RICE UNIVERSITY

# MAP Detection with Soft Information in an Estimate and Forward Relay Network

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A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

**Master of Science**

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ABSTRACT

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One proven solution for improving the reliability of a wireless channel is to use relays. We analyze the three-node relay network, focusing on the estimate-and-forward (EF) relay protocol, where the relay node estimates the source symbol and then forwards it to the destination.

For an uncoded system in a path-loss model for the cooperative three-node wireless network, we show that EF with minimum mean square error (MMSE) estimate outperforms other common relay protocols such as amplify-and-forward (AF) and detect-and-forward (DF). Since the probability density function required by the optimal maximum a posteriori (MAP) detector at the destination does not have an analytical form, we provide a solution to bypass its numerical approximation. We show that EF with a piecewise linear approximation of the MMSE estimate provides an analytical form of the detector at the destination and its performance is similar to the case when a MMSE estimate is used.

# Acknowledgements

I would like to first thank my advisor, Dr. Behnaam Aazhang, for accepting me as a graduate student. I came in with a computer science background and due to your patience and understanding I made it to the day when I defended my Master Thesis. Thank you for being such an inspiring mentor throughout this journey.

I would also like to thank Dr. Jorma Lilleberg for offering me the opportunity to work on such an interesting problem. I will never forget the first day we met and started talking about the MMSE estimate.

A big thank you to all my colleagues here at Rice, who also became my friends (Sam, Ryan, Kanes, Jason, Matt, Drew, Pedro, Evan, Achal, David and the list goes on). Thank you all for your support, help and encouragement.

The warmest thanks goes to my family: my dad, Cezar, my mum, Mariana and my brother, Andrei. I am who I am because of you all. Thank you for your love and support throughout all these years.

Last but not least, I would like to thank Cosmin, my husband-to-be. Thank you for believing in me no matter what. I am where I am now because of all your love and trust. You are the one who made me push my limits and overcome my fears. Thank you for being there for me.

To my grandpa, Efrem Serediuc, who I wish was here to see me, I will never forget you.

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# Chapter 1

# Introduction

In recent years, wireless communications have evolved tremendously. One important difference from wired communications is that in the wireless environment the communication medium is shared among users. Thus, other nodes in the neighborhood of the transmitting node are able to hear the transmission. Due to the availability of information at nodes in the vicinity of the source, the idea of using these nodes to help with communication emerged. This technique became known as *cooperative communications.* The information at the receiving node depends not only on the signal received from the source, but also on the signals received from neighboring nodes. This thesis analyzes the type of information the neighboring nodes should send to aid the communication and provides a strategy for the best use of the additional information at the destination node.

The maximum amount of information that can be sent over the shared medium in a wireless network is constrained by the interference among nodes. For long distance communications, the transmitted signal suffers attenuation and also produces interference to more nodes in the system. In contrast, short distance communications produce interference to a smaller number of neighbors, allowing more communication channels to be active in the wireless network. This advantage of short distance wireless communications motivated the use of multihop networks. In such a network, nodes pass information to the next ones, until the data reaches its destination node.

For example, in Figure 1.1 we named as relays the neighboring nodes of the source

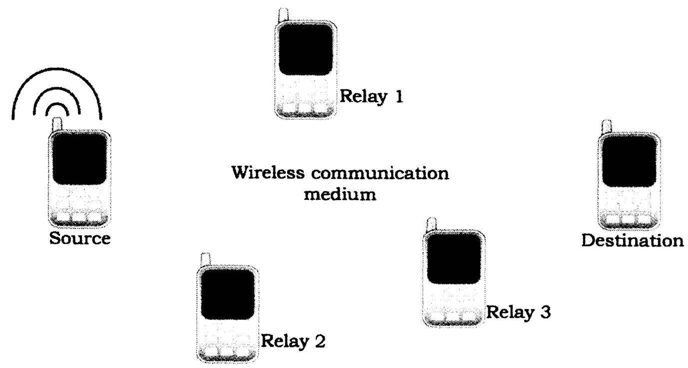


Figure 1.1 : Wireless cooperative network setup.

which can hear the full transmission. When the source transmits, the relays can help communication in several ways. One possible multihop link can be formed with relays 2 and 3 by sending the signal from one to the another until it reaches the destination. Also, a one-hop link can be formed with relay 1, which retransmits the signal directly to the destination. Thus, if the direct communication between the source and the destination is weak, then the destination can combine the signal received from the source with the signals received from relays 1 and 3 to correctly decode the information.

We focus on a source-destination wireless network with a single relay. This represents the fundamental building block for wireless cooperative networks, and is called *the three-node channel*, or more commonly *the relay channel*. In this setup, we have one node transmitting the information (the source node S), one neighboring node (the relay R), and one node for which the signal is intended (the destination D), as depicted in Figure 1.2.

The data communication occurs as follows. First, the source broadcasts the signal,

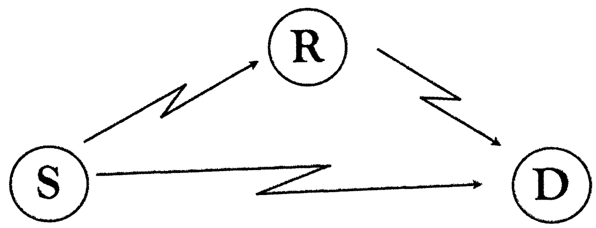


Figure 1.2 : Fundamental building block for wireless cooperative networks.

which is heard by both the relay and the destination. Secondly, the relay processes its received signal and forwards it to the destination. During this time, the source may be silent, such that only the relay transmits, or it may also broadcast new data. This forces the relay to perform in full duplex mode, receiving and transmitting at the same time. The importance of cooperative communications becomes clear when information is retrieved at the destination. In a point-to-point network (no relays), when the source-destination link is bad, the information may not be decoded accurately. However, in a cooperative setting, the destination can use additional information from the relays, resulting in an improved decoding.

One major factor which influences the accuracy of the decoding performed at the destination is the type of information sent by the relay. Next, we present several well known relaying strategies which use different types of information forwarding by the relay.

## 1.1 Relaying strategies

A relay protocol is defined by the type of processing used for the received signal at the relay. There are three main classes of relaying protocols:

- **Decode-and-Forward (DF).** The relay decodes and re-encodes the received signal, then it forwards it to the destination. This processing of the signal at the relay is also know as making a *hard* decision, as the information sent by the relay does not include any additional information about the reliability of the source-relay link.

- **Amplify-and-Forward (AF)**. In this case, the relay just amplifies its received signal, maintaining a fixed average transmit power.

- **Estimate-and-Forward(EF)**. This protocol is also known as Compress-and-Forward or Quantize-and-Forward. At the relay, a transformation is applied to the received signal, which provides an estimate of the source signal. This estimate is called *soft* information, and it is forwarded to the destination.

AF is a straightforward scheme for both coded and uncoded systems. On the other hand, DF comes with the option of choosing a specific type of coding which can improve the overall performance. Both protocols have been extensively investigated using capacity, achievable rates and also different metrics such as outage probability, bit-error-rate (BER) or throughput. These two schemes present a complementary behavior which depends on the signal to noise ratio regimes.

## 1.2 Main contribution of this thesis

While AF and DF have well defined signal processing at the relay, the EF relay protocol includes a broad variety of estimates which can be sent to the destination to improve detection. Thus, since EF is not restricted to only one type of processing at the relay, it has raised a lot of interest on what should be the best estimate to use, or how should this estimate best be used at the destination. In this work, we address these questions and show how EF can be used to improve the performance of the system over AF and DF.

Previous work on EF has shown that there are optimal estimates which minimize specific performance criteria. Motivated by this, we represent the soft information at the relay by the minimum mean squared error(MMSE) estimate. We show that EF with the MMSE estimate provides superior performance to both AF and DF, with respect to bit-error-rate(BER). We propose using a piecewise linear approximation of the MMSE estimate to obtain a closed form detector at the destination.

# Chapter 2

# Related work

One effective solution for improving communication between a source (S) and a destination (D) pair is to use a relay (R), as presented in Figure 2.1. This *three-terminal communication channel*, first introduced in [1], represents the fundamental scheme in cooperative communications.

The performance of this scheme has been extensively analyzed from different perspectives by many authors. There exist several well crystallized approaches for investigating the relay channel, including information theoretical analysis such as achievable rates and capacity bounds. The first to explore the capacity for the relay channel were Cover and Gamal in [2]. This seminal work was later followed by a more generalized analysis on the capacity of the relay channel, done by Kramer et. al in [3] and others in [4] and [5] . Furthermore, the user cooperation diversity was explored in [6] and [7]; while other performance metrics which recently attracted attention include outage probability [8], [9] and diversity-multiplexing tradeoff [10], [11].

The common conclusion of all these previous papers is that adding a relay improves performance significantly. However, we have noticed that the approach of analysing *symbol error rates* has received little attention in the past. In this work, we shall focus on analyzing the symbol error rate for the relay network.

As previously stated, a relay protocol is defined by the processing method used for the received signal at the relay. We term the processing at the relay as the *relay*

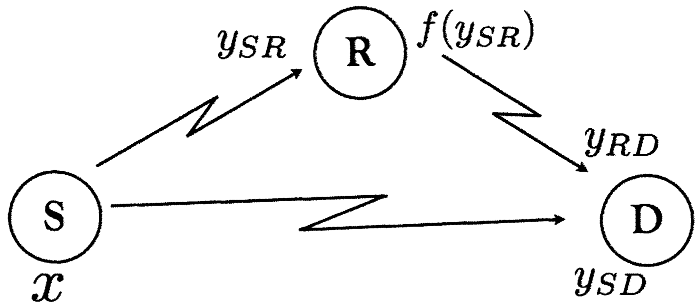


Figure 2.1 : Three-node cooperative wireless network

*function* $f(y_{SR})$, where $y_{SR}$ denotes the received signal at the relay.

Two basic relay protocols were first proposed by Cover and El Gamal in [2]. These protocols are decode-and-forward (DF) and compress-and-forward (CF) also known as estimate-and-forward (EF). Another important protocol, called amplify-and-forward (AF) has been well investigated by Laneman in [9, 8, 12] and others [13, 14, 15]. Furthermore, other protocols can be derived from these basic ones, such as quantized-and-forward (QF) [16].

One of the main features of the AF and DF protocols is that the relay function is well defined. Thus, a lot of attention has been directed towards these protocols, see [3]-[15]. On the other hand, the analysis of the EF protocol can prove to be rather tedious, since the EF relay function is defined to be *any* type of estimate of the source signal which could aid the detection at the destination. As a result, among the three most prominent relaying protocols, EF has received the least attention in literature.

When investigating the EF protocol, besides the metric used to showcase the results, there are several characteristics which make each analysis unique. They include the network topology, the relay function, and the criterion on which the EF relay function is chosen. Another important aspect of EF is the type of information each node is required to have. In addition, most of the existing EF work completely ignores the direct link between the source and the destination, and little to no attention has been given to the detector at the destination side.

For the two-hop transmission network topology, pioneering contributions in EF relaying were made by Abou-Faycal and Médard in [17], where they focused on obtaining the relay function which minimizes the probability of error. A network topology similar to the one in Fig.2.1 was considered, but with no direct link from S to D. Channel state information was assumed to be known at each receiver (CSIR at R and D) and in addition to the R to D channel, the S to R channel state was assumed to be known at the receiver D. With these assumptions, the optimal relay function which minimizes the probability of error was found to be the Lambert W, which is defined as the solution to $\mathrm{W}(x)e^{\mathrm{W}(x)} = x$. The results were derived for uncoded antipodal source signaling in conjunction with an additive white Gaussian noise (AWGN) assumption on each of the two channels. This function is non-analytical and requires analytical approximation or table look up. Therefore, the detector at the destination is not tractable.

Using a similar setup, Gomadam and Jafar proved in [18] that forwarding the minimum mean squared error (MMSE) estimate maximizes the generalized signal to noise ratio (SNR) at the destination. The network topology was extended to ones involving multiple relays in a parallel and serial topology arrangements. These results were derived for the relaying network without exploiting the direct link between S and D. In addition, in [19], the same authors showed numerically that the MMSE estimate is *capacity optimal* for BPSK modulation in a two hop relay network.

The work presented [17] was later extended by Cui et. al [20] for the two way relay channel. For binary antipodal signaling, in a network which included the direct link, they showed that a Lambert W function minimizes the probability of error. The Lambert W function and the MMSE estimate are the only EF relay functions to optimize a criterion such as the probability of error or SNR. These functions are specific for the uncoded system for the relay network. For a coded relay network, the functions encountered in literature are approximations of the mean squared error (MSE) estimate as the log likelihood functions (LLRs) [21, 22] or approximations of the LLRs, [23]. However, our focus in this paper shall be solely on an uncoded system, which uses the MMSE estimate as the relay function.

In previous work on EF, the relay function depends only on the received symbol and not on previous symbols. This methodology was recently denoted as *instantaneous relaying* by Khormuji in [24], where they performed analysis for the Gaussian relay channel with a direct link and a perfect S to R link. Khormuji and Skoglund numerically showed that a relay protocol with parametric piecewise linear mapping improves the performance of the relay network over the AF relay protocol. In this paper we propose a more generalized method which employs a piecewise linear approximation of the relay function.

Once the relay function is defined for a relay protocol, the detector at the destination is critical for the practical implementation of the protocol. In the communications literature, sparse attention has been given to the detector techniques which could be used at the destination. Seminal work was done by Brennan in [33], where a comprehensive analysis is done on the practical maximum-ratio-combining (MRC) detector applied to both AF and DF. Later, Wang et. al [25] introduced cooperative-MRC (C-MRC) and showed that for a DF relay protocol maximum possible diversity is achieved. Most recently, a novel soft-symbol estimate-and-forward (SEF) with the MRC detector technique is proposed by Hu and Lilleberg in [26]. They show numerically that significant power gain is obtained when a posteriori probability is used to weight the soft information sent by the relay.

We argue that in a wireless environment, the direct link may still provide reliable information and warrant processing at the destination. The MMSE estimate has been shown to be optimum in several scenarios of the three-node network without the direct link. Thus, in this work we analyze the performance of the EF protocol when a conditional expectation is the relay function in a three-node wireless network, and we take advantage of the existence of the direct link. We provide a generalized method consisting of using a piecewise linear approximation of the MMSE. We also address the important issue of the detection algorithm at the destination. To the best of our knowledge, we are among the first to provide a closed form solution of the optimal detector.

It has also come to our attention that recently, similar work has been done independently and in parallel by Tian et al in [27]. They use a specific three-segment piecewise linear approximation in a system with BPSK modulation.

# Chapter 3

# Preliminaries

In this chapter, we define the system model and two of the most well know protocols, amplify-and-forward (AF) and decode-and-forward (DF). Their performance will later be compared with the performance of the proposed EF protocol.

## 3.1 System model

Considering the general setup (Figure 3.1 ), data is to be transmitted from the source node (S) to the destination (D). A third node called the relay (R) will help the communication between the two nodes. The scope of the relay is to only assist in the communication to the destination, it does not have data of its own to be transmitted. In this setup, we assume that each node has only one antenna.

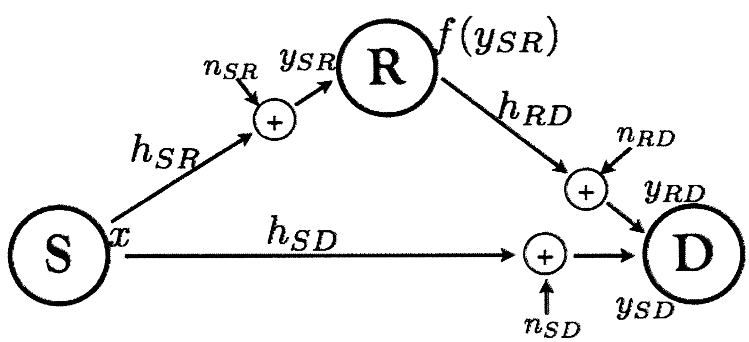


Figure 3.1 : Notations on the three-node cooperative wireless network.

We use $x$ to denote the symbol transmitted by the source, while the relay function is represented by $f(y_{SR})$, where $y_{SR}$ denotes the received signal at the relay from the source. Similarly, we define $y_{RD}$ and $y_{SD}$ as the received signals at the destination.

According to the type of function used at the relay, we can classify the the signal transmitted by the relay as soft or hard information. Results presented throughout this thesis will be given for the general case with fading channels, and specific examples will be shown for different modulation schemes. The fading coefficient of S-R, R-D and S-D channels are denoted by $h_{SR}$, $h_{RD}$ and $h_{SD}$, respectively. Every channel in the network is degraded by independent AWGN as $n_{SR} \sim \mathcal{N}(0,1)$, $n_{SD} \sim \mathcal{N}(0,1)$ and $n_{RD} \sim \mathcal{N}(0,1)$. It follows that the system model is described by the following equations:

$$y_{SR} = h_{SR} \cdot x + n_{SR} \tag{3.1}$$

$$y_{SD} = h_{SD} \cdot x + n_{SD} \tag{3.2}$$

$$y_{RD} = h_{RD} \cdot f(y_{SR}) + n_{RD} \tag{3.3}$$

To showcase the performance of the relay protocol we propose, our simulation results are presented for a path-loss system model, that is, the relay is located on the direct line, between the source and the destination, as shown in Figure 3.2. The S to R distance normalized by the S to D one is denoted by $d$, and for the path-loss model the fading gain of the S to R channel is given by $|h_{SR}|^2 = d^{-\alpha}$. Here, $\alpha$ represents the path-loss exponent, which usually is set to have values between 2 and 4. The fading gains for the R to D and S to D channels are defined in a similar manner by $|h_{RD}|^2 = (1-d)^{-\alpha}$ and $|h_{SD}|^2 = 1$, respectively. This system model covers a wide range of possible locations of the relay, which will ease the interpretation of the results.

The main results of this work are presented for a general setup, that is, inde-

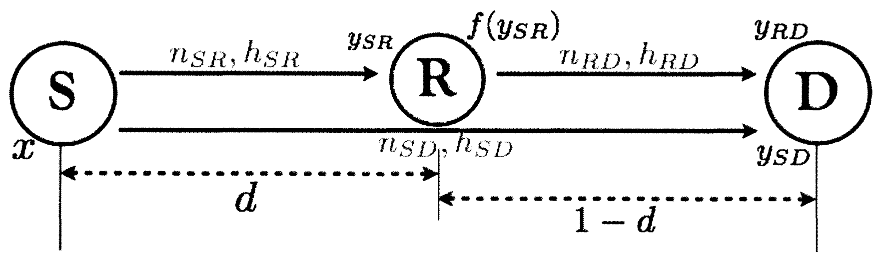


Figure 3.2 : Pathloss model. Relay inline with source and destination.

pendent of the modulation order chosen for the source symbols. Examples are given for specific modulations like uncoded binary-phase-shift-keying (BPSK) and M-ary QAM (quadrature amplitude modulation), such as 16 QAM. We assume fixed transmit power for the source ($P_S$) and the relay ($P_R$). Their sum is equal to the source transmit power in a S to D only model, that is, the single link model with no relays.

There are several medium access protocols that can be used in such a network topology. A detailed description of these protocols can be found in [29].

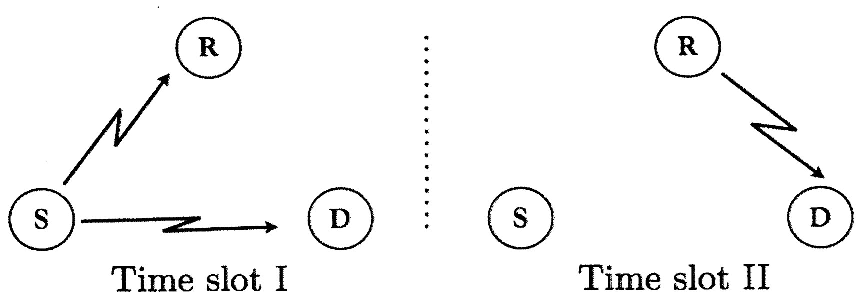


Figure 3.3 : Transmission protocol. First time slot - Broadcast phase; Second time slot - MAC phase - usually both the source and the relay transmit in this phase, but in our case, only the relay will be transmitting.

The setup of the medium access protocol used in this work is such that the relay works in half-duplex mode, that is, it will not receive and transmit at the same time. This assumption implies that, with help from the relay, it will take two time slots to complete the transmission from S to D, as shown in Figure 3.3. In the first time slot, also know as the broadcast phase, S transmits, while R is silent and both R and D receive the signal. In the second time slot, S is silent and only R transmits to D.

We also impose the CSIR assumption, that is, channel state information is known at the receiver. This a realistic assumption, since training pilots can be used to estimate the channel characteristics and frequency offset correction can be applied at the receiver to recover from constellation rotation.

An important aspect of any relaying protocol is the detector at the destination. The decision at the destination is done based on all received signals, $y_{SD}$ and $y_{RD}$. In the next section we introduce the optimal detector to be used at the destination.

## 3.2 Detector at destination

Once the two signals $y_{SD}$, $y_{RD}$ are received at D, the optimal detector is used in the form of a maximum a posteriori probability (MAP) detector. The MAP detector is given by,

$$\widehat{x}_D = \max_x \mathrm{p}_{X|Y_{SD}Y_{RD}}(x|y_{SD}, y_{RD}) \tag{3.4}$$

where $\widehat{x}_D$ denotes the symbol resulted from the decision made at the destination side. From now on we drop the subscripts, which are obvious by inspection of the arguments of the probability density function. The subscripts will be shown only in those cases when they are essential for clarity.

Applying Bayes' Rule[30] we obtain.

$$\widehat{x}_D = \max_x \frac{\mathrm{p}(y_{SD}, y_{RD}|x)\mathrm{p}(x)}{\mathrm{p}(y_{SD}, y_{RD})}. \tag{3.5}$$

The decision at the destination is made for given values of $y_{SD}$ and $y_{RD}$, thus we can ignore $\mathrm{p}(y_{SD}, y_{RD})$ , as it is independent of $x$ and it does not affect the end result of the detector. We assume the symbols are equally likely, therefore $\mathrm{p}(x)$ has the same value for any given symbol and thus it does not influence the outcome of the detector. With these assumptions, the general form of the detector can be further simplified as

$$\widehat{x}_D = \max_x \mathrm{p}(y_{SD}, y_{RD}|x) \tag{3.6}$$

$$= \max_x \mathrm{p}(y_{SD}|x)\mathrm{p}(y_{RD}|x). \tag{3.7}$$

The detector given in (3.6) is the well known maximum likelihood (ML) detector. The MAP and ML detector are the same for this system, mainly because equally likely symbols are used for transmission.

The transition from (3.6) to (3.7) is possible because of the following system properties. The first is that the noise is i.i.d. on all channels, due to the nature of the wireless communication medium. Combining this property with the fact that the two probability density functions are conditioned on $x$, we get that the two random variables representing the received signals at the destination, $y_{SD}$ and $y_{RD}$ given $x$ are independent.

The proof for the conditional independence of the two variables is independent on the modulation order and it is given in the following subsection. Thus, the form of the detector given in (3.7) is specific to this model setup of the network and it will be used throughout the rest of this work and exemplified for specific modulations.

### 3.2.1 Proof of independence

Applying Bayes' rule on the two variables $y_{SD}$ and $y_{RD}$ from the detector given in (3.5) we obtain:

$$x_D = \max_x \mathrm{p}(y_{SD}, y_{RD}|x) = \tag{3.8}$$

$$= \max_x \mathrm{p}(y_{RD}|y_{SD}, x)\mathrm{p}(y_{SD}|x) \tag{3.9}$$

$$= \max_x \mathrm{p}(y_{RD}|x)\mathrm{p}(y_{SD}|x), \tag{3.10}$$

where going from 3.9 to 3.10 is possible due to the following result:

$$\mathrm{p}(y_{RD}|x, y_{SD}) = \mathrm{p}(h_{RD}f(y_{SR}) + n_{RD}|x, h_{SD}x + n_{SD}) \tag{3.11}$$

$$= \mathrm{p}(h_{RD}f(h_{SR}x + n_{SR}) + n_{RD}|x, h_{SD}x + n_{SD}) \tag{3.12}$$

$$= \mathrm{p}(h_{RD}f(h_{SR}x + n_{SR}) + n_{RD}|x) \tag{3.13}$$

$$= \mathrm{p}(y_{RD}|x). \tag{3.14}$$

In (3.12), we expand the received signals $y_{SD}$ and $y_{SR}$ according to the system model given in (3.1)-(3.3). The noise on each channel is independent, thus $n_{SD}$ is independent of $n_{RD}$ and $n_{SR}$. Therefore in (3.13) conditioning on $n_{SD}$ gives the same probability value as not conditioning on it. In the last step (3.14), we return to the compact form of $y_{SD}$ given in (3.2).

For example, for BPSK modulation the detector given in (3.7) becomes a simple comparison between two probability values

$$\mathrm{p}(y_{RD}|x = 1)\mathrm{p}(y_{SD}|x = 1) \underset{H_0}{\overset{H_1}{\gtrless}} \mathrm{p}(y_{RD}|x = 1)\mathrm{p}(y_{SD}|x = 1) \tag{3.15}$$

where the hypotheses $H_0$ and $H_1$ are given by $H_0 = \{x = -1\}$ and $H_1 = \{x = 1\}$.

Given the system model in (3.2) and the channel and noise characteristics, we can apply the natural logarithm in 3.15 and simplify to get

$$\frac{1}{2} h_{SD} y_{SD} \underset{H_0}{\overset{H_1}{\gtrless}} \ln(\mathrm{p}(y_{RD}|x = -1)) - \ln(\mathrm{p}(y_{RD}|x = 1)). \tag{3.16}$$

As each relay protocol depends on the specificity of the relay function, this influences the form of the probability density function (pdf) $\mathrm{p}(y_{RD}|x)$. The density of the received signal at the destination from the source $y_{SD}$ is not affected by the change in the relay function. Thus, the conditional probability density function of $y_{RD}$ plays an important role in defining the decision rule at the destination.

For 16 QAM, the detector in (3.7) can be reduced to having several comparisons between the 16 possible symbols, similar in form to the ones for BPSK, (3.15).

Now that the detector under study has been identified we will turn our attention to the relay function. Next, we introduce the most common relaying protocols by investigating the associated relay functions. The relay function is the defining element of a relay protocol and the system performance is directly affected by the "quality" of the signal received at the relay. Throughout this thesis we investigate three forms of relaying and we consider their symbol error performance.

## 3.3 Decode-and-Forward

In the DF protocol the relay decodes the received signal, re-encodes it and then forwards it to the destination. This type of signal processing is also known as making a hard decision at the relay

$$f_{DF}(y_{SR}) = \widehat{x} \tag{3.17}$$

where $\widehat{x}$ represents the decoded symbol at the relay. For the case when uncoded signals are considered, the protocol is known as *detect-and-forward.*

For BPSK modulation the DF relay function is

$$f_{DF}(y_{SR}) = \text{sgn}(y_{SR}) \tag{3.18}$$

where *sgn* is the sign function. Notice that this relay function depends only on the received signal $y_{SR}$, and does not incorporate any information about the quality of the S-R channel, or about how accurate the relay decoding is.

After the relay transmits, the destination makes a decision based on the detector given in (3.15) with the two received signals from R and D. The probability density function of $y_{RD}$ is that of a transmitted discrete symbol received in Gaussian noise at the destination, $y_{RD} \sim \mathcal{N}(h_{RD}\widehat{x}, 1)$. Replacing the density $\text{p}(y_{RD}|x)$ in (3.16), the DF decision rule at the destination for BPSK modulation becomes:

$$h_{SD}y_{SD} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(y_{RD}+h_{RD})^2}{2}\right\}}{\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(y_{RD}-h_{RD})^2}{2}\right\}}\right) \tag{3.19}$$

$$h_{SD}y_{SD} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(e^{-2h_{RD}y_{RD}}\right) \tag{3.20}$$

$$h_{SD}y_{SD} + h_{RD}y_{RD} \underset{H_0}{\overset{H_1}{\gtrless}} 0 \tag{3.21}$$

where, as stated before, the two hypotheses $H_0$ and $H_1$ for the two possible models of the system are given by $H_0 = \{x = -1\}$ and $H_1 = \{x = 1\}$.

As seen from the decision rule, the destination needs to know channel information only for the R to D and S to D links.

The quality of the decision made at the relay affects the overall performance of the system. However, for the DF protocol the quality of the decision at the relay is not taken into consideration on the destination side (no information about S to R is required at node D). This property can be harmful to the performance of the system. For example, for a bad S-R channel, a lot of wrong decisions will be made at the relay. Even if the R-D channel is very good, the destination will not be able tell the accuracy of the signal sent by the relay, $f_{DF}(y_{SR})$. On the other hand, suppose the S-R channel is good and assume that symbol 1 was sent. If a weak signal is received at the relay, for example $y_{SR} = 0.1$, when it is forwarded to the destination it will be amplified to the symbol power $P_R \cdot 1$. This means a correct decision has been made and a strong signal has been sent for this decision.

These are some of the most important advantages and disadvantages of DF. We will later see that EF with MMSE estimate incorporates most of the advantages of DF.

## 3.4 Amplify-and-Forward

The AF relay function is an amplification of the received signal,

$$f_{AF}(y_{SR}) = \beta y_{SR} \tag{3.22}$$

with $\beta$ the relay transmit average power constraint coefficient. The coefficient $\beta$ ensures that the average transmit power at the relay is constant and equal to $P_R$, therefore $\beta$ is derived in a similar way to the one obtained by Laneman in [9]:

$$E[|f(y_{SR})|^2] \leq P_R \tag{3.23}$$

$$E[|\beta y_{SR}|^2] \leq P_R \tag{3.24}$$

$$\beta^2 E[|h_{SR}x + n_{SR}|^2] \leq P_R \tag{3.25}$$

$$\beta \leq \sqrt{\frac{P_R}{h_{SR}^2 E[|x|^2] + E[|n_{SR}|^2]}} \tag{3.26}$$

and for Gaussian noise with unit variance and $Es$, the energy of the symbol becomes

$$\beta \leq \sqrt{\frac{P_R}{h_{SR}^2 Es + 1}}. \tag{3.27}$$

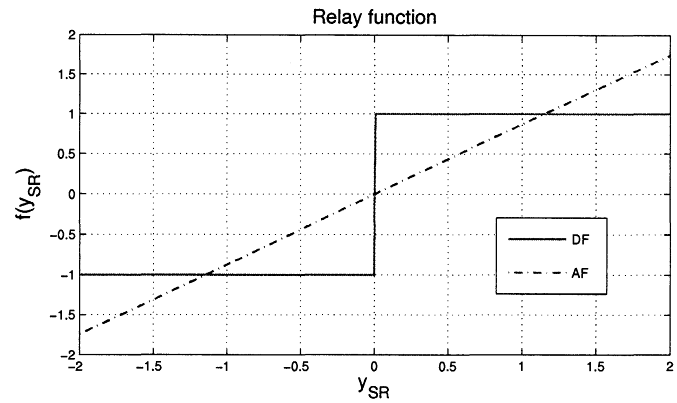


Figure 3.4 : Relay functions for AF and DF; $y_{SR}$ is the received signal at the relay and $f(y_{SR})$ is the relay function. For this plot, we have considered BPSK modulation, $SNR_{SR} = 3dB$ and an average transmit power constraint at the relay $P_R = 1$.

The form of the AF relay function given in (3.22) is independent of the modulation scheme of the source symbol. Since the transmitted signal at the relay is linear in $y_{SR}$, as seen in Figure 3.4, for higher values of $y_{SR}$ the relay requires more instantaneous transmit power than the case when the DF relay function is used.

The probability density function of $y_{SD}$ is the same as in the DF relaying protocol, while the signal $y_{RD}$ is

$$\begin{aligned} y_{RD} &= h_{RD}\beta y_{SR} + n_{RD} = h_{RD}\beta(h_{SR}x + n_{SR}) + n_{RD} \\ &= \beta h_{SR}h_{RD}x + h_{RD}\beta n_{SR} + n_{RD}. \end{aligned} \tag{3.28}$$

Therefore, given the independence of the noise and using the linear property of Gaussian random variables, the conditional probability density function of $y_{RD}$ is

$$p(y_{RD}|x) = \mathcal{N}(\beta h_{SR}h_{RD}x, h_{RD}^2\beta^2 + 1). \tag{3.29}$$

This probability density function together with the one for $y_{SD}$, are replaced in the detector given in (3.7) and a simplified formulation is obtained. Note, that until now, all the results shown for AF are independent on the modulation order.

For example, for BPSK modulation, replacing 3.29 in the detector form given in (3.16), we get that the detector at the destination is given by

$$h_{SD}y_{SD} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \left( \frac{\frac{1}{\sqrt{2\pi(h_{RD}^2\beta^2+1)}} \exp\left\{ -\frac{(y_{RD}+\beta h_{SR}h_{RD})^2}{2h_{RD}^2\beta^2+1} \right\}}{\frac{1}{\sqrt{2\pi(h_{RD}^2\beta^2+1)}} \exp\left\{ -\frac{(y_{RD}-\beta h_{SR}h_{RD})^2}{2(h_{RD}^2\beta^2+1)} \right\}} \right) \quad (3.30)$$

$$h_{SD}y_{SD} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \left( \exp\left\{ -\frac{4\beta h_{SR}h_{RD}y_{RD}}{2(h_{RD}^2\beta^2+1)} \right\} \right) \quad (3.31)$$

$$h_{SD}y_{SD} + \frac{\beta h_{SR}h_{RD}}{h_{RD}^2\beta^2+1} y_{RD} \underset{H_0}{\overset{H_1}{\gtrless}} 0. \quad (3.32)$$

In addition to the DF decision rule, for which the destination is required to know only S to D and R to D link characteristics, AF requires extra information about the S to R link. This implies that for the AF relay protocol the destination decision rule takes into consideration the quality of the channel between S and R. Therefore, one might assume that AF should perform better than the DF protocol. However, we show in the simulation results that this is not always true; the performance of different relay protocols is directly influenced by the quality of the channels, and more precisely by the position of the relay. One argument for this behavior is that, for the AF protocol, when the relay amplifies the received signal it also amplifies the noise. For very good channel conditions, the DF protocol makes the right decision, eliminating the noise, while the AF relay protocol, amplifies the noise. Thus, for high SNR on the S to R channel, a clean signal as the one provided by the DF protocol is preferred at the destination, rather than the noisy signal resulted from the AF relay function.

In the next chapter we introduce the EF relay protocol and its characteristics for specific relay functions.

# Chapter 4

# Estimate-and-Forward

The class of estimate-and-forward relay protocols includes *any* "useful" side information which the relay can provide to the destination in support of its detection and decoding of the information from the source. The EF relay function is often referred to as soft information.

There are many other types of "side" information that the relay can forward to destination to help with decoding. For example, any type of signal estimate is a possible soft information. From the broad range of possible relay functions, only few improve the overall performance of EF. In [17], Abou-Faycal and Médard showed that the Lambert $W$ function minimizes the probability of error in a system without the direct link. Others, such as Gomadam and Jafar, proved that the MMSE maximizes the receiver SNR and numerically showed that it is capacity optimal, again for a system without the direct link, [18]. These two functions are very similar in form. Log-likelihood ratio (LLR) functions have also been used as soft information, as an approximation of the MMSE estimate, but proved to encounter difficulties at the detector located at destination, [21]. Hu and Lilleberg in [26] used a weighted type of soft information which mimics the behavior of the MMSE estimate. In this work, we use the MMSE estimate as the relay function because of the properties this transformation has, and because it efficiently incorporates information about the quality of the S to R channel.

The next section introduces the MMSE estimate as a relay function and presents the properties which give intuition on the expected behavior of the EF relay protocol.

## 4.1 Relay function

We assume the relay function for the EF protocol is the MMSE estimate, which is the conditional expectation of the source sent symbol $x$ given the received signal at the relay $y_{SR}$

$$f_{EF}(y_{SR}) = k\mathrm{E}[x|y_{SR}] = k\sum_{x} x\mathrm{p}(x|y_{SR}). \tag{4.1}$$

After applying Bayes' rule, we get

$$f_{EF}(y_{SR}) = k\sum_{x} x\frac{\mathrm{p}(y_{SR}|x)\mathrm{p}(x)}{\mathrm{p}(y_{SR})} = k\frac{\sum_{x} x\mathrm{p}(y_{SR}|x)p(x)}{\sum_{x} p(y_{SR}|x)p(x)}, \tag{4.2}$$

where $k$ is the average transmit power constraint coefficient for the relay. Therefore, for an average transmit power at the relay $P_R$ we can obtain the value of $k$ similarto the AF case:

$$E[|k * E[x|y_{SR}]|^2] \leq P_R \tag{4.3}$$

$$k \leq \sqrt{\frac{P_R}{E[|E[x|y_{SR}]|^2]}} \tag{4.4}$$

where $|\cdot|$ represents the absolute value.

Next, we investigate its characteristics by applying it to different modulation schemes and by comparing it to the common relay functions presented so far. The conditional density function of $y_{SR}$ given the source symbol $x$ is the same as for the DF and AF relay protocols, that is $\mathrm{p}(y_{SR}|x) = \mathcal{N}(h_{SR}x, 1)$. Introducing the conditional probability density function of $y_{SR}$ in (4.2), the EF relay function is further expanded as

$$f_{EF}(y_{SR}) = k\ \frac{\sum\limits_{x} x\mathcal{N}(h_{SR}x, 1)\text{p}(x)}{\sum\limits_{x} \mathcal{N}(h_{SR}x, 1)\text{p}(x)} \tag{4.5}$$

$$= k\ \frac{\sum\limits_{x} x\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(y-h_{SR}x)^2}{2}\right\}\text{p}(x)}{\sum\limits_{x} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(y-h_{SR}x)^2}{2}\right\}\text{p}(x)}. \tag{4.6}$$

### 4.1.1 BSPK modulation

When the source uses antipodal signals, $x \in \{-1, 1\}$, also known as BPSK modulation, the soft information of the EF protocol shown in (4.6) becomes

$$f_{EF}(y_{SR}) = k\ \frac{\text{e}^{h_{SR}y_{SR}} - \text{e}^{-h_{SR}y_{SR}}}{\text{e}^{h_{SR}y_{SR}} + \text{e}^{-h_{SR}y_{SR}}} = k\ \tanh\left(y_{SR}h_{SR}\right). \tag{4.7}$$

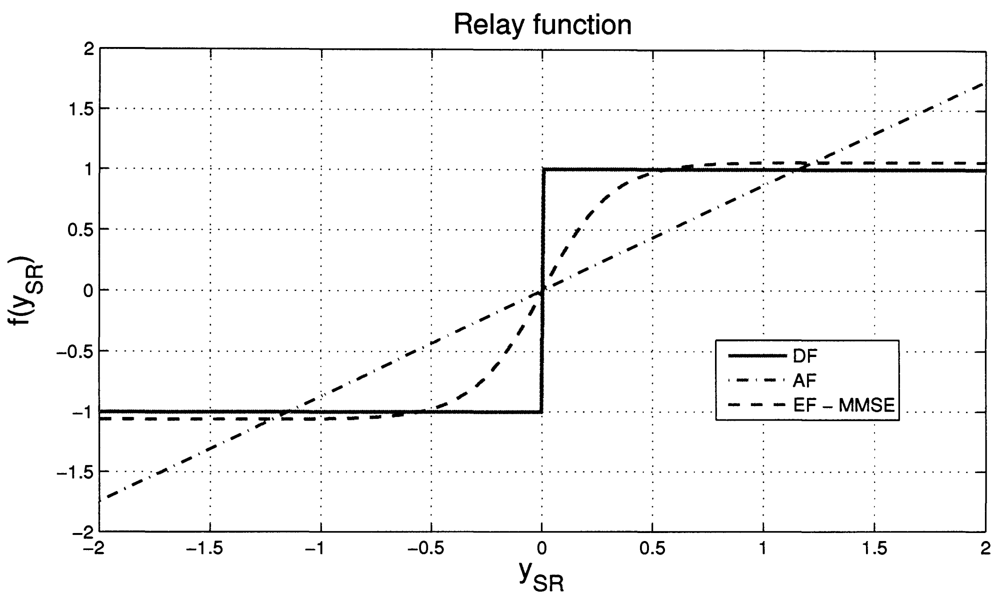


Figure 4.1 : Relay functions for AF, DF and EF; $y_{SR}$ is the received signal at the relay and $f(y_{SR})$ is the relay function. For this plot, we have considered BPSK modulation, $SNR_{SR} = 3dB$ and an average transmit power constraint at the relay $P_R = 1$.

The EF relay function specific for BPSK modulation is shown in Figure 4.1, where both the AF and DF relay functions are also plotted. The MMSE as the EF relay function is denoted as *EF - MMSE*. One important observation made on this graph is that the EF relay function is in between the AF and DF ones. This attracted our attention on the specific form of the MMSE estimate.

We noticed that the quality of the S to R link has great influence on the performance of the EF protocol, as seen in Figure 4.2. We define $SNR$ to represent signal to noise ratio at the receiver. For high values of the $SNR_{SR}$, when the relay is closer to the source, the MMSE estimate becomes very steep and similar to the DF hard decision relay function. While, for low values of the $SNR_{SR}$ (the relay is closer to the

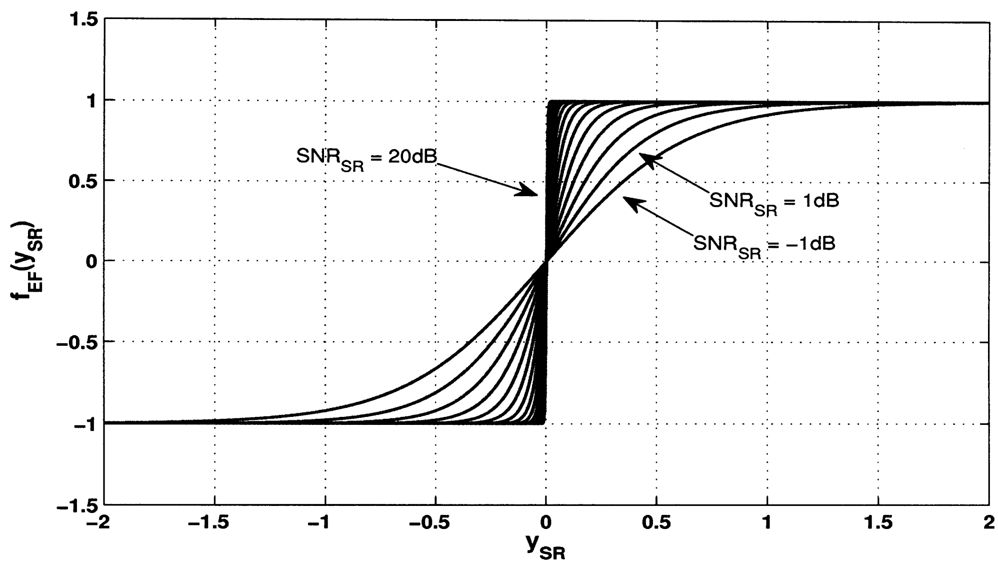


Figure 4.2 : The relay function $f_{EF}(y_{SR})$ plotted for different values of $SNR_{SR}$ from $-1$dB to 20dB.

destination), the EF relay function is similar to the AF one. Thus, we expect that for high SNR on the S to R link, EF will perform as good as DF and for a degraded S to R channel, the performance of the system will be similar to AF.

With this type of soft information, we expect the EF relay protocol to perform as the best of AF and DF. Furthermore, the MMSE estimate inherits one important property from the DF protocol: the instantaneous transmit power at the relay is slightly higher than for DF, but it is much lower than the one the AF protocol requires. This can be observed in Figure 4.1; for $y_{SR} \geq 1.3$ the AF relay protocol requires more instantaneous transmit power than all other schemes.

### 4.1.2 M-QAM modulation

A variety of communications protocols implement quadrature amplitude modulation (QAM), for example current protocols such as 802.11b wireless Ethernet (Wi-Fi) and digital video broadcast (DVB) use 64-QAM modulation. Next, we present the general EF relay function from (4.2) for the case when QAM modulation is used for the source symbol $x$. Specific examples are given for rectangular QAM, such as 16-QAM.

Rectangular QAM is considered when the amplitudes of the carriers are a set of discrete values. This is equivalent to having two orthonormal signals represented with pulse-amplitude modulation (PAM) for the real and imaginary parts of the source transmitted signal. Such representations imply that the real and imaginary parts can be considered as two independent signals. As a result, the relay function applies to each part individually and independently

$$f_{EF}(y_{SR}) = k\mathrm{E}[x|y_{SR}] \tag{4.8}$$

$$= k(\mathrm{E}[\mathrm{Re}(x)|\mathrm{Re}(y_{SR})] + i\mathrm{E}[\mathrm{Im}(x)|\mathrm{Im}(y_{SR})]) \tag{4.9}$$

$$= k\left(f_{EF/RePAM}(\mathrm{Re}(y_{SR})) + i f_{EF/ImPAM}(\mathrm{Im}(y_{SR}))\right) \tag{4.10}$$

where $f_{EF/RePAM}(\mathrm{Re}(y_{SR}))$ and $f_{EF/ImPAM}(\mathrm{Im}(y_{SR}))$ are the notations for the EF relay function applied to the real and imaginary part representation of the sent signal $x$ given the real and imaginary part representation of the received signal at the relay, $y_{SR}$.

We introduce the EF relay function for the real part, which means $f_{EF}$ is applied to PAM modulation. The imaginary part is treated in a similar way. The probability density function of the real part of the received signal at the relay is the same as in the AF and DF relay protocols, that is, $\mathcal{N}(h_{SR}\mathrm{Re}(x), 1)$. Applying it to the EF relay function given in (4.2), it becomes

$$f_{EF}(\mathrm{Re}(y_{SR})) = k\frac{\sum\limits_{\mathrm{Re}(x)} \mathrm{Re}(x)\mathrm{p}(\mathrm{Re}(y_{SR})|\mathrm{Re}(x))\mathrm{p}(\mathrm{Re}(x))}{\sum\limits_{\mathrm{Re}(x)} \mathrm{p}(\mathrm{Re}(y_{SR})|\mathrm{Re}(x))\mathrm{p}(\mathrm{Re}(x))} \tag{4.11}$$

$$= k\frac{\sum\limits_{x} x\mathcal{N}(h_{SR}x, 1)\mathrm{p}(x)}{\sum\limits_{x}\mathcal{N}(h_{SR}x, 1)\mathrm{p}(x)} \tag{4.12}$$

$$= k\frac{\sum\limits_{x} x\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(y-h_{SR}x)^2}{2}\right\}\mathrm{p}(x)}{\sum\limits_{x}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(y-h_{SR}x)^2}{2}\right\}\mathrm{p}(x)} \tag{4.13}$$

where in (4.12) and (4.13) we dropped the $Re()$ notation for clarity, such that $x$ denotes $Re(x)$ and $y_{SR}$ denotes $Re(y_{SR})$. For 16QAM modulation, 4PAM modulation is used for each of the rectangular coordinates which are also known as the In-phase (I) and Quadrature (Q) coordinates of the signal. Applying 4PAM modulation to each coordinate implies that $Re(x) \in \{-3, -1, 1, 3\}$, and similarly for $Im(x)$. Expanding the relay function given in (4.13) for the 4PAM modulation, we obtain the relay function specific for each coordinate that preserves the characteristics of the MMSE estimate obtained for BPSK modulation (4.7) :

$$f_{4PAM}(y) = \sum_{x} x\frac{p(y|x)p(x)}{p(y)} \tag{4.14}$$

$$= \frac{-3\mathrm{e}^{-6h_{SR}y-9h_{SR}^2} - \mathrm{e}^{-2h_{SR}y-h_{SR}^2} + \mathrm{e}^{2hy-h^2} + 3\mathrm{e}^{6h_{SR}y-9h_{SR}^2}}{\mathrm{e}^{-6h_{SR}y-9h_{SR}^2} + \mathrm{e}^{-2h_{SR}y-h_{SR}^2} + \mathrm{e}^{2h_{SR}y+h_{SR}^2} + \mathrm{e}^{6h_{SR}y-9h_{SR}^2}} \tag{4.15}$$

where $y$ represents either the imaginary or real part of $y_{SR}$, and $x$ the equivalent part for source sent symbol.

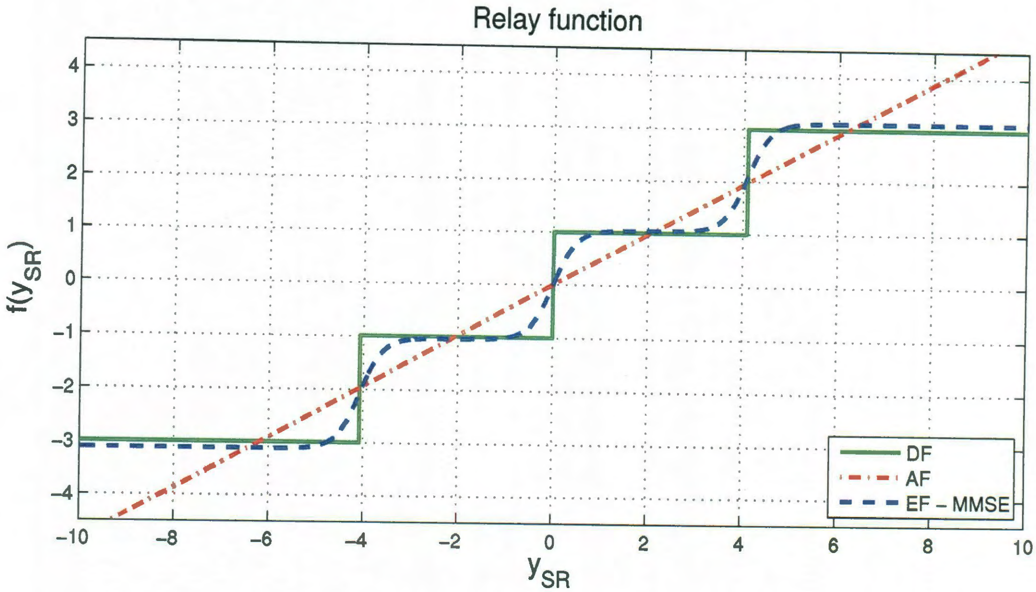


Figure 4.3 : Relay functions for AF, DF and EF; $y_{SR}$ is the received signal at the relay and $f(y_{SR})$ is the relay function. For this plot we have considered 4PAM modulation. The path-loss model is considered with $d = 0.7$ and $\alpha = 4$.

As it can be seen in Figure 4.3, the MMSE estimate for 4PAM modulation has the same characteristics as for BPSK modulation, in comparison with the relay functions for AF and DF. Therefore, we showed that using higher order modulation for the source symbol $x$ maintains the MMSE estimate properties.

Next, we turn our attention to the detector at the destination and its characteristics.

## 4.2 Detector at the destination

As stated previously, the detector is the same for all cases analyzed. Therefore, the general form of the MAP detector for the EF protocol is independent on the modulation of the source signal and it is given in (3.7), with its specific form for BPSK shown in (3.16). The detector at the destination requires the density $p(y_{SD}|x)$, which is the same for all three relaying protocols analyzed. The conditional probability density function of $y_{RD}$ is dependent on the specificity of the relay function and introduces differences in the destination detector.

For the EF relaying protocol, as a result of (3.3), one way to obtain $\mathrm{p}(y_{RD}|x)$ is through the convolution of the probability density function of the relay function and the Gaussian density of the noise, as shown bellow

$$\mathrm{p}(y_{RD}|x) = \mathrm{p}(f_{EF}(y_{SR})|x) * \mathrm{p}(n_{RD}|x). \tag{4.16}$$

Once the two densities are defined, a decision rule can be implemented at the destination following the form of the MAP detector given in (3.7). However, there is no analytical form for the conditional probability density function of $y_{RD}$ and therefore no analytical form for the detector at the destination. Next, we analyze the probability density function of the MMSE estimate to gain insight into its characteristics.

### 4.2.1 Pdf of the MMSE estimate

One can obtain the probability density function of the signal sent by the relay, $\mathrm{p}(f_{EF}(y_{SR})|x)$, using the formula for the density of a function of a random variable shown in equation 4.18 (also see chapter x from [30]). Using this method, the conditional probability density function of the conditional expectation is derived.

For example, for BPSK modulation with the notation

$$z = E[x|y_{SR}] = f(y_{SR}) \quad \Rightarrow \quad y_{SR} = f^{-1}(z) \tag{4.17}$$

we derive

$$\begin{aligned} \mathrm{p}_{Z|X}(z|x) &= \mathrm{p}_{Y_{SR}|X}(f^{-1}(z)|x) \cdot (f^{-1})'(z) && (4.18)\\ &= \mathrm{p}_{Y_{SR}|X}\left(\frac{1}{2h_{SR}} \ln\left(\frac{1+z}{1-z}\right)\middle| x\right) \frac{1}{h_{SR}} \frac{1}{1-z^2}. && (4.19) \end{aligned}$$

Now, $p_{Z|X}(z|x)$ is defined in terms of the probability density function of $y_{SR}$, which is know. Therefore, we obtain the following closed form for the conditional pdf of the MMSE estimate for BPSK modulation

$$\mathrm{p}_{Z|X}(z|x) = \frac{1}{h_{SR}(1-z^2)} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left[ \frac{1}{2h_{SR}} \ln\left( \frac{1+z}{1-z} \right) - h_{SR}x \right]^2 \right\} \quad (4.20)$$

where, as denoted in (4.17), $z = \mathrm{E}[x|y_{SR}]$.

The probability density function given in (4.20) has finite support and is presented in Figure 4.4, where we have only shown the corresponding pdf for $x = 1$, as the pdf of $\mathrm{E}[x|y_{SR}]$ for $x = -1$ is symmetric with respect to the y-axis. Notice that for lower $SNR$ values of the S - R channel, the probability density function has a Gaussian form, but as $SNR_{SR}$ increases, the Gaussian similarities disappear.

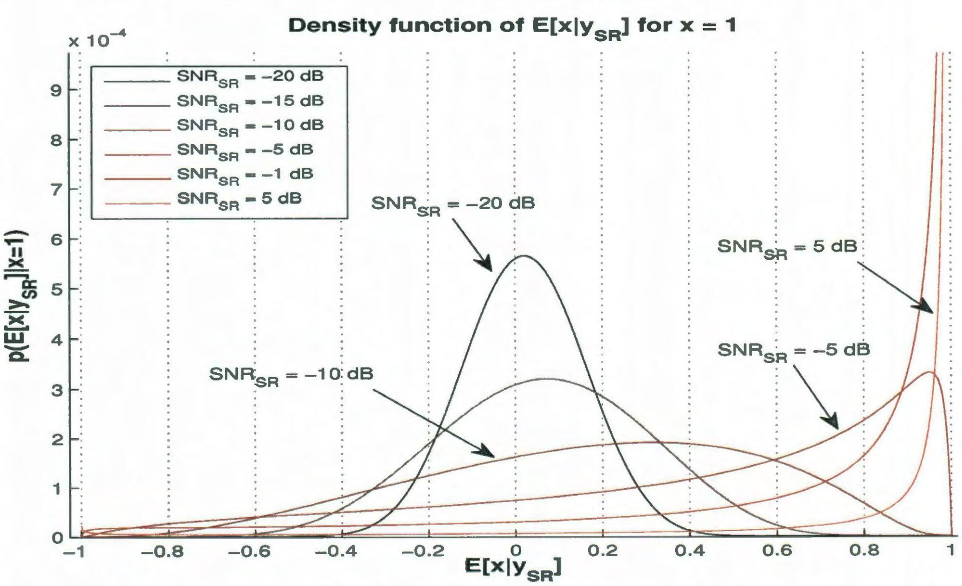


Figure 4.4 : Probability density function of $\mathrm{E}[x|y_{SR}]$ for $x = 1$, for different values of $SNR_{SR}$.

Using a similar approach, the probability density function of the MMSE estimate for higher order modulation can be obtained.

The similarities between p($f_{EF}(y_{SR})|x$) and a Gaussian probability density function explain why when a Gaussian approximation has been used for this probability density function, good performance was obtained withing a specific range of the $SNR_{SR}$. The range of $SNR_{SR}$ on which a Gaussian approximation might perform well can be determined by analyzing the values of the 4th and 5th moment generating functions of the pdf.

However, the conditional probability density function of $y_{RD}$ has to be computed numerically and thus for BPSK modulation the detector at the destination can not be reduced further than the general form given in (3.16), that is,

$$\frac{1}{2}h_{SD}y_{SD} \underset{H_0}{\overset{H_1}{\gtrless}} \ln(p(y_{RD}|x=-1)) - \ln(p(y_{RD}|x=1)). \qquad (4.21)$$

The same remarks hold for higher order modulation. Therefore, the convolution needs to be done numerically, which implies that a numerical approximation of the density is required at the destination side.

Furthermore, to use such a detector, the destination will need to know information about the R to D and S to D links. It will also need to know extra information about the S to R link, similar to the AF protocol. From (4.16), the density p($y_{RD}|x$) is the convolution of the relay function and the noise. The density of the EF relay function depends on the S to R channel characteristics. Thus, the quality of the S to R channel is embedded in the density of the received signal at the relay, $y_{RD}$. Therefore the detector for EF requires channel state information about all links in the network. This overhead is more than it is required for the DF protocol but it is the same as for the AF relay protocol.

The EF detector at the destination needs to have a numerically computed pdf $p(y_{RD}|x)$, which makes this approach less appealing for a practical implementation.

Therefore, in the following, we propose a solution to bypass the need of a numerical probability density function.

## 4.3 Piecewise linear approximation

The solution we propose for overcoming the need of a numerical density function is to use a piecewise linear approximation of the conditional expectation. The soft information sent by the relay is composed of multiple linear functions, defined as:

$$f_{lin}(y_{SR}) = k \cdot \begin{cases} a_1 y_{SR} + b_1, & y_{SR} \in (-\infty, y_1) \\ a_2 y_{SR} + b_2, & y_{SR} \in (y_1, y_2) \\ \dots & \\ a_i y_{SR} + b_i, & y_{SR} \in (y_{i-1}, y_i) \\ \dots & \\ a_{n+1} y_{SR} + b_{n+1}, & y_{SR} \in (y_n, +\infty) \end{cases} \tag{4.22}$$

where $k$ is the transmit power constraint coefficient

$$k \leq \sqrt{\frac{P_R}{E\left[|f_{lin}(y_{SR})|^2\right]}}. \tag{4.23}$$

For $i \in (2, n+1)$, the coefficients $a_i$ and $b_i$ are dependent on $y_{i-1}, y_i, f_{EF}(y_{i-1})$ and $f_{EF}(y_i)$ as follows:

$$a_i = \frac{f_{EF}(y_i) - f_{EF}(y_{i-1})}{y_i - y_{i-1}} \tag{4.24}$$

$$b_i = \frac{f_{EF}(y_{i-1})y_i - f_{EF}(y_i)y_{i-1}}{y_i - y_{i-1}}. \tag{4.25}$$

In (4.22), (4.24) and (4.25), the points $y_{i-1}$ and $y_i$ belong to the set of $n$ points $\{y_1 < y_2 < \dots < y_n\}$ on the real line, called *knots* (see [32], lecture 11). The knots are chosen to minimize the $\mathbb{L}_1$ norm, which is equivalent to minimizing the area

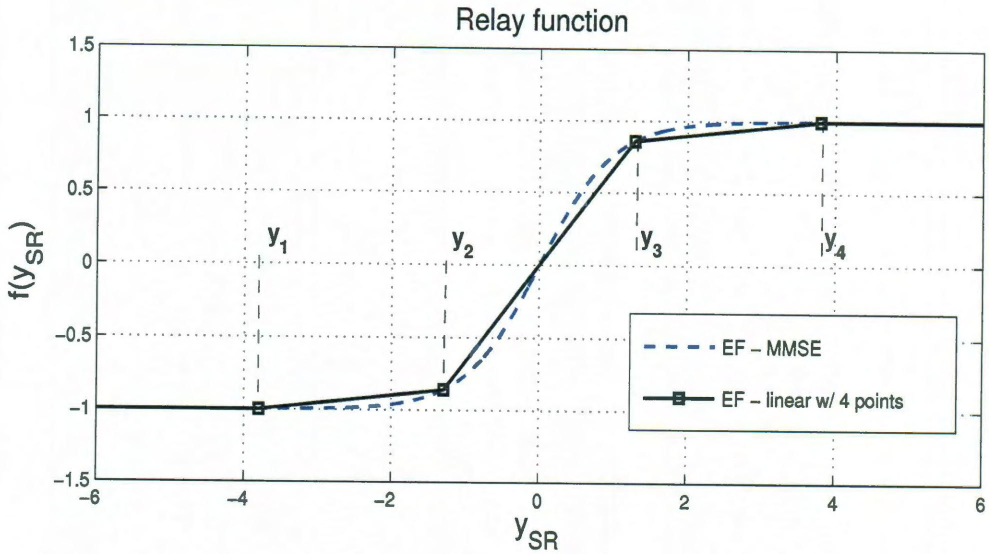


Figure 4.5 : Relay functions for EF with MMSE estimate and EF with linear approximation; $y_{SR}$ is the received signal at the relay and $f(y_{SR})$ is the relay function. For this plot, we have considered BPSK modulation, $SNR_{SR} = 3dB$ and an average transmit power constraint at the relay $P_R = 1$.

between the two relay functions:

$$\min_{\text{set}\{y_i\}_{i=1\ldots n}} \int_{-\infty}^{+\infty} |f_{EF}(y_{SR}) - f_{lin}(y_{SR})| \, \mathrm{d}y_{SR} \tag{4.26}$$

The accuracy of the approximation depends on the number of knots chosen. For example, for BPSK modulation, the relay function $f_{lin}(y_{SR})$ is shown in Figure 4.5, for a four-knot (or five-segment) approximation.

The same function can be compared with the corresponding functions for AF and DF, see Figure 4.6. Since it is a good approximation of the MMSE estimate, we expect the behavior of the system to be similar. Compared to the conditional expectation, this relay function requires less resources at the relay, as it is just a linear transformation of the received signal.

Due to the linearity of the approximation, the relay function maintains the Gaus-

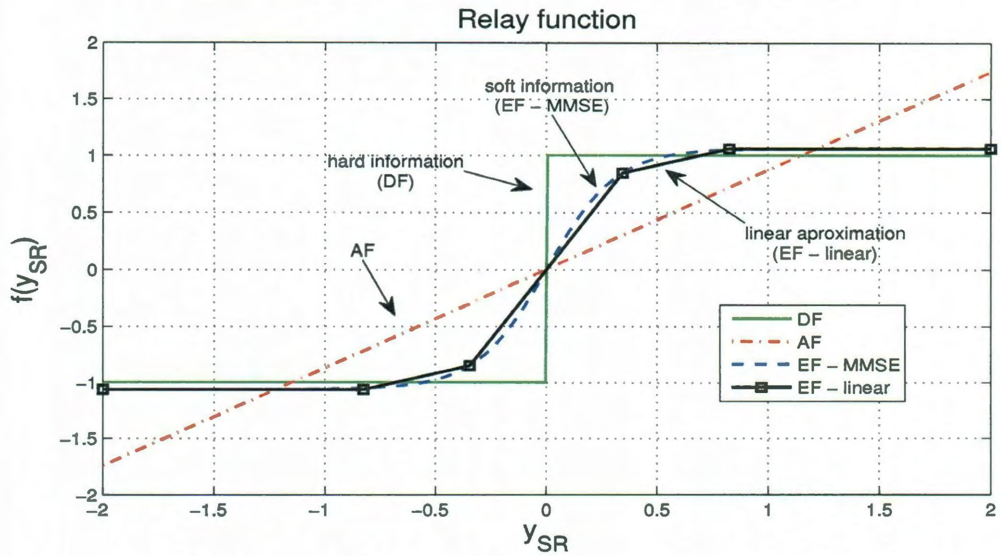


Figure 4.6 : Relay functions for AF, DF, EF - MMSE and EF - linear; $y_{SR}$ is the received signal at the relay and $f(y_{SR})$ is the relay function. For this plot, we have considered BPSK modulation, $SNR_{SR} = 3dB$ and an average transmit power constraint at the relay $P_R = 1$.

sian property of $y_{SR}$. The received symbol at the destination becomes:

$$y_{RD} = \begin{cases} h_{RD}k(a_1 y_{SR} + b_1) + n_{RD}, & y_{SR} \in (-\infty, y_1) \\ \dots & \\ h_{RD}k(a_i y_{SR} + b_i) + n_{RD}, & y_{SR} \in (y_{i-1}, y_i) \\ \dots & \\ h_{RD}k(a_{n+1} y_{SR} + b_{n+1}) + n_{RD}, & y_{SR} \in (y_n, +\infty) \end{cases} \tag{4.27}$$

with $i \in (2 \dots n)$. Since the system uses instantaneous relaying and the relay function $f_{lin}(y_{SR})$ is deterministic, the density $\mathrm{p}(y_{RD}|x)$ is given by:

$$\mathrm{p}(y_{RD}|x) = \int_{-\infty}^{+\infty} \mathrm{p}(y_{RD}|y_{SR})\mathrm{p}(y_{SR}|x)\mathrm{d}y_{SR}. \tag{4.28}$$

The signal $y_{RD}$ has specific forms for different regions with respect to $y_{SR}$, as seen in (4.27). This imposes a similar form for the density of $y_{RD}$ given that $y_{SR}$ is known.

When the noise is AWGN this density is:

$$\mathrm{p}(y_{RD}|y_{SR}) = \begin{cases} \mathcal{N}(m_1, 1), & y_{SR} \in (-\infty, y_1) \\ \dots & \\ \mathcal{N}(m_i, 1), & y_{SR} \in (y_{i-1}, y_i) \\ \dots & \\ \mathcal{N}(m_{n+1}, 1), & y_{SR} \in (y_n, +\infty) \end{cases} \tag{4.29}$$

where the means of the Gaussian densities $m_i$ are given by:

$$m_i = h_{RD} k(a_i y_{SR} + b_i). \tag{4.30}$$

In (4.28), we substitute both the density $\mathrm{p}(y_{RD}|y_{SR})$ given in (4.29) and the probability density function $\mathrm{p}(y_{SR}|x)$ given by $\mathcal{N}(h_{SR}x, 1)$. We can now compute the analytical form of the density $\mathrm{p}(y_{RD}|x)$. After simplifying and rearranging the variables (intermediate steps are shown in Appendix A), the probability of the signal sent from the relay and received at the destination becomes*

$$\begin{aligned} \mathrm{p}(y_{RD} \mid x) = \\ &= \frac{1}{\sqrt{2\pi\sigma^2_{rd(1)}}} \exp\left\{-\frac{\left(y_{RD} - m_{rd(1)}\right)^2}{2\sigma^2_{rd(1)}}\right\} Q\left(-\frac{y_1 - m_{sr(1)}}{\sigma_{sr(1)}}\right) \\ &+ \sum_{i=2}^{n} \frac{1}{\sqrt{2\pi\sigma^2_{rd(i)}}} \exp\left\{-\frac{\left(y_{RD} - m_{rd(i)}\right)^2}{2\sigma^2_{rd(i)}}\right\} \left[Q\left(\frac{y_{i-1} - m_{sr(i)}}{\sigma_{sr(i)}}\right) - Q\left(\frac{y_i - m_{sr(i)}}{\sigma_{sr(i)}}\right)\right] \\ &+ \frac{1}{\sqrt{2\pi\sigma^2_{rd(n+1)}}} \exp\left\{-\frac{\left(y_{RD} - m_{rd(n+1)}\right)^2}{2\sigma^2_{rd(n+1)}}\right\} Q\left(\frac{y_n - m_{sr(n+1)}}{\sigma_{sr(n+1)}}\right) \end{aligned} \tag{4.31}$$

where the means $m_{sr(i)}$ and $m_{rd(i)}$ and variances $\sigma^2_{sr(i)}$ and $\sigma^2_{rd(i)}$ depend on the cor-

*We define the $Q$-function to be $Q(\alpha) = \int_{\alpha}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}} d\omega$.

responding intervals through $a_i$ and $b_i$ as shown bellow:

$$m_{sr(i)} = \frac{h_{RD}ka_i(y_{RD} - h_{RD}kb_i) + h_{SR}x}{1 + h_{RD}^2k^2a_i^2} \tag{4.32}$$

$$\sigma_{sr(i)}^2 = \frac{1}{1 + h_{RD}^2k^2a_i^2} \tag{4.33}$$

$$m_{rd(i)} = h_{RD}k(b_i + h_{SR}xa_i) \tag{4.34}$$

$$\sigma_{rd(i)}^2 = 1 + h_{RD}^2k^2a_i^2. \tag{4.35}$$

In previous notations we used the subscript $rd(i)$ to denote that the mean/variance which pertains to a Gaussian density function of the random variable indicated by the subscript, in this case that is $y_{RD}$. Also, the subscript $(i)$ denotes that the notation is dependent on the interval $(y_{i-1}, y_i)$. Furthermore, the form of the conditional pdf given in (4.31) can be interpreted as a sum of the probability of $y_{RD}$ being received at the destination, given that the signal sent by the relay is in a specific region $(y_{SR} \in (y_{i-1}, y_i))$, scaled by the probability of being in that interval. The destination will need to be synchronized with the relay such that it will know exactly what knots are chosen for the approximation.

The detector for the EF protocol shown in (4.21) has analytical form and can be implemented using the density of the piecewise linear approximation given in (4.31).

A higher performance requirement implies the need for a better approximation, which can be obtained using a higher number of knots. However, a higher number of knots increases the complexity of the density and thus of the detector at the destination. Depending on the system requirements, there will always be a trade-off between the complexity of the detector and the accuracy of the approximation.

# Chapter 5

# Simulation results

In this chapter, we evaluate the proposed relay protocol for the three-node network with the pathloss system model described in Chapter 3. We use the Symbol-error-rate (SER) as the metric to evaluate the performance of each relaying protocol.

Simulation results for the EF protocol with MMSE estimate were possible as the probability density function p($y_{RD}|x$) was evaluated numerically. Although this pdf does not have an analytical form, using Monte Carlo simulations we obtained a good numerical approximation of it. The numerical form of this density gave us further insight into the influence of the MMSE estimate on the performance of the EF relaying protocol.

For example, for BPSK modulation, the conditional probability density function of $y_{RD}$ is presented in Figure 5.1. The shape of the density is directly influenced by the position of the relay relative to the source to destination distance. As the relay gets closer to the source, the shape of the density becomes closer in form to a Gaussian density, as seen for densities for $d = \{0.37, 0.46\}$. In addition, as the relay is farther away from the source, the pdf of $y_{RD}$ takes the form of a heavy tail density and it losses its Gaussian shape; notice the density plotted for $d = 0.75$. As it can be seen in detail in Figure 5.1, apart from the heavy tail form, this density also seems to have a bump. This form of the density motivated the search for an approximation, which came from using the piecewise linear approximation.

The same characteristics of the probability density function p($y_{RD}|x$) are observed

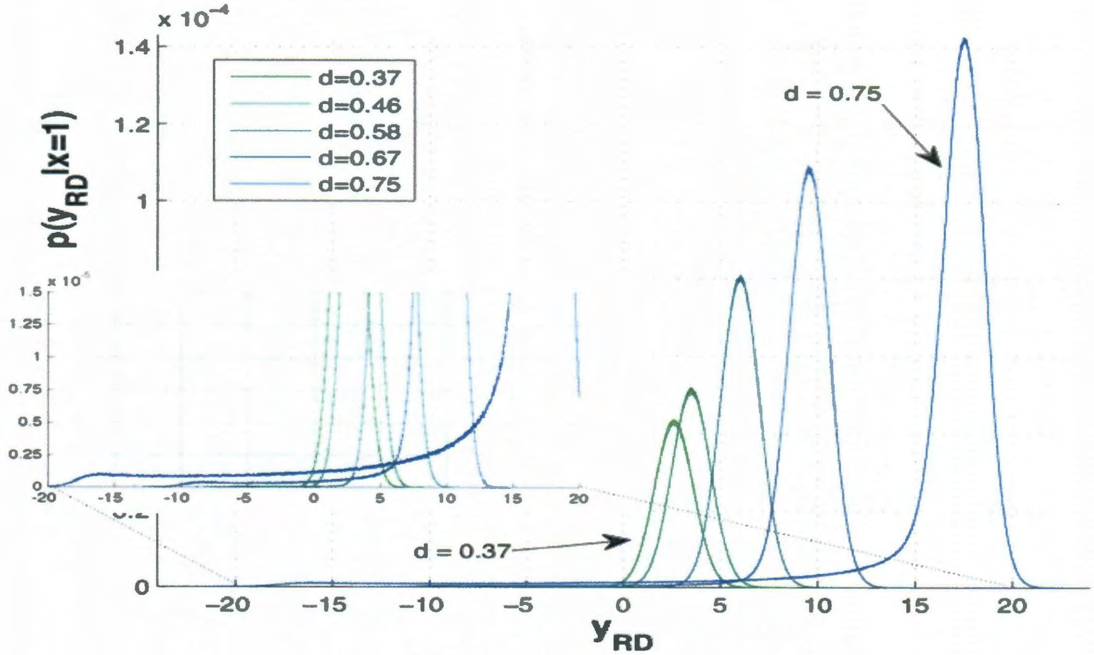


Figure 5.1 : Pdf p($y_{RD}|x = 1$) for the EF relay protocol with the MMSE estimate. The density is shown for different positions of the relay with respect to the source (different $d$).

for the case when the source modulation is increased to 16 QAM.

When the source uses BPSK modulation the symbol-error-rate (SER) is the same as the bit-error-rate (BER), as BPSK modulation implies that one bit is used to represent one symbol. In Figure 5.2, the BER for the presented protocols has been plotted on a logarithmic scale versus $d$ - the relative distance between source and relay. First, notice that compared with the results for the direct link, adding a relay to the communication improves the performance of the system. Secondly, the BER curve for EF with MMSE, denoted as *EF* - $E[x|y_{SR}]$ confirms the intuition suggested by the behavior of the EF relay function - $f_{EF}$, seen in Figure 4.2. For a good channel S to R, which means the relay is closer to the source ($d \in (0, 0.6)$), EF performs as well as the DF protocol. When the S to R channel is not so good, meaning the relay

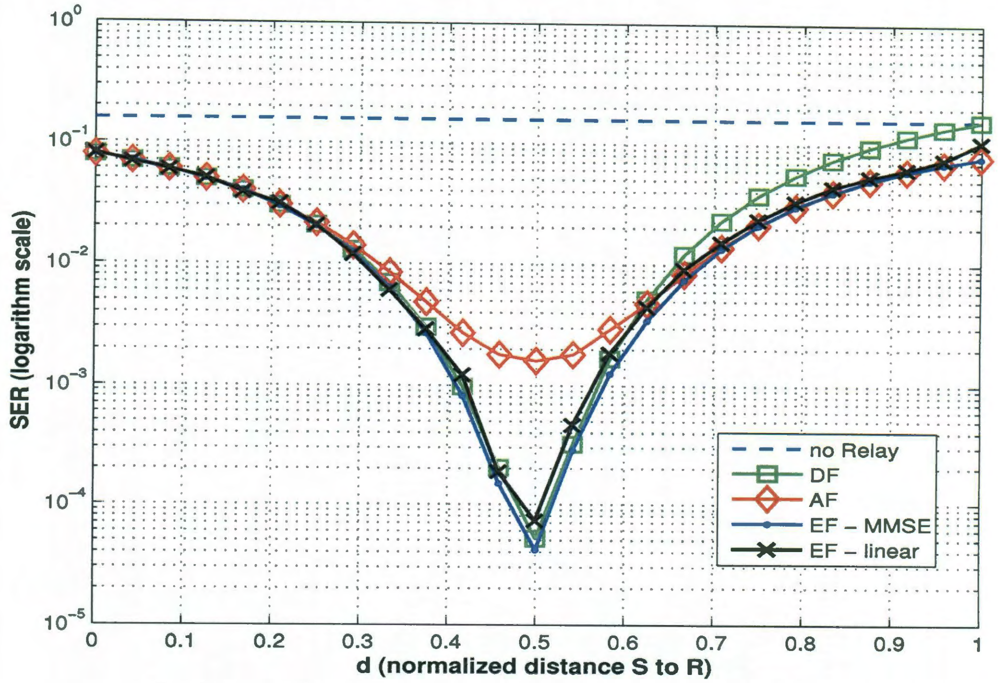


Figure 5.2 : BER for AF, DF and EF protocols for BPSK modulation for the pathloss model. On the x-axis, $d$ represents the normalized distance between S and R.

is closer to the destination ($d > 0.6$), EF behaves like the AF protocol. *Therefore, with only one scheme (EF), the system performs as well as the best of the two other schemes (AF and DF).*

In Figure 5.2, the curve *EF linear* denotes the EF relaying protocol with the relay function $f_{EF}$, where the density used for the detector is the one obtained from the piecewise linear approximation. The density $p(y_{RD}|x)$ used for the BER curve for the *EF linear* was obtained for a piecewise linear approximation with four knots (or five segments). With our linear approximation, the performance of the system remains similar to the case of EF with MMSE using a numerical approximation of the probability density function of $y_{RD}$.

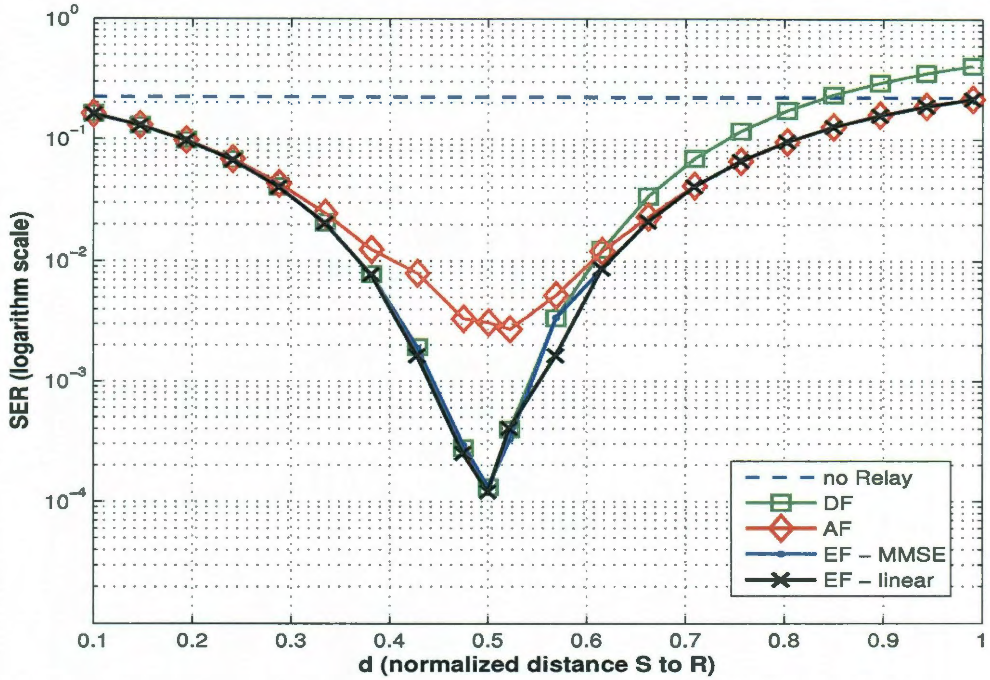


Figure 5.3 : SER for AF, DF and EF protocols for 16QAM modulation for the pathloss model. On the x-axis, $d$ represents the normalized distance between S and R.

There is a small loss in performance for the case when R is closer to D, when $d > 0.7$. This is due to using an approximation of the density $p(y_{RD}|x)$. The performance of the system is strictly related to the accuracy of the piecewise linear approximation used to derive the density of $y_{RD}$.

In Figure 5.3, the SER is plotted for all analyzed relaying protocols. The line denoted as *no Relay*, represents the SER obtained for the direct link, when no cooperation is employed. For fairness of the comparison, we assumed the power used by the source is equal to the total power used in the cooperative case, that is, equal to source power($P_S$) plus relay power($P_R$). In the case of increasing the source modulation order to 16 QAM, the same trends are observed for the *EF - MMSE* and *EF -*

*linear* as when BPSK modulation is used. The piecewise linear approximation used for the 16 QAM SER results employed a 20-knot discretization.

The properties of the MMSE estimate are preserved for higher order modulation as expected from the way the MMSE behaves, see Figure 4.3. Also, the linear piecewise approximation of the MMSE estimate preserves the properties of the MMSE, its overall performance follows the best of the AF and DF.

# Chapter 6

# Conclusions

In this work we have investigated the performance of the EF protocol with MMSE estimate as the relay function. In a pathloss model and for both BPSK and 16 QAM modulation, EF with this type of soft information performs as well as the best of AF and DF. When R is close to S, EF performs like DF, while when R is closer to D, it performs like AF. However, this performance comes at a price; the conditional probability density function of the received signal at the source from the relay, $p(y_{RD}|x)$ does not have an analytical form. Therefore, the detector at the destination can be implemented only by using a density obtained through numerical computation. We have shown that the need for this density can be avoided by employing a piecewise linear approximation of the MMSE estimate as the relay function. With this linear approximation, the density of the received symbol at the destination has an analytical form and thus, so does the detector. In conclusion, we do not only offer another type of soft information which can be used in the EF relay protocol, we also provide an analytical form of the optimal detector at the destination. Simulation results with SER as the performance metric confirmed the theoretical results.

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# Appendix A

# Derivation of $\mathbf{p}(y_{RD}|x)$

In this Appendix, we focus on presenting in detail the steps for deriving the final form of the pdf $\mathrm{p}(y_{RD}|x)$. This density has slightly different forms depending on the region of integration.

Using the conditional probability density function of $y_{RD}$ given $y_{SR}$ from equation (4.29) in (4.28) we obtain the density,

$$\begin{aligned}\mathrm{p}(y_{RD}|x) = &\int_{-\infty}^{y_1} \mathcal{N}(m_1, 1)\mathcal{N}(h_{SR}x, 1)\mathrm{d}y_{SR} \\ &+\sum_{i=2}^{n}\int_{y_{i-1}}^{y_i} \mathcal{N}(m_i, 1)\mathcal{N}(h_{SR}x, 1)\mathrm{d}y_{SR} \\ &+\int_{y_n}^{+\infty} \mathcal{N}(m_{n+1}, 1)\mathcal{N}(h_{SR}x, 1)\mathrm{d}y_{SR}\end{aligned} \tag{A.1}$$

where $m_i$ is the mean given in (4.30). This density is split into three different cases, depending on the region of integration. Two cases are unique, the ones corresponding to the extremities, $y_{SR} \in (-\inf, y_1)$ and $y_{SR} \in (y_n, +\inf)$, which are particular cases of the most general one given by $y_{SR} \in (y_{i-1}, y_i)$. We focus on the case when $y_{SR}$ is between two knots, that is $y_{SR} \in (y_{i-1}, y_i)$. This case is the most general one, and all other regions represent particular cases of this one. We focus on the general case, which is equivalent to the following part of the density given in (A.1), where we used the mean $m_i$ given in (4.30) :

$$\mathrm{p}(y_{RD}|x) = \int_{y_{i-1}}^{y_i} \mathcal{N}(h_{RD}k(a_iy_{SR} + b_i), 1)\mathcal{N}(h_{SR}x, 1)\mathrm{d}y_{SR}, \text{ for } y_{SR} \in (y_{i-1}, y_i). \tag{A.2}$$

Next, we expand the Gaussian densities:

$$p(\ y_{RD}|x\ ) = \tag{A.3}$$
$$\int_{y_{i-1}}^{y_i} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(y_{RD} - h_{RD}k(a_i y_{SR} + b_i))^2}{2}\right\} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(y_{SR} - h_{SR}x)^2}{2}\right\} dy_{SR}$$
$$= \int_{y_{i-1}}^{y_i} \left(\frac{1}{\sqrt{2\pi}}\right)^2 \exp\left\{-\frac{1}{2}\left[y_{RD}^2 - 2y_{RD}h_{RD}k(a_i y_{SR} + b_i) + h_{RD}^2 k^2 (a_i y_{SR} + b_i)^2 \right.\right.$$
$$\left.\left. + \left(y_{SR}^2 - 2h_{SR}x y_{SR} + h_{SR}^2 x^2\right)\right]\right\} dy_{SR}. \tag{A.4}$$

The next step is to group the terms with respect to the powers of $y_{SR}$ and use the following notation :

$$A_i = \frac{1}{h_{RD}^2 k^2 a_i^2 + 1} \tag{A.5}$$
$$B_i(y_{RD}, x) = h_{RD} k a_i (y_{RD} - h_{RD} k b_i) + h_{SR} x \tag{A.6}$$
$$C_i(y_{RD}, x) = (y_{RD} - h_{RD} k b_i)^2 + h_{SR}^2 x^2. \tag{A.7}$$

With this notation, the probability density function for this region becomes:

$$p(y_{RD}|x) = \tag{A.8}$$
$$\int_{y_{i-1}}^{y_i} \left(\frac{1}{\sqrt{2\pi}}\right)^2 \exp\left\{-\frac{1}{2}C_i(y_{RD}, x)\right\} \exp\left\{-\frac{1}{2}\left(\frac{1}{A_i} y_{SR}^2 - 2B_i(y_{RD}, x) y_{SR}\right)\right\} dy_{SR}.$$

Since the exponent of the first term does not depend on $y_{SR}$, it can be taken outside the integral and we complete the square with respect to variable $y_{SR}$:

$$p(y_{RD}|x) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \exp\left\{-\frac{C_i(y_{RD}, x)}{2}\right\} \tag{A.9}$$
$$\cdot \int_{y_{i-1}}^{y_i} \exp\left\{-\frac{(y_{SR} - B_i(y_{RD}, x) A_i)^2}{2A_i} + \frac{B_i^2(y_{RD}, x) A_i}{2}\right\} dy_{SR}.$$

The second term of the exponential does not depend on $y_{SR}$, so it will be taken outside the integral. We are left with a finite integral of a Gaussian density, thus we can express it with respect to the $Q$-function*:

$$\mathrm{p}(y_{RD}|x) \overset{y_{SR}\in(y_{i-1},y_i)}{=} \sqrt{\frac{A_i}{2\pi}} \exp\left\{-\frac{C_i(y_{RD},x) - B_i^2(y_{RD},x)A_i}{2}\right\} \cdot \left[Q\left(\frac{y_{i-1} - B_i(y_{RD},x)A_i}{\sqrt{A_i}}\right) - Q\left(\frac{y_i - B_i(y_{RD},x)A_i}{\sqrt{A_i}}\right)\right]. \tag{A.10}$$

Once we expand the notation introduced in (A.5) to (A.7) and rearrange terms accordingly, we obtain the analytical density of the probability density function

$$\begin{aligned}\mathrm{p}(y_{RD}\,|\,x) &= \\ &= \frac{1}{\sqrt{2\pi\sigma^2_{rd(1)}}} \exp\left\{-\frac{\left(y_{RD} - m_{rd(1)}\right)^2}{2\sigma^2_{rd(1)}}\right\} Q\left(-\frac{y_1 - m_{sr(1)}}{\sigma_{sr(1)}}\right) \\ &+ \sum_{i=2}^{n} \frac{1}{\sqrt{2\pi\sigma^2_{rd(i)}}} \exp\left\{-\frac{\left(y_{RD} - m_{rd(i)}\right)^2}{2\sigma^2_{rd(i)}}\right\} \left[Q\left(\frac{y_{i-1} - m_{sr(i)}}{\sigma_{sr(i)}}\right) - Q\left(\frac{y_i - m_{sr(i)}}{\sigma_{sr(i)}}\right)\right] \\ &+ \frac{1}{\sqrt{2\pi\sigma^2_{rd(n+1)}}} \exp\left\{-\frac{\left(y_{RD} - m_{rd(n+1)}\right)^2}{2\sigma^2_{rd(n+1)}}\right\} Q\left(\frac{y_n - m_{sr(n+1)}}{\sigma_{sr(n+1)}}\right)\end{aligned} \tag{A.11}$$

where the means $m_{sr(i)}$ and $m_{rd(i)}$ and variances $\sigma^2_{sr(i)}$ and $\sigma^2_{rd(i)}$ depend on the corresponding intervals through $a_i$ and $b_i$, as shown bellow:

$$m_{sr(i)} = \frac{h_{RD}ka_i(y_{RD} - h_{RD}kb_i) + h_{SR}x}{1 + h^2_{RD}k^2a_i^2} \tag{A.12}$$

$$\sigma^2_{sri} = \frac{1}{1 + h^2_{RD}k^2a_i^2} \tag{A.13}$$

$$m_{rd(i)} = h_{RD}k(b_i + h_{SR}xa_i) \tag{A.14}$$

$$\sigma^2_{rd(i)} = 1 + h^2_{RD}k^2a_i^2. \tag{A.15}$$

The cases for the end intervals are derived from the general case, by setting the corresponding upper or lower bound of the $y_{SR}$ interval to $+\infty$ or $-\infty$.

*We define the $Q$-function to be $Q(\alpha) = \int_\alpha^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}} \mathrm{d}\omega$.