$$V_i^p(c) = \tau_m(c) - \tau_{m-1}(c_{-i}) \text{ for all } i \text{ and } c. \tag{3.1}$$

We can simplify equation (3.1) as $V_i^p(c) = c_i$ if $c_i \leq c^{*(m-1)}$ or $V_i^p(c) = c^{*m}$ if $c_i \geq c^{*m}$. Given cost profile $c \in \mathcal{R}_+^N$, the pivotal mechanism generates a budget surplus of:

$$ps(c) = \sum_{i \in N} V_i^p - \tau_m(c) = (n-m)c^{*m}.$$

Whether a mechanism under our consideration generates budget surplus or not, it is convenient to write the function $t_i(c_{-i})$ as $t_i(c_{-i}) = -\tau_{m-1}(c_{-i}) - r(i; c_{-i})$, where $r(i; c_{-i})$ is a ***redistribution scheme*** for agent i. Thus, the general form of VCG mechanisms is given as:

$$V_i(c) = \tau_m(c) - \tau_{m-1}(c_{-i}) - r(i; c_{-i}) = V_i^p(c) - r(i; c_{-i}) \text{ for all } c \in \mathcal{R}_+^N.$$

Interpreting budget imbalance as an implementation cost, our VCG mechanisms ask the residual claimant to first run the pivotal mechanism. Then, the residual claimant distributes a suitable rebate to each agent if there is a budget surplus, or charges agents of additional tax if there is a deficit. We rewrite the budget imbalance of a

VCG mechanism with a redistribution scheme r as:

$$\Delta(c,r) = ps(c) - \sum_{i=1}^{n} r(i;c_{-i}) = (n-m)c^{*m} - \sum_{i=1}^{n} r(i;c_{-i}).$$

Now we will use the worst case analysis to measure the performance of any VCG mechanism. The worst case performance index of a mechanism will be defined as the largest budget imbalance relative to a meaningful measure of "efficient surplus" over all cost profiles.

Drawing on the concept of opportunity cost, we notice that implementing a VCG mechanism actually saves costs when performing tasks. To perform tasks, a VCG mechanism will spend the efficient cost while a random assignment, as the primitive benchmark, will spend average cost. The saved cost garnered by the VCG mechanism is the difference between the average cost and the efficient cost. Thus, we define ***efficient surplus*** (es) as follows:

$$es(c) = \frac{m}{n}c_N - \tau_m(c)$$

where $c_N = \sum_{i \in N} c_i$.

We define ***efficiency loss*** as the performance measurement of a VCG mechanism with the redistribution scheme r. It is written as the following number $\lambda_{n,m}$, $0 < \lambda_{n,m} \leq \infty$

$$\lambda_{n,m}(r) = \sup_{c \in \mathcal{R}_+^N} \frac{|\Delta(c,r)|}{es(c)}$$

for the case of n agents and m objects. If $\Delta(c,r) > 0$ and $es(c) = 0$ given a cost profile $c \in \mathcal{R}_{+}^{N}$, we set $\lambda_{n,m}(r) = \infty$ conventionally. An *optimal VCG mechanism* is a VCG mechanism with a redistribution scheme r^* which has the smallest efficiency loss $\lambda_{n,m}^* = \lambda_{n,m}(r^*) \leq \lambda_{n,m}(r)$ for any redistribution scheme r.

Another natural estimator of efficient surplus is the spread between maximal cost and efficient cost $(\sum_{k=n-m+1}^{n} c^{*k} - \sum_{k=1}^{m} c^{*k})$. Using this estimator, Moulin (2010) performs the worst-case analysis when the object is a single costly task. The corresponding index of efficiency loss is smaller due to an increase in the denominator. It is, however, difficult to write a general formula for the optimal loss when $m \geq 2$. Alternatively, we might think that we can use efficient cost as an estimator of efficient surplus, but this ultimately fails as Moulin (2010) proves that the index would be at least $n-1$ for $m = 1$.

Using efficiency loss as a performance measure, we compute the efficiency loss of the pivotal mechanism (which does not redistribute anything) as follows:

$$\begin{aligned}\lambda_{n,m}(0) &= \sup_{c \in \mathcal{R}_{+}^{N}} \frac{(n-m)c^{*m}}{\frac{m}{n}\left[\sum_{i=1}^{m} c^{*i} + \sum_{i=m+1}^{n} c^{*i}\right] - \sum_{i=1}^{m} c^{*i}} \\ &= \sup_{c \in \mathcal{R}_{+}^{N}} \frac{(n-m)c^{*m}}{\frac{m-n}{n}\sum_{i=1}^{m-1} c^{*i} + \frac{m-n}{n}c^{*m} + \frac{m}{n}\sum_{i=m+1}^{n} c^{*i}} = \frac{n}{m-1}.\end{aligned}$$

The last equality holds since the worst case occurs when $c^{*1}, \cdots, c^{*(m-1)}, c^{*(m+1)}, \cdots, c^{*n}$ are as small as possible. By setting $c^{*1} = \cdots = c^{*(m-1)} = 0$ and $c^{*m} = c^{*(m+1)} = \cdots = c^{*n}$, we find the efficiency loss of the pivotal mechanism. If $m = 1$, the pivotal mechanism has infinite efficiency loss. Given m, $m \geq 2$, its efficiency loss is increasing in n. Since $\lambda_{m+1,m} = 1 + \frac{2}{m-1}$, the smallest efficiency loss in n is already greater than 1. With this, the implementation cost of the pivotal mechanism is too large compared to the benefit it creates. Therefore, the pivotal mechanism is not attractive to use, and so we must construct redistribution schemes.

3.2 Main Results

We denote by $\binom{n}{k}$ the binomial coefficient. The notation $f(n) \simeq g(n)$ means

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

The notation $r^*_{n,m}$ denotes the optimal redistribution scheme when there are n agents and m objects. Likewise, $\lambda^*_{n,m}$ denotes the optimal efficiency loss for the case of n agents and m objects.

3.2.1 Optimal Non-Deficit VCG Mechanisms

The residual claimant is not required to create financial inflow, so the redistribution scheme should be designed to satisfy the following *non-deficit* constraint:

Non-Deficit **(ND)**: given r, $\Delta(c, r) \geq 0$ for all $c \in \mathcal{R}^N_+$.

Theorem 1.1 *Let $m = 1$ and $n \geq 3$. the optimal efficiency loss of any non-deficit VCG mechanism is given as:*

$$\lambda^*_{n,1} = \frac{n-1}{2^{n-2}-1}.$$

The following linear redistribution scheme defines an optimal mechanism:

$$r^*_{3,1}(c_{-i}) = (c_{-i})^{*1} - \frac{1}{3}(c_{-i})^{*2};$$
$$r^*_{4,1}(c_{-i}) = (c_{-i})^{*1} - \frac{1}{4}(c_{-i})^{*2};$$
$$r^*_{5,1}(c_{-i}) = (c_{-i})^{*1} - \frac{23}{105}(c_{-i})^{*2} + \frac{1}{21}(c_{-i})^{*3} - \frac{1}{35}(c_{-i})^{*4};$$

and for n, $n \geq 6$,

$$r^*_{n,1}(c_{-i}) = \sum_{k=1}^{3} \alpha^*_k (c_{-i})^{*k} + \sum_{k=4}^{n-2} \beta^*_k (c_{-i})^{*k} + \omega^*_{n-1}(c_{-i})^{*(n-1)}$$

where

$$\alpha^*_1 = 1, \ \alpha^*_2 = \frac{n^2 - 2^{n-2}n - 2n + 2}{(2^{n-2} - 1)(n-2)n}, \ \alpha^*_3 = -\frac{n^2 - 2^{n-1} - 3n + 4}{(n-2)(n-3)(2^{n-2} - 1)},$$

$$\beta^*_k = \frac{\lambda^*_{n,1}}{n} + \frac{\lambda^*_{n,1}}{n-1} \cdot \frac{\sum_{j=1}^{k-2} \binom{n-2}{j}}{\binom{n-2}{k-1}} - \frac{1}{\binom{n-2}{k-1}} \quad \textit{if } k \textit{ is even};$$
$$\beta^*_k = -\frac{\lambda^*_{n,1}}{(n-k)} - \frac{\lambda^*_{n,1}}{n-1} \frac{\sum_{j=1}^{k-3} \binom{n-2}{j}}{\binom{n-2}{k-1}} + \frac{1}{\binom{n-2}{k-1}} \quad \textit{if } k \textit{ is odd};$$

$$\omega^*_{n-1} = -\frac{1}{n\,(2^{n-2} - 1)} \quad \textit{if } n \textit{ is odd};$$
$$\omega^*_{n-1} = 0 \quad \textit{if } n \textit{ is even.}$$

Remark 1 If the spread between maximal cost and efficient cost is used as an estimator of efficient surplus, the optimal efficiency loss for $m = 1$ is $\lambda^*_{n,1} = \frac{n-1}{2^{n-1}-1}$ when n is odd, and $\lambda^*_{n,1} = \frac{n-1}{2^{n-1}-2}$ when n is even (Moulin (2010)). As we mentioned in Section 2, this index is smaller than our optimal efficiency loss.

Theorem 1.2 *For* $2 \leq m \leq n-2$, (i) *the optimal efficiency loss of any non-deficit VCG mechanism is given as:*

$$\lambda^*_{n,m} = \frac{(n-m)\binom{n-1}{m-1}}{(n-m)\sum_{k=0}^{m-2}\binom{n-2}{k} + m\sum_{k=m}^{n-2}\binom{n-2}{k}}$$

and (ii) *for a fixed* m,

$$\lambda^*_{n,m} \simeq \frac{n^m}{m!2^{n-2}}.$$

We provide the optimal redistribution schemes corresponding to Theorem 1.2 in Appendix 2.5.2 (Lemma 2 and Lemma 3).

Theorem 1.3 *For* $n \geq 3$, *the optimal efficiency loss of any non-deficit VCG mechanism is written as:*

$$\lambda^*_{n,n-1} = \frac{n-1}{2^{n-2}-1}.$$

Notice that the optimal efficiency loss for $m = n-1$ is the same as the optimal efficiency loss for $m = 1$.

If we abandon non-deficit requirement, and find the optimal redistribution scheme

$r^{\#}$, the corresponding efficiency loss is:

$$\lambda^{\#}_{n,m} = \sup_{c \in \mathcal{R}^N_+} \frac{|\Delta(c, r^{\#})|}{es(c)}.$$

The following result shows that the total optimal loss of any VCG mechanism is almost the same as the optimal loss of any non-deficit VCG mechanism.

Proposition 1 *The optimal efficiency loss $\lambda^{\#}_{n,m}$ of any VCG mechanism satisfies $\lambda^{\#}_{n,m} \simeq \frac{1}{2}\lambda^{*}_{n,m}$ for all n and m.*

Even if we discard non-deficit constraint, and request the residual claimant to finance the mechanism, there is no essential change in the total optimal loss. The optimal loss from budget surplus under a non-deficit mechanism is equally split into *surplus loss* (efficiency loss due to surplus) and *deficit loss* (efficiency loss due to deficit).

3.2.2 Optimal Individually Rational VCG Mechanisms

In this section, we will restrict our discussion to anonymous mechanisms.

Anonymity **(AN)**: A VCG mechanism with the redistribution scheme r is *anonymous* if $r(i; c_{-i}) = r(c_{-i})$ for all $i \in N$.

A VCG mechanism is expected to cause each agent to have a net disutility less than or equal to his disutility under random assignment.[9] This notion is expressed in the following test:

Unanimity upper bound **(UUB)**: $V_i(c) \leq \frac{m}{n} c_i$ for all $i \in N$ and $c \in \mathcal{R}^N_+$.

[9] While the random assignment is simple to implement, and strategy-proof, it is not efficient

Unfortunately, this test is not compatible with the non-deficit property in our model.[10]

Proposition 2 *There exists no anonymous linear VCG mechanism that satisfies unanimity upper bound and non-deficit.*

A weaker constraint for unanimity upper bound is individual rationality. Individual rationality implies that participation in the mechanism will cost any agent less than or equal to what it would cost them if they were to perform the task alone.

Individual Rationality **(IR)**: $V_i \leq c_i$ for all $i \in N$.

Proposition 2 and Corollary 1 below show that non-deficit requirement is very restrictive, and therefore makes the implementation of VCG mechanisms unattractive. When there is a single bad, we cannot improve upon the pivotal mechanism that has an infinite efficiency loss. Therefore, we will investigate VCG mechanisms that allow for a budget deficit.

With budget deficit permitted, the ratio of budget surplus to efficient surplus is bounded by λ and the absolute ratio of budget deficit to efficient surplus is bounded by μ:

$$\text{if } \Delta(c) > 0,\ 0 < \frac{\Delta(c)}{es(c)} \leq \lambda \ \text{ and } \ \text{if } \Delta(c) < 0,\ 0 < -\frac{\Delta(c)}{es(c)} \leq \mu.$$

This two-way worst case constraint is written as:

$$ps(c) - \lambda \cdot es(c) \leq \sum_{i \in N} r(c_{-i}) \leq ps(c) + \mu \cdot es(c) \ \text{ for all } c \in \mathcal{R}_+^N. \tag{3.2}$$

[10] Moulin (2010) proves a similar but more universal point for the case of a single bad. For $m = 1$, the unanimity upper bound test fails under the non-deficit constraint for general strategyproof mechanisms.

A pair of (λ, μ) is said to be *feasible* if it satisfies constraint (3.2) along with individual rationality. Let Λ be the set of all feasible pairs (λ, μ).

For two pairs (λ', μ') and (λ, μ) in Λ, if $\lambda' \geq \lambda$ with $\mu' > \mu$ holds or $\lambda' > \lambda$ with $\mu' \geq \mu$ holds, then (λ, μ) *dominates* (λ', μ'). When $\lambda' > \lambda$ and $\mu' > \mu$, (λ, μ) *strictly dominates* (λ', μ'). If a pair (λ^*, μ^*) in Λ is not dominated by any pairs in Λ, the pair is said to be ***optimal***. We denote the set of all optimal pairs by using $\partial\Lambda$ and call $\partial\Lambda$ the ***optimal frontier***. A VCG mechanism is said to be *optimal* if its redistribution scheme r^* generates an optimal pair (λ^*, μ^*) in $\partial\Lambda$.

With this new definition of optimality, we provide the optimal VCG mechanisms for $m = 1$ in Theorem 2.1.

Theorem 2.1 *For the case of $m = 1$, $n \geq 3$, there are two optimal anonymous and individually rational VCG mechanisms. One is the pivotal mechanism whose $\lambda^*_{n,1} = \infty$ and $\mu^*_{n,1} = 0$. For the other, $\mu^*_{n,1} = 1$ with $\lambda^*_{n,1} = 0$, and its linear redistribution scheme is $r^*_{n,1}(c_{-i}) = \frac{n-1}{n}(c_{-i})^{*1}$ for all $i \in N$.*

Corollary 1 *For the case of $m = 1$, $n \geq 3$, the pivotal mechanism is the optimal anonymous VCG mechanism that satisfies individual rationality and non-deficit.*

Remark 2 According to Theorem 2.1, there are only two extreme pairs of $\mu^*_{n,1}$ and $\lambda^*_{n,1}$ for the case of $m = 1$. The pivotal mechanism has infinite efficiency loss due strictly to surplus, and therefore generates no deficit. This phenomenon is unique for the case of $m = 1$, while there are infinitely many pairs of $\mu^*_{n,m}$ and $\lambda^*_{n,m}$ for $m \geq 2$. In addition, as the other optimal mechanism has $\mu^*_{n,1} = 1$ with $\lambda^*_{n,1} = 0$ (generating no surplus), its optimal efficiency loss due to deficit is relatively small, compared to the infinite efficiency loss due to surplus of the pivotal mechanism. This implies that by allowing deficit, we can save a great deal of efficiency loss. We will observe that this property holds true for $m \geq 2$ in the following Theorem 2.2 and Theorem 2.3:

Theorem 2.2 *For the case of* $m = 2$, $n \geq 3$, *the optimal frontier of any individually rational VCG mechanism is given as follows:*

$$\frac{\lambda^*_{n,2}}{A(n,2)} + \frac{\mu^*_{n,2}}{B(n,2)} = 1$$

where

$$A(n,2) = n-1 \quad \textit{and} \quad B(n,2) = \frac{\binom{n-1}{2}}{2^{n-2}-1}.$$

$B(n,2)$ *is strictly decreasing in* n *and* $B(n,2) \simeq \frac{n^2}{2^{n-1}}$.

Remark 3 The function $P(n,2) = B(n,2)/A(n,2)$ is strictly decreasing in n. As the number of agents increases, deficit becomes much more inexpensive than surplus. For instance $P(3,2) = 0.5$ implies that unit surplus loss can be replaced with 0.5 unit deficit loss when there are three agents. Computing $P(4,2) = 0.33$, $P(5,2) = 0.21$, and $P(6,2) = 0.13$, we observe that when more agents participate, the shrinking deficit loss can replace unit surplus loss.

Here we illustrate the optimal redistribution schemes corresponding to Theorem 2.2. If $\mu^*_{n,2} = 0$ (non-deficit), the optimal redistribution scheme is $r^*(c_{-i}) = \frac{n-2}{n}(c_{-i})^{*1}$. For the opposite case, $\lambda^*_{n,2} = 0$ (deficit only), the redistribution scheme of the optimal individually rational VCG mechanism is given as follows:

$$r^*(c_{-i}) = \sum_{k=1}^{6} \alpha^*_k (c_{-i})^{*k} + \sum_{k=7}^{n-1} \beta^*_k (c_{-i})^{*k}$$

where

$$\alpha_1^* = 0;\ \alpha_2^* = 1 - \frac{2\binom{n-1}{2}}{n(2^{n-2}-1)};\ \alpha_3^* = \frac{2\left[\binom{n-1}{2} - (2^{n-2}-1)\right]}{(2^{n-2}-1)(n-3)};$$
$$\alpha_4^* = -\frac{12\binom{n-1}{4} + 9\binom{n}{3} - 3n(2^{n-2}-1)}{n(2^{n-2}-1)\binom{n-3}{2}};$$
$$\alpha_5^* = \frac{4\left[\binom{n-1}{2} + \binom{n-1}{4} - (2^{n-2}-1)\right]}{(2^{n-2}-1)\binom{n-3}{3}};$$
$$\alpha_6^* = -\frac{2\binom{n-1}{2}}{(n-5)(2^{n-2}-1)} - \frac{2\binom{n-1}{2}\binom{n-1}{3}}{n(2^{n-2}-1)} - \frac{5\left[\binom{n-1}{2} - (2^{n-2}-1)\right]}{\binom{n-3}{4}(2^{n-2}-1)};$$

$$\beta_k^* = -\frac{2\binom{n-1}{2}\sum_{l=k}^{n-2}\binom{n-2}{l}}{(2^{n-2}-1)\binom{n-1}{k-1}(n-k)} \quad \textit{if } k \textit{ is odd;}$$
$$\beta_k^* = -\frac{2\binom{n-1}{2}}{(2^{n-2}-1)(n-k+1)} - \frac{2\binom{n-1}{2}\left[\frac{k}{n}\binom{n-1}{k-2} - \sum_{l=k-3}^{n-2}\binom{n-2}{l}\right]}{(2^{n-2}-1)\binom{n-1}{k-1}(n-k)} \quad \textit{if } k \textit{ is even.}$$

In addition, the optimal redistribution schemes for any $\mu_{n,2}^* > 0$ are provided in Appendix 2.5.2 (Lemma 4).

Theorem 2.3 *For* m, $3 \leq m \leq n-1$, *the optimal frontier of any anonymous and individually rational VCG mechanism is given as:*

$$\frac{\lambda_{n,m}^*}{A(n,m)} + \frac{\mu_{n,m}^*}{B(n,m)} = 1$$

where

$$A(n,m) = \frac{\binom{n-1}{m-1}}{\sum_{k=0}^{m-2}\binom{n-2}{k}} \simeq \frac{n}{m-1};$$
$$B(n,m) = \frac{\binom{n-1}{m-1}}{\sum_{k=0}^{m-3}\binom{n-2}{k} + \frac{m}{n-m}\sum_{k=m-1}^{n-2}\binom{n-2}{k}} \simeq \frac{n^m}{m!2^{n-2}}.$$

Remark 4 We conjecture that for a fixed m, the function $P(n, m) = \frac{B(n,m)}{A(n,m)}$ is strictly decreasing in n as is $P(n, 2)$. This implies that the more agents participate, the smaller deficit loss that results can replace unit surplus loss. Because $P(n, m) \simeq \frac{n^{m-1}}{m(m-2)!2^{n-2}}$, more participation enables this replacement to be effective: the deficit becomes much more inexpensive than surplus as the number of agents increases. This behavior is not present in the problem of assigning economic goods. As Moulin (2009) discusses, individual rationality does not affect the relationship between surplus loss and deficit loss. For the case of economic goods, unit surplus loss can be replaced with unit deficit loss regardless of individual rationality.

Remark 5 Recall that the optimal loss of the pivotal mechanism is $\frac{n}{m-1}$. $A(n, m) \simeq \frac{n}{m-1}$ in Theorem 2.3 tells us that the optimal mechanism converges to the pivotal mechanism if deficit is not allowed. Again, the efficiency loss of the pivotal mechanism increases as more agents participate and its implementation cost always exhausts the entirety of efficient surplus.

3.3 Conclusion

Contrary to expectations, individual rationality significantly changes the characteristics of optimal mechanisms when facing the problem of assigning bads. Additionally, we need to run further equity tests on our optimal mechanisms. Although we provide a partial answer in A1, a more systematic analysis of the relationship between different performance measures could raise interesting questions.

3.4 Appendices

3.4.1 Discussion

We will illustrate that the alternative performance measure in Guo and Conitzer (2009) does not work in the problem of assigning bads. According to Guo and Conitzer (2009), the index is defined as:

$$\eta_{n,m}(r) = \sup_{c \in \mathcal{R}_+^N} \frac{|\Delta(c,r)|}{ps(c)}.$$

The *optimal "GC" mechanism* is a VCG mechanism with a redistribution scheme r^* that generates $\eta^*_{n,m} = \eta_{n,m}(r^*) \leq \eta_{n,m}(r)$ for any redistribution scheme r. The following propositions show that this measure is inappropriate since its optimal mechanism cannot even achieve its original goal.

Proposition 3 below presents the optimal "GC" mechanism and the corresponding index. Proposition 4 proves that the "GC" optimality fails to achieve its original objective.

Proposition 3 *The optimal non-deficit linear "GC" mechanism has the index:*

$$\eta^*_{n,m} = \frac{\binom{n-1}{m-1}}{\sum_{j=0}^{m-1}\binom{n-1}{j}}.$$

If $m = 1$, the mechanism redistributes nothing. For $m \geq 2$, its redistribution scheme r is written as $r^*(c_{-i}) = \sum_{k=1}^{m-1} a^*_k (c_{-i})^{*k}$. If m is odd, $a^*_k = (-1)^k a_k$ and if m is even,

$a_k^* = (-1)^{k-1} a_k$. *Here we write:*

$$a_k = \frac{(n-m)\sum_{j=0}^{k-1}\binom{n-1}{j}}{k\binom{n-1}{k}}\eta_{n,m}^* = \frac{(n-m)\binom{n-1}{m-1}\sum_{j=0}^{k-1}\binom{n-1}{j}}{k\binom{n-1}{k}\sum_{j=0}^{m-1}\binom{n-1}{j}}.$$

Proof. The worst case constraint is as follows:

$$\eta_{n,m} \geq 1 - \frac{\sum_{i=1}^{n} r(i; c_{-i})}{(n-m)c^{*m}}.$$

The non-deficit and worst case constraints are written together as:

$$(n-m)c^{*m} \geq \sum_{i \in N} r(i; c_{-i}) \geq (1 - \eta_{n,m})(n-m)c^{*m}.$$

Again, the system of inequalities is symmetric across all variables, so we will construct a symmetric redistribution scheme $r(c_{-i})$. We can write:

$$\begin{aligned}\sum_{i \in N} r(c_{-i}) = na_0 + (n-1)a_1 \cdot c^{*1} + (a_1 + (n-2)a_2)c^{*2} + (2a_2 + (n-3)a_3)c^{*3} + \cdots \\ + ((n-3)a_{n-3} + 2a_{n-2})c^{*(n-2)} + ((n-2)a_{n-2} + a_{n-1})c^{*(n-1)} \\ + (n-1)a_{n-1} \cdot c^{*n}.\end{aligned}$$

Step 1: We first show that the non-deficit and worst case constraints imply $a_m = a_{m+1} = \cdots = a_{n-1} = 0$ and $a_0 = 0$. For cost profile $c^{*1} = c^{*2} = \cdots = c^{*n} = 0$, non-deficit and worst case constraints imply $0 \geq na_0 \geq 0$, that is, $a_0 = 0$. For a cost profile $c^{*1} = c^{*2} = \cdots = c^{*(n-1)} = 0$ and $c^{*n} = 1$, the two constraints imply $0 \geq (n-1)a_{n-1} \geq 0$, so $a_{n-1} = 0$. For a cost profile $c^{*1} = \cdots = c^{*(n-2)} = 0$ and

$c^{*(n-1)} = 1$, the constraints imply $0 \geq (n-2)a_{n-2} \geq 0$, so $a_{n-2} = 0$. Likewise, we can conclude that $a_m = \cdots = a_{n-1} = 0$.

Then, the non-deficit and worst case constraints are written as:

$$\begin{aligned}0 &\geq (n-1)a_1 \cdot c^{*1} + (a_1 + (n-2)a_2)c^{*2} + (2a_2 + (n-3)a_3)c^{*3} + \cdots \\ &+ ((m-2)a_{m-2} + (n-m+1)a_{m-1})c^{*(m-1)} + ((m-1)a_{m-1} + (n-m)(-1))c^{*m}\end{aligned}$$

and

$$\begin{aligned}0 &\leq (n-1)a_1 \cdot c^{*1} + (a_1 + (n-2)a_2)c^{*2} + (2a_2 + (n-3)a_3)c^{*3} + \cdots \\ &+ ((m-2)a_{m-2} + (n-m+1)a_{m-1})c^{*(m-1)} \\ &+ ((m-1)a_{m-1} + (n-m)(-1+\eta_{n,m}))c^{*m}.\end{aligned}$$

Applying Lemma 1, we transform the original optimization problem into a linear program. We aim to minimize $\eta_{n,m}$ satisfying the non-deficit and worst case constraints as follows:

$$\begin{aligned}(n-m) &\geq \quad (m-1)a_{m-1} \quad \geq (n-m)(1-\eta_{n,m}) \\ (n-m) &\geq (m-2)a_{m-2} + na_{m-1} \geq (n-m)(1-\eta_{n,m}) \\ (n-m) &\geq (m-3)a_{m-3} + n(a_{m-2} + a_{m-1}) \geq (n-m)(1-\eta_{n,m}) \\ &\vdots \\ (n-m) &\geq a_1 + n(a_2 + a_3 + \cdots + a_{m-1}) \geq (n-m)(1-\eta_{n,m}) \\ (n-m) &\geq n(a_1 + a_2 + a_3 + \cdots + a_{m-1}) \geq (n-m)(1-\eta_{n,m}).\end{aligned}$$

Step 2: Let a redistribution scheme $\hat{r}(c_{-i}) = \sum_{k=1}^{m-1} \hat{a}_k (c_{-i})^{*k}$ generate $\hat{\eta}_{n,m}$. Suppose that $\hat{\eta}_{n,m} \leq \eta^*_{n,m}$. Let $\hat{x}_k = \sum_{j=k}^{m-1} \hat{a}_j$ and let $x^*_k = \sum_{j=k}^{m-1} a^*_j$ for $k = 1, \cdots, m-1$. If m is odd, observe that

$$
\begin{aligned}
(m-1)x^*_{m-1} &= (n-m)(1-\eta^*_{n,m}) \\
(m-2)x^*_{m-2} + (n-m+2)x^*_{m-1} &= (n-m) \\
(m-3)x^*_{m-3} + (n-m+3)x^*_{m-2} &= (n-m)(1-\eta^*_{n,m}) \\
(m-4)x^*_{m-4} + (n-m+4)x^*_{m-3} &= (n-m) \\
&\vdots \\
2x^*_2 + (n-2)x{*}_3 &= (n-m)(1-\eta^*_{n,m}) \\
x^*_1 + (n-1)x^*_2 &= (n-m) \\
nx^*_1 &= (n-m)(1-\eta^*_{n,m}).
\end{aligned}
$$

Since the redistribution scheme $\hat{r}$ satisfies non-deficit and worst case constraints, we have $(m-1)\hat{x}_{m-1} \geq (n-m)(1-\hat{\eta}_{n,m})$. In addition, we have $\hat{\eta}_{n,m} \leq \eta^*_{n,m}$ and $(m-1)x^*_{m-1} = (n-m)(1-\eta^*_{n,m})$. We can then conclude $\hat{x}_{m-1} \geq x^*_{m-1}$. The constraints also give $(n-m) \geq (m-2)\hat{x}_{m-2} + (n-m+2)\hat{x}_{m-1}$, and the previous observation gives $(m-2)x^*_{m-2} + (n-m+2)x^*_{m-1} = (n-m)$. With $\hat{x}_{m-1} \geq x^*_{m-1}$, we conclude $\hat{x}_{m-2} \leq x^*_{m-2}$. Applying the same logic from the third to the $(m-1)$th constraints and observation, we know $\hat{x}_{m-3} \geq x^*_{m-3}$, $\hat{x}_{m-4} \leq x^*_{m-4}$, $\cdots$, $\hat{x}_1 \leq x^*_1$ (the direction of inequality is alternating). Finally, the m^{th} constraints give $n\hat{x}_1 \geq (n-m)(1-\hat{\eta}_{n,m})$ and the observation gives $(n-m)(1-\eta^*_{n,m}) = nx^*_1$, so $\hat{x}_1 \geq x^*_1$. Concluding $\hat{x}_1 = x^*_1$ and $\hat{\eta}_{n,m} = \eta^*_{n,m}$, we have $\hat{x}_i = x^*_i$ for $i = 1, \cdots, m-1$, and this implies that $\hat{a}_k = a^*_k$ for $k = 1, \cdots, m-1$. Therefore, $\eta^*_{n,m}$ is optimal, and r^* is a unique optimal redistribution scheme. ■

Proposition 4

(i) *For m fixed,* $\eta^*_{n,m}$ *increases in n and it converges to 1.*

(ii) *For n fixed,* $\eta^*_{n,m}$ *decreases in m.*

(iii) *For m fixed, the largest ratio of budget imbalance to efficient surplus (efficiency loss) of the optimal "GC" mechanism diverges in n if m is even:* $\eta^*_{n,m} \simeq n$ *and it is infinite if m is odd.*

Proof.

(i) We define $h(n) = \frac{\binom{n-1}{m-1}}{\sum_{j=0}^{m-1}\binom{n-1}{j}}$. For $k \geq m+1$, we have

$$h(k+1) - h(k) = \frac{\binom{k}{m-1}\sum_{j=0}^{m-1}\binom{k-1}{j} - \binom{k-1}{m-1}\sum_{j=0}^{m-1}\binom{k}{j}}{\sum_{j=0}^{m-1}\binom{k}{j}\cdot\sum_{j=0}^{m-1}\binom{k-1}{j}}$$
$$= \frac{\binom{k-1}{m-1}}{(k-m+1)\sum_{j=0}^{m-1}\binom{k}{j}\cdot\sum_{j=0}^{m-1}\binom{k-1}{j}}\sum_{j=0}^{m-1}(m-1-j)\binom{k}{j} > 0.$$

which implies that h is increasing in n. Finally, $\lim_{n\to\infty}\eta^*_{n,m} = 1$. This is because $\binom{n-1}{m-1} \simeq \frac{n^{m-1}}{(m-1)!}$ and $\sum_{j=0}^{m-1}\binom{n-1}{j} \simeq \frac{n^{m-1}}{(m-1)!}$.

(ii) Proposition 3 states that the optimal "GC" mechanism generates $\eta^*_{n,m} = \frac{\binom{n-1}{m-1}}{\sum_{j=0}^{m-1}\binom{n-1}{j}} = 1 - \frac{\sum_{j=0}^{m-2}\binom{n-1}{j}}{\sum_{j=0}^{m-1}\binom{n-1}{j}}$. We fix n and define $l(m) = \frac{\sum_{j=0}^{m-2}\binom{n-1}{j}}{\sum_{j=0}^{m-1}\binom{n-1}{j}}$ for $2 \leq m \leq n-1$. $l(2) = \frac{1}{n}$ and $l(3) = \frac{2n}{n^2-n+2}$ which leads $l(3)-l(2) = \frac{n(n+1)-2}{n(n(n-1)+2)} > 0$. For $2 \leq k \leq n-2$, we have

$$\begin{aligned}
l(k+1)-l(k) =&\left(\sum_{j=0}^{k-1}\binom{n-1}{j}\right)^2-\left(\sum_{j=0}^{k}\binom{n-1}{j}\right)\left(\sum_{j=0}^{k-2}\binom{n-1}{j}\right)\\
=&\left(\sum_{j=0}^{k-1}\binom{n-1}{j}\right)\left(-\binom{n-1}{k}+\binom{n-1}{k-1}\right)+\binom{n-1}{k}\binom{n-1}{k-1}\\
=&\frac{(n-1)!}{(k-1)!(n-k-1)!}\left(-\frac{1}{k}+\frac{1}{n-k}\right)\left(\sum_{j=0}^{k-1}\binom{n-1}{j}\right)\\
&+\binom{n-1}{k}\binom{n-1}{k-1}\\
=&\frac{(n-1)!}{k!(n-k)!}\left((-n+2k)\sum_{j=0}^{k-1}\binom{n-1}{j}+k\binom{n-1}{k}\right)\\
=&\frac{(n-1)!}{k!(n-k)!}\left(k\sum_{j=0}^{k}\binom{n-1}{j}+(-n+k)\sum_{j=0}^{k-1}\binom{n-1}{j}\right).
\end{aligned}$$

Define $L(k)=k\sum_{j=0}^{k}\binom{n-1}{j}+(-n+k)\sum_{j=0}^{k-1}\binom{n-1}{j}$. Then, we have

$$\begin{aligned}
L(k+1)-L(k) =&(k+1)\sum_{j=0}^{k+1}\binom{n-1}{j}+(-n+k+1)\sum_{j=0}^{k}\binom{n-1}{j}\\
&-k\sum_{j=0}^{k}\binom{n-1}{j}-(-n+k)\sum_{j=0}^{k-1}\binom{n-1}{j}\\
=&k\binom{n-1}{k+1}+\sum_{j=0}^{k+1}\binom{n-1}{j}+\binom{n-1}{k}(-n+k)+\sum_{j=0}^{k}\binom{n-1}{j}\\
=&-\binom{n-1}{k}+2\sum_{j=0}^{k}\binom{n-1}{j}>0
\end{aligned}$$

and thus, $L(k)$ is increasing in k. With $L(2)>0$, $L(k)$ is positive for any $k\geq 2$, so $l(k)$ increases in k. Therefore, $\eta^*_{n,m}$ decreases in m.

(iii) Suppose m is even. Then, $l\cdot a_l+(n-l-1)\cdot a_{l+1}=\frac{(-1)^l(n-m)\binom{n-1}{m-1}}{\sum_{j=0}^{m-1}\binom{n-1}{j}}$ for

$0 \leq l \leq m-2$ and $(m-1)a_{m-1} = (n-m)\frac{\sum_{j=0}^{m-2}\binom{n-1}{j}}{\sum_{j=0}^{m-1}\binom{n-1}{j}}$. Since we have

$$\sum_{i \in N} r^*(c_{-i}) = \sum_{l=0}^{m-2} \frac{(-1)^l (n-m)\binom{n-1}{m-1}}{\sum_{j=0}^{m-1}\binom{n-1}{j}} c^{*(l+1)} + (n-m)\frac{\sum_{j=0}^{m-2}\binom{n-1}{j}}{\sum_{j=0}^{m-1}\binom{n-1}{j}} c^{*m},$$

the efficiency loss of the optimal "GC" mechanism is

$$\lambda = \frac{(n-m)\binom{n-1}{m-1}}{\sum_{j=0}^{m-1}\binom{n-1}{j}} \cdot \sup_{c \in \mathcal{R}_+^N} \frac{n\left[\sum_{\substack{k=2 \\ even}}^{m} c^{*k} - \sum_{\substack{k=1 \\ odd}}^{m-1} c^{*k}\right]}{m\sum_{k=m+1}^{n} c^{*k} - (n-m)\sum_{k=1}^{m} c^{*k}}.$$

Observing that the ratio increases as $c^{*(m+1)}, c^{*(m+2)}, \cdots, c^{*n}$ decrease, we write

$$\begin{aligned}
&\sup_{c \in \mathcal{R}_+^N} \frac{n\left[\sum_{\substack{k=2 \\ even}}^{m} c^{*k} - \sum_{\substack{k=1 \\ odd}}^{m-1} c^{*k}\right]}{m\sum_{k=m+1}^{n} c^{*k} - (n-m)\sum_{k=1}^{m} c^{*k}} \\
&= \sup_{c \in \mathcal{R}_+^N} \frac{n\left[c^{*m} + \sum_{\substack{k=2 \\ even}}^{m-2} c^{*k} - \sum_{\substack{k=1 \\ odd}}^{m-1} c^{*k}\right]}{(n-m)\left[(m-1)c^{*m} - \sum_{\substack{k=1 \\ odd}}^{m-1} c^{*k} - \sum_{\substack{k=2 \\ even}}^{m-2} c^{*k}\right]}.
\end{aligned}$$

Notice that given $c^{*1}, c^{*3}, \cdots, c^{*(m-1)}$, the ratio increases as $c^{*2}, c^{*4}, \cdots, c^{*(m-2)}$ increase. Thus, the expression is written as

$$\begin{aligned}
&\sup_{c \in \mathcal{R}_+^N} \frac{n\left[c^{*m} + \sum_{\substack{k=3 \\ odd}}^{m-1} c^{*k} - \sum_{\substack{k=1 \\ odd}}^{m-1} c^{*k}\right]}{(n-m)\left[(m-1)c^{*m} - \sum_{\substack{k=1 \\ odd}}^{m-1} c^{*k} - \sum_{\substack{k=3 \\ odd}}^{m-1} c^{*k}\right]} \\
&= \sup_{c \in \mathcal{R}_+^N} \frac{n(c^{*m} - c^{*1})}{(n-m)\left[(m-1)c^{*m} - c^{*1} - 2\sum_{\substack{k=3 \\ odd}}^{m-1} c^{*k}\right]} \\
&= \sup_{c \in \mathcal{R}_+^N} \frac{n(c^{*m} - c^{*1})}{(n-m)\left[(m-1)c^{*m} - c^{*1} - (m-2)c^{*m}\right]} = \frac{n}{n-m}.
\end{aligned}$$

The second last equality holds since the ratio increases as $c^{*3}, c^{*5}, \cdots, c^{*(m-1)}$ increase. Thus, $\lambda = \frac{n\binom{n-1}{m-1}}{\sum_{j=0}^{m-1}\binom{n-1}{j}}$. We know that $n\binom{n-1}{m-1} \simeq \frac{n^m}{(m-1)!}$ and $\sum_{j=0}^{m-1}\binom{n-1}{j} \simeq \frac{n^{m-1}}{(m-1)!}$.

Therefore, $\lambda \simeq n$ if m is even. Similarly, if m is odd, $m \geq 3$,

$$\lambda = \frac{(n-m)}{\sum_{j=0}^{m-1}\binom{n-1}{j}} \sup_{c \in \mathcal{R}_+^N} \frac{n\Big\{\big[\binom{n-1}{m-1} + 2\sum_{j=0}^{m-2}\binom{n-1}{k}\big]c^{*m} + \binom{n-1}{m-1}\big[\sum_{\substack{k=1\\ odd}}^{m-2} c^{*k} - \sum_{\substack{k=2\\ even}}^{m-1} c^{*k}\big]\Big\}}{m\sum_{k=m+1}^{n} c^{*k} - (n-m)\sum_{k=1}^{m} c^{*k}}$$

$$= \frac{(n-m)}{\sum_{j=0}^{m-1}\binom{n-1}{j}} \sup_{c \in \mathcal{R}_+^N} \frac{n\Big\{\big[\binom{n-1}{m-1} + 2\sum_{j=0}^{m-2}\binom{n-1}{k}\big]c^{*m} + \binom{n-1}{m-1}\big[\sum_{\substack{k=1\\ odd}}^{m-2} c^{*k} - \sum_{\substack{k=2\\ even}}^{m-1} c^{*k}\big]\Big\}}{(n-m)\big[(m-1)c^{*m} - \sum_{\substack{k=1\\ odd}}^{m-2} c^{*k} - \sum_{\substack{k=2\\ even}}^{m-1} c^{*k}\big]}.$$

The last equality holds since the ratio increases as $c^{*(m+1)}, \cdots, c^{*n}$ decrease. Observing that the ratio increase as $c^{*1}, c^{*3}, \cdots, c^{*(m-2)}$ increase, we write

$$\lambda = \frac{(n-m)}{\sum_{j=0}^{m-1}\binom{n-1}{j}} \sup_{c \in \mathcal{R}_+^N} \frac{n\big[\binom{n-1}{m-1} + 2\sum_{j=0}^{m-2}\binom{n-1}{j}\big]c^{*m}}{(n-m)\big[(m-1)c^{*m} - 2\sum_{\substack{k=2\\ even}}^{m-1} c^{*k}\big]}$$

$$= \frac{(n-m)}{\sum_{j=0}^{m-1}\binom{n-1}{j}} \sup_{c \in \mathcal{R}_+^N} \frac{n\big[\binom{n-1}{m-1} + 2\sum_{j=0}^{m-2}\binom{n-1}{j}\big]c^{*m}}{(n-m)\big[(m-1)c^{*m} - (m-1)c^{*m}\big]} = \infty.$$

The second last equality holds since the ratio increases as $c^{*2}, c^{*4}, \cdots, c^{*(m-1)}$ increase. If $m = 1$, we know the pivotal is optimal and its efficiency loss is infinite. ■

The statement (i) points out that the pivotal mechanism becomes optimal as the number of agents increases. The optimal mechanism fails to redistribute any of the budget surplus of the pivotal mechanism. The statement (iii) shows that the efficiency loss of the optimal "GC" mechanism diverges in n or is infinite. Therefore, throughout this paper, we insist that we measure the performance of a mechanism by a worst case ratio whose denominator is efficient surplus.

3.4.2 Proofs

We will use notations as follows:

$$B_s^{t,t'} = \sum_{k=t}^{t'} \binom{s}{k}, \quad B_s^t = B_s^{t,s}, \quad B_s^{\to t} = B_s^{0,t}$$

$$\sum_{\substack{j=2\\ even}} x_k = x_2 + x_4 + \cdots \quad \text{and} \quad \sum_{\substack{j=1\\ odd}} x_k = x_1 + x_3 + \cdots .$$

Lemma 1.

(i) $b_1c_1 + \cdots + b_nc_n \leq 0$ *for* $0 \leq c_1 \leq \cdots \leq c_n$ *if and only if* $\sum_{j=k}^n b_j \leq 0$ *for* $k = 1, \cdots, n$.

(ii) $b_1c_1 + \cdots + b_nc_n \geq 0$ *for* $0 \leq c_1 \leq \cdots \leq c_n$ *if and only if* $\sum_{j=k}^n b_j \geq 0$ *for* $k = 1, \cdots, n$.

Proof. (i) Let $d_1 = c_1$, $d_2 = c_2 - c_1$, $\cdots$, $d_n = c_n - c_{n-1}$. Then, $b_1c_1 + \cdots + b_nc_n \leq 0$ for $0 \leq c_1 \leq \cdots \leq c_n$ if and only if $b_nd_n + (\sum_{j=n-1}^n b_j)d_{n-1} + (\sum_{j=n-2}^n b_j)d_{n-2} + \cdots + (\sum_{j=1}^n b_j)d_1 \leq 0$ for all $d_i \geq 0$, $i \in N$. Setting for each $i \in N$, $d_i = 1$ and $d_j = 0$ for all $j \in N$, $j \neq i$, we have the statement proven. (ii) can be proven in the same way.
■

Proof of Theorem 1.2

Statement (i)

Firstly, we will show the statement for the case of m, $4 \leq m \leq n-2$.

<u>Case 1</u>. *m is odd:*

The worst case constraint is as follows:

$$\lambda \geq \frac{(n-m)c^{*m} - \sum_{i=1}^{n} r(i; c_{-i})}{\frac{m}{n}\sum_{i=1}^{n} c_i - \sum_{i=1}^{m} c^{*i}}.$$

The non-deficit and worst case constraints are characterized by a system of linear inequalities as follows:

$$(n-m)c^{*m} \geq \sum_{i=1}^{n} r(i; c_{-i}) \geq (n-m)c^{*m} - \lambda\left(\frac{m}{n}\sum_{i=1}^{n} c_i - \sum_{i=1}^{m} c^{*i}\right).$$

In the inequalities above, both sides of $\sum_{i=1}^{n} r(i; c_{-i})$ is symmetric in all variables. If every $r(i; c_{-i})$ for $i \in N$ satisfies all inequalities, we can construct a symmetric scheme $\bar{r}$ meeting the inequalities. The symmetric scheme is written as $\bar{r}(c_{-i}) = \frac{1}{n!}\sum_{i \in N, \pi \in \Pi} r(i; c_{-i}^{\pi})$ where Π is the set of all permutations of $N \setminus \{i\}$ and c_{-i}^{π} results from permuting the coordinates of c_{-i} accordingly. Therefore, it is natural to restrict our discussion to symmetric redistribution schemes. $r(i; c_{-i})$ will be denoted by $r(c_{-i})$ from now on.

Let $e^{n-k} = (0, 0, \cdots, 0, 1, 1, \cdots, 1)$, $e^{n-k} \in \mathcal{R}^n$ for $k = 0, 1, \cdots, n$ where $\sum_{i=1}^{n} (e^{n-k})_i = n - k$. Let $\epsilon^{n-1-k} = (0, 0, \cdots, 0, 1, 1, \cdots, 1)$, $\epsilon^{n-1-k} \in \mathcal{R}^{n-1}$ for $k = 0, 1, \cdots, n-1$ where $\sum_{i=1}^{n-1} (\epsilon^{n-1-k})_i = n - 1 - k$. Define $\rho_k = r(\epsilon^{n-1-k})$ for $k = 0, \cdots, n-1$. The set $\{e^0, e^1, \cdots, e^n\}$ is a basis of C, $C = \{c \in R_{+}^{n} \mid c_1 \leq c_2 \leq \cdots \leq c_n\}$. Each $c^* \in C$ is uniquely written as a linear combination of elements of the basis.

Since $c_{-i} = (\epsilon^{n-1} - \epsilon^{n-2})(c_{-i})^{*1} + (\epsilon^{n-2} - \epsilon^{n-3})(c_{-i})^{*2} + \cdots + (\epsilon^3 - \epsilon^2)(c_{-i})^{*(n-3)} + (\epsilon^2 - \epsilon^1)(c_{-i})^{*(n-2)} + \epsilon^1 (c_{-i})^{*(n-1)}$, the redistribution scheme is written as $r(c_{-i}) = (\rho_0 - \rho_1)(c_{-i})^{*1} + (\rho_1 - \rho_2)(c_{-i})^{*2} + \cdots + (\rho_{n-3} - \rho_{n-2})(c_{-i})^{*(n-2)} + \rho_{n-2}(c_{-i})^{*(n-1)}$.

Recall that $ps(c) = (n-m)c^{*m}$ and $es(c) = \frac{m}{n}\sum_{i \in N} c_i - \sum_{i=1}^{m} c^{*i}$. For a cost profile e^{n-k}, we notice that if $0 \leq k \leq m-1$, $es(c) = \frac{k}{n}(n-m)$ and $ps(c) = n-m$ and if $m \leq k \leq n$, $es(c) = \frac{m}{n}(n-k)$ and $ps(c) = 0$.

Now we will apply e^{n-k} for various k's. When $k = 0$, the non-deficit and worst case constraints are written as $n-m \leq n\rho_0 \leq n-m$, so $\rho_0 = \frac{n-m}{n}$. When $k = n$, the two constrains are written as $0 \leq n\rho_{n-1} \leq 0$, so $\rho_{n-1} = 0$. Applying e^{n-k} for other k, $1 \leq k \leq n-1$, the non-deficit and worst case constraints are written as follows:

$$
\begin{aligned}
(n-m)\left(1-\frac{1}{n}-\frac{\lambda}{n}\right) &\leq (n-1)\rho_1 \leq (n-m)\left(1-\frac{1}{n}\right) \\
(n-m)\left(1-\frac{2}{n}\lambda\right) &\leq 2\rho_1 + (n-2)\rho_2 \leq n-m \\
(n-m)\left(1-\frac{3}{n}\lambda\right) &\leq 3\rho_2 + (n-3)\rho_3 \leq n-m \\
&\vdots \\
(n-m)\left(1-\frac{m-1}{n}\lambda\right) &\leq (m-1)\rho_{m-2} + (n-m+1)\rho_{m-1} \leq n-m \\
-\frac{m(n-m)}{n}\lambda &\leq m\rho_{m-1} + (n-m)\rho_m \leq 0 \\
-\frac{m(n-m-1)}{n}\lambda &\leq (m+1)\rho_m + (n-m-1)\rho_{m+1} \leq 0 \\
&\vdots \\
-\frac{2m}{n}\lambda &\leq (n-2)\rho_{n-3} + 2\rho_{n-2} \leq 0 \\
-\frac{m}{n}\lambda &\leq (n-1)\rho_{n-2} \leq 0.
\end{aligned}
$$

We will use the notations M and ρ as follows:

$$M = \begin{pmatrix} (n-1) & 0 & 0 & \cdots & 0 & 0 \\ 2 & (n-2) & 0 & \cdots & 0 & 0 \\ 0 & 3 & (n-3) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & 0 \\ 0 & 0 & 0 & \cdots & (n-2) & 2 \\ 0 & 0 & 0 & \cdots & 0 & (n-1) \end{pmatrix} \text{ and } \rho = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{n-3} \\ \rho_{n-2} \end{pmatrix}.$$

Then, M is a $(n-1) \times (n-2)$ matrix and $\rho \in \mathcal{R}^{n-2}$. Using the notations of M and ρ, the central part of above inequalities is written as the follows: $(M\rho)_1 = (n-1)\rho_1$, $(M\rho)_2 = 2\rho_1 + (n-2)\rho_2$, $(M\rho)_3 = 3\rho_2 + (n-3)\rho_3$, $\cdots$, $(M\rho)_{n-2} = (n-2)\rho_{n-3} + 2\rho_{n-2}$ and $(M\rho)_{n-1} = (n-1)\rho_{n-2}$.

By computing the null space of the transposed M, we find the hyperplane of $\mathcal{R}^{n-1}$ as the range of M. For X in the range of M, the hyperplane is presented as

$$\binom{n}{1}X_1 + \binom{n}{3}X_3 + \cdots = \binom{n}{2}X_2 + \binom{n}{4}X_4 + \cdots$$

and the last term $\binom{n}{n-1}X_{n-1}$ appears in either side depending on whether n is odd or even. The no deficit and worst case constraints imply that

$$(n-m)\left(1 - \frac{1}{n} - \frac{\lambda}{n}\right) \le X_1 \le (n-m)\left(1 - \frac{1}{n}\right),$$

and

$$(n-m)\left(1 - \frac{k}{n}\lambda\right) \le X_k \le n - m$$

for $2 \leq k \leq m-1$. And

$$-\lambda \frac{m(n-k)}{n} \leq X_k \leq 0$$

for $m \leq k \leq n-1$.

When n is odd, the non-deficit and worst case constraints imply that

$$(n-m)\left[\sum_{\substack{k=1\\ odd}}^{m-2} \binom{n}{k} - 1\right] \geq$$

$$\binom{n}{1} X_1 + \sum_{\substack{k=3\\ odd}}^{m-2} \binom{n}{k} X_k + \sum_{\substack{k=m\\ odd}}^{n-2} \binom{n}{k} X_k = \sum_{\substack{k=2\\ even}}^{m-1} \binom{n}{k} X_k + \sum_{\substack{k=m+1\\ even}}^{n-1} \binom{n}{k} X_k$$

$$\geq (n-m) \sum_{\substack{k=2\\ even}}^{m-1} \binom{n}{k} \left(1 - \frac{k}{n}\lambda\right) - \lambda \sum_{\substack{k=m+1\\ even}}^{n-1} \binom{n}{k} \frac{m}{n}(n-k).$$

Then, we have

$$\lambda \geq \frac{(n-m)\left[\sum_{\substack{k=0\\ even}}^{m-1} \binom{n}{k} - \sum_{\substack{k=1\\ odd}}^{m-2} \binom{n}{k}\right]}{(n-m)\sum_{\substack{k=2\\ even}}^{m-1} \frac{k}{n}\binom{n}{k} + \sum_{\substack{k=m+1\\ even}}^{n-1} \frac{m}{n}(n-k)\binom{n}{k}}.$$

Likewise, when n is even, the non-deficit and worst case constraints imply that

$$(n-m)\left[\sum_{\substack{k=1\\ odd}}^{m-2} \binom{n}{k} - 1\right] \geq$$

$$\binom{n}{1} X_1 + \sum_{\substack{k=3\\ odd}}^{m-2} \binom{n}{k} X_k + \sum_{\substack{k=m\\ odd}}^{n-1} \binom{n}{k} X_k = \sum_{\substack{k=2\\ even}}^{m-1} \binom{n}{k} X_k + \sum_{\substack{k=m+1\\ even}}^{n-2} \binom{n}{k} X_k$$

$$\geq (n-m) \sum_{\substack{k=2\\ even}}^{m-1} \binom{n}{k} \left(1 - \frac{k}{n}\lambda\right) - \lambda \sum_{\substack{k=m+1\\ even}}^{n-2} \binom{n}{k} \frac{m}{n}(n-k).$$

Then, we have

$$\lambda \geq \frac{(n-m)\left[\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k} - \sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\right]}{(n-m)\sum_{\substack{k=2\\ even}}^{m-1}\frac{k}{n}\binom{n}{k} + \sum_{\substack{k=m+1\\ even}}^{n-2}\frac{m}{n}(n-k)\binom{n}{k}}.$$

The optimal efficiency loss is written as follows:

$$\lambda^*_{n,m} = \frac{(n-m)\left[\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k} - \sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\right]}{(n-m)\sum_{\substack{k=2\\ even}}^{m-1}\frac{k}{n}\binom{n}{k} + \sum_{\substack{k=m+1\\ even}}^{\tilde{n}}\frac{m}{n}(n-k)\binom{n}{k}}.$$

From $\sum_{j=0}^{k}(-1)^j\binom{n}{j} = (-1)^k\binom{n-1}{k}$ for k, $0 \leq k \leq n-1$, we write

$$\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k} - \sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k} = \sum_{k=0}^{m-1}(-1)^k\binom{n}{k} = (-1)^{m-1}\binom{n-1}{m-1}.$$

From $k\binom{n}{k} = n\binom{n-1}{k-1}$, we write

$$\begin{aligned}
\sum_{\substack{k=2\\ even}}^{m-1} k\binom{n}{k} &= n\sum_{\substack{k=2\\ even}}^{m-1}\binom{n-1}{k-1} = n\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n-1}{k} \\
&= n\sum_{\substack{k=1\\ odd}}^{m-2}\left[\binom{n-2}{k} + \binom{n-2}{k-1}\right] = n\left[\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n-2}{k} + \sum_{\substack{k=0\\ even}}^{m-3}\binom{n-2}{k}\right] \\
&= n\sum_{k=0}^{m-2}\binom{n-2}{k}
\end{aligned}$$

and

$$\sum_{\substack{k=m+1\\ even}}^{\tilde{n}} \frac{m}{n}(n-k)\binom{n}{k} = \frac{m}{n}\Bigg[n\sum_{\substack{k=m+1\\ even}}^{\tilde{n}}\binom{n}{k} - \sum_{\substack{k=m+1\\ even}}^{\tilde{n}} k\binom{n}{k}\Bigg]$$

$$= m\Bigg[\sum_{\substack{k=m+1\\ even}}^{\tilde{n}}\binom{n}{k} - \sum_{\substack{k=m+1\\ even}}^{\tilde{n}}\binom{n-1}{k-1}\Bigg]$$

$$= m\Bigg[\sum_{\substack{k=m+1\\ even}}^{\tilde{n}}\binom{n}{k} - \sum_{\substack{k=m\\ odd}}^{\tilde{n}-1}\binom{n-1}{k}\Bigg]$$

$$= m\Bigg\{\sum_{\substack{k=m+1\\ even}}^{\tilde{n}}\Bigg[\binom{n-1}{k} + \binom{n-1}{k-1}\Bigg]$$

$$- \sum_{\substack{k=m\\ odd}}^{\tilde{n}-1}\Bigg[\binom{n-2}{k} + \binom{n-2}{k-1}\Bigg]\Bigg\}$$

$$= m\Bigg[\sum_{k=m}^{\tilde{n}}\binom{n-1}{k} - \sum_{k=m-1}^{\tilde{n}-1}\binom{n-2}{k}\Bigg].$$

Therefore, the optimal efficiency loss is written as

$$\lambda^*_{n,m} = \frac{(n-m)\binom{n-1}{m-1}}{(n-m)\sum_{k=0}^{m-2}\binom{n-2}{k} + m\big[\sum_{k=m}^{\tilde{n}}\binom{n-1}{k} - \sum_{k=m-1}^{\tilde{n}-1}\binom{n-2}{k}\big]}.$$

<u>Case 2</u>. *m is even:*

When n is odd, the non-deficit and worst case constraints imply that

$$n(n-m)\left(1-\frac{1}{n}-\frac{\lambda}{n}\right)+(n-m)\sum_{\substack{k=3\\ odd}}^{m-1}\binom{n}{k}\left(1-\frac{k}{n}\lambda\right)-\lambda\sum_{\substack{k=m+1\\ odd}}^{n-2}\binom{n}{k}\frac{m}{n}(n-k)$$

$$\leq\binom{n}{1}X_1+\sum_{\substack{k=3\\ odd}}^{m-1}\binom{n}{k}X_k+\sum_{\substack{k=m+1\\ odd}}^{n-2}\binom{n}{k}X_k=\sum_{\substack{k=2\\ even}}^{m-2}\binom{n}{k}X_k+\sum_{\substack{k=m\\ even}}^{n-1}\binom{n}{k}X_k$$

$$\leq(n-m)\sum_{\substack{k=2\\ even}}^{m-2}\binom{n}{k}.$$

Then, we have

$$\lambda\geq\frac{(n-m)\left[\sum_{\substack{k=1\\ odd}}^{m-1}\binom{n}{k}-\sum_{\substack{k=0\\ even}}^{m-2}\binom{n}{k}\right]}{(n-m)\sum_{\substack{k=1\\ odd}}^{m-1}\frac{k}{n}\binom{n}{k}+\sum_{\substack{k=m+1\\ odd}}^{n-2}\frac{m}{n}(n-k)\binom{n}{k}}.$$

Likewise, when n is even, the non-deficit and worst case constraints imply that

$$n(n-m)\left(1-\frac{1}{n}-\frac{\lambda}{n}\right)+(n-m)\sum_{\substack{k=3\\ odd}}^{m-1}\binom{n}{k}\left(1-\frac{k}{n}\lambda\right)-\lambda\sum_{\substack{k=m+1\\ odd}}^{n-1}\binom{n}{k}\frac{m}{n}(n-k)$$

$$\leq\binom{n}{1}X_1+\sum_{\substack{k=3\\ odd}}^{m-1}\binom{n}{k}X_k+\sum_{\substack{k=m+1\\ odd}}^{n-1}\binom{n}{k}X_k=\sum_{\substack{k=2\\ even}}^{m-2}\binom{n}{k}X_k+\sum_{\substack{k=m\\ even}}^{n-2}\binom{n}{k}X_k$$

$$\leq(n-m)\sum_{\substack{k=2\\ even}}^{m-2}\binom{n}{k}.$$

Then, we have

$$\lambda\geq\frac{(n-m)\left[\sum_{\substack{k=1\\ odd}}^{m-1}\binom{n}{k}-\sum_{\substack{k=0\\ even}}^{m-2}\binom{n}{k}\right]}{(n-m)\sum_{\substack{k=1\\ odd}}^{m-1}\frac{k}{n}\binom{n}{k}+\sum_{\substack{k=m+1\\ odd}}^{n-1}\frac{m}{n}(n-k)\binom{n}{k}}.$$

The optimal efficiency loss is written as follows:

$$\lambda^*_{n,m} = \frac{(n-m)\left[\sum_{\substack{k=1\\odd}}^{m-1}\binom{n}{k} - \sum_{\substack{k=0\\even}}^{m-2}\binom{n}{k}\right]}{(n-m)\sum_{\substack{k=1\\odd}}^{m-1}\frac{k}{n}\binom{n}{k} + \sum_{\substack{k=m+1\\odd}}^{\bar{n}}\frac{m}{n}(n-k)\binom{n}{k}}.$$

We write $\sum_{\substack{k=1\\odd}}^{m-1}\binom{n}{k} - \sum_{\substack{k=0\\even}}^{m-2}\binom{n}{k} = -\sum_{k=0}^{m-1}(-1)^k\binom{n}{k} = (-1)^m\binom{n-1}{m-1} = \binom{n-1}{m-1}$ since m is even. And we write

$$\begin{aligned}\frac{n-m}{n}\sum_{\substack{k=1\\odd}}^{m-1}k\binom{n}{k} &= (n-m)\sum_{\substack{k=1\\odd}}^{m-1}\binom{n-1}{k-1} = (n-m)\sum_{\substack{k=0\\even}}^{m-2}\binom{n-1}{k}\\ &= (n-m)\left[1+\sum_{\substack{k=2\\even}}^{m-2}\left[\binom{n-2}{k}+\binom{n-2}{k-1}\right]\right] = (n-m)\sum_{k=0}^{m-2}\binom{n-2}{k}\end{aligned}$$

and

$$\begin{aligned}\frac{m}{n}\sum_{\substack{k=m+1\\odd}}^{\bar{n}}(n-k)\binom{n}{k} &= m\left[\sum_{\substack{k=m+1\\odd}}^{\bar{n}}\binom{n}{k} - \sum_{\substack{k=m+1\\odd}}^{\bar{n}}\binom{n-1}{k-1}\right]\\ &= m\left[\sum_{\substack{k=m+1\\odd}}^{\bar{n}}\binom{n}{k} - \sum_{\substack{k=m\\even}}^{\bar{n}-1}\binom{n-1}{k}\right]\\ &= m\left\{\sum_{\substack{m+1\\odd}}^{\bar{n}}\left[\binom{n-1}{k}+\binom{n-1}{k-1}\right] - \sum_{\substack{k=m\\even}}^{\bar{n}-1}\left[\binom{n-2}{k}+\binom{n-2}{k-1}\right]\right\}\\ &= m\left[\sum_{k=m}^{\bar{n}}\binom{n-1}{k} - \sum_{k=m-1}^{\bar{n}-1}\binom{n-2}{k}\right].\end{aligned}$$

Therefore, the optimal efficiency loss is written as

$$\lambda^*_{n,m} = \frac{(n-m)\binom{n-1}{m-1}}{(n-m)\sum_{k=0}^{m-2}\binom{n-2}{k} + m\left[\sum_{k=m}^{\bar{n}}\binom{n-1}{k} - \sum_{k=m-1}^{\bar{n}-1}\binom{n-2}{k}\right]}.$$

We rewrite the optimal efficiency loss as follows:

$$\lambda^*_{n,m} = \frac{(n-m)\binom{n-1}{m-1}}{(n-m)\sum_{k=0}^{m-2}\binom{n-2}{k} + m\big[\sum_{k=m}^{\ddot{n}}\binom{n-1}{k} - \sum_{k=m-1}^{\ddot{n}-1}\binom{n-2}{k}\big]}$$

where $\ddot{n} = \tilde{n}$ if m is odd and $\ddot{n} = \bar{n}$ if m is even. $\tilde{n} = n-1$ if n is odd and $\tilde{n} = n-2$ if n is even. $\bar{n} = n-2$ if n is odd and $\bar{n} = n-1$ if n is even.

<u>*Case 3*</u>*. $m = 2$:* the non-deficit and worst case constraints are characterized by

$$(n-2)c^{*2} \geq \sum_{i\in N} r(c_{-i}) \geq (n-2)c^{*2} - \lambda\big[\frac{2}{n}\sum_{i\in N} c_i - c^{*1} - c^{*2}\big].$$

We apply e^{n-k} to the system above. Again $\rho_0 = \frac{n-2}{n}$ and $\rho_{n-1} = 0$. For $k = 1$,

$$(n-2)(1-\frac{1}{n}) \geq (n-1)\rho_1 \geq (n-2)\big(1-\frac{1}{n}-\frac{\lambda}{n}\big)$$

and for $n-1 \geq k \geq 2$,

$$0 \geq k\rho_{k-1} + (n-k)\rho_k \geq -\lambda\cdot\frac{2}{n}(n-k).$$

Then, we have

$$(n-2)(n-1-\lambda) - \lambda\sum_{\substack{k=3\\ odd}}\binom{n}{k}(n-k)\frac{2}{n} \leq 0$$

and find the optimal loss.

Case 4. $m = 3$: the non-deficit and worst constraints are written as

$$(n-3)c^{*3} \geq \sum_{i \in N} r(c_{-i}) \geq (n-3)c^{*3} - \lambda\left[\frac{3}{n}\sum_{i \in N} c_i - c^{*1} - c^{*2} - c^{*3}\right].$$

Applying e^{n-k} for $0 \leq k \leq n$, we have $\rho_0 = \frac{n-3}{n}$, $\rho_{n-1} = 0$ and for $k = 1$,

$$(n-3)\left(1 - \frac{1}{n}\right) \geq (n-1)\rho_1 \geq (n-3)\left(1 - \frac{1}{n}\right) - \lambda\left(\frac{3}{n}(n-1) - 2\right)$$

for $k = 2$,

$$(n-3) \geq 2\rho_1 + (n-2)\rho_2 \geq (n-3) - \lambda\left(\frac{3}{n}(n-2) - 1\right)$$

and for $n - 1 \geq k \geq 3$,

$$0 \geq k\rho_{k-1} + (n-k)\rho_k \geq -\lambda\frac{3}{n}(n-k).$$

Finding

$$\binom{n}{1}(n-3)\left(1 - \frac{1}{n}\right) \geq \binom{n}{2}\left[(n-3) - \lambda\left(\frac{3}{n}(n-2) - 1\right)\right] - \lambda\sum_{\substack{k=4 \\ even}}\binom{n}{k}(n-k)\frac{3}{n}$$

gives

$$\lambda\left\{\frac{3}{n}\sum_{\substack{k=2 \\ even}}(n-k)\binom{n}{k} - \binom{n}{2}\right\} \geq \frac{n(n-3)^2}{2}.$$

Statement (ii)

Case 1. *m is odd:*

We rewrite the optimal efficiency loss as

$$\lambda^*_{n,m} = \frac{(n-m)\binom{n-1}{m-1}}{(n-m)B^{\rightarrow(m-2)}_{n-2} + m\big[B^{m,\tilde{n}}_{n-1} - B^{m-1,\tilde{n}-1}_{n-2}\big]}.$$

If n is even, we write

$$\begin{aligned} B^{m,\tilde{n}}_{n-1} - B^{m-1,\tilde{n}-1}_{n-2} &= \sum_{k=m}^{n-2}\left[\binom{n-1}{k} - \binom{n-2}{k-1}\right] = \sum_{k=m}^{n-2}\binom{n-2}{k} \\ &= 2^{n-2} - B^{\rightarrow(m-1)}_{n-2} \end{aligned}$$

and if n is odd, we write

$$\begin{aligned} B^{m,\tilde{n}}_{n-1} - B^{m-1,\tilde{n}-1}_{n-2} &= \sum_{k=m-1}^{n-2}\left[\binom{n-1}{k+1} - \binom{n-2}{k}\right] = (n-2) + \sum_{k=m-1}^{n-3}\binom{n-2}{k+1} \\ &= (n-2) + 2^{n-2} - B^{\rightarrow(m-1)}_{n-2}. \end{aligned}$$

Then, we have $\binom{n-1}{m-1} \simeq \frac{n^{m-1}}{(m-1)!}$. Note that $B^{\rightarrow(m-2)}_{n-2}$ is a polynomial of degree $m-2$ and $B^{\rightarrow(m-1)}_{n-2}$ is a polynomial of degree $m-1$. Thus, we have

$$\begin{aligned} &(n-m)B^{\rightarrow(m-2)}_{n-2} + m\big(2^{n-2} - B^{\rightarrow(m-1)}_{n-2}\big) \simeq m2^{n-2} \\ &(n-m)B^{\rightarrow(m-2)}_{n-2} + m\big((n-2) + 2^{n-2} - B^{\rightarrow(m-1)}_{n-2}\big) \simeq m2^{n-2}. \end{aligned}$$

Therefore, we conclude

$$\lambda^*_{n,m} \simeq \frac{(n-m)}{m!}\cdot\frac{n^{m-1}}{2^{n-2}}.$$

<u>*Case 2*</u>. *m is even:*

We rewrite the optimal efficiency loss as

$$\lambda^*_{n,m} = \frac{(n-m)\binom{n-1}{m-1}}{(n-m)B^{\rightarrow(m-2)}_{n-2} + m\left[B^{m,\bar{n}}_{n-1} - B^{m-1,\bar{n}-1}_{n-2}\right]}.$$

Since $\binom{n-1}{m-1} \simeq \frac{n^{m-1}}{(m-1)!}$ and

$$B^{m,\bar{n}}_{n-1} - B^{m-1,\bar{n}-1}_{n-2} = \sum_{k=m-1}^{\bar{n}-1}\left[\binom{n-1}{k+1} - \binom{n-2}{k}\right] = \sum_{k=m-1}^{n-3}\binom{n-2}{k+1} = 2^{n-2} - B^{\rightarrow(m-1)}_{n-2},$$

we conclude $\lambda^*_{n,m} \simeq \frac{(n-m)}{m!}\frac{n^{m-1}}{2^{n-2}}$. ■

We provides optimal redistribution schemes for $m \geq 3$ odd, corresponding to Theorem 1.2 in the following lemma.

Lemma 2 *The optimal redistribution scheme for m odd is as follows:*

$$\begin{aligned} r^*_{n,m}(c_{-i}) = &\sum_{k=1}^{3}\alpha^*_k(c_{-i})^{*k} + \sum_{k=4}^{m-1}\beta^*_k(c_{-i})^{*k} + \sum_{k=m}^{m+3}\gamma^*_k(c_{-i})^{*k} \\ &+ \sum_{k=4}^{n-m-2}\xi^*_{m+k}(c_{-i})^{*(m+k)} + \omega^*_{n-1}(c_{-i})^{*(n-1)} \end{aligned}$$

where

$$\alpha^*_1 = 0;\ \alpha^*_2 = \frac{2\lambda^*_{n,m}(n-m)}{n(n-2)};\ \alpha^*_3 = \frac{-2\lambda^*_{n,m}(n-m)}{(n-3)(n-2)};$$

$$\beta_k^* = \frac{(n-m)}{k} \cdot \frac{n - k\lambda_{n,m}^*}{n} - \frac{n}{k}\binom{n-m}{n-k}\left\{\frac{n-k\lambda_{n,m}^*}{n} - \frac{\binom{k}{2}}{\binom{n}{2}}\frac{\sum_{j=2}^{k-1}\binom{n-1}{j}}{\binom{n-2}{k-2}}\right.$$
$$\left. + \frac{2\binom{k}{2}}{n\binom{n-2}{k-2}} \sum_{\substack{j=2 \\ even}}^{k-2} \frac{(n-j\lambda_{n,m}^*)\binom{n-2}{j-2}}{j!} - \frac{2\binom{k}{2}}{n\binom{n-2}{k-2}}\right\} \text{ if } k \text{ is even;}$$

$$\beta_k^* = -\frac{n-m}{n-k} + \frac{n}{n-k}\binom{n-m}{n-k+1}\left\{\frac{n-(k-1)\lambda_{n,m}^*}{n} - \frac{\binom{k-1}{2}}{\binom{n}{2}}\sum_{j=2}^{k-2}\frac{\binom{n-1}{j}}{\binom{n-2}{k-3}}\right.$$
$$\left. + \frac{2\binom{k-1}{2}}{n\binom{n-2}{k-3}} \sum_{\substack{j=2 \\ even}}^{k-3} \frac{(n-j\lambda_{n,m}^*)\binom{n-2}{j-2}}{j!} - \frac{2\binom{k-1}{2}}{n\binom{n-2}{k-3}}\right\} \text{ if } k \text{ is odd;}$$

Letting $L_{n,m} = \frac{n-m}{n} - \sum_{k=1}^{3}\alpha_k^* - \sum_{k=4}^{m-1}\beta_k^*$,

$$\gamma_m^* = \frac{n}{n-m}L_{n,m};\ \gamma_{m+1}^* = \frac{m}{n}\lambda_{n,m}^* - \frac{nmL_{n,m}}{2\binom{n-m}{2}};$$
$$\gamma_{m+2}^* = -\frac{m}{(n-(m+2))}\lambda_{n,m}^* + \frac{n\binom{m+1}{2}L_{n,m}}{3\binom{n-m}{3}};$$
$$\gamma_{m+3}^* = \frac{m}{n}\lambda_{n,m}^* + \frac{m(m+2)}{2\binom{n-m-2}{2}}\lambda_{n,m}^* - \frac{n\binom{m+2}{3}L_{n,m}}{4\binom{n-m}{4}};$$

$$\xi_{m+k}^* = -\frac{n}{m+k}\left\{\frac{m(m+k)}{n(n-(m+k))}\lambda_{n,m}^* + \frac{m\lambda_{n,m}^*}{n\binom{n-1}{m+k}}\sum_{j=0}^{k-3}\binom{n-2}{m+j} - \frac{m}{n-m}\frac{\binom{n-1}{m}}{\binom{n-1}{m+k}}L_{n,m}\right\}$$

if k is even;

$$\xi_{m+k}^* = \frac{m}{n}\lambda_{n,m}^* + \frac{n}{n-(m+k)}\left\{\frac{m(m+k-1)}{n(n-(m+k-1))}\lambda_{n,m}^* + \frac{m}{n}\lambda_{n,m}^*\sum_{j=0}^{k-4}\frac{\binom{n-2}{m+j}}{\binom{n-1}{m+k-1}}\right.$$
$$\left. - \frac{m}{n-m}\frac{\binom{n-1}{m}}{\binom{n-1}{m+k-1}}L_{n,m}\right\} \text{ if } k \text{ is odd;}$$

$$\omega^*_{n-1} = -\frac{m}{n(n-1)}\lambda^*_{n,m} \quad \textit{if } n \textit{ is odd;}$$

$$\omega^*_{n-1} = 0 \quad \textit{if } n \textit{ is even.}$$

Proof. To find the optimal redistribution scheme, notice that when $\lambda = \lambda^*_{n,m}$, we have $X_1 = (n-m)(n-1)/n$. When $2 \le k \le m-1$, $X_k = n-m$ if k is an odd number and $X_k = (n-m)\left(1 - \frac{k}{n}\lambda^*_{n,m}\right)$ if k is an even number. When $m \le k \le n-1$, $X_k = 0$ if k is an odd number and $X_k = -\lambda^*_{n,m}\frac{m(n-k)}{n}$ if k is an even number. Recalling $X_i = (M\rho)_i$, we can find the optimal redistribution scheme has coefficients.

The last term is given as

$$\rho_{n-2} = -\frac{m}{n(n-1)}\lambda^*_{n,m} \text{ if } n \text{ is odd and } 0 \text{ if } n \text{ is even.}$$

The first three terms are given as

$$\begin{aligned}
\rho_0 - \rho_1 &= 0 \\
\rho_1 - \rho_2 &= \frac{2\lambda^*_{n,m}(n-m)}{n(n-2)} \\
\rho_2 - \rho_3 &= \frac{-2\lambda^*_{n,m}(n-m)}{(n-3)(n-2)}.
\end{aligned}$$

We will find the coefficient $\rho_k - \rho_{k+1}$ for all k, $3 \le k \le m-2$. We have

$$\begin{aligned}
\rho_{2h} = &\left(\frac{n-m}{n-2h}\right)\left\{\frac{n - 2h\lambda^*_{n,m}}{n} - \sum_{j=0}^{h-2}\frac{2h(2h-1)\binom{n-2}{2j+1}}{(2j+2)(2j+3)\binom{n-2}{2h-2}}\right. \\
&\left. + \sum_{j=1}^{h-1}\frac{(n-2j\lambda^*_{n,m})(2h)(2h-1)\binom{n-2}{2j-2}}{n(2j)!\binom{n-2}{2h-2}} - \frac{2h(2h-1)}{n\binom{n-2}{2h-2}}\right\}
\end{aligned}$$

for $2 \le h \le \frac{m-1}{2}$ and

$$\rho_{2h+1} = \frac{n-m}{n-2h-1} - \frac{2h+1}{n-2h-1}\rho_{2h}$$

for $2 \le h \le \frac{m-3}{2}$. Since we can write

$$\rho_{2h-1} - \rho_{2h} = \frac{(n-m)}{2h} \cdot \frac{n-2h\lambda^*_{n,m}}{n} - \frac{n}{2h}\rho_{2h}$$

for $2 \le h \le \frac{m-1}{2}$ and

$$\rho_{2h} - \rho_{2h+1} = -\frac{n-m}{n-2h-1} + \frac{n}{n-2h-1}\rho_{2h}$$

for $2 \le h \le \frac{m-3}{2}$, we conclude that

$$\begin{aligned}\rho_{2h-1} - \rho_{2h} = & \frac{(n-m)}{2h} \cdot \frac{n-2h\lambda^*_{n,m}}{n} \\ & - \frac{n}{2h}\left(\frac{n-m}{n-2h}\right)\left\{\frac{n-2h\lambda^*_{n,m}}{n} - \sum_{j=0}^{h-2} \frac{2h(2h-1)\binom{n-2}{2j+1}}{(2j+2)(2j+3)\binom{n-2}{2h-2}}\right. \\ & \left. + \sum_{j=1}^{h-1} \frac{(n-2j\lambda^*_{n,m})(2h)(2h-1)\binom{n-2}{2j-2}}{n(2j)!\binom{n-2}{2h-2}} - \frac{2h(2h-1)}{n\binom{n-2}{2h-2}}\right\}\end{aligned}$$

for $2 \le h \le \frac{m-1}{2}$ and

$$\begin{aligned}\rho_{2h} - \rho_{2h+1} = & -\frac{n-m}{n-2h-1} \\ & + \frac{n}{n-2h-1}\left(\frac{n-m}{n-2h}\right)\left\{\frac{n-2h\lambda^*_{n,m}}{n} - \sum_{j=0}^{h-2} \frac{2h(2h-1)\binom{n-2}{2j+1}}{(2j+2)(2j+3)\binom{n-2}{2h-2}}\right. \\ & \left. + \sum_{j=1}^{h-1} \frac{(n-2j\lambda^*_{n,m})(2h)(2h-1)\binom{n-2}{2j-2}}{n(2j)!\binom{n-2}{2h-2}} - \frac{2h(2h-1)}{n\binom{n-2}{2h-2}}\right\}\end{aligned}$$

for $2 \le h \le \frac{m-3}{2}$. From $\sum_{k=0}^{m-2} \rho_k - \rho_{k+1} = \rho_0 - \rho_{m-1}$ and $\rho_0 = \frac{n-m}{n}$, we can compute ρ_{m-1}.

Now we will find the remaining coefficients $\rho_k - \rho_{k+1}$ for $m-1 \le k \le n-3$. We have the first four terms as

$$\begin{aligned}
\rho_{m-1} - \rho_m =& \frac{n}{n-m}\rho_{m-1} \\
\rho_m - \rho_{m+1} =& \frac{m}{n}\lambda^*_{n,m} - \frac{nm}{(n-m)(n-(m+1))}\rho_{m-1} \\
\rho_{m+1} - \rho_{m+2} =& -\frac{m}{(n-(m+2))}\lambda^*_{n,m} + \frac{nm(m+1)}{(n-m)(n-(m+1))(n-(m+2))}\rho_{m-1} \\
\rho_{m+2} - \rho_{m+3} =& \frac{m}{n}\lambda^*_{n,m} + \frac{m(m+2)}{(n-(m+3))(n-(m+2))}\lambda^*_{n,m} \\
&- \frac{nm(m+1)(m+2)}{(n-m)(n-(m+3))(n-(m+2))(n-(m+1))}\rho_{m-1}.
\end{aligned}$$

We can write

$$\begin{aligned}
\rho_{m+2h} =& \frac{m(m+2h)}{n(n-(m+2h))}\lambda^*_{n,m} + \frac{m}{n}\lambda^*_{n,m}\sum_{j=0}^{h-2}\left(\frac{m+2j+2}{m+2h+1}\right)\frac{\binom{n}{m+2j+2}}{\binom{n}{m+2h+1}} \\
&- \left(\frac{m}{n-m}\right)\left(\frac{m+1}{m+2h+1}\right)\frac{\binom{n}{m+1}}{\binom{n}{m+2h+1}}\rho_{m-1}
\end{aligned}$$

for $2 \le h \le \frac{n-m-2}{2}$. With

$$\rho_{m+2h-1} = -\frac{n-(m+2h)}{m+2h}\rho_{m+2h},$$

we have

$$\rho_{m+2h-1} - \rho_{m+2h} = -\frac{n}{m+2h}\left\{ \frac{m(m+2h)}{n(n-(m+2h))}\lambda^*_{n,m} + \frac{m}{n}\lambda^*_{n,m}\sum_{j=0}^{h-2}\left(\frac{m+2j+2}{m+2h+1}\right)\frac{\binom{n}{m+2j+2}}{\binom{n}{m+2h+1}} - \left(\frac{m}{n-m}\right)\left(\frac{m+1}{m+2h+1}\right)\frac{\binom{n}{m+1}}{\binom{n}{m+2h+1}}\rho_{m-1} \right\}$$

for $2 \leq h \leq \frac{n-m-2}{2}$. With

$$\rho_{m+2h+1} = -\frac{m}{n}\lambda^*_{n,m} - \frac{m+2h+1}{n-(m+2h+1)}\rho_{m+2h},$$

$$\rho_{m+2h} - \rho_{m+2h+1} = \frac{m}{n}\lambda^*_{n,m} + \frac{n}{n-(m+2h+1)}\left\{ \frac{m(m+2h)}{n(n-(m+2h))}\lambda^*_{n,m} + \frac{m}{n}\lambda^*_{n,m}\sum_{j=0}^{h-2}\left(\frac{m+2j+2}{m+2h+1}\right)\frac{\binom{n}{m+2j+2}}{\binom{n}{m+2h+1}} - \left(\frac{m}{n-m}\right)\left(\frac{m+1}{m+2h+1}\right)\frac{\binom{n}{m+1}}{\binom{n}{m+2h+1}}\rho_{m-1} \right\}$$

for $2 \leq h \leq \frac{n-m-4}{2}$. ■

The following lemma provides optimal redistribution schemes for $m = 2$, corresponding to Theorem 1.2.

Lemma 3 *The following linear redistribution scheme defines an optimal mechanism for $m = 2$:*

$$r^*_{n,2}(c_{-i}) = \sum_{k=1}^{5}\alpha^*_k(c_{-i})^{*k} + \sum_{k=6}^{n-1}\beta^*_k(c_{-i})^{*k}$$

where

$$\alpha_1^* = \frac{n-2}{n(n-1)}\lambda_{n,2}^*;\ \alpha_2^* = 1 - \frac{\lambda_{n,2}^*}{n-1};\ \alpha_3^* = -\frac{2}{n-3} + \frac{2\lambda_{n,2}^*}{n}\frac{n^2-4n+6}{(n-1)(n-3)};$$

$$\alpha_4^* = \frac{3}{\binom{n-3}{2}} - \frac{\lambda_{n,2}^*(n^2-4n+6)}{(n-1)\binom{n-3}{2}};$$

$$\alpha_5^* = -\frac{4}{\binom{n-3}{3}} + \frac{\lambda_{n,2}^*(n^4-9n^3+43n^2-83n+60)}{6\binom{n}{2}\binom{n-3}{3}};$$

$$\beta_k^* = \frac{\binom{n}{k}\left[2\binom{n-1}{2} + \lambda_{n,2}^*[n - 2\sum_{l=0}^{k-3}\binom{n-2}{l}]\right]}{n\binom{n-1}{k-1}\binom{n-1}{k}} - \frac{2\lambda_{n,2}^*}{n-k} \quad \textit{if } k \textit{ is even;}$$

$$\beta_k^* = \frac{\binom{n}{k}\left[(n-2)(\lambda_{n,2}^* - n + 1) + 2\lambda_{n,2}^*\sum_{l=2}^{k-4}\binom{n-2}{l}\right]}{n\binom{n-1}{k-1}\binom{n-1}{k}} + \frac{2\lambda_{n,2}^*}{n}\left[\frac{n}{n-k+1} + \frac{\binom{n-2}{k-2}}{\binom{n-1}{k}}\right]$$

if k is odd.

Proof of Theorem 1.1

When $m = 1$, $ps(c) = (n-1)c^{*1}$ and $es(c) = \frac{1}{n}\sum_{i\in N} c_i - c^{*1}$. Applying e^{n-k} with $k = 0$ and $k = n$, the no deficit and worst case constraints give $\rho_0 = \frac{n-1}{n}$ and $\rho_{n-1} = 0$. With $k = 1$, the constraints are $-(1+\lambda)\left(\frac{n-1}{n}\right) \leq (n-1)\rho_1 \leq -\frac{n-1}{n}$. And for k, $2 \leq k \leq n-1$, the constraints give $-\lambda\frac{n-k}{n} \leq k\rho_{k-1} + (n-k)\rho_k \leq 0$. Setting M and ρ as before, we find the same hyperplane and

$$-(n-1) \geq \binom{n}{1}X_1 + \binom{n}{3}X_3 + \cdots = \binom{n}{2}X_2 + \binom{n}{4}X_4 + \cdots \geq -\lambda\sum_{\substack{k=2 \\ even}}^{\hat{n}}\binom{n}{k}\left(\frac{n-k}{n}\right)$$

leads to the following inequality:

$$\lambda \geq \frac{n-1}{\sum_{\substack{k=2 \\ even}}^{\hat{n}} \binom{n}{k} \binom{\frac{n-k}{n}}{}}$$

where $\hat{n}$ is $n-1$ if n is odd and $\hat{n}$ is $n-2$ if n is even. Therefore, the optimal efficiency loss to efficient surplus is given as $\lambda^*_{n,1} = \frac{n-1}{2^{n-2}-1}$.

The optimal redistribution mechanism is as follows: If $n = 3$, $\rho_0 - \rho_1 = 1$ and $\rho_1 = -\frac{1}{3}$. If $n = 4$, $\rho_0 - \rho_1 = 1$, $\rho_1 - \rho_2 = -\frac{1}{4}$, and $\rho_2 = 0$. If $n \geq 5$, the first three terms are given as

$$\begin{aligned}
\rho_0 - \rho_1 &= 1 \\
\rho_1 - \rho_2 &= \frac{8 - 8n - 2^n n + 4n^2}{(-4+2^n)(-2+n)n} \\
\rho_2 - \rho_3 &= -\frac{2(8 - 2^n - 6n + 2n^2)}{(-4+2^n)(-3+n)(-2+n)}
\end{aligned}$$

and the last term is given as

$$\rho_{n-2} = -\frac{4}{(-4+2^n)\,n}$$

if n is odd and $\rho_{n-2} = 0$ if n is even. The residual terms are computed as

$$\rho_{2h+1} - \rho_{2h+2} = \frac{4(n-1)}{n(2^n-4)} + \frac{4}{2^n-4} \cdot \frac{\sum_{l=1}^{h} \binom{n-1}{2l}}{\binom{n-2}{2h+1}} - \frac{1}{\binom{n-2}{2h+1}}$$

for $1 \leq h \leq \frac{n-5}{2} \cdot 1_{\{n:odd\}} + \frac{n-4}{2} \cdot 1_{\{n:even\}}$, and

$$\rho_{2h+2} - \rho_{2h+3} = -\frac{4(n-1)}{(n-2h-3)(2^n-4)} - \frac{4}{2^n-4} \frac{\sum_{l=1}^{h} \binom{n-1}{2l}}{\binom{n-2}{2h+2}} + \frac{1}{\binom{n-2}{2h+2}}$$

for $1 \leq h \leq \frac{n-5}{2} \cdot 1_{\{n:odd\}} + \frac{n-6}{2} \cdot 1_{\{n:even\}}$. ■

Proof of Theorem 1.3

When $m = n-1$ and $m \geq 2$, $ps(c) = c^{*(n-1)}$ and $es(c) = -\frac{1}{n}\sum_{i \in N} c_i + c^{*n}$. When cost profile is e^{n-k}, $ps(c) = 1$ for $0 \leq k \leq n-2$ and $ps(c) = 0$ for $n-1 \leq k \leq n$. Likewise, $es(c) = 1 - \frac{1}{n}(n-k)$ for $0 \leq k \leq n-1$ and $es(c) = 0$ for $k = n$. At each profile e^{n-k}, the no deficit and worst case constraints give $1 - \frac{1}{n} - \frac{\lambda}{n} \leq (n-1)\rho_1 \leq 1 - \frac{1}{n}$ for $k = 1$, $1 - \lambda\frac{k}{n} \leq k\rho_{k-1} + (n-k)\rho_k \leq 1$ for $2 \leq k \leq n-2$ and $-\lambda(1 - \frac{1}{n}) \leq (n-1)\rho_{n-2} \leq 0$ for $k = n-1$. Using the same M and ρ and finding the hyperplane, we have

$$(n-1-\lambda) + \sum_{\substack{k=3 \\ odd}}^{n-2} \left(1 - \lambda\frac{k}{n}\right)\binom{n}{k} \leq \sum_{\substack{k=1 \\ odd}}^{n-2} \binom{n}{k} X_k = \sum_{\substack{k=2 \\ even}}^{n-1} \binom{n}{k} X_k \leq \sum_{\substack{k=2 \\ even}}^{n-3} \binom{n}{k}$$

if n is odd and

$$(n-1) + \sum_{\substack{k=3 \\ odd}}^{n-3} \binom{n}{k} \geq \sum_{\substack{k=1 \\ odd}}^{n-1} \binom{n}{k} X_k = \sum_{\substack{k=2 \\ even}}^{n-2} \binom{n}{k} X_k \geq \sum_{\substack{k=2 \\ even}}^{n-2} \binom{n}{k}\left(1 - \lambda\frac{k}{n}\right)$$

if n is even. Then, we have the following inequality:

$$\lambda \geq \frac{\sum_{\substack{k=1 \\ odd}}^{n-2} \binom{n}{k} - \sum_{\substack{k=0 \\ even}}^{n-3} \binom{n}{k}}{\sum_{\substack{k=1 \\ odd}}^{n-2} \frac{k}{n}\binom{n}{k}}$$

when n is odd, and

$$\lambda \geq \frac{\sum_{\substack{k=0 \\ even}}^{n-2} \binom{n}{k} - \sum_{\substack{k=1 \\ odd}}^{n-3} \binom{n}{k}}{\sum_{\substack{k=2 \\ even}}^{n-2} \frac{k}{n}\binom{n}{k}}$$

when n is even. Then, the optimal efficiency loss to efficient surplus is written as $\lambda^*_{n,n-1} = \frac{n-1}{2^{n-2}-1}$. ■

Proof of Proposition 1

Case 1. $m = 1$: when deficit is allowed, the two way worst case constraints are written as

$$(n-1)c^{*1} - \lambda\left(\frac{c_N}{n} - c^{*1}\right) \leq \sum_{i \in N} r(c_{-i}) \leq (n-1)c^{*1} + \lambda\left(\frac{c_N}{n} - c^{*1}\right).$$

Applying e^{n-k}, for $k = 1$, we have

$$(-1-\lambda)\frac{n-1}{n} \leq (n-1)\rho_1 \leq (-1+\lambda)\frac{n-1}{n}$$

and for $n-1 \geq k \geq 2$

$$-\lambda\frac{n-k}{n} \leq k\rho_{k-1} + (n-k)\rho_k \leq \lambda\frac{n-k}{n}.$$

Then, we have

$$(n-1)(-1+\lambda) + \frac{\lambda}{n}\sum_{\substack{k=3 \\ odd}} \binom{n}{k}(n-k) \geq -\frac{\lambda}{n}\sum_{\substack{k=2 \\ even}} \binom{n}{k}(n-k)$$

giving

$$\lambda \geq \frac{n-1}{2^{n-1}-1}.$$

Thus, $\lambda^{\#} = \frac{n-1}{2^{n-1}-1}$. Recalling $\lambda^* = \frac{n-1}{2^{n-2}-1}$, $\frac{\lambda^{\#}}{\lambda^*} = \frac{2^{n-2}-1}{2^{n-1}-1} \simeq \frac{1}{2}$.

Case 2. $n-1 \geq m \geq 2$: when deficit is allowed, the two way constraints are given as

$$(n-m)c^{*m} - \lambda\left[\frac{m}{n}c_N - \tau_m(c)\right] \leq \sum_{i\in N} r(c_{-i}) \leq (n-m)c^{*m} + \lambda\left[\frac{m}{n}c_N - \tau_m(c)\right].$$

Applying e^{n-k}, for $k=1$, this is written as

$$\left[1-\frac{1}{n}\right](n-m) - \lambda\left[1-\frac{m}{n}\right] \leq (n-1)\rho_1 \leq (n-m)\left[1-\frac{1}{n}\right] + \lambda\left[1-\frac{m}{n}\right]$$

and $m-1 \geq k \geq 2$,

$$\begin{aligned}(n-m) - \lambda\left[\frac{m}{n}(n-k)-(m-k)\right] &\leq k\rho_{k-1} + (n-k)\rho_k \\ &\leq (n-m) + \lambda\left[\frac{m}{n}(n-k)-(m-k)\right].\end{aligned}$$

For $n-1 \geq k \geq m$,

$$-\lambda\frac{m}{n}(n-k) \leq k\rho_{k-1} + (n-k)\rho_k \leq \lambda\frac{m}{n}(n-k).$$

If m is odd (if m is even, computation is the same),

$$\begin{aligned}&(n-m)(n-1) + \lambda(n-m) + \sum_{\substack{k=3\\ odd}}^{m-2}\binom{n}{k}\left[(n-m)+\lambda\left(\frac{m}{n}(n-k)-(m-k)\right)\right] \\ &+\lambda\frac{m}{n}\sum_{\substack{k=m\\ odd}}\binom{n}{k}(n-k) \geq \sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}\left[n-m-\lambda\left(\frac{m}{n}(n-k)-(m-k)\right)\right] \\ &-\lambda\frac{m}{n}\sum_{\substack{k=m+1\\ even}}\binom{n}{k}(n-k).\end{aligned}$$

This is rearranged as

$$(n-m)\binom{n-1}{m-1} \le \lambda\Big\{(n-m)\sum_{k=0}^{m-2}\binom{n-1}{k} + m\sum_{k=m}^{n-1}\binom{n-1}{k}\Big\}$$

and thus,

$$\lambda^{\#} = \frac{(n-m)\binom{n-1}{m-1}}{(n-m)\sum_{k=0}^{m-2}\binom{n-1}{k} + m\sum_{k=m}^{n-1}\binom{n-1}{k}}$$

for $n-1 \ge m \ge 2$. Recalling $\lambda^* = \frac{(n-m)\binom{n-1}{m-1}}{(n-m)\sum_{k=0}^{m-2}\binom{n-2}{k}+m\sum_{k=m}^{n-2}\binom{n-2}{k}}$ for $n-2 \ge m \ge 2$, we write

$$\frac{\lambda^{\#}}{\lambda^*} = \frac{(n-m)\sum_{k=0}^{m-2}\binom{n-2}{k} + m\sum_{k=m}^{n-2}\binom{n-2}{k}}{(n-m)\sum_{k=0}^{m-2}\binom{n-1}{k} + m\sum_{k=m}^{n-1}\binom{n-1}{k}} \simeq \frac{1}{2}$$

for $n-2 \ge m \ge 2$. For $m = n-1$,

$$\frac{\lambda^{\#}}{\lambda^*} = \frac{2^{n-2}-1}{2^{n-1}-1} \simeq \frac{1}{2}. \quad \blacksquare$$

Proof of Proposition 2

We will apply cost profile e^{n-k} for k, $0 \le k \le n$. Recall for $0 \le k \le m-1$, $es(c) = \frac{k}{n}(n-m)$ with $ps(c) = n-m$ and for $m \le k \le n$, $es(c) = \frac{m}{n}(n-k)$ with $ps(c) = 0$.

$V_i \le \frac{m}{n}c_i$ with anonymity requires $\frac{n-m}{n}c^{*m} \le r(c^{*1}, \cdots, c^{*(m-1)}, c^{*(m+1)}, \cdots, c^{*n})$.

This is written as

$$(\rho_0 - \rho_1)c^{*1} + \cdots + (\rho_{m-2} - \rho_{m-1})c^{*(m-1)} - \frac{n-m}{n}c^{*m}$$
$$+ (\rho_{m-1} - \rho_m)c^{*(m+1)} + \cdots + \rho_{n-2}c^{*n} \geq 0.$$

This inequality holds if and only if $\rho_k \geq 0$ for all k, $m \leq k \leq n-2$ and $\rho_k \geq \frac{n-m}{m}$ for all k, $0 \leq k \leq m-1$. The non-deficit constraint implies

$$(n-m)c^{*m} \geq (n-1)(\rho_0 - \rho_1)c^{*1} + [(\rho_0 - \rho_1) + (n-2)(\rho_1 - \rho_2)]c^{*2}$$
$$+ \cdots + [(m-1)(\rho_{m-2} - \rho_{m-1}) + (n-m)(\rho_{m-1} - \rho_m)]c^{*m} + \cdots$$
$$+ [(n-2)(\rho_{n-3} - \rho_{n-2}) + \rho_{n-2}]c^{*(n-1)} + (n-1)\rho_{n-2}c^{*n}$$

and by Lemma 1, this holds if and only if $\rho_{n-2} \leq 0$, $(n-2)\rho_{n-3} + 2\rho_{n-2} \leq 0$, $\cdots$, $m\rho_{m-1} + (n-m)\rho_m \leq 0$, $(m-1)\rho_{m-2} + (n-m+1)\rho_{m-1} \leq n-m$, $\cdots$, $\rho_0 + (n-1)\rho_1 \leq n-m$ and $n\rho_0 \leq n-m$. With unanimity upper bound, this implies $\rho_{n-2} = \cdots = \rho_m = 0$. Since $\rho_m = 0$, the non-deficit constraint gives $m\rho_{m-1} \leq 0$ but this contradicts $\rho_{m-1} \geq \frac{n-m}{n}$ given by unanimity upper bound. Therefore, there is no anonymous linear VCG mechanism satisfying unanimity upper bound and non-deficit. ■

Proof of Theorem 2.1

We know that the pivotal mechanism is anonymous and individually rational. It generates no deficit, that is, $\mu^*_{n,1} = 0$, but its efficiency loss is $\lambda^*_{n,1} = \infty$. If λ is restricted to be finite, we have that for $k = 0$, $es(c) = 0$ and $ps(c) = n-1$, so the worst case constraint implies $\rho_0 = \frac{n-1}{n}$. For $k = n$, the worst case constraint implies

$\rho_{n-1} = 0$. For k, $1 \leq k \leq n-1$, the worst case constraint gives

$$-\lambda\frac{1}{n}(n-k) \leq k\rho_{k-1} + (n-k)\rho_k \leq \mu\frac{1}{n}(n-k)$$

and individual rationality implies $r(c_{-i}) \geq 0$, which is $\rho_k \geq 0$ for all $0 \leq k \leq n-1$. The worst case and individual rationality constraints are together written as

$$0 \leq k\rho_{k-1} + (n-k)\rho_k \leq \mu\frac{1}{n}(n-k)$$

for $1 \leq k \leq n-1$. For X in the range of M, we have

$$0 \leq X_1 \leq \frac{n-1}{n}(\mu - 1)$$

and for $2 \leq k \leq n-1$

$$0 \leq X_k \leq \mu\frac{m}{n}(n-k).$$

Note that from $0 \leq X_1 \leq \frac{n-1}{n}(\mu-1)$, we should have $\mu \geq 1$. Let $\mu = 1$. Set $\rho_k = 0$ for all k, $1 \leq k \leq n-2$, then the inequality constraints are satisfied. With $\rho_0 = \frac{n-1}{n}$, we can set $r(c_{-i}) = \frac{n-1}{n}(c_{-i})^{*1}$ and compute the efficiency loss of this redistribution scheme. $\Delta(c) = \frac{n-1}{n}(c^{*1} - c^{*2})$ and $es(c) = \frac{\sum_{i\in N} c_i}{n} - c^{*1}$.

$$\mu = \sup_{c\in\mathcal{R}_+^N} \frac{|\Delta(c)|}{es(c)} = \frac{n-1}{n} \sup_{c\in\mathcal{R}_+^N} \frac{c^{*2} - c^{*1}}{\frac{\sum_{i\in N} c_i}{n} - c^{*1}} = \frac{n-1}{n} \sup_{c\in\mathcal{R}_+^N} \frac{c^{*2} - c^{*1}}{\frac{c^{*1}+(n-1)c^{*2}}{n} - c^{*1}} = 1.$$

Therefore, the optimal $\mu = 1$ with $\lambda = 0$, and the optimal redistribution scheme is $r(c_{-i}) = \frac{n-1}{n}(c_{-i})^{*1}$. ■

Proof of Theorem 2.2

We found $\rho_0 = \frac{n-2}{n}$ and $\rho_{n-1} = 0$.

$$\max\left\{0, \frac{n-2}{n}(n-1-\lambda)\right\} \leq X_1 \leq \frac{n-2}{n}(n-1+\mu).$$

For $2 \leq k \leq n-1$, $0 \leq X_k \leq \mu\frac{2}{n}(n-k)$. Suppose $n-1 \geq \lambda$. Then, we have

$$(n-2)(n-1-\lambda) \leq \binom{n}{1}X_1 + \sum_{\substack{k=3 \\ odd}}^{\tilde{n}} \binom{n}{k} X_k = \sum_{\substack{k=2 \\ even}}^{\hat{n}} \binom{n}{k} X_k \leq \mu\frac{2}{n}\sum_{\substack{k=2 \\ even}}^{\hat{n}} \binom{n}{k}(n-k)$$

and

$$(n-2)(n-1) = (n-2)\lambda^*_{n,2} + 2\sum_{k=0}^{n-3}\binom{n-2}{k}\mu^*_{n,2}.$$

The maximal $\lambda^*_{n,2} = n-1$, so $\lambda^*_{n,2}$ satisfies $\lambda \leq n-1$.

Recall $B(n,2) = \frac{(n-1)(n-2)}{2^{n-1}-2}$. For $n \geq 5$, we have

$$B(n,2) - B(n-1,2) = \frac{n-2}{2}\left[\frac{n-1}{2^{n-2}-1} - \frac{n-3}{2^{n-3}-1}\right] = -\frac{(n-2)[(n-5)2^{n-3}+2]}{2(2^{n-2}-1)(2^{n-3}-1)} < 0,$$

so $B(n,2)$ is strictly decreasing in n. $B(4,2) = \frac{3}{n-1}$ gives the result. ■

We provide the optimal redistribution schemes corresponding to Theorem 2.2 in the following lemma.

Lemma 4 *For any* $\mu^*_{n,2} > 0$ *chosen, the optimal redistribution scheme is as* *follows:*

$$r^*(c_{-i}) = \sum_{k=1}^{6} \alpha_k^*(c_{-i})^{*k} + \sum_{k=7}^{n-1} \beta_k^*(c_{-i})^{*k}$$

where

$$\alpha_1^* = -\frac{\mu_{n,2}^*(2^{n-1}-2) - 2\binom{n-1}{2}}{n(n-1)}$$

$$\alpha_2^* = \frac{\mu_{n,2}^* n(2^{n-2}-1)^2 - 2\binom{n-1}{2}^2}{n\binom{n-1}{2}(2^{n-2}-1)}$$

$$\alpha_3^* = \frac{2\left[\binom{n-1}{2}^2 - \mu_{n,2}^*(2^{n-2}-1)^2\right]}{(2^{n-2}-1)\binom{n-1}{2}(n-3)}$$

$$\alpha_4^* = -\frac{2\binom{n-1}{2}}{n(2^{n-2}-1)} - \frac{2\left[\binom{n-1}{2}^2 - \mu_{n,2}^*(2^{n-2}-1)^2\right]}{(2^{n-2}-1)\binom{n-1}{3}(n-4)}$$

$$\alpha_5^* = \frac{2}{(n-5)(2^{n-2}-1)}\left[\binom{n-1}{2} + \frac{\binom{n-1}{2}^2 - \mu_{n,2}^*(2^{n-2}-1)^2}{\binom{n-1}{4}}\right]$$

$$\alpha_6^* = -\frac{2\binom{n-1}{2}}{(2^{n-2}-1)(n-5)} - \frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\frac{\binom{n-1}{3}}{\binom{n-1}{6}} - \frac{\binom{n-1}{2}^2 - \mu_{n,2}^*(2^{n-2}-1)^2}{3\binom{n-1}{6}(2^{n-2}-1)};$$

$$\beta_k^* = \frac{2\left[\binom{n-1}{2}\sum_{l=1}^{k-1}\binom{n-2}{l} - \mu_{n,2}^*(2^{n-2}-1)^2\right]}{(2^{n-2}-1)\binom{n-1}{k-1}(n-k)} \quad \textit{if } k \textit{ is odd;}$$

$$\beta_k^* = -\frac{2\left\{\binom{n-1}{2}\left[n\sum_{l=1}^{k-4}\binom{n-2}{l} + k\binom{n-1}{k-2}\right] - n\mu_{n,2}^*(2^{n-2}-1)^2\right\}}{n(2^{n-2}-1)\binom{n-1}{k-1}(n-k)} - \frac{2\binom{n-1}{2}}{(2^{n-2}-1)(n-k+1)} \quad \textit{if } k \textit{ is even.}$$

Proof. If $\lambda^*_{n,2} = 0$, then

$$\mu^*_{n,2} = \frac{(n-2)(n-1)}{2\sum_{k=0}^{n-3}\binom{n-2}{k}} = \frac{\binom{n-1}{2}}{2^{n-2}-1}.$$

From $X_1 = \frac{n-2}{n}(n-1)$, $\rho_1 = \frac{n-2}{n}$. $X_k = 0$ for k odd, $3 \le k \le \tilde{n}$ and $X_k = \frac{2}{n}(n-k)\frac{\binom{n-1}{2}}{2^{n-2}-1}$ for k even, $2 \le k \le \hat{n}$. We have $\rho_0 = \rho_1 = \frac{n-2}{n}$ and $\rho_{n-1} = 0$. Recall that $\lambda^*_{n,2}$ and $\mu^*_{n,2}$ satisfy

$$2\binom{n-1}{2} = (n-2)\lambda^*_{n,2} + 2\sum_{k=0}^{n-3}\binom{n-2}{k}\mu^*_{n,2}.$$

Let $C(n,m) = 2\binom{n-1}{2}$, $A(n,m) = n-2$ and $B(n,m) = 2\sum_{k=0}^{n-3}\binom{n-2}{k}$. Let $L = \frac{C(n,m)}{B(n,m)}$ $= \frac{\binom{n-1}{2}}{2^{n-2}-1}$. For k even, $2 \le k \le \hat{n}$,

$$\rho_k = \frac{2L}{n}\cdot 1_{\{k\ge 4\}} + \frac{2L}{n}\frac{\sum_{l=3}^{k-2}\binom{n-2}{l}}{\binom{n-1}{k}}\cdot 1_{\{k\ge 6\}} + \frac{2\binom{n-1}{2}}{\binom{n-1}{k}}\frac{L-1}{n}$$

and for k odd, $\tilde{n} \ge k \ge 3$,

$$\rho_k = -\frac{k}{n-k}\left\{\frac{2L}{n}\cdot 1_{\{k\ge 5\}} + \frac{2L}{n}\frac{\sum_{l=3}^{k-3}\binom{n-2}{l}}{\binom{n-1}{k-1}}\cdot 1_{\{k\ge 7\}} + \frac{2\binom{n-1}{2}}{\binom{n-1}{k-1}}\frac{L-1}{n}\right\}.$$

Therefore, $a^*_1 = \rho_0 - \rho_1 = 0$ and $a^*_2 = \rho_1 - \rho_2 = 1 - \frac{2L}{n}$. For $k \ge 3$, we have

$$\begin{aligned}
a^*_k =& \rho_{k-1} - \rho_k = \frac{n}{n-k}\left\{\frac{2L}{n}\cdot 1_{\{k\ge 5\}} + \frac{2L}{n}\frac{\sum_{l=3}^{k-3}\binom{n-2}{l}}{\binom{n-1}{k-1}}\cdot 1_{\{k\ge 7\}} + \frac{2\binom{n-1}{2}}{\binom{n-1}{k-1}}\frac{L-1}{n}\right\} \\
=& \frac{n}{n-k}\left\{\frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\cdot 1_{\{k\ge 5\}} + \frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\frac{\sum_{l=3}^{k-3}\binom{n-2}{l}}{\binom{n-1}{k-1}}\cdot 1_{\{k\ge 7\}}\right. \\
&\left. + \frac{2\binom{n-1}{2}}{\binom{n-1}{k-1}}\frac{\binom{n-1}{2} - (2^{n-2}-1)}{n(2^{n-2}-1)}\right\}
\end{aligned}$$

if k is odd and

$$a_k^* = -\frac{2L}{n}\left\{1 + \frac{k-1}{(n-k+1)} \cdot 1_{\{k\geq 6\}}\right\} - \frac{2L}{n}\left\{\frac{\sum_{l=3}^{k-4}\binom{n-2}{l}}{\binom{n-1}{k-1}} \cdot 1_{\{k\geq 8\}}\right.$$
$$\left. + \frac{\sum_{l=3}^{k-2}\binom{n-2}{l}}{\binom{n-1}{k}} \cdot 1_{\{k\geq 6\}}\right\} - 2\binom{n-1}{2}\frac{L-1}{n}\frac{\binom{n}{k}}{\binom{n-1}{k-1}\binom{n-1}{k}}$$
$$= -\frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\left\{1 + \frac{k-1}{n-k+1} \cdot 1_{\{k\geq 6\}}\right\} - \frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\left\{\frac{\sum_{l=3}^{k-4}\binom{n-2}{l}}{\binom{n-1}{k-1}} \cdot 1_{\{k\geq 8\}}\right.$$
$$\left. + \frac{\sum_{l=3}^{k-2}\binom{n-2}{l}}{\binom{n-1}{k}} \cdot 1_{\{k\geq 6\}}\right\} - 2\binom{n-1}{2}\frac{\binom{n-1}{2} - (2^{n-2}-1)}{n(2^{n-2}-1)}\frac{\binom{n}{k}}{\binom{n-1}{k-1}\binom{n-1}{k}}$$

if k is even.

Given any $\mu_{n,2}^*$, $0 < \mu_{n,2}^* \leq \frac{C(n,m)}{B(n,m)}$, we can find the corresponding redistribution scheme. Let $T = \frac{\mu_{n,2}^* B(n,m) - C(n,m)}{A(n,m)(n-1)}$. Now $X_1 = \frac{(n-1)(n-2)}{n}(1+T)$ instead of $X_1 = \frac{(n-1)(n-2)}{n}$ for the $\mu_{n,2}^* = L$ case. For k odd, $k \geq 3$, still we have $X_k = 0$ and for k even, $k \geq 2$, we have $X_k = \mu^* \frac{2}{n}(n-k)$ instead of $L\frac{2}{n}(n-k)$ for the $\mu_{n,2}^* = L$ case. Then, $\rho_1 = \frac{n-2}{n}[1+T]$ with $\rho_0 = \frac{n-2}{n}$. For k even, $2 \leq k \leq \hat{n}$,

$$\rho_k = \frac{2L}{n} \cdot 1_{\{k\geq 4\}} + \frac{2L}{n}\frac{\sum_{l=3}^{k-2}\binom{n-2}{l}}{\binom{n-1}{k}} \cdot 1_{\{k\geq 6\}} + \frac{2\binom{n-1}{2}}{\binom{n-1}{k}}\frac{L-1-T}{n}$$

and for k odd, $\tilde{n} \geq k \geq 3$,

$$\rho_k = -\frac{k}{n-k}\left\{\frac{2L}{n} \cdot 1_{\{k\geq 5\}} + \frac{2L}{n}\frac{\sum_{l=3}^{k-3}\binom{n-2}{l}}{\binom{n-1}{k-1}} \cdot 1_{\{k\geq 7\}} + \frac{2\binom{n-1}{2}}{\binom{n-1}{k-1}}\frac{L-1-T}{n}\right\}.$$

Therefore, $a_1 = \rho_0 - \rho_1 = -\frac{n-2}{n}T$ and $a_2 = \rho_1 - \rho_2 = 1 + T - \frac{2L}{n}$. For $k \geq 3$, we have

$$a_k = \rho_{k-1} - \rho_k = \frac{n}{n-k}\left\{\frac{2L}{n} \cdot 1_{\{k\geq 5\}} + \frac{2L}{n}\frac{\sum_{l=3}^{k-3}\binom{n-2}{l}}{\binom{n-1}{k-1}} \cdot 1_{\{k\geq 7\}} + \frac{2\binom{n-1}{2}}{\binom{n-1}{k-1}}\frac{L-1-T}{n}\right\}$$

if k is odd and

$$\begin{aligned} a_k = &- \frac{2L}{n}\left\{1_{\{k\geq 4\}} + \frac{k-1}{(n-k+1)}\cdot 1_{\{k\geq 6\}}\right\} \\ &- \frac{2L}{n}\left\{\frac{\sum_{l=3}^{k-4}\binom{n-2}{l}}{\binom{n-1}{k-1}}\cdot 1_{\{k\geq 8\}} + \frac{\sum_{l=3}^{k-2}\binom{n-2}{l}}{\binom{n-1}{k}}\cdot 1_{\{k\geq 6\}}\right\} \\ &- 2\binom{n-1}{2}\frac{L-1-T}{n}\frac{\binom{n}{k}}{\binom{n-1}{k-1}\binom{n-1}{k}} \end{aligned}$$

if k is even. From $L = \frac{\binom{n-1}{2}}{2^{n-2}-1}$, we have

$$T = \frac{\mu^*_{n,2}}{L} - 1 = \frac{\mu^*_{n,2}(2^{n-2}-1) - \binom{n-1}{2}}{\binom{n-1}{2}}$$

and

$$L - 1 - T = \frac{\binom{n-1}{2}}{2^{n-2}-1} - \mu^*_{n,2}\cdot\frac{2^{n-2}-1}{\binom{n-1}{2}} = \frac{\binom{n-1}{2}^2 - \mu^*_{n,2}\cdot(2^{n-2}-1)^2}{(2^{n-2}-1)\binom{n-1}{2}}.$$

Plugging the functional forms of L, T and $L-1-T$ in n and m into a_k's, we have the coefficient a^*_k for $3 \leq k \leq n-1$ as follows:

$$\begin{aligned} a^*_k = &\frac{n}{n-k}\left\{\frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\cdot 1_{\{k\geq 5\}} + \frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\frac{\sum_{l=3}^{k-3}\binom{n-2}{l}}{\binom{n-1}{k-1}}\cdot 1_{\{k\geq 7\}}\right. \\ &\left. + \frac{2\left[\binom{n-1}{2}^2 - \mu^*_{n,2}\cdot(2^{n-2}-1)^2\right]}{n(2^{n-2}-1)\binom{n-1}{k-1}}\right\} \end{aligned}$$

if k is odd and

$$a_k^* = -\frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\left\{1_{\{k\geq 4\}} + \frac{k-1}{(n-k+1)}\cdot 1_{\{k\geq 6\}}\right\}$$
$$-\frac{2\binom{n-1}{2}}{n(2^{n-2}-1)}\left\{\frac{\sum_{l=3}^{k-4}\binom{n-2}{l}}{\binom{n-1}{k-1}}\cdot 1_{\{k\geq 8\}} + \frac{\sum_{l=3}^{k-2}\binom{n-2}{l}}{\binom{n-1}{k}}\cdot 1_{\{k\geq 6\}}\right\}$$
$$-\frac{2\binom{n}{k}}{\binom{n-1}{k-1}\binom{n-1}{k}}\frac{\binom{n-1}{2}^2 - \mu_{n,2}^*\cdot(2^{n-2}-1)^2}{n(2^{n-2}-1)}$$

if k is even where $0 < \mu_{n,2}^* \leq \frac{\binom{n-1}{2}}{2^{n-2}-1}$. ■

Proof of Theorem 2.3

Let $\tilde{n} = n-1$ and $\hat{n} = n-2$ if n is even and $\tilde{n} = n-2$ and $\hat{n} = n-1$ if n is odd.

Firstly, we will show for any m, $3 \leq m \leq n-2$ in Case 1 and 2:

Case 1. m is odd:

Again $\rho_0 = \frac{n-m}{n}$ and $\rho_{n-1} = 0$. For $1 \leq k \leq n-1$, individual rationality and the worst case constraint require

$$\max\left\{0, (n-m)\left(1-\frac{\lambda}{n}-\frac{1}{n}\right)\right\} \leq X_1 \leq (n-m)\left(\frac{\mu}{n}+\frac{n-1}{n}\right)$$

and for $m-1 \geq k \geq 2$

$$\max\left\{0, (n-m)\left(1-\frac{\lambda k}{n}\right)\right\} \leq X_k \leq (n-m)\left(1+\mu\frac{k}{n}\right)$$

and for $n-1 \geq k \geq m$,

$$0 \leq X_k \leq \mu\frac{m}{n}(n-k).$$

If $\lambda \leq \frac{n}{m-1}$, then, $\max\left\{0, (n-m)\left(1-\frac{\lambda k}{n}\right)\right\} = (n-m)\left(1-\frac{\lambda k}{n}\right)$ and $\max\left\{0, (n-m)\left(1-\frac{\lambda}{n}-\frac{1}{n}\right)\right\} = (n-m)\left(1-\frac{\lambda}{n}-\frac{1}{n}\right)$.

We have

$$
\begin{aligned}
&(n-m)(n+\mu-1) + (n-m)\sum_{\substack{k=3\\ odd}}^{m-2}\binom{n}{k}\left(1+\mu\frac{k}{n}\right) + \sum_{\substack{k=m\\ odd}}^{\tilde{n}}\binom{n}{k}\mu\frac{m}{n}(n-k)\\
&\geq \sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}X_k + \sum_{\substack{k=m\\ odd}}^{\tilde{n}}\binom{n}{k}X_k = \sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}X_k + \sum_{\substack{k=m+1\\ even}}^{\hat{n}}\binom{n}{k}X_k\\
&\geq (n-m)\sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}\max\left\{0, 1-\frac{\lambda k}{n}\right\}.
\end{aligned}
$$

Assuming that $\lambda \leq \frac{n}{m-1}$, we have

$$
\begin{aligned}
&\lambda\left[(n-m)\sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}\frac{k}{n}\right] + \mu\left[(n-m)\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\frac{k}{n} + \sum_{\substack{k=m\\ odd}}^{\tilde{n}}\binom{n}{k}(n-k)\frac{m}{n}\right]\\
&\geq (n-m)\left\{\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k} - \sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\right\}
\end{aligned}
$$

and thus,

$$
\begin{aligned}
&\lambda^*_{n,m}\left[(n-m)\sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}\frac{k}{n}\right] + \mu^*_{n,m}\left[(n-m)\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\frac{k}{n} + \sum_{\substack{k=m\\ odd}}^{\tilde{n}}\binom{n}{k}(n-k)\frac{m}{n}\right]\\
&= (n-m)\left\{\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k} - \sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\right\}.
\end{aligned}
$$

This is rewritten as

$$\mu^*_{n,m}\Bigg[(n-m)\sum_{k=0}^{m-3}\binom{n-2}{k}+m\Bigg[\sum_{k=m-1}^{\tilde{n}}\binom{n-1}{k}-\sum_{k=m-2}^{\tilde{n}-1}\binom{n-2}{k}\Bigg]\Bigg]$$
$$+\lambda^*_{n,m}\Bigg[(n-m)\sum_{k=0}^{m-2}\binom{n-2}{k}\Bigg]=(n-m)\binom{n-1}{m-1}.$$

Now we will check if $\lambda^*_{n,m}$ satisfies the assumption $\lambda \leq \frac{n}{m-1}$. Let $\mu^*_{n,m} = \delta\lambda^*_{n,m}$ for some $\delta \geq 0$. Then,

$$\lambda^*_{n,m}=\frac{(n-m)\Big\{\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k}-\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\Big\}}{(n-m)\Big\{\sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}\frac{k}{n}+\sum_{\substack{k=1\\ odd}}^{m-2}\delta\binom{n}{k}\frac{k}{n}\Big\}+\delta\sum_{\substack{k=m\\ odd}}^{\tilde{n}}\binom{n}{k}(n-k)\frac{m}{n}}.$$

$\lambda^*_{n,m} \leq \frac{n}{m-1}$ holds if and only if

$$(n-m)(m-1)\Bigg\{\sum_{\substack{k=0\\ even}}^{m-1}\binom{n}{k}-\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k}\Bigg\}$$
$$\leq n\Bigg\{(n-m)\Bigg\{\sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}\frac{k}{n}+\sum_{\substack{k=1\\ odd}}^{m-2}\delta\binom{n}{k}\frac{k}{n}\Bigg\}+\delta\sum_{\substack{k=m\\ odd}}^{\tilde{n}}\binom{n}{k}(n-k)\frac{m}{n}\Bigg\}.$$

Since $\binom{n}{k}k = n\binom{n-1}{k-1}$ and $\binom{n}{k}(n-k) = n\binom{n-1}{k}$, the right hand side of the previous inequality is written as

$$n\Bigg\{(n-m)\Bigg\{\sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k}\frac{k}{n}+\sum_{\substack{k=1\\ odd}}^{m-2}\delta\binom{n}{k}\frac{k}{n}\Bigg\}+\delta\sum_{\substack{k=m\\ odd}}^{\tilde{n}}\binom{n}{k}(n-k)\frac{m}{n}\Bigg\}$$
$$=(n-m)\Bigg\{n\sum_{\substack{k=2\\ even}}^{m-1}\binom{n-1}{k-1}+n\delta\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n-1}{k-1}\Bigg\}+nm\delta\sum_{\substack{k=m\\ odd}}^{\tilde{n}}\binom{n-1}{k}.$$

Thus, $\lambda^*_{n,m} \leq \frac{n}{m-1}$ if and only if

$$(n-m)\Bigg\{(m-1)\Bigg(\sum_{\substack{k=0\\even}}^{m-1}\binom{n}{k} - \sum_{\substack{k=1\\odd}}^{m-2}\binom{n}{k}\Bigg) - n\sum_{\substack{k=2\\even}}^{m-1}\binom{n-1}{k-1}\Bigg\}$$
$$\leq \delta\Bigg\{n(n-m)\sum_{\substack{k=1\\odd}}^{m-2}\binom{n-1}{k-1} + mn\sum_{\substack{k=m\\odd}}^{\tilde{n}}\binom{n-1}{k}\Bigg\}.$$

The right hand side of the inequality is nonnegative. We will show that the left hand side is always negative and thus, the inequality holds. Let

$$A(n) = (m-1)\Bigg(\sum_{\substack{k=0\\even}}^{m-1}\binom{n}{k} - \sum_{\substack{k=1\\odd}}^{m-2}\binom{n}{k}\Bigg) - n\sum_{\substack{k=2\\even}}^{m-1}\binom{n-1}{k-1}.$$

Observe that $A(n) < 0$ if $m = 3$. For $m \geq 5$, we check first

$$\begin{aligned} A(m+1) &= (m-1)\Bigg(\sum_{\substack{k=0\\even}}^{m-1}\binom{m+1}{k} - \sum_{\substack{k=1\\odd}}^{m-2}\binom{m+1}{k}\Bigg) - (m+1)\sum_{\substack{k=2\\even}}^{m-1}\binom{m}{k-1} \\ &= (m-1)m - (m+1)\sum_{\substack{k=2\\even}}^{m-1}\binom{m}{k-1} \leq (m-1)m - m(m+1) < 0 \end{aligned}$$

and show that $A(n)$ is decreasing in n.

$$\begin{aligned} A(n) - A(n+1) =&(m-1)\Bigg(\sum_{\substack{k=0\\even}}^{m-1}\binom{n}{k} - \sum_{\substack{k=1\\odd}}^{m-2}\binom{n}{k}\Bigg) - n\sum_{\substack{k=2\\even}}^{m-1}\binom{n-1}{k-1} \\ &- (m-1)\Bigg(\sum_{\substack{k=0\\even}}^{m-1}\binom{n+1}{k} - \sum_{\substack{k=1\\odd}}^{m-2}\binom{n+1}{k}\Bigg) + (n+1)\sum_{\substack{k=2\\even}}^{m-1}\binom{n}{k-1}. \end{aligned}$$

Since $\binom{n-1}{k-1} - \binom{n}{k-1} = -\binom{n-1}{k-2}$, $\binom{n}{k} - \binom{n+1}{k} = -\binom{n}{k-1}$ and $n\binom{n-1}{k-2} + \binom{n}{k-1} = k\binom{n}{k-1}$, we have

$$A(n) - A(n+1) = (m-1)\Bigg[\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k-1} - \sum_{\substack{k=2\\ even}}^{m-1}\binom{n}{k-1}\Bigg] + \sum_{\substack{k=2\\ even}}^{m-1} k\binom{n}{k-1}.$$

Finally, we write

$$\begin{aligned} A(n) - A(n+1) &= (m-1)\Bigg[\sum_{\substack{k=1\\ odd}}^{m-2}\binom{n}{k-1} - \sum_{\substack{k=2\\ even}}^{m-3}\binom{n}{k-1}\Bigg] + \sum_{\substack{k=2\\ even}}^{m-3} k\binom{n}{k-1} \\ &= (m-1)\Bigg[\sum_{\substack{k=0\\ even}}^{m-3}\binom{n}{k} - \sum_{\substack{k=1\\ odd}}^{m-4}\binom{n}{k}\Bigg] + \sum_{\substack{k=2\\ even}}^{m-3} k\binom{n}{k-1} > 0 \end{aligned}$$

and thus, $A(n)$ is decreasing in n. We conclude that $A(n) < 0$, that is, the desired inequality holds for $\lambda^*_{n,m}$.

Case 2. *m is even:*

We have

$$\begin{aligned} &\max\left\{0, n(n-m)\left(1-\frac{\lambda}{n}-\frac{1}{n}\right)\right\} + \sum_{\substack{k=3\\ odd}}^{m-1}\binom{n}{k}\max\left\{0,(n-m)\left(1-\frac{\lambda k}{n}\right)\right\} \\ &\leq \sum_{\substack{k=1\\ odd}}^{m-1}\binom{n}{k}X_k + \sum_{\substack{k=m+1\\ odd}}^{\tilde{n}}\binom{n}{k}X_k = \sum_{\substack{k=2\\ even}}^{m-2}\binom{n}{k}X_k + \sum_{\substack{k=m\\ even}}^{\hat{n}}\binom{n}{k}X_k \\ &\leq (n-m)\sum_{\substack{k=2\\ even}}^{m-2}\binom{n}{k}\left(1+\mu\frac{k}{n}\right) + \mu\sum_{\substack{k=m\\ even}}^{\hat{n}}\binom{n}{k}\frac{m}{n}(n-k) \end{aligned}$$

Given m, the optimal surplus loss λ^* can be considered as a function of n. We should find $\hat{k}$ such that $\hat{k} = \max\{k,\ 3 \leq k \leq m-1, \text{odd} \mid \frac{n}{k} \geq \lambda^*(n) \text{ for all } n\}$. Here $\lambda^*(n)$ is the optimal surplus loss computed with assuming $n/\hat{k} \geq \lambda^*$.

Assuming $n/\hat{k} \geq \lambda$, we have

$$\max\left\{0, (n-m)\left(1-\frac{\lambda k}{n}\right)\right\} = (n-m)\left(1-\frac{\lambda k}{n}\right)$$

only for $3 \leq k \leq \hat{k}$. Then,

$$\begin{aligned}
&(n-m)(n-\lambda-1) + (n-m)\sum_{\substack{k=3\\ odd}}^{\hat{k}} \binom{n}{k}\left(1-\frac{\lambda k}{n}\right) \\
&\leq \sum_{\substack{k=1\\ odd}}^{m-1} \binom{n}{k} X_k + \sum_{\substack{k=m+1\\ odd}}^{\tilde{n}} \binom{n}{k} X_k = \sum_{\substack{k=2\\ even}}^{m-2} \binom{n}{k} X_k + \sum_{\substack{k=m\\ even}}^{\hat{n}} \binom{n}{k} X_k \\
&\leq (n-m)\sum_{\substack{k=2\\ even}}^{m-2} \binom{n}{k}\left(1+\mu\frac{k}{n}\right) + \frac{m}{n}\mu \sum_{\substack{k=m\\ even}}^{\hat{n}} \binom{n}{k}(n-k)
\end{aligned}$$

gives

$$\begin{aligned}
&\lambda\left[(n-m)\sum_{\substack{k=1\\ odd}}^{\hat{k}} \binom{n}{k}\frac{k}{n}\right] + \mu\left[(n-m)\sum_{\substack{k=2\\ even}}^{m-2} \binom{n}{k}\frac{k}{n} + \frac{m}{n}\sum_{\substack{k=m\\ even}}^{\hat{n}} \binom{n}{k}(n-k)\right] \\
&\geq (n-m)\left[\sum_{\substack{k=1\\ odd}}^{\hat{k}} \binom{n}{k} - \sum_{\substack{k=0\\ even}}^{m-2} \binom{n}{k}\right]
\end{aligned}$$

and thus,

$$\begin{aligned}
&\lambda^*_{n,m}\left[(n-m)\sum_{\substack{k=1\\ odd}}^{\hat{k}} \binom{n}{k}\frac{k}{n}\right] + \mu^*_{n,m}\left[(n-m)\sum_{\substack{k=2\\ even}}^{m-2} \binom{n}{k}\frac{k}{n} + \frac{m}{n}\sum_{\substack{k=m\\ even}}^{\hat{n}} \binom{n}{k}(n-k)\right] \\
&= (n-m)\left[\sum_{\substack{k=1\\ odd}}^{\hat{k}} \binom{n}{k} - \sum_{\substack{k=0\\ even}}^{m-2} \binom{n}{k}\right].
\end{aligned}$$

Let $\mu^*_{n,m} = \delta\lambda^*_{n,m}$ for some $\delta \geq 0$. Then, the optimal surplus loss computed with assuming $n/\hat{k} \geq \lambda^*_{n,m}$ is

$$\lambda^*_{n,m} = \frac{(n-m)\left[\sum_{\substack{k=1\\odd}}^{\hat{k}}\binom{n}{k} - \sum_{\substack{k=0\\even}}^{m-2}\binom{n}{k}\right]}{(n-m)\left[\sum_{\substack{k=1\\odd}}^{\hat{k}}\binom{n}{k}\frac{k}{n} + \delta\sum_{\substack{k=2\\even}}^{m-2}\binom{n}{k}\frac{k}{n}\right] + \frac{m}{n}\delta\sum_{\substack{k=m\\even}}^{\hat{n}}\binom{n}{k}(n-k)}$$

and this $\lambda^*_{n,m}$ should not contradict the assumption, that is,

$$\frac{n}{\hat{k}} \geq \frac{(n-m)\left[\sum_{\substack{k=1\\odd}}^{\hat{k}}\binom{n}{k} - \sum_{\substack{k=0\\even}}^{m-2}\binom{n}{k}\right]}{(n-m)\left[\sum_{\substack{k=1\\odd}}^{\hat{k}}\binom{n}{k}\frac{k}{n} + \delta\sum_{\substack{k=2\\even}}^{m-2}\binom{n}{k}\frac{k}{n}\right] + \frac{m}{n}\delta\sum_{\substack{k=m\\even}}^{\hat{n}}\binom{n}{k}(n-k)}$$

for $3 \leq \hat{k} \leq m-1$. This inequality is equivalent to

$$\delta\left[\sum_{\substack{k=2\\even}}^{m-2}\binom{n}{k}k + \frac{m}{n-m}\sum_{\substack{k=m\\even}}^{\hat{n}}\binom{n}{k}(n-k)\right] \geq \hat{k}\left[\sum_{\substack{k=1\\odd}}^{\hat{k}}\binom{n}{k} - \sum_{\substack{k=0\\even}}^{m-2}\binom{n}{k}\right] - \sum_{\substack{k=1\\odd}}^{\hat{k}}\binom{n}{k}k.$$

Since the left hand side of the inequality is always nonnegative for any n, $n \geq m+1$, we like to have the right hand side negative for any n, $n \geq m+1$. Our objective is to find maximal $\hat{k}$, $3 \leq \hat{k} \leq m-1$ satisfying

$$\hat{k}\left[\sum_{\substack{k=1\\odd}}^{\hat{k}}\binom{n}{k} - \sum_{\substack{k=0\\even}}^{m-2}\binom{n}{k}\right] - \sum_{\substack{k=1\\odd}}^{\hat{k}}\binom{n}{k}k \leq 0. \tag{3.3}$$

Let $\hat{k} = m-1$. Then, inequality (3.3) is

$$\sum_{\substack{k=1\\odd}}^{m-1}\binom{n}{k}k \geq (m-1)\left[\sum_{\substack{k=1\\odd}}^{m-1}\binom{n}{k} - \sum_{\substack{k=0\\even}}^{m-2}\binom{n}{k}\right].$$

This is rewritten as

$$\sum_{\substack{k=1\\ odd}}^{m-3} \binom{n}{k}(m-1-k) - (m-1)\sum_{\substack{k=0\\ even}}^{m-2} \binom{n}{k} \leq 0.$$

Let $S(n) = \sum_{\substack{k=1\\ odd}}^{m-3} \binom{n}{k}(m-1-k) - (m-1)\sum_{\substack{k=0\\ even}}^{m-2} \binom{n}{k}$. Firstly observe that

$$S(m+1) = 1 + m^2 - 2^{m-1}(m+1) < 0$$

for m, $m \geq 4$. We can show that $S(n)$ is decreasing in n for $n \geq m+1$. Using $\binom{n}{k} - \binom{n+1}{k} = -\binom{n}{k-1}$, we write

$$\begin{aligned} S(n) - S(n+1) &= -(m-1)\sum_{k=1}^{m-2}(-1)^{k-1}\binom{n}{k-1} + k\sum_{\substack{k=1\\ odd}}^{m-3}\binom{n}{k-1} \\ &= -(m-1)\sum_{k=0}^{m-3}(-1)^{k}\binom{n}{k} + k\sum_{\substack{k=1\\ odd}}^{m-3}\binom{n}{k-1}. \end{aligned}$$

From $\sum_{j=0}^{k}(-1)^j\binom{n}{j} = (-1)^k\binom{n-1}{k}$ for $0 \leq k \leq n-1$,

$$S(n) - S(n+1) = (m-1)(-1)^{m-2}\binom{n-1}{m-3} + k\sum_{\substack{k=1\\ odd}}^{m-3}\binom{n}{k-1}.$$

Since m is even, we have $S(n) - S(n+1) > 0$ implying that $S(n)$ is decreasing in n given m. Therefore, we conclude $S(n) < 0$ for all n, $n \geq m+1$ as desired. That is, $\hat{k} = m-1$ works without contradiction and the optimal efficiency loss must be computed with $\hat{k} = m-1$.

Plugging $\hat{k} = m - 1$, we have

$$\mu^*_{n,m}\left[(n-m)\sum_{k=0}^{m-3}\binom{n-2}{k} + m\left[\sum_{k=m-1}^{\hat{n}}\binom{n-1}{k} - \sum_{k=m-2}^{\hat{n}-1}\binom{n-2}{k}\right]\right]$$
$$+ \lambda^*_{n,m}\left[(n-m)\sum_{k=0}^{m-2}\binom{n-2}{k}\right] = (n-m)\binom{n-1}{m-1}.$$

Now we will show the statement for the case of $m = n - 1$ for $m \geq 3$:

The two way worst case constraints are written as

$$c^{*(n-1)} - \lambda\left[\frac{n-1}{n}c^{*n} - \frac{1}{n}\sum_{i=1}^{n-1}c^{*i}\right] \leq \sum_{i\in N} r(c_{-i}) \leq c^{*(n-1)} + \mu\left[\frac{n-1}{n}c^{*n} - \frac{1}{n}\sum_{i=1}^{n-1}c^{*i}\right].$$

Applying e^{n-k} with individual rationality, we have $\rho_0 = \frac{1}{n}$ and $\rho_{n-1} = 0$. For $k = 1$,

$$\max\{0, \frac{n-1-\lambda}{n}\} \leq (n-1)\rho_1 \leq \frac{n-1+\mu}{n}$$

and for $n - 2 \geq k \geq 2$,

$$\max\{0, 1 - \frac{k}{n}\lambda\} \leq k\rho_{k-1} + (n-k)\rho_k \leq 1 + \frac{k}{n}\mu.$$

For $k = n - 1$, we have

$$0 \leq (n-1)\rho_{n-2} \leq \mu\frac{n-1}{n}.$$

If n is odd and $\frac{n}{n-2} \geq \lambda$,

$$(n-1-\lambda) + \sum_{\substack{k=3 \\ odd}}^{n-2} \binom{n}{k}\left(1 - \frac{k}{n}\lambda\right) \leq \sum_{\substack{k=2 \\ even}}^{n-3} \binom{n}{k}\left(1 + \frac{k}{n}\mu\right) + \mu(n-1)$$

which is equivalent to

$$(n-1) \leq \lambda(2^{n-2} - 1) + \mu \cdot 2^{n-2}.$$

Thus, the optimal frontier is

$$(n-1) = \lambda^*(2^{n-2} - 1) + \mu^* \cdot 2^{n-2}.$$

We can easily check that the maximal λ^* on the optimal frontier satisfies $\lambda^* \leq \frac{n}{n-2}$. Likewise, if n is even and $\lambda \leq \frac{n}{n-2}$,

$$(n-1+\mu) + \sum_{\substack{k=3 \\ odd}}^{n-3} \binom{n}{k}\left(1 + \frac{k}{n}\mu\right) + \binom{n}{n-1}\mu\frac{n-1}{n} \geq \sum_{\substack{k=2 \\ even}}^{n-2} \binom{n}{k}\left(1 - \frac{k}{n}\lambda\right).$$

From this, we have

$$\lambda(2^{n-2} - 1) + \mu 2^{n-2} \geq n - 1$$

and the optimal frontier is

$$\lambda^*(2^{n-2} - 1) + \mu^* 2^{n-2} = n - 1.$$

The maximal λ^* on the optimal frontier doesn't contradict $\lambda \leq \frac{n}{n-2}$. ■

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