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A coupled finite volume and discontinuous Galerkin method for convection-diffusion problems

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ABSTRACT

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This work formulates and analyzes a new coupled finite volume (FV) and discontinuous Galerkin (DG) method for convection-diffusion problems. DG methods, though costly, have proved to be accurate for solving convection-diffusion problems and capable of handling discontinuous and tensor coefficients. FV methods have proved to be very efficient but they are only of first order accurate and they become ineffective for tensor coefficient problems. The coupled method takes advantage of both the accuracy of DG methods in the regions containing heterogeneous coefficients and the efficiency of FV methods in other regions. Numerical results demonstrate that this coupled method is able to resolve complicated coefficient problems with a decreased computational cost compared to DG methods. This work can be applied to problems such as the transport of contaminant underground, the CO_2 sequestration and the transport of cells in the body.

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Contents

Illustrations

Tables

Chapter 1

Introduction

This work presents and analyzes the coupling of finite volume (FV) and discontinuous Galerkin (DG) methods for convection-diffusion equations. DG methods have proved to be accurate for convection-diffusion problems during the last fifteen years because DG methods have many good properties such as locality, local mass conservativity and high accuracy, but DG methods are indeed costly. FV methods have been widely used for convection-diffusion problems thanks to their efficiency and locally mass conservative property. However, they become ineffective for more complicated problems that have tensor or discontinuous coefficients. I use DG methods to gain accuracy in the heterogeneous coefficient region and use FV methods to reduce total computational cost.

1.1 DG overview

In 1973, Reed and Hill [1] first introduced the Discontinuous Galerkin method to solve the neutron transport equation which is a hyperbolic equation. Since then, the DG methods for hyperbolic and nearly hyperbolic equations were developed rigorously. The stabilization and error estimates of linear hyperbolic systems were studied by many people such as Delfour, Hager and Trochu [2], Bottcher and Rannacher [3]. At the same time, the DG methods for nonlinear hyperbolic systems were developed. Examples are the slope limiter method introduced by Chavent and Cockburn [4], the

Runge Kutta Discontinuous Galerkin (RKDG) method by Cockburn and Shu [5], the DG method of Allmaras [6] and the DG method of Halt [7]. With the satisfactory results of the DG methods for hyperbolic problems, people started to explore the DG methods on convection problems with non-negligible diffusion.

In addition, Galerkin methods for elliptic and parabolic equations using discontinuous finite element methods were introduced also in the 1970s. These Galerkin methods use penalties to guarantee stability and impose continuity and are named interior penalty discontinuous Galerkin (IPDG) methods. Examples are found in Babuška [8], Baker [9], Wheeler [10] and Arnold [11]. IPDG methods were developed independently of the DG methods for hyperbolic problems for many years. Then in the 1990s when DG methods for hyperbolic problems were extended to elliptic problems, some authors began to realize the similarity between the newly developed DG and the old IPDG methods. A unified framework of the DG methods for elliptic problems were given by Arnold, Brezzi, Cockburn and Marini [12] in 2000. They also compared the properties such as consistency, conservativity, stability and error in H^1, L^2 spaces of almost all DG methods up to then [13].

I now give an overview of the DG methods for the convection-diffusion equations, based on the DG schemes for the convection terms in hyperbolic equations and the diffusion terms in elliptic equations. In 1991, Dawson [14] introduced the upwind-mixed finite element methods to solve advection-diffusion equations. The DG approximation is obtained by mixed formulations which use auxiliary variables to cast the second order equation to first order equation systems. This method has a disadvantage of adding as many equations as the dimension of the problem. This methodology was also used by Arbogast and Wheeler [15], Bassi and Rebay [16] for compressible Navier-Stokes equations, and Warburton, Lomtev, Kirby and Karniadakis [17] for the Navier-Stokes equations, and the error estimates were given by Brezzi, Manzini, Marini, Pietra and Russo [18]. Following the same approach, Cockburn and Shu [19] presented the LDG method for time-dependent convection-diffusion systems. Then in 1999, Baumann and Oden [20] introduced a OBB-DG method that does not use auxiliary variables. They compared their method with the continuous Galerkin method and their method turned out to be much more robust. They also presented priori error estimates for one-dimensional problems and for polynomials with degree larger than two. In 1999, Riviere, Wheeler, Girault [21] presented error estimates in higher dimensions. In 2003, the incomplete interior penalty Galerkin method (IIPG) was introduced in [22, 23]. Up to then there were four members in the primal DG methods, including OBB-DG, symmetric interior penalty method (SIPG) [10],non-symmetric interior penalty method (NIPG) [24] and IIPG. In 2006, Sun and Wheeler [25] analyzed three primal DG methods with penalty for solving reactive transport problems in porous media. They built up the DG schemes and derived error estimates in $L^2(H^1)$ and $L^2(L^2)$ for SIPG, NIPG and IIPG. They also numerically investigated the $h-$ and $p-$convergence behaviors.

Much more work on DG methods for convection-diffusion problems was done during the last ten years. DG methods prove to be robust because they can be used on unstructured meshes and they have some properties such as locality, local conservation, high accuracy and are capable of handling discontinuous coefficients. In 2001, the priori error analysis of the LDG method for elliptic problems was studied and tested by numerical experiments by Castillo, Cockburn, Perugia and Schötzau

[26]. The convergence properties of the hp-version of the LDG method for 1 dimensional convection-diffusion problems was analyzed by Castillo, Cockburn, Schötzau and Schwab [27] in the same year. In 2005, a new stabilized mixed discontinuous Galerkin method for Darcy flow was presented and analyzed by Hughes, Masud and Wan [28]. The framework for the construction and the analysis of the newly developed discontinuous Galerkin method for the elliptic problems was also proposed. Examples are Brezzi, Cockburn, Marini and Suli [29], and Cockburn, Gopalakrishnan and Lazarov [30]. In 2009, Proft and Riviere [31] developed and analyzed a new family of DG methods for time-dependent convection-diffusion equations with highly varying or even vanishing diffusion coefficients. They did not use slope limiting techniques or streamline-diffusion stabilization. Instead, their methods, which were based on NIPG/SIPG, used special fluxes. Their methods would automatically choose the right flux to maintain stability according to the variability of the convection-diffusion ratio. In my paper, I use their IPDG scheme with some modifications to suit my problem better.

Though DG methods are very accurate, they are also very costly. Meanwhile, FV methods are well developed for their efficiency and local mass conservation property. We now have an overview of FV methods.

1.2 FV overview

Finite volume methods for convection-diffusion equations were first introduced in the early 1960s by Tichonov and Samarskii [32, 33]. However, the stability and convergence rates of FV methods in various dimensions were not studied until the eighties and nineties. FV methods can be divided into vertex-centered FV schemes and cell-centered FV schemes by the position of the points concerned with fluxes. Since cell-centered grids are very attractive when it comes to physical discontinuities and internal boundaries, I will only consider the cell-center schemes here.

In 1988, quadratic convergence rates of the approximate solution for elliptic problems were confirmed by Forsyth and Sammon [34], and both the approximate solution and its first derivatives were proved to be second-order convergent for discrete L^2 norm for all nonuniform rectangular grids by Weiser and Wheeler [35]. In the 1990s, Vassilevski, Petrova and Lazarov [36] established the FV scheme for elliptic equations on triangular cell-centered grids and found the second order superconvergence rate in H^1 norm. A similar superconvergence rate study was done by Arbogast, Wheeler and Yotov [36]. FV methods for convection-diffusion problems were studied on rectangular grids by Lazarov, Mishev and Vassilevski [37]. They established and analyzed the upwind scheme with first order accuracy with respect to H^1 and L^2 norms and the modified upwind scheme with second order accuracy. FV schemes and error estimates for nonlinear convection-diffusion problems were also studied. Examples are Feistauer and Felcman [38], and Eymard, Gallouet and Herbin [39]. In 2000, Gallouet, Herbin and Vignal [40] established the error estimates for convection-diffusion equations with three general boundary conditions which are Dirichlet, Neuman and Robin. They proved that FV schemes are first order accurate with respect to the H^1 and L^2 norms for the admissible meshes including the voronoi and triangular meshes. Their work is the basis of this paper because of the first order accuracy with respect to discrete H^1 norm. We do not want to use 0 degree polynomials in p-DG methods because we will have zero order accuracy for H^1 norm where we use 0 degree polynomials.

In summary, DG methods can handle well for convection-diffusion equations with heterogeneous coefficients, but they are indeed costly. FV methods are very efficient but they can not handle complicated coefficient problems. Thus, it is worth looking for a coupled method which can utilize the advantages of the DG methods and reduce computational cost at the same time. In 2010, Chidyagwai and Riviere [41] introduced a coupled method of finite volume and DG methods for elliptic problems. They used the IPDG method in the regions containing complicated features and the highly efficient FV method in other regions to reduce computational cost. They proved the convergence of the error with respect to the energy norm. Inspired by their work, I use the coupled FV and DG method on the convection-diffusion equations, and I show theoretically and numerically that the error of the new scheme also converges.

This paper is organized as follows. In Chapter 2, I present the 1D steady state convection-diffusion model problem. Then I give the coupled scheme and prove the error has first order accuracy. At last, I show some numerical results to verify my proof. In Chapter 3, I introduce some notations and the steady state convection-diffusion model problem. Then I show the coupled scheme and prove that the scheme has a unique solution. I also prove that the error is is bounded by $C(h_{FV}+h_{DG}^p)$, where C is a constant and h_{FV}, h_{DG} are the size of the FV and DG meshes respectively. At last, I give some numerical examples. In Chapter 4, I introduce the time-dependent convection-diffusion model problem. I give the coupled scheme and analyze it. In Chapter 5, I give the conclusion and an overview of my future work.

Chapter 2

The coupled FV and DG method for 1D convection-diffusion problem

2.1 Model problem

Consider the two point boundary value problem:

$$\begin{aligned} &-(K(x)u'(x))' + \beta u'(x) = f(x), \quad x \in (a,b) \\ &u(a) = g_1,\ u(b) = g_2 \end{aligned} \tag{2.1}$$

where $K \in H^1([a,b]), f \in L^2([a,b])$, and β is a constant. Assume that

$$0 < k_0 \leq K(x) \leq k_1 < \infty, x \in (a,b).$$

2.2 Scheme

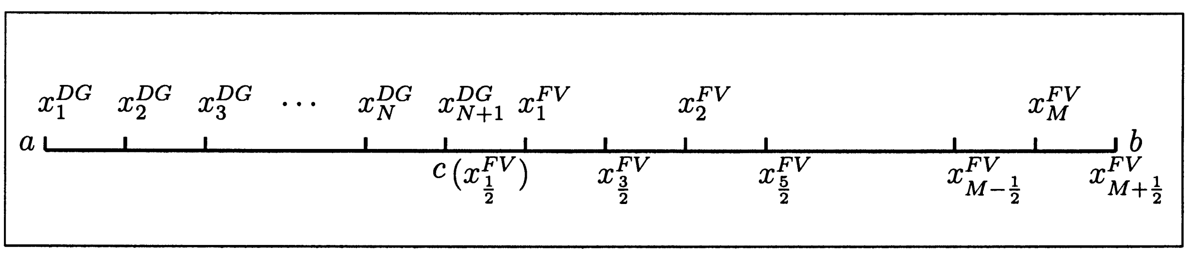

Figure 2.1 : 1D mesh

Let c be a point between a and b. Let $a = x_1^{DG} < x_2^{DG} < \cdots < x_{N+1}^{DG} = c$ be a partition of $[a,c]$ (see Figure 2.1) and let I_n^{DG} denote the interval $[x_n^{DG}, x_{n+1}^{DG}]$.

Define $h_n^{DG} = x_{n+1}^{DG} - x_n^{DG}$, $\tilde{h}_n^{DG} = \max(h_n^{DG}, h_{n+1}^{DG})$ and $h_{DG} = \max_{1\leq n\leq N}\left(h_n^{DG}\right)$. We require that there is a constant θ such that

$$1 \leq \frac{h_{DG}}{\min_{1\leq n\leq N}(h_n^{DG})} \leq \theta < \infty.$$

Define the jump as $[v(x)] = v(x^-) - v(x^+)$, for $a < x < c$, $[v(a)] = -v(a^+)$, $[v(c)] = v(c^-) - v(x_1^{FV})$.

Define the average as $\{v(x)\} = \frac{1}{2}(v(x^-) + v(x^+))$, for $a < x < c$, $\{v(a)\} = v(a^+)$, $\{v(c)\} = \frac{1}{2}\left(v(c^-) + v(x_1^{FV})\right)$.

Define the upwind as

$$\begin{aligned} u^{\uparrow}(x) &= u(x^-), \quad \text{if } \beta \geq 0, \quad \forall x \in (a,c), \\ u^{\uparrow}(x) &= u(x^+), \quad \text{if } \beta < 0, \quad \forall x \in (a,c), \\ u^{\uparrow}(c) &= u(x^-), \quad \text{if } \beta \geq 0, \\ u^{\uparrow}(c) &= u(x_1^{FV}), \quad \text{if } \beta < 0. \end{aligned} \tag{2.2}$$

Then the exact solution $u|_{[a,c]}$ satisfies:

$$a^{DG}(u,v) - K(c)u'(c^-)v(c^-) + \beta u(c^-)v(c^-) = L^{DG}(v), \tag{2.3}$$

where

$$\begin{aligned} & a^{DG}(u,v) \\ &= -\sum_{n=1}^{N}\int_{x_n^{DG}}^{x_{n+1}^{DG}} (\beta u - Ku')v' + \sum_{n=2}^{N} \beta u^{\uparrow}(x_n^{DG})[v(x_n^{DG})] - \sum_{n=2}^{N}\{Ku'(x_n^{DG})\}[v(x_n^{DG})] \\ &+ \varepsilon\sum_{n=2}^{N}\{Kv'(x_n^{DG})\}[u(x_n^{DG})] + \sum_{n=2}^{N}(\tilde{h}_n^{DG})^{-1}\sigma_n[u(x_n^{DG})][v(x_n^{DG})] + \frac{1}{\tilde{h}_1^{DG}}\sigma_1 u(a^+)v(a^+) \\ &+ K(a)u'(a^+)v(a^+) - \varepsilon K(a)v'(a^+)u(a^+) - \frac{1}{2}\beta u(a^+)v(a^+) \end{aligned}$$

$$L^{DG}(v) = \int_a^c fv - \varepsilon K(a)g_1 v'(a^+) + \frac{1}{2}\beta g_1 v(a^+) + (\tilde{h}_1^{DG})^{-1}\sigma_1 g_1 v(a^+)$$

for all v piecewise discontinuous polynomial of degree r on the interval $[a, c]$:

$$v \in \{v : v|_{I_n^{DG}} \in \mathbb{P}_r(I_n^{DG})\}.$$

The parameters in the bilinear form a^{DG} are the penalty parameters. σ_n is a nonnegative real number that penalizes the jumps in the solution u and the symmetrization parameter ε takes the value -1 or $+1$. The proof of (2.3) is found in the proof of Lemma 2.2.

Let $c = x_{\frac{1}{2}}^{FV} < x_1^{FV} < x_{\frac{3}{2}}^{FV} < x_2^{FV} < \cdots < x_M^{FV} < x_{M+\frac{1}{2}}^{FV} = b$ be a partition of $[c, b]$ (see Figure 2.1). Define $I_i^{FV} = [x_{i-\frac{1}{2}}^{FV}, x_{i+\frac{1}{2}}^{FV}]$, and let $h_{i+\frac{1}{2}} = x_{i+1}^{FV} - x_i^{FV}, i = 1, \cdots, M-1$. Let $h_0 = x_1^{FV} - c$ and $h_M = b - x_M^{FV}$. We define

$$h_{FV} = \max_{1 \le i \le M-1} (h_{i+\frac{1}{2}}, h_0, h_M).$$

Define the jump as $[v](x_{i+\frac{1}{2}}^{FV}) = v(x_i^{FV}) - v(x_{i+1}^{FV}),\ i = 1, \cdots, M-1$.

Define the upwind as

$$\forall i \ge 1,\ v^{\uparrow}(x_{i+\frac{1}{2}}^{FV}) = v(x_i^{FV}),\ \text{if } \beta \ge 0,$$
$$\forall i \ge 1,\ v^{\uparrow}(x_{i+\frac{1}{2}}^{FV}) = v(x_{i+1}^{FV}),\ \text{if } \beta < 0.$$

Multiplying (2.1) by v, a piecewise constant function over the partition $c = x_{\frac{1}{2}}^{FV} < x_{\frac{3}{2}}^{FV} < \cdots < x_{M+\frac{1}{2}}^{FV} = b$, integrating over every interval I_i^{FV}, using integration by part and summing over all intervals, we obtain

$$\begin{aligned}
&\sum_{i=2}^{M-1} \Big(K(x_{i-\frac{1}{2}}^{FV}) u'(x_{i-\frac{1}{2}}^{FV}) v_i - K(x_{i+\frac{1}{2}}^{FV}) u'(x_{i+\frac{1}{2}}^{FV}) v_i \Big) \\
&- K(x_{\frac{3}{2}}^{FV}) u'(x_{\frac{3}{2}}^{FV}) v_1 + K(c) u'(c) v_1 + K(x_{M-\frac{1}{2}}^{FV}) u'(x_{M-\frac{1}{2}}^{FV}) v_M - K(b) u'(b) v_M \\
&+ \sum_{i=2}^{M-1} \Big(-\beta u(x_{i-\frac{1}{2}}^{FV}) v_i + \beta u(x_{i+\frac{1}{2}}^{FV}) v_i \Big) \\
&+ \beta u(x_{\frac{3}{2}}^{FV}) v_1 - \beta u(c) v_1 - \beta u(x_{M-\frac{1}{2}}^{FV}) v_M + \beta u(b) v_M \\
= &\int_c^b f v.
\end{aligned} \tag{2.4}$$

We will use the notation $v_i = v(x_i^{FV})$ throughout the text.

Next we replace the derivative terms with finite difference approximations and the terms having β as the coefficient with the upwind approximations. But we keep the terms evaluated at the point c unchanged. We obtain the forms:

$$\begin{aligned}
a^{FV}(u,v) &= \sum_{i=2}^{M-1}\left(K(x_{i-\frac{1}{2}}^{FV})\frac{u(x_i^{FV})-u(x_{i-1}^{FV})}{h_{i-\frac{1}{2}}}v_i - K(x_{i+\frac{1}{2}}^{FV})\frac{u(x_{i+1}^{FV})-u(x_i^{FV})}{h_{i+\frac{1}{2}}}v_i\right)\\
&\quad - K(x_{\frac{3}{2}}^{FV})\frac{u(x_2^{FV})-(x_1^{FV})}{h_{\frac{3}{2}}}v_1 + K(x_{M-\frac{1}{2}}^{FV})\frac{u(x_M^{FV})-u(x_{M-1}^{FV})}{h_{M-\frac{1}{2}}}v_M\\
&\quad + K(b)\frac{u(x_M^{FV})}{h_M}v_M + \sum_{i=2}^{M-1}\left(-\beta u^{\uparrow}(x_{i-\frac{1}{2}}^{FV})v_i + \beta u^{\uparrow}(x_{i+\frac{1}{2}}^{FV})v_i\right)\\
&\quad + \beta u^{\uparrow}(x_{\frac{3}{2}}^{FV})v_1 - \beta u^{\uparrow}(x_{M-\frac{1}{2}}^{FV})v_M + \frac{1}{2}\beta u(x_M^{FV})v_M\\
L^{FV}(v) &= \int_c^b fv + K(b)\frac{g_2}{h_M}v_M - \frac{1}{2}\beta g_2 v_M
\end{aligned} \tag{2.5}$$

for all v, piecewise constant over $[c,b]$.

We now consider the coupling of DG and FV at the interface c. We obtain two different schemes by choosing two different interface forms. If we replace the derivative at the interface with the finite difference approximations and use the average in the β terms, we get the average interface scheme:

$$a_{DF}^{\{\}}(u,v) = K(c)\frac{u(x_1^{FV})-u(c^-)}{h_0}\left(v(x_1^{FV})-v(c^-)\right) + \beta\{u(c)\}\left(v(c^-)-v(x_1^{FV})\right),$$

where we define $\{u(c)\} = \frac{1}{2}(u(c^-)+u(x_1^{FV}))$.

Average scheme : Combining a_D, a_F and $a_{DF}^{\{\}}$ terms, we now write the coupled DG-finite volume scheme: find $u_h \in \mathbb{X}^h$ such that

$$a^{\{\}}(u_h,v) = a^{DG}(u_h,v) + a^{FV}(u_h,v) + a_{DF}^{\{\}}(u_h,v) = L^{DG}(v) + L^{FV}(v), \tag{2.6}$$

for all $v \in \mathbb{X}^h = \{v : v|_{I_n^{DG}} \in \mathbb{P}_r(I_n^{DG}), v|_{I_i^{FV}} \in \mathbb{P}_0(I_i^{FV})\}$.

If we replace the derivative at the interface with finite difference approximations and use the upwind in the β terms, we get the upwind interface scheme:

$$a^{\uparrow}_{DF}(u_h, v) = K(c)\frac{u_h(x_1^{FV}) - u_h(c^-)}{h_0}\left(v(x_1^{FV}) - v(c^-)\right) + \beta u_h^{\uparrow}(c)\left(v(c^-) - v(x_1^{FV})\right).$$

We recall that $u^{\uparrow}(c)$ is defined by (2.2).

Upwind scheme : Combining a_D, a_F and $a^{\uparrow}_{DF}$ terms, we obtain the coupled DG-finite volume scheme: find $u_h \in \mathbb{X}^h$ such that

$$a^{\uparrow}(u_h, v) = a^{DG}(u_h, v) + a^{FV}(u_h, v) + a^{\uparrow}_{DF}(u_h, v) = L^{DG}(v) + L^{FV}(v), \tag{2.7}$$

for all $v \in \mathbb{X}^h = \{v : v|_{I_n^{DG}} \in \mathbb{P}_r(I_n^{DG}), v|_{I_i^{FV}} \in \mathbb{P}_0(I_i^{FV})\}$.

2.3 Existence and uniqueness

First, let us define some norms. Define

$$\|u\|_{\widetilde{DG}} = \sqrt{\sum_{n=1}^{N}\int_{x_n^{DG}}^{x_{n+1}^{DG}}(K^{\frac{1}{2}}u')^2 + \sum_{n=2}^{N}(\tilde{h}_n)^{-1}\sigma_n[u(x_n^{DG})]^2 + (\tilde{h}_1^{DG})^{-1}\sigma_1 u(a^+)^2},$$

and define the DG norm

$$\|u\|_{DG} = \sqrt{\|u\|^2_{\widetilde{DG}} + \frac{1}{2}|\beta|\sum_{n=2}^{N}[u(x_n^{DG})]^2}.$$

Define the FV norm to be

$$\|u\|_{FV} = \sqrt{\sum_{i=1}^{M-1}K(x_{i+\frac{1}{2}}^{FV})\frac{(u(x_{i+1}^{FV}) - u(x_i^{FV}))^2}{h_{i+\frac{1}{2}}} + K(b)\frac{u^2(x_M^{FV})}{h_M} + \frac{1}{2}|\beta|\sum_{i=2}^{M}[u(x_{i-\frac{1}{2}}^{FV})]^2}.$$

Define the energy norm

$$\|u\|_{\mathcal{E}} = \sqrt{\|u\|^2_{DG} + \|u\|^2_{FV} + K(c)\frac{(u(x_1^{FV}) - u(c^-))^2}{h_0}}.$$

It is easy to see that $\|\cdot\|_{\mathcal{E}}$ is indeed a norm for $\mathbb{X}^h$. Now we prove existence and uniqueness of the solution to the multinumeric scheme.

Theorem 2.1

There exists a unique solution $u_h \in \mathbb{X}^h$ satisfying (2.6) and a unique solution $u_h \in \mathbb{X}^h$ satisfying (2.7).

Proof 2.1

We only need to prove uniqueness, since the problem is linear and finite dimensional. Let the boundary conditions and f be zero. We will show that the solutions to the schemes are zero everywhere. Define

$$\tilde{a}^{DG}(u,v) = \sum_{n=1}^{N} \int_{x_n^{DG}}^{x_{n+1}^{DG}} Ku'v' - \sum_{n=2}^{N} \{Ku'(x_n^{DG})\}[v(x_n^{DG})] + \varepsilon \sum_{n=2}^{N} \{Kv'(x_n^{DG})\}[u(x_n^{DG})]$$
$$+ K(a)u'(a^+)v(a^+) - \varepsilon K(a)v'(a^+)u(a^+)$$
$$+ \sum_{n=2}^{N} (\tilde{h}_n^{DG})^{-1}\sigma_n[u(x_n^{DG})][v(x_n^{DG})] + (\tilde{h}_1^{DG})^{-1}\sigma_1 u(a^+)v(a^+)$$

Then

$$a^{DG}(u,v) = \tilde{a}^{DG}(u,v) - \sum_{n=1}^{N} \int_{x_n^{DG}}^{x_{n+1}^{DG}} \beta u v' + \sum_{n=2}^{N} \beta u^{\uparrow}(x_n^{DG})[v(x_n^{DG})] - \frac{1}{2}\beta u(a^+)v(a^+)$$

It is well known that $\tilde{a}^{DG}(v,v)$is coercive with respect to $\|v\|_{\widetilde{DG}}$norm, i.e. there is a constant $\alpha \geq 0$, such that

$$\tilde{a}^{DG}(v,v) \geq \alpha \|v\|_{\widetilde{DG}}^2 \quad \text{for all } v \in \mathbb{X}^h. \tag{2.8}$$

Indeed (2.8) is trivially true if $\varepsilon = 1$. In the case $\varepsilon = -1$, one can show that (2.8) holds if the penalty parameter σ is large enough [42].

When $\beta \geq 0$, we have

$$a^{DG}(v,v) = \tilde{a}^{DG}(v,v) - \sum_{n=1}^{N} \int_{x_n^{DG}}^{x_{n+1}^{DG}} \beta v v' + \sum_{n=2}^{N} \beta v^{\uparrow}(x_n^{DG})[v(x_n^{DG})] - \frac{1}{2}\beta v(a^+)v(a^+)$$
$$= \tilde{a}^{DG}(v,v) + \frac{1}{2}\beta\left(-\sum_{n=1}^{N} v^2(x_{n+1}^{DG-}) + \sum_{n=1}^{N} v^2(x_n^{DG+})\right)$$
$$+ \beta\sum_{n=2}^{N}\left(v^2(x_n^{DG-}) - v(x_n^{DG-})v(x_n^{DG+})\right) - \frac{1}{2}\beta v^2(a^+)$$
$$= \tilde{a}^{DG}(v,v) - \frac{1}{2}\beta v^2(x_{N+1}^{DG-}) - \frac{1}{2}\beta\sum_{n=2}^{N} v^2(x_n^{DG-}) + \frac{1}{2}\beta v^2(a^+)$$
$$+ \frac{1}{2}\beta\sum_{n=2}^{N} v^2(x_n^{DG+}) + \beta\sum_{n=2}^{N} v^2(x_n^{DG-}) - \beta\sum_{n=2}^{N} v(x_n^{DG-})v(x_n^{DG+}) - \frac{1}{2}\beta v^2(a^+)$$
$$= \tilde{a}^{DG}(v,v) - \frac{1}{2}\beta v^2(c^-)$$
$$+ \frac{1}{2}\beta\sum_{n=2}^{N}\left(v^2(x_n^{DG-}) - 2v(x_n^{DG-})v(x_n^{DG+}) + v^2(x_n^{DG+})\right)$$
$$= \tilde{a}^{DG}(v,v) - \frac{1}{2}\beta v^2(c^-) + \frac{1}{2}\beta\sum_{n=2}^{N}[v(x_n^{DG})]^2.$$

Similarly, when $\beta < 0$, we have

$$a^{DG}(v,v) = \tilde{a}^{DG}(v,v) - \frac{1}{2}\beta v^2(c^-) - \frac{1}{2}\beta\sum_{n=2}^{N}[v](x_n^{DG})^2.$$

Thus

$$a^{DG}(v,v) = \tilde{a}^{DG}(v,v) - \frac{1}{2}\beta v^2(c^-) + \frac{1}{2}|\beta|\sum_{n=2}^{N}[v(x_n^{DG})]^2. \tag{2.9}$$

Now let us consider the FV part. We separate the diffusive and convective terms and write $a^{FV}(v,v) = D + B$, where

$$D = \sum_{i=2}^{M-1}\left(K(x_{i-\frac{1}{2}}^{FV})\frac{v(x_i^{FV}) - v(x_{i-1}^{FV})}{h_{i-\frac{1}{2}}}v(x_i^{FV}) - K(x_{i+\frac{1}{2}}^{FV})\frac{v(x_{i+1}^{FV}) - v(x_i^{FV})}{h_{i+\frac{1}{2}}}v(x_i^{FV})\right)$$
$$- K(x_{\frac{3}{2}}^{FV})\frac{v(x_2^{FV}) - (x_1^{FV})}{h_{\frac{3}{2}}}v(x_1^{FV}) + K(x_{M-\frac{1}{2}}^{FV})\frac{v(x_M^{FV}) - v(x_{M-1}^{FV})}{h_{M-\frac{1}{2}}}v(x_M^{FV})$$
$$+ K(b)\frac{v(x_M^{FV})}{h_M}v(x_M^{FV}),$$

and

$$B = \sum_{i=2}^{M-1} \Big(-\beta v^{\uparrow}(x_{i-\frac{1}{2}}^{FV}) v(x_i^{FV}) + \beta v^{\uparrow}(x_{i+\frac{1}{2}}^{FV}) v(x_i^{FV}) \Big) \\ + \beta v^{\uparrow}(x_{\frac{3}{2}}^{FV}) v(x_1^{FV}) - \beta v^{\uparrow}(x_{M-\frac{1}{2}}^{FV}) v(x_M^{FV}) + \frac{1}{2}\beta v(x_M^{FV}) v(x_M^{FV}).$$

We compute

$$\begin{aligned} D &= \sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV}) \frac{v(x_{i+1}^{FV}) - v(x_i^{FV})}{h_{i+\frac{1}{2}}} v(x_{i+1}^{FV}) - \sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV}) \frac{v(x_{i+1}^{FV}) - v(x_i^{FV})}{h_{i+\frac{1}{2}}} v(x_i^{FV}) \\ &+ K(b) \frac{v(x_M^{FV})}{h_M} v(x_M^{FV}) \\ &= \sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV}) \frac{v^2(x_{i+1}^{FV}) - 2v^2(x_i^{FV}) v^2(x_{i+1}^{FV}) + v^2(x_i^{FV})}{h_{i+\frac{1}{2}}} + K(b) \frac{v^2(x_M^{FV})}{h_M} \\ &= \sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV}) \frac{(v(x_{i+1}^{FV}) - v(x_i^{FV}))^2}{h_{i+\frac{1}{2}}} + K(b) \frac{v^2(x_M^{FV})}{h_M}. \end{aligned} \tag{2.10}$$

When $\beta \geq 0$, we note that $v^{\uparrow}(x_{i+\frac{1}{2}}^{FV}) = v(x_i^{FV})$. Then we obtain

$$\begin{aligned} B &= \sum_{i=2}^{M-1} \left(-\beta v(x_{i-1}^{FV}) v(x_i^{FV}) + \beta v(x_i^{FV}) v(x_i^{FV}) \right) \\ &+ \beta v(x_1^{FV}) v(x_1^{FV}) - \beta v(x_{M-1}^{FV}) v(x_M^{FV}) + \frac{1}{2}\beta v(x_M^{FV})(x_M^{FV}) \\ &= \sum_{i=2}^{M} \left(-\beta v(x_{i-1}^{FV}) v(x_i^{FV}) \right) + \sum_{i=1}^{M-1} \left(\beta v^2(x_i^{FV}) \right) + \frac{1}{2}\beta v^2(x_M^{FV}) \\ &= \frac{1}{2} \sum_{i=2}^{M} \left(-2\beta v(x_{i-1}^{FV}) v(x_i^{FV}) + \beta v^2(x_i^{FV}) + \beta v^2(x_{i-1}^{FV}) \right) + \frac{1}{2}\beta v^2(x_1^{FV}) \\ &= \frac{1}{2}\beta \sum_{i=2}^{M} [v(x_{i-\frac{1}{2}}^{FV})]^2 + \frac{1}{2}\beta v^2(x_1^{FV}) \end{aligned}$$

When $\beta < 0$, we note that $v^{\uparrow}(x_{i+\frac{1}{2}}^{FV}) = v(x_{i+1}^{FV})$. Then we obtain similarly

$$
\begin{aligned}
B &= \sum_{i=2}^{M-1} \left(-\beta v(x_i^{FV})v(x_i^{FV}) + \beta v(x_{i+1}^{FV})v(x_i^{FV})\right) \\
&+ \beta v(x_2^{FV})v(x_1^{FV}) - \beta v^2(x_M^{FV}) + \frac{1}{2}\beta v(x_M^{FV})v(x_M^{FV}) \\
&= \sum_{i=2}^{M-1} \left(-\beta v^2(x_i^{FV})\right) + \sum_{i=1}^{M-1} \left(\beta v(x_{i+1}^{FV})v(x_i^{FV})\right) - \frac{1}{2}\beta v^2(x_M^{FV}) \\
&= \frac{1}{2}\sum_{i=2}^{M} \left(2\beta v(x_i^{FV})v(x_{i-1}^{FV}) - \beta v^2(x_i^{FV}) - \beta v^2(x_{i-1}^{FV})\right) + \frac{1}{2}\beta v^2(x_1^{FV}) \\
&= -\frac{1}{2}\beta \sum_{i=2}^{M}[v(x_{i-\frac{1}{2}}^{FV})]^2 + \frac{1}{2}\beta v^2(x_1^{FV}).
\end{aligned}
$$

Therefore, we obtain

$$
B = \frac{1}{2}|\beta| \sum_{i=2}^{M}[v(x_{i-\frac{1}{2}}^{FV})]^2 + \frac{1}{2}\beta v^2(x_1^{FV}). \tag{2.11}
$$

Combining (2.10) and (2.11), we have

$$
\begin{aligned}
a_F(v,v) = \sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV})\frac{(v(x_{i+1}^{FV}) - v(x_i^{FV}))^2}{h_{i+\frac{1}{2}}} + K(b)\frac{v^2(x_M^{FV})}{h_M} \\
+ \frac{1}{2}|\beta| \sum_{i=2}^{M}[v(x_{i-\frac{1}{2}}^{FV})]^2 + \frac{1}{2}\beta v^2(x_1^{FV}).
\end{aligned}
\tag{2.12}
$$

Next, we assume u_h is the solution of the scheme (2.6). Taking (2.9) and (2.12) into (2.6), we have

$$
\begin{aligned}
&a^{\{\}}(u_h, u_h) \\
&= \tilde{a}^{DG}(u_h, u_h) - \frac{1}{2}\beta u_h^2(c^-) + \frac{1}{2}|\beta| \sum_{n=2}^{N} [u_h(x_n^{DG})]^2 \\
&+ \sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV}) \frac{(u_h(x_{i+1}^{FV}) - u_h(x_i^{FV}))^2}{h_{i+\frac{1}{2}}} + K(b)\frac{u_h^2(x_M^{FV})}{h_M} + \frac{1}{2}|\beta| \sum_{i=2}^{M} [u_h(x_{i-\frac{1}{2}}^{FV})]^2 \\
&+ \frac{1}{2}\beta u_h^2(x_1^{FV}) + K(c)\frac{(u_h(x_1^{FV}) - u_h(c^-))^2}{h_0} + \frac{1}{2}\beta u_h^2(c^-) - \frac{1}{2}\beta u_h^2(x_1^{FV}) \\
&= \tilde{a}^{DG}(u_h, u_h) + \frac{1}{2}|\beta| \sum_{n=2}^{N} [u_h(x_n^{DG})]^2 + \sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV}) \frac{(u_h(x_{i+1}^{FV}) - u_h(x_i^{FV}))^2}{h_{i+\frac{1}{2}}} \\
&+ K(b)\frac{u_h^2(x_M^{FV})}{h_M} + \frac{1}{2}|\beta| \sum_{i=2}^{M} [u_h(x_{i-\frac{1}{2}}^{FV})]^2 + K(c)\frac{(u_h(x_1^{FV}) - u_h(c^-))^2}{h_0} = 0.
\end{aligned}
$$

By (2.8), we obtain

$$
\begin{aligned}
&\alpha \|u_h\|_{\widetilde{DG}}^2 + \frac{1}{2}|\beta| \sum_{n=2}^{N} [u_h(x_n^{DG})]^2 + \sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV}) \frac{(u_h(x_{i+1}^{FV}) - u_h(x_i^{FV}))^2}{h_{i+\frac{1}{2}}} \\
&+ K(b)\frac{u_h^2(x_M^{FV})}{h_M} + \frac{1}{2}|\beta| \sum_{i=2}^{M} [u_h(x_{i-\frac{1}{2}}^{FV})]^2 + K(c)\frac{(u_h(x_1^{FV}) - u_h(c^-))^2}{h_0} \le 0.
\end{aligned} \tag{2.13}
$$

This implies that u_h is identically zero, and therefore there is a unique solution to (2.6).

Next, we consider the solution u_h to scheme (2.7) with zero f and boundary conditions.

When $\beta \ge 0$, we have

$$
a_{DF}^{\uparrow}(u_h, u_h) = K(c)\frac{(u_h(x_1^{FV}) - u_h(c^-))^2}{h_0} + \beta(u_h^2(c^-) - u_h(c^-)u_h(x_1^{FV})). \tag{2.14}
$$

Adding (2.9), (2.12) and (2.14) together, we obtain

$$\begin{aligned}
a^{\dagger}(u_h,u_h) &= a_D(u_h,u_h) + a_F(u_h,u_h) + a^{\dagger}_{DF}(u_h,u_h) \\
&= \tilde{a}^{DG}(u_h,u_h) + \frac{1}{2}|\beta|\sum_{n=2}^{N}[u_h(x_n^{DG})]^2 + \sum_{i=1}^{M-1} K(x^{FV}_{i+\frac{1}{2}})\frac{(u_h(x^{FV}_{i+1}) - u_h(x^{FV}_i))^2}{h_{i+\frac{1}{2}}} \\
&+ K(b)\frac{u_h^2(x^{FV}_M)}{h_M} + \frac{1}{2}|\beta|\sum_{i=2}^{M}[u_h(x^{FV}_{i-\frac{1}{2}})]^2 + K(c)\frac{(u_h(x^{FV}_1) - u_h(c^-))^2}{h_0} + \frac{1}{2}\beta[u_h(c)]^2.
\end{aligned} \tag{2.15}$$

Similarly, when $\beta < 0$, we have

$$a^{\dagger}_{DF}(u_h,u_h) = K(c)\frac{(u_h(x^{FV}_1) - u_h(c^-))^2}{h_0} + \beta(u_h(x^{FV}_1)u_h(c^-) - u_h^2(x^{FV}_1)). \tag{2.16}$$

Adding (2.9), (2.12) and (2.16) together, we obtain

$$\begin{aligned}
a^{\dagger}(u_h,u_h) &= a_D(u_h,u_h) + a_F(u_h,u_h) + a^{\dagger}_{DF}(u_h,u_h) \\
&= \tilde{a}^{DG}(u_h,u_h) + \frac{1}{2}|\beta|\sum_{n=2}^{N}[u_h(x_n^{DG})]^2 + \sum_{i=1}^{M-1} K(x^{FV}_{i+\frac{1}{2}})\frac{(u_h(x^{FV}_{i+1}) - u_h(x^{FV}_i))^2}{h_{i+\frac{1}{2}}} \\
&+ K(b)\frac{u_h^2(x^{FV}_M)}{h_M} + \frac{1}{2}|\beta|\sum_{i=2}^{M}[u_h(x^{FV}_{i-\frac{1}{2}})]^2 + K(c)\frac{(u_h(x^{FV}_1) - u_h(c^-))^2}{h_0} - \frac{1}{2}\beta[u_h(c)]^2.
\end{aligned} \tag{2.17}$$

According to (2.15) and (2.17), we obtain

$$\begin{aligned}
&a^{\dagger}(u_h,u_h) \\
&= \tilde{a}^{DG}(u_h,u_h) + \frac{1}{2}|\beta|\sum_{n=2}^{N}[u_h(x_n^{DG})]^2 + \sum_{i=1}^{M-1} K(x^{FV}_{i+\frac{1}{2}})\frac{(u_h(x^{FV}_{i+1}) - u_h(x^{FV}_i))^2}{h_{i+\frac{1}{2}}} \\
&+ K(b)\frac{u_h^2(x^{FV}_M)}{h_M} + \frac{1}{2}|\beta|\sum_{i=2}^{M}[u_h(x^{FV}_{i-\frac{1}{2}})]^2 + K(c)\frac{(u_h(x^{FV}_1) - u_h(c^-))^2}{h_0} + \frac{1}{2}|\beta|[u_h(c)]^2 \\
&= 0.
\end{aligned} \tag{2.18}$$

Using the coercivity (2.8) of $\tilde{a}^{DG}$, we can conclude that (2.18) implies $u_h = 0$ everywhere.□

2.4 Error estimate

We first show that the exact solution satisfies the scheme up to a consistency error that is of first order.

Lemma 2.1

Define the residuals for any $u \in H^2([c,b])$.

$$R_{i-\frac{1}{2}} = K(x_{i-\frac{1}{2}}^{FV})\frac{u(x_i^{FV}) - u(x_{i-1}^{FV})}{h_{i-\frac{1}{2}}^{FV}} - K(x_{i-\frac{1}{2}}^{FV})u'(x_{i-\frac{1}{2}}^{FV}),\ 2 \le i \le M, \tag{2.19}$$

$$R_M = K(b)\frac{u(b) - u(x_M^{FV})}{h_M^{FV}} - K(b)u'(b), \tag{2.20}$$

$$R_{i-\frac{1}{2}}^{\beta} = \beta(u^{\dagger}(x_{i-\frac{1}{2}}^{FV}) - u(x_{i-\frac{1}{2}}^{FV})),\ 1 \le i \le M, \tag{2.21}$$

$$R_M^{\beta} = \frac{1}{2}\beta(u(x_M^{FV}) - u(b)), \tag{2.22}$$

$$R_0 = K(c)\left(\frac{u(x_1^{FV}) - u(c)}{h_0^{FV}} - u'(c)\right), \tag{2.23}$$

$$R_1 = \beta\left(u(c) - \frac{u(c^-) + u(x_1^{FV})}{2}\right). \tag{2.24}$$

Then there exists a constant C depending on u and independent of h_{FV}, such that

$$|R_{i-\frac{1}{2}}| \le k_1(h_{i-\frac{1}{2}}^{FV})^{\frac{1}{2}}\|u''\|_{L^2([x_{i-1}^{FV}, x_i^{FV}])},\ 2 \le i \le M, \tag{2.25}$$

$$|R_0| \le k_1(h_0^{FV})^{\frac{1}{2}}\|u''\|_{L^2([c, x_1^{FV}])} \tag{2.26}$$

$$|R_M| \le k_1(h_M^{FV})^{\frac{1}{2}}\|u''\|_{L^2([x_M^{FV}, b])}, \tag{2.27}$$

$$|R_{i-\frac{1}{2}}^{\beta}| \le Ch_{FV},\ 2 \le i \le M, \tag{2.28}$$

$$|R_1| + |R_M^{\beta}| \le Ch_{FV}. \tag{2.29}$$

Proof 2.2

For $R_{i-\frac{1}{2}}, 2 \le i \le M$, by Taylor's expansion we obtain

$$u(x_i^{FV}) = u(x_{i-\frac{1}{2}}^{FV}) + u'(x_{i-\frac{1}{2}}^{FV})(x_i^{FV} - x_{i-\frac{1}{2}}^{FV}) + \int_0^1 u''(tx_i^{FV} + (1-t)x_{i-\frac{1}{2}}^{FV})(x_i^{FV} - x_{i-\frac{1}{2}}^{FV})^2 t dt. \tag{2.30}$$

$$u(x_{i-1}^{FV}) = u(x_{i-\frac{1}{2}}^{FV}) + u'(x_{i-\frac{1}{2}}^{FV})(x_{i-1}^{FV} - x_{i-\frac{1}{2}}^{FV}) + \int_0^1 u''(tx_{i-1}^{FV} + (1-t)x_{i-\frac{1}{2}}^{FV})(x_{i-1}^{FV} - x_{i-\frac{1}{2}}^{FV})^2 tdt. \tag{2.31}$$

Substracting one equation from the other and dividing by $h_{i-\frac{1}{2}}^{FV}$, we obtain

$$\frac{u(x_i^{FV}) - u(x_{i-1}^{FV})}{h_{i-\frac{1}{2}}^{FV}} - u'(x_{i-\frac{1}{2}}^{FV})) = \frac{1}{h_{i-\frac{1}{2}}^{FV}} \left(\int_0^1 u''(tx_i^{FV} + (1-t)x_{i-\frac{1}{2}}^{FV})(x_i^{FV} - x_{i-\frac{1}{2}}^{FV})^2 tdt \right)$$
$$- \frac{1}{h_{i-\frac{1}{2}}^{FV}} \left(\int_0^1 u''(tx_{i-1}^{FV} + (1-t)x_{i-\frac{1}{2}}^{FV})(x_{i-1}^{FV} - x_{i-\frac{1}{2}}^{FV})^2 tdt \right). \tag{2.32}$$

Set $tx_i^{FV} + (1-t)x_{i-\frac{1}{2}}^{FV} = x$ and we obtain

$$\int_0^1 u''(tx_i^{FV} + (1-t)x_{i-\frac{1}{2}}^{FV})(x_i^{FV} - x_{i-\frac{1}{2}}^{FV})^2 tdt = \int_{x_{i-\frac{1}{2}}^{FV}}^{x_i^{FV}} u''(x)(x - x_{i-\frac{1}{2}}^{FV})dx.$$

Similarly, we have

$$\int_0^1 u''(tx_{i-1}^{FV} + (1-t)x_{i-\frac{1}{2}}^{FV})(x_{i-1}^{FV} - x_{i-\frac{1}{2}}^{FV})^2 tdt = \int_{x_{i-\frac{1}{2}}^{FV}}^{x_{i-1}^{FV}} u''(x)(x - x_{i-\frac{1}{2}}^{FV})dx.$$

Thus we obtain the following inequality which is (2.25).

$$\begin{aligned}
&\left|\frac{u(x_i^{FV}) - u(x_{i-1}^{FV})}{h_{i-\frac{1}{2}}^{FV}} - u'(x_{i-\frac{1}{2}}^{FV})\right| \\
&= \frac{1}{h_{i-\frac{1}{2}}^{FV}} \left| \int_{x_{i-\frac{1}{2}}^{FV}}^{x_i^{FV}} u''(x)(x - x_{i-\frac{1}{2}}^{FV})dx - \int_{x_{i-\frac{1}{2}}^{FV}}^{x_{i-1}^{FV}} u''(x)(x - x_{i-\frac{1}{2}}^{FV})dx \right| \\
&= \frac{1}{h_{i-\frac{1}{2}}^{FV}} \left| \int_{x_{i-1}^{FV}}^{x_i^{FV}} u''(x)(x - x_{i-\frac{1}{2}}^{FV})dx \right| \\
&\le \frac{1}{h_{i-\frac{1}{2}}^{FV}} \left(\int_{x_{i-1}^{FV}}^{x_i^{FV}} u''^2(x))dx \right)^{\frac{1}{2}} \left(\int_{x_{i-1}^{FV}}^{x_i^{FV}} (x - x_{i-\frac{1}{2}}^{FV})^2 dx \right)^{\frac{1}{2}} \\
&\le \frac{1}{h_{i-\frac{1}{2}}^{FV}} \left(\int_{x_{i-1}^{FV}}^{x_i^{FV}} u''^2(x))dx \right)^{\frac{1}{2}} \left(\int_{x_{i-1}^{FV}}^{x_i^{FV}} (h_{i-\frac{1}{2}}^{FV})^2 dx \right)^{\frac{1}{2}} \\
&= (h_{i-\frac{1}{2}}^{FV})^{\frac{1}{2}} \|u''\|_{L^2([x_{i-1}^{FV}, x_i^{FV}])}.
\end{aligned} \tag{2.33}$$

Now let us prove (2.26). Thanks to Taylor's expansion, we obtain

$$u(x_1^{FV}) = u(c) + u'(c)h_0^{FV} + \int_0^1 u''(tx_1^{FV} + (1-t)c)(h_0^{FV})^2 t dt.$$

Setting $x = tx_1^{FV} + (1-t)c$, we obtain

$$\frac{u(x_1^{FV}) - u(c)}{h_0^{FV}} - u'(c) = \frac{1}{h_0^{FV}} \int_c^{x_1^{FV}} u''(x)(x-c)dx.$$

Bounding $(x-c)$ by h_0^{FV} and using Cauchy-Schwarz's inequality, we obtain

$$\left|\frac{u(x_1^{FV}) - u(c)}{h_0^{FV}} - u'(c)\right| \leq (h_0^{FV})^{\frac{1}{2}} \|u''\|_{L^2([c,x_1^{FV}])}.$$

The proof for (2.27) is similar.

Since $u \in H^2([c,b])$, we have $u \in C^1([c,b])$ by the Sobolev imbedding theorem. Therefore, $u'(x) \forall x \in [c,b]$ is bounded by some constant M. Thus by Taylor's expansion, we obtain (2.28):

$$|R_{i-\frac{1}{2}}^{\beta}| \leq |\beta| M h_{FV}.$$

Similarly, we obtain (2.29). □

Lemma 2.2

Let $u \in H^1(a,b)$ be the solution to problem (2.1) and assume $Ku' \in C(a,b)$. Then u satisfies

$$a^{DG}(u,v) + a^{FV}(u,v) + a_{DF}^{\{\}}(u,v) = L^{DG}(v) + L^{FV}(v) + R_{FV}(v) + R_I(v) \qquad (2.34)$$

where

$$\begin{aligned} R_{FV}(v) = &\sum_{i=2}^{M} R_{i-\frac{1}{2}} v(x_i^{FV}) - \sum_{i=1}^{M-1} R_{i+\frac{1}{2}} v(x_i^{FV}) - R_M v(x_M^{FV}) \\ &- \sum_{i=2}^{M} R_{i-\frac{1}{2}}^{\beta} v(x_i^{FV}) + \sum_{i=1}^{M-1} R_{i+\frac{1}{2}}^{\beta} v(x_i^{FV}) + R_M^{\beta} v(x_M^{FV}), \end{aligned}$$

$$R_I(v) = R_0(v(x_1^{FV}) - v(c^-)) - R_1(v(c^-) - v(x_1^{FV})).$$

Proof 2.3

Multiplying (2.1) by $v \in \{v : v|_{I_n^{DG}} \in \mathbb{P}_r(I_n^{DG})\}$ and integrating over every interval I_n^{DG}, we obtain

$$\int_{x_n^{DG}}^{x_{n+1}^{DG}} -(Ku')'v + \int_{x_n^{DG}}^{x_{n+1}^{DG}} \beta u'v = \int_{x_n^{DG}}^{x_{n+1}^{DG}} fv.$$

Integrating by parts, we obtain

$$\begin{aligned}
&K(x_n^{DG})u'(x_n^{DG})v(x_n^{DG+}) - K(x_{n+1}^{DG})u'(x_{n+1}^{DG})v(x_{n+1}^{DG\,-}) + \int_{x_n^{DG}}^{x_{n+1}^{DG}} Ku'v' \\
&+ \beta u(x_{n+1}^{DG})v(x_{n+1}^{DG\,-}) - \beta u(x_n^{DG})v(x_n^{DG+}) - \int_{x_n^{DG}}^{x_{n+1}^{DG}} \beta uv' \\
&= \int_{x_n^{DG}}^{x_{n+1}^{DG}} fv,
\end{aligned}$$

for $n = 1, 2, \cdots, N$. Summing up over all the intervals, we obtain

$$\begin{aligned}
&-\sum_{n=1}^{N} \int_{x_n^{DG}}^{x_{n+1}^{DG}} (\beta u - Ku')v' - \sum_{n=2}^{N} Ku'(x_n^{DG})[v(x_n^{DG})] + \sum_{n=2}^{N} \beta u(x_n^{DG})[v(x_n^{DG})] \\
&+ K(a)u'(a)v(a^+) - K(c)u'(c^-)v(c^-) + \beta u(c)v(c^-) - \beta u(a)v(a^+) \\
&= \int_a^b fv.
\end{aligned}$$

Since Ku' and u are continuous over (a, b), we have $\{Ku'(x)\} = Ku'(x)$, $u^{\uparrow}(x) = u(x)$ and $[u(x)] = 0 \quad \forall x \in (a, b)$. We can then, without changing the equality, replace Ku' in the term $\sum_{n=2}^{N} Ku'(x_n^{DG})[v(x_n^{DG})]$ with the average and u in the term $\sum_{n=2}^{N} \beta u(x_n^{DG})[v(x_n^{DG})]$ with the upwind and add the stabilization terms. Therefore, we obtain (2.35).

$$a^{DG}(u, v) - K(c^-)u'(c^-)v(c^-) + \beta u(c^-)v(c^-) = L^{DG}(v),\ \forall v|_{I_n^{DG}} \in \mathbb{P}_r(I_n^{DG}), \quad (2.35)$$

From (2.4) and (2.5), we obtain (2.36).

$$a^{FV}(u, v) + K(c^+)u'(c^+)v_1 - \beta u(c^+)v_1 = L^{FV}(v) + R_{FV}(v), \forall v|_{I_n^{FV}} \in \mathbb{P}_0(I_n^{FV}). \quad (2.36)$$

Adding (2.35) and (2.36), we obtain

$$a^{DG}(u,v)+a^{FV}(u,v)-K(c^-)u'(c^-)v(c^-)+\beta u(c^-)v(c^-)+K(c^+)u'(c^+)v(x_1^{FV})$$
$$-\beta u(c^+)v(x_1^{FV})=L^{DG}(v)+L^{FV}(v)+R_{FV}(v). \tag{2.37}$$

In the interface, using $\frac{u(x_1^{FV})-u(c^-)}{h_0}$ to approximate $u'(c)$ and $\{u(c)\}$ to approximate $u(c)$, we can easily get the result.□

Theorem 2.2

We assume that u satisfies the assumptions in Lemma 2.2 and also assume that $u\in H^{r+1}(I_n^{DG})$, $\forall 1\le n\le N$ and $u\in H^2[c,b]$. Let u_h be the solution of the schemes (2.6) and (2.7). Then there exists a constant C independent of h_{DG} and h_{FV} such that

$$\|u_h-u\|_{\mathcal{E}}\le C(h_{DG}^r+h_{FV}).$$

Proof 2.4

We recall that the solution u_h of the scheme (2.6) satisfies

$$a^{DG}(u_h,v)+a^{FV}(u_h,v)+a_{DF}^{\{\}}(u_h,v)=L^{DG}(v)+L^{FV}(v). \tag{2.38}$$

We substract (2.34) from (2.38):

$$a^{DG}(u_h-u,v)+a^{FV}(u_h-u,v)+a_{DF}^{\{\}}(u_h-u,v)=-R_{FV}(v)-R_I(v). \tag{2.39}$$

Let

$$\hat{u}(x)=\begin{cases} w_h(x), & a\le x<c \\ u(x_i^{FV}), & x_{i-\frac{1}{2}}^{FV}<x<x_{i+\frac{1}{2}}^{FV},\ 1\le i\le M, \end{cases}$$

where w_h satisfies ([42])

$$\|w_h-u\|_{L^2(I_n^{DG})}\le C(h_n^{DG})^{r+1}|u|_{H^{r+1}(I_n^{DG})}, \tag{2.40a}$$

$$\|w_h'-u'\|_{L^2(I_n^{DG})}\le C(h_n^{DG})^{r}|u|_{H^{r+1}(I_n^{DG})}. \tag{2.40b}$$

Define $u_h - \hat{u} = \chi$ and $u - \hat{u} = \xi$. Then we have $u_h - u = u_h - \hat{u} + \hat{u} - u = \chi - \xi$. Substitute it in (2.39) and choose v equal to χ:

$$a^{DG}(\chi - \xi, \chi) + a^{FV}(\chi - \xi, \chi) + a^{\{\}}_{DF}(\chi - \xi, \chi) = -R_{FV}(\chi) - R_I(\chi). \tag{2.41}$$

Since $\xi(x_i^{FV}) = 0$ for all $1 \leq i \leq M$, we have $a^{FV}(\xi, \chi) = 0$. Tidy (2.42) up and we obtain

$$a^{DG}(\chi,\chi) + a^{FV}(\chi,\chi) + a^{\{\}}_{DF}(\chi,\chi) \ = a^{DG}(\xi,\chi) + a^{\{\}}_{DF}(\xi,\chi) - R_{FV}(\chi) - R_I(\chi). \tag{2.42}$$

We now bound the terms in the right-hand side of (2.42).

$$\begin{aligned} a^{DG}(\xi,\chi) = &-\sum_{n=1}^{N}\int_{x_n^{DG}}^{x_{n+1}^{DG}} \beta\xi\chi' + \sum_{n=1}^{N}\int_{x_n^{DG}}^{x_{n+1}^{DG}} K\xi'\chi' + \sum_{n=2}^{N}\beta\xi^{\uparrow}(x_n^{DG})[\chi(x_n^{DG})] \\ &- \sum_{n=2}^{N}\{K\xi'(x_n^{DG})\}[\chi(x_n^{DG})] + \varepsilon\sum_{n=2}^{N}\{K\chi'(x_n^{DG})\}[\xi(x_n^{DG})] \\ &+ \sum_{n=2}^{N}(\tilde{h}_n^{DG})^{-1}\sigma_n[\xi(x_n^{DG})][\chi(x_n^{DG})] + (\tilde{h}_1^{DG})^{-1}\sigma_1\xi(a^+)\chi(a^+) \\ &+ K(a)\xi'(a^+)\chi(a^+) - \varepsilon K(a)\chi'(a^+)\xi(a^+) - \frac{1}{2}\beta\xi(a^+)\chi(a^+) \end{aligned} \tag{2.43}$$

In the remainder of the proof, C is a generic constant that takes many values at many places, and that is independent of the mesh size.

The first term in the right-hand side of (2.43) is bounded by using Cauchy-Schwarz's and Young's inequalities, and by the approximation result (2.40a).

$$\begin{aligned} \left| -\sum_{n=1}^{N}\int_{x_n^{DG}}^{x_{n+1}^{DG}} \beta\xi\chi'\right| &\leq \sum_{n=1}^{N}|\beta|k_1^{-1}\left(\int_{x_n^{DG}}^{x_{n+1}^{DG}}\xi^2\right)^{\frac{1}{2}}\left(\int_{x_n^{DG}}^{x_{n+1}^{DG}}K\chi'^2\right)^{\frac{1}{2}} \\ &\leq |\beta|^2k_0^{-2}\alpha^{-1}\|\xi\|^2_{L^2([a,c])} + \frac{\alpha}{8}\sum_{n=1}^{N}\left(\int_{x_n^{DG}}^{x_{n+1}^{DG}}K\chi'^2\right) \\ &\leq |\beta|^2k_0^{-2}\alpha^{-1}Ch_{DG}^{2r+2}|u|^2_{H^{r+1}([a,c])} + \frac{\alpha}{8}\sum_{n=1}^{N}\left(\int_{x_n^{DG}}^{x_{n+1}^{DG}}K\chi'^2\right). \end{aligned}$$

We recall that α is the coercivity constant in (2.8). The second term in the right-hand side of (2.43) is handled similarly.

$$
\begin{aligned}
|\sum_{n=1}^{N}\int_{x_n^{DG}}^{x_{n+1}^{DG}} K(x)\xi'\chi'| &\leq \sum_{n=1}^{N}(\int_{x_n^{DG}}^{x_{n+1}^{DG}} K(x)\xi'^2)^{\frac{1}{2}}(\int_{x_n^{DG}}^{x_{n+1}^{DG}} K(x)\chi'^2)^{\frac{1}{2}} \\
&\leq k_1\alpha^{-1}\|\xi'\|^2_{L^2([a,c])} + \frac{\alpha}{8}\sum_{n=1}^{N}(\int_{x_n^{DG}}^{x_{n+1}^{DG}} K(x)\chi'^2) \\
&\leq k_1\alpha^{-1}Ch_{DG}^{2r}|u|^2_{H^{r+1}([a,c])} + \frac{\alpha}{8}\sum_{n=1}^{N}(\int_{x_n^{DG}}^{x_{n+1}^{DG}} K(x)\chi'^2).
\end{aligned}
$$

To bound the third term in the right-hand side of (2.43), we use the following trace inequality:

$$
\forall v \in H^1(I_n^{DG}),\ |v(x_n^{DG+})| \leq C\left((h_n^{DG})^{-\frac{1}{2}}\|v\|_{L^2(I_n^{DG})} + (h_n^{DG})^{\frac{1}{2}}\|v'\|_{L^2(I_n^{DG})}\right). \quad (2.44)
$$

$$
\begin{aligned}
&|\sum_{n=2}^{N}\beta\xi^{\uparrow}[\chi(x_n^{DG})]| \\
&\leq 2\sum_{n=2}^{N}|\beta|\xi^{\uparrow 2}(x_n^{DG}) + \frac{1}{8}\sum_{n=2}^{N}|\beta|[\chi(x_n^{DG})]^2 \\
&\leq 2|\beta|\sum_{n=1}^{N}(\xi^2(x_n^{DG-}) + \xi^2(x_n^{DG+})) + \frac{1}{8}|\beta|\sum_{n=2}^{N}[\chi]^2(x_n^{DG}) \\
&\leq C|\beta|\sum_{n=1}^{N}\left((h_n^{DG})^{-1}\|\xi\|^2_{L^2(I_n^{DG})} + h_n^{DG}\|\xi'\|^2_{L^2(I_n^{DG})}\right) + \frac{1}{8}|\beta|\sum_{n=2}^{N}[\chi]^2(x_n^{DG}) \\
&\leq C|\beta|h_{DG}^{2r+1}|u|^2_{H^{r+1}([a,c])} + \frac{1}{8}|\beta|\sum_{n=2}^{N}[\chi]^2(x_n^{DG}).
\end{aligned}
$$

To bound the fourth term in the right-hand side of (2.43), we apply the bound (2.44) to the function ξ'.

$$
\begin{aligned}
&|-\sum_{n=2}^{N}\{K\xi'(x_n^{DG})\}[\chi(x_n^{DG})]| \\
&\leq \sum_{n=2}^{N}\left(2\alpha^{-1}\tilde{h}_n^{DG}\sigma_n^{-1}\{K\xi'(x_n^{DG})\}^2+\frac{1}{8}\alpha\sum_{n=2}^{N}(\tilde{h}_n^{DG})^{-1}\sigma_n[\chi(x_n^{DG})]^2\right) \\
&\leq 2\alpha^{-1}\sum_{n=2}^{N}k_1^2\tilde{h}_n^{DG}\sigma_n^{-1}(\xi'^2(x_n^{DG-})+\xi'^2(x_n^{DG+}))+\frac{1}{8}\alpha\sum_{n=2}^{N}(\tilde{h}_n^{DG})^{-1}\sigma_n[\chi(x_n^{DG})]^2) \\
&\leq C(\min_n \sigma_n)^{-1}k_1^2\sum_{n=2}^{N}h_{DG}\left(h_n^{DG-1}\|\xi'\|^2_{L^2(I_n^{DG})}+h_n^{DG}\|\xi''\|^2_{L^2(I_n^{DG})}\right) \\
&+\frac{1}{8}\alpha\sum_{n=2}^{N}(\tilde{h}_n^{DG})^{-1}\sigma_n[\chi(x_n^{DG})]^2 \\
&\leq C(\min_n(\sigma_n))^{-1}k_1^2(h_{DG})^{2r}|u|^2_{H^{r+1}([a,c])}+\frac{1}{8}\alpha\sum_{n=2}^{N}(\tilde{h}_n^{DG})^{-1}\sigma_n[\chi(x_n^{DG})]^2.
\end{aligned}
$$

To bound the fifth term in the right-hand side of (2.43), we apply (2.44) to ξ and the trace inequality

$$
\forall v\in\mathbb{P}_r(I_n^{DG}),\ |v(x_n^{DG+})|\leq C(h_n^{DG})^{-\frac{1}{2}}\|v\|_{L^2(I_n^{DG})}, \tag{2.45}
$$

to χ'.

$$
\begin{aligned}
&|\varepsilon\sum_{n=2}^{N}\{K\chi'(x_n^{DG})\}[\xi(x_n^{DG})]| \\
&\leq \sum_{n=2}^{N}\frac{1}{2}|K(x_n^{DG+})\chi'(x_n^{DG+})+K(x_n^{DG-})\chi'(x_n^{DG-})||(\xi(x_n^{DG-})-\xi(x_n^{DG+})| \\
&\leq C\sum_{n=1}^{N}\left(k_1k_0^{-\frac{1}{2}}(h_n^{DG})^{-\frac{1}{2}}\|K^{\frac{1}{2}}\chi'\|_{L^2(I_n^{DG})}\right)\left((h_n^{DG})^{-\frac{1}{2}}\|\xi\|_{L^2(I_n^{DG})}+(h_n^{DG})^{\frac{1}{2}}\|\xi'\|_{L^2(I_n^{DG})}\right) \\
&\leq Ck_0k_1^{-2}\sum_{n=1}^{N}\left((h_n^{DG})^{-1}\|\xi\|^2_{L^2(I_n^{DG})}+h_n^{DG}\|\xi'\|^2_{L^2(I_n^{DG})}\right)+\frac{\alpha}{8}\sum_{n=1}^{N}\|K^{\frac{1}{2}}\chi'\|^2_{L^2(I_n^{DG})} \\
&\leq Ck_0k_1^{-2}h_{DG}^{2r}|u|^2_{H^{r+1}([a,c])}+\frac{\alpha}{8}\sum_{n=1}^{N}\|K^{\frac{1}{2}}\chi'\|^2_{L^2(I_n^{DG})}.
\end{aligned}
$$

The sixth term in the right-hand side of (2.43) is bounded similarly to the third term.

$$
\begin{aligned}
&|\sum_{n=2}^{N}(\tilde{h}_n^{DG})^{-1}\sigma_n[\xi(x_n^{DG})][\chi(x_n^{DG})]| \\
&\leq 2\alpha^{-1}\sum_{n=2}^{N}(\tilde{h}_n^{DG})^{-1}\sigma_n[\xi(x_n^{DG})]^2 + \frac{1}{8}\alpha\sum_{n=2}^{N}(\tilde{h}_n^{DG})^{-1}\sigma_n[\chi]^2(x_n^{DG}) \\
&\leq 4\alpha^{-1}\sum_{n=2}^{N}(\tilde{h}_n^{DG})^{-1}\sigma_n(\xi^2(x_n^{DG-}) + \xi^2(x_n^{DG+})) + \frac{1}{8}\alpha\sum_{n=2}^{N}(\tilde{h}_n^{DG})^{-1}\sigma_n[\chi(x_n^{DG})]^2 \\
&\leq C\max_n(\sigma_n)\sum_{n=1}^{N}(h_n^{DG})^{-1}\left(h_n^{DG^{-1}}\|\xi\|^2_{L^2(I_n^{DG})} + h_n^{DG}\|\xi'\|^2_{L^2(I_n^{DG})}\right) \\
&+\frac{1}{8}\alpha\sum_{n=2}^{N}(\tilde{h}_n^{DG})^{-1}\sigma_n[\chi(x_n^{DG})]^2 \\
&\leq C\max_n(\sigma_n)(h_{DG})^{2r}|u|^2_{H^{r+1}([a,c])} + \frac{1}{8}\alpha\sum_{n=2}^{N}(\tilde{h}_n^{DG})^{-1}\sigma_n[\chi(x_n^{DG})]^2.
\end{aligned}
$$

The seventh and eighth terms in the right-hand side of (2.43) are easily bounded as follows:

$$
|(\tilde{h}_1^{DG})^{-1}\sigma_1\xi(a^+)\chi(a^+)| \leq C\sigma_1(\tilde{h}_{DG})^{2r}|u|^2_{H^{r+1}([a,c])} + \frac{1}{8}\alpha(h_1^{DG})^{-1}\sigma_1\chi^2(a+),
$$

$$
\begin{aligned}
|K(a)\xi'(a^+)\chi(a^+)| &\leq 2k_1^2(h_1^{DG})\sigma_1^{-1}(\xi')^2(a^+) + \frac{1}{8}\alpha(h_1^{DG})^{-1}\sigma_1\chi^2(a^+) \\
&\leq C{k_1}^2\sigma_1^{-1}(h_{DG})^{2r}|u|^2_{H^{r+1}([a,c])} + \frac{1}{8}\alpha(h_1^{DG})^{-1}\sigma_1\chi^2(a^+).
\end{aligned}
$$

The ninth term is bounded using (2.44) and (2.45).

$$
\begin{aligned}
|-\varepsilon K(a)\chi'(a^+)\xi(a^+)| &\leq Ck_1k_0^{-\frac{1}{2}}(h_1^{DG})^{-\frac{1}{2}}\|K^{\frac{1}{2}}\chi'\|_{L^2(I_1^{DG})}|\xi(a)| \\
&\leq C{k_1}^2k_0^{-1}(h_{DG})^{2r}|u|^2_{H^{r+1}([a,c])} + \frac{1}{8}\alpha\|K^{\frac{1}{2}}\chi'\|^2_{L^2(I_1^{DG})}.
\end{aligned}
$$

Finally, the last term in the right-hand side of (2.43) is bounded as follows:

$$
|-\frac{1}{2}\beta\xi(a^+)\chi(a^+)| \leq C\beta^2\sigma_1^{-2}(h_{DG})^{2r+2}|u|^2_{H^{r+1}(I_1^{DG})} + \frac{1}{8}\alpha(h_1^{DG})^{-1}\sigma_1\chi^2(a^+)
$$

Now let us consider the interface terms in the right-hand side of (2.42).

$$
\begin{aligned}
&|K(c)\frac{\chi(x_1^{FV})-\chi(c^-)}{h_0}(\xi(x_1^{FV})-\xi(c^-))| \\
&\le K(c)\frac{(\xi(x_1^{FV})-\xi(c^-))^2}{h_0}+K(c)\frac{(\chi(x_1^{FV})-\chi(c^-))^2}{4h_0} \\
&= K(c)\frac{\xi(c^-)^2}{h_0}+K(c)\frac{(\chi(x_1^{FV})-\chi(c^-))^2}{4h_0} \\
&\le Ck_1h_0^{-1}(h_N^{DG})^{2r+1}|u|^2_{H^{r+1}(I_N^{DG})}+K(c)\frac{(\chi(x_1^{FV})-\chi(c^-))^2}{4h_0} \\
&\le Ck_1(h_0^{-1}h_N^{DG})(h_{DG})^{2r}|u|^2_{H^{r+1}(I_N^{DG})}+K(c)\frac{(\chi(x_1^{FV})-\chi(c^-))^2}{4h_0},
\end{aligned}
$$

and

$$
\begin{aligned}
&|\beta\frac{\xi(c^-)+\xi(x_1^{FV})}{2}(\chi(c^-)-\chi(x_1^{FV}))| \\
&\le 4K(c)^{-1}\beta^2\xi^2(c^-)h_0+K(c)\frac{(\chi(c^-)-\chi(x_1^{FV}))^2}{4h_0} \\
&\le Ck_0^{-1}\beta^2h_0h_{DG}^{2r+1}|u|^2_{H^{r+1}(I_N^{DG})}+K(c)\frac{(\chi(c^-)-\chi(x_1^{FV}))^2}{4h_0}.
\end{aligned}
$$

Now let us consider the error term R_{FV} in (2.42).

By Lemma 2.1 and Lemma 2.2, we obtain

$$
\begin{aligned}
|R_{FV}(\chi)| &= \Big|\sum_{i=2}^{M}R_{i-\frac{1}{2}}\chi(x_i^{FV})-\sum_{i=1}^{M-1}R_{i+\frac{1}{2}}\chi(x_i^{FV})-R_M\chi(x_M^{FV}) \\
&\quad -\sum_{i=2}^{M}R^{\beta}_{i-\frac{1}{2}}\chi(x_i^{FV})+\sum_{i=1}^{M-1}R^{\beta}_{i+\frac{1}{2}}\chi(x_i^{FV})+R^{\beta}_M\chi(x_M^{FV})\Big| \\
&\le \Big|\sum_{i=1}^{M-1}R_{i+\frac{1}{2}}(\chi(x_{i+1}^{FV})-\chi(x_i^{FV}))-R_M\chi(x_M^{FV})\Big| \\
&+\Big|\sum_{i=1}^{M-1}R^{\beta}_{i+\frac{1}{2}}(\chi(x_i^{FV})-\chi(x_{i+1}^{FV}))+R^{\beta}_M\chi(x_M^{FV})\Big|
\end{aligned}
$$

$$\leq \left(\sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV}) \frac{(\chi(x_{i+1}^{FV}) - \chi(x_i^{FV}))^2}{h_{i+\frac{1}{2}}^{FV}} + K(b) \frac{\chi(x_M^{FV})^2}{h_M^{FV}} \right)^{\frac{1}{2}} \cdot$$

$$\left(\sum_{i=1}^{M-1} \frac{h_{i+\frac{1}{2}}^{FV}}{K(x_{i+\frac{1}{2}}^{FV})} R_{i+\frac{1}{2}}^2 + \frac{h_M^{FV}}{K(b)} R_M^2 \right)^{\frac{1}{2}}$$

$$+ \left(\sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV}) \frac{(\chi(x_{i+1}^{FV}) - \chi(x_i^{FV}))^2}{h_{i+\frac{1}{2}}^{FV}} + K(b) \frac{\chi(x_M^{FV})^2}{h_M^{FV}} \right)^{\frac{1}{2}} \cdot$$

$$\left(\sum_{i=1}^{M-1} \frac{h_{i+\frac{1}{2}}^{FV}}{K(x_{i+\frac{1}{2}}^{FV})} (R_{i+\frac{1}{2}}^{\beta})^2 + \frac{h_M^{FV}}{K(b)} (R_M^{\beta})^2 \right)^{\frac{1}{2}}$$

$$\leq h_{FV}^{\frac{1}{2}} k_0^{-\frac{1}{2}} \left(\sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV}) \frac{(\chi(x_{i+1}^{FV}) - \chi(x_i^{FV}))^2}{h_{i+\frac{1}{2}}^{FV}} + K(b) \frac{\chi(x_M^{FV})^2}{h_M^{FV}} \right)^{\frac{1}{2}} \left(\sum_{i=1}^{M-1} R_{i+\frac{1}{2}}^2 + R_M^2 \right)^{\frac{1}{2}}$$

$$+ C h_{FV} k_0^{-\frac{1}{2}} \left(\sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV}) \frac{(\chi(x_{i+1}^{FV}) - \chi(x_i^{FV}))^2}{h_{i+\frac{1}{2}}^{FV}} + K(b) \frac{\chi(x_M^{FV})^2}{h_M^{FV}} \right)^{\frac{1}{2}} \left(\sum_{i=1}^{M-1} h_{i+\frac{1}{2}}^{FV} + h_M^{FV} \right)^{\frac{1}{2}}$$

$$\leq h_{FV}^{\frac{1}{2}} k_0^{-\frac{1}{2}} \left(\sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV}) \frac{(\chi(x_{i+1}^{FV}) - \chi(x_i^{FV}))^2}{h_{i+\frac{1}{2}}^{FV}} + K(b) \frac{\chi(x_M^{FV})^2}{h_M^{FV}} \right)^{\frac{1}{2}} \cdot$$

$$\left(\sum_{i=1}^{M-1} k_1^2 h_{i+\frac{1}{2}}^{FV} \|u''\|_{L^2([x_i^{FV}, x_{i+1}^{FV}])}^2 + k_1^2 h_M^{FV} \|u''\|_{L^2([x_M^{FV}, b])}^2 \right)^{\frac{1}{2}}$$

$$+ C h_{FV} k_0^{-\frac{1}{2}} (b-a)^{\frac{1}{2}} \left(\sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV}) \frac{(\chi(x_{i+1}^{FV}) - \chi(x_i^{FV}))^2}{h_{i+\frac{1}{2}}^{FV}} + K(b) \frac{\chi(x_M^{FV})^2}{h_M^{FV}} \right)^{\frac{1}{2}}$$

$$\leq h_{FV} k_0^{-\frac{1}{2}} k_1 \left(\|u''\|_{L^2([c,b])} + C \right) \left(\sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV}) \frac{(\chi(x_{i+1}^{FV}) - \chi(x_i^{FV}))^2}{h_{i+\frac{1}{2}}^{FV}} + K(b) \frac{\chi(x_M^{FV})^2}{h_M^{FV}} \right)^{\frac{1}{2}}$$

$$\leq C h_{FV}^2 + \frac{1}{2} \left(\sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV}) \frac{(\chi(x_{i+1}^{FV}) - \chi(x_i^{FV}))^2}{h_{i+\frac{1}{2}}^{FV}} + K(b) \frac{\chi(x_M^{FV})^2}{h_M^{FV}} \right)$$

For the residual term R_I, we use the fact that $|R_0| + |R_1| \leq Ch_{FV}$.

$$\begin{aligned}
|R_I(\chi)| &= |R_0(\chi(x_1^{FV}) - \chi(c^-)) - R_1(\chi(c^-) - \chi(x_1^{FV}))| \\
&= |(R_0 + R_1)(\chi(x_1^{FV}) - \chi(c^-))| \\
&\leq Ch_{FV}|\chi(x_1^{FV}) - \chi(c^-)| \\
&\leq Ch_{FV}^3 k_0^{-1} + K(c)\frac{(\chi(x_1^{FV}) - \chi(c^-))^2}{4h_0}.
\end{aligned} \tag{2.46}$$

Combining the bounds above, we obtain

$$\begin{aligned}
&|a^{DG}(\xi,\chi) + a_{DF}^{\{\}}(\xi,\chi) - R_{FV}(\chi) - R_I(\chi)| \\
&\leq \frac{\alpha}{8}\sum_{n=1}^{N}(\int_{x_n^{DG}}^{x_{n+1}^{DG}} K(x)\chi'^2) + \frac{\alpha}{8}\sum_{n=1}^{N}(\int_{x_n^{DG}}^{x_{n+1}^{DG}} K(x)\chi'^2) + \frac{1}{8}|\beta|\sum_{n=2}^{N}[\chi(x_n^{DG})]^2 \\
&+ \frac{\alpha}{8}\sum_{n=2}^{N}(\tilde{h}_n^{DG})^{-1}\sigma_n[\chi(x_n^{DG})]^2 + \frac{\alpha}{8}\sum_{n=1}^{N}\|K^{\frac{1}{2}}\chi'\|_{L^2(I_n^{DG})}^2 + \frac{\alpha}{8}\sum_{n=2}^{N}(\tilde{h}_n^{DG})^{-1}\sigma_n[\chi(x_n^{DG})]^2 \\
&+ \frac{\alpha}{8}(\tilde{h}_1^{DG})^{-1}\sigma_1\chi^2(a^+) + \frac{\alpha}{8}(\tilde{h}_1^{DG})^{-1}\sigma_1\chi^2(a^+) + \frac{\alpha}{8}\|K^{\frac{1}{2}}\chi'\|^2(I_1^{DG}) + \frac{\alpha}{8}(\tilde{h}_1^{DG})^{-1}\sigma_1\chi^2(a^+) \\
&+ K(c)\frac{(\chi(x_1^{FV}) - \chi(c^-))^2}{4h_0} + \frac{1}{2}\left(\sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV})\frac{(\chi(x_{i+1}^{FV}) - \chi(x_i^{FV}))^2}{h_{i+\frac{1}{2}}^{FV}} + K(b)\frac{\chi(x_M^{FV})^2}{h_M^{FV}}\right) \\
&+ K(c)\frac{(\chi(x_1^{FV}) - \chi(c^-))^2}{4h_0} + \tilde{C}_1 h_{DG}^{2r} + \tilde{C}_2 h_{FV}^2.
\end{aligned} \tag{2.47}$$

Using the coercivity of $\tilde{a}^{DG}$, we obtain, as in (2.13),

$$\begin{aligned}
&\alpha\|\chi\|_{\widetilde{DG}}^2 + \frac{1}{2}|\beta|\sum_{n=2}^{N}[\chi(x_n^{DG})]^2 + \sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV})\frac{(\chi(x_{i+1}^{FV}) - \chi(x_i^{FV}))^2}{h_{i+\frac{1}{2}}} \\
&+ K(b)\frac{\chi^2(x_M^{FV})}{h_M} + \frac{1}{2}|\beta|\sum_{i=2}^{M}[\chi(x_{i-\frac{1}{2}}^{FV})]^2 + K(c)\frac{(\chi(x_1^{FV}) - \chi(c^-))^2}{h_0} \\
&\leq a^{DG}(\chi,\chi) + a^{FV}(\chi,\chi) + a_{DF}^{\{\}}(\chi,\chi).
\end{aligned}$$

Using the bound (2.47) and taking the right terms to the left, we obtain

$$\frac{1}{2}\alpha\|\chi\|_{\tilde{DG}} + \frac{3}{8}|\beta|\sum_{n=2}^{N}[\chi(x_n^{DG})]^2 + \frac{1}{2}\sum_{i=1}^{M-1} K(x_{i+\frac{1}{2}}^{FV})\frac{(\chi(x_{i+1}^{FV}) - \chi(x_i^{FV}))^2}{h_{i+\frac{1}{2}}}$$
$$+ \frac{1}{4}K(b)\frac{\chi^2(x_M^{FV})}{h_M} + \frac{1}{2}|\beta|\sum_{i=2}^{M}[\chi(x_{i-\frac{1}{2}}^{FV})]^2 + \frac{1}{2}K(c)\frac{(\chi(x_1^{FV}) - \chi(c^-))^2}{h_0}$$
$$\le \tilde{C}_1 h_{DG}^{2r} + \tilde{C}_2 h_{FV}^2.$$

Thus there exists $\tilde{C}$, such that

$$\|\chi\|_{\mathcal{E}} \le \tilde{C}(h_{DG}^r + h_{FV})$$

Also $\|\xi\|_{\mathcal{E}} \le C(h_{DG}^r + h_{FV})$. Therefore,

$$\|u_h - u\|_{\mathcal{E}} = \|\chi - \xi\|_{\mathcal{E}} \le \|\chi\|_{\mathcal{E}} + \|\xi\|_{\mathcal{E}} \le C(h_{DG}^r + h_{FV})$$

for some constant C depending on the real solution.

The error estimate proof for the scheme (2.7) is very similar to the above proof with a little modification for the interface term. The only difference lies in $R_I(v)$. For $a^{\dagger}(u,v)$ scheme, we denote $R_I(v)$ by $R_I^{\dagger}(v)$ and it satisfies

$$R_I^{\dagger}(v) = \begin{cases} R_0(v(x_1^{FV}) - v(c^-)), & \text{if } \beta \ge 0, \\ R_0(v(x_1^{FV}) - v(c^-)) + \beta(u(x_1^{FV}) - u(c))(v(c^-) - v(x_1^{FV})), & \text{if } \beta < 0. \end{cases}$$

We bound $R_I^{\dagger}(v)$ the same way as (2.46) and obtain the same bound as (2.46) with a different C. □

2.5 Numerical results

In this section, we present some numerical results which verify the error estimates given in the last section. For all the examples, we use a uniform mesh with size h

and choose the interval $(0, 1)$ with $c = 0.5$. We let $\varepsilon = -1$ and $\sigma_n = 10\ \forall n$. We use piecewise quadratic approximation over the DG interval. The integrals are evaluated by Gauss-Legendre quadrature with two nodes.

We first consider a constant solution to demonstrate the consistency of the coupled DG-FV method. Then we consider other examples.

Consider the boundary value problem:

$$\begin{aligned} -(u'(x))' + u' &= 0, \quad x \in [0, 1] \\ u(0) = 1,\ u(1) &= 1 \end{aligned} \tag{2.48}$$

The exact solution is $u(x) = 1$. Table 2.1 shows the numerical results. Since the solution is a constant, we expect the energy error to be zero by our error estimates.

h	$\|\|e\|\|_{\mathcal{E}}$ of $a^{\{\}}$	$\|\|e\|\|_{\mathcal{E}}$ of $a^{\uparrow}$
0.250000	2.5718e-15	1.8927e-15
0.125000	1.4463e-14	1.0035e-14
0.062500	1.3894e-14	9.6184e-15
0.031250	7.5633e-14	5.7359e-14
0.015625	1.0497e-13	7.1137e-14

Table 2.1 : Numerical results of the average and upwind schemes for Problem (2.48).

Second, we repeat the experiment for the following problem:

$$\begin{aligned} -(u'(x))' + u' &= 0, \quad x \in [0, 1] \\ u(0) = 1,\ u(1) &= e \end{aligned} \tag{2.49}$$

h	$\|e\|_{\tilde{DG}}$	CR	$\|e\|_{FV}$	CR	$\|e\|_{\mathcal{E}}$ of $a^{\{\}}$	CR
0.2500000	5.5976e-02	—	9.8069e-02	—	1.1627e-01	—
0.1250000	4.5703e-02	0.2925	5.1981e-02	0.9158	7.0830e-02	0.7151
0.0625000	2.7557e-02	0.7298	2.5709e-02	1.0157	3.8204e-02	0.8906
0.0312500	1.4982e-02	0.8792	1.2667e-02	1.0211	1.9764e-02	0.9509
0.0156250	7.7950e-03	0.9427	6.2725e-03	1.0140	1.0043e-02	0.9766
0.0078125	3.9737e-03	0.9720	3.1192e-03	1.0079	5.0614e-03	0.9886

Table 2.2 : Numerical results of the average scheme for Problem (2.49).

h	$\|e\|_{\tilde{DG}}$	CR	$\|e\|_{FV}$	CR	$\|e\|_{\mathcal{E}}$ of $a^{\uparrow}$	CR
0.2500000	7.2312e-02	—	9.3990e-02	—	1.1872e-01	—
0.1250000	4.9832e-02	0.5372	5.0627e-02	0.8926	7.1084e-02	0.7399
0.0625000	2.8584e-02	0.8019	2.5331e-02	0.9990	3.8233e-02	0.8947
0.0312500	1.5238e-02	0.9076	1.2568e-02	1.0111	1.9767e-02	0.9517
0.0156250	7.8585e-03	0.9553	6.2470e-03	1.0085	1.0044e-02	0.9768
0.0078125	3.9896e-03	0.9780	3.1127e-03	1.0050	5.0615e-03	0.9886

Table 2.3 : Numerical results of the upwind scheme for Problem (2.49).

The exact solution is $u(x) = e^x$. Table 2.2 and Table 2.3 show the numerical results.

Now let us test a function $u \in C([0,1])$ such that u' and K are not continuous, but $Ku' \in C([0,1])$. We choose $\beta = 1$, and

$$K(x) = \begin{cases} 1 & x \notin (\frac{1}{8}, \frac{2}{8}) \\ 2 & x \in (\frac{1}{8}, \frac{2}{8}) \end{cases} \qquad f(x) = \begin{cases} 0 & x \in [0, \frac{1}{8}] \\ (-x - \frac{7}{8})e^x & x \in (\frac{1}{8}, \frac{2}{8}) \\ -\frac{1}{8}e^x & x \in [\frac{2}{8}, 1] \end{cases}$$

The exact solution is

$$u(x) = \begin{cases} -2e^x \\ (x - \frac{17}{8})e^x \\ (\frac{1}{8}x - \frac{61}{32})e^x \end{cases} \quad D(x)u'(x) = \begin{cases} -2e^x & x \in [0, \frac{1}{8}] \\ 2(x - \frac{9}{8})e^x & x \in (\frac{1}{8}, \frac{2}{8}) \\ -(\frac{1}{8}x - \frac{53}{32})e^x & x \in [\frac{2}{8}, 1] \end{cases} \tag{2.50}$$

The errors and rates are given on Table 2.4 and Table 2.5.

h	$\|e\|_{\tilde{DG}}$	CR	$\|e\|_{FV}$	CR	$\|e\|_{\mathcal{E}}$ of $a^{\{\}}$	CR
0.1250000	1.0601e+01	—	6.8996e-01	—	1.0626e+01	—
0.0625000	4.7727e-02	7.7952	4.2685e-02	4.0147	6.4947e-02	7.3541
0.0312500	2.5926e-02	0.8804	2.0961e-02	1.0260	3.3594e-02	0.9511
0.0156250	1.3484e-02	0.9431	1.0360e-02	1.0167	1.7071e-02	0.9766
0.0078125	6.8730e-03	0.9722	5.1464e-03	1.0093	8.6034e-03	0.9886
0.0039062	3.4693e-03	0.9863	2.5644e-03	1.0049	4.3186e-03	0.9944

Table 2.4 : Numerical results of the average scheme for Problem (2.50).

h	$\|e\|_{\tilde{DG}}$	CR	$\|e\|_{FV}$	CR	$\|e\|_{\mathcal{E}}$ of $a^{\dagger}$	CR
0.1250000	1.0605e+01	—	6.9401e-01	—	1.0630e+01	—
0.0625000	4.9432e-02	7.7451	4.1959e-02	4.0479	6.4917e-02	7.3553
0.0312500	2.6350e-02	0.9077	2.0770e-02	1.0145	3.3580e-02	0.9510
0.0156250	1.3590e-02	0.9553	1.0311e-02	1.0104	1.7067e-02	0.9764
0.0078125	6.8994e-03	0.9780	5.1340e-03	1.0060	8.6023e-03	0.9884
0.0039062	3.4759e-03	0.9891	2.5613e-03	1.0032	4.3183e-03	0.9943

Table 2.5 : Numerical results of the upwind scheme for Problem (2.50).

Chapter 3

The coupled FV and DG method for higher dimensional convection-diffusion problem

FV and DG methods are two methods that are well suited for convection-diffusion problems. They are prefered over other numerical methods which (such as classical finite element method) give poor numerical solutions with wiggles or crosswind effects. Also FV and DG methods can be used on unstructured meshes and are locally mass conservative. Therefore, they are good choices for convection-diffusion equations. Because DG methods are more accurate and their meshes are much easier to be refined to capture local features than FV methods, I want to use DG methods in the regions where I want more accuracy. Because FV methods are more efficient than DG methods, I want to use FV method where less accuracy is needed.

In this chapter, I introduce the convection-diffusion model problem and establish the coupled FV and DG scheme in 2D and 3D. Then I prove the existence and uniqueness of the solution to the scheme. And finally I give the order of the error with respect to the energy norm.

3.1 Model problem and scheme

In this section, I first present the model problem. Then I introduce the FV and DG meshes I use for this model. After that, I establish the coupled FV and DG scheme on the meshes. Finally, I prove the existence and uniqueness of the multinumeric scheme.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded polygonal domain subdivided into non overlapping subdomains Ω_F^i and Ω_D^i and let $\Omega_F = \cup_i \Omega_F^i$ and $\Omega_D = \cup_i \Omega_D^i$. The multinumerical method uses a finite volume method on Ω_F and a discontinuous Galerkin method on Ω_D. The solution u of the convection-diffusion problem satisfies

$$-\nabla \cdot (K\nabla u - \boldsymbol{\beta} u) = f, \quad \text{in} \quad \Omega. \tag{3.1}$$

The function f belongs to $L^2(\Omega)$. The coefficient K is bounded above and below by positive constants k_1 and k_0 respectively. The vector $\boldsymbol{\beta}$ is divergence-free: $\nabla \cdot \boldsymbol{\beta} = 0$. Let $\mathcal{E}_D^h$ (resp. $\mathcal{E}_F^h$) be a subdivision of Ω_D (resp. Ω_F), made of cells V (Voronoi cells in Ω_F and either triangles/tetrahedra/hexahedra or Voronoi cells in Ω_D). We also denote by h_F (resp. h_D) the maximum diameter over all cells in Ω_F (resp. Ω_D) and we let $h = \max(h_F, h_D)$. We assume that the meshes match at the interface $\Gamma_{DF} = \partial\Omega_D \cap \partial\Omega_F$.

The definition of the mesh $\mathcal{E}_F^h$ requires further notation. We assume that $\mathcal{E}_F^h$ is an admissible finite volume mesh, in the following sense:

1. There is a family of nodes $\{x_V\}_{V \in \mathcal{E}_F^h}$ such that $x_V \in \overline{V}$ and if a face γ is such that $\gamma = \partial V \cap \partial W$ with $W \neq V$, it is assumed that $x_W \neq x_V$ and that the straight line going through x_V and x_W is orthogonal to γ.

2. For any boundary face $\gamma = \partial V \cap \partial\Omega$ with $V \in \mathcal{E}_F^h$, it is assumed that $x_V \notin \gamma$. However this condition can be relaxed (see Remark 1 in Section 3.2). Let y_γ be the (non-empty) intersection between the straight line going through x_V and orthogonal to γ. See Figure 3.1 .

We denote by $\Gamma_F^{h,\mathcal{I}}$ the set of faces that belong to the interior of Ω_F and by $\Gamma_F^{h,\partial}$ the set of boundary faces that belong to $\partial\Omega_F \cap \partial\Omega$. Similarly, the sets of faces that belong to

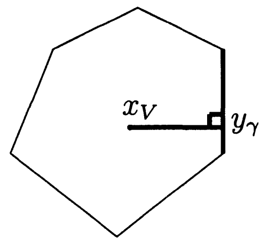

Figure 3.1 : A Voronoi cell on the interface or the boundary

the interior of Ω_D and boundary faces that belong to $\partial\Omega_D \cap \partial\Omega$ are denoted by $\Gamma_D^{h,\mathcal{I}}$ and $\Gamma_D^{h,\partial}$ respectively. Let $\boldsymbol{n}$ be the unit normal vector outward of Ω. We decompose those boundaries into inflow and outflow boundaries as follows:

$$\Gamma_F^{h,\partial+} = \{x \in \Gamma_F^{h,\partial}, \quad \boldsymbol{\beta} \cdot \boldsymbol{n} > 0\}, \quad \Gamma_F^{h,\partial-} = \Gamma_F^{h,\partial} \setminus \Gamma_F^{h,\partial+}, \tag{3.2}$$

$$\Gamma_D^{h,\partial+} = \{x \in \Gamma_D^{h,\partial}, \quad \boldsymbol{\beta} \cdot \boldsymbol{n} > 0\}, \quad \Gamma_D^{h,\partial-} = \Gamma_D^{h,\partial} \setminus \Gamma_D^{h,\partial+}. \tag{3.3}$$

The boundary condition is of Dirichlet type:

$$u = g \quad \text{on } \partial\Omega. \tag{3.4}$$

We also define

$$\Gamma_F^h = \Gamma_F^{h,\mathcal{I}} \cup \Gamma_F^{h,\partial}, \quad \Gamma_D^h = \Gamma_D^{h,\mathcal{I}} \cup \Gamma_D^{h,\partial}.$$

There remains the set of faces that belong to the interface Γ_{DF}; this particular set is denoted by Γ_{DF}^h.

We now define a parameter d_γ that is associated to each face in the FV mesh. Let V and W be two cells in the FV region such that $\gamma = \partial V \cap \partial W$ is an interior face. We define the parameter d_γ to be the Euclidean distance between the nodes x_V and x_W.

$$d_\gamma = d(x_V, x_W).$$

If the face γ is a boundary face (i.e. belongs to $\partial V \cap \partial\Omega$) the parameter d_γ is the distance between the node x_V and the face γ.

$$d_\gamma = d(x_V, \gamma) = d(x_V, y_\gamma).$$

Next, assume that a face γ is the intersection of a FV cell V and a DG cell W. The parameter d_γ is defined to be the distance between the node x_V and the point y_γ, which is (as in the boundary case) the intersection between the straight line going through x_V and orthogonal to γ. Here, we have made the assumption that x_V does not lie on the interface γ. Assume there is some $\theta > 0$ such that

$$\forall\gamma \in \Gamma_F^{h,\mathcal{I}}, \quad \gamma = \partial V \cap \partial W, \quad d_\gamma \geq \theta \max(h_V, h_W),$$

$$\forall\gamma \in \Gamma_F^{h,\partial}, \quad \gamma = \partial V \cap \partial\Omega, \quad d_\gamma \geq \theta h_V,$$

$$\forall\gamma \in \Gamma_{DF}^h, \quad \gamma = \partial V \cap \partial W, \quad V \in \mathcal{E}_F^h, \quad W \in \mathcal{E}_D^h, \quad d_\gamma \geq \theta h_V.$$

Finally, we define the harmonic average of the diffusion coefficient:

$$\forall\gamma \in \Gamma_F^{h,\mathcal{I}}, \quad \gamma = \partial V \cap \partial W, \quad K_\gamma = d_\gamma \left| \int_{x_V}^{x_W} \frac{ds}{K(s)} \right|^{-1},$$

$$\forall\gamma \in \Gamma_F^{h,\partial}, \quad \gamma = \partial V \cap \partial\Omega, \quad K_\gamma = d_\gamma \left| \int_{x_V}^{y_\gamma} \frac{ds}{K(s)} \right|^{-1},$$

$$\forall\gamma \in \Gamma_{DF}^h, \quad \gamma = \partial V \cap \partial W, \quad V \in \mathcal{E}_F^h, \quad W \in \mathcal{E}_D^h, \quad K_\gamma = d_\gamma \left| \int_{x_V}^{y_\gamma} \frac{ds}{K(s)} \right|^{-1}.$$

It is easy to see that K_γ is also bounded above and below by k_1 and k_0 respectively.

We denote by $|\gamma|$ the length of a face γ. The finite dimensional space consists of piecewise polynomials of degree less than or equal to r in the DG region and of degree equal to zero in the FV region.

$$\mathbb{X}^h = \{v \in L^2(\Omega) : v|_V \in \mathbb{P}_r(V) \quad \forall V \in \mathcal{E}_D^h, \quad v|_V \in \mathbb{P}_0(V) \quad \forall V \in \mathcal{E}_F^h\}.$$

We define the jump of a function in $\mathbb{X}^h$. For any face γ we fix a unit normal vector $\boldsymbol{n}_\gamma$ to γ. We assume that if γ is a boundary face (belongs to $\partial\Omega$), then $\boldsymbol{n}_\gamma$ points outward of $\partial\Omega$. If γ belongs to the interface Γ^h_{DF}, then we assume that $\boldsymbol{n}_\gamma$ points from the DG region into the FV region. Let us denote by V and W the mesh elements so that the vector $\boldsymbol{n}_\gamma$ points from ∂V into ∂W. We now define the jump of a function $v \in \mathbb{X}^h$.

$$
\begin{aligned}
&\gamma \in \Gamma_F^{h,\mathcal{I}}, \quad [v]|_\gamma = v(x_V) - v(x_W),\\
&\gamma \in \Gamma_D^{h,\mathcal{I}}, \quad [v]|_\gamma = v|_V - v|_W,\\
&\gamma \in \Gamma_{DF}^{h}, \quad [v]|_\gamma = v|_{\Omega_D}(y_\gamma) - v|_{\Omega_F}(x_W),\\
&\gamma \in \Gamma_F^{h,\partial}, \quad [v]|_\gamma = v(x_V),\\
&\gamma \in \Gamma_D^{h,\partial}, \quad [v]|_\gamma = v|_V.
\end{aligned}
$$

We remark that the quantity $[v]|_\gamma$ is a number except for the faces $\gamma \in \Gamma^h_D$.

We define the upwind and the downwind of a function $v \in \mathbb{X}^h$.

$$
\begin{aligned}
&\gamma \in \Gamma_F^{h,\mathcal{I}}, \begin{cases} v^\uparrow|_\gamma = v(x_V), \\ v^\uparrow|_\gamma = v(x_W), \end{cases} \begin{cases} v^\downarrow|_\gamma = v(x_W), & \text{if } \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma \geq 0, \\ v^\downarrow|_\gamma = v(x_V), & \text{otherwise.} \end{cases}\\
&\gamma \in \Gamma_D^{h,\mathcal{I}}, \begin{cases} v^\uparrow|_\gamma = v|_V, \\ v^\uparrow|_\gamma = v|_W, \end{cases} \begin{cases} v^\downarrow|_\gamma = v|_W, & \text{if } \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma \geq 0, \\ v^\downarrow|_\gamma = v|_V, & \text{otherwise.} \end{cases}\\
&\gamma \in \Gamma_{DF}^{h}, \begin{cases} v^\uparrow|_\gamma = v|_{\Omega_D}(y_\gamma), \\ v^\uparrow|_\gamma = v|_{\Omega_F}(x_W), \end{cases} \begin{cases} v^\downarrow|_\gamma = v|_{\Omega_F}(x_W), & \text{if } \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma \geq 0, \\ v^\downarrow|_\gamma = v|_{\Omega_D}(y_\gamma), & \text{otherwise.} \end{cases}\\
&\gamma \in \Gamma_F^{h,\partial+}, \quad v^\uparrow|_\gamma = v(x_V).\\
&\gamma \in \Gamma_D^{h,\partial+}, \quad v^\uparrow|_\gamma = v|_V.
\end{aligned}
\tag{3.5}
$$

The DG method requires additional notation. Let $\{v\}$ denote the average of a function

$v \in \mathbb{X}^h$.

$$\gamma \in \Gamma_D^{h,\mathcal{I}}, \quad \gamma = \partial V \cap \partial W, \quad \{v\}|_\gamma = 0.5(v|_V + v|_W),$$

$$\gamma \in \Gamma_D^{h,\partial}, \quad \gamma \in \partial V, \qquad\qquad \{v\}|_\gamma = v.$$

Let $\sigma > 0$ denote the penalty parameter and $\epsilon \in \{-1, 1\}$ be the symmetrization parameter. For a given face γ shared by two mesh elements V and W, let $h_\gamma = \max(\mathrm{diam}(V), \mathrm{diam}(W))$. The DG bilinear form is for all $u, v \in \mathbb{X}^h$

$$\begin{aligned} a_D(u,v) = &\sum_{V \in \mathcal{E}_D^h} \int_V K\nabla u \cdot \nabla v - \sum_{\gamma \in \Gamma_D^{h,\mathcal{I}}} \int_\gamma \{K\nabla u \cdot \boldsymbol{n}_\gamma\}[v] \\ &+ \epsilon \sum_{\gamma \in \Gamma_D^{h,\mathcal{I}}} \int_\gamma \{K\nabla v \cdot \boldsymbol{n}_\gamma\}[u] + \sum_{\gamma \in \Gamma_D^{h,\mathcal{I}}} \frac{\sigma}{h_\gamma} \int_\gamma [u][v] \\ &- \sum_{\gamma \in \Gamma_D^{h,\partial}} \int_\gamma K\nabla u \cdot \boldsymbol{n}_\gamma v + \epsilon \sum_{\gamma \in \Gamma_D^{h,\partial}} \int_\gamma K\nabla v \cdot \boldsymbol{n}_\gamma u + \sum_{\gamma \in \Gamma_D^{h,\partial}} \frac{\sigma}{h_\gamma} \int_\gamma uv \\ &- \sum_{V \in \mathcal{E}_D^h} \int_V \boldsymbol{\beta} u \cdot \nabla v + \sum_{\gamma \in \Gamma_D^{h,\mathcal{I}}} \int_\gamma \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma u^\uparrow [v] + \sum_{\gamma \in \Gamma_D^{h,\partial +}} \int_\gamma \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma uv. \end{aligned} \tag{3.6}$$

The cell-centered finite volume method is defined by the following bilinear form for all $u, v \in \mathbb{X}^h$

$$a_F(u,v) = \sum_{\gamma \in \Gamma_F^{h,\mathcal{I}} \cup \gamma \in \Gamma_F^{h,\partial}} \frac{|\gamma|}{d_\gamma} K_\gamma [u][v] + \sum_{\gamma \in \Gamma_F^{h,\mathcal{I}} \cup \Gamma_F^{h,\partial +}} \beta_\gamma u^\uparrow [v], \tag{3.7}$$

where

$$\beta_\gamma = \int_\gamma \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma.$$

Our scheme uses the overall bilinear form for all $u, v \in \mathbb{X}^h$

$$a(u,v) = a_D(u,v) + a_F(u,v) + a_{DF}(u,v), \tag{3.8}$$

where a_{DF} is the coupling form at the interface Γ_{FD}^h:

$$a_{DF}(u,v) = \sum_{\gamma \in \Gamma_{FD}^h} \frac{|\gamma|}{d_\gamma} K_\gamma [u][v] + \frac{1}{2} \sum_{\gamma \in \Gamma_{FD}^h} \int_\gamma \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma (u|_{\Omega_D} + u|_{\Omega_F})(v|_{\Omega_D} - v|_{\Omega_F}). \tag{3.9}$$

The source functions and inflow boundary condition are taken into account in the form

$$\forall v \in \mathbb{X}^h, \quad \ell(v) = \int_\Omega fv - \sum_{\gamma \in \Gamma_F^{h,\partial-} \cup \Gamma_D^{h,\partial-}} \int_\gamma \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma gv + \epsilon \sum_{\gamma \in \Gamma_D^{h,\partial}} \int_\gamma K \nabla v \cdot \boldsymbol{n}_\gamma g$$
$$+ \sum_{\gamma \in \Gamma_D^{h,\partial}} \frac{\sigma}{h_\gamma} \int_\gamma gv + \sum_{\gamma \in \Gamma_F^{h,\partial}} \frac{|\gamma|}{d_\gamma} K_\gamma g(y_\gamma) v. \tag{3.10}$$

The numerical scheme is: find $u_h \in \mathbb{X}^h$ satisfying

$$\forall v \in \mathbb{X}^h, \quad a(u_h, v) = \ell(v). \tag{3.11}$$

We next define some norms, that naturally arise from the bilinear forms above:

$$\|v\|_{DG} = \Big(\sum_{V \in \mathcal{E}_D^h} \|K^{1/2} \nabla v\|^2_{L^2(V)} + \sum_{\gamma \in \Gamma_D^h} \frac{\sigma}{h_\gamma} \|[v]\|^2_{L^2(\gamma)}$$
$$+ \frac{1}{2} \sum_{\gamma \in \Gamma_D^{h,\mathcal{I}}} \||\boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma|^{1/2} [v]\|^2_{L^2(\gamma)} + \frac{1}{2} \sum_{\gamma \in \Gamma_D^{h,\partial}} \||\boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma|^{1/2} v\|^2_{L^2(\gamma)} \Big)^{1/2}, \tag{3.12}$$

$$\|v\|_{FV} = \left(\sum_{\gamma \in \Gamma_F^h} \frac{|\gamma|}{d_\gamma} K_\gamma [v]^2 + \frac{1}{2} \sum_{\gamma \in \Gamma_F^{h,\mathcal{I}}} |\beta_\gamma| [v]^2 + \frac{1}{2} \sum_{\gamma \in \Gamma_F^{h,\partial}} |\beta_\gamma| v^2 \right)^{1/2}, \tag{3.13}$$

$$\|v\|_{\mathcal{E}} = \left(\|v\|^2_{DG} + \|v\|^2_{FV} + \sum_{\gamma \in \Gamma_{FD}^h} \frac{|\gamma| K_\gamma}{d_\gamma} [v]^2 \right)^{1/2}. \tag{3.14}$$

We now give some important properties of the bilinear forms.

Lemma 3.1

There exists a positive constant α independent of h such that

$$\forall v \in \mathbb{X}^h, \quad a(v, v) \geq \alpha \|v\|^2_{\mathcal{E}}. \tag{3.15}$$

Proof 3.1

First we show that

$$a_D(v, v) \geq \kappa \|v\|^2_{DG} - \frac{1}{2} \sum_{\gamma \in \Gamma_{DF}^h} \int_\gamma \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma v|^2_{\Omega_D}. \tag{3.16}$$

Second we show that

$$a_F(v,v) = \|v\|_{FV}^2 + \frac{1}{2}\sum_{\gamma\in\Gamma_{DF}^h}\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma v|_{\Omega_F}^2. \tag{3.17}$$

Since

$$a_{DF}(v,v) = \sum_{\gamma\in\Gamma_{DF}^h}\frac{|\gamma|K_\gamma}{d_\gamma}[v]^2 + \frac{1}{2}\sum_{\gamma\in\Gamma_{DF}^h}\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma(v_{\Omega_D}^2 - v_{\Omega_F}^2), \tag{3.18}$$

the result easily follows.

$$\begin{aligned}
a_D(v,v) = &\sum_{V\in\mathcal{E}_D^h}\int_V K\nabla v\cdot\nabla v - \sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\int_\gamma\{K\nabla v\cdot\boldsymbol{n}_\gamma\}[v] \\
&+\epsilon\sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\int_\gamma\{K\nabla v\cdot\boldsymbol{n}_\gamma\}[v] + \sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\frac{\sigma}{h_\gamma}\int_\gamma[v]^2 \\
&-\sum_{\gamma\in\Gamma_D^{h,\partial}}\int_\gamma(K\nabla v\cdot\boldsymbol{n}_\gamma)v + \epsilon\sum_{\gamma\in\Gamma_D^{h,\partial}}\int_\gamma(K\nabla v\cdot\boldsymbol{n}_\gamma)v + \sum_{\gamma\in\Gamma_D^{h,\partial}}\frac{\sigma}{h_\gamma}\int_\gamma v^2 \\
&-\sum_{V\in\mathcal{E}_D^h}\int_V\boldsymbol{\beta}v\cdot\nabla v + \sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\int_\gamma\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma v^\uparrow[v] + \sum_{\gamma\in\Gamma_D^{h,\partial+}}\int_\gamma\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma v^2.
\end{aligned} \tag{3.19}$$

Denote

$$\begin{aligned}
A_1 = &\sum_{V\in\mathcal{E}_D^h}\int_V K\nabla v\cdot\nabla v - \sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\int_\gamma\{K\nabla v\cdot\boldsymbol{n}_\gamma\}[v] \\
&+\epsilon\sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\int_\gamma\{K\nabla v\cdot\boldsymbol{n}_\gamma\}[v] + \sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\frac{\sigma}{h_\gamma}\int_\gamma[v]^2 \\
&-\sum_{\gamma\in\Gamma_D^{h,\partial}}\int_\gamma(K\nabla v\cdot\boldsymbol{n}_\gamma)v + \epsilon\sum_{\gamma\in\Gamma_D^{h,\partial}}\int_\gamma(K\nabla v\cdot\boldsymbol{n}_\gamma)v + \sum_{\gamma\in\Gamma_D^{h,\partial}}\frac{\sigma}{h_\gamma}\int_\gamma v^2 \\
A_2 = &-\sum_{V\in\mathcal{E}_D^h}\int_V\boldsymbol{\beta}v\cdot\nabla v + \sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\int_\gamma\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma v^\uparrow[v] + \sum_{\gamma\in\Gamma_D^{h,\partial+}}\int_\gamma\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma v^2.
\end{aligned}$$

We know from [42] there is a constant $\tilde{\kappa} > 0$ such that

$$A_1 \geq \tilde{\kappa}\left(\sum_{V\in\mathcal{E}_D^h}\|K^{1/2}\nabla v\|_{L^2(V)}^2 + \sum_{\gamma\in\Gamma_D^h}\frac{\sigma}{h_\gamma}\|[v]\|_{L^2(\gamma)}^2\right).$$

We use Green's first identity for the first equality in the following equations.

$$A_2 = -\frac{1}{2}\sum_{V\in\mathcal{E}_D^h}\int_{\partial V}\boldsymbol{\beta}\cdot\boldsymbol{n}v^2 + \sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma v^\uparrow[v] + \sum_{\gamma\in\Gamma_D^{h,\partial+}}\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma v^2$$

$$= -\frac{1}{2}\sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma[v^2] - \frac{1}{2}\sum_{\gamma\in\Gamma_D^{h,\partial}\cup\Gamma_{FD}^h}\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma v|^2_{\Omega_D}$$

$$+\sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma v^\uparrow[v] + \sum_{\gamma\in\Gamma_D^{h,\partial+}}\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma v^2$$

$$= \sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma(v^\uparrow[v]-\frac{1}{2}[v^2]) - \frac{1}{2}\sum_{\gamma\in\Gamma_{FD}^h}\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma v|^2_{\Omega_D} + \frac{1}{2}\sum_{\gamma\in\Gamma_D^{h,\partial}}\int_\gamma |\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma|v^2.$$

But $\quad (v^\uparrow[v]-\frac{1}{2}[v^2])\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma = \frac{1}{2}|\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma|[v]^2,$

then $\quad A_2 = \frac{1}{2}\sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\int_\gamma|\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma|[v]^2 + \frac{1}{2}\sum_{\gamma\in\Gamma_D^{h,\partial}}\int_\gamma|\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma|v^2 - \frac{1}{2}\sum_{\gamma\in\Gamma_{FD}^h}\int_\gamma\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma v|^2_{\Omega_D}.$

Therefore, we have

$$a_D(v,v) = A_1 + A_2 \geq \min(\tilde{\kappa},1)\|v\|^2_{DG} - \frac{1}{2}\sum_{\gamma\in\Gamma_{FD}^h}\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma v|^2_{\Omega_D}$$

$$\text{i.e. } a_D(v,v) \geq \kappa\|v\|^2_{DG} - \frac{1}{2}\sum_{\gamma\in\Gamma_{FD}^h}\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma v|^2_{\Omega_D}.$$

Now let us show that

$$a_F(v,v) = \|v\|^2_{FV} + \frac{1}{2}\sum_{\gamma\in\Gamma_{DF}^h}\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma v|^2_{\Omega_F}.$$

We recall that

$$a_F(v,v) = \sum_{\gamma\in\Gamma_F^h}\frac{|\gamma|}{d_\gamma}K_\gamma[v]^2 + \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}}\beta_\gamma v^\uparrow[v] = B_1 + B_2.$$

Using the definition (3.5), we have

$$\forall\gamma\in\Gamma_F^{h,I},\quad \beta_\gamma v^\uparrow[v] = |\beta_\gamma|v^\uparrow(v^\uparrow - v^\downarrow)$$

$$= \frac{1}{2}|\beta_\gamma|\left((v^\uparrow - v^\downarrow)^2 + v^{\uparrow 2} - v^{\downarrow 2}\right)$$

$$= \frac{1}{2}|\beta_\gamma|[v]^2 + \frac{1}{2}|\beta_\gamma|(v^{\uparrow 2} - v^{\downarrow 2})$$

Thus $\sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}} \beta_\gamma v^\uparrow[v] = \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}} \frac{1}{2}|\beta_\gamma|[v]^2 + \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}} \frac{1}{2}|\beta_\gamma|({v^\uparrow}^2 - {v^\downarrow}^2).$

Since $\nabla\cdot\boldsymbol{\beta} = 0$ and v is a piecewise constant on each cell, we have

$$-\sum_{V\in\mathcal{E}_F^h} \int_{\partial V} \boldsymbol{\beta}\cdot\boldsymbol{n}_V v^2 = -\sum_{V\in\mathcal{E}_F^h} v^2|_V \int_V \nabla\cdot\boldsymbol{\beta} = 0. \tag{3.20}$$

Noticing that

$$\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_V v^2 + \int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_W v^2 = \int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma [v^2] \quad \forall\gamma\in V\cap W,\ V \text{ and } W\in\mathcal{E}_F^h,$$

we collect the terms in (3.20) according to the faces and obtain

$$\begin{aligned} 0 &= -\sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}} \int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma[v^2] - \sum_{\gamma\in\Gamma_F^{h,\partial+}\cup\Gamma_F^{h,\partial-}} \beta_\gamma v^2 + \sum_{\gamma\in\Gamma_{DF}^h} \beta_\gamma v|^2_{\Omega_F} \\ \text{or}\quad 0 &= -\sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}} |\beta_\gamma|({v^\uparrow}^2 - {v^\downarrow}^2) - \sum_{\gamma\in\Gamma_F^{h,\partial+}\cup\Gamma_F^{h,\partial-}} \beta_\gamma v^2 + \sum_{\gamma\in\Gamma_{DF}^h} \beta_\gamma v|^2_{\Omega_F}. \end{aligned} \tag{3.21}$$

Adding B_2 and $\frac{1}{2}\times$(3.21) together, we obtain

$$\begin{aligned} B_2 &= \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}} \beta_\gamma v^\uparrow[v] + \sum_{\gamma\in\Gamma_F^{h,\partial+}} \beta_\gamma v^2 \\ &\quad + \frac{1}{2}\left(-\sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}} |\beta_\gamma|({v^\uparrow}^2 - {v^\downarrow}^2) - \sum_{\gamma\in\Gamma_F^{h,\partial+}\cup\Gamma_F^{h,\partial-}} \beta_\gamma v^2 + \sum_{\gamma\in\Gamma_{DF}^h} \beta_\gamma v|^2_{\Omega_F} \right) \\ &= \frac{1}{2}\sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}} |\beta_\gamma|[v]^2 + \frac{1}{2}\sum_{\gamma\in\Gamma_F^{h,\partial+}\cup\Gamma_F^{h,\partial-}} |\beta_\gamma| v^2 + \frac{1}{2}\sum_{\gamma\in\Gamma_{DF}^h} \beta_\gamma v|^2_{\Omega_F}. \end{aligned} \tag{3.22}$$

Therefore,

$$a_F(v,v) = \|v\|^2_{FV} + \frac{1}{2}\sum_{\gamma\in\Gamma_{DF}^h} \int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma v|^2_{\Omega_F}.$$

Theorem 3.1

There exists a unique solution $u_h \in \mathbb{X}^h$ satisfying (3.11).

Proof 3.2

It suffices to show uniqueness of u_h satisfying (3.11) with $f = g = 0$. Take $v = u_h$ in (3.11), and use coercivity of a. This implies that $\|u_h\|_{\mathcal{E}} = 0$ and thus $u_h = 0$ in $\mathbb{X}^h$.

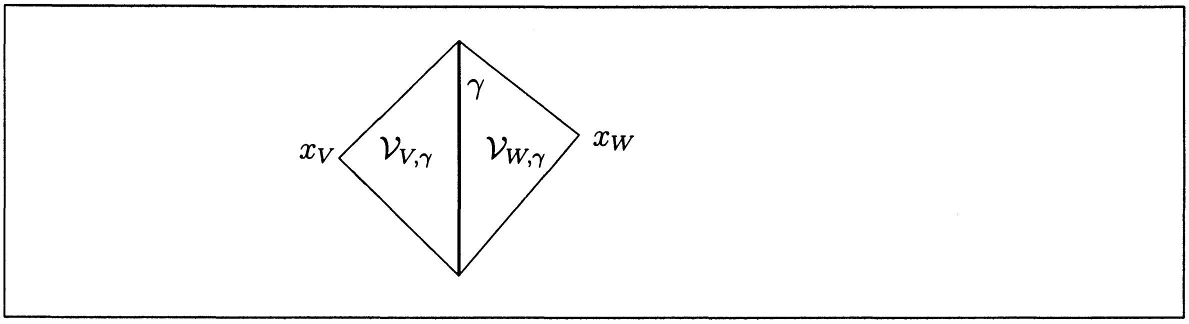

Figure 3.2 : $\mathcal{V}_{W,\gamma}$ and $\mathcal{V}_{V,\gamma}$

3.2 Error analysis

In this section, I obtain an error estimate with respect to the energy norm.

The proofs are given in the case where there are N DG regions and M FV regions.

For each face γ in the FV region, we define a subdomain $\mathcal{V}_\gamma$ as follows. Assume that $\gamma \in \Gamma_F^{h,\mathcal{I}}$ with $\gamma = \partial V \cap \partial W$. Define (see Figure 3.2)

$$\mathcal{V}_{W,\gamma} = \{tx_V + (1-t)x, \quad x \in \gamma, \quad t \in [0,1]\},$$

and let

$$\mathcal{V}_\gamma = \mathcal{V}_{W,\gamma} \cup \mathcal{V}_{V,\gamma}.$$

Assume now that $\gamma \in \Gamma_F^{h,\partial}$ with $\gamma \subset \partial W$, then $\mathcal{V}_\gamma = \mathcal{V}_{W,\gamma}$. Finally if $\gamma \in \Gamma_{DF}^h$ with $\gamma = \partial V \cap \partial W$, and $W \in \mathcal{E}_F^h$, then $\mathcal{V}_\gamma = \mathcal{V}_{W,\gamma}$.

Lemma 3.2

Define the residuals for any $u \in H^2(\Omega)$.

$$\forall \gamma \in \Gamma_F^{h,\mathcal{I}}, \qquad R_\gamma(u) = -\frac{|\gamma|}{d_\gamma} K_\gamma [u] - \int_\gamma K \nabla u \cdot \boldsymbol{n}_\gamma, \tag{3.23}$$

$$\forall \gamma \in \Gamma_F^{h,\partial}, \qquad R_\gamma(u) = -\frac{|\gamma|}{d_\gamma} K_\gamma (u(x_V) - g(y_\gamma)) - \int_\gamma K \nabla u \cdot \boldsymbol{n}_\gamma, \tag{3.24}$$

$$\forall \gamma \in \Gamma_F^{h,\mathcal{I}} \cup \Gamma_F^{h,\partial +}, \qquad Q_\gamma(u) = -\beta_\gamma u^\uparrow + \int_\gamma \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma u, \tag{3.25}$$

$$\forall \gamma \in \Gamma_{DF}^h, \qquad R_\gamma(u) = -K \nabla u \cdot \boldsymbol{n}_\gamma - \frac{K_\gamma}{d_\gamma}[u], \tag{3.26}$$

$$\forall \gamma \in \Gamma_{DF}^h, \qquad Q_\gamma(u) = \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma (u|_{\Omega_d} - u(x_{y_\gamma})). \tag{3.27}$$

Let $\mathbf{H}(u)$ denote the Hessian matrix of u. Assume K is a positive constant. Then, there exist a constant C_1 only dependent on θ and a constant C_2 only dependent on θ, d, $\boldsymbol{\beta}$, and p, such that

$$\gamma \in \Gamma_F^h, \quad |R_\gamma(u)|^2 \le C_1 \frac{h_F^2 |\gamma|}{d_\gamma} \int_{\mathcal{V}_\gamma} |\mathbf{H}(u)|^2, \tag{3.28}$$

$$\gamma \in \Gamma_{DF}^h, \quad \left(\int_\gamma |R_\gamma(u)| \right)^2 \le C_1 \frac{h_F^2 |\gamma|}{d_\gamma} \int_{\mathcal{V}_\gamma} |\mathbf{H}(u)|^2, \tag{3.29}$$

$$\gamma \in \Gamma_F^{h,\mathcal{I}} \cup \Gamma_F^{h,\partial +}, \quad |Q_\gamma(u)| \le C_2 h_F |\gamma|^{\frac{p-2}{p}} (\frac{|\gamma|}{d_\gamma})^{\frac{1}{p}} \|u\|_{W^{1,p}_{(\mathcal{V}_\gamma)}}, \tag{3.30}$$

$$\gamma \in \Gamma_{DF}^h, \quad \int_\gamma |Q_\gamma(u)| \le C_2 h_F |\gamma|^{\frac{p-2}{p}} (\frac{|\gamma|}{d_\gamma})^{\frac{1}{p}} \|u\|_{W^{1,p}_{(\mathcal{V}_\gamma)}}, \tag{3.31}$$

for all $p > d$ and such that $p < \infty$ if $d = 2$ and $p \le 6$ if $d = 3$.

Proof 3.3

Inequalities (3.28) and (3.29) can be found in [43], and inequalities (3.30) and (3.31) can be found in [44]. For completeness, we recall the proofs in Appendix A.

The following result shows that there is a consistency error only due to the FV discretization. In the DG regions, there is no consistency error.

Lemma 3.3

Let $u \in H^1(\Omega) \cap H^2(\mathcal{E}_D^h) \cap H^2(\Omega_F)$ be the solution to problem (3.1)-(3.4). Then u satisfies

$$\begin{aligned}\forall v \in \mathbb{X}^h, \quad a(u,v) = \ell(v) - \sum_{\gamma\in\Gamma_F^h} R_\gamma(u)[v] - \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}} Q_\gamma(u)[v] \\ - \sum_{\gamma\in\Gamma_{DF}^h} [v]\int_\gamma R_\gamma(u) + \sum_{\gamma\in\Gamma_{DF}^h}\int_\gamma K\nabla u\cdot \boldsymbol{n}_\gamma(v|_{\Omega_D} - v|_{\Omega_D}(y_\gamma)) \\ - \sum_{\gamma\in\Gamma_{DF}^h}\frac{1}{2}\int_\gamma Q_\gamma(u)(v|_{\Omega_D} - v_{\Omega_D}(y_\gamma)) - \sum_{\gamma\in\Gamma_{DF}^h}\frac{1}{2}[v]\int_\gamma Q_\gamma(u).\end{aligned} \tag{3.32}$$

Proof 3.4

Let $V \in \mathcal{E}_F^h$ and let $v \in \mathbb{X}^h$ such that $v|_V = 1$ and $v = 0$ elsewhere. Denote by $\boldsymbol{n}_V$ the outward unit normal to V. Multiply (3.1) by v and integrate on V by parts:

$$-\int_{\partial V} K\nabla u\cdot\boldsymbol{n}_V v + \int_{\partial V}\boldsymbol{\beta}\cdot\boldsymbol{n}_V uv = \int_V fv,$$

or

$$-\sum_{\gamma\in\partial V}\int_\gamma K\nabla u\cdot\boldsymbol{n}_V v + \sum_{\gamma\in\partial V}\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_V uv = \int_V fv. \tag{3.33}$$

Summing (3.33) over all FV cells, we obtain

$$\begin{aligned}-\sum_{\gamma\in\Gamma_F^{h,I}}\int_\gamma K\nabla u\cdot\boldsymbol{n}_\gamma[v] - \sum_{\gamma\in\Gamma_F^{h,\partial}}\int_\gamma K\nabla u\cdot\boldsymbol{n}_\gamma v + \sum_{\gamma\in\Gamma_F^{h,I}}\int_\gamma\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma u[v] \\ + \sum_{\gamma\in\Gamma_F^{h,\partial}}\int_\gamma\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma uv + \sum_{\gamma\in\Gamma_{DF}^h}\int_\gamma(K\nabla u - \boldsymbol{\beta}u)\cdot\boldsymbol{n}_\gamma v|_{\Omega_F} = \int_{\Omega_F} fv.\end{aligned} \tag{3.34}$$

Using the residual definitions, from (3.34) we obtain for all $v \in \mathbb{X}^h$

$$\begin{aligned}a_F(u,v) + \sum_{\gamma\in\Gamma_{DF}^h}\int_\gamma(K\nabla u - \boldsymbol{\beta}u)\cdot\boldsymbol{n}_\gamma v|_{\Omega_F} + \sum_{\gamma\in\Gamma_F^h} R_\gamma(u)[v] + \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}} Q_\gamma(u)[v] \\ = \int_{\Omega_F} fv - \sum_{\gamma\in\Gamma_F^{h,\partial-}}\int_\gamma\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma gv + \sum_{\Gamma_F^{h,\partial}}\frac{|\gamma|}{d_\gamma}K_\gamma g(y_\gamma)v.\end{aligned} \tag{3.35}$$

Next, we consider $V \in \mathcal{E}_D^h$, multiply (3.1) by $v \in \mathbb{X}^h$ and integrate by parts:

$$\int_V K\nabla u \cdot \nabla v - \int_{\partial V} K\nabla u \cdot \boldsymbol{n}_V v - \int_V \boldsymbol{\beta} u \cdot \nabla v + \int_{\partial V} \boldsymbol{\beta} u \cdot \boldsymbol{n}_V v = \int_V fv.$$

We sum over all V in the DG regions and collect the terms according to the faces:

$$\begin{aligned}\sum_{V\in\mathcal{E}_D^h} \int_V K\nabla u \cdot \nabla v - \sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}} \int_\gamma K\nabla u \cdot \boldsymbol{n}_\gamma [v] - \sum_{\gamma\in\Gamma_D^{h,\partial}\cup\Gamma_{DF}^h} \int_\gamma K\nabla u \cdot \boldsymbol{n}_\gamma v \\ - \sum_{V\in\mathcal{E}_D^h} \int_V \boldsymbol{\beta} u \cdot \nabla v + \sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}} \int_\gamma \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma u[v] + \sum_{\gamma\in\Gamma_D^{h,\partial}\cup\Gamma_{DF}^h} \int_\gamma \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma uv = \int_{\Omega_F} fv.\end{aligned} \tag{3.36}$$

We then replace $K\nabla u \cdot \boldsymbol{n}_\gamma$ in the second term of (3.36) with the average $\{K\nabla u \cdot \boldsymbol{n}_\gamma\}$ and u in the fifth term of (3.36) with the upwind $u^\uparrow$. We also use the boundary conditions and add the stabilization terms to obtain

$$\begin{aligned}a_D(u,v) - \sum_{\gamma\in\Gamma_{DF}^h} \int_\gamma (K\nabla u - \boldsymbol{\beta} u) \cdot \boldsymbol{n}_\gamma v|_{\Omega_D} \\ = \int_{\Omega_D} fv - \sum_{\gamma\in\Gamma_D^{h,\partial-}} \int_\gamma \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma gv + \epsilon \sum_{\gamma\in\Gamma_D^{h,\partial}} \int_\gamma K\nabla v \cdot \boldsymbol{n}_\gamma g + \sum_{\gamma\in\Gamma_D^{h,\partial}} \frac{\sigma}{h_\gamma} \int_\gamma gv.\end{aligned} \tag{3.37}$$

We now add (3.35) and (3.37) together:

$$a_F(u,v) + a_D(u,v) + T = \ell(v) - \sum_{\gamma\in\Gamma_F^h} R_\gamma(u)[v] - \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}} Q_\gamma(u)[v],$$

where T corresponds to the terms involving integrals on the interface Γ_{DF}. We can write using the regularity of the solution u (namely the fact that $u \in H^2(\Omega)$):

$$
\begin{aligned}
T = &- \sum_{\gamma \in \Gamma^h_{DF}} \int_\gamma (K\nabla u - \boldsymbol{\beta} u) \cdot \boldsymbol{n}_\gamma (v|_{\Omega_D} - v|_{\Omega_F}) \\
= &- \sum_{\gamma \in \Gamma^h_{DF}} \int_\gamma K\nabla u \cdot \boldsymbol{n}_\gamma (v|_{\Omega_D} - v|_{\Omega_F}) + \sum_{\gamma \in \Gamma^h_{DF}} \int_\gamma \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma u (v|_{\Omega_D} - v|_{\Omega_F}) \\
= &\sum_{\gamma \in \Gamma^h_{DF}} \frac{|\gamma| K_\gamma}{d_\gamma} [u][v] - \sum_{\gamma \in \Gamma^h_{DF}} \int_\gamma \left(K\nabla u \cdot \boldsymbol{n}_\gamma + \frac{K_\gamma}{d_\gamma}[u] \right) [v] \\
&- \sum_{\gamma \in \Gamma^h_{DF}} \int_\gamma K\nabla u \cdot \boldsymbol{n}_\gamma \, (v|_{\Omega_D} - v(y_\gamma)) \\
&+ \frac{1}{2} \sum_{\gamma \in \Gamma^h_{DF}} \int_\gamma \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma u (v|_{\Omega_D} - v|_{\Omega_F}) + \frac{1}{2} \sum_{\gamma \in \Gamma^h_{DF}} \int_\gamma \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma u (v_{\Omega_D} - v|_{\Omega_F}).
\end{aligned}
$$

Using the definition of the residual in Lemma 3.2,we obtain

$$
\begin{aligned}
T = &\sum_{\gamma \in \Gamma^h_{DF}} \frac{|\gamma| K_\gamma}{d_\gamma} [u][v] + \sum_{\gamma \in \Gamma^h_{DF}} [v] \int_\gamma R_\gamma(u)[v] - \sum_{\gamma \in \Gamma^h_{DF}} \int_\gamma K\nabla u \cdot \boldsymbol{n}_\gamma (v|_{\Omega_D} - v(y_\gamma)) \\
&+ \frac{1}{2} \sum_{\gamma \in \Gamma^h_{DF}} \int_\gamma \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma u (v|_{\Omega_D} - v_F) + \frac{1}{2} \sum_{\gamma \in \Gamma^h_{DF}} \int_\gamma (Q_\gamma(u) + \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma u|_{\Omega_F}(x_V))(v|_{\Omega_D} - v_F),
\end{aligned}
$$

or

$$
\begin{aligned}
T = &\, a_{DF}(u, v) + \sum_{\gamma \in \Gamma^h_{DF}} \int_\gamma R_\gamma(u)[v] - \sum_{\gamma \in \Gamma^h_{DF}} \int_\gamma K\nabla u \cdot \boldsymbol{n}_\gamma (v|_{\Omega_D} - v(y_\gamma)) \\
&+ \frac{1}{2} \sum_{\gamma \in \Gamma^h_{DF}} \int_\gamma Q_\gamma(u)(v|_{\Omega_D} - v|_{\Omega_D}(y_\gamma)) + \frac{1}{2} \sum_{\gamma \in \Gamma^h_{DF}} [v] \int_\gamma Q_\gamma(u).
\end{aligned}
$$

Thus we can conclude.

Theorem 3.2

Assume that $u \in H^2(\Omega)$ and that $u|_{\Omega_D} \in H^{r+1}(\mathcal{E}^h_D)$ for $r \geq 1$. Then there exists a constant C independent of h_D and h_F such that

$$
\|u_h - u\|_{\mathcal{E}} \leq C(h^r_D + h^{\frac{1}{2}}_D + h_F).
$$

Proof 3.5

We can write

$$
u_h - u = \chi - \xi, \quad \chi = u_h - \tilde{u}, \quad \xi = u - \tilde{u}.
$$

The function $\tilde{u} \in \mathbb{X}^h$ is chosen so that

$$\forall V \in \mathcal{E}_F^h, \quad \tilde{u}|_V = u(x_V). \tag{3.38}$$

On the DG regions $\tilde{u}$ is assumed to satisfy the usual approximation properties:

$$\|\tilde{u} - u\|_{H^q(V)} \leq Ch_V^{r+1-q}|u|_{H^{r+1}(V)}.$$

Using the definition of the scheme (3.11), Lemma 3.1 and Lemma 3.3, we obtain an error equation:

$$\begin{aligned}
\alpha\|\chi\|_{\mathcal{E}}^2 \leq a(\chi,\chi) = a(\xi,\chi) &+ \sum_{\gamma\in\Gamma_F^h} R_\gamma(u)[\chi] + \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}} Q_\gamma(u)[\chi] \\
&+ \sum_{\gamma\in\Gamma_{DF}^h} [\chi]\int_\gamma R_\gamma(u) - \sum_{\gamma\in\Gamma_{DF}^h}\int_\gamma K\nabla u\cdot \boldsymbol{n}_\gamma(\chi|_{\Omega_D} - \chi|_{\Omega_D}(y_\gamma)) \\
&+ \sum_{\gamma\in\Gamma_{DF}^h} \frac{1}{2}\int_\gamma Q_\gamma(u)(\chi|_{\Omega_D} - \chi|_{\Omega_D}(y_\gamma)) + \sum_{\gamma\in\Gamma_{DF}^h}\frac{1}{2}[\chi]\int_\gamma Q_\gamma(u).
\end{aligned}$$

Let us estimate the terms in the right hand side. Since $\xi(x_V) = 0$ for all nodes $x_V \in \Omega_F$, we have

$$a(\xi,\chi) = a_D(\xi,\chi) + a_F(\xi,\chi) + a_{DF}(\xi,\chi) = a_D(\xi,\chi) + a_{DF}(\xi,\chi).$$

We can use standard techniques to bound $a_D(\xi,\chi)$. We list the bounds for the terms in $a_D(\xi,\chi)$ here without proof.

$$\sum_{V\in\mathcal{E}_D^h}\int_V K\nabla\xi\cdot\nabla\chi \le \frac{1}{16}\alpha\sum_{V\in\mathcal{E}_D^h}\|K^{\frac{1}{2}}\nabla\xi\|^2_{L^2(V)} + Ch_D^{2r}|u|^2_{H^{r+1}(\mathcal{E}_D^h)}, \tag{3.39}$$

$$-\sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\int_\gamma\{K\nabla\xi\cdot\boldsymbol{n}_\gamma\}[\chi] \le \frac{1}{16}\alpha\sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\int_\gamma[\chi]^2 + Ch_D^{2r}|u|^2_{H^{r+1}(\mathcal{E}_D^h)}, \tag{3.40}$$

$$\epsilon\sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\int_\gamma\{K\nabla\chi\cdot\boldsymbol{n}_\gamma\}[\xi] \le \frac{1}{16}\alpha|\epsilon|\sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\|K^{\frac{1}{2}}\nabla\chi\|^2_{L^2(V_\gamma)} + Ch_D^{2r}|u|^2_{H^{r+1}(\mathcal{E}_D^h)}, \tag{3.41}$$

$$\sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\frac{\sigma}{h_\gamma}\int_\gamma[\xi][\chi] \le \frac{1}{16}\alpha\sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\frac{\sigma}{h_\gamma}\int_\gamma[\chi]^2 + Ch_D^{2r}|u|^2_{H^{r+1}(\mathcal{E}_D^h)}, \tag{3.42}$$

$$-\sum_{\gamma\in\Gamma_D^{h,\partial}}\int_\gamma K\nabla\xi\cdot\boldsymbol{n}_\gamma\chi \le \frac{1}{16}\alpha\sum_{\gamma\in\Gamma_D^{h,\partial}}\frac{\sigma}{h_\gamma}\int_\gamma\chi^2 + Ch_D^{2r}|u|^2_{H^2(\mathcal{E}_D^h)}, \tag{3.43}$$

$$\epsilon\sum_{\gamma\in\Gamma_D^{h,\partial}}\int_\gamma K\nabla\chi\cdot\boldsymbol{n}_\gamma\xi \le \frac{1}{16}\alpha|\epsilon|\sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\|K^{\frac{1}{2}}\nabla\chi\|^2_{L^2(V_\gamma)} + Ch_D^{2r}|u|^2_{H^{r+1}(\mathcal{E}_D^h)}, \tag{3.44}$$

$$\sum_{\gamma\in\Gamma_D^{h,\partial}}\frac{\sigma}{h_\gamma}\int_\gamma[\xi][\chi] \le \frac{1}{16}\alpha\sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\|K^{\frac{1}{2}}\nabla\chi\|^2_{L^2(V_\gamma)} + Ch_D^{2r}|u|^2_{H^{r+1}(\mathcal{E}_D^h)}, \tag{3.45}$$

$$-\sum_{V\in\mathcal{E}_D^h}\int_V\boldsymbol{\beta}\xi\cdot\nabla\chi \le \frac{1}{16}\alpha\sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\|K^{\frac{1}{2}}\nabla\chi\|^2_{L^2(V_\gamma)} + Ch_D^{2r+2}|u|^2_{H^{r+1}(\mathcal{E}_D^h)}, \tag{3.46}$$

$$\sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\int_\gamma\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma\xi^\uparrow[\chi] \le \frac{1}{16}\alpha\sum_{\gamma\in\Gamma_D^{h,\mathcal{I}}}\int_\gamma|\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma|[\chi]^2 + Ch_D^{2r+1}|u|^2_{H^{r+1}(\mathcal{E}_D^h)}, \tag{3.47}$$

$$\sum_{\gamma\in\Gamma_D^{h,\partial+}}\int_\gamma\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma\xi\chi \le \frac{1}{16}\alpha\sum_{\gamma\in\Gamma_D^{h,\partial+}}\|(\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma)^{\frac{1}{2}}\chi\|^2_{L^2(\gamma)} + Ch_D^{2r+1}|u|^2_{H^{r+1}(\mathcal{E}_D^h)}. \tag{3.48}$$

The other term reduces to

$$a_{DF}(\xi,\chi) = \sum_{\gamma\in\Gamma_{DF}^h}\frac{|\gamma|K_\gamma}{d_\gamma}[\chi]\xi|_{\Omega_D}(y_\gamma) + \sum_{\gamma\in\Gamma_{DF}^h}\int_\gamma\boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma 0.5\xi|_{\Omega_D}(\chi|_{\Omega_D}-\chi|_{\Omega_F}).$$

We claim that we can choose the approximation $\tilde{u}$ such that $\xi|_{\Omega_D}(y_\gamma)=0$. In that case we have

$$a_{DF}(\xi,\chi) = \frac{1}{2}\sum_{\gamma\in\Gamma_{DF}^h}\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma\xi|_{\Omega_D}(\chi|_{\Omega_D}-\chi|_{\Omega_F})$$
$$= \frac{1}{2}\sum_{\gamma\in\Gamma_{DF}^h}\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma\xi|_{\Omega_D}(\chi|_{\Omega_D}-\chi|_{\Omega_D}(y_\gamma)) + \frac{1}{2}\sum_{\gamma\in\Gamma_{DF}^h}[\chi]\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma\xi|_{\Omega_D}.$$

Let us fix a face $\gamma\in\Gamma_{DG}^h$ with $\gamma=\partial V\cap\partial W$, and $V\in\mathcal{E}_D^h$. Let us denote by $\eta=\chi|_V-\chi|_V(y_\gamma)$. Then we have by [24] and trace and inverse inequalities:

$$\|\eta\|_{L^\infty(\gamma)} \le Ch_D^{-1/2}\|\eta\|_{L^2(\gamma)} \le C\|\nabla\eta\|_{L^2(V)}. \tag{3.49}$$

Therefore, we obtain

$$\begin{aligned}a_{DF}(\xi,\chi) \le& \frac{1}{2}\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}\sum_{\gamma\in\Gamma_{DF}^h}\|\chi|_{\Omega_D}-\chi|_{\Omega_D}(y_\gamma)\|_{L^\infty(\gamma)}\int_\gamma|\xi|_{\Omega_D}| \\ &+\frac{1}{2}\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}\sum_{\gamma\in\Gamma_{DF}^h}|[\chi]|\int_\gamma|\xi|_{\Omega_D}| \\ &\le C\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}\sum_{\gamma\in\Gamma_{DF}^h}\|\nabla\chi|_{\Omega_D}\|_{L^2(V_\gamma)}\int_\gamma|\xi|_{\Omega_D}| + \frac{1}{2}\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}\sum_{\gamma\in\Gamma_{DF}^h}|[\chi]|\int_\gamma|\xi|_{\Omega_D}| \\ &\le \frac{\alpha}{16}\sum_{\gamma\in\Gamma_{DF}^h}\|K^{\frac{1}{2}}\nabla\chi|_{\Omega_D}\|_{L^2(V_\gamma)}^2 + Ck_0^{-1}\alpha^{-1}\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}^2\sum_{\gamma\in\Gamma_{DF}^h}(\int_\gamma|\xi|_{\Omega_D}|)^2 \\ &+\frac{\alpha}{16}\sum_{\gamma\in\Gamma_{DF}^h}\frac{|\gamma|}{d_\gamma}K_\gamma[\chi]^2 + 2\alpha^{-1}k_0^{-1}\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}^2\sum_{\gamma\in\Gamma_{DF}^h}\frac{d_\gamma}{|\gamma|}(\int_\gamma|\xi|_{\Omega_D}|)^2.\end{aligned} \tag{3.50}$$

Using the inequalities $\int_\gamma|\xi|_{\Omega_D}| \le |\gamma|^{\frac{1}{2}}(\int_\gamma|\xi|_{\Omega_D}|^2)^{\frac{1}{2}}$, $|\gamma|\le h_D^{d-1}$ and $d_\gamma\le 2h_D$, we obtain

$$a_{DF}(\xi,\chi) \le \frac{\alpha}{16}\|\chi\|_{DG}^2 + \frac{\alpha}{16}\sum_{\gamma\in\Gamma_{DF}^h}\frac{|\gamma|}{d_\gamma}K_\gamma[\chi]^2 + C\alpha^{-1}k_0^{-1}\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}^2 h_D\|\xi\|_{L^2(\Gamma_{DF}^h)}^2.$$

Using the approximation property of $\tilde{u}$ and trace inequalities, we get

$$a_{DF}(\xi,\chi) \le \frac{\alpha}{16}\|\chi\|_{DG}^2 + \frac{\alpha}{16}\sum_{\gamma\in\Gamma_{DF}^h}\frac{|\gamma|}{d_\gamma}K_\gamma[\chi]^2 + Ch_D^{2r+2}|u|_{H^{r+1}(\mathcal{E}_D^h)}^2. \tag{3.51}$$

The first consistency error term is bounded as follows:

$$\sum_{\gamma\in\Gamma_F^h} R_\gamma(u)[\chi] \leq \frac{1}{16}\alpha \sum_{\gamma\in\Gamma_F^h} \frac{|\gamma|K_\gamma}{d_\gamma}[\chi]^2 + 4\alpha^{-1}\sum_{\gamma\in\Gamma_F^h}\frac{d_\gamma}{|\gamma|}(R_\gamma(u))^2.$$

Using the bound (3.28) and denoting by $\mathbf{H}(u)$ the Hessian matrix of u, we have

$$\sum_{\gamma\in\Gamma_F^h} R_\gamma(u)[\chi] \leq \frac{1}{16}\alpha \sum_{\gamma\in\Gamma_F^h} \frac{|\gamma|K_\gamma}{d_\gamma}[\chi]^2 + Ch_F^2\int_{\Omega_F}|\mathbf{H}(u)|^2. \tag{3.52}$$

The second consistency error term is as:

$$\sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}} Q_\gamma(u)[\chi] \leq \frac{1}{16}\alpha \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}} \frac{|\gamma|}{d_\gamma}K_\gamma[\chi]^2 + 4\alpha^{-1}\sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}}\frac{d_\gamma}{|\gamma|}K_\gamma^{-1}\left(Q_\gamma(u)\right)^2.$$

Using Cauchy-Schwarz's inequality and the bound (3.30), we obtain

$$\begin{aligned}&\sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}} Q_\gamma(u)[\chi]\\ \leq&\frac{1}{16}\alpha \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}} \frac{|\gamma|}{d_\gamma}K_\gamma[\chi]^2 + C\alpha^{-1}k_0^{-1}h_F^2 \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}} |\gamma|^{1-\frac{2}{p}}d_\gamma^{1-\frac{2}{p}}\|u\|^2_{W^{1,p}(v_\gamma)}.\end{aligned} \tag{3.53}$$

Using Hölder's inequality, we obtain

$$\begin{aligned}&\sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}} Q_\gamma(u)[\chi]\\ &\leq \frac{1}{16}\alpha \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}} \frac{|\gamma|}{d_\gamma}K_\gamma[\chi]^2\\ &+ Ch_F^2\left(\sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}} (|\gamma|^{1-\frac{2}{p}}d_\gamma^{1-\frac{2}{p}})^{\frac{p}{p-2}}\right)^{1-\frac{2}{p}}\left(\sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}}\|u\|^p_{W^{1,p}(V_\gamma)}\right)^{\frac{2}{p}}\\ &\leq \frac{1}{16}\alpha \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}} \frac{|\gamma|}{d_\gamma}K_\gamma[\chi]^2 + Ch_F^2\left(\sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}}|\gamma|d_\gamma\right)^{1-\frac{2}{p}}\|u\|^2_{W^{1,p}(\mathcal{E}_F^h)}\\ &\leq \frac{1}{16}\alpha \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}} \frac{|\gamma|}{d_\gamma}K_\gamma[\chi]^2 + Ch_F^2|\Omega_F|^{1-\frac{2}{p}}\|u\|^2_{W^{1,p}(\mathcal{E}_F^h)}\\ &\leq \frac{1}{16}\alpha \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}} \frac{|\gamma|}{d_\gamma}K_\gamma[\chi]^2 + Ch_F^2\max(|\Omega_F|,1)\|u\|^2_{W^{1,p}(\mathcal{E}_F^h)}.\end{aligned} \tag{3.54}$$

The third consistency error term is as:

$$\sum_{\gamma\in\Gamma_{DF}^h}[\chi]\int_\gamma R_\gamma(u)\leq\frac{1}{16}\alpha\sum_{\gamma\in\Gamma_{DF}^h}\frac{|\gamma|K_\gamma}{d_\gamma}[\chi]^2+4\alpha^{-1}\sum_{\gamma\in\Gamma_{DF}^h}\frac{d_\gamma}{|\gamma|}(\int_\gamma R_\gamma(u))^2.$$

which with the bound (3.29) gives:

$$\sum_{\gamma\in\Gamma_{DF}^h}[\chi]\int_\gamma R_\gamma(u)\leq\frac{1}{16}\alpha\sum_{\gamma\in\Gamma_{DF}^h}\frac{|\gamma|K_\gamma}{d_\gamma}[\chi]^2+Ch_F^2\int_{\Omega_F}|\mathbf{H}(u)|^2. \tag{3.55}$$

The fourth consistency error term is bounded as below using Cauchy-Schwarz's inequality, the bound (3.49) and Young's Inequality.

$$\begin{aligned}
&-\sum_{\gamma\in\Gamma_{DF}^h}\int_\gamma K\nabla u\cdot\boldsymbol{n}_\gamma(\chi|_{\Omega_D}-\chi|_{\Omega_D}(y_\gamma))\\
&\leq\sum_{\gamma\in\Gamma_{DF}^h}\left(\int_\gamma|K\nabla u\cdot\boldsymbol{n}_\gamma|^2\right)^{\frac{1}{2}}\left(\int_\gamma(\chi|_{\Omega_D}-\chi|_{\Omega_D}(y_\gamma))^2\right)^{\frac{1}{2}}\\
&\leq Ck_1^{\frac{1}{2}}\sum_{\gamma\in\Gamma_{DF}^h}\|\nabla u\|_{L^2(\gamma)}h_D^{\frac{1}{2}}\|\nabla\chi\|_{L^2(V_\gamma)}\\
&\leq\frac{\alpha}{16}\|\chi\|_{DG}^2+Ch_D\|\nabla u\|_{L^2(\Gamma_{DF}^h)}^2.
\end{aligned} \tag{3.56}$$

Using the same argument and the bound (3.31), we bound the fifth consistent error:

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_{DF}^h}\frac{1}{2}\int_\gamma Q_\gamma(u)(\chi|_{\Omega_D}-\chi_{\Omega_D}(y_\gamma))\\
&\leq C\sum_{\gamma\in\Gamma_{DF}^h}\|\nabla\chi\|_{L^2(V_\gamma)}\int_\gamma|Q_\gamma(u)|\\
&\leq\frac{\alpha}{16}\sum_{\gamma\in\Gamma_{DF}^h}\|\nabla\chi\|_{L^2(V_\gamma)}^2+Ch_F^2\sum_{\gamma\in\Gamma_{DF}^h}|\gamma|^{\frac{2(p-2)}{p}}(\frac{|\gamma|}{d_\gamma})^{\frac{2}{p}}\|u\|_{W^{1,p}(V_\gamma)}^2\\
&\leq\frac{1}{16}\alpha\|\chi\|_{DG}^2+Ch_F^2\left(\sum_{\gamma\in\Gamma_{DF}^h}|\gamma|^{\frac{2(p-2)}{p}}(\frac{|\gamma|}{d_\gamma})^{\frac{p}{p-2}}\right)^{1-\frac{2}{p}}\left(\sum_{\gamma\in\Gamma_{DF}^h}\|u\|_{W^{1,p}(V_\gamma)}^p\right)^{\frac{2}{p}}.
\end{aligned}$$

Since $|\gamma| \le h_V^{d-1}, d_\gamma \ge \theta h_V$, we have $\frac{|\gamma|}{d_\gamma} \le \frac{h_\gamma^{d-2}}{\theta}$. Therefore, we obtain

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_{DF}^h} \frac{1}{2}\int_\gamma Q_\gamma(u)(\chi|_{\Omega_D} - \chi_{\Omega_D}(y_\gamma)) \\
&\le \frac{1}{16}\alpha\|\chi\|_{DG}^2 + Ch_F^2 h_F^{d-2}\left(\sum_{\gamma\in\Gamma_{DF}^h}|\gamma|d_\gamma\right)^{1-\frac{2}{p}}\left(\sum_{\gamma\in\Gamma_{DF}^h}\|u\|_{W^{1,p}(V_\gamma)}^p\right)^{\frac{2}{p}} \qquad (3.57)\\
&\le \frac{1}{16}\alpha\|\chi\|_{DG}^2 + Ch_F^2 h_F^{d-2}|\Omega_F|^{1-\frac{2}{p}}\|u\|_{W^{1,p}(\mathcal{E}_F^h)}^2 \\
&\le \frac{1}{16}\alpha\|\chi\|_{DG}^2 + Ch_F^2 h_F^{d-2}\max(|\Omega_F|,1)\|u\|_{W^{1,p}(\mathcal{E}_F^h)}^2.
\end{aligned}$$

Using the same skills as for (3.53) and the bound (3.31), we bound the last term:

$$\sum_{\gamma\in\Gamma_{DF}^h}\frac{1}{2}[\chi]\int_\gamma Q_\gamma(u) \le \frac{1}{16}\alpha\sum_{\gamma\in\Gamma_{DF}^h}\frac{|\gamma|}{d_\gamma}K_\gamma[\chi]^2 + Ch_F^2\|u\|_{W^{1,p}(\mathcal{E}_F^h)}^2. \qquad (3.58)$$

Combining all the bounds, we finally obtain

$$\frac{1}{16}\alpha\|\chi\|_{\mathcal{E}}^2 \le Ch_D^{2r}\left|u\right|_{H^{r+1}(\mathcal{E}_D^h)}^2 + Ch_F^2\int_{\Omega_F}\left|\mathbf{H}(u)\right|^2 + Ch_F^2\|u\|_{W^{1,p}(\mathcal{E}_F^h)}^2 + Ch_D\|\nabla u\|_{L^2(\Gamma_{DF}^h)}^2. \qquad (3.59)$$

We can then conclude.

Remark 1: The results of Theorem 3.2 are still valid if there are some nodes x_V located on boundary edges $\gamma \in \Gamma_F^{h,\partial}$. Let denote by $\Gamma_F^{h,0}$ the set of such edges. The coupled scheme is slightly modified. The discrete space is the set $\mathbb{Y}^h$ of functions $v \in \mathbb{X}^h$ such that $v(x_V) = 0$ for all $x_V \in \Gamma_F^{h,0}$. The bilinear form a_F and linear form ℓ become

$$a_F(u,v) = \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup(\Gamma_F^{h,\partial}\setminus\Gamma_F^{h,0})}\frac{|\gamma|}{d_\gamma}K_\gamma[u][v] + \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup(\Gamma_F^{h,\partial+}\setminus\Gamma_F^{h,0})}\beta_\gamma u^\uparrow[v] \qquad (3.60)$$

$$\begin{aligned}
\ell(v) &= \int_\Omega fv - \sum_{\gamma\in(\Gamma_F^{h,\partial-}\setminus\Gamma_F^{h,0})\cup\Gamma_D^{h,\partial-}}\int_\gamma \boldsymbol{\beta}\cdot\boldsymbol{n}_\gamma gv + \epsilon\sum_{\gamma\in\Gamma_D^{h,\partial}}\int_\gamma K\nabla v\cdot\boldsymbol{n}_\gamma g \\
&\quad + \sum_{\gamma\in\Gamma_D^{h,\partial}}\frac{\sigma}{h_\gamma}\int_\gamma gv + \sum_{\gamma\in\Gamma_F^{h,\partial}\setminus\Gamma_F^{h,0}}\frac{|\gamma|}{d_\gamma}K_\gamma g(y_\gamma)v. \qquad (3.61)
\end{aligned}$$

Remark 2: The error can be bounded by $C(h_D + h_F)$ if we assume that there exists a constant C_1 independent of h_D such that

$$\left(\sum_{\gamma\in\Gamma_{DF}, V_\gamma\in\mathcal{E}_D^h} \|\nabla u\|^2_{L^2(V_\gamma)}\right)^{\frac{1}{2}} \le C_1 h_D \|\nabla u\|_{L^2(\Omega_D)}. \tag{3.62}$$

We only need to focus on (3.56). Applying the trace inequality to (3.56), we obtain

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_{DF}} \Big| \int_\gamma K\nabla u\cdot \boldsymbol{n}_\gamma \left(v(y_\gamma) - v|_{\Omega_D}\right) \Big| \\
&\le Ch_D \sum_{\gamma\in\Gamma_{DF}} \|\nabla u\cdot \boldsymbol{n}_\gamma\|^2_{L^2(\gamma)} + \frac{\alpha}{16}\sum_{\gamma\in\Gamma_{DF}} \|\nabla v|_{\Omega_D}\|^2_{L^2(V_\gamma)} \\
&\le Ch_D \sum_{\gamma\in\Gamma_{DF}} h_D^{-1}\left(\|\nabla u\|^2_{L^2(V_\gamma)} + h_D^2\|\nabla^2 u\|^2_{L^2(V_\gamma)}\right) + \frac{\alpha}{16}\sum_{\gamma\in\Gamma_{DF}} \|\nabla v|_{\Omega_D}\|^2_{L^2(V_\gamma)}.
\end{aligned} \tag{3.63}$$

By the assumption (3.62), we obtain

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_{DF}} \Big| \int_\gamma K\nabla u\cdot \boldsymbol{n}_\gamma \left(v(y_\gamma) - v|_{\Omega_D}\right) \Big| \\
&\le Ch_D^2\left(\|\nabla u\|^2_{L^2(\Omega_D)} + \|\nabla^2 u\|^2_{L^2(\Omega_D)}\right) + \frac{\alpha}{16}\sum_{\gamma\in\Gamma_{DF}} \|\nabla v|_{\Omega_D}\|^2_{L^2(V_\gamma)}.
\end{aligned} \tag{3.64}$$

Thus we obtain error bounded by $C(h_D + h_F)$.

Let us see why the assumption (3.62) is reasonable. Let us review the motivation of the coupled FV-DG method. We want to use DG method for accuracy and FV method for less computational cost. Thus we expect the true solution to have more variations in DG region than in FV region. Therefore, when we choose the interface, we want the true solution to vary less in FV region and near the interface than in DG region. In other words, we want the average of the change of u near the interface is less than the average of the change of u on the DG region:

$$\frac{\left(\sum_{\gamma\in\Gamma_{DF}} \|\nabla u_{DG}\|^2_{L^2(V_\gamma)}\right)^{\frac{1}{2}}}{\left|\cup_{\gamma\in\Gamma_{DF}} V_\gamma\right|} \le L\frac{\|\nabla u\|_{L^2(\Omega_D)}}{|\Omega_D|}, \tag{3.65}$$

where L is some constant. Since

$$\big|\cup_{\gamma\in\Gamma_{DF}} V_\gamma\big| = \sum_{\gamma\in\Gamma_{DF}} \frac{1}{d}|\gamma| h_V \le \frac{1}{d} h_D \sum_{\gamma\in\Gamma_{DF}} |\gamma| \le \frac{1}{d} h_D \big(diam(\Omega_D)\big)^{d-1},$$

we have

$$\left(\sum_{\gamma\in\Gamma_{DF}} \|\nabla u_{DG}\|^2_{L^2(V_\gamma)}\right)^{\frac{1}{2}} \le \frac{LM(diam(\Omega_D))^{d-1}}{d|\Omega_D|} h_D \|\nabla u\|_{L^2(\Omega_D)}. \tag{3.66}$$

Thus, we get the assumption.

3.3 Numerical results

In this section, we present some numerical results which verify the error estimates given in the last section. For all the examples, we choose the Ω to be $(0,1)\times(0,1)$ with the interface to be $[0.5,0]\times[0.5,1]$, FV domain to be $[0,0.5]\times[0,1]$ and DG domain to be $[0.5,1]\times[0,1]$. We let $\varepsilon=-1$ and $\sigma=1$. We use piecewise quadratic approximation over the DG domain. The integrals on the cells are evaluated by Dunavant Gaussian quadrature with 7 nodes and the integrals on the faces are evaluated by Gauss-Legendre quadrature with 12 nodes. In the tables, e denotes the error and CR means the convergence rate. $|||e|||_{L^2(\Omega_F)}$ is defined to be $\left(\sum_{V\in\mathcal{E}_F^h} |V|\big(u(x_V)-u_h(x_V)\big)^2\right)^{\frac{1}{2}}$ and $|||e|||_{H^1}(\Omega_F)$ to be $\left(\sum_{\gamma\in\Gamma_F^h} \frac{|\gamma|}{d_\gamma} K_\gamma [e]^2\right)^{\frac{1}{2}}$. The interface error $|||e|||_{\Gamma_{DF}}$ is defined to be $\left(\sum_{\gamma\in\Gamma_{FD}^h} \frac{|\gamma| K_\gamma}{d_\gamma}[v]^2\right)^{\frac{1}{2}}$.

We first present a constant solution to demonstrate the consistency of the coupled DG-FV method. Consider the boundary value problem:

$$\begin{aligned} &-\nabla\cdot(\nabla u(x,y)-\boldsymbol{\beta} u(x,y)) = 0, \quad (x,y)\in\Omega,\ \boldsymbol{\beta}=(-1,5) \\ &u(x,y)=10, \quad \text{on } \partial\Omega. \end{aligned} \tag{3.67}$$

The exact solution is $u = 10$. Table 3.1 shows the numerical results. Since the solution is a constant, we expect the error to be zero by our error estimates.

h	$\|\|\|e\|\|\|_{L^2(\Omega_F)}$	$\|\|\|e\|\|\|_{H^1(\Omega_F)}$	$\|\|e\|\|_{L^2(\Omega_D)}$	$\|\|e\|\|_{H^1(\Omega_D)}$	$\|\|\|e\|\|\|_{\Gamma_{DF}}$	$\|\|e\|\|_{\mathcal{E}}$
0.30000	9.0568e-13	7.6680e-12	3.8056e-12	6.2773e-11	1.1533e-11	9.4935e-11
0.15000	5.8430e-13	6.0879e-12	2.1246e-12	5.6248e-11	5.6548e-12	8.9238e-11
0.07500	3.9842e-13	4.1519e-12	1.1481e-12	5.4747e-11	3.0654e-12	8.8464e-11
0.03750	7.7991e-10	9.0686e-09	2.8969e-08	3.8870e-06	7.8226e-10	4.1526e-06
0.01875	1.3290e-11	2.4848e-10	2.3210e-09	6.2895e-07	7.2830e-12	6.7208e-07

Table 3.1 : The numerical results for Problem (3.67).

Second, we repeat the experiment for another problem. Consider the following problem:

$$
\begin{aligned}
&-\nabla \cdot (\nabla u - \boldsymbol{\beta} u) = sin(x) + 0.5cos(x) + 2cos(2y) - 1.2sin(2y), (x,y) \in \Omega,\\
&\boldsymbol{\beta} = (0.5, 1.2),\\
&u = cos(y)^2, \quad (x,y) \in [0,0] \times [0,1],\\
&u = sin(x) + 1, \quad (x,y) \in [0,0] \times [1,0],\\
&u = sin(1) + cos(y)^2, \quad (x,y) \in [1,0] \times [1,1],\\
&u = sin(x) + cos(1)^2, \quad (x,y) \in [0,1] \times [1,1].
\end{aligned}
\tag{3.68}
$$

The true solution is $u = sin(x) + cos(y)^2$. The numerical results are showed in Table 3.2.

The errors on FV regions and the interface

(a)

h	$\|\|\|e\|\|\|_{L^2(\Omega_F)}$	CR	$\|\|\|e\|\|\|_{H^1(\Omega_F)}$	CR	$\|\|\|e\|\|\|_{\Gamma_{DF}}$	CR
0.30000	2.9060e-03	–	2.2188e-02	–	9.7080e-02	–
0.15000	1.7728e-03	0.7130	1.2438e-02	0.8350	3.5607e-02	1.4470
0.07500	1.0368e-03	0.7739	7.2345e-03	0.7818	1.5729e-02	1.1787
0.03750	5.2614e-04	0.9786	3.5621e-03	1.0222	7.4419e-03	1.0797
0.01875	2.6859e-04	0.9700	1.7762e-03	1.0039	3.6378e-03	1.0326

The errors on DG regions and the energy error

(b)

h	$\|\|e\|\|_{L^2(\Omega_D)}$	CR	$\|\|e\|\|_{H^1(\Omega_D)}$	CR	$\|\|e\|\|_{\mathcal{E}}$	CR
0.30000	2.7990e-03	–	3.9176e-02	–	1.0856e-01	–
0.15000	1.7189e-03	0.7034	2.1646e-02	0.8559	4.4168e-02	1.2974
0.07500	9.5880e-04	0.8421	1.2185e-02	0.8290	2.1616e-02	1.0309
0.03750	4.8525e-04	0.9825	7.6697e-03	0.6679	1.1541e-02	0.9053
0.01875	2.4547e-04	0.9832	5.0791e-03	0.5946	6.6832e-03	0.7882

Table 3.2 : The numerical results for Problem (3.68).

Let us test a third problem.

$$
\begin{aligned}
&-\nabla \cdot (\nabla u(x,y) - \boldsymbol{\beta} u(x,y)) = (14y-9)(x^2-x) + (1-6x)(y^2-y), \quad (x,y) \in \Omega, \\
&\boldsymbol{\beta} = (-3,7), \\
&u(x,y) = 0, \quad \text{on } \partial\Omega.
\end{aligned}
\tag{3.69}
$$

The true solution is $u = x(x-1)y(y-1)$. See Table 3.3 for the numerical results.

The errors on FV regions and the interface

(a)

h	$\|\|\|e\|\|\|_{L^2(\Omega_F)}$	CR	$\|\|\|e\|\|\|_{H^1(\Omega_F)}$	CR	$\|\|\|e\|\|\|_{\Gamma_{DF}}$	CR
0.30000	8.2516e-04	–	5.7522e-03	–	8.4777e-03	–
0.15000	1.9218e-04	2.1022	1.5502e-03	1.8917	5.1240e-03	0.7264
0.07500	3.5398e-05	2.4407	5.0068e-04	1.6305	2.5175e-03	1.0253
0.03750	3.5822e-06	3.3048	2.0792e-04	1.2679	1.2665e-03	0.9911
0.01875	4.1005e-06	0.1950	9.8329e-05	1.0803	6.3269e-04	1.0013

The errors on DG regions and the energy error

(b)

h	$\|e\|_{L^2(\Omega_D)}$	CR	$\|e\|_{H^1(\Omega_D)}$	CR	$\|e\|_{\mathcal{E}}$	CR
0.30000	5.6181e-04	–	4.4700e-03	–	1.1363e-02	–
0.15000	1.6505e-04	1.7672	1.5636e-03	1.5154	5.6295e-03	1.0133
0.07500	5.9141e-05	1.4807	7.8468e-04	0.9947	2.7119e-03	1.0537
0.03750	1.8031e-05	1.7137	4.4124e-04	0.8305	1.3775e-03	0.9773
0.01875	6.0853e-06	1.5671	2.7864e-04	0.6631	7.1558e-04	0.9449

Table 3.3 : The numerical results for Problem (3.69).

Here is the fourth example.

$$\begin{aligned}
&-\nabla\cdot(\nabla u(x,y)-\boldsymbol{\beta}u(x,y)) = (-4x^2-6x+4)e^{x^2}e^{y}, \quad (x,y)\in\Omega,\\
&\boldsymbol{\beta}=(-3,7),\\
&u=e^{y}, \quad (x,y)\in[0,0]\times[0,1],\\
&u=e^{x^2}, \quad (x,y)\in[0,0]\times[1,0],\\
&u=e^{y+1}, \quad (x,y)\in[1,0]\times[1,1],\\
&u=e^{x^2+1}, \quad (x,y)\in[0,1]\times[1,1].
\end{aligned} \tag{3.70}$$

The true solution is $u = e^{x^2}e^y$. See Table 3.4 for numerical results.

The errors on FV regions and the interface

(a)

h	$\|\|\|e\|\|\|_{L^2(\Omega_F)}$	CR	$\|\|\|e\|\|\|_{H^1(\Omega_F)}$	CR	$\|\|\|e\|\|\|_{\Gamma_{DF}}$	CR
0.30000	1.0986e-02	–	1.2007e-01	–	2.0655e-01	–
0.15000	7.5485e-03	0.5414	9.8318e-02	0.2883	1.1150e-01	0.8894
0.07500	5.5050e-03	0.4554	7.1179e-02	0.4660	5.6243e-02	0.9873
0.03750	3.0863e-03	0.8349	3.8702e-02	0.8790	2.8541e-02	0.9786
0.01875	1.6375e-03	0.9144	1.9898e-02	0.9598	1.4374e-02	0.9896

The errors on DG regions and the energy error

(b)

h	$\|\|e\|\|_{L^2(\Omega_D)}$	CR	$\|\|e\|\|_{H^1(\Omega_D)}$	CR	$\|\|e\|\|_{\mathcal{E}}$	CR
0.30000	7.2179e-03	–	1.1440e-01	–	2.8217e-01	–
0.15000	2.0094e-03	1.8448	5.6285e-02	1.0233	1.6776e-01	0.7502
0.07500	5.1764e-04	1.9567	3.2555e-02	0.7899	1.0060e-01	0.7378
0.03750	2.3605e-04	1.1329	2.0413e-02	0.6734	5.3663e-02	0.9066
0.01875	1.7179e-04	0.4584	1.3591e-02	0.5868	2.8632e-02	0.9063

Table 3.4 : The numerical results for Problem (3.70).

The problem here is an example of Remark 2 in Section 3.2. The example has first order convergence rate.

$$\begin{aligned}
&-\nabla \cdot (\nabla u(x,y) - \boldsymbol{\beta} u(x,y)) = -1 - 6x + 14y, \quad (x,y) \in \Omega, \\
&\boldsymbol{\beta} = (-3, 7), \\
&u = 0.25 + y^2, \quad (x,y) \in [0,0] \times [0,1], \\
&u = (x - 0.5)^2, \quad (x,y) \in [0,0] \times [1,0], \\
&u = 0.25 + y^2, \quad (x,y) \in [1,0] \times [1,1], \\
&u = (x - 0.5)^2 + 1, \quad (x,y) \in [0,1] \times [1,1].
\end{aligned} \tag{3.71}$$

The true solution is $u = (x - 0.5)^2 + y^2$. See Table 3.5 for numerical results.

We see from the tables that the convergence rates are greater than 0.5 which match our error estimate. From Table 3.2, we notice that the convergence rate of $\|e\|_{H^1(\Omega_D)}$ is going close to 0.5. Since $\|e\|_{H^1(\Omega_D)}$ is a part of the energy error, we expect the convergence rate of the energy error to be 0.5 if the mesh size is small enough. From Table 3.5, we see that the convergence rate is 1. This is because our true solution satisfies Remark 2. Thus we get first order convergence rate.

Figure 3.3 shows the exact solution on the voronoi mesh with h=0.0375. Figure 3.4 shows the DG solution and the error on the same mesh. The time used to calculate the DG solution is 22 seconds. Figure 3.5 shows the FV solution and the error on the same mesh. The time used to calculate the FV solution is 5 seconds. Figure 3.6 shows the coupled FV and DG solution on the same mesh. The time used for this solution is 13 seconds. From these figures, we can see that the coupled method uses less time than the DG method, but it gains the same accuracy as the DG method. Though the FV method uses less time, the error is bigger than the DG and the coupled methods.

The errors on FV regions and the interface

(a)

h	$\|\|\|e\|\|\|_{L^2(\Omega_F)}$	CR	$\|\|\|e\|\|\|_{H^1(\Omega_F)}$	CR	$\|\|\|e\|\|\|_{\Gamma_{DF}}$	CR
0.30000	2.7076e-02	–	1.9896e-01	–	1.3964e-01	–
0.15000	1.6876e-02	0.6820	1.2675e-01	0.6505	4.9746e-02	1.4891
0.07500	9.7086e-03	0.7976	7.5608e-02	0.7454	2.1099e-02	1.2374
0.03750	5.2047e-03	0.8994	4.0809e-02	0.8897	9.4969e-03	1.1517
0.01875	2.6504e-03	0.9736	2.0781e-02	0.9736	4.4527e-03	1.0928

The errors on DG regions and the energy error

(b)

h	$\|\|e\|\|_{L^2(\Omega_D)}$	CR	$\|\|e\|\|_{H^1(\Omega_D)}$	CR	$\|\|e\|\|_{\mathcal{E}}$	CR
0.30000	7.6043e-03	–	5.6859e-02	–	2.8501e-01	–
0.15000	6.4438e-03	0.2389	4.5141e-02	0.3330	1.5869e-01	0.8448
0.07500	3.7446e-03	0.7831	2.7050e-02	0.7388	8.8089e-02	0.8492
0.03750	1.9947e-03	0.9086	1.4895e-02	0.8608	4.5892e-02	0.9407
0.01875	1.0126e-03	0.9781	7.7677e-03	0.9393	2.3012e-02	0.9959

Table 3.5 : The numerical results for Problem (3.71).

The Figures 3.7, 3.8, 3.9 and 3.10 are the true, DG, FV and the coupled solutions on the voronoi mesh with h=0.01875. The figures show that for this problem, the coupled FV-DG solution is as accurate as the DG solution but with much lower cost and is more accurate than the FV solution. Figure 3.11 shows the error of the FV solution and the coupled FV-DG solution on different meshes. The figure shows that the coupled method can gain the same accuracy as the FV method on a much courser mesh and, therefore, leads to a smaller or equal cost.

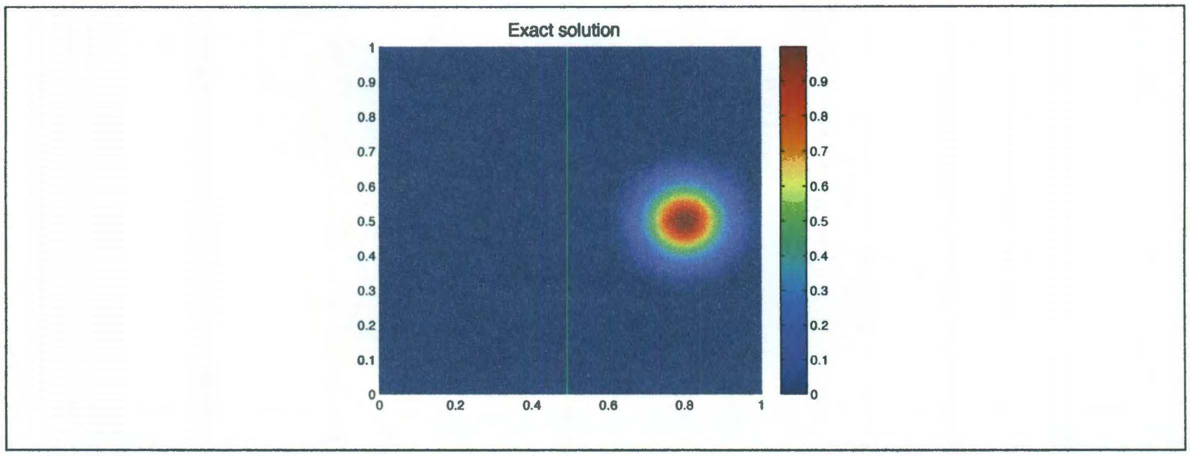

Figure 3.3 : Exact solution on the voronoi mesh with h=0.0375.

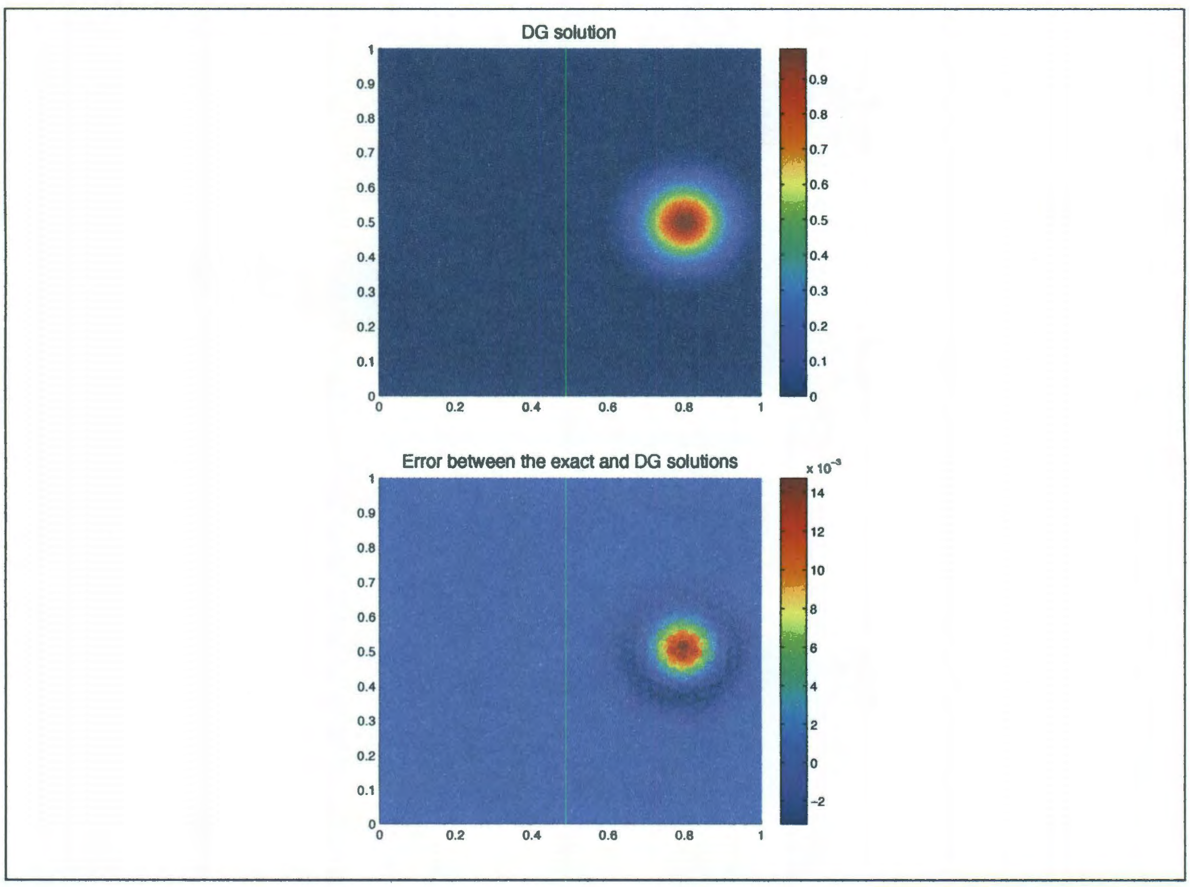

Figure 3.4 : Voronoi mesh with h=0.0375; time used=22 seconds.

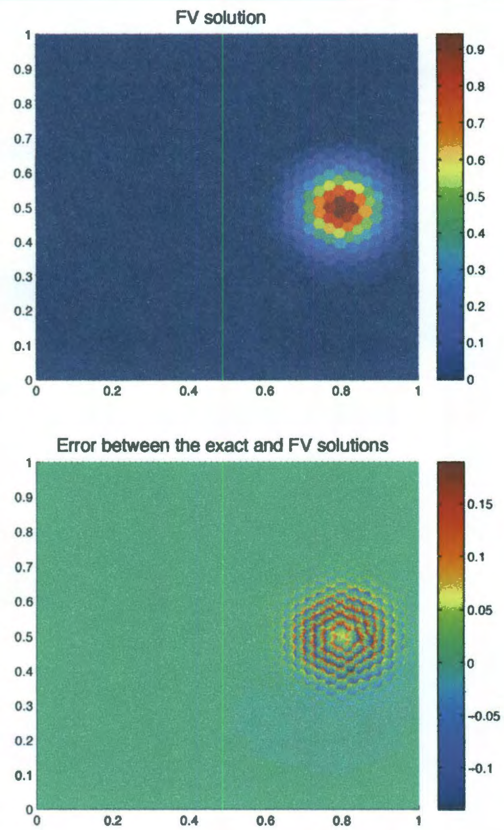

Figure 3.5 : Voronoi mesh with h=0.0375; time used=5 seconds.

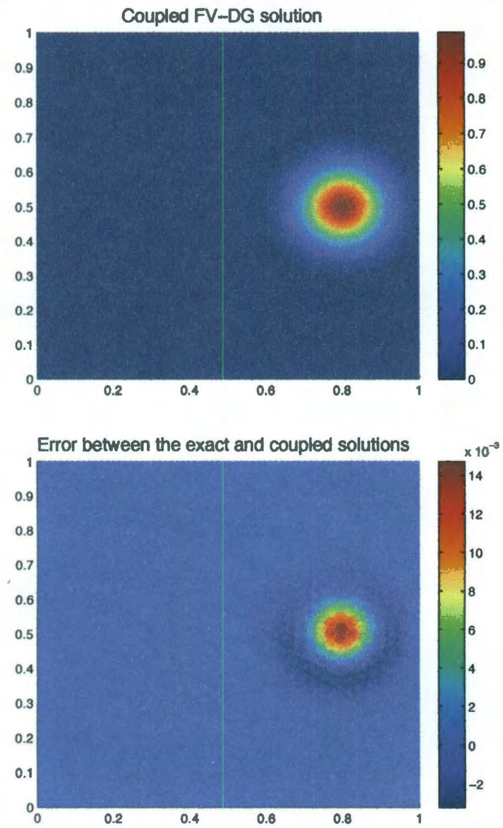

Figure 3.6 : Voronoi mesh with h=0.0375; time used=13 seconds.

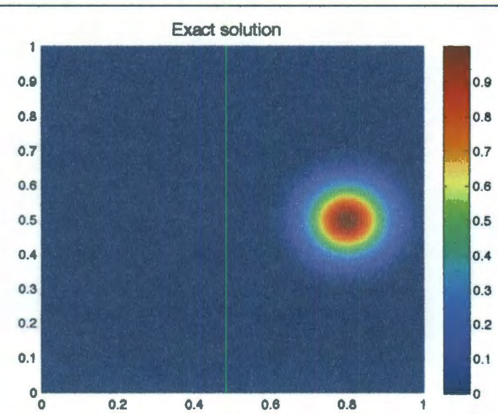

Figure 3.7 : Exact solution on the voronoi mesh with h=0.01875.

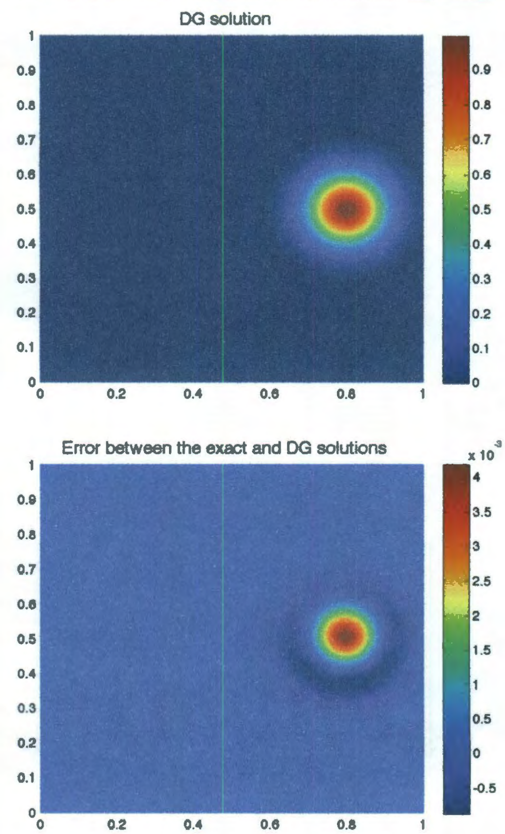

Figure 3.8 : Voronoi mesh with h=0.01875; time used=394 seconds.

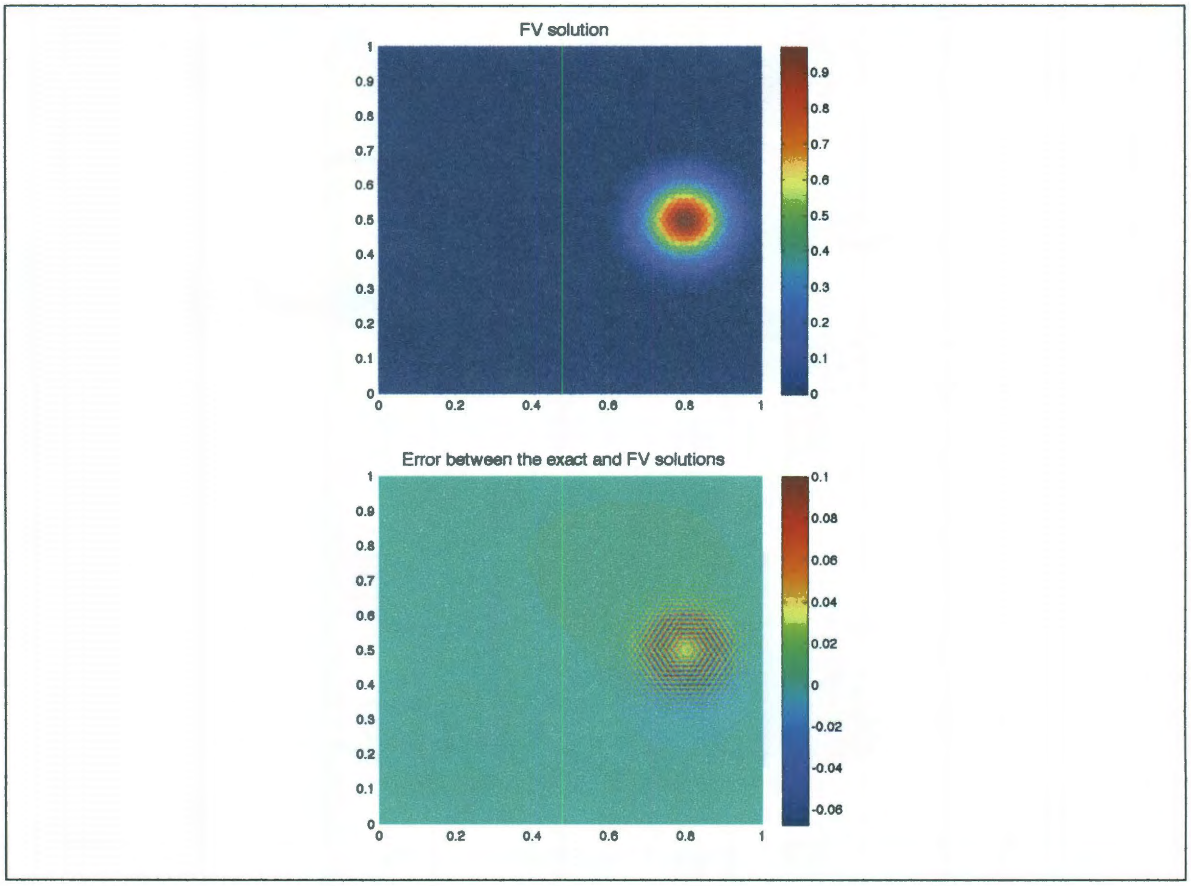

Figure 3.9 : Voronoi mesh with h=0.01875; time used=7 seconds.

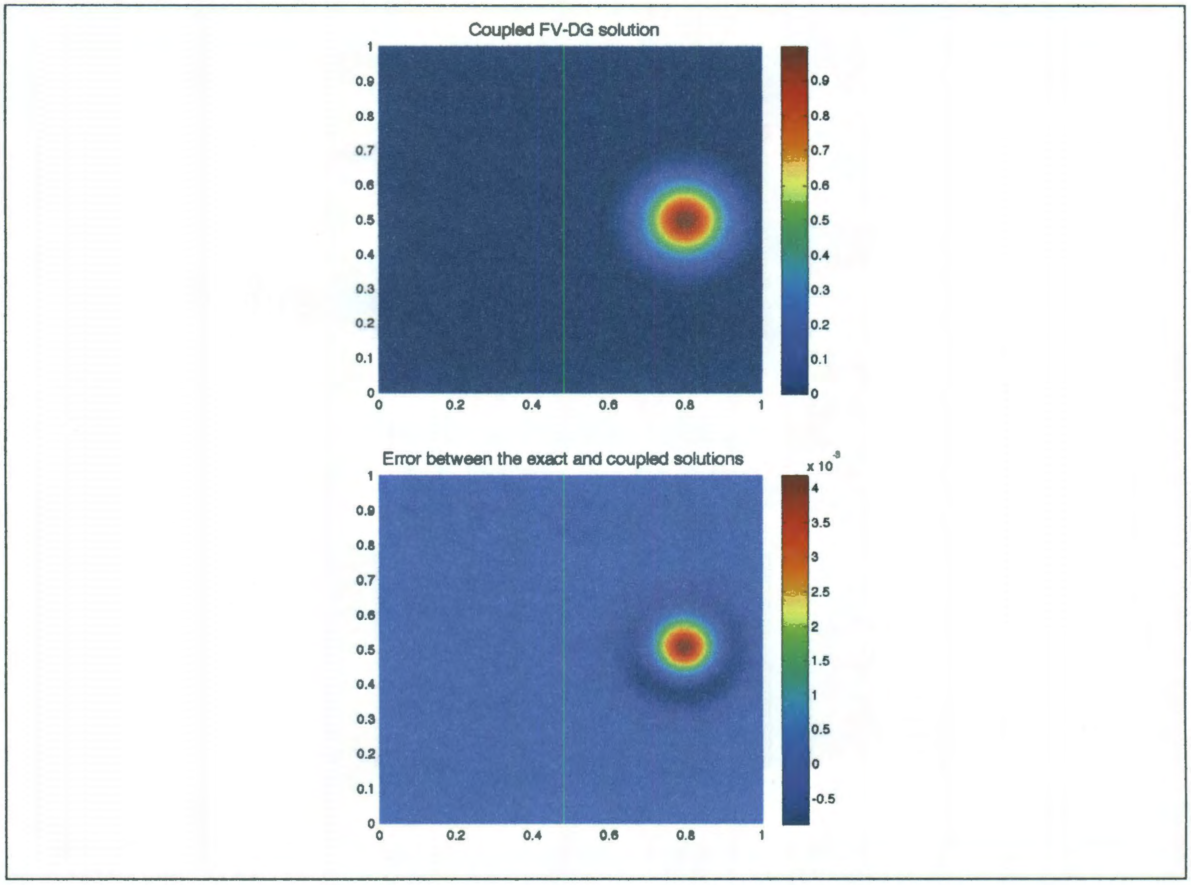

Figure 3.10 : Voronoi mesh with h=0.01875; time used=99 seconds.

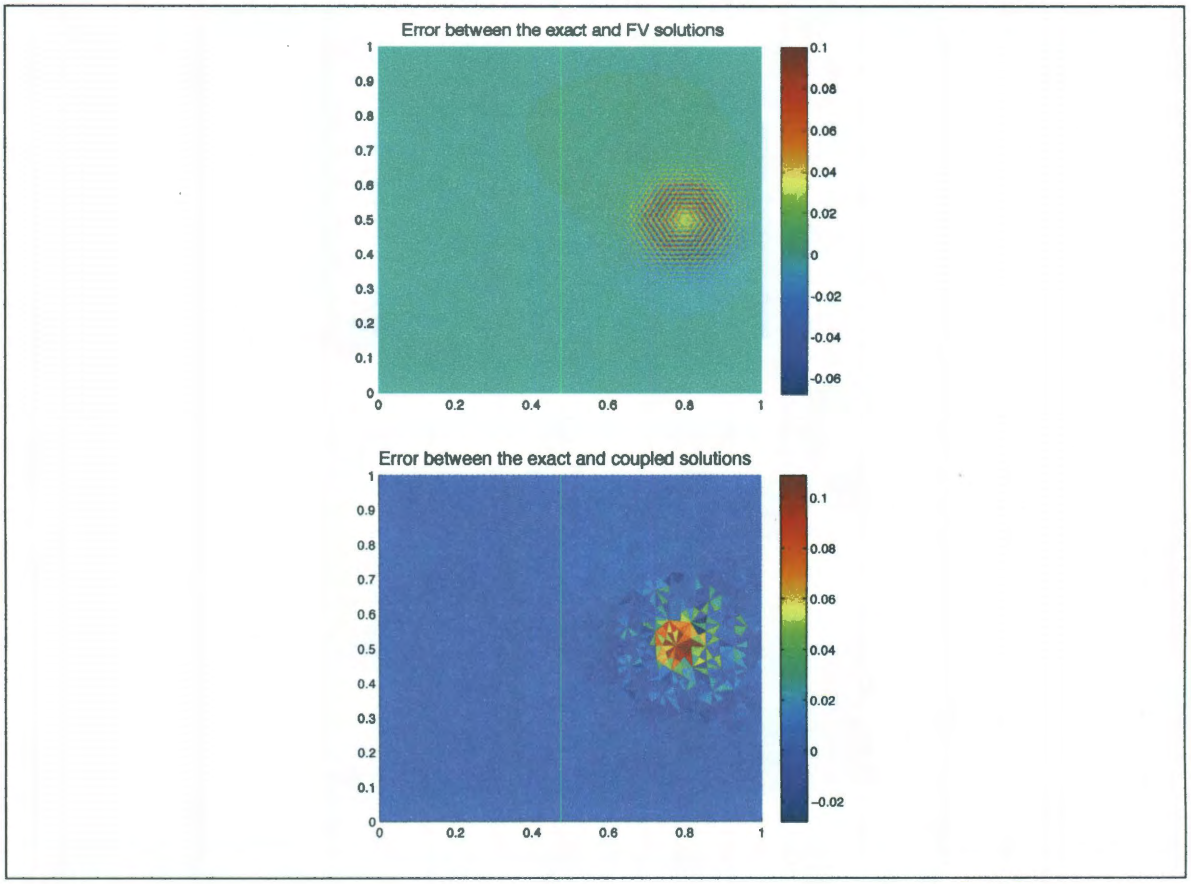

Figure 3.11 : Voronoi mesh with h=0.0185; time used=7 seconds (top). Voronoi mesh with h=0.1500; time used=6 seconds (bellow).

Chapter 4

The coupled FV and DG method for higher dimensional time dependent convection-diffusion problem

In the previous chapter, we presented and analyzed the scheme for steady state convection-diffusion problem. In this chapter, we extend the method to the time dependent problem. We present the scheme and give the error estimate.

4.1 Model problem and scheme

Consider this problem

$$\partial_t u - \nabla \cdot (K \nabla u - \boldsymbol{\beta} u) = f, \quad \text{in} \quad \Omega \times (0, T), \tag{4.1}$$

supplemented with initial and boundary conditions

$$u(x,t) = u_0(x), \quad x \in \Omega, t = 0, \tag{4.2}$$

$$u(x,t) = g(x,t), \quad x \in \partial\Omega, t \geq 0. \tag{4.3}$$

The domain Ω is a bounded polygonal domain. The function f belongs to $L^2\big(0,T;L^2(\Omega)\big)$ and u_0 belongs to $L^2(\Omega)$. The spatially dependent coefficient K is bounded above and below by positive constants k_1 and k_0 respectively. The vector $\boldsymbol{\beta}$ is divergence-free: $\nabla \cdot \boldsymbol{\beta} = 0$. We use the same meshes and notations as in Chapter 3.

4.2 Analysis tools

We use the following inequalities in the error analysis. The bound (4.4) and (4.5) can be found in [42]. There exist a constant C independent of h_V ($V \in \mathcal{E}_D^h$) and a function $u^*(t) \in \mathbb{X}^h$ satisfying

$$\forall t \in (0,T),\ \forall V \in \mathcal{E}_D^h, \quad \|u(t) - u^*(t)\|_{H^q(V)} \leq C h_D^{r+1-q} |u(t)|_{H^{r+1}(V)}, \tag{4.4}$$

and

$$\forall t \in (0,T),\ \forall V \in \mathcal{E}_D^h, \quad \|\partial_t u(t) - \partial_t u^*(t)\|_{L^2(V)} \leq C h_D^r |\partial_t u(t)|_{H^r(V)}, \tag{4.5}$$

where $q = 0, 1, 2$.

Lemma 4.1

Assume $u \in C^2([0,T] \times \overline{\Omega_F})$. Then we have the following inequalities:

$$\|u(t,x) - u(t,x_V)\|_{L^2(V)} \leq h_F |V|^{\frac{1}{2}} \sup_{0 \leq t \leq T, x \in \Omega_F} \|\nabla u(t,x)\|, \tag{4.6}$$

$$\|\partial_t u(t,x) - \partial_t u(t,x_V)\|_{L^2(V)} \leq h_F |V|^{\frac{1}{2}} \sup_{0 \leq t \leq T, x \in \Omega_F} \|\nabla u_t(t,x)\|. \tag{4.7}$$

Proof 4.1

Thanks to the Taylor's expansion with integral remainder, we can easily obtain (4.6) and (4.7). □

4.3 Semi-discrete scheme

The space discretization is the same as in Chapter 3. The semi-discrete scheme for problem (4.1) is: find, for any $t \geq 0$, the continuous in time function $u_h(t) \in \mathbb{X}^h$ satisfying

$$\forall v \in \mathbb{X}^h, \quad (\partial_t u_h, v)_\Omega + a(u_h, v) = l(v), \tag{4.8}$$

$$\forall v \in \mathbb{X}^h, \quad (u_h(0), v)_\Omega = (u_0, v)_\Omega, \tag{4.9}$$

where $a(u_h, v)$ and $l(v)$ are defined in (3.8) and (3.10) respectively.

4.4 Fully discrete scheme and analysis

Let Δt denote a positive time step and let t^j denote the time at the j^{th} step. We denote by v^j the function v evaluated at time t^j. We define $\ell^j(v) : \mathbb{X}^h \to \mathbb{R}$:

$$\forall v \in \mathbb{X}^h, \quad \ell^j(v) = \int_\Omega f^j v - \sum_{\gamma \in \Gamma_F^{h,\partial -} \cup \Gamma_D^{h,\partial -}} \int_\gamma \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma g^j v + \epsilon \sum_{\gamma \in \Gamma_D^{h,\partial}} \int_\gamma K \nabla v \cdot \boldsymbol{n}_\gamma g^j$$
$$+ \sum_{\gamma \in \Gamma_D^{h,\partial}} \frac{\sigma}{h_\gamma} \int_\gamma g^j v + \sum_{\gamma \in \Gamma_F^{h,\partial}} \frac{|\gamma|}{d_\gamma} K_\gamma g^j(y_\gamma) v. \tag{4.10}$$

We choose the backward Euler discretization for time. Our scheme is: find $\{u_h^j\}$, $j = 0, 1, \cdots, \frac{T}{\Delta t} = P$, satisfying

$$\forall v \in \mathbb{X}^h, \quad (\frac{u_h^{j+1} - u_h^j}{\Delta t}, v)_\Omega + a(u_h^{j+1}, v) = \ell^{j+1}(v), \; j = 0, 1, \cdots, P-1, \tag{4.11a}$$

$$\forall v \in \mathbb{X}^h, \quad (u_h^0, v)_\Omega = (u_0, v)_\Omega. \tag{4.11b}$$

We now derive a stability bound.

Theorem 4.1

Let $(u_h^j)_j$ be the discrete solution in $\mathbb{X}^h$ to (4.11a) and (4.11b). Assume that the penalty σ is large enough and is equal to a constant number and assume that the boundary datum g is 0. Then there exists $\Delta t_0 > 0$ independent of h_F, h_D and Δt, such that for all $\Delta t \le \Delta t_0$, $(u_h^j)_j$ satisfies the bound:

$$\|u_h^n\|^2_{L^2(\Omega)} + \Delta t \sum_{j=1}^n \|u_h^j\|^2_{\mathcal{E}} \le C \|u_h^0\|^2_{L^2(\Omega)} + C \Delta t \sum_{j=1}^n \|f^j\|^2_{L^2(\Omega)}. \tag{4.12}$$

for all $n > 0$. Here C is a constant independent of h_F, h_D and Δt.

Proof 4.2

Recall that $(u_h^j)_j$ satisfies (4.11a). Choose v to be u_h^{j+1} and we obtain

$$\frac{1}{\Delta t}\left((u_h^{j+1}, u_h^{j+1})_\Omega - (u_h^{j+1}, u_h^j)_\Omega\right) + a(u_h^{j+1}, u_h^{j+1}) = \ell^{j+1}(u_h^{j+1}).$$

Since

$$(u_h^{j+1}, u_h^j)_\Omega \le \|u_h^{j+1}\|_{L^2(\Omega)} \|u_h^j\|_{L^2(\Omega)} \le \frac{1}{2}\left(\|u_h^{j+1}\|^2_{L^2(\Omega)} + \|u_h^j\|^2_{L^2(\Omega)} \right), \tag{4.13}$$

We have

$$\frac{1}{2\Delta t}\left(\|u_h^{j+1}\|^2_{L^2(\Omega)} - \|u_h^j\|^2_{L^2(\Omega)} \right) \le \frac{1}{\Delta t}\left((u_h^{j+1}, u_h^{j+1})_\Omega - (u_h^{j+1}, u_h^j)_\Omega \right). \tag{4.14}$$

Now let us bound the right-hand side. By assumption $g = 0$, we obtain

$$|\ell^{j+1}(u_h^{j+1})| = \left| \int_\Omega f^{j+1} u_h^{j+1} \right| \le \frac{1}{4}\|f^{j+1}\|^2_\Omega + \|u_h^{j+1}\|^2_\Omega. \tag{4.15}$$

Recall that $a(v, v)$ is coercive satisfying (3.15). Therefore, we obtain by (4.14) and (4.15)

$$\frac{1}{2\Delta t}\left(\|u_h^{j+1}\|^2_{L^2(\Omega)} - \|u_h^j\|^2_{L^2(\Omega)} \right) + \alpha \|u_h^{j+1}\|^2_{\mathcal{E}} \le \frac{1}{4}\|f^{j+1}\|^2_{L^2(\Omega)} + \|u_h^{j+1}\|^2_{L^2(\Omega)}. \tag{4.16}$$

Summing up over $j = 0, 1, \cdots, n-1$ and multiplying by $2\Delta t$, we obtain

$$\begin{aligned} &\|u_h^n\|^2_{L^2(\Omega)} - \|u_h^0\|^2_{L^2(\Omega)} + 2\alpha \Delta t \sum_{j=1}^n \|u_h^j\|^2_{\mathcal{E}} \\ &\le \frac{1}{2}\Delta t \sum_{j=1}^n \|f^j\|^2_{L^2(\Omega)} + 2\Delta t \sum_{j=1}^n \|u_h^j\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.17}$$

By the discrete Gronwall's inequality, we obtain

$$\begin{aligned} &\forall n \ge 1, \quad \|u_h^n\|^2_{L^2(\Omega)} + 2\alpha \Delta t \sum_{j=1}^n \|u_h^j\|^2_{\mathcal{E}} \\ &\le e^{2(n+1)\Delta t} \left(\|u_h^0\|^2_{L^2(\Omega)} + \frac{1}{2}\Delta t \sum_{j=1}^n \|f^j\|^2_{L^2(\Omega)} \right), \end{aligned} \tag{4.18}$$

where $2\Delta t \le 1$. Since $(n+1)\Delta t \le (P+1)\Delta t \le T + \frac{1}{2}$, we have

$$\begin{aligned} &\forall n \ge 1, \quad \|u_h^n\|^2_{L^2(\Omega)} + 2\alpha \Delta t \sum_{j=1}^n \|u_h^j\|^2_{\mathcal{E}} \\ &\le e^{2T+1} \left(\|u_h^0\|^2_{L^2(\Omega)} + \frac{1}{2}\Delta t \sum_{j=1}^n \|f^j\|^2_{L^2(\Omega)} \right). \end{aligned} \tag{4.19}$$

Thus we can conclude. □

Now let us present the error estimate.

Lemma 4.2

Let $u(t) \in H^1(\Omega) \cap H^2(\mathcal{E}^h)$ for all $t \geq 0$ be the solution to problem (4.1)-(4.3). Then u satisfies

$$\forall v \in \mathbb{X}^h, \quad (\partial_t u, v)_\Omega + a(u,v) = \ell(v) - \sum_{\gamma \in \Gamma_F^{h,\mathcal{I}}} R_\gamma(u)[v] - \sum_{\gamma \in \Gamma_F^{h,\mathcal{I}} \cup \Gamma_F^{h,\partial+}} Q_\gamma(u)[v]$$
$$- \sum_{\gamma \in \Gamma_{DF}^h} [v] \int_\gamma R_\gamma(u) - \sum_{\gamma \in \Gamma_{DF}^h} \int_\gamma K \nabla u \cdot \boldsymbol{n}_\gamma (v|_{\Omega_D} - v|_{\Omega_D}(y_\gamma))$$
$$- \sum_{\gamma \in \Gamma_{DF}^h} \frac{1}{2} \int_\gamma Q_\gamma(u)(v|_{\Omega_D} - v_{\Omega_D}(y_\gamma)) - \sum_{\gamma \in \Gamma_{DF}^h} \frac{1}{2}[v] \int_\gamma Q_\gamma(u). \quad (4.20)$$

Proof 4.3

The result (4.20) can be easily obtained by Lemma 3.3. □

Theorem 4.2

Let u be the solution to problem (4.1)-(4.3) and $(u_h^j)_j$ be the discrete solution in $\mathbb{X}^h$ to (4.11a)-(4.11b). Assume that $u \in L^2\big(0,T; H^{r+1}(\mathcal{E}_D^h)\big) \cap C^2\big([0,T] \times \overline{\Omega_F}\big)$ and $\partial_t u, \partial_{tt} u \in L^2\big(0,T; H^r(\mathcal{E}_D^h)\big)$. Then there exists $\Delta t_0 > 0$ such that for all $\Delta t \leq \Delta t_0$, we have

$$\|u^n - u_h^n\|_{L^2(\Omega)}^2 + \Delta t \sum_{j=1}^n \|u^j - u_h^j\|_{\mathcal{E}}^2 \leq C \Delta t^2 \int_0^T \|\partial_{tt} \tilde{u}\|_{L^2(\Omega)}^2$$
$$+ C h_D^{2r+2} \big|u^0\big|_{H^{r+1}(\Omega_D)}^2 + C h_F^2 \big|u^0\big|_{H^1(\Omega_F)}^2 + C h_F^2 + C h_D^{2r+2} \big|u^n\big|_{H^{r+1}(\Omega_D)}^2$$
$$+ C h_D^{2r} \left(\Delta t \sum_{j=1}^n \big|\partial_t u^j\big|_{H^r(\mathcal{E}_D^h)}^2 + \Delta t \sum_{j=1}^n \|u^j\|_{H^{r+1}(\mathcal{E}_D^h)}^2 \right) + C T h_F^2$$
$$+ C h_F^2 \left(\Delta t \sum_{j=1}^n \int_{\Omega_F} \big|\mathbf{H}(u^j)\big|^2 + \Delta t \sum_{j=1}^n \|u^j\|_{W^{1,p}(\mathcal{E}_F^h)}^2 \right) + C h_D \Delta t \sum_{j=1}^n \|\nabla u^j\|_{L^2(\Gamma_{DF}^h)}^2, \quad (4.21)$$

where C is a constant independent of h_D, h_F and Δt.

Proof 4.4

Let

$$\tilde{u}(t) = \begin{cases} u^*(t,x) & x \in \Omega_D, \\ u(t,x_V) & x \in V \in \mathcal{E}_F^h. \end{cases} \tag{4.22}$$

Let $u_h^j - u^j = \chi^j - \xi^j$ for $j = 0, 1, \cdots, P$, where $\chi^j = u_h^j - \tilde{u}^j$ and $\xi^j = u^j - \tilde{u}^j$. Subtracting (4.20) from (4.11a), we obtain

$$\begin{aligned}
\forall v \in \mathbb{X}^h, \quad & (\frac{u_h^{j+1} - u_h^j}{\Delta t} - \partial_t u^{j+1}, v)_\Omega + a(u_h^{j+1} - u^{j+1}, v) \\
&= \sum_{\gamma \in \Gamma_F^{h,\mathcal{I}}} R_\gamma(u^{j+1})[v] + \sum_{\gamma \in \Gamma_F^{h,\mathcal{I}} \cup \Gamma_F^{h,\partial+}} Q_\gamma(u^{j+1})[v] \\
&+ \sum_{\gamma \in \Gamma_{DF}^h} [v] \int_\gamma R_\gamma(u^{j+1}) + \sum_{\gamma \in \Gamma_{DF}^h} \int_\gamma K \nabla u^{j+1} \cdot \boldsymbol{n}_\gamma (v|_{\Omega_D} - v|_{\Omega_D}(y_\gamma)) \\
&+ \sum_{\gamma \in \Gamma_{DF}^h} \frac{1}{2} \int_\gamma Q_\gamma(u^{j+1})(v|_{\Omega_D} - v_{\Omega_D}(y_\gamma)) + \sum_{\gamma \in \Gamma_{DF}^h} \frac{1}{2}[v] \int_\gamma Q_\gamma(u^{j+1}).
\end{aligned} \tag{4.23}$$

We transform the first term on the left-hand side of (4.23):

$$\begin{aligned}
(\frac{u_h^{j+1} - u_h^j}{\Delta t} - \partial_t u^{j+1}, v)_\Omega &= \left(\frac{u_h^{j+1} - u_h^j}{\Delta t} - \partial_t \tilde{u}^{j+1} - (-\partial_t \tilde{u}^{j+1} + \partial_t u^{j+1}), v \right)_\Omega \\
= \left(\frac{u_h^{j+1} - u_h^j}{\Delta t} - \frac{\tilde{u}^{j+1} - \tilde{u}^j}{\Delta t} \right. &+ \left. (\frac{\tilde{u}^{j+1} - \tilde{u}^j}{\Delta t} - \partial_t \tilde{u}^{j+1}) - (\partial_t u^{j+1} - \partial_t \tilde{u}^{j+1}), v \right)_\Omega \\
&= \left(\frac{\chi^{j+1} - \chi^j}{\Delta t} + (\frac{\tilde{u}^{j+1} - \tilde{u}^j}{\Delta t} - \partial_t \tilde{u}^{j+1}) - (\partial_t u^{j+1} - \partial_t \tilde{u}^{j+1}), v \right)_\Omega .
\end{aligned}$$

By choosing $v = \chi^{j+1}$,we obtain

$$
\begin{aligned}
&(\frac{\chi^{j+1}-\chi^j}{\Delta t},\chi^{j+1})_\Omega + a(\chi^{j+1}-\xi^{j+1},\chi^{j+1}) + \left((\frac{\tilde{u}^{j+1}-\tilde{u}^j}{\Delta t}-\partial_t\tilde{u}^{j+1}),\chi^{j+1}\right)_\Omega \\
&= \left(\partial_t\xi^{j+1},\chi^{j+1}\right)_\Omega + \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}} R_\gamma(u^{j+1})[\chi^{j+1}] \\
&+ \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}} Q_\gamma(u^{j+1})[\chi^{j+1}] + \sum_{\gamma\in\Gamma_{DF}^h}[\chi^{j+1}]\int_\gamma R_\gamma(u^{j+1}) \\
&+ \sum_{\gamma\in\Gamma_{DF}^h}\int_\gamma K\nabla u^{j+1}\cdot\boldsymbol{n}_\gamma(\chi^{j+1}|_{\Omega_D}-\chi^{j+1}|_{\Omega_D}(y_\gamma)) \\
&+ \sum_{\gamma\in\Gamma_{DF}^h}\frac{1}{2}\int_\gamma Q_\gamma(u^{j+1})(\chi^{j+1}|_{\Omega_D}-\chi^{j+1}|_{\Omega_D}(y_\gamma)) + \sum_{\gamma\in\Gamma_{DF}^h}\frac{1}{2}[\chi^{j+1}]\int_\gamma Q_\gamma(u^{j+1}).
\end{aligned}
\tag{4.24}
$$

By (4.13), we obtain

$$
-(\chi^j,\chi^{j+1})_\Omega \geq -\frac{1}{2}\left(\|\chi^{j+1}\|^2_{L^2(\Omega)} + \|\chi^j\|^2_{L^2(\Omega)}\right). \tag{4.25}
$$

Applying (4.25) to the first term in (4.24) and moving the term $a(\xi^{j+1},\chi^{j+1})$ on the left-hand side of (4.24) to the right-hand side, we obtain

$$
\begin{aligned}
&\frac{1}{2\Delta t}\left(\|\chi^{j+1}\|^2_{L^2(\Omega)} - \|\chi^j\|^2_{L^2(\Omega)}\right) + a(\chi^{j+1},\chi^{j+1}) \\
&\leq -\left((\frac{\tilde{u}^{j+1}-\tilde{u}^j}{\Delta t}-\partial_t\tilde{u}^{j+1}),\chi^{j+1}\right)_\Omega + \left(\partial_t\xi^{j+1},\chi^{j+1}\right)_\Omega \\
&+ a(\xi^{j+1},\chi^{j+1}) + \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}} R_\gamma(u^{j+1})[\chi^{j+1}] + \sum_{\gamma\in\Gamma_F^{h,\mathcal{I}}\cup\Gamma_F^{h,\partial+}} Q_\gamma(u^{j+1})[\chi^{j+1}] \\
&+ \sum_{\gamma\in\Gamma_{DF}^h}[\chi^{j+1}]\int_\gamma R_\gamma(u^{j+1}) + \sum_{\gamma\in\Gamma_{DF}^h}\int_\gamma K\nabla u^{j+1}\cdot\boldsymbol{n}_\gamma(\chi^{j+1}|_{\Omega_D}-\chi^{j+1}|_{\Omega_D}(y_\gamma)) \\
&+ \sum_{\gamma\in\Gamma_{DF}^h}\frac{1}{2}\int_\gamma Q_\gamma(u^{j+1})(\chi^{j+1}|_{\Omega_D}-\chi^{j+1}|_{\Omega_D}(y_\gamma)) + \sum_{\gamma\in\Gamma_{DF}^h}\frac{1}{2}[\chi^{j+1}]\int_\gamma Q_\gamma(u^{j+1}).
\end{aligned}
\tag{4.26}
$$

Let us now bound the first term on the right-hand side. Using Cauchy-Schwarz's inequality and Young's inequality, we obtain

$$
\left|\left((\frac{\tilde{u}^{j+1}-\tilde{u}^j}{\Delta t}-\partial_t\tilde{u}^{j+1}),\chi^{j+1}\right)_\Omega\right| \leq \|\chi^{j+1}\|^2_{L^2(\Omega)} + \frac{1}{4}\|\frac{\tilde{u}^{j+1}-\tilde{u}^j}{\Delta t}-\partial_t\tilde{u}^{j+1}\|^2_{L^2(\Omega)}.
$$

Using Taylor's expansion with integral remainder, we obtain ([42])

$$\|\frac{\tilde{u}^{j+1}-\tilde{u}^j}{\Delta t} - \partial_t \tilde{u}^{j+1}\|^2_{L^2(\Omega)}$$
$$\leq \frac{1}{2\Delta t^2}\int_{t^j}^{t^{j+1}} (s-t^j)^2 ds \int_{t^j}^{t^{j+1}} \|\partial_{tt}\tilde{u}\|^2_{L^2(\Omega)} ds \leq \frac{\Delta t}{6}\int_{t^j}^{t^{j+1}} \|\partial_{tt}\tilde{u}\|^2_{L^2(\Omega)}.$$

Therefore,

$$\left| \left((\frac{\tilde{u}^{j+1}-\tilde{u}^j}{\Delta t} - \partial_t \tilde{u}^{j+1}), \chi^{j+1}\right)_\Omega \right| \leq \|\chi^{j+1}\|^2_{L^2(\Omega)} + \frac{\Delta t}{24}\int_{t^j}^{t^{j+1}} \|\partial_{tt}\tilde{u}\|^2_{L^2(\Omega)}. \quad (4.27)$$

We use (4.5) and (4.7) and obtain the bound for the second term:

$$\left(\partial_t \xi(t^{j+1}), \chi^{j+1}\right)_\Omega \leq \|\chi^{j+1}\|^2_{L^2(\Omega)} + \frac{1}{4}\|\partial_t \xi^{j+1}\|^2_{L^2(\Omega)}$$
$$\leq \|\chi^{j+1}\|^2_{L^2(\Omega)} + Ch_D^{2r}\left|\partial_t u^{j+1}\right|^2_{H^r(\mathcal{E}_D^h)} + \sup_{0\leq t\leq T, x\in\Omega_F} \|\nabla u_t(t,x)\|^2 |\Omega_F| h_F^2. \quad (4.28)$$

We have already bounded the rest of the terms in the proof of Theorem 3.2. We now use directly the result (3.59), and we obtain

$$\frac{1}{2\Delta t}\left(\|\chi^{j+1}\|^2_{L^2(\Omega)} - \|\chi^j\|^2_{L^2(\Omega)}\right) + \frac{1}{16}\alpha\|\chi^{j+1}\|^2_{\mathcal{E}}$$
$$\leq 2\|\chi^{j+1}\|^2_{L^2(\Omega)} + \frac{\Delta t}{24}\int_{t^j}^{t^{j+1}} \|\partial_{tt}\tilde{u}\|^2_{L^2(\Omega)} + Ch_D^{2r}\left|\partial_t u^{j+1}\right|^2_{H^r(\mathcal{E}_D^h)} + Ch_F^2$$
$$+Ch_D^{2r}\left|u^{j+1}\right|^2_{H^{r+1}(\mathcal{E}_D^h)} + Ch_F^2\int_{\Omega_F}\left|\mathbf{H}(u^{j+1})\right|^2 + Ch_F^2\|u^{j+1}\|^2_{W^{1,p}(\mathcal{E}_F^h)} + Ch_D\|\nabla u^{j+1}\|^2_{L^2(\Gamma_{DF}^h)}. \quad (4.29)$$

Summing up over $j = 0, 1, \cdots, n-1$ and multiplying by $2\Delta t$, we obtain

$$\|\chi^n\|^2_{L^2(\Omega)} - \|\chi^0\|^2_{L^2(\Omega)} + \frac{1}{8}\alpha\Delta t\sum_{j=1}^n \|\chi^j\|^2_{\mathcal{E}}$$
$$\leq 4\Delta t\sum_{j=1}^n \|\chi^j\|^2_{L^2(\Omega)} + \frac{\Delta t^2}{12}\sum_{j=0}^{n-1}\int_{t^j}^{t^{j+1}} \|\partial_{tt}\tilde{u}\|^2_{L^2(\Omega)}$$
$$+ Ch_D^{2r}\Delta t\sum_{j=1}^n \left|\partial_t u^j\right|^2_{H^r(\mathcal{E}_D^h)} + C\Delta t\sum_{j=1}^n h_F^2 + C\Delta t\sum_{j=1}^n h_D^{2r}\left|u^j\right|^2_{H^{r+1}(\mathcal{E}_D^h)}$$
$$+ C\Delta t\sum_{j=1}^n h_F^2\int_{\Omega_F}\left|\mathbf{H}(u^j)\right|^2 + C\Delta t\sum_{j=1}^n h_F^2\|u^j\|^2_{W^{1,p}(\mathcal{E}_F^h)} + C\Delta t\sum_{j=1}^n h_D\|\nabla u^j\|^2_{L^2(\Gamma_{DF}^h)}. \quad (4.30)$$

Thanks to the discrete Gronwall's inequality, we obtain

$$\begin{aligned}
&\|\chi^n\|^2_{L^2(\Omega)} + \frac{1}{8}\alpha\Delta t \sum_{j=1}^{n} \|\chi^j\|^2_{\mathcal{E}} \le C\Delta t^2 \int_0^T \|\partial_{tt}\tilde{u}\|^2_{L^2(\Omega)} \\
&+ C\|\chi^0\|^2_{L^2(\Omega)} + Ch_D^{2r}\left(\Delta t \sum_{j=1}^{n} \left|\partial_t u^j\right|^2_{H^r(\mathcal{E}_D^h)} + \Delta t \sum_{j=1}^{n} \left|u^j\right|^2_{H^{r+1}(\mathcal{E}_D^h)}\right) + CTh_F^2 \\
&+ Ch_F^2\left(\Delta t \sum_{j=1}^{n} \int_{\Omega_F} \left|\mathbf{H}(u^j)\right|^2 + \Delta t \sum_{j=1}^{n} \|u^j\|^2_{W^{1,p}(\mathcal{E}_F^h)}\right) + Ch_D\Delta t \sum_{j=1}^{n} \|\nabla u^j\|^2_{L^2(\Gamma_{DF}^h)},
\end{aligned} \tag{4.31}$$

where $4\Delta t < 1$.

Now we bound $\|\chi^0\|^2_{L^2(\Omega)}$. Since u_h^0 satisfies (4.11b), we have

$$\|u_h^0 - u^0\|^2_{L^2(\Omega)} \le Ch_D^{2r+2}\|u^0\|^2_{H^{r+1}(\Omega_D)} + Ch_F^2\|u^0\|^2_{H^1(\Omega_F)}.$$

Together with (4.4) and (4.6), we obtain

$$\begin{aligned}
\|\chi^0\|^2_{L^2(\Omega)} &\le 2\left(\|u_h^0 - u^0\|^2_{L^2(\Omega)} + \|u^0 - \tilde{u}^0\|^2_{L^2(\Omega)}\right) \\
&\le Ch_D^{2r+2}\left|u^0\right|^2_{H^{r+1}(\Omega_D)} + Ch_F^2\left|u^0\right|^2_{H^1(\Omega_F)} + Ch_F^2.
\end{aligned} \tag{4.32}$$

Taking (4.32) into (4.31), we obtain

$$\begin{aligned}
&\|\chi^n\|^2_{L^2(\Omega)} + \frac{1}{8}\alpha\Delta t \sum_{j=1}^{n} \|\chi^j\|^2_{\mathcal{E}} \le C\Delta t^2 \int_0^T \|\partial_{tt}\tilde{u}\|^2_{L^2(\Omega)} + Ch_D^{2r+2}\left|u^0\right|^2_{H^{r+1}(\Omega_D)} \\
&+ Ch_F^2\left|u^0\right|^2_{H^1(\Omega_F)} + Ch_F^2 + Ch_D^{2r}\left(\Delta t \sum_{j=1}^{n} \left|\partial_t u^j\right|^2_{H^r(\mathcal{E}_D^h)} + \Delta t \sum_{j=1}^{n} \left|u^j\right|^2_{H^{r+1}(\mathcal{E}_D^h)}\right) + CTh_F^2 \\
&+ Ch_F^2\left(\Delta t \sum_{j=1}^{n} \int_{\Omega_F} \left|\mathbf{H}(u^j)\right|^2 + \Delta t \sum_{j=1}^{n} \|u^j\|^2_{W^{1,p}(\mathcal{E}_F^h)}\right) + Ch_D\Delta t \sum_{j=1}^{n} \|\nabla u^j\|^2_{L^2(\Gamma_{DF}^h)}.
\end{aligned} \tag{4.33}$$

Next, we bound $u - u_h$. By triangle inequality, we obtain

$$\begin{aligned}
&\|u^n - u_h^n\|^2_{L^2(\Omega)} + \Delta t \sum_{j=1}^{n} \|u^j - u_h^j\|^2_{\mathcal{E}} \\
&\le \|\chi^n\|^2_{L^2(\Omega)} + \Delta t \sum_{j=1}^{n} \|\chi^j\|^2_{\mathcal{E}} + \|\xi^n\|^2_{L^2(\Omega)} + \Delta t \sum_{j=1}^{n} \|\xi^j\|^2_{\mathcal{E}}.
\end{aligned} \tag{4.34}$$

The first two terms on the right-hand side of (4.34) can be bounded by (4.31) and the third term can be bounded by (4.4) and (4.6). We only need to bound the fourth term. By the definition of ξ, we have $\xi(x_V) = 0$ for any $V \in \mathcal{E}_F^h$. We also can choose $u^*(t, y_\gamma) = u(t, y_\gamma)$ .Therefore, we obtain

$$\|\xi^j\|_{\mathcal{E}}^2 = \|\xi^j\|_{DG}^2.$$

Thus by trace inequality and (4.4), we obtain

$$\|\xi^j\|_{\mathcal{E}}^2 \leq Ch_D^{2r}\|u^j\|_{H^{r+1}(\mathcal{E}_D^h)}^2. \tag{4.35}$$

Thus, we can conclude. □

Chapter 5

Conclusion and future work

In this thesis, the coupled of finite volume and discontinuous Galerkin method is proposed to solve the convection-diffusion equations.

In Chapter 2, we present the coupled finite volume and discontinuous Galerkin scheme for the one dimensional steady state convection-diffusion problem. We prove the uniqueness and existence of the solution to the scheme. We prove that the error is bounded in the rate of $O(h_D + h_F)$ and show some numerical examples which verify the error estimate.

In Chapter 3, we present the coupled scheme for the 2D and 3D steady state convection-diffusion problem. We prove the uniqueness and existence of the solution to the scheme and show that the error is bounded in the rate of $O(h_D^{\frac{1}{2}} + h_F$ theoretically and numerically. Furthermore, if the interface of the finite volume domain and the DG domain is chosen properly so that the average gradient of the true solution near the interface is bounded by the average gradient of the true solution on DG domain, then the convergence rate is of first order.

In Chapter 4, we present the 2D and 3D time dependent convection-diffusion problem and use the backward Euler method for the time discretization and coupled FV-DG method for the space discretization. We show the stability bound and error estimate. In the future, we will show some numerical examples for the time-dependent scheme to verify our error estimate. For the 2D and 3D convection-diffusion problems, the convergence rate is lost by one half as shown in this thesis. In order to get a better convergence rate, we will try to use high order finite volume method near the interface of the FV domain and DG domain. We also want to extend the coupled FV and DG method to the nonlinear time-dependent convection-diffusion equations and apply this method to the CO_2 sequestration problem. Now let me present the model for CO_2 sequestration problem.

5.1 CO_2 sequestration model

Carbon dioxide disposal into deep aquifer has been an important venue to trap excess gas emission, not only is this technology economical, it provides a promising media to trap large capacity of residual gas. In the CO_2 sequestration process, we have two components (i.e. CO_2 and H_2O) and two phases (i.e. liquid (L) and vapor (v) phase). The mathematical model of this compositional problem can be described by a set of mass conservation equations and thermodynamic equilibrium formulae. Specifically, the conservation equations used to describe the transport phenomenon of the fluid are established by Sasaki to account for the CO_2 dissolution effect into water. (Cited from [45].)

This model has two phases: liquid (L), vapor (v), and two components: gas (g), water (w).

Here are some notation.

1. The primary variable is $p := p_L$, the pressure of the liquid phase and the saturation S_L of the liquid phase. 2. Capillary pressure corresponds to the difference in pressure between the phases and it is given by:

$$p_{cv} = p_v - p.$$

3. The capillary pressure p_{cv} is a function of saturation: $p_{cv} = p_{cv}(S_L)$.

4. Relative permeabilities for water and gas are functions of saturation: $k_{rL} = k_{rL}(S_L)$, $k_{rv} = k_{rv}(S_L)$. $\mathbf{k}$ is the absolute permeability tensor of the porous medium.

5. Viscosities for liquid and gas phases are functions of pressure: $\mu_L = \mu_L(p)$, $\mu_v = \mu_v(p)$.

Define $\alpha_r = \frac{k_{r\alpha}\mathbf{k}}{\mu_r}$.

6. Porosity ϕ is the measure of the pore space of the rock. It is defined to be the ratio of the volume of pores to the total volume, i.e.

$$\phi = \frac{V_{pore}}{V_{total}},$$

where V stands for volume. p^0 is the reference pressure and ϕ^0 is the porosity at p^0.

7. The rock comprssibility c_R is given by $c_R = \frac{1}{\phi}\frac{\partial \phi}{\partial p}$.

8. Density of phase α is a function of pressure: $\rho_\alpha = \rho_\alpha(p)$.

9. $x_{m,\alpha}$ is the mole fraction of component m in phase α and q_α is the source term of phase α.

10. Fugacity is a thermodynamic property that describes the tendency of a gas to escape. $f_{m,\alpha}$ denotes the fugacity of component m in phase α. τ_m is the fugacity coefficient of component m.

11. $K_{m,t,p}$ is the thermodynamic equilibrium constant of component m at temperature t and pressure p with

$$K_{H_2O,T,P} = \frac{f_{H_2O,v}}{1 - x_{CO_2,L}} \tag{5.1}$$

$$K_{CO_2,T,P} = \frac{f_{CO_2,v}}{55.508 x_{CO_2,L}} \tag{5.2}$$

12. $\bar{V}_m$ is the average partial molar volume of component m and R is the gas constant. Denote

$$A = \frac{K_{H_2O,T,p^0}}{\tau_{H_2O}\ p} exp\left(\frac{(p - p^0)\bar{V}_{H_2O}}{RT}\right) \tag{5.3}$$

$$B = \frac{r_{CO_2} p}{55.508 K_{CO_2,T,p^0}} exp\left(-\frac{(p-p^0)\bar{V}_{CO_2}}{RT}\right) \tag{5.4}$$

The model below is a closed system for the CO_2 sequestration processes. There are six unknown variables: p, $x_{H_2O,v}$, $x_{H_2O,L}$, $x_{CO_2,v}$, $x_{CO_2,L}$ and S_L. The other variables are either functions of the unknown variables or can be obtained. The transport equations for this two phase two component flow in porous media are

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\phi^0(1+c_R(p-p^0))\left(x_{CO_2,L}\rho_L S_L + x_{CO_2.v}\rho_v(1-S_L)\right)\right) \\ &-\nabla\cdot\left(x_{CO_2,L}\rho_L\alpha_L\nabla p + x_{CO_2,v}\rho_v\alpha_v\nabla(p_{cv}+p)\right) \\ &+x_{CO_2,L}\rho_L q_L + x_{H_2O,v}\rho_v q_v = 0, \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\phi^0(1+c_R(p-p^0))\left(x_{H_2O,L}\rho_L S_L + x_{H_2O.v}\rho_v(1-S_L)\right)\right) \\ &-\nabla\cdot\left(x_{H_2O,L}\rho_L\alpha_L\nabla p + x_{H_2O,v}\rho_v\alpha_v\nabla(p_{cv}+p)\right) \\ &+x_{H_2O,L}\rho_L q_L + x_{H_2O,v}\rho_v q_v = 0. \end{aligned} \tag{5.6}$$

Phase constraints are

$$x_{CO_2,L} + x_{H_2O,L} = 1, \tag{5.7}$$

$$x_{CO_2,v} + x_{H_2O,v} = 1. \tag{5.8}$$

Fugacity equations are

$$x_{H_2O,v} = \frac{1-B}{(1/A)-B}, \tag{5.9}$$

$$x_{C_2O,L} = B(1-x_{H_2O,v}). \tag{5.10}$$

We will apply the FV-DG scheme to (5.5) and (5.6) and test the scheme with some numerical implementations.

Appendix A

Theorem A.1

Define the residuals for any $u \in H^2(\Omega)$.

$$\forall \gamma \in \Gamma_F^h, \qquad R_\gamma(u) = -\frac{|\gamma|}{d_\gamma} K_\gamma [u] - \int_\gamma K \nabla u \cdot \boldsymbol{n}_\gamma, \tag{A.1}$$

$$\forall \gamma \in \Gamma_F^{h,\partial+}, \qquad R_\gamma(u) = -\frac{|\gamma|}{d_\gamma} K_\gamma (u(x_V) - g(y_\gamma)) - \int_\gamma K \nabla u \cdot \boldsymbol{n}_\gamma, \tag{A.2}$$

$$\forall \gamma \in \Gamma_F^{h,\partial-}, \qquad R_\gamma(u) = -\frac{|\gamma|}{d_\gamma} K_\gamma (g(y_\gamma) - u(x_V)) - \int_\gamma K \nabla u \cdot \boldsymbol{n}_\gamma, \tag{A.3}$$

$$\forall \gamma \in \Gamma_F^{h,\mathcal{I}} \cup \Gamma_F^{h,\partial+}, \qquad Q_\gamma(u) = -\boldsymbol{\beta}_\gamma u^\uparrow + \int_\gamma \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma u, \tag{A.4}$$

$$\forall \gamma \in \Gamma_{DF}^h, \qquad R_\gamma(u) = -K \nabla u \cdot \boldsymbol{n}_\gamma - \frac{K_\gamma}{d_\gamma}[u], \tag{A.5}$$

$$\forall \gamma \in \Gamma_{DF}^h, \qquad Q_\gamma(u) = \boldsymbol{\beta} \cdot \boldsymbol{n}_\gamma (u|_{\Omega_D} - u|_{\Omega_F}). \tag{A.6}$$

Let $\mathbf{H}(u)$ denote the Hessian matrix of u. Assume K is a positive constant and $\nabla \cdot \boldsymbol{\beta} = 0$. Then, there exist a constant C_1 only dependent on θ and a constant C_2 only dependent on θ, d, $\boldsymbol{\beta}$, and p, such that

$$\gamma \in \Gamma_F^h, \quad |R_\gamma(u)|^2 \leq C_1 \frac{h_F^2 |\gamma|}{d_\gamma} \int_{\mathcal{V}_\gamma} |\mathbf{H}(u)|^2, \tag{A.7}$$

$$\gamma \in \Gamma_{DF}^h, \quad \left(\int_\gamma |R_\gamma(u)| \right)^2 \leq C_1 \frac{h_F^2 |\gamma|}{d_\gamma} \int_{\mathcal{V}_\gamma} |\mathbf{H}(u)|^2, \tag{A.8}$$

$$\gamma \in \Gamma_F^{h,\mathcal{I}} \cup \Gamma_F^{h,\partial+}, \quad |Q_\gamma(u)| \leq C_2 h_F |\gamma|^{\frac{p-2}{p}} (\frac{|\gamma|}{d_\gamma})^{\frac{1}{p}} \|u\|_{W^{1,p}_{(\mathcal{V}_\gamma)}}, \tag{A.9}$$

$$\gamma \in \Gamma_{DF}^h, \quad \int_\gamma |Q_\gamma(u)| \leq C_2 h_F |\gamma|^{\frac{p-2}{p}} (\frac{|\gamma|}{d_\gamma})^{\frac{1}{p}} \|u\|_{W^{1,p}_{(\mathcal{V}_\gamma)}}, \tag{A.10}$$

for all $p > d$ and such that $p < \infty$ if $d = 2$ and $p \leq 6$ if $d = 3$.

Proof A.1

The theorem and the proof is as same as in [44] except a little modification.

First note that thanks to Sobolev's imbeddings, if $u \in H^2(\Omega)$, then $u \in W^{1,p}(\Omega)$ for all p such that $1 \le p < \infty$ if $d = 2$ and such that $1 \le p \le 6$ if $d = 3$. Then (3.30) and (3.31) are well defined.

Let $\gamma \in \Gamma_F^h$. Since $u \in H^2(\Omega)$, the restriction of u to $\mathcal{V}_\gamma$ belongs to $H^2(\mathcal{V}_\gamma)$. The space $C^2(\bar{\mathcal{V}_\gamma})$ is dense in $H^2(\mathcal{V}_\gamma)$. Then, using a density argument, one needs only to prove (3.28), (3.29), (3.30) and (3.31) for $u \in C^2(\bar{\mathcal{V}_\gamma})$. Therefore let us first assume that $u \in C^2(\bar{\mathcal{V}_\gamma})$.

First, we prove (3.28) if $\gamma \in \Gamma_F^{h,I}$. Let W and L be the two control volumes such that $\gamma = W \cap L$. Define $d_{W,\gamma} = d(x_W, \gamma)$. It is possible to assume, for simplicity of notation and without loss of generality, that $\gamma = 0 \times \tilde{\gamma}$, with some $\tilde{\gamma} \subset \mathbb{R}^{d-1}$, and $x_W = (-d_{W,\gamma}, 0)^t, x_L = (d_{L,\gamma}, 0)^t$.

A Taylor expansion using $u \in C^2(\bar{\mathcal{V}_\gamma})$ gives, for a.e. (for the (d-1)-Lebesgue measure) $x = (0, \tilde{x}) \in \gamma$,

$$u(x_L) - u(x) = \nabla u(x) \cdot (x_L - x) + \int_0^1 H(u)(tx + (1-t)x_L)(x_L - x) \cdot (x_L - x)tdt,$$

where $H(u)(z)$ denotes the Hessian matrix of u at point z, and

$$u(x_W) - u(x) = \nabla u(x) \cdot (x_W - x) + \int_0^1 H(u)(tx + (1-t)x_W)(x_W - x) \cdot (x_W - x)tdt.$$

Note that $x_L - x_W = \boldsymbol{n}_\gamma$; substracting one equation off the other and integrating over γ yields $|R_\gamma(u)| \le B_{W,\gamma} + B_{L,\gamma}$, with, for some C_3 depending on d and K,

$$B_{W,\gamma} = \frac{C_3}{d_\gamma} \int_\gamma \int_0^1 |H(u)(tx + (1-t)x_W)| x_W - x|^2 tdtd\gamma(x),$$

where $|H(u)(x)|^2 = \sum_{i,j=1}^d |D_i D_j u(z)|^2$.

The quantity $B_{L,\gamma}$ is obtained from $B_{W,\gamma}$ by changing W in L. One uses a change of variables in $B_{W,\gamma}$. Indeed, one sets $z = tx + (1-t)x_W$. Since $|x_W - x| \le h_F$ and $dz = t^{d-1} d_{W,\gamma} dt d\gamma(x)$, one obtains, using $z_1 = (t-1)d_{W,\gamma}$, $z = (z_1, \bar{z})^t$ with $\bar{z} \in \mathbb{R}^{d-1}$,

$$B_{W,\gamma} \le \frac{C_3 h_F^2}{d_\gamma} \int_{\mathcal{V}_{W,\gamma}} |H(u)(z)| \frac{(d_{W,\gamma})^{d-2}}{d_{W,\gamma}(z_1 + d_{W,\gamma})^{d-2}} dz.$$

This gives with the Cauchy-Schwarz inequality,

$$\begin{aligned} B_{W,\gamma} \le & \frac{C_3 (d_{W,\gamma})^{d-3} h_F^2}{d_\gamma} \\ & \times \left(\int_{\mathcal{V}_{W,\gamma}} |H(u)(z)|^2 dz \right)^{\frac{1}{2}} \left(\int_{\mathcal{V}_{W,\gamma}} \frac{1}{(z_1 + d_{W,\gamma})^{2(d-2)}} dz \right)^{\frac{1}{2}}. \end{aligned} \tag{A.11}$$

For $d = 2$, noting that $|\mathcal{V}_{W,\gamma}| = \frac{d_{W,\gamma}|\gamma|}{2}$, (A.11) gives

$$B_{W,\gamma} \le \frac{C_3 h_F^2 |\gamma|^{\frac{1}{2}}}{\sqrt{2} d_\gamma d_{W,\gamma}^{\frac{1}{2}}} \left(\int_{\mathcal{V}_{W,\gamma}} |H(u)(z)|^2 dz \right)^{\frac{1}{2}}.$$

A similar estimate holds on $B_{L,\gamma}$ by changing W in L and $d_{W,\gamma}$ in $d_{L,\gamma}$. Since $d_{W,\gamma}, d_{L,\gamma} \ge \theta_1 h_F$ and $d_\gamma = d_{W,\gamma} + d_{L,\gamma} \ge 2\theta_1 h_F$, these estimates on $B_{W,\gamma}$ and $B_{L,\gamma}$ yield (3.28) for some C_1 only depending on d and θ_1.

For $d = 3$,

$$\begin{aligned} B_{W,\gamma} &\le \frac{C_3 h_F^2 |\gamma|^{\frac{1}{2}}}{d_\gamma d_{W,\gamma}^{\frac{1}{2}}} \left(\int_{\mathcal{V}_{W,\gamma}} |H(u)(z)|^2 dz \right)^{\frac{1}{2}} \\ &\le \frac{C_3 h_F |\gamma|^{\frac{1}{2}}}{\sqrt{2}\theta_1 d_\gamma^{\frac{1}{2}}} \|H(u)\|_{L^2(\mathcal{V}_{W,\gamma})}. \end{aligned}$$

With a similar estimate on $B_{L,\gamma}$, this yields (3.28) for some C_1 only depending on d and θ_1. Now we prove (3.28) if $\gamma \in \Gamma_F^{h,\partial}$. Let W be the control volume such that $\gamma \in W$. One can assume, without loss of generality, that $x_W = 0$ and $\gamma = d_{W,\gamma} \times \tilde{\sigma}$ with $\tilde{\sigma} \subset \mathbb{R}^{d-1}$. The above proof gives, with some C_4 only depending on d and θ_1,

$$\begin{aligned} & \left| \frac{(u(y_\gamma) - u(x_W))}{d_{W,\gamma}} - \frac{1}{|\hat{\gamma}|} \int_{\hat{\gamma}} \nabla u(x) \cdot \boldsymbol{n}_{W,\gamma} d\gamma(x) \right|^2 \\ & \le C_4 \frac{h_F^2}{|\gamma| d_\gamma} \int_{\mathcal{V}_{\hat{\gamma}}} |H(u)(z)|^2 dz \end{aligned} \tag{A.12}$$

with $\hat{\gamma} = \{(\frac{d_{W,\gamma}}{2}, \frac{\tilde{x}}{2}), \tilde{x} \in \tilde{\gamma}\}$ and $\mathcal{V}_{\hat{\gamma}} = \{ty_\gamma + (1-t)x, x \in \hat{\gamma}, t \in [0,1]\} \cup \{tx_W + (1-t)x, x \in \hat{\gamma}, t \in [0,1]\}$. Note that $|\hat{\gamma}| = \frac{|\gamma|}{2^{d-1}}$ and that $\mathcal{V}_{\hat{\gamma}} \subset \mathcal{V}_\gamma$. One must now compare $I_\gamma = \frac{1}{|\gamma|}\int_\gamma \nabla u \cdot \boldsymbol{n}_{W,\gamma} d\sigma(x)$ with $I_{\hat{\gamma}} = \frac{1}{|\hat{\gamma}|}\int_{\hat{\gamma}} \nabla u \cdot \boldsymbol{n}_{W,\gamma} d\sigma(x)$.

Let $y = 2x, \forall x \in \hat{\gamma}$, we obtain

$$I_{\hat{\gamma}} = \frac{1}{|\hat{\gamma}|}\int_\gamma \nabla u(\frac{y}{2}) \cdot \boldsymbol{n}_{W,\gamma} 2^{-(d-1)} d\sigma(y) = \frac{1}{|\gamma|}\int_\gamma \nabla u(\frac{x}{2}) \cdot \boldsymbol{n}_{W,\gamma} d\sigma(x)$$

By $\frac{x_W}{2} = x_W = 0$, $s = \frac{t}{2}$ and Taylor expansion, we obtain

$$\begin{aligned} &\forall x \in \gamma, \nabla u(\frac{x}{2}) \cdot \boldsymbol{n}_{W,\gamma} \\ &= \nabla u(\frac{x_W}{2}) \cdot \boldsymbol{n}_{W,\gamma} + \int_0^1 H(u)(x_W + t(\frac{x}{2} - \frac{x_W}{2}))(x - x_W) \cdot \boldsymbol{n}_{W,\gamma} 2^{1-d} dt \\ &= \nabla u(x_W) \cdot \boldsymbol{n}_{W,\gamma} + \int_0^{\frac{1}{2}} H(u)(x_W + s(x - x_W))(x - x_W) \cdot \boldsymbol{n}_{W,\gamma} ds. \end{aligned}$$

Thus

$$I_{\hat{\gamma}} = \frac{1}{|\gamma|}\int_\gamma \left(\nabla u(x_W) \cdot \boldsymbol{n}_{W,\gamma} + \int_0^{\frac{1}{2}} H(u)(x_W + t(x - x_W))(x - x_W) \cdot \boldsymbol{n}_{W,\gamma} dt\right) d\sigma(x) \tag{A.13}$$

Taylor expansion gives

$$\forall x \in \gamma, \nabla u(x) \cdot \boldsymbol{n}_{W,\gamma} = \nabla u(x_W) \cdot \boldsymbol{n}_{W,\gamma} + \int_0^1 H(u)(x_W + t(x - x_W))(x - x_W) \cdot \boldsymbol{n}_{W,\gamma} dt.$$

Thus

$$I_\gamma = \frac{1}{|\gamma|}\int_\gamma \left(\nabla u(x_W) \cdot \boldsymbol{n}_{W,\gamma} + \int_0^1 H(u)(x_W + t(x - x_W))(x - x_W) \cdot \boldsymbol{n}_{W,\gamma} dt\right) d\sigma(x). \tag{A.14}$$

Subtract (A.13) from (A.14):

$$I_\gamma - I_{\hat{\gamma}} = \frac{1}{|\gamma|}\int_\gamma \int_{\frac{1}{2}}^1 H(u)(x_W + t(x - x_W))(x - x_W) \cdot \boldsymbol{n}_{W,\gamma} dt d\sigma(x). \tag{A.15}$$

The change of variables in this last integral $z = x_W + t(x - x_W)$, which gives $dz = 2d_{W,\gamma} t^{d-1} dt d\sigma(x)$, yields, with $E_\gamma = \{tx + (1-t)x_W, x \in \gamma, t \in [\frac{1}{2}, 1]\}$ and some C_5

only depending on d (note that $t \geq \frac{1}{2}$),

$$|I_\gamma - I_{\hat{\gamma}}| \leq \frac{C_5}{|\gamma| d_{W,\gamma}} \int_{E_\gamma} |H(u)(z)||x - x_W| dz.$$

Then, using once more the Cauchy-Schwarz inequality and $|x - x_W| \leq h_F$,

$$\begin{aligned} |I_\gamma - I_{\hat{\gamma}}|^2 &\leq \frac{C_6 h_F^2}{|\gamma| d_\gamma} \int_{E_\gamma} |H(u)(z)|^2 dz \\ &\leq \frac{C_6 h_F^2}{|\gamma| d_\gamma} \int_{\mathcal{V}_\gamma} |H(u)(z)|^2 dz \end{aligned} \tag{A.16}$$

with some C_6 only depending on d. Inequalities (A.12) and (A.16) yield (3.28) for some C_1 only depending on d and θ_1 for $u \in C^2(\bar{\mathcal{V}}_\gamma)$. Taking C_1 convenient for $\gamma \in \Gamma_F^{h,I}$ and $\Gamma_F^{h,\partial}$ gives (3.28) for all $\gamma \in \Gamma_F^h$.

Now for the density argument, let $u \in H^2(\mathcal{V}_\gamma)$ and let $(u_n)_{n \in \mathbb{N}} \subset C^2(\bar{\mathcal{V}}_\gamma)$ be a sequence which converges to u in the $H^2(\mathcal{V}_\gamma)$ norm. Thanks to the previous result, one has

$$\begin{aligned} &\left|\frac{u_n(x_L) - u_n(x_W)}{d_\gamma} - \frac{1}{|\gamma|} \int_\gamma \nabla u_n(x) \cdot \boldsymbol{n}_{W,\gamma} d\sigma(x)\right| \\ &\leq C_1 h_F (|\gamma| d_\gamma)^{-\frac{1}{2}} \|u_n\|_{H^2(\mathcal{V}_\gamma)}. \end{aligned}$$

Thanks to Sobolev imbeddings the sequence $(u_n)_{n \in \mathbb{N}} \subset C^2(\bar{\mathcal{V}}_\gamma)$ converges to $u \in H^2(\mathcal{V}_\gamma)$ uniformly and the sequence $(\nabla u_n \cdot \boldsymbol{n}_{W,\gamma}) \subset L^2(\gamma)$ converges to $\nabla u \cdot \boldsymbol{n}_{W,\gamma}$ in $L^2(\gamma)$ and therefore in $L^1(\gamma)$. Passing to the limit in the latter inequality yields (3.28) for some C_1 only depending on d and θ_1 for $u \in H^2(\mathcal{V}_\gamma)$.

The proof for (3.29) is the same as the proof for (3.28) when γ is on the boundary. Let us now prove (3.30) in the case $\gamma \in \Gamma_F^{h,I}$; let $\gamma = W|L$ with $W, L \in \mathcal{E}_F^h$. We assume $\boldsymbol{\beta}_\gamma \geq 0$ (the case $\boldsymbol{\beta}_\gamma < 0$ works in the same way) and $\boldsymbol{n}_\gamma = \boldsymbol{n}_{W,\gamma}$ so

$$|Q_\gamma(u)| = \left| \int_\gamma \boldsymbol{\beta} \cdot \boldsymbol{n}_{W,\gamma} (u(x) - u(x_W)) d\sigma(x) \right|.$$

It is possible to assume, for simplicity of notation and without loss of generality, that $\gamma = 0 \times \tilde{\gamma}$, with some $\tilde{\gamma} \subset \mathbb{R}^{d-1}$, and $x_W = (-d_{W,\gamma}, 0)^t$. A Taylor expansion using

$u \in C^1(\bar{\mathcal{V}_\gamma})$ gives with $x = (0, \tilde{x})^t \in \gamma$

$$|Q_\gamma(u)| \leq \sup_{x\in\bar{\Omega}} |\boldsymbol{\beta}(x)| h_F \int_{\tilde{\gamma}} \int_0^1 |\nabla u((t-1)d_{W,\gamma}, t\tilde{x})| dt d\tilde{x}.$$

Let $p > d$ be such that $p < \infty$ if $d = 2$ and $p \leq 6$ if $d = 3$; let q be its conjugate exponent, that is, $\frac{1}{p} + \frac{1}{q} = 1$. Thanks to Hölder's inequality,

$$\begin{aligned}|Q_\gamma(u)| \leq & \sup_{x\in\bar{\Omega}} |\boldsymbol{\beta}(x)| h_F \left(\int_{\tilde{\gamma}} \int_0^1 |\nabla u((t-1)d_{W,\gamma}, t\tilde{x})|^p t^{d-1} d_{W,\gamma} dt d\tilde{x} \right)^{\frac{1}{p}} \\ & \times \left(\int_{\tilde{\gamma}} \int_0^1 \frac{1}{(t^{d-1} d_{W,\gamma})^{\frac{q}{p}}} dt d\tilde{x} \right)^{\frac{1}{q}}.\end{aligned}$$

Using a change of variables such that $(\tilde{x}, t) \mapsto z = ((t-1)d_{W,\gamma}, t\tilde{x})$ and noting that $\frac{q}{p}(d-1) = (q-1)(d-1) < 1$ since $p > d$, one obtains

$$\begin{aligned}|Q_\gamma(u)| &\leq \sup_{x\in\bar{\Omega}} |\boldsymbol{\beta}(x)| h_F \|u\|_{W^{1,p}(\mathcal{V}_{W,\gamma})} |\gamma| (|\gamma| d_{W,\gamma})^{-\frac{1}{p}} \left(\int_0^1 \frac{1}{(t^{(q-1)(d-1)}} dt \right)^{\frac{1}{q}} \\ &= \frac{\sup_{x\in\bar{\Omega}} |\boldsymbol{\beta}(x)|}{(1-(q-1)(d-1))^{\frac{1}{q}}} \|u\|_{W^{1,p}(\mathcal{V}_{W,\gamma})} h_F |\gamma| (|\gamma| d_{W,\gamma})^{-\frac{1}{p}}.\end{aligned}$$

Noting that $d_\gamma = d_{W,\gamma} + d_{L,\gamma} \geq 2\theta_1 h_F \geq 2\theta_1 d_{W,\gamma}$ one obtains (3.30) for some C_2 only depending on $\boldsymbol{\beta}, \theta_1$, and p.

Now for $\gamma \in \Gamma_F^{h,\partial+}$, $\boldsymbol{\beta}_\gamma \geq 0$, the proof is identical to the case $\gamma \in \Gamma_F^{h,I}$.

The proof for (3.31) is similar to the proof for (3.30). □

Bibliography

[1] W. Reed and T. Hill, "Triangular mesh methods for the neutron transport equation," *Los Alamos Report LA-UR-73-479*, 1973.

[2] M. Delfour, W. Hager, and F. Trochu, "Discontinuous Galerkin methods for ordinary differential equations," *Mathematics of Computation*, vol. 36, no. 154, pp. 455–473, 1981.

[3] K. Boettcher and R. Rannacher, "Adaptive error control in solving ordinary differential equations by the discontinuous Galerkin method," *Report, Institute of Applied Mathematics, University of Heidelberg*, 1996.

[4] G. Chavent and B. Cockburn, "The local projection $P^0 - P^1$-discontinuous-Galerkin finite element method for scalar conservation laws," *RAIRO Modél. Math. Anal. Numér*, vol. 23, pp. 565–592, 1989.

[5] B. Cockburn and C. Shu, "TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws II: general framework," *Mathematics of Computation*, vol. 52, no. 186, pp. 411–435, 1989.

[6] S. Allmaras, *A coupled Euler/Navier-Stokes algorithm for 2-D unsteady transonic shock/boundary-layer interaction.* Massachusetts Institute of Technology, Dept. of Aeronautics and Astronautics, 1989.

[7] D. Halt, "A compact high-order Euler solver for unstructured grids," 1992.

[8] I. Babuska, "The finite element method with penalty," *Mathematics of Computation*, vol. 27, no. 122, pp. 221–228, 1973.

[9] G. Baker, "Finite element methods for elliptic equations using nonconforming elements," *Mathematics of Computation*, vol. 31, no. 137, pp. 45–59, 1977.

[10] M. Wheeler, "An elliptic collocation-finite element method with interior penalties," *SIAM Journal on Numerical Analysis*, vol. 15, no. 1, pp. 152–161, 1978.

[11] D. Arnold, "An interior penalty finite element method with discontinuous elements," *SIAM Journal on Numerical Analysis*, vol. 19, no. 4, pp. 742–760, 1982.

[12] D. Arnold, F. Brezzi, B. Cockburn, and D. Marini, "Discontinuous Galerkin methods for elliptic problems," *Lecture notes in computational science and engineering*, vol. 11, pp. 89–102, 2000.

[13] D. Arnold, F. Brezzi, B. Cockburn, and L. Marini, "Unified analysis of discontinuous Galerkin methods for elliptic problems," *SIAM Journal on Numerical Analysis*, vol. 39, no. 5, pp. 1749–1779, 2002.

[14] C. Dawson, "Godunov-mixed methods for advective flow problems in one space dimension," *SIAM Journal on Numerical Analysis*, vol. 28, no. 5, pp. 1282–1309, 1991.

[15] T. Arbogast and M. Wheeler, "A characteristics-mixed finite element method for advection-dominated transport problems," *SIAM Journal on Numerical Analysis*, vol. 32, no. 2, pp. 404–424, 1995.

[16] F. Bassi and S. Rebay, "A High-Order accurate discontinuous finite element method for the numerical solution of the compressible Navier-Stokes equations,"

Journal of Computational Physics, vol. 131, no. 2, pp. 267–279, 1997.

[17] T. Warburton, I. Lomtev, R. Kirby, and G. Karniadakis, "A discontinuous Galerkin method for the Navier-Stokes equations on hybrid grids," *Center for Fluid Mechanics*, vol. 13, p. 14, 1997.

[18] F. Brezzi, G. Manzini, D. Marini, P. Pietra, and A. Russo, "Discontinuous galerkin approximations for elliptic problems," *Numerical Methods for Partial Differential Equations*, vol. 16, no. 4, pp. 365–378, 2000.

[19] B. Cockburn and C. Shu, "The local discontinuous Galerkin method for time-dependent convection-diffusion systems," *SIAM Journal on Numerical Analysis*, pp. 2440–2463, 1998.

[20] C. Baumann and J. Oden, "A discontinuous hp finite element method for convection–diffusion problems," *Computer Methods in Applied Mechanics and Engineering*, vol. 175, no. 3-4, pp. 311–341, 1999.

[21] B. Riviere, M. Wheeler, and V. Girault, "Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems. Part I," *Computational Geosciences*, vol. 3, no. 3, pp. 337–360, 1999.

[22] S. Sun, "Discontinuous Galerkin methods for reactive transport in porous media," 2003.

[23] C. Dawson, S. Sun, and M. Wheeler, "Compatible algorithms for coupled flow and transport," *Computer Methods in Applied Mechanics and Engineering*, vol. 193, no. 23-26, pp. 2565–2580, 2004.

[24] B. Riviere, M. Wheeler, and V. Girault, "A priori error estimates for finite element methods based on discontinous approximation spaces for elliptic problems," *SIAM Journal on Numerical Analysis*, vol. 39, pp. 902–931, 2001.

[25] S. Sun and M. Wheeler, "Symmetric and nonsymmetric discontinuous Galerkin methods for reactive transport in porous media," *SIAM Journal on Numerical Analysis*, vol. 43, no. 1, pp. 195–219, 2006.

[26] P. Castillo, B. Cockburn, I. Perugia, and D. Schötzau, "An a priori error analysis of the local discontinuous galerkin method for elliptic problems," *SIAM Journal on Numerical Analysis*, pp. 1676–1706, 2001.

[27] P. Castillo, B. Cockburn, D. Schotzau, and C. Schwab, "Optimal a priori error estimates for the hp-version of the local discontinuous galerkin method for convection-diffusion problems," *Mathematics of Computation*, vol. 71, no. 238, pp. 455–478, 2001.

[28] T. Hughes, A. Masud, and J. Wan, "A stabilized mixed discontinuous galerkin method for darcy flow," *Computer Methods in Applied Mechanics and Engineering*, vol. 195, no. 25-28, pp. 3347–3381, 2006.

[29] F. Brezzi, B. Cockburn, L. Marini, and E. Suli, "Stabilization mechanisms in discontinuous galerkin finite element methods," *Computer Methods in Applied Mechanics and Engineering*, vol. 195, no. 25-28, pp. 3293–3310, 2006.

[30] B. Cockburn, J. Gopalakrishnan, and R. Lazarov, "Unified hybridization of discontinuous galerkin, mixed and continuous galerkin methods for second order elliptic problems," *SIAM J. Numer. Anal*, vol. 47, no. 2, pp. 1319–1365, 2009.

[31] J. Proft and B. Riviere, "Discontinuous Galerkin methods for convection-diffusion equations for varying and vanishing diffusivity," *International Journal of Numerical Analysis and Modeling*, vol. 6, no. 4, pp. 533–561, 2009.

[32] A. Tikhonov and A. Samarskii, "Homogeneous difference schemes on non-uniform nets," *USSR Computational Mathematics and Mathematical Physics*, vol. 2, no. 5, pp. 927–953, 1963.

[33] A. Samarskiy, "On the monotone difference schemes for elliptical and parabolic equations in the case of not self-conjugate elliptical operator," *Zh. Vychisl. Mat. Mat. Fiz*, vol. 5, pp. 548–551, 1965.

[34] P. Forsyth, "Quadratic convergence for cell-centered grids," *Applied Numerical Mathematics*, vol. 4, no. 5, pp. 377–394, 1988.

[35] A. Weiser and M. Wheeler, "On convergence of block-centered finite differences for elliptic problems," *SIAM Journal on Numerical Analysis*, vol. 25, no. 2, pp. 351–375, 1988.

[36] P. Vassilevski, S. Petrova, and R. Lazarov, "Finite difference schemes on triangular cell-centered grids with local refinement," *SIAM Journal on Scientific and Statistical Computing*, vol. 13, p. 1287, 1992.

[37] R. Lazarov, I. Mishev, and P. Vassilevski, "Finite volume methods for convection-diffusion problems," *SIAM Journal on Numerical Analysis*, vol. 33, no. 1, pp. 31–55, 1996.

[38] M. Feistauer, J. Felcman, and M. Lukáčová-Medvid'ová, "On the convergence of a combined finite volume-finite element method for nonlinear convection-diffusion

problems," *Numerical Methods for Partial Differential Equations*, vol. 13, no. 2, pp. 163–190, 1997.

[39] R. Eymard, T. Gallouët, and R. Herbin, "Convergence of finite volume schemes for semilinear convection diffusion equations," *Numerische Mathematik*, vol. 82, no. 1, pp. 91–116, 1999.

[40] T. Gallouët, R. Herbin, and M. Vignal, "Error estimates on the approximate finite volume solution of convection diffusion equations with general boundary conditions," *SIAM Journal on Numerical Analysis*, vol. 37, no. 6, pp. 1935–1972, 2000.

[41] P. Chidyagwai, I. Mishev, and B. Riviere, "On the coupling of finite volume and discontinuous Galerkin method for elliptic problems," *Journal of Computational and Applied Mathematics*, vol. 231, pp. 2193–2204, 2010.

[42] B. Riviere, *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation.* SIAM, 2008.

[43] R. Eymard, T. Gallouët, and R. Herbin, *Handbook for Numerical Analysis Spaces.* North Holland, 2000.

[44] T. Gallouët, R. Herbin, and M. Vignal, "Error estimates on the approximate finite volume solution of convection diffusion equations with general boundary conditions," *SIAM Journal on Numerical Analysis*, vol. 37, no. 6, pp. 1935–1972, 2000.

[45] C. Ho, "Discontinuous Galerkin formulation for multi-component multiphase flow," 2010.