BOREL-CANTELLI SEQUENCES

By

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Abstract. A sequence $\{x_n\}_1^\infty$ in the unit interval $[0, 1) = \mathbb{R}/\mathbb{Z}$ is called **Borel-Cantelli**, or **BC**, if for all non-increasing sequences of positive real numbers $\{a_n\}$ with $\sum_{i=1}^\infty a_i = \infty$, the set

$$\{x \in [0, 1) \mid |x - x_n| < a_n \text{ for infinitely many } n \geq 1\}$$

has full Lebesgue measure. (Speaking informally, BC sequences are sequences for which a natural converse to the Borel-Cantelli Theorem holds).

The notion of BC sequences is motivated by the monotone shrinking target property for dynamical systems, but our approach is from a geometric rather than dynamical perspective. A sufficient condition, a necessary condition and a necessary and sufficient condition for a sequence to be BC are established. A number of examples of BC sequences and sequences that are not BC are also presented.

The property of a sequence to be BC is a delicate Diophantine property. For example, the orbits of a pseudo-Anosoff IET (interval exchange transformation) are BC, while the orbits of a "generic" IET are not.

The notion of BC sequences is extended from [0, 1) to sequences in Ahlfors regular spaces.

1 Set up

Denote by $\mathcal{I} = [0, 1) = \mathbb{R}/\mathbb{Z}$ the unit interval and by λ Lebesgue measure on $\mathcal{I}$. For $r > 0$ and $a \in \mathcal{I}$, denote by $B(a, r)$ the r-ball around a (taken mod 1, so that $\lambda(B(c, r)) = \min(2r, 1)$). For $c \in \mathbb{R}$, let $\langle\langle c \rangle\rangle = c - \lfloor c \rfloor \in \mathcal{I}$ denote the fractional part of c (or c mod 1).

By a **standard sequence**, we mean a non-increasing sequence $\mathbf{a} = \{a_n\}_1^\infty$ of positive real numbers such that $\sum_{n=1}^\infty a_n = \infty$.

Definition 1. A sequence $\mathbf{x} = \{x_n\}_1^\infty$ in $\mathcal{I} = [0, 1)$ is called **Borel-Cantelli** or BC, if $\lambda\big(\limsup_{n\to\infty} B(x_n, a_n)\big) = 1$ for every standard sequence $\mathbf{a} = \{a_n\}_1^\infty$.

Recall that $\limsup_{n\to\infty} B(x_n, a_n) = \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty B(x_n, a_n)$ denotes the set of points lying in infinitely many $B(x_n, a_n)$.

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Observe that if $\sum_{i=1}^{\infty} a_i < \infty$, then $\lambda(\limsup_{n\to\infty} B(x_n, a_n)) = 0$ by the Borel-Cantelli Theorem. Also note that if we fix a dense sequence $\mathbf{x}$ and let

$$a_i = \begin{cases} 1/5, & \text{if } 2/5 < x_i < 3/5, \\ 0, & \text{otherwise,} \end{cases}$$

then $\sum_{i=1}^{\infty} a_i = \infty$ but $\lambda(\limsup_{n\to\infty} B(x_n, a_n)) = \lambda((1/5, 4/5)) < 1$. This example shows that to obtain a meaningful definition of Borel-Cantelli sequences, one must restrict the choice of radii in some way beyond the standard condition in the Borel-Cantelli Theorem.

The approach we follow (of restricting to non-increasing radii, or targets) works: the added restriction is mild and, on the other hand, many sequences are shown to be Borel-Cantelli. Note that this approach is also natural in the context of dynamical systems, as seen by the monotone shrinking target property (MSTP); see the survey paper [1].

Definition 2. A λ measure preserving map $T\colon \mathcal{I} \to \mathcal{I}$ is said to satisfy the **monotone shrinking target property (MSTP)** if for any standard sequence $\mathbf{a}$ and any $y \in \mathcal{I}$, $\lambda\big(\limsup_{n\to\infty} T^{-n}(B(y, a_n))\big) = 1$.

We can consider a dual property. We say a map $T\colon \mathcal{I} \to \mathcal{I}$ is **absolutely Borel-Cantelli (ABC)** if the forward orbit $\{T^n x\}_{n\ge 0}$ of every $x \in \mathcal{I}$ is BC, i.e., for any $x \in X$ and any standard sequence $\mathbf{a} = \{a_k\}_1^{\infty}$, $\lambda\big(\limsup_{n\to\infty} B(T^n x, a_n)\big) = 1$.

The emphasis in our paper is on abstract sequences not necessarily originating from dynamical systems. We focus on the Borel-Cantelli property (for sequences) as a version of the ABC property (for maps). We do not have a natural candidate for the notion of MSTP for abstract sequences.

Approximation of points in a space by sets has also been considered in the context of regular systems [2] and ubiquitous systems [4]. Some of our results (sufficient conditions Theorems 1 and 4) have been proved more generally in these contexts. In particular, the study of ubiquitous systems considers approximation by sets (instead of just points) and allows for more general targets. This generality leads to definitions much more involved (than BC). At least one natural example of approximation by sets can also be handled by BC sequences (Example 2 in the next section can be thought of as describing approximation of irrationals by rationals based on denominator).

The BC (Borel Cantelli) property is quite delicate, as the following examples suggest.

Let $\alpha \in \mathbb{R}$. If $x_n = \langle\!\langle n\alpha \rangle\!\rangle$, then $\mathbf{x}$ is BC if and only if α is a badly approximable irrational, i.e., if the terms in its continued fraction expansion are bounded; see

[17] and also Example 5. If $x_n = \langle\langle \alpha \log(n) \rangle\rangle$, then $\mathbf{x}$ is BC for any $\alpha \neq 0$; see Example 2. If $x_n = \langle\langle \alpha \sqrt{\log(n)} \rangle\rangle$, then $\mathbf{x}$ is never BC; see Corollary 4.

We also show that a number of natural sequences are BC. These include sequences given by some (but not all) independent identically distributed random variables; see Examples 1, 8 and 13. The Farey sequence of rationals (taken in the natural order) is also BC. (This observation recovers a classic theorem of Khinchin on approximation of irrationals by rationals; see Example 4.) Additionally, $x_n = \langle\langle \sqrt{n} \rangle\rangle$ is BC by the results in [13] concerning the distribution of gaps of this sequence and $x_n = \langle\langle n^2\alpha \rangle\rangle$ is BC for almost every α by weaker results on gaps in [19]; see Remark 5. On the other hand, the same sequence $x_n = \langle\langle n^2\alpha \rangle\rangle$ fails to be BC for a residual set of α, in particular, for all α satisfying $\inf_{n\geq 1} n^3 \langle\langle \alpha n \rangle\rangle = 0$.

We conjecture, and some computer computations suggest, that a large class of sequences like $\{\langle\langle \sqrt[3]{n} \rangle\rangle\}$, $\{\langle\langle n \log(n) \rangle\rangle\}$, and $\{\langle\langle (\log(n))^2 \rangle\rangle\}$ are BC; however, we lack the rigorous methods to validate this conjecture.

In the context of dynamical systems, we show that a mild quantitative rigidity condition makes almost every orbit not BC; see Corollary 3. It follows that for almost every IET T, almost all orbits are not BC; see [9, Theorem 7].

On the other hand, linearly recurrent systems are ABC; see Example 11. The result implies that some exceptional IETs (like pseudo-Anosoff, or self-similar ones) are ABC; see Example 3. In particular, all minimal IETs over quadratic number fields are ABC (because these reduce to a pseudo-Anosoff IET on a subinterval by [6, Proposition 1]).

The main results of the paper are

(1) a frequently checkable sufficient condition for a sequence to be BC (Theorems 1 and 4),
(2) a frequently checkable necessary condition (Theorems 2 and 5), which is phrased as a sufficient condition for a sequence not to be BC,
(3) a necessary and sufficient condition (Theorems 3 and 6).

The first two results and their corollaries help determine whether or not many sequences are Borel-Cantelli. The last condition provides some properties of BC sequences and identifies the properties that govern whether or not a sequence is BC; see Remark 2. These results are proven for $\mathcal{I}$ and then generalized to Ahlfors regular spaces (Section 3). The methods used in this paper are robust and can be applied to other related situations; see Remark 7 and Section 4.

The plan for this paper is to address first the results for sequences in $[0, 1)$ (which are most developed in the dynamical side of the literature). This is done in Section 2. The Borel-Cantelli status of many natural sequences is also addressed in this section. In Section 3, we generalize these results from the unit interval to

Ahlfors regular spaces. We generalize these results to some weaker properties in the Section 4. Then we present some classification results in Section 5. The main tools we use are density point arguments and covering arguments. Throughout the paper, explicit constants are found, though they are not optimal.

2 [0,1) and Lebesgue measure

The following theorem provides a checkable sufficient condition for a sequence to be BC (Borel-Cantelli). The condition is used in the proof of Theorem 3.

Theorem 1. *Let* $\mathbf{x} = \{x_n\}_1^\infty$ *be a sequence in* $\mathfrak{I}$ *and assume that there exists* $d > 0$ *such that* $\liminf_{N\to\infty} \lambda\big(\bigcup_{i=1}^n B(x_i, 1/N) \cap J\big) \geq d\lambda(J)$ *for all intervals* J*. Then* $\mathbf{x}$ *is BC.*

The proof is given after Corollary 6.

Remark 1. This result is analogous to results for regular systems by V. Beresnevich [3]. We include the proof for completeness.

Example 1. If $\{R_n\}$ is a sequence of independent random variables, all distributed according to a probability measure μ that has Radon-Nikodym derivative bounded away from 0, then for $\mu^{\mathbb{N}}$ almost every ζ, the sequence $\{R_n(\zeta)\}$ is BC; see also Example 8 for a more precise result.

It is classical and not hard to show that for any particular sequence of positive reals $\mathbf{a} = \{a_n\}$ (not necessarily monotone) with $\sum_{n=1}^\infty a_n = \infty$, almost every sequence $\{R_n(\zeta)\}$ satisfies $\lambda\big(\limsup_{n\to\infty} B(R_n(\zeta), a_n)\big) = 1$. Theorem 1 states that a full measure set works *simultaneously* for all standard sequences.

Corollary 1. *Let* $\mathbf{x} = \{x_n\}_1^\infty$ *be a sequence in* $\mathfrak{I}$*. If there exists* $D > 0$ *such that the sets* $\mathbf{X_n} = \{x_k \mid 1 \leq k \leq n\}$ *are* D/n*-dense in* $\mathfrak{I}$ *for all large enough* n*, then* $\mathbf{x}$ *is BC.*

Example 2. It follows that if $x_i = \langle\langle \log_c(i) \rangle\rangle$, then $\mathbf{x}$ is BC since the $\mathbf{X_n}$ are roughly $1/(nc\ln c)$ dense.

Example 3. It follows from Corollary 1 that linear recurrent IETs are ABC, i.e., every forward orbit is BC. An interval exchange transformation is called **linearly recurrent** if its symbolic coding is a linearly recurrent subshift; see [12] for an introduction to and basic properties of linearly recurrent subshifts.

Corollary 2. *If* $\mathbf{x}$ *is uniformly distributed and there exists* c *such that*

$$|x_n - x_m| > \frac{c}{\max\{n, m\}},$$

then $\mathbf{x}$ *is BC.*

Example 4. Define the Farey sequence by the rational numbers in $\mathcal{I}$ arranged in the order

$$\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \dots\right\}.$$

Corollary 2 implies that the Farey sequence is BC. (Note that p/q is the $O(q^2)$-th term in this sequence). The fact that the Farey sequence is BC easily implies Khinchin's classic theorem which states that

$$\lambda\left(\left\{\alpha \in \mathcal{I} : \left|\alpha - \frac{p}{q}\right| < \frac{a_q}{q} \text{ for infinitely many } q\right\}\right) = 1$$

for any standard sequence $\mathbf{a} = \{a_k\}$; see, e.g., [16, Theorem 32] for a slightly weaker result. The reduction is based on the observation that $\sum_{i=1}^{\infty} ia_i = \infty$ if and only if $\sum_{i=1}^{\infty} a_{\lfloor\sqrt{i}\rfloor} = \infty$. Indeed,

$$\sum_{i=1}^{\infty} a_{\lfloor\sqrt{i}\rfloor} = \sum_{n=1}^{\infty}\left(\sum_{k=n^2}^{n^2+2n} a_{\lfloor\sqrt{k}\rfloor}\right) = \sum_{n=1}^{\infty}(2n+1)a_n.$$

The following is a checkable necessary condition for a sequence to be BC (and hence a sufficient condition for a sequence not to be BC). It is a *partial* converse to Theorem 1.

Theorem 2. *Let* $\mathbf{x} = \{x_i\}_1^{\infty}$ *be a sequence in* $\mathcal{I}$. *If there exists an interval* J *such that for every* $\epsilon > 0$, *there exists arbitrarily large* N_ϵ *such that* $\lambda\left(\bigcup_{i=1}^{N_\epsilon} B(x_i, 1/N_\epsilon) \cap J\right) < \epsilon\lambda(J)$, *then* $\mathbf{x}$ *is not BC.*

Proof. Select $M_i = N_{2^{-i}}$ so that $M_{i+1} > 2M_i$. Let

$$a_j = \begin{cases} 1, & \text{if } j < M_1, \\ 1/M_{i+1}, & \text{if } M_i \le j < M_{i+1}. \end{cases}$$

Since $\sum_{j=M_i}^{M_{i+1}-1} a_j = (M_{i+1} - M_i)/M_{i+1} > 1/2$, it follows that $\mathbf{a}$ is a standard sequence.

On the other hand, for all $i \ge 1$,

$$\lambda\left(\bigcup_{i=M_i}^{M_{i+1}-1} B(x_i, a_i) \cap J\right) \le \lambda\left(\bigcup_{i=1}^{M_{i+1}} B(x_i, 1/M_{i+1}) \cap J\right) < 2^{-(i+1)}\lambda(J),$$

and therefore, by Borel-Cantelli Theorem, $\lambda(\limsup_{n\to\infty} B(x_n, a_n) \cap J) = 0$. □

The following general result for dynamical systems with a mild quantitative rigidity assumption follows from Theorem 2.

Corollary 3. *Assume that* $T\colon [0,1) \to [0,1)$ *is* λ *measure preserving and that* $\liminf_{n\to\infty} n\int_0^1 |T^n x - x|dx = 0$. *Then almost every forward orbit* $\{T^n(x)\}$ *is not BC, i.e., the sequence* $\{T^n(x)\}$ *is not BC for almost all* $x \in \mathfrak{I}$.

Proof. Choose n_i such that $\int_0^1 |T^{n_i}x - x|dx < 1/(20^i n_i)$. Observe that for each j, $\lambda(\{x : |T^{n_i+j}(x) - T^j(x)| > 1/(2^i n_i)\}) < 10^{-i}$. Thus

$$\lambda\Big(\Big\{x : |T^{kn_i+j}(x) - T^j(x)| > \frac{k}{2^i\, n_i},\quad \text{for each } 1 \le k \le i\Big\}\Big) < 10^{-i}\, i.$$

Therefeore, the Borel-Cantelli Theorem implies that for almost every x, the sequence $\{T^n(x)\}$ satisfies the condition of Theorem 2. □

It follows from Corollary 3 and Veech's generic rigidity result for IETs [21, Part I, Theorem 1.4] that for almost every IET T and almost every initial point x, the orbit sequence $\{T^n(x)\}$ is not BC; see [**?**, Theorem 7] for details. One can tweak the argument to get that in this case, *every* orbit is not BC.

Example 5. An immediate consequence of Corollaries 1 and 3 is that $\{\langle\!\langle n\alpha\rangle\!\rangle\}$ is BC if and only if the real α is a badly approximable irrational, i.e., the terms of its continued fraction expansion are uniformly bounded. This follows from the fact that rotations are isometries and $\min_{0\le i<j\le N} d(R^i x, R^j x)$ is proportional to $1/(c_{r+1}q_r)$. Here, q_r denotes the largest denominator of a convergent of α that is at most N and c_{r+1} is the $(r+1)^{\text{st}}$ term in the continued fraction expansion of α. This is proportional to $1/N$ for all N if and only if the sequence (c_r) is uniformly bounded. This example is a restatement of a result originally proven by J. Kurzweil in [17].

Corollary 4. *If* $\mathbf{x} = \{x_n\}$ *is a sequence in* $\mathfrak{I}$ *such that* $\limsup\limits_{n\to\infty} nd(x_n, x_{n+1}) = 0$, *then* $\mathbf{x}$ *is not BC.*

The corollary follows easily from Theorem 2.

Example 6. In particular, the sequences $\{\langle\!\langle \ln(\ln(3{+}n))\rangle\!\rangle\}$ and $\{\langle\!\langle (\ln(2{+}n))^{0.99}\rangle\!\rangle\}$ are not BC.

In order to state our necessary and sufficient condition for a sequence to be Borel-Cantelli, we need the following definition.

Definition 3. Let $A = \{N_n\}$ be an infinite increasing sequence of natural numbers. Given $\mathbf{x} = \{x_n\}$, define

$$f_A(z) := \liminf_{r\to 0^+} \limsup_{N\in A} \frac{\lambda\big(\bigcup_{i=1}^N B(x_i, 1/N) \cap B(z,r)\big)}{\lambda(B(z,r))}.$$

Lemma 1. f_A *is measurable.*

Proof. Let $f_{A,r}(z) = \limsup_{N\in A} \lambda\big(\bigcup_{i=1}^{N} B(x_i, 1/N) \cap B(z, r)\big)/\lambda(B(z, r))$. Note that $f_{A,r}$ is continuous; also, $f_{A,r+\epsilon}(z)+2\epsilon \geq f_{A,r}(z) \geq f_{A,r+\epsilon}(z)r/(r+\epsilon)$. Therefore, $f_A(z) = \liminf_{r\in\mathbb{Q},r\to 0^+} f_{A,r}(z)$ is measurable. □

Theorem 3. *A sequence* $\mathbf{x} = \{x_n\}$ *is not BC if and only if* $\lambda(f_A^{-1}(0)) > 0$ *for some sequence A.*

We defer the proof of this theorem to the end of the section and first state some consequences.

Remark 2. Theorem 3 shows that the BC property can be detected by sequences of the form $a_j = 1/N_i$ for $N_{i-1} < j \leq N_i$. For the purposes of testing of the BC property, we need not bother with the many standard sequences for which $\limsup_{n\to\infty} na_n = 0$ (such as $a_n = 1/(n\ln(n))$).

Remark 3. If we were to define

$$\tilde{f}(z) = \limsup_{r\to 0^+} \liminf_{N\to\infty} \frac{\lambda\big(\bigcup_{i=1}^{N} B(x_i, 1/N) \cap B(z, r)\big)}{\lambda(B(z, r))},$$

we would obtain BC sequences such that $\tilde{f}(z) = 0$ for almost every z.

Remark 4. For $A = \{1, 2, 3, \ldots\}$, there exist non-BC sequences such that $f_A = 1$ almost everywhere.

Example 7. It follows that a non-uniquely ergodic IET has orbits that are not BC. Additionally, if $T : [0, 1) \to [0, 1)$ is a continuous, λ measure preserving transformation that is not λ ergodic, then λ almost every orbit is not BC. Indeed, any ergodic measure μ gives 0 weight to some set U of positive Lebesgue measure. Then $\limsup_{r\to 0} \mu(B(z, r))/(2r) = 0$, for Lebesgue almost any $z \in U$. Then, for a μ typical point y, the sequence of measures

$$\left\{\delta_y, \frac{\delta_y + \delta_{Ty}}{2}, \frac{\delta_y + \delta_{Ty} + \delta_{T^2y}}{3}, \ldots\right\}$$

converges to μ in the weak* topology. (Here, δ_u stands for the point mass at u.) Thus, for the sequence $\{T^i y\}_{i=1}^{\infty}$ and Lebesgue almost every $z \in U$, we have $f_{\mathbb{N}}(z) = 0$.

Example 8. It follows that if $\{R_n\}$ is a sequence of independent random variables, each distributed according to a measure μ, then $\{R_1(\zeta), R_2(\zeta), \ldots\}$ is BC for $\mu^{\mathbb{N}}$ almost every ζ if and only if $\lambda \ll \mu$. This follows similarly to the previous example because for $\mu^{\mathbb{N}}$ almost every ζ, the sequence of measures $\{\delta_{R_1(\zeta)}, (\delta_{R_1(\zeta)} + \delta_{R_2(\zeta)})/2, \ldots\}$ converges to μ in the weak* topology.

Remark 5. Fix a uniformly distributed mod 1 sequence **x**. The first n points define a partition of the $[0, 1)$ into segments of length $\delta_1^{(n)}, \ldots, \delta_{n+1}^{(n)}$. It follows from Theorem 3 that if for any $\epsilon > 0$, there exists a constant $s_\epsilon > 0$ such that for large n, all but ϵn of the $\delta_i^{(n)}$ are bigger than s_ϵ/n, then **x** is Borel-Cantelli.

The above (sufficient) criterion enables us to conclude that the square root sequence $\{\langle\langle\sqrt{k}\rangle\rangle\}_{k\geq 1}$ is BC. The validation of the criterion is based on the gap distribution results for this sequence by N. Elkies and C. McMullen [13, Theorem 1.1]. Likewise, the sequence $\{\langle\langle k^r\alpha\rangle\rangle\}_{k\geq 1}$ is BC for any integer $r \geq 2$ and almost every α by a result by Z. Rudnick and P. Sarnak [19, Theorem 1].

The following sufficient condition for a sequence **x** to be BC is both stronger than the one in Remark 5 and easier to apply in some situations.

Remark 6. Let $\mathbf{x} = \{x_n\}_i^\infty$ be a sequence uniformly distributed in $\mathcal{I}$ and

$$X_n(u) = \{(p, q) \mid 1 \leq p < q \leq n,\ |x_p - x_q| < u\}, \quad \text{for } n \geq 1, u > 0.$$

Assume that for any $\epsilon > 0$, there exists a constant $s_\epsilon > 0$ such that for all large n, $\mathrm{card}(X_n(s_\epsilon/n)) \leq \epsilon n$. Then **x** is Borel-Cantelli.

We now begin the proofs of Theorems 1 and 3 with the key lemma of this paper. It is distilled from the proof of [17, Lemma 4].

Lemma 2. *Let $M \in \mathbb{N}$, $c > 0$ and $e > 0$ be constants, $\mathbf{x} = \{x_k\}_i^\infty$ a sequence in $\mathcal{I}$, and* **a** *a standard sequence. If for all $r \in \mathbb{N}$, at least cM^r of the points in $\{x_{M^{r-1}}, x_{M^{r-1}+1}, \ldots, x_{M^r}\}$ are e/M^r separated from each other, then there exists $\delta > 0$, depending only on c and e such that $\lambda(\limsup_{n\to\infty} B(x_n, a_n)) > \delta$. In particular, δ is independent of* **a** *(so long as* **a** *is standard).*

Remark 7. By imposing stricter conditions on **a**, one can prove versions of this lemma with weaker hypotheses. For instance, if one requires **a** to be standard and ia_i to be monotone, then one only needs to assume that cM^r of the points $\{x_{M^{r-1}}, x_{M^{r-1}+1}, \ldots, x_{M^r}\}$ are e/M^r separated from each other for a positive (lower) density set of r. Call such a sequence a **Khinchin sequence**. Our approach of restricting attention to Khinchin sequences of radii is carried out to prove [9, Theorem 8]. One can prove versions of Theorems 1, 2 and 3 in this context. In particular, for Khinchin sequences, one can obtain the analogue to Theorem 3 by letting

$$f_A(z) = \liminf_{r\to 0^+} \limsup_{N\in A} \frac{1}{N} \sum_{k=1}^{N} \frac{\lambda\big(\bigcup_{i=1}^{2^k} B(x_i, 1/2^k) \cap B(z, r)\big)}{2r}.$$

In the more involved direction, the proof is similar (using the fact that for Khinchin sequences, one may apply Lemma 3 twice), and in the other direction, let $a_i = 1/i \log(N_j)$ for $N_{j-1} \leq i < N_j$. This can also be carried out in the Ahlfors regular setting.

The following well-known and simple fact is used in the proof of Lemma 2.

Lemma 3. *Let $M \geq 2$ be an integer and* **a** *a non-increasing sequence. Then $\sum_{i=0}^{\infty} a_i$ diverges if and only if $\sum_{i=0}^{\infty} M^i a_{M^i}$ diverges.*

Proof of Lemma 2. Without loss of generality, assume that $c > 2/M$. (If not, replace M with some power M^k; then the new c is $(M^k - 1/M^k)c$, which is greater than $2/M^k$ for large enough k).

To ease computation, we replace $a_1, a_2, \ldots$ with $b_1, b_2, \ldots$, where

$$b_i = \min\left\{a_{M^j}, \frac{e}{2M^j}\right\} \quad \text{for } M^{j-1} \leq i < M^j.$$

It suffices to show that for any k_0, $\lambda\left(\bigcup_{i=k_0}^{\infty} B(x_i, b_i)\right) > \delta := ec/2$.

Observe that $B(z, r + e/M^j)$ can contain at most $\lceil 2rM^j/e \rceil + 1$ points that are e/M^j separated (because this is an open interval). So at most $u(M^j/e) + 1$ points x_i that are e/M^j separated can have $B(x_i, e/M^j)$ intersect non-trivially an interval of measure u. Thus, if $\lambda\left(\bigcup_{i=k_0}^{M^{j-1}} B(x_i, b_i)\right) < \delta$, then at most $M^{j-1} + (M^j/e)\delta = M^{j-1} + (c/2)M^j$ of the e/M^j separated points from the list $\{x_{M^{j-1}}, x_{M^{j-1}+1}, \ldots, x_{M^j}\}$ satisfy $B(x_t, b_t) \cap \bigcup_{i=k_0}^{M^{j-1}} B(x_i, b_i) \neq \varnothing$. This leaves at least $(c/2 - 1/M)M^j$ separated points. This is a positive number since $c > 2/M$. Observing that $b_i = b_{M^j} \leq e/2M^j$ for $M^{j-1} < i \leq M^j$, we obtain

$$\lambda\Big(\bigcup_{i=M^{j-1}}^{M^j} B(x_i, b_i) \setminus \bigcup_{i=k_0}^{M^{j-1}} B(x_i, b_i)\Big) \geq \Big(\frac{c}{2} - \frac{1}{M}\Big) M^j b_{M^j}.$$

Therefore, if $\lambda\left(\bigcup_{i=g}^{M^{j-1}} B(x_i, b_i)\right) < \delta$, then $\lambda\left(\bigcup_{i=M^{j-1}}^{M^j} B(x_i, b_i) \setminus \bigcup_{i=g}^{m^{j-1}} B(x_i, b_i)\right)$ is proportional to $M^j b_{M^j}$. Since $\sum_{i=1}^{\infty} b_i = \infty$, we have

$$\sum_{i=g}^{\infty} \left(\Big(\frac{c}{2} - \frac{1}{M}\Big) \sum_{i=M^{j-1}}^{M^j} b_i\right) > \delta$$

for all g, We conclude that $\lambda\left(\bigcup_{i=g}^{\infty} B(x_i, a_i)\right) > \delta$ for any g. Because $\lambda(\mathcal{I}) = 1 < \infty$, it follows that $\lambda\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B(x_i, a_i)\right) = \lim_{n\to\infty} \lambda(\bigcup_{i=n}^{\infty} B(x_i, a_i)) \geq \delta$. □

The following local version is an immediate corollary.

Corollary 5. *Let $M \in \mathbb{N}$, $c > 0$, and $e > 0$ be constants,* **x** *a sequence in $\mathcal{I}$, and* **a** *a standard sequence. If there exists an interval J such that for all $r \in \mathbb{N}$, at* *least $c\lambda(J)M^r$ of the points in the set $\{x_{M^{r-1}}, x_{M^{r-1}+1}, \dots, x_{M^r}\}$ are e/M^r separated from each other and lie in J, then there exists $\delta > 0$ depending only on c and e such that $\lambda(\limsup_{n\to\infty} B(x_n, a_n) \cap J) > \delta\lambda(J)$.*

Lemma 4. *Let $J \subset \mathfrak{I}$ be an interval and assume that for some $d > 0$,*

$$\liminf_{N\to\infty} \lambda\Big(\bigcup_{i=1}^{N} B(x_i, \frac{1}{N}) \cap J\Big) \geq d\lambda(J).$$

Then

$$\lambda(\limsup_{n\to\infty} B(x_n, a_n) \cap J) > \frac{1}{12} d\lambda(J).$$

Proof. For simplicity, for each N, we ignore the effect of the at most two different $B(x_i, 1/N)$, where $B(x_i, 1/N) \cap J \neq \varnothing$ but $B(x_i, 1/N) \not\subset J$. Notice that if $\{y_1, y_2, \dots, y_k\}$ does not contain two points that are $1/N$ separated, then $\lambda\big(\bigcup_{i=1}^{k} B(y_i, 1/N)\big) \leq 3/N$. This is because the points $y_1, \dots, y_k$ are contained in an interval of length at most $1/N$.

It follows that if $\lambda\big(\bigcup_{i=1}^{N} B(x_i, 1/N)\big)/3 \geq g$, the set $\{x_1, x_2, \dots, x_N\}$ contains at least gN points that are $1/N$ separated.

By our assumption, for any $\epsilon > 0$ and all large enough N, $\{x_1, x_2, \dots, x_N\} \cap J$ contains at least $N(d\lambda(J) - \epsilon)$ points. Let $M \geq d\lambda(J)/2$. For large enough r, the set $\{x_1, \dots, x_{M^r}\}$ contains at least $M^r(d\lambda(J) - \epsilon)$ points that are $1/M^r$ separated. Thus the set $\{x_{M^{r-1}}, x_{M^{r-1}+1}, \dots, x_{M^r}\}$ contains at least $M^r(d\lambda(J) - \epsilon) - M^{r-1} \geq 4M^{r-1} - M^{r-1} = 3M^{r-1}$ points that are $1/M^r$ separated. To deal with the various ϵ, let $c = d\lambda(J)/6$, $M = d\lambda(J)/2$, and $e = 1$. Then apply Lemma 2. □

Corollary 6. *Given a finite union of intervals J, if there exists $d > 0$ such that $\liminf_{N\to\infty} \lambda\big(\bigcup_{i=1}^{N} B(x_i, 1/N) \cap J\big) \geq d\lambda(J)$, then $\lambda(\limsup_{n\to\infty} B(x_n, a_n) \cap J) > d\lambda(J)//12$.*

We now use Lemma 4 to prove Theorem 1.

Proof of Theorem 1. Corollary 6 shows that every y satisfies

$$\frac{1}{2r} \lambda(\limsup_{n\to\infty} B(x_n, a_n) \cap B(y, r)) > \frac{1}{12} d > 0.$$

This implies that $\lambda(\limsup_{n\to\infty} B(x_n, a_n)) = 1$ because its complement has no density points. □

Remark 8. This proof also gives a local version of Theorem 1. To be exact, let $f(z) := \limsup_{r\to 0^+} \liminf_{N\to\infty} \lambda\big(\bigcup_{i=1}^{N} B(x_i, 1/N) \cap B(z, r)\big)/\lambda(B(z, r))$. If $f(z) > 0$ for almost every z, then $\mathbf{x}$ is BC.

Using this remark, one can construct sequences that are BC for $\mathbb{R}$.

Proof of Theorem 3. Assume $\mathbf{x}$ is not BC. Then there exists a standard sequence $\mathbf{a}$ such that $\lambda((\limsup_{n\to\infty} B(x_n, a_n))^c) > 0$. We consider the following sets:

$$\begin{aligned} S &= \limsup_{n\to\infty} B(x_n, a_n), \\ R_{t,\delta} &= \{y \in S^c : \lambda(B(y, \delta') \cap S^c) > 2\delta' t, \quad \text{for all } \delta' < \delta\}. \end{aligned}$$

By the Lebesgue Density Theorem, $\lambda\big(S^c \cap \bigcup_{n=1}^{\infty} R_{t,1/n}\big) = \lambda(S^c)$ for any $t < 1$. Choose δ small enough so that $R_{999/1000,\delta} \neq \varnothing$. Let $y_1 \in R_{999/1000,\delta}$. By Lemma 4, there exist infinitely many N such that $\lambda\big(\bigcup_{i=1}^{N} B(x_i, 1/N) \cap B(y_1, \delta)\big)/12 \leq \lambda(B(y_1, \delta))/1000$. Pick one such N and denote it by N_1. Now cover most of $B(y_1, \delta)$.

There exist points, $y_2^{(1)}, y_2^{(2)}, \ldots, y_2^{(t_2)}$ and corresponding radii, $r_2^{(1)}, r_2^{(2)}, \ldots, r_2^{(t_2)}$ such that

(1′) $y_2^{(i)} \in R_{9999/10000, r_2^{(i)}}$,
(2′) $B(y_2^{(i)}, r_2^{(i)}) \subset B(y_1, \delta)$ for all i,
(3′) $\lambda(B(y_1, \delta) \cap \bigcup_{i=1}^{t_2} B(y_2^{(i)}, r_2^{(i)})) > (99/100) \cdot 2\delta$,
(4′) the $B(y_2^{(i)}, r_2^{(i)})$ are all disjoint.

To see that these conditions can be met, first notice that

$$\lim_{n\to\infty} \lambda\left(S^c \cap B(y_1, \delta) \cap R_{t,\epsilon}\right) = \lambda\left(S^c \cap B(y_1, \delta)\right) \geq (1 - 10^{-3})2\delta$$

and therefore

$$\lambda\left(B(y_1, \delta) \setminus \bigcup_{n=1}^{\infty} \bigcup_{\substack{y \in R_{1-\epsilon,1/n} \\ B(y,1/n) \subset B(y_1,\delta)}} B(y, 1/n)\right) \leq (1 - 10^{-3})2\delta.$$

By Theorem 9 (which gives disjointness of $B(y_2^{(i)}, r_2^{(i)})$), it is possible to cover $B(y_1, \delta)$ up to a set of measure $2 \cdot 10^{-3}\delta$ by a countable number of $B(y_2^{(i)}, r_2^{(i)})$ satisfying Conditions 1′-4′. Therefore, we can cover all but a set measure $1 - 10^{-2}$ of $B(y_1, \delta) \cap S^c$ by a finite number of balls $B(y_2^{(i)}, r_2^{(i)})$ satisfying Conditions 1′–4′.

By Condition 1′ and Corollary 6, we cannot have

$$\frac{1}{12}\lambda\left(\bigcup_{i=1}^{N} B(x_i, 1/N) \cap \bigcup_{j=1}^{t_2} B(y_2^{(j)}, r_2^{(j)})\right) > 10^{-4}\lambda\left(\bigcup_{j=1}^{t_2} B(y_2^{(j)}, r_2^{(j)})\right)$$

for all but finitely many N. This implies that for infinitely many N, there exists $H_N \subset \{1, \ldots, t_2\}$ with $\lambda\big(\bigcup_{i\in H_N} B(y_2^{(i)}, r_2^{(i)})\big) > (99/100)\lambda\big(\bigcup_{i=1}^{t_2} B(y_2^{(i)}, r_2^{(i)})\big)$ such that for each $j \in H_N$,

$$\frac{1}{12}\lambda\Bigl(\bigcup_{i=1}^{N} B(x_i, 1/N) \cap B(y_2^{(j)}, r_2^{(j)})\Bigr) < 10^{-4} \cdot 10^2 \cdot 2\lambda(B(y_2^{(i)}, r_2^{(j)})).$$

(Our choice of 2 depends on how closely we can divide up the space by the balls, i.e., the smallness of the largest $r_2^{(i)}$. One could chose it arbitrarily close to 1 by making the $r_2^{(i)}$ small enough.) Pick one of these N times N_2, and the corresponding collection $\mathcal{U}_2$. Notice that for any $z \in B(y_2^{(i)}, (1-1/16)r_2^{(i)})$, where $i \in \mathcal{U}_2$, we have

$$\lambda\Bigl(\bigcup_{j=1}^{N_2} B(x_j, 1/N_2) \cap B(z, r_2^{(i)} - |z - y_2^{(i)}|)\Bigr)$$
$$< 12 \cdot 16 \cdot 10^{-4} \cdot 100 \cdot 2\lambda(B(z, r_2^{(i)} - |z - y_2^{(i)}|)).$$

This estimate follows by assuming the worst case scenario

$$\bigcup_{j=1}^{N_2} B(x_j, 1/N_2) \cap B(y_2^{(i)}, r_2^{(i)}) \subset B(z, r_2^{(i)} - |y_2^{(i)} - z|).$$

Also, by Condition 3′, $\lambda\bigl(\bigcup_{i=1}^{N_2} B(x_i, 1/N_2) \cap B(y_1, \delta)\bigr) > 10^{-2} + 12 \cdot 10^{-4} \cdot 2\delta$.

We now proceed inductively, choosing t_k points, $y_k^{(1)}, \dots, y_k^{(t_k)}$, with corresponding radii, $r_k^{(1)}, \dots, r_k^{(t_k)}$, such that

1. $y_k^{(i)} \in R_{1-10^{-2k}, r_k^{(i)}}$,
2. $B(y_k^{(j)}, r_k^{(j)}) \subset \bigcup_{i=1}^{t_{k-1}} B(y_{k-1}^{(i)}, r_{k-1}^{(i)})$ for all j,
3. $\lambda(B(y_{k-1}^{(i)}, r_{k-1}^{(i)}) \cap \bigcup_{i=1}^{t_k} B(y_k^{(i)}, r_k^{(i)})\bigr) > (1-10^{-k})\lambda(B(y_{k-1}^{(i)}, r_{k-1}^{(i)}))$ for each $y_{k-1}^{(i)}$,
4. the $B(y_k^{(i)}, r_k^{(i)})$ are disjoint.

This is done analogously to the earlier construction of $y_2^{(i)}, r_2^{(i)}$ satisfying Conditions $(1') - (4')$ in the beginning of the proof of Theorem 3.

By Condition 1 and Corollary 6, we cannot have

$$\frac{1}{12}\lambda\Bigl(\bigcup_{i=1}^{N} B(x_i, 1/N) \cap \bigcup_{j=1}^{t_k} B(y_k^{(j)}, r_k^{(j)})\Bigr) > 10^{-2k} \cdot \lambda\Bigl(\bigcup_{j=1}^{t_k} B(y_k^{(j)}, r_k^{(j)})\Bigr)$$

for all but finitely many N. This implies that for infinitely many N, there exists $H_N \subset \{1, \dots, t_k\}$ with $\lambda\bigl(\bigcup_{i\in H_N} B(y_k^{(i)}, r_k^{(i)})\bigr) > (1-10^{-k})\lambda\bigl(\bigcup_{i=1}^{t_k} B(y_k^{(i)}, r_k^{(i)})\bigr)$ such that for each $j \in H_N$,

$$\lambda\Bigl(\bigcup_{i=1}^{N} B(x_i, 1/N) \cap B(y_k^{(j)}, r_k^{(j)})\Bigr) < 12 \cdot 10^{-2k} \cdot 10^k \cdot 2\lambda(B(y_k^{(j)}, r_k^{(j)})).$$

(As before, our choice of 2 depends on how closely we can divide up the measure.) Pick one of these N times N_k, and the corresponding collection $\mathfrak{U}_k$. Notice that for any $z \in B(y_k^{(i)}, (1-4^{-k})r_k^{(i)})$, where $y_k^{(i)} \in \mathfrak{U}_k$, we have

$$(1) \quad \lambda\left(\bigcup_{j=1}^{N_k} B(x_j, 1/N_k) \cap B(z, r_k^{(i)} - |z - y_k^{(i)}|)\right) < 4^k \cdot 12 \cdot 10^{-2k} \cdot 10^k \cdot 2(r_k^{(i)} - |z - y_k^{(i)}|).$$

This estimate follows by assuming the worst case scenario

$$\bigcup_{j=1}^{N_k} B(x_j, 1/N_k) \cap B(y_k^{(i)}, r_i^{(i)}) \subset B(z, r_k^{(i)} - |z - y_k^{(i)}|).$$

Choose $A = \{N_1, N_2, \ldots\}$. We show that

$$f_A^{-1}(0) \supset \bigcup_{r=1}^{\infty} \bigcap_{k=r}^{\infty} \bigcup_{i \in \mathfrak{U}} B(y_k^{(i)}, (1-4^{-k})r_k^{(i)}).$$

This set has positive measure because at each step at most 10^{-k} of the measure is eliminated by the choice of $y_k^{(i)}$, 10^{-k} is avoided by the choice of $\mathfrak{U}_k$, and 4^{-k} is avoided by the excluded annuli. If $z \in \bigcup_{r=1}^{\infty} \bigcap_{k=r}^{\infty} \bigcup_{i=1}^{t_k} B(y_k^{(i)}, (1-4^{-k})r_k^{(i)})$, then for all sufficiently large k, there exists i such that $|y_k^{(i)} - z| < (1-4^{-k})r_k^{(i)}$. A sequence of radii tending to zero is given by $r_k^{(i)} - |y_k^{(i)} - z|$. The following lemma and its corollary show that $f_A(z) = 0$ in this case.

Lemma 5. *Given $\epsilon > 0$, there exists m_ϵ such that for all $m > m_\epsilon$,*

$$\lambda\left(\bigcup_{i=1}^{N_m} B(x_i, 1/N_m) \cap B(y_k^{(i)}, r_k^{(i)})\right) < \left(\frac{10}{9} \cdot 10^{-k} + \epsilon\right) 2r_k^{(i)}.$$

Proof. Condition 3 implies that

$$\lambda\left(\bigcup_{i=1}^{t_m} B(y_m^{(i)}, r_m^{(i)}) \cap B(y_k^{(i)}, r_k^{(i)})\right) > \left(1 - \frac{10}{9} \cdot 10^{-k}\right) 2r_k^{(i)}$$

for any $m > k$ and that for large enough m, $12 \cdot 10^{-m} \cdot 2 \cdot 2r_k^{(i)} + 10^{-m} < 2\epsilon r_k^{(i)}$. (Notice that 10^{-m} is greater than or equal to what the choice of $\mathfrak{U}_m$ excludes.) The lemma follows from 1 and assuming that the portion of $B(y_k^{(i)}, r_k^{(i)})$ not covered by $B(y_m^{(1)}, r_m^{(1)}), \ldots, B(y_m^{(t_m)}, r_m^{(t_m)})$ is as large as possible. □

The following corollary is immediate.

Corollary 7. *If $z \in B(y_k^{(i)}, (1-4^{-k})r_k^{(i)})$, then for all sufficiently large m,*

$$\lambda\Big(\bigcup_{I=1}^{N_m} B(x_i, 1/N_m) \cap B(z, r_k^{(i)} - |z - y_k^{(i)}|)\Big) < 2 \cdot 4^k \Big(\frac{10}{9}(10^{-k} + \epsilon)\Big)(r_k^{(i)} - |z - y_k^{(i)}|).$$

With one direction of the proof of Theorem 3 completed, we proceed with the other direction. We have to show that if there exist $A = \{N_1, N_2, \ldots\}$ such that $\lambda(f_A^{-1}\{0\}) > 0$, the sequence $\mathbf{x}$ is not BC.

By definition of f_A, for each $y \in f_A^{-1}\{0\}$, there exist $r_i(y)$ such that

$$\lambda\Big(\bigcup_{j=1}^{N_k} B(x_j, 1/N_k) \cap B(y, r_i(y))\Big) < 2 \cdot 4^{-i} r_i(y)$$

for all $k > k_i(y)$. There exists $l_i \in \mathbb{N}$ such that $\mathcal{V}_i = \{y \in f^{-1}\{0\} : k_i(y) < l_i\}$ has $\lambda(\mathcal{V}_i) > (1 - 10^{-i})\lambda(f^{-1}\{0\})$. The sequence $\{a_i\}$ is defined by $a_i = 1/N_{l_j}$ for $N_{l_{j-1}} < i \leq N_{k_j}$. By our choice of $a_1, a_2, \ldots$, it follows that $\bigcap_{i=1}^{\infty} \mathcal{V}_i$ are not density points for $\bigcap_{r=1}^{\infty} \bigcup_{i=r}^{\infty} B(x_i, a_i)$. Also, $\lambda\big(\bigcap_{i=1}^{\infty} \mathcal{V}_i\big) \geq (8/9)\lambda(f^{-1}\{0\})$. □

Remark 9. The conditions imposed throughout the proof are by no means optimal. Moreover, easier conditions are possible in this case (or the case of $\mathbb{R}^k$). However, the conditions of this proof generalize to the Ahlfors regular case.

Remark 10. Given a standard sequence $\mathbf{a}$, one can modify the argument to find A such that $\lambda(f_A^{-1}\{0\}) > (1-\epsilon)\,\lambda(\limsup_{n\to\infty} B(x_n, a_n))$. Likewise, given A, one can modify the argument to find a standard sequence $\mathbf{a}$ such that

$$\lambda(\limsup_{n\to\infty} B(x_n, a_n)) > (1-\epsilon)\,\lambda(f_A^{-1}\{0\}).$$

3 Generalizations

We now generalize results for $[0, 1)$ to another setting. The results are parallel to the case of $[0, 1)$ with Lebesgue measure.

Definition 4. A complete metric space (X, d) with a measure μ is called **Ahlfors regular of dimension** ω if there exists a constant C such that for all $y \in X$ and r with $\operatorname{diam}(X) > r > 0$, $C^{-1} r^{\omega} < \mu(B(y, r)) < C r^{\omega}$.

For further references on Ahlfors regular spaces, see [18] or [11].

Example 9. A linear recurrent subshift on a finite alphabet (see Example 3 or [12]) with metric $\bar{d}(\mathbf{x}, \mathbf{y}) = (1 + \min\{i : x_i \neq y_i\})^{-1}$ and measure μ given by the unique measure under which the shift map is ergodic is an Ahlfors regular space (of dimension 1). All minimal substitution dynamical systems are linear recurrent.

Example 10. Hausdorff $\log 2/\log 3$ measure on the middle thirds Cantor set is Ahlfors regular of dimension $\log 2/\log 3$ with respect to the usual metric on $\mathcal{I}$.

Definition 5. Let (X, d, μ) be an ω Ahlfors regular metric space. A sequence $\mathbf{x} = \{x_n\}$ in X is called **Borel-Cantelli (in** X**)** if for any standard sequence $\mathbf{a}$,

$$\mu\left(X \setminus \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B(x_n, a_n^{1/\omega})\right) = 0.$$

Remark 11. The Ahlfors regular condition ensures that

$$\sum_{i=1}^{\infty} \mu(B(x_i, a_i^{1/\omega})) = \infty \quad \text{if and only if} \sum_{i=1}^{\infty} a_i = \infty.$$

The following is a sufficient condition for a sequence to be BC in X. It is the version of Theorem 1 in this (more general) setting. It is also used in the proof of the necessary and sufficient condition (Theorem 6) in this setting.

Theorem 4. *If* $\mathbf{x}$ *is a sequence in* X *and there exists* $d > 0$ *such that for every ball* J*,* $\liminf_{N\to\infty} \mu\left(\bigcup_{i=1}^{N} B(x_i, (1/N)^{1/\omega}) \cap J\right) \geq d\mu(J)$*, then* $\mathbf{x}$ *is BC in* X*.*

We defer the proof of this theorem to later in the section, after Remark 14.

Remark 12. Theorem 4 also follows from a result of by V. Beresnevich, D. Dickson and S. Velani [4, Corollary 2], with $u(n) = 2^n$, $l_n = 2^{n-1}$, $\delta = \omega$, $\gamma = 0$, and $\rho(r) = (1/r)^{1/\omega}$. Its proof is included for completeness, and the theorem itself is used in the proof of Theorem 6.

Example 11. The systems in Example 9 are ABC, i.e., they have the property that all forward orbits are BC.

Example 12. The endpoints of the middle thirds Cantor set K (i.e., the one sided limit points) enumerated by increasing denominator form a BC sequence (for K).

Next, we provide a sufficient condition for a sequence not to be Borel-Cantelli. It is this setting's version of Theorem 2.

Theorem 5. *If there exists a ball* J *such that for every* $\epsilon > 0$ *there exists arbitrarily large* N_ϵ *with* $\mu\left(\bigcup_{i=1}^{N_\epsilon} B(x_i, (1/N_\epsilon)^{1/\omega}) \cap J\right) < \epsilon\mu(J)$*, then* $\mathbf{x}$ *is not BC.*

The proof is parallel to Theorem 2.

Definition 6. Let $A = \{N_1, N_2, \ldots\}$ be an infinite increasing sequence of natural numbers. Given $\mathbf{x}$, define

$$f_A(z) = \liminf_{r\to 0^+} \limsup_{N\in A} \frac{\mu\left(\bigcup_{i=1}^N B(x_i, (1/N)^{1/\omega}) \cap B(z,r)\right)}{\mu(B(z,r))}.$$

Question 1. Is f_A measurable? Note that $f_A^{-1}(0)$ is a measurable set.

Theorem 6. $\mathbf{x}$ *is not BC if and only if* $f_A^{-1}(0)$ *contains a set of positive measure for some* A.

We defer the proof of this theorem to the end of the section.

Remark 13. If one considers [4, Corollary 2] for $\rho(r) = (1/r)^{1/\omega}$, $u_n = 2^n$, $l_n = 2^{n-1}$, and R_α one point sets, Theorem 6 provides a necessary and sufficient condition for Theorem 4 to hold.

Example 13. Let $R_1, R_2, \ldots$ be independent random variables, all distributed according to a probability measure ν. The sequence $\{R_1(\zeta), R_2(\zeta), \ldots\}$ is BC for $\nu^{\mathbb{N}}$ almost every ζ if and only if $\mu \ll \nu$.

Example 14. If $T : X \to X$ is continuous, μ measure preserving, and not μ ergodic, then μ almost every orbit is not Borel-Cantelli.

Our next result a more general version of [8, Lemma 9], and the proof is largely the same.

Theorem 7. *Let* $\mathbf{s}$ *be a sequence of real numbers such that* $\lim_{n\to\infty} s_n = \infty$, *and let* $\mathbf{x}$ *be a sequence in* X. *For almost every* y, $\liminf_{n\to\infty} s_n d(x_n, y)$ *is either zero or infinity, i.e.,* $\mu(\{y : \liminf_{n\to\infty} s_n d(x_n, y) \in (0,\infty)\}) = 0$.

To prove this theorem, we use a version of the Lebesgue Density Theorem for an Ahlfors regular space which is also used to prove Theorems 4 and 6.

Theorem 8 ([15], Theorem 1.8)**.** *For any* μ *measurable* A, *there exists* $\bar{A}$ *such that* $\mu(A \Delta \bar{A}) = 0$ *and for every* $x \in \bar{A}$, $\lim_{r\to 0^+} \mu(A \cap B(x,r))/\mu(B(x,r)) = 1$.

Proof of Theorem 7. It suffices to show that

$$A_n = \{y : \liminf_{n\to\infty} s_n d(x_n, y) \in (a, 2a) \text{ and } \inf_{n>N} s_n d(x_n, y) > a\}$$

has measure 0. If $s_n d(x_n, y) < 2a$, then $B(x_n, a/s_n) \subset B(y, 3a/s_n)$. Notice that if $n > N$, $B(x_n, a/s_n) \cap A_N = \varnothing$. But $\mu(B(x_n, a/s_n)) \geq 1/(C^2 3^\omega)\mu(B(x_n, 3a/s_n))$, implying that A_N has no density points. This is because $\lim_{n\to\infty} a/s_n = 0$. By Theorem 8, A_N has measure 0. $\square$

We begin the proof of Theorems 4 and 6 with this section's key lemma, which is analogous to Lemma 2.

Lemma 6. *Let $M \in \mathbb{N}$, $c > 0$, $e > 0$ be constants,* $\mathbf{x}$ *a sequence in X, and* $\mathbf{a}$ *a standard sequence. If there exists a ball J such that for all $r \in \mathbb{N}$ at least $c\mu(J)M^r$ points of the set $\{x_{M^{r-1}}, x_{M^{r-1}+1}, \dots, x_{M^r}\}$ lie in J and are $(e/M^r)^{1/\omega}$ separated from each other, then there exists $\delta > 0$, depending only on c and e, such that $\mu(\limsup_{n\to\infty} B(x_n, (a_n)^{1/\omega}) \cap J) > \delta\mu(J)$.*

Proof. The proof is similar to that of Lemma 2. As done there, we assume $c\mu(J) > 2/M$. Let $b_i = \min\{(a_{M^j})^{1/\omega}, (1/4)(e/M^j)^{1/w}\}$ for $M^{j-1} \le i < M^j$. Let $\delta = ec\mu(J)/(2^{2\omega+1}C)$. If $\mu = \bigcup_{i=k_0}^{M^{j-1}} B(x_i, a_i^{1/\omega} \cap J)) < \delta$, then we examine $\bigcup_{i=M^{j-1}}^{M_j} B(x_i, b_i)$.

By the definition of an Ahlfors regular measure, any ball of measure m contains at most $mM^rC2^{2\omega}/e$ disjoint balls of radius $(e/M^r)^{1/\omega}/4$. Note that if $\{y_1, \dots, y_r\}$ is a maximal δ-separated set contained in J, then $\bigcup_{i=1}^r B(y_i, 2\delta)$ covers J. It follows that a $(e/M^r)^{1/\omega}/4$ neighborhood of a ball of measure m contains at most $mM^rC2^{2\omega}/e$ points that are $3(e/M^r)^{1/\omega}/4$ separated. Therefore, at most $\delta M^jC2^{2\omega}/e + M^{j-1} \le 3M^{j-1}/2$ of the separated points are within $(e/M^j)^{1/\omega}/4$ of $\bigcup_{i=k_0}^{M^{j-1}} B(x_i, a_i^{1/\omega} \cap J)$ (if y_0 and y_1 are ϵ separated, then $B(y_0, \epsilon/4)$ is $\epsilon/2$ separated from $B(y_1, \epsilon/4)$). This leaves at least $(c\mu(J)/2 - 1/M)M^j$ separated points left. This is positive because $c\mu(J) > 2/M$. It follows that if $\mu\big(\bigcup_{i=g}^{M^{j-1}} B(x_i, b_i) \cap J\big) < \delta$, then $\mu\big(\bigcup_{i=M^{j-1}}^{M^j} B(x_i, b_i) \cap J \setminus \bigcup_{i=g}^{M^{j-1}} B(x_i, b_i)\big)$ is at least proportional to $b_{M^j}^{\omega} M^j$. Notice that $(c\mu(J)/2 - 1/M)\sum_{i=M^{j-1}}^{\infty} b_i^{\omega}$ is a divergent series. Thus, for any g, we have $\mu\big(\bigcup_{i=g}^{\infty} B(x_i, a_i^{1/\omega}) \cap J\big) > \delta$. By the finiteness of $\mu(J)$, it follows that $\mu\big(\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} B(x_i, a_i^{1/\omega})\big) \ge \delta$. □

The following lemma is helpful because the hypotheses of Theorem 4 imply its hypotheses.

Lemma 7. *There exists a function $\beta\colon \mathbb{R}^+ \to \mathbb{R}^+$ with the property that if there exists $d > 0$ such that for all N, $\mu\big(\bigcup_{i=1}^N B(x_i, 1/N^{1/\omega})\big) \ge d$, then for any standard sequence* $\mathbf{a}$, *$\mu(\limsup_{n\to\infty} B(x_n, (a_n)^{1/\omega})) > \beta(d)$. Moreover,*

$$\beta(d) > \left(\left(\frac{1}{2}\right)^{2\omega+1}\left(\frac{1}{C}\right)\right)\left(\frac{1}{2}\right)^{\omega}\frac{d}{2C}.$$

Proof. First observe that $\mu\big(\bigcup_{i=N}^{2CN/d} B(x_i, \big(d/(2CN)\big)^{1/\omega})\big) \ge d/2$ because $\mu\big(\bigcup_{i=1}^N B(x_i, (d/(2CN))^{1/\omega})\big) \le d/2$.

We claim that at least $(1/2)^{\omega}N$ points in $\{x_N, \dots, x_{2CN/d}\}$ are $\big(d/(2CN)\big)^{1/\omega}$ separated. To see this, observe that if $y_1, \dots, y_k$ is a collection of points lying within $\big(d/(2CN)\big)^{1/\omega}$ of each other, then

$$\mu\left(\bigcup_{i=1}^{k} B\big(y_i, (d/(2CN))^{1/\omega}\big)\right) \leq 2^{\omega}\frac{d}{2N}.$$

(This union is contained in a ball of radius $2\big(d/(2CN)\big)^{1/\omega}$.) Therefore, there must be at least $(1/2)^{\omega}N$ points in the set $\{x_N, \dots, x_{2CN/d}\}$ that are $\big(d/(2CN)\big)^{1/\omega}$ separated. To summarize, this argument shows that $c = (1/2)^{\omega}d/(2C)$ and $e = 1$. □

The local version follows by realizing that the the previous proof works with $M = 2C/\big(d\mu(J)\big)$, $c = (1/2)^{\omega}d/(2C)$, and $e = 1$.

Corollary 8. *There exists a function* $\alpha\colon \mathbb{R}^+ \to \mathbb{R}^+$ *such that if there exist a ball* J *and* $d > 0$ *for which for all* N $\mu\big(\bigcup_{i=1}^{N} B(x_i, (1/N)^{1/\omega}) \cap J\big) \geq d\mu(J)$, *then for any standard sequence* $\mathbf{a}$, $\mu\big(\limsup_{n\to\infty} B(x_n, (a_n)^{1/\omega}) \cap J\big) > \alpha(d)\mu(J)$. *Moreover,* $\alpha(d) > (1/2)^{2\omega+1}(1/C)(1/2)^{\omega}\big(d/(2C)\big)$.

Remark 14. For ease of notation, let $\alpha^{-1}(s) = \inf\{d > 0 : \alpha(d) > s\}$.

Proof of Theorem 4. By Lemma 6, the conditions of Theorem 4 imply that for any standard sequence $\mathbf{a}$, the complement of $\limsup_{n\to\infty} B(x_n, (a_n)^{1/\omega})$ has no density points. Theorem 8 implies that $\limsup_{n\to\infty} B(x_n, (a_n)^{1/\omega})$ has full measure. □

Remark 15. A sequence $\{x_1, x_2, \dots\}$ in $[0, 1)$ is BC if for all $\delta > 0$ there are constants $\epsilon > 0$ and M_ϵ such that $\mu(X\backslash M_\epsilon) < \delta$ and

$$\limsup_{r\to 0}\liminf_{N\to\infty} \mu\left(\bigcup_{i=1}^{N} B(x_i, (1/N)^{1/\omega}) \cap B(z, r)\right) > \epsilon\mu(B(z, r))$$

for $z \in M_\epsilon$.

We now continue to the proof of Theorem 6, which requires a covering theorem.

Theorem 9 ([15] Theorem 1.6)**.** *Let* A *be a measurable set and* F *be a family of balls* F *such that* $\liminf\{r > 0 : b(a, r) \in F\} = 0$ *for all* $a \in A$. *Almost all of* A *can be covered with a disjoint countable collection of balls in* F.

Corollary 9. *Let* A *be a measurable set such that* $\mu(A) < \infty$, *and let* $\epsilon > 0$. *There exists a finite number* $N_{A,\epsilon}$ *such that all but* ϵ *of* A *can be covered with* $N_{A,\epsilon}$ *disjoint balls in* F.

Proof of Theorem 6. Assume $\mathbf{x}$ is not BC. Thus, there exists a standard sequence $\mathbf{a}$ such that $\mu((\limsup_{n\to\infty} B(x_n,(a_n)^{1/\omega}))^c) > 0$. We define the following sets:

$$S = \limsup_{n\to\infty} B(x_n,(a_n)^{1/\omega}),$$
$$R_{t,\delta} = \{y \in S^c : \mu(B(y,\delta')\cap S^c) > \mu(B(y,\delta))t \text{ for all } \delta' < \delta\}.$$

By the existence of density points (Theorem 8), $\mu\left(S^c \cap \bigcup_{n=1}^{\infty} R_{t,1/n}\right) = \mu(S^c)$ for any $t < 1$.

We also define families of balls (which we use for covering arguments):

$$F_{t,s} = \left\{B(y,r)\colon y \in R_{t,r},\ \mu(B(y,r)\backslash B(y,(1-s)r)) < 2^{\omega+1}\frac{Cr^{\omega}-(r/2)^{\omega}/C}{1/(2s)}\right\}.$$

Remark 16. For t, s sufficiently close to 0, and μ Ahlfors, this family satisfies the hypothesis of Theorem 9 because of Theorem 8 and the fact that there are at least $(1/(2s)) - 1$ disjoint annuli $B(y,r')\backslash B(y,(1-s)r')$ between $B(y,r/2)$ and $B(y,r)$. The ball contained in these annuli has radius at least $r/2$. (For more sophisticated coverings along this line, see [10, Appendix 1], which describes a generalization of dyadic cubes.) This small boundary condition is needed to obtain this setting's version of Corollary 7.

Armed with the small boundary condition on balls in $F_{t,s}$, we proceed with an argument similar to that used in the case of $\mathcal{I}$ with λ measure.

Choose δ so small that $R_{999/1000,\delta} \neq \varnothing$. Let $y_1 \in R_{999/1000,\delta}$. By Corollary 8 and the definition of α^{-1} in Remark 14, there exist infinitely many N such that $\mu\left(\bigcup_{i=1}^{N} B(x_i,(1/N)^{1/\omega}) \cap B(y_1,\delta)\right) \leq \mu(B(y_1,\delta))/(1000\alpha)$. Pick one such N and denote it by N_1.

For the inductive step, cover most of $B\left(y_{k-1}^{(1)}, r_{k-1}^{(1)}\right), \ldots, B\left(y_{k-1}^{(t_{k-1})}, r_{k-1}^{(t_{k-1})}\right)$ by choosing t_k points $y_k^{(1)}, \ldots, y_k^{(t_k)}$ with corresponding radii, $r_k^{(1)}, \ldots, r_k^{(t_k)}$ that satisfy the following.

1. $B(y_k^{(i)}, t_k^{(i)}) \in F_{1-10^{-2k},(C^{4/\omega}4^{-k})^{1/2}}$.
2. $B\left(y_k^{(j)}, r_k^{(j)}\right) \subset \bigcup_{i=1}^{t_{k-1}} B\left(y_{k-1}^{(i)}, r_{k-1}^{(i)}\right)$ for all j.
3. $\mu\left(B(y_{k-1}^{(i)}, r_{k-1}^{(i)}) \cap \bigcup_{i=1}^{t_k} B\left(y_k^{(i)}, r_k^{(i)}\right)\right) > (1-10^{-k})\mu\left(B(y_{k-1}^{(i)}, r_{k-1}^{(i)})\right)$ for each $y_{k-1}^{(i)}$.
4. The $B(y_k^{(i)}, r_k^{(i)})$ are all disjoint.

To see that balls can be chosen that satisfy these conditions, first observe that for all $s < 1$, we have

$$\lim_{n\to\infty} \mu\left(S^c \cap \bigcup_{i=1}^{t_{k-1}} B\left(y_{k-1}^{(i)}, r_{k-1}^{(i)}\right) \cap R_{s,1/n}\right) = \mu\left(S^c \cap \bigcup_{i=1}^{t_{k-1}} B\left(y_{k-1}^{(i)}, r_{k-1}^{(i)}\right)\right),$$

which, by induction, is greater than $(1-10^{-2(k-1)})\mu\big(\bigcup_{i=1}^{t_{k-1}} B(y_{k-1}^{(i)}, r_{k-1}^{(i)})\big)$ if $k>2$ and greater than $(1-10^{-4})\mu(B(y_1,\delta))$ if $k=2$. Therefore,

$$\mu\bigg(\bigcup_{i=1}^{t_{k-1}} B(y_{k-1}^{(i)}, r_{k-1}^{(i)})\Big\backslash \bigcup_{n=1}^{\infty}\bigcup_{z\in W_n} B\Big(z,\frac{1}{n}\Big)\bigg) \le \begin{cases}(1-10^{-2(k-1)})\mu\big(\bigcup_{i=1}^{t_{k-1}} B(y_{k-1}^{(i)}, r_{k-1}^{(i)})\big) & \text{if } k>2,\\ (1-10^{-4})\mu(B(y_1,\delta)) & \text{if } k=2,\end{cases}$$

where

$$W_n=\Big\{x\in R_{1-\epsilon,1/n} | B(x,1/n)\subset \bigcup_{i=1}^{t_{k-1}} B\big(y_{k-1}^{(i)}, r_{k-1}^{(i)}\big)\Big\}.$$

By Theorem 9 (which gives the disjointness of $B(y_k^{(i)}, r_k^{(i)})$), if $k>2$, it is possible to cover all but a subset of measure $(1-10^{-2(k-1)})\mu\big(\bigcup_{i=1}^{t_{k-1}} B(y_{k-1}^{(i)}, r_{k-1}^{(i)})\big)$ of $\bigcup_{i=1}^{t_{k-1}} B(y_{k-1}^{(i)}, r_{k-1}^{(i)})$ and if $k=2$, all but a subset of measure $(1-10^{-4})\mu(B(y_1,\delta))$ of this same set by a countable number of $B(y_k^{(i)}, r_k^{(i)})$ satisfying Conditions 1–4. Therefore, we can cover all but a set of measure $(1-10^{-k})\mu\big(\bigcup_{i=1}^{t_{k-1}} B(y_{k-1}^{(i)}, r_{k-1}^{(i)})\big)$ by a finite number of $B(y_k^{(i)}, r_k^{(i)})$ satisfying Conditions 1–4.

By Condition 1 and Corollary 7, we cannot have

$$\mu\bigg(\bigcup_{i=1}^{N} B(x_i,(1/N)^{1/\omega})\cap \bigcup_{j=1}^{t_k} B(y_k^{(j)}, r_k^{(j)})\bigg) > \alpha^{-1}(10^{-2k})\mu\bigg(\bigcup_{i=1}^{t_k} B(y_k^{(i)}, r_k^{(i)})\bigg)$$

for all but finitely many N. This implies that for infinitely many N, there exists a subset $H_N\subset\{1,\dots,t_k\}$, $\lambda\big(\bigcup_{i\in H_N} B(y_k^{(i)}, r_k^{(i)})\big) > (1-10^{-k})\lambda\big(\bigcup_{i=1}^{t_k} B(y_k^{(i)}, r_k^{(i)})\big)$, such that for each $j\in H_N$,

$$\mu\bigg(\bigcup_{i=1}^{N} B(x_i,(1/N)^{1/\omega})\cap B(y_k^{(j)}, r_k^{(j)})\bigg) < \alpha^{-1}(2\cdot 10^{-k})\mu\big(B(y_k^{(j)}, r_k^{(j)})\big). \tag{2}$$

(As before, our choice of 2 depends on how closely we can divide up the measure.) Pick one of these N times N_k, and the corresponding collection $\mathcal{U}_k$. Note that for any $z\in B(y_k^{(i)}, (1-(C^{4/\omega}4^{-k/\omega})^{1/2})r_k^{(i)})$, where $y_k^{(i)}\in\mathcal{U}_k$, we have

$$\mu\bigg(\bigcup_{j=1}^{N_k} B(x_j,(1/N_k)^{1/\omega})\cap B(z, r_k^{(i)}-d(z,y_k^{(i)}))\bigg) < 4^{k/2}\cdot\alpha^{-1}(2\cdot 10^{-k})\cdot\mu\big(B(z, r_k^{(i)}-d(z,y_k^{(i)}))\big).$$

This is obtained by assuming the worst possible case

$$\bigcup_{j=1}^{N_k} B\big(x_j,(1/N_k)^{1/\omega}\big)\cap B\big(y_k^{(i)}, r_k^{(i)}\big)\subset B(z, r_k^{(i)}-d(z,y_k^{(i)})),$$

$\mu(B(y_k^{(i)}, r_k^{(i)}))$ is as large as possible, and $\mu\big(B(z, r_k^{(i)} - d(z, y_k^{(i)}))\big)$ is as small as possible.

Our set of times is $A = \{N_1, N_2, \ldots\}$. We show that

$$f_A^{-1}(0) \supset \bigcup_{r=1}^{\infty} \bigcap_{k=r}^{\infty} \bigcup_{i \in \mathcal{U}_k} B\big(y_k^{(i)}, (1 - (C^{4/\omega} 4^{-k/\omega})^{1/2}) r_k^{(i)}\big).$$

This has positive measure because at each step, at most 10^{-k} of the measure is removed, the choice of $\mathcal{U}_k$ avoids at most 10^{-k} of the measure, and the annuli only avoids at most $C^2 2^{\omega} (C^{4/\omega} 4^{-k/\omega})^{1/2}$ of the measure (by the definition of $F_{t,s}$). If

$$z \in \bigcup_{r=1}^{\infty} \bigcap_{k=r}^{\infty} \bigcup_{i=1}^{t_k} B(y_k^{(i)}, 1 - (C^{4/\omega} 4^{-k/\omega})^{1/2} r_k^{(i)}),$$

then for all sufficiently large k, there exists i such that

$$|y_k^{(i)} - z| < (1 - (C^{4/\omega} 4^{-k/\omega})^{1/2}) r_k^{(i)}.$$

A sequence of radii tending to zero is given by $r_k^{(i)} - |y_k^{(i)} - z|$. The following lemma and its corollary show that, in this case, $f_A(z) = 0$.

Lemma 8. *Given $\epsilon > 0$, there exists m_ϵ such that for all $m > m_\epsilon$,*

$$\mu\left(\bigcup_{i=1}^{N_m} B(x_i, (1/N_m)^{1/\omega}) \cap B(y_k^{(i)}, r_k^{(i)})\right) < \left(\frac{10}{9} \cdot 10^{-k} + \epsilon\right) \mu\big(B(y_k^{(i)}, r_k^{(i)})\big).$$

Proof. This follows from the fact that by Condition 3,

$$\mu\left(\bigcup_{i=1}^{t_m} B(y_m^{(i)}, r_m^{(i)}) \cap B(y_k^{(i)}, r_k^{(i)})\right) > \frac{10}{9} \cdot 10^{-k} \mu(B(y_k^{(i)}, r_k^{(i)}))$$

for any $m > k$ and that for large enough m, $\alpha^{-1}(10^{-m}(r_k^{(i)})) + 10^{-m} \mu\big(B(y_1, \delta)\big) < \epsilon \mu\big(B(y_k^{(i)}, r_k^{(i)})\big)$. (Notice that $\mathcal{U}_m$ excludes at most $10^{-m} \mu\big(B(y_1, \delta)\big)$. The lemma follows from (2) by assuming the worst possible estimate on the portion not covered by $B(y_m^{(1)}, r_m^{(1)}), \ldots, B(y_m^{(t_m)}, r_m^{(t_m)})$. □

The following corollary is immediate.

Corollary 10. *If $z \in B\big(y_k^{(i)}, (1 - (C^{4/\omega} 4^{-k/\omega})^{1/2}) r_k^{(i)}\big)$, then for sufficiently large m,*

$$\mu\left(\bigcup_{i=1}^{N_m} B(x_i, (1/N_m)^{1/\omega}) \cap B(z, r_k^{(i)} - d(z, y_k^{(i)})\right) < \mu(B(z, r_k^{(i)} - d(z, y_k^{(i)}))) 4^{k/2} \left(\frac{10}{9} \cdot 10^{-k} + \epsilon\right).$$

We now turn to proving the other direction of Theorem 6. Assume there exist $A = \{N_1, N_2, \ldots\}$ such that $f_A^{-1}\{0\} \supset B$, where $\mu(B) > 0$. By definition of f_A, for each $y \in B$, there exist $r_i(y)$ such that

$$\mu\left(\bigcup_{j=1}^{N_k} B(x_j, (1/N_k)^{1/\omega}) \cap B(y, r_i(y))\right) < \frac{1}{4^i}\mu\big(B(y, r_i(y))\big)$$

for all $k > k_i(y)$. There exists $l_i \in \mathbb{N}$ such that $\mathcal{V}_i = \{y \in B : k_i(y) < l_i)$ satisfies $\mu(\mathcal{V}_i) > (1 - 10^{-i})\mu(B)$. The sequence a_i is defined by $a_i = 1/N_{l_j}$ for $N_{l_{j-1}} < i \leq N_{l_j}$. By our choice of $a_1, a_2, \ldots$, we have that points in $\bigcap_{i=1}^{\infty} \mathcal{V}_i$ are not density points for $\limsup_{n\to\infty} B(x_n, (a_n)^{1/\omega})$. Also, $\mu\left(\bigcap_{i=1}^{\infty} \mathcal{V}_i\right) \geq 8/9\mu(B)$. □

4 s-BC

We now define a modification of Borel-Cantelli sequences which is related to the s-monotone shrinking target property introduced in [20].

Definition 7. A sequence $\mathbf{x} \subset X$ is called **s-BC** if for every monotonic sequence $\{a_i\}_i^{\infty}$ such that $\sum_{i=1}^{\infty} a_i^s = \infty$, $\lambda(\limsup_{n\to\infty} B(x_n, a_n)) = 1$.

Remark 17. This property is interesting in the case $s > 1$, in which case s-BC is a weaker condition than BC; i.e., any BC sequence is s-BC for all $s \geq 1$.

Lemma 9. *Suppose $s > 1$ and $a_1, a_2, \ldots$ is a decreasing sequence such that $\sum_{i=1}^{\infty} a_i^s = \infty$. Then the sequence $\mathbf{b}$ given by $b_i = a_{\lfloor i^s \rfloor}$ is standard.*

Proof. Because the sequence $\{a_i\}$ is decreasing, $\sum_{i=1}^{\infty} a_i^s = \infty$ if and only if for any $M \in \mathbb{N}$, $\sum_{i=1}^{\infty} M^{i-1} a_{M^i}^s = \sum_{i=1}^{\infty}((M^{1/s})^{i-1} a_{M^i})^s = \infty$. Because $s > 1$,

$$\infty = \sum_{i=1}^{\infty}(M^{1/s})^{i-1} a_{M^i} \leq \sum_{i=1}^{\infty}(M^{1/s})^{i-1} b_{(M^{1/s})^i}.$$

□

The proofs of theorems in this section follow from results in Section 1 after passing to an appropriate subsequence, and are therefore omitted.

Theorem 10. *Suppose there exists $d > 0$ such that for every interval J, $\liminf_{N\to\infty} \lambda\left(\bigcup_{i=1}^{N^s} B(x_i, 1/N) \cap J\right) \geq d\lambda(J)$. Then $\mathbf{x}$ is s-Borel Cantelli.*

The key lemma in this setting is the following result.

Lemma 10. *Let $M \in \mathbb{N}$, $c > 0$, and $e > 0$ be constants, $\mathbf{x}$ a sequence in $\mathcal{I}$, and $\mathbf{a}$ a decreasing sequence such that $\sum_{n=1}^{\infty} a_n^s = \infty$. Suppose that for* *all $r \in \mathbb{N}$, at least cM^r of the points in $\{x_{M^{(r-1)s}}, x_{M^{(r-1)s}+1}, \dots, x_{M^{sr}}\}$ are e/M^r separated from each other. Then there exists $\delta > 0$, depending only on c and e, such that $\lambda(\limsup_{n\to\infty} B(x_n, a_n)) > \delta$.*

Definition 8. Let $A = \{N_1, N_2, \dots\}$ be an infinite increasing sequence of natural numbers. Given $\mathbf{x}$, define

$$f_A(z) := \liminf_{r\to 0^+} \limsup_{N\in A} \frac{\lambda\left(\bigcup_{i=1}^{N^s} B(x_i, 1/N) \cap B(z, r)\right)}{\lambda(B(z, r))}.$$

Theorem 11. *A sequence is not s-BC if and only if there exists a sequence $A = \{N_1, N_2, \dots\}$ such that $\lambda(f_A^{-1}(\{0\})) > 0$.*

Remark 18. We could have rephrased the theorems in this section substituting N^s for N. Similarly, the theorems in the previous section could be rephrased substituting N^ω for N. One can treat s-Borel-Cantelli in Ahlfors regular spaces of dimension ω.

5 Properties

For completeness, we include some basic properties of BC sequences.

Proposition 1. *Suppose that $\{k_1, k_2, \dots\} \subset \mathbb{N}$ has positive lower density and $\{x_{k_1}, x_{k_2}, \dots\}$ is BC. Then so is $\{x_1, x_2, \dots\}$.*

Proposition 2. *If $x_1, x_2, \dots$ is Borel-Cantelli and $\{k_1, k_2, \dots\} \subset \mathbb{N}$ has density 1, then $x_{k_1}, x_{k_2}, \dots$ is Borel-Cantelli.*

Remark 19. Proposition 2 states that the property of being Borel-Cantelli survives the deletion of any sequence of density 0. The same need not be true for sequences of positive upper density (even if they have lower density 0).

Definition 9. Given a sequence $\mathbf{x}$, we say a measure ν is a weak-* limit point of $\mathbf{x}$ if it is a weak-* limit point of the sequence of measures

$$\left\{\delta_{x_1}, \frac{1}{2}(\delta_{x_1} + \delta_{x_2}), \dots, \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}, \dots\right\},$$

where δ_z denotes the point mass at z.

Proposition 3. *If $\mathbf{x}$ is a Borel-Cantelli sequence in $\mathfrak{I}$, then Lebesgue measure is absolutely continuous with respect to its weak-* limit points.*

Proof. Pick a sequence $\{N_1, N_2, \ldots\}$ so that $\sum_{i=1}^{N_j} \delta_{x_i}$ weak-* converges to a measure ν that does not have full support. Pick a set S such that $\nu(S) = 1$ but $\lambda(S) < 1$. By the Lebesgue Density Theorem and the definition of weak-* convergence, if $A = \{N_1, N_2, \ldots\}$, then $\lambda(f_A^{-1}(\{0\}) \cap S^c) = \lambda(S^c)$. □

Remark 20. Proposition 3 holds for Ahlfors regular spaces as well.

Definition 10. Given a sequence $\mathbf{x}$ in an ω Ahlfors regular space (X, μ), we say $\{y_1, y_2, \ldots\}$ is an l^p **perturbation of** $\bar{x}$ if $\sum_{i=1}^{\infty} d(x_i, y_i)^p$ converges.

Proposition 4. *If* $\mathbf{x}$ *is a Borel-Cantelli sequence in an* ω *Ahlfors regular space* (X, d, μ)*, then any* l^p *perturbation of* $\mathbf{x}$ *for* $p \leq 1/\omega$ *is also a Borel-Cantelli sequence.*

Proof. If $\mathbf{x}$ is not Borel-Cantelli, then there exist $A \subset \mathbb{N}$ and a measurable set S with $\mu(S) > 1$ and $f_A(S) = 0$. Let $\mathbf{y}$ be a $l^{1/\omega}$ perturbation of $\mathbf{x}$. The same A and S show that $\mathbf{y}$ is not Borel-Cantelli. □

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