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Disjoint mixing operators

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Abstract

Chan and Shapiro showed that each (non-trivial) translation operator $f(z) \overset{T_\lambda}{\mapsto} f(z+\lambda)$ acting on the Fréchet space of entire functions endowed with the topology of locally uniform convergence supports a universal function of exponential type zero. We show the existence of d-universal functions of exponential type zero for arbitrary finite tuples of pairwise distinct translation operators. We also show that every separable infinite-dimensional Fréchet space supports an arbitrarily large finite and commuting disjoint mixing collection of operators. When this space is a Banach space, it supports an arbitrarily large finite disjoint mixing collection of C_0-semigroups. We also provide an easy proof of the result of Salas that every infinite-dimensional Banach space supports arbitrarily large tuples of dual d-hypercyclic operators, and construct an example of a mixing Hilbert space operator T so that (T, T^2) is not d-mixing.

Keywords: Hypercyclic operators; Mixing operators; Disjoint mixing

1. Introduction

This paper deals with two themes of linear dynamics – the existence of universal functions of slow growth, and the existence of hypercyclic operators and of operator semigroups on Fréchet

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spaces – and on how they extend to the setting of disjointness in linear dynamics introduced in [6] and [8].

The study of the first theme was initiated by Duyos-Ruiz [17], who extended a classical result of Birkhoff [10] by showing the existence of entire functions f of arbitrary small growth order whose set of translates

$$f(z), \quad f(z+1), \quad f(z+2), \quad \ldots$$

is dense in the space $H(\mathbb{C})$ of entire functions on the complex plane and endowed with the compact open topology. Chan and Shapiro [13] showed that each (non-trivial) translation operator

$$f(z) \overset{T_\lambda}{\mapsto} f(z+\lambda)$$

is not only hypercyclic on $H(\mathbb{C})$ but on Hilbert spaces of entire functions of growth order one and of exponential type zero. Analogue extensions were done to Seidel and Walsh's [27] hypercyclicity result on non-Euclidean translations acting on the space $H(\mathbb{D})$ of holomorphic functions on the unit disc: Bourdon and Shapiro [12] and Gallardo-Gutiérrez and Montes-Rodríguez [18] showed the existence of slow-growth universal functions by showing the hypercyclicity of these non-Euclidean translations on the Hardy space and on weighted Dirichlet spaces.

With respect to the second theme, it is well known that every separable, infinite-dimensional Fréchet space X supports a hypercyclic operator T, thanks to the works of Ansari [1], Bernal-González [5], and Bonet and Peris [11]. Indeed, Grivaux [19] further showed that we may obtain such T to be mixing. Also, Salas [25] showed that if X is a Banach space with separable dual X^*, then one may obtain T to be dual-hypercyclic, that is, so that T is hypercyclic on X and its adjoint T^* is hypercyclic on X^*. Bermúdez, Bonilla, Conejero, and Peris [4] showed that every separable infinite-dimensional Banach space supports a topologically mixing holomorphic uniformly continuous semigroup of operators.

Several of the above mentioned results have been extended to the setting of disjointness (cf. Section 1.3 for definitions) by Bernal-González [6], Salas [26], and the authors [8,29,7,9]. Any k-tuple $(T_1, \ldots, T_k)$ of different (non-trivial) translations is d-mixing on $H(\mathbb{C})$ [6, Proposition 5.5], [8, Theorem 3.1]. When the k-tuple consists of non-Euclidean translations, it is d-mixing on $H(\mathbb{D})$ [6, Proposition 5.6], [9, Theorem 4], and also on the Hardy space and on certain weighted Dirichlet spaces as well [9, Theorem 3]. With regards to the second theme, we know that every separable, infinite-dimensional Fréchet space X supports a d-hypercyclic k-tuple of operators $(T_1, \ldots, T_k)$ of arbitrary length [26, Theorem 3.2], [29, Theorem D]. When X is a Banach space with separable dual X^*, it supports arbitrary long dual d-hypercyclic k-tuples $(T_1, \ldots, T_k)$, that is, so that $(T_1, \ldots, T_k)$ and $(T_1^*, \ldots, T_k^*)$ are d-hypercyclic on X and X^*, respectively [26, Theorem 3.4].

1.1. Main results

In this paper we extend the earlier mentioned result of Chan and Shapiro by showing that any finite collection of different translation operators acting on Hilbert spaces of growth order one and of exponential type zero is d-mixing (Corollary 2.2). Indeed, we show that this is a consequence of Theorem 1.3 below, a rather general result about finite k-tuples of operators induced by series of powers of a "backward shift" operator. To state this precisely, we need the following definitions.

Definition 1.1. An operator T on a topological vector space X is called a *backward space shift* provided there exists a sequence $\{X_n\}_{n\in\mathbb{Z}_+}$ of linear subspaces of X such that their sum is dense in X, $X_0=\{0\}$ and $T(X_{n+1})$ is a dense subspace of X_n for each $n\in\mathbb{Z}_+$.

For instance, if X is a space of scalar sequences $\{x_n\}_{n\in\mathbb{Z}_+}$ such that the sequences with finite support form a dense subspace of X, then any continuous backward weighted shift on X is a backward space shift. It is also worth noticing that, without loss of generality, the sequence $\{X_n\}$ in Definition 1.1 may be assumed to be increasing. Indeed, nothing changes if we pass from $\{X_n\}$ to $\{X_0+\cdots+X_n\}$.

Definition 1.2. Let X be a topological vector space over the real or complex scalar field $\mathbb{K}$ and $T\in L(X)$. We say that T is *exponentiable* if for any $z\in\mathbb{K}$ and $x\in X$, the series

$$e^{zT}x=\sum_{j=0}^{\infty}\frac{z^j}{j!}T^jx$$

converges in X and defines a strongly continuous group $\{e^{zT}\}_{z\in\mathbb{K}}$ of continuous linear operators.

Clearly, any continuous linear operator on a Banach space is exponentiable, and so is any nilpotent continuous linear operator on a topological vector space.

Theorem 1.3. *Let X be a topological vector space, $T\in L(X)$ a backward space shift, and $T_1,\dots,T_m\in L(X)$ be given by the formulae*

$$T_j=\sum_{l=0}^{\infty}a_{j,l}T^l \quad \text{with } a_{j,l}\in\mathbb{K},$$

where the series in the right-hand side converges pointwise. Suppose that $a_{1,0},\dots,a_{m,0}$ have modulus one, and that $a_{1,1}/a_{1,0},\dots,a_{m,1}/a_{m,0}$ are non-zero and pairwise distinct. Then the m-tuple $(T_1,\dots,T_m)$ is d-mixing. In particular, given pairwise distinct non-zero scalars $z_1,\dots,z_m$, the m-tuple $(I+z_1T,\dots,I+z_mT)$ is d-mixing. If additionally, T is exponentiable, then the m-tuple $\{e^{z_1zT},\dots,e^{z_mzT}\}_{z\in\mathbb{K}}$ of operator groups is d-mixing.

As another application of Theorem 1.3 we obtain a short proof of Salas' result on the existence of dual d-hypercyclic tuples $(T_1,\dots,T_r)$, with the added bonus of $T_1,\dots,T_r$ being pairwise commuting.

Theorem 1.4. *Let X be an infinite-dimensional Banach space with separable dual. Then there exists $T\in L(X)$ such that $(T,T^2,\dots,T^r)$ and $(T^*,T^{*2},\dots,T^{*r})$ are d-transitive on X and X^* respectively for every $r\in\mathbb{N}$.*

We stress that there is no hope of replacing d-transitivity by d-mixing in the above theorem. Indeed, it is easy to show that for a dual hypercyclic operator T, neither T nor T^* can be mixing.

We also provide results on the existence of d-mixing finite collections of operators and of operator semigroups acting on separable, infinite-dimensional Fréchet spaces. In particular, the case $r=1$ of Theorem 1.5 below extends the earlier mentioned result of Bermúdez et al.

[4, Theorem 2.4]. Recall that the Fréchet space ω denotes the product of countably many copies of $\mathbb{K}$.

Theorem 1.5. *Let X be a separable infinite-dimensional Fréchet space non-isomorphic to ω and $r \in \mathbb{N}$. Then there exist pairwise commuting exponentiable operators $S_1, \ldots, S_r$ on X such that the r-tuple $\{e^{zS_1}, \ldots, e^{zS_r}\}_{z\in\mathbb{K}}$ of operator groups is strongly d-mixing.*

The condition of X being non-isomorphic to ω in Theorem 1.5 is necessary. Indeed, there is no transitive strongly continuous operator semigroup $\{T_t\}_{t\geqslant 0}$ on ω [14]. However, this condition can be dropped for discrete semigroups.

Corollary 1.6. *Let X be a separable infinite-dimensional Fréchet space and $r \in \mathbb{N}$. Then there exist pairwise commuting operators $S_1, \ldots, S_r$ on X such that the r-tuple $(S_1, \ldots, S_r)$ is strongly d-mixing.*

Finally, we also study conditions for r-tuples of the form $(T, T^2, \ldots, T^r)$ to be d-mixing. When T is a mixing weighted shift on $\ell^2(\mathbb{Z})$ or a mixing composition operator on either $H(\mathbb{D})$ or a weighted Dirichlet space, its r-tuples are always d-mixing, see Remark 3.5. In contrast, we construct a mixing Hilbert space operator for which the tuple (T, T^2) is not d-mixing (Theorem 3.8).

1.2. Organization of the paper

The paper is organized as follows: In Section 2 we show how Theorem 1.3 gives that a finite collection of different translations on Hilbert spaces of entire functions of growth order one and of exponential type 0 is necessarily d-mixing. We also provide many examples of backward space shifts to showcase the applicability of Theorem 1.3. Section 3 is devoted to studying d-mixing on tuples of operators of the form $(T, T^2, \ldots, T^r)$. In Section 4 we prove the commutative version of Salas' result on the existence of dual d-hypercyclic tuples (Theorem 1.4). In Section 5 we introduce the concept of backward space shift tuples, which we need to obtain Theorem 5.7, a general result from which on later sections we derive Theorem 1.5 and Corollary 1.6. Since most of our techniques are based on finite-dimensional matrices, we face these basic results in Section 6. The proofs of the main results are then given in Sections 7 and 8. Appendix A contains technical results on Sobolev spaces we use for showing Theorem 3.8, and calculations of special determinants appearing in the proof of Theorem 1.3.

We conclude this introduction with a subsection on notation and definitions. For general background on linear dynamics and hypercyclicity, we refer the reader to the recent books by Bayart and Matheron [3] and by Grosse-Erdmann and Peris Manguillot [20].

1.3. Notation and definitions

All vector spaces in this article are over the field $\mathbb{K}$, being either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers and all topological spaces *are assumed to be Hausdorff*. As usual, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{Z}$ is the set of integers, $\mathbb{Z}_+$ is the set of non-negative integers, $\mathbb{N}$ is the set of positive integers and $\mathbb{N}_n = \{1, \ldots, n\}$ for $n \in \mathbb{N}$. We denote by

$$\mathbb{K}^{[m]} = \{z \in \mathbb{K}^m\colon\ z_j \neq 0 \text{ for } 1 \leqslant j \leqslant m \text{ and } z_j \neq z_k \text{ for } 1 \leqslant j < k \leqslant m\}. \tag{1.1}$$

The symbol $L(X)$ stands for the space of continuous linear operators on a topological vector space X and X^* is the space of continuous linear functionals on X. For a subset A of a vector space X, $\operatorname{span}(A)$ stands for the linear span of A and $\overline{\operatorname{span}}(A)$ is the closure of $\operatorname{span}(A)$. Recall that an $\mathcal{F}$-space is a complete metrizable topological vector space and that a Fréchet space is a locally convex $\mathcal{F}$-space.

Definition 1.7. A sequence $\{(T_{1,n},\dots,T_{k,n})\}_{n\in\mathbb{Z}_+}$ of k-tuples of continuous self-maps on a topological space X is called *d-mixing* (respectively, *d-transitive*), where 'd' stands for *diagonal* or *disjoint*, if for any non-empty open subsets $V_0,\dots,V_k$ of X,

$$V_0\cap\bigcap_{j=1}^{k}T_{j,n}^{-1}(V_j)\neq\emptyset$$

for any sufficiently large n (respectively, for infinitely many n's). We say that $\{(T_{1,n},\dots,T_{k,n})\}_{n\in\mathbb{Z}_+}$ is *d-universal* if there is $x\in X$ such that the orbit $\{(T_{1,n}x,\dots,T_{k,n}x):\ n\in\mathbb{Z}_+\}$ is dense in X^k. Such an x is called a *d-universal element for* $\{(T_{1,n},\dots,T_{k,n})\}_{n\in\mathbb{Z}_+}$. If the set of d-universal elements for $\{(T_{1,n},\dots,T_{k,n})\}_{n\in\mathbb{Z}_+}$ is dense in X, we say that $\{(T_{1,n},\dots,T_{k,n})\}_{n\in\mathbb{Z}_+}$ is *densely d-universal*.

A k-tuple $(T_1,\dots,T_k)$ of continuous self-maps on X is called *d-mixing* (respectively, *d-transitive*) if the sequence $\{(T_1^n,\dots,T_k^n)\}_{n\in\mathbb{Z}_+}$ is d-mixing (respectively, d-transitive). If additionally X is a topological vector space and $T_j\in L(X)$, then d-universality goes under the name *d-hypercyclicity*.

Remark 1.8. An application of the Baire theorem shows that if X is Baire and second countable, then a sequence $\{(T_{1,n},\dots,T_{k,n})\}_{n\in\mathbb{Z}_+}$ is d-transitive if and only it is densely d-universal. Clearly, a sequence $\{(T_{1,n},\dots,T_{k,n})\}_{n\in\mathbb{Z}_+}$ is d-mixing if and only if its every subsequence is d-transitive.

If $\{(T_{1,k},\dots,T_{m,k})\}_{k\in\mathbb{Z}_+}$ is a sequence of m-tuples of continuous self-maps on a topological space X, we denote

$$\Sigma\bigl(\{T_{1,k},\dots,T_{m,k}\}_{k\in\mathbb{Z}_+}\bigr)\tag{1.2}$$

to be the set of $(x_0,\dots,x_m)\in X^{m+1}$ for which there exists a sequence $\{u_k\}_{k\in\mathbb{N}}$ in X such that $u_k\to x_0$ and $T_{j,k}u_k\to x_j$ for $1\leqslant j\leqslant m$ as $k\to\infty$. If $T_1,\dots,T_m$ are continuous self-maps on a topological space X, we write $\Sigma(T_1,\dots,T_m)$ instead of $\Sigma(\{T_1^k,\dots,T_m^k\}_{k\in\mathbb{Z}_+})$.

Remark 1.9. It is worth noting that if X is second countable, then $\Sigma(\{T_{1,k},\dots,T_{m,k}\}_{k\in\mathbb{Z}_+})$ is closed in X^{m+1}. It is also easy to see that if X is a topological vector space and $T_{j,k}$ are continuous linear operators on X, then $\Sigma(\{T_{1,k},\dots,T_{m,k}\}_{k\in\mathbb{Z}_+})$ is a linear subspace of X^{m+1}.

We also consider the following two modifications of the notion of d-mixing sequences adapted for operator semigroups.

Definition 1.10. Let A be an additive submonoid of $\mathbb{R}^m$ and let $\{T_{1,z}\}_{z\in A},\dots,\{T_{k,z}\}_{z\in A}$ be strongly continuous operator semigroups on a topological vector space X. A k-tuple

$\{T_{1,z},\dots,T_{k,z}\}_{z\in A}$ of operator semigroups is called *d-mixing* if for any non-empty open subsets $V_0,\dots,V_k$ of X, there is $r>0$ such that

$$V_0\cap\bigcap_{j=1}^{k}T_{j,z}^{-1}(V_j)\neq\emptyset\quad\text{for any } z\in A,\ |z|\geqslant r.$$

Equivalently, $\{T_{1,z},\dots,T_{k,z}\}_{z\in A}$ is d-mixing if for any sequence $\{z_n\}_{n\in\mathbb{Z}_+}$ in A satisfying $|z_n|\to\infty$, the sequence $\{(T_{1,z_n},\dots,T_{k,z_n})\}_{n\in\mathbb{Z}_+}$ is d-mixing.

We say that the k-tuple $\{T_{1,z},\dots,T_{k,z}\}_{z\in A}$ of operator semigroups is *strongly d-mixing* if for any non-empty open subsets $V_0,\dots,V_k$ of X, there exists $r>0$ such that

$$V_0\cap\bigcap_{j=1}^{k}T_{j,z_j}^{-1}(V_j)\neq\emptyset\quad\text{for any } z_1,\dots,z_k\in A\text{ satisfying }\min_{1\leqslant j\leqslant k}|z_j|\geqslant r.$$

Equivalently, $\{T_{1,z},\dots,T_{k,z}\}_{z\in A}$ is strongly d-mixing if for any sequence $\{z_n=(z_{n,1},\dots,z_{n,k})\}_{n\in\mathbb{Z}_+}$ in A^k satisfying $\min_{1\leqslant j\leqslant k}|z_{n,j}|\to\infty$, the sequence $\{(T_{1,z_{n,1}},\dots,T_{k,z_{n,k}})\}_{n\in\mathbb{Z}_+}$ is d-mixing. Finally, we say that $(T_1,\dots,T_k)\in L(X)^k$ is *strongly d-mixing* if $\{T_1^n,\dots,T_k^n\}_{n\in\mathbb{Z}_+}$ is strongly d-mixing.

Of course, any strongly d-mixing tuple of semigroups is d-mixing, but the converse is not true. Indeed, let B be any mixing unilateral backward weighted shift on ℓ_2. Then $\{T_{1,n},T_{2,n}\}_{n\in\mathbb{Z}_+}=\{B^n,B^{2n}\}_{n\in\mathbb{Z}_+}$ is d-mixing [8, Section 4.1], but it is not strongly d-mixing: the sequence $\{z_n=(2n,n)\}_{n\in\mathbb{Z}_+}$ in $\mathbb{Z}_+^2$ satisfies $\min_{1\leqslant j\leqslant 2}|z_{n,j}|\to\infty$, but $\{T_{1,z_{n,1}},T_{2,z_{n,2}}\}_{n\in\mathbb{Z}_+}=\{B^{2n},B^{2n}\}_{n\in\mathbb{Z}_+}$ is not d-transitive.

2. Space shifts and translations on Hilbert spaces of entire functions

While we postpone the proof of Theorem 1.3 to Section 7, we show here some of its consequences. Given a sequence $v=\{v_n\}_{n\in\mathbb{Z}_+}$ of positive weights and $1\leqslant p<\infty$, we consider the Banach space

$$\ell^p(v)=\left\{x=(x_j)\in\mathbb{K}^{\mathbb{Z}_+}:\ \|x\|^p=\sum_{i=0}^{\infty}|x_j|^p v_j<\infty\right\}.$$

The backward shift $\{x_j\}\overset{B}{\mapsto}\{x_{j+1}\}$ is a well-defined continuous operator on $\ell^p(v)$ if and only if the sequence $\{\frac{v_n}{v_{n+1}}\}_{n\in\mathbb{Z}_+}$ is bounded. B is a backward space shift provided it is continuous.

Similarly, we may consider backward shifts on function spaces on $\mathbb{R}_+$. Given a Lebesgue measurable almost everywhere positive function $w:\mathbb{R}_+\to\mathbb{R}_+$, we can consider the space $L^p(w)$ for $1\leqslant p<\infty$ being the space $L^p(\mathbb{R}_+,\mu_w)$, where μ_w is the measure on $[0,\infty)$ with the density w with respect to the Lebesgue measure. We recall that the norm on $L^p(w)$ is defined by the formula

$$\|f\|^p=\int_0^{\infty}\big|f(x)\big|^p w(x)\,dx.$$

As in the sequence space case, it is easy to see that the backward shift $f(x) \overset{B}{\mapsto} f(x+1)$ is continuous on $L^p(w)$ if and only if the function $x \mapsto \frac{w(x)}{w(x+1)}$ is essentially bounded on $[0,\infty)$. In the latter case B is a backward space shift. Indeed, for $n \in \mathbb{N}$ we consider the space X_n of $f \in L^p(w)$, whose support is contained in $[n-1,n]$. Then the sum of X_n is dense in $L^p(w)$, $B(X_1) = \{0\}$ and $B(X_n)$ is a dense subspace of X_{n-1} for $n \geqslant 2$.

We are ready to state an immediate consequence of Theorem 1.3.

Corollary 2.1. *Let X be either $\ell^p(v)$ or $L^p(w)$, $1 \leqslant p < \infty$, and assume that the backward shift B acts continuously on X. Then for each pair I_1, I_2 of disjoint, finite subsets of $\mathbb{K} \setminus \{0\}$, the operator tuple $\{e^{zB}\colon z \in I_1\} \cup \{I + zB\colon z \in I_2\}$ is d-mixing. Moreover, for any finitely many pairwise different non-zero scalars $z_1, \dots, z_m$, the tuple $\{e^{z_1 zB}, \dots, e^{z_m zB}\}_{z \in \mathbb{K}}$ of operator groups is d-mixing.*

We apply Corollary 2.1 to prove the existence of functions of growth order one and of exponential type zero that are d-mixing for translations. We recall the following terminology from [13]. We say that an entire function $\gamma(z) = \sum_{n=0}^{\infty} \gamma_n z^n$ is an *admissible comparison function* provided $\gamma_n > 0$ for $n \in \mathbb{Z}_+$ and the sequence $\{\frac{(n+1)\gamma_{n+1}}{\gamma_n}\}_{n \in \mathbb{Z}_+}$ is decreasing. Each admissible comparison function induces a Hilbert space of entire functions

$$E^2(\gamma) = \left\{ f = \sum_{n=0}^{\infty} \hat{f}(n) z^n\colon \|f\|_{2,\gamma}^2 = \left|\hat{f}(n)\right|^2 \gamma_n^{-2} < \infty \right\},$$

and the functions in this space have growth order one and of exponential type at most $\lim_{n\to\infty} \frac{(n+1)\gamma_{n+1}}{\gamma_n}$ [13, Propositions 1.3 and 1.4]. Chan and Shapiro, extending a result by Birkhoff [10], showed that any non-trivial translation operator is hypercyclic on each of these spaces [13, Theorem 2.1].

Theorem CS. *For each admissible comparison function γ, the translation operator $T_\lambda f(z) = f(z-\lambda)$ is hypercyclic on $E^2(\gamma)$ for every $0 \neq \lambda \in \mathbb{C}$.*

We now extend Chan and Shapiro's result to the setting of disjointness.

Corollary 2.2. *For any pairwise distinct non-zero complex numbers $\lambda_1, \dots, \lambda_r$, the translation operators $T_{\lambda_1}, \dots, T_{\lambda_r}$ are d-mixing on $E^2(\gamma)$.*

Proof. Let $v = \{v_n\}_{n \in \mathbb{Z}_+}$ be an increasing sequence of positive numbers such that the sequence $\{v_n^{-1}(n!\gamma_n)^{-2}\}_{n \in \mathbb{N}}$ is bounded. Then the linear map

$$\Phi : \ell^2(v) \to E^2(\gamma), \qquad \Phi c = \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n$$

is continuous and has dense range. Moreover, it is easy to see that $T_\lambda \Phi = \Phi e^{\lambda B}$ for each $\lambda \in \mathbb{C}$. By Corollary 2.1, the operators $e^{\lambda_1 B}, \dots, e^{\lambda_r B}$ are d-mixing on $\ell^2(v)$. The equality $T_\lambda \Phi = \Phi e^{\lambda B}$ and a standard quasisimilarity argument show that $T_{\lambda_1}, \dots, T_{\lambda_r}$ must be d-mixing on $E^2(\gamma)$. $\square$

To justify the generality in which Theorem 1.3 is stated, the remaining of this section is devoted to providing examples of backward space shifts. We start with the following general observation.

Lemma 2.3. *Let X be a topological vector space and $T \in L(X)$ such that $T(X) = X$ and $\ker^*(T) = \bigcup_{n=1}^{\infty} \ker T^n$ is dense in X. Then T is a backward space shift.*

Proof. Denote $X_n = \ker T^n$ for $n \in \mathbb{N}$. Since T is surjective, we have $T(X_{n+1}) = X_n$ for each $n \in \mathbb{N}$. Obviously, $T(X_1) = \{0\}$. By Definition 1.1, T is a backward space shift on X. $\square$

Corollary 2.4. *Let X be a Banach space and $T \in L(X)$ such that $T(X) = X$ and $\bigcap_{n=1}^{\infty} (T^*)^n(X^*) = \{0\}$. Then T is a backward space shift.*

Proof. The equality $\bigcap_{n=1}^{\infty} (T^*)^n(X^*) = \{0\}$ implies that $\ker^*(T)$ is dense in X. It remains to apply Lemma 2.3. $\square$

Lemma 2.3 and Corollary 2.4 can be applied to various classes of operators, including, for instance, *transfer operators* [2] of non-invertible chaotic measure preserving maps. Instead of giving lengthy definitions, we just provide two examples of such operators. Let $\mathcal{H}_0$ be the closed hyperplane in $L^2[0, 1]$, consisting of functions with zero Lebesgue integral. Then for any $n \in \mathbb{N}$, $n \geqslant 2$ the operator $U_n \in L(\mathcal{H}_0)$,

$$U_n f(x) = \frac{1}{n} \sum_{j=0}^{n-1} f\left(\frac{x + j}{n}\right)$$

is a backward space shift. The above operator is known as the Frobenius–Perron operator of the Renyi map. Now we consider the measure μ on $[-1, 1]$ with the density $\rho(x) = (1 - x)/2$ with respect to the Lebesgue measure. Let $\mathcal{H}_1$ be the hyperplane in $L^2(\mu)$, consisting of functions with zero integral with respect to the measure μ. The Frobenius–Perron operator $U \in L(\mathcal{H}_1)$ of the cusp map $x \mapsto 1 - 2\sqrt{|x|}$ on the interval $[-1, 1]$ is given by the formula

$$Uf(x) = \frac{1}{2}\left(1 - \frac{(1-x)^2}{4}\right) f\left(\frac{(1-x)^2}{4}\right) + \frac{1}{2}\left(1 + \frac{(1-x)^2}{4}\right) f\left(-\frac{(1-x)^2}{4}\right).$$

Applying Corollary 2.4, one can easily see that U is a backward space shift.

We next note that many weighted composition operators on L^p-spaces are backward space shifts. This holds not only for Banach L^p-spaces, but also in the case $0 \leqslant p < 1$. For the sake of completeness we recall the definition. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with μ being σ-finite. Recall that if $0 < p < 1$, then $L^p(\Omega, \mu)$ consists of (classes of equivalence up to being equal almost everywhere with respect to μ of) measurable functions $f : \Omega \to \mathbb{K}$ satisfying $q_p(f) = \int_\Omega |f|^p \, d\mu < \infty$ with the topology defined by the metric $d_p(f, g) = q_p(f - g)$. The space $L^0(\Omega, \mu)$ consists of (equivalence classes of) all measurable functions $f : \Omega \to \mathbb{K}$ with the topology defined by the metric $d_0(f, g) = q_0(f - g)$, where $q_0(h) = \sum_{n=0}^{\infty} \frac{2^{-n}}{\mu(\Omega_n)} \int_{\Omega_n} \frac{|f|}{1+|f|} \, d\mu$ and $\{\Omega_n\}_{n \in \mathbb{Z}_+}$ is a sequence of measurable subsets of Ω such that $\mu(\Omega_n) < \infty$ for each $n \in \mathbb{Z}_+$ and Ω is the union of Ω_n. Although d_0 depends on the choice of $\{\Omega_n\}$, the topology defined by this metric does not depend on this choice. If Ω is a subset of $\mathbb{R}^k$ of positive Lebesgue measure and μ is the restriction of the Lebesgue measure to Ω, we omit the notation for the underlying measure and σ-algebra and simply write $L^p(\Omega)$.

Proposition 2.5. *Let $0 \leqslant p < \infty$, $(\Omega, \mathcal{A}, \mu)$ be a measure space with μ being σ-finite and let $\varphi : \Omega \to \Omega$, $\alpha : \Omega \to \mathbb{K}$ be measurable maps such that the formula $Tf(x) = \alpha(x) f(\varphi(x))$ defines a continuous linear operator on $X = L^p(\Omega, \mathcal{A}, \mu)$. Assume also that $\alpha(x) \neq 0$ almost everywhere, φ is injective, $\varphi(\Omega) \in \mathcal{A}$, $\varphi^{-1} : \varphi(\Omega) \to \Omega$ is measurable and $\mu(\bigcap_{n=1}^{\infty} \varphi^n(\Omega)) = 0$. Then T is a backward space shift.*

Proof. Let X_n be the closed subspace of X consisting of functions vanishing almost everywhere on $\varphi^n(\Omega)$. The conditions imposed on φ and α imply that $T(X_{n+1})$ is a dense subspace of X_n for each $n \in \mathbb{N}$ and that the union of the increasing sequence $\{X_n\}$ is dense in X. Obviously $T(X_1) = \{0\}$. By Definition 1.1, T is a backward space shift. $\square$

It is worth noting that the condition in Proposition 2.5 of T being well-defined and continuous has double purpose. Apart from allowing us to speak of the operator T, it prohibits anomalies such as φ sending a set of positive measure to a set of measure zero. Proposition 2.5 is a generalization of the fact observed above that the backward shift operator on a weighted $L^p(\mathbb{R}_+)$ is always a backward space shift. The following example illustrates Proposition 2.5.

Example 2.6. Let $\varphi : [0, 1] \to [0, 1]$ and $\alpha \in L^\infty[0, 1]$ be such that φ is absolutely continuous and strictly increasing, the essential infimum of φ' is positive, $\alpha(x) \neq 0$ almost everywhere on $[0, 1]$ and $\varphi(x) < x$ for $0 < x \leqslant 1$. Then for any $p \in [0, \infty)$, the operator $T \in L(L^p[0, 1])$, $Tf(x) = \alpha(x) f(\varphi(x))$ is a backward space shift.

Proof. The conditions imposed upon φ and α ensure that T is well-defined and continuous. The inequality $\varphi(x) < x$ for $0 < x \leqslant 1$ guarantees that $\bigcap_{n=1}^{\infty} \varphi^n([0, 1]) = \{0\}$. It remains to apply Proposition 2.5. $\square$

We stress that the scope of Theorem 1.3 goes beyond locally convex spaces. Indeed, note that if $X = L^p(\Omega, \mathcal{A}, \mu)$ with $0 \leqslant p < 1$ and μ has no atoms, then $X^* = \{0\}$ (see e.g. [21]). In particular, $L^p[0, 1]$ for $0 \leqslant p < 1$ has trivial dual. The above example and Theorem 1.3 provide a supply of d-mixing tuples of any size of operators on $\mathcal{F}$-spaces $L^p[0, 1]$ for $0 \leqslant p < 1$, with trivial dual. We also note with Example 2.7 below that the concept of a backward space shift goes beyond backward shifts on sequence and function spaces.

Example 2.7. Let $1 \leqslant p < \infty$, $\varphi : [0, 1] \to [0, 1]$ be a continuous strictly increasing function such that $\varphi(x) < x$ for $0 < x \leqslant 1$ and let $\alpha : [0, 1] \to \mathbb{K}$ be a bounded measurable function almost everywhere different from zero. Then the operator $T \in L(L^p[0, 1])$ defined by the formula

$$Tf(x) = \int_0^{\varphi(x)} \alpha(t) f(t)\, dt$$

is a backward space shift.

Proof. Consider the sequence $\{c_n\}_{n\in\mathbb{N}}$ in $(0,1]$ defined by $c_1 = \varphi(1)$ and $c_{n+1} = \varphi(c_n)$ for $n \in \mathbb{N}$. Clearly $\{c_n\}$ is strictly decreasing and $c_n \to 0$. Now let X_n for $n \in \mathbb{N}$ be the space of $f \in L^p[0,1]$ vanishing almost everywhere on $[0, c_n]$. It is straightforward to verify that the union of X_n is dense in $L^p[0,1]$, $T(X_1) = \{0\}$ and $T(X_{n+1})$ is a dense subspace of X_n for any $n \in \mathbb{N}$. Thus T is a backward space shift. □

3. d-mixing powers of one operator

In this section we consider the question of when the tuple $(T, T^2, \dots, T^r)$ is d-mixing for a continuous linear operator T on a topological vector space X.

Theorem 3.1. *Let X be a topological vector space and $T \in L(X)$ be such that $T - I$ is a backward space shift on X. Then $(T, T^2, \dots, T^r)$ is d-mixing for each $r \in \mathbb{N}$.*

Proof. Let $S = T - I$. Then S is a backward space shift and $T^j = (I + S)^j = I + jS + S^2 p_j(S)$ for $j \in \mathbb{N}$, where p_j are polynomials. By Theorem 1.3, $(T, T^2, \dots, T^r)$ is d-mixing for each $r \in \mathbb{N}$. □

Remark 3.2. Theorem 3.1 holds true if we replace the condition of $T - I$ being a backward space shift by the weaker condition that $T - zI$ is a backward space shift for some $z \in \mathbb{K}$ satisfying $|z| = 1$. The proof requires only slight modifications.

Recall that a continuous linear operator T on a topological vector space X is said to *satisfy the Kitai Criterion* if there exist dense subsets E and F of X and a map $S: F \to F$ such that $TSy = y$, $T^n x \to 0$ and $S^n y \to 0$ for each $y \in F$ and $x \in E$. The point of such operators is that they are all mixing [15]. The above definition differs slightly from the original formulation by Carol Kitai [22], who also assumed that $E = F$. In the latter case, we say that T *satisfies the Original Kitai Criterion*.

Lemma 3.3. *Let $\{(T_{1,k}, \dots, T_{m,k})\}_{k\in\mathbb{Z}_+}$ be a sequence of m-tuples of continuous self-maps on a topological space X such that $\Sigma = \Sigma(\{T_{1,k}, \dots, T_{m,k}\}_{k\in\mathbb{Z}_+})$ is dense in X^{m+1}. Then $\{T_{1,k}, \dots, T_{m,k}\}_{k\in\mathbb{Z}_+}$ is d-mixing.*

Proof. Let $U_0, \dots, U_m$ be non-empty open subsets of X. Since Σ is dense in X^{m+1}, we can find $x_j \in U_j$ for $0 \leqslant j \leqslant m$ such that $(x_0, \dots, x_m) \in \Sigma$. By definition of Σ, there is a sequence $\{u_k\}_{k\in\mathbb{Z}_+}$ in X such that $u_k \to x_0$ and $T_{j,k}u_k \to x_j$ for $1 \leqslant j \leqslant m$. Hence we can pick $r \in \mathbb{Z}_+$ for which $u_k \in U_0$ and $T_{j,k}u_k \in U_j$ for $1 \leqslant j \leqslant m$ whenever $k \geqslant r$. Hence $u_k \in U_0 \cap T_{1,k}^{-1}(U_1) \cap \dots \cap T_{m,k}^{-1}(U_r)$ for $k \geqslant r$. Thus the last intersection is non-empty if $k \geqslant r$. That is, $\{T_{1,k}, \dots, T_{m,k}\}_{k\in\mathbb{Z}_+}$ is d-mixing. □

Theorem 3.4. *Let X be a topological vector space and $T \in L(X)$. Assume also that T satisfies the Original Kitai Criterion. Then $(T, T^2, \dots, T^r)$ is d-mixing for each $r \in \mathbb{N}$.*

Proof. Since T satisfies the Original Kitai Criterion, there is a dense subset E of X and a map $S: E \to E$ such that $TSx = x$, $T^n x \to 0$ and $S^n x \to 0$ for each $x \in E$. Fix $r \in \mathbb{N}$ and let $u_0, \dots, u_r \in E$. Consider the sequence $\{x_k\}_{k \in \mathbb{N}}$ in X defined by the formula

$$x_k = u_0 + S^k u_1 + S^{2k} u_2 + \dots + S^{rk} u_r.$$

For $0 \leqslant j \leqslant r$, using the equality $TSx = x$ for $x \in E$, we obtain

$$T^{jk} x_k = u_j + \sum_{l<j} T^{(j-l)k} u_l + \sum_{m>j} S^{(m-j)k} u_m.$$

Since $T^n x \to 0$ and $S^n x \to 0$ for each $x \in E$, $T^{jk} x_k \to u_j$ for $0 \leqslant j \leqslant r$. Hence $(u_0, \dots, u_r) \in \Sigma(T, \dots, T^r)$. Since $u_0, \dots, u_r$ were arbitrary elements of E, we have $E^{r+1} \subseteq \Sigma(T, \dots, T^r)$. Since E is dense in X, $\Sigma(T, \dots, T^r)$ is dense in X^{r+1}. By Lemma 3.3, $(T, T^2, \dots, T^r)$ is d-mixing. □

Remark 3.5. Theorem 3.4 implies that for any mixing bilateral weighted shift T on $\ell_p(\mathbb{Z})$ for $1 \leqslant p < \infty$ or on $c_0(\mathbb{Z})$ and any $r \in \mathbb{N}$, $(T, T^2, \dots, T^r)$ is d-mixing. Indeed, it is easy to verify that any mixing bilateral weighted shift satisfies the Original Kitai Criterion with E being the space of sequences with finite support and S being the inverse of the restriction of T to E. In general, for a bilateral weighted shift T, the r-tuple $(T, T^2, \dots, T^r)$ is hereditarily densely d-hypercyclic with respect to a sequence $\{n_k\}_{k \in \mathbb{N}}$ if and only if the direct sum $T \oplus \dots \oplus T^r$ is hereditarily hypercyclic with respect to $\{n_k\}$, see [8]. Also, any hypercyclic composition operator T on either $H(\mathbb{D})$ or a weighted Dirichlet space satisfies that $(T, T^2, \dots, T^r)$ is d-mixing [9, Corollary 23].

Theorems 3.1 and 3.4 and Remark 3.5 may make one wonder whether $(T, T^2, \dots, T^r)$ must be d-mixing for any mixing T. It turns out that such a conjecture is false. Surprisingly, a counterexample is rather hard to come by. We start with a non-linear example, which is considerably easier.

Example 3.6. Let $M = \{0, 1\}^{\mathbb{Z}}$ be endowed with the metric $d(a, b) = \sum_{n=-\infty}^{+\infty} 2^{-|n|} |a_n - b_n|$. Then d defines the product topology on M with $\{0, 1\}$ naturally carrying the discrete topology. Thus (M, d) is a compact metric space. Let $S: M \to M$ be the shift: $(Ma)_k = a_{k+1}$. Obviously S is invertible and S, S^{-1} are Lipschitz and therefore continuous. Let

$$X = \{a \in M: \ a_k + a_{k+n} + a_{k+2n} \leqslant 2 \text{ for any } k \in \mathbb{Z} \text{ and } n \in \mathbb{N}\}.$$

It is easy to see that X is a closed subset of M and $S(X) = X$. Thus X is a compact metric space and the restriction $T: X \to X$ of S to X is a homeomorphism from X onto itself. Then T is mixing and (T, T^2) is not d-transitive.

Proof. First, observe that $B = \{a \in X: \ \{n \in \mathbb{Z}: \ a_n = 1\} \text{ is finite}\}$ is a dense subset of X. Let $a, b \in B$. It is easy to verify that $c_n = a + T^{-n} b$ belongs to X for any sufficiently large $n \in \mathbb{N}$, where $+$ stands for the coordinate-wise sum of sequences. Moreover, $c_n \to a$ and $T^n c_n = b + T^n a \to b$ as $n \to \infty$. Hence $(a, b) \in \Sigma(T)$, where $\Sigma(T) = \Sigma(\{T^n\})$ is defined in (1.2). Since a and b are arbitrary elements of B, $B^2 \subseteq \Sigma(T)$ and therefore $\Sigma(T)$ is dense in X^2. By Lemma 3.3, T is mixing. Using the definition of X and T, we have

$$\{(x, T^n x, T^{2n} x):\ n \in \mathbb{N},\ x \in X\} \subseteq Q = \{(a, b, c) \in X^3:\ a_k + b_k + c_k \leqslant 2 \text{ for } k \in \mathbb{Z}\}.$$

It is easy to see that Q is closed and nowhere dense in X^3. Hence (T, T^2) is not d-transitive. □

The linear case is far more difficult. We did not manage to construct a mixing continuous linear operator T for which (T, T^2) is not d-transitive. That is, the following question remains open.

Question 3.7. *Does there exist a mixing continuous linear operator T on a separable Banach space, such that (T, T^2) is not d-transitive*?

We note that replacing the word "mixing" by "hypercyclic" in Question 3.7 was asked by Bernal-González [6, Problem 1] and has a simple answer in the affirmative [9, p. 855]. The following theorem provides an answer to another weaker version of Question 3.7.

Theorem 3.8. *There exists $T \in L(\ell_2)$ such that T is mixing and the sequence $\{2T^n - T^{2n}\}_{n\in\mathbb{N}}$ is non-mixing. In particular, (T, T^2) is not d-mixing.*

We need some preparation. Recall that for $1 \leqslant p < \infty$, $-\infty < a < b < +\infty$ and $k \in \mathbb{N}$, the Sobolev space $W^{k,p}[a, b]$ is the space of functions $f \in C^{k-1}[a, b]$ such that $f^{(k-1)}$ is absolutely continuous and $f^{(k)} \in L^p[a, b]$. The space $W^{k,p}[a, b]$ endowed with the norm

$$\|f\|_{W^{k,p}[a,b]} = \left(\int_a^b \left(\sum_{j=0}^{k} \left|f^{(j)}(x)\right|^p \right) dx \right)^{1/p}$$

is a Banach space isomorphic to $L^p[0, 1]$. Clearly $W^{k,2}[a, b]$ is a separable infinite-dimensional Hilbert space for each $k \in \mathbb{N}$. We consider a family of operators on separable complex Hilbert spaces built from a single operator. Let $M \in L(W^{2,2}[-\pi, \pi])$ be defined by the formula

$$M : W^{2,2}[-\pi, \pi] \to W^{2,2}[-\pi, \pi], \qquad Mf(x) = e^{ix} f(x). \tag{3.1}$$

Denote $\mathcal{H} = W^{2,2}[-\pi, \pi]$. In our context, it is more convenient to speak of the dual operator M^* rather than the Hilbert space adjoint $M^\star$. By the Riesz representation theorem, $\mathcal{H}^*$ is also a separable infinite-dimensional Hilbert space. Since $M \in L(\mathcal{H})$, we have $M^* \in L(\mathcal{H}^*)$. For each $t \in [-\pi, \pi]$, the functional $\delta_t : \mathcal{H} \to \mathbb{C}$, $\delta_t(f) = f(t)$ belongs to $\mathcal{H}^*$. It is easy to see that the map $t \mapsto \delta_t$ from $[-\pi, \pi]$ to $\mathcal{H}^*$ is norm-continuous. For a non-empty compact subset K of $[-\pi, \pi]$, we denote

$$X_K = \overline{\text{span}}\,\{\delta_t:\ t \in K\}, \tag{3.2}$$

where the closure of span$\{\delta_t\colon t \in K\}$ is taken with respect to the norm of $\mathcal{H}^*$. Clearly, the functionals δ_t are linearly independent. Hence X_K is always a separable Hilbert space and X_K is infinite-dimensional if and only if K is infinite. It is easy to see that

$$M^*\delta_t = e^{it}\delta_t \quad \text{for each } t \in [-\pi, \pi]. \tag{3.3}$$

Hence each X_K is an invariant subspace for M^*. This allows us to consider

$$Q_K \in L(X_K), \qquad Q_K = M^*\big|_{X_K}. \tag{3.4}$$

First, we figure out when Q_K is mixing or transitive.

Proposition 3.9. *Let K be a non-empty compact subset of $[-\pi, \pi]$. If K has no isolated points, then Q_K is mixing. If K has an isolated point, then Q_K is non-transitive.*

Proof. If K has an isolated point s, then we can pick $f \in \mathcal{H}$ such that $f(s) = 1$ and $f(t) = 0$ for each $t \in K \setminus \{s\}$. Now we consider $\varphi \in X_K^*$ defined by the formula $\varphi(y) = y(f)$. It is easy to see that $Q_K^*\varphi = e^{is}\varphi$. Hence Q_K^* has non-empty point spectrum and therefore Q_K is non-transitive.

It remains to consider the case when K has no isolated points. For each $t \in [-\pi, \pi]$ we consider the functional $\delta_t' \in \mathcal{H}^*$ defined by the formula $\delta_t'(f) = f'(t)$. It is easy to see that $\frac{\delta_s - \delta_t}{s-t}$ converges to δ_t' in the norm of $\mathcal{H}^*$ as $s \to t$. Since K has no isolated points, $\delta_t' \in X_K$ for each $t \in K$. Direct computation shows that $Q_K\delta_t' = M^*\delta_t' = e^{it}(\delta_t' + i\delta_t)$ for each $t \in K$. Using this equality and (3.3), we have

$$Q_K^n\delta_t = e^{int}\delta_t \quad \text{and} \quad Q_K^n\delta_t' = e^{int}\bigl(\delta_t' + in\delta_t\bigr) \quad \text{for any } n \in \mathbb{Z}_+ \text{ and } t \in K. \tag{3.5}$$

Now let $u, v \in E = \operatorname{span}\{\delta_t\colon t \in K\}$. We can choose a finite subset A of K such that

$$u = \sum_{t\in A} \alpha_t\delta_t \quad \text{and} \quad v = \sum_{t\in A} \beta_t\delta_t, \tag{3.6}$$

where $\alpha_t, \beta_t \in \mathbb{C}$. For each $n \in \mathbb{N}$, we consider $x_n \in X_K$ defined by the formula

$$x_n = \sum_{t\in A}\biggl(\frac{e^{-int}(\beta_t - \alpha_t)}{in}\delta_t' + \alpha_t\delta_t\biggr).$$

According to the last two displays $x_n \to u$. Using (3.5), we have

$$Q_K^n x_n = \sum_{t\in A}\biggl(\frac{\beta_t - \alpha_t}{in}\delta_t' + \beta_t\delta_t\biggr),$$

which together with (3.6) implies that $Q_K^n x_n \to v$. Thus $E^2 \subseteq \Sigma(Q_K)$, where $\Sigma(Q_K) = \Sigma(\{Q_K^n\}_{n\in\mathbb{Z}_+})$ is defined in (1.2). By Lemma 3.3, Q_K is mixing. $\square$

According to Proposition 3.9, Theorem 3.8 will be proven if we find a non-empty compact set $K \subset [-\pi, \pi]$ with no isolated points such that the sequence $\{2Q_K^n - Q_K^{2n}\}_{n\in\mathbb{N}}$ is non-mixing. A few technical lemmas are needed, which are included in Appendix A.

Consider the set

$$K = \left\{ \sum_{n=1}^{\infty} 2\pi \varepsilon_n 2^{-6^n}\colon\ \varepsilon \in \{0,1\}^{\mathbb{N}} \right\}. \tag{3.7}$$

It is easy to see that K is a compact subset of $[-\pi, \pi]$ with no isolated points. By Proposition 3.9, the operator $Q_K \in L(X_K)$ is mixing.

Proposition 3.10. *Let K be the compact subset of $[-\pi,\pi]$ defined in (3.7). Then the sequence $\{2Q_K^{k_n} - Q_K^{2k_n}\}_{n\in\mathbb{N}}$ of continuous linear operators on X_K is non-universal, where $k_n = 2^{6^n}$ for $n \in \mathbb{N}$.*

Proof. For $f \in W^{2,2}[-\pi,\pi]$, we can consider $\Phi_f \in X_K^*$ defined by the formula $\Phi_f(y) = y(f)$ (recall that X_K consists of linear functionals on $W^{2,2}[-\pi,\pi]$). It is easy to see that $\|\Phi_f\| \leqslant \|f\|_{W^{2,2}[-\pi,\pi]}$ and that $\Phi_f = \Phi_g$ if $f|_K = g|_K$. Thus

$$\|\Phi_f\| \leqslant \inf\bigl\{\|g\|_{W^{2,2}[-\pi,\pi]}\colon\ f|_K = g|_K\bigr\}. \tag{3.8}$$

We use the symbol $\mathbf{1}$ to denote the constant 1 function in the space $W^{2,2}[-\pi,\pi]$. Clearly, the functional $\Phi_{\mathbf{1}} \in X_K^*$ is non-zero. Indeed, $\Phi_{\mathbf{1}}(\delta_t) = 1$ for each $t \in K$. Denote $T_n = 2Q_K^{k_n} - Q_K^{2k_n} \in L(X_K)$. We shall estimate $\|T_n^*\Phi_{\mathbf{1}}\|$. By definition of Q_K, we have $Q_K^*\Phi_f = \Phi_{Mf}$ for each $f \in W^{2,2}[-\pi,\pi]$, where M is the multiplication operator defined in (3.1). It follows that $T_n^*\Phi_{\mathbf{1}} = \Phi_{h_{k_n}}$, where $h_n(t) = 2e^{int} - e^{2int}$. By Lemma A.3, there is a bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ in the Hilbert space $W^{2,2}[-\pi,\pi]$ such that $f_n(t) = h_n(t)$ whenever $|t - \frac{2\pi k}{n}| \leqslant \frac{2}{n^5}$ for some $k \in \mathbb{Z}$. Now let $t \in K$. Then $t = \sum_{j=1}^{\infty} \frac{2\pi\varepsilon_j}{k_j}$ for some $\varepsilon \in \{0,1\}^{\mathbb{N}}$. For each $n \in \mathbb{N}$, we have $t = y + u$, where $y = \sum_{j=1}^{n} \frac{2\pi\varepsilon_j}{k_j}$ and $u = \sum_{j=n+1}^{\infty} \frac{2\pi\varepsilon_j}{k_j}$. Clearly, $y = \frac{2\pi m}{k_n}$ for some $m \in \mathbb{N}$ and

$$0 \leqslant u \leqslant 2\pi \sum_{j=n+1}^{\infty} \frac{1}{k_j} < 2\pi \sum_{j=6^{n+1}}^{\infty} 2^{-j} = 4\pi 2^{-6^{n+1}} = 4\pi k_n^{-6} < 2k_n^{-5}.$$

Hence $|t - \frac{2\pi m}{k_n}| = u < 2k_n^{-5}$. Thus $f_{k_n}(t) = h_{k_n}(t)$ for each $t \in K$ and $n \in \mathbb{N}$. By (3.8) $\|\Phi_{h_{k_n}}\| \leqslant \|f_{k_n}\|_{W^{2,2}[-\pi,\pi]}$ for $n \in \mathbb{N}$. Since $\{f_n\}$ is bounded in $W^{2,2}[-\pi,\pi]$, the sequence $\{\|\Phi_{h_{k_n}}\|\}$ is bounded. That is, there is $C > 0$ such that $\|\Phi_{h_{k_n}}\| \leqslant C$ for each $n \in \mathbb{N}$. Since $T_n^*\Phi_{\mathbf{1}} = \Phi_{h_{k_n}}$, it follows that $|\Phi_{\mathbf{1}}(T_n x)| = |T_n^*\Phi_{\mathbf{1}}(x)| \leqslant C\|x\|$ for each $x \in X_K$. Since $\Phi_{\mathbf{1}}$ is a non-zero continuous linear functional on X_K, $\{T_n x\colon\ n \in \mathbb{N}\}$ cannot be dense in X_K for any given $x \in X_K$. That is, $\{T_n\colon\ n \in \mathbb{N}\}$ is non-universal. $\square$

Proof of Theorem 3.8. Let K be the compact subset of $[\pi,\pi]$ from (3.7) and $Q_K \in L(X_K)$ be the operator defined in (3.4). By Proposition 3.9, Q_K is a mixing operator on the separable infinite-dimensional Hilbert space X_K. By Proposition 3.10, $\{2Q_K^{k_n} - Q_K^{2k_n}\}_{n\in\mathbb{N}}$ is non-universal and therefore non-transitive for some strictly increasing sequence $\{k_n\}_{n\in\mathbb{N}}$ of positive integers. Hence $\{2Q_K^n - Q_K^{2n}\}_{n\in\mathbb{Z}_+}$ is non-mixing and therefore (Q_K, Q_K^2) is not d-mixing. Since all separable infinite-dimensional Hilbert spaces are isomorphic to ℓ_2, there is $T \in L(\ell_2)$ such that T is mixing, $\{2T^n - T^{2n}\}_{n\in\mathbb{N}}$ is non-mixing and (T, T^2) is not d-mixing. $\square$

4. Existence of dual d-mixing tuples on Banach spaces

We recall that Salas' result on the existence of dual d-mixing tuples on any Banach space with separable dual required a long, technical proof [26]. We provide here a short proof of Theorem 1.4, a slight enhancement of Salas' result. We first show the following lemma dealing with bilateral shifts.

Lemma 4.1. *Let X be a separable Banach space and let $\{x_n\}_{n\in\mathbb{Z}}$, $\{f_n\}_{n\in\mathbb{Z}}$ be bounded sequences in X and X^* respectively such that* $\operatorname{span}\{x_n\colon n\in\mathbb{Z}\}$ *is dense in X, $f_n(x_m)=0$ whenever $n\neq m$ and $f_n(x_n)\neq 0$ for each $n\in\mathbb{Z}$. For $a\in\ell_1(\mathbb{Z})$, let $T_a\in L(X)$ be defined by the formula*

$$T_a x = x+\sum_{n\in\mathbb{Z}} a_n f_{n+1}(x)x_n \tag{4.1}$$

(boundedness of $\{x_n\}$ and $\{f_n\}$ and summability of $\{|a_n|\}$ imply absolute convergence of the above series and continuity of T_a). Then the set

$$\Pi=\bigl\{a\in\ell_1(\mathbb{Z})\colon\ \bigl(T_a,T_a^2,\dots,T_a^k\bigr)\text{ is d-transitive for each }k\in\mathbb{N}\bigr\} \tag{4.2}$$

is a dense G_δ-subset of $\ell_1(\mathbb{Z})$.

Proof. Since X is a separable Banach space, we can pick a sequence $\{U_j\}_{j\in\mathbb{N}}$ of non-empty open subsets of X, which form a basis of the topology of X. It is straightforward to see that $(T_a,\dots,T_a^k)$ is d-transitive if and only if for any $m=(m_0,\dots,m_k)\in\mathbb{N}^{k+1}$, there exists $n\in\mathbb{N}$ such that $\bigcap_{j=0}^k (T_a^{jn})^{-1}(U_{m_j})\neq\emptyset$. That is, the set $\Pi_k=\{a\in\ell_1(\mathbb{Z})\colon (T_a,\dots,T_a^k)$ is d-transitive$\}$ can be written in the following way

$$\Pi_k=\bigcap_{m\in\mathbb{N}^{k+1}}\bigcup_{n\in\mathbb{N}} M_{m,n,k},\quad\text{where } M_{m,n,k}=\left\{a\in\ell_1(\mathbb{Z})\colon \bigcap_{j=0}^{k}\bigl(T_a^{jn}\bigr)^{-1}(U_{m_j})\neq\emptyset\right\}.$$

Hence

$$\Pi=\bigcap_{k=1}^{\infty}\Pi_k=\bigcap_{k\in\mathbb{N}}\bigcap_{m\in\mathbb{N}^{k+1}} N_{m,k},\quad\text{where } N_{m,k}=\bigcup_{n\in\mathbb{N}} M_{m,n,k}.$$

It is easy to see that the map $a\mapsto T_a$ from $\ell_1(\mathbb{Z})$ to $L(X)$ is norm-continuous (even Lipschitz). It follows that each $M_{m,n,k}$ is open in $\ell_1(\mathbb{Z})$. Hence each $N_{m,k}$ is open in $\ell_1(\mathbb{Z})$. According to the last display and the Baire theorem, in order to show that Π is a dense G_δ-subset of $\ell_1(\mathbb{Z})$, it suffices to verify that the open sets $N_{m,k}$ are dense in $\ell_1(\mathbb{Z})$.

Let $k\in\mathbb{N}$ and $m=(m_0,\dots,m_k)\in\mathbb{N}^{k+1}$. Pick a non-empty open subset V of $\ell_1(\mathbb{Z})$. For $j\in\mathbb{N}$, let $X_j=\operatorname{span}\{x_{-j},\dots,x_j\}$ and $E_j=\operatorname{span}\{e_{-j},\dots,e_j\}$, where $\{e_n\}_{n\in\mathbb{Z}}$ is the canonical basis of ℓ_1. Since $\operatorname{span}\{x_n\colon n\in\mathbb{Z}\}$ is dense in X and $\operatorname{span}\{e_n\colon n\in\mathbb{Z}\}$ is dense in $\ell_1(\mathbb{Z})$, there is $j\in\mathbb{N}$ such that $E_j\cap V\neq\emptyset$ and $X_j\cap U_{m_l}\neq\emptyset$ for $0\leqslant l\leqslant k$. Since $E_j\cap V\neq\emptyset$, we can pick $a\in V$ such that $a_l=0$ for $l<-j$ and $a_l\neq 0$ for $l\geqslant -j$. Consider $Y=\overline{\operatorname{span}}\{x_l\colon l\geqslant -j\}$. Since $a_l=0$ for $l<-j$, $T_a(Y)\subseteq Y$. Moreover, $(T_a-I)x_{-j}=0$ and $(T_a-I)x_l=c_l x_{l-1}$ for $l\geqslant -j+1$ with $c_l\in\mathbb{K}\setminus\{0\}$. Hence $(T_a-I)|_Y$ is a backward space shift on Y. By Theorem 3.1,

$(T_a|_Y, \dots, T_a^k|_Y)$ is d-mixing on Y. Since $X_j \cap U_{m_l} \neq \emptyset$ for $0 \leqslant l \leqslant k$ and $X_j \subseteq Y$, the sets $V_l = U_{m_l} \cap Y$ are non-empty open subsets of Y for $0 \leqslant l \leqslant k$. Since $(T_a|_Y, \dots, T_a^k|_Y)$ is d-mixing on Y, we can find $n \in \mathbb{N}$ and $x \in V_0$ such that $T_a^{jn} x \in V_j$ for $1 \leqslant j \leqslant k$. Since $a \in V$ and $V_l \subseteq U_{m_l}$, we have $(a, x, T_a^n x, \dots, T_a^{kn} x) \in V \times U_{m_0} \times \dots \times U_{m_k}$. Hence $a \in M_{m,n,k} \subseteq N_{m,k}$. Since V is an arbitrary non-empty open subset of $\ell_1(\mathbb{Z})$, $N_{m,k}$ is dense in $\ell_1(\mathbb{Z})$. $\square$

Under the conditions of Lemma 4.1, the dual of T_a defined in (4.1) acts according to the formula

$$T_a^* f = f + \sum_{n \in \mathbb{Z}} a_{n-1} f(x_{n-1}) f_n.$$

Denoting $g_n = f_{-n}$ and considering $h_n \in X^{**}$ defined as $h_n(f) = f(x_{-n})$, we can rewrite the above display in the following way

$$T_a^* f = f + \sum_{n \in \mathbb{Z}} a_{-1-n} h_{n+1}(f) g_n.$$

Clearly $\{g_n\}$ and $\{h_n\}$ are bounded in X^* and in X^{**} respectively, $h_n(g_m) = f_{-m}(x_{-n}) = 0$ if $n \neq m$ and $h_n(g_n) = f_{-n}(x_{-n}) \neq 0$ for $n \in \mathbb{Z}$. If we additionally assume that $\operatorname{span}\{f_n\colon n \in \mathbb{Z}\}$ is dense in X^*, then T_a^* has the same shape as defined in (4.1) with the sequence a replaced by $\tilde{a} = \{a_{-1-n}\}_{n \in \mathbb{Z}}$. Now, observing that the map $a \mapsto \tilde{a}$ is a homeomorphism from $\ell_1(\mathbb{Z})$ onto itself, we can apply Lemma 4.1 to conclude that the set

$$\Pi' = \left\{a \in \ell_1(\mathbb{Z})\colon \left(T_a^*, T_a^{*2}, \dots, T_a^{*k}\right) \text{ is d-transitive for each } k \in \mathbb{N}\right\} \tag{4.3}$$

is a dense G_δ-subset of $\ell_1(\mathbb{Z})$ provided $\operatorname{span}\{f_n\colon n \in \mathbb{Z}\}$ is dense in X^*. Since the intersection of two dense G_δ-sets in a Baire topological space is again a dense G_δ-set, we obtain the following corollary.

Corollary 4.2. *Let X be a separable Banach space with separable dual and let $\{x_n\}_{n \in \mathbb{Z}}$, $\{f_n\}_{n \in \mathbb{Z}}$ be bounded sequences in X and X^* respectively such that* $\operatorname{span}\{x_n\colon n \in \mathbb{Z}\}$ *is dense in X,* $\operatorname{span}\{f_n\colon n \in \mathbb{Z}\}$ *is dense in X^*, $f_n(x_m) = 0$ whenever $n \neq m$ and $f_n(x_n) \neq 0$ for each $n \in \mathbb{Z}$. Then the set of $a \in \ell_1(\mathbb{Z})$ for which $(T_a, T_a^2, \dots, T_a^k)$ is d-transitive on X and $(T_a^*, T_a^{*2}, \dots, T_a^{*k})$ is d-transitive on X^* for any $k \in \mathbb{N}$ is a dense G_δ-subset of $\ell_1(\mathbb{Z})$, where $T_a \in L(X)$ are defined in* (4.1).

Proof of Theorem 1.4. According to Pelczynski [23], we can pick sequences $\{x_n\}_{n \in \mathbb{Z}}$ and $\{f_n\}_{n \in \mathbb{Z}}$ in X and X^* respectively such that $\operatorname{span}\{x_n\colon n \in \mathbb{Z}\}$ is dense in X, $\operatorname{span}\{f_n\}_{n \in \mathbb{Z}}$ is dense in X^*, $f_n(x_m) = \delta_{n,m}$ for each $m, n \in \mathbb{Z}$ and $\|x_n\| \leqslant 2$, $\|f_n\| \leqslant 2$ for each $n \in \mathbb{Z}$. By Corollary 4.2, there is $a \in \ell_1(\mathbb{Z})$ such that $(T, T^2, \dots, T^k)$ is d-transitive on X and $(T^*, T^{*2}, \dots, T^{*k})$ is d-transitive on X^* for any $k \in \mathbb{N}$, where $T = T_a \in L(X)$ is defined in (4.1). $\square$

5. Backward space shift tuples

The purpose in this section is to introduce tuples of backward shifts in different 'directions', which will allow us to state Theorem 5.7, from which Theorem 1.5 and Corollary 1.6 will be derived. We use the following notation. For $m \in \mathbb{N}$, $1 \leqslant j \leqslant m$, let

$$\mathrm{e}_j \in \mathbb{Z}^m, \qquad (\mathrm{e}_j)_k = \delta_{j,k}, \tag{5.1}$$

where $\delta_{j,k}$ is the Kronecker delta. For $a, b \in \mathbb{Z}^m$, we write $a \leqslant b$ if $a_j \leqslant b_j$ for $1 \leqslant j \leqslant m$. Thus $\leqslant$ is a partial ordering which is not a total ordering if $m \geqslant 2$. Also, let $m, n \in \mathbb{N}$ and a topological vector space X be given. We say that a family $\{X_a\}_{a\in J}$ of linear subspaces of X, labeled by $J = \mathbb{N}^m$ or by $J = \mathbb{N}_n^m$ is a *filtration* of X if $X_b \subseteq X_a$ whenever $b \leqslant a$ and $\bigcup_{a\in J} X_a$ is dense in X.

Definition 5.1. Let $m, n \in \mathbb{N}$, X a topological vector space and $\{X_a\}_{a\in\mathbb{N}_n^m}$ a filtration of X. We say that $(T_1, \dots, T_m) \in L(X)^m$ is a *backward shift m-tuple with respect to* $\{X_a\}_{a\in\mathbb{N}_n^m}$ if for $1 \leqslant j \leqslant m$ and $a \in \mathbb{N}_n^m$, $T_j(X_a) = \{0\}$ when $a_j = 1$, and $T_j(X_a) = X_{a-\mathrm{e}_j}$ when $a_j \geqslant 2$.

What we are really interested in is in the following related notion.

Definition 5.2. Let $m \in \mathbb{N}$, X a topological vector space and $(T_1, \dots, T_m) \in L(X)^m$. We say that $(T_1, \dots, T_m)$ is a *backward space shift m-tuple* if there is a filtration $\{X_a\}_{a\in\mathbb{N}^m}$ in X and for any $n \in \mathbb{N}$, there is a filtration $\{X_a^{(n)}\}_{a\in\mathbb{N}_n^m}$ of $X_{(n,\dots,n)}$ such that $Y_n = X_{(n,\dots,n)}^{(n)}$ is invariant for each T_j and $(T_1|_{Y_n}, \dots, T_m|_{Y_n}) \in L(Y_n)^m$ is a backward shift m-tuple with respect to $\{X_a^{(n)}\}_{a\in\mathbb{N}_n^m}$.

Remark 5.3. Note that in the case $m = 1$, Definition 5.2 recovers the concept of a backward space shift. Also, the concept of a backward space shift tuple of operators admits a simpler formulation in the case when the operators commute. Namely, let $m \in \mathbb{N}$, X a topological vector space and $T_1, \dots, T_m \in L(X)$ be pairwise commuting. Then $(T_1, \dots, T_m)$ is a backward space shift m-tuple if and only if there is a filtration $\{X_a\}_{a\in\mathbb{N}^m}$ of X such that $T_j(X_a) = \{0\}$ if $a_j = 1$ and $T_j(X_a)$ is a dense subspace of $X_{a-\mathrm{e}_j}$ otherwise. Indeed, for $n \in \mathbb{N}$ and $a \in \mathbb{N}_n^m$, it is enough to consider $X_a^{(n)} = T_1^{n-a_1} \cdots T_m^{n-a_m}(X_{n,\dots,n})$ and observe that all conditions of Definition 5.2 are satisfied.

At this point an example will be in order.

Example 5.4. Let $X = \mathcal{H}(\mathbb{C}^m)$ be the Fréchet space of entire functions of m complex variables $z_1, \dots, z_m$ and for $a \in \mathbb{N}^m$, let X_a be the subspace consisting of polynomials whose z_j-degree is less than a_j for $1 \leqslant j \leqslant m$. Clearly $\{X_a\}_{a\in\mathbb{N}^m}$ is a filtration of X. Let now $T_j = \frac{\partial}{\partial z_j}$ be the derivation operator with respect to the jth variable. It is easy to see that $T_j(X_a) = \{0\}$ if $a_j = 1$ and $T_j(X_a) = X_{a-\mathrm{e}_j}$ otherwise. Thus all conditions of Definition 5.2 are satisfied with $X_a^{(n)} = X_a$ and therefore $(T_1, \dots, T_m)$ is a backward space shift m-tuple.

Remark 5.5. Let $m \in \mathbb{N}$, $1 \leqslant j \leqslant m$, X a topological vector space and $\{e_a\}_{a\in\mathbb{N}^m}$ a linearly independent sequence in X with dense linear span. We say that $(T_1, \dots, T_m) \in L(X)^m$ is a *backward shift m-tuple with respect to* $\{e_a\}$ if $T_j e_a = 0$ whenever $a_j = 1$ and $T_j e_a$ is a linear combination of $e_{a-\mathrm{e}_j}, \dots, e_{a-(a_j-1)\mathrm{e}_j}$ with the coefficient near $e_{a-\mathrm{e}_j}$ being non-zero otherwise. It is straightforward to see that a backward shift m-tuple $(T_1, \dots, T_m)$ with respect to $\{e_a\}$ is also a backward space shift m-tuple. Indeed, one has just to take $X_a = X_a^{(n)} = \operatorname{span}\{e_b\colon b \leqslant a\}$.

With Example 5.6 we note that a backward space shift r-tuple of operators need not come as a backward shift tuple with respect to a linear independent sequence labeled by $\mathbb{N}^r$.

Example 5.6. Let $m \in \mathbb{N}$, $0 \leqslant p < \infty$ and $X = L^p(\mathbb{R}_+^m)$. Let also $\alpha_1, \dots, \alpha_m \in L^\infty(\mathbb{R}_+^m)$ be such that each α_j is non-zero almost everywhere. Consider $T_1, \dots, T_m \in L(X)$ defined by the formula

$$\begin{aligned} &T_j f(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_m) \\ &\quad = \alpha_j(x_1, \dots, x_m) f(x_1, \dots, x_{j-1}, x_j + 1, x_{j+1}, \dots, x_m), \quad 1 \leqslant j \leqslant m. \end{aligned}$$

Then $(T_1, \dots, T_m)$ is a backward space shift m-tuple of operators on X.

Proof. For $a \in \mathbb{N}^m$, we set X_a to be the space of $f \in X$ supported on $[0, a_1] \times \cdots \times [0, a_m]$. It is an easy exercise to verify that all conditions of Definition 5.2 are satisfied. $\square$

We state the main result of this section.

Theorem 5.7. *Let $m \in \mathbb{N}$, X a topological vector space and $(T_1, \dots, T_m) \in L(X)^m$ a backward space shift m-tuple. Then the m-tuple $(I + T_1, \dots, I + T_m)$ is strongly d-mixing. If additionally, T_j is exponentiable for $1 \leqslant j \leqslant m$, then the m-tuple $\{e^{zT_1}, \dots, e^{zT_m}\}_{z \in \mathbb{K}}$ of operator groups is strongly d-mixing.*

Noting that $L^p(\mathbb{R}_+^m)$ is isomorphic to $L^p[0,1]$ for each $m \in \mathbb{N}$ and $p \in [0, \infty)$, we see that $L^p[0,1]$ supports a backward space shift m-tuple of operators for any $m \in \mathbb{N}$. Combining this remark with Theorem 5.7, we get the following corollary.

Corollary 5.8. *Let $0 \leqslant p < 1$ and $m \in \mathbb{N}$. Then there exist $T_1, \dots, T_m \in L(L^p[0,1])$ such that $(T_1, \dots, T_m)$ is strongly d-mixing.*

We finish the section by exhibiting with Proposition 5.9 more examples of backward space shift tuples. We note that the proposition admits a number of generalizations. First, we can consider a wider class of spaces. Second, we can consider different topologies on the tensor products.

Proposition 5.9. *Let $Y_1, \dots, Y_m$ be Banach spaces, $S_j \in L(Y_j)$ backward space shifts for $1 \leqslant j \leqslant m$ and let $X = Y_1 \widehat{\otimes} \cdots \widehat{\otimes} Y_m$ be the completion of the tensor product of Y_j with respect to the projective topology. Let also $T_j \in L(X)$ for $1 \leqslant j \leqslant m$ be defined by the formula $T_j = I \otimes \cdots \otimes I \otimes S_j \otimes I \otimes \cdots \otimes I$, where S_j sits in the jth. Then $(T_1, \dots, T_m)$ is a backward space shift tuple.*

Proof. Since S_j are backward space shifts, we can pick increasing sequences $\{Y_{j,n}\}_{n \in \mathbb{N}}$ of linear subspaces of Y_j for $1 \leqslant j \leqslant m$ such that $S_j(Y_{j,1}) = \{0\}$ and $S_j(Y_{j,n+1})$ is a dense subspace of $Y_{j,n}$ for $n \in \mathbb{N}$. For $a \in \mathbb{N}^m$, denote $X_a = Y_{1,a_1} \otimes \cdots \otimes Y_{m,a_m}$. It is easy to see that $\{X_a\}_{a \in \mathbb{N}^m}$ is a filtration of X, $T_j(X_a) = \{0\}$ if $a_j = 1$ and $T_j(X_a) = X_{a-\mathrm{e}_j}$ if $a_j \geqslant 2$. It remains to notice that T_j are pairwise commuting and apply Remark 5.3. $\square$

6. Shifts on finite-dimensional spaces

The core of several of our main results is actually based on finite-dimensional matrices. We intend to present in this section the corresponding notions and results, with the only exception of certain determinants that are calculated in Appendix A.

Definition 6.1. We say that an $n \times n$ matrix $A = \{a_{j,l}\}_{j,l=1}^{n}$ with entries in $\mathbb{K}$ is *nicely upper triangular* if $a_{j,l} = 0$ for $j \geqslant l$ and $a_{j,j+1} = 1$ for $1 \leqslant j \leqslant n-1$. That is, A is upper triangular, has zero main diagonal and has the diagonal immediately above the main one filled with 1's.

The following lemma is our main tool. It is a much stronger form of a lemma by Salas [24] that he used to prove that any perturbation of the identity by adding a backward weighted shift on ℓ_1 is hypercyclic. To be more precise, we obtain a multi-approximation version of Salas' lemma, with fine estimates.

Lemma 6.2. *Let $n, m \in \mathbb{N}$, $\{e_1, \dots, e_q\}$ a basis in a q-dimensional Banach space X with $q \geqslant n(m+1)$, $z \in \mathbb{K}^{[m]}$ and $S_1, \dots, S_m$ linear operators on X with nicely upper triangular matrices with respect to the basis $\{e_1, \dots, e_q\}$. Then for any bounded subset B of $E = \operatorname{span}\{e_1, \dots, e_n\}$,*

$$\sup_{\substack{w\in\mathbb{K},\ |w|\geqslant 1\\ u_0,\dots,u_m\in B}} \inf_{x\in X} |w| \cdot \max\bigl\{|w|^{q-n(m+1)}\|u_0 - x\|, \bigl\|u_1 - e^{wz_1S_1}x\bigr\|, \dots, \bigl\|u_m - e^{wz_mS_m}x\bigr\|\bigr\} < \infty. \tag{6.1}$$

Proof. For $x \in X$ we denote the coefficients of x decomposed by the basis $\{e_1, \dots, e_q\}$ as $x_1, \dots, x_q$. That is, $x = x_1e_1 + \dots + x_qe_q$. Fix a bounded subset B of E and assume that $u_0, \dots, u_m \in B$.

For $w \in \mathbb{K}$, we attempt to find $y \in X$ such that

$$y_l = 0 \quad \text{for } 1 \leqslant l \leqslant q - nm \quad \text{and} \quad \bigl(e^{wz_rS_r}(u_0 + y)\bigr)_j = (u_r)_j \quad \text{for } 1 \leqslant r \leqslant m \text{ and } 1 \leqslant j \leqslant n. \tag{6.2}$$

Since each S_r has a nicely upper triangular matrix with respect to the basis $\{e_1, \dots, e_q\}$, we easily see that

$$\bigl(e^{tS_r}x\bigr)_j = \sum_{l=j}^{q} p_{l,j,r}(t)\frac{x_l}{(l-j)!} \quad \text{for } x \in X,\ t \in \mathbb{K},\ 1 \leqslant j \leqslant q \text{ and } 1 \leqslant r \leqslant m, \tag{6.3}$$

where $p_{l,j,r}$ is a polynomial in t of degree $l - j$ with the leading coefficient 1.

According to (6.3), (6.2) is equivalent to the following system of nm linear equations with nm variables:

$$\sum_{l=1}^{nm} \frac{p_{q-nm+l,j,r}(wz_r)}{(q-mn+l-j)!} y_{q-nm+l} = v_{j,r} \quad \text{for } 1 \leqslant j \leqslant n \text{ and } 1 \leqslant r \leqslant m,$$

$$\text{where } v_{j,r} = (u_r)_j - \sum_{t=j}^{n} \frac{p_{t,j,r}(wz_r)}{(t-j)!}(u_0)_t \tag{6.4}$$

and $p_{l,j,r}$ are the polynomials from (6.3). Enumerating the set $\mathbb{N}_m \times \mathbb{N}_n$ by elements of $\mathbb{N}_{nm}$ as specified in (A.18), we can rewrite the system (6.4) in the following way:

$$\sum_{l=1}^{nm} p_{j,l}(w) y_{q-nm+l} = v_j(w) \quad \text{for } 1 \leqslant j \leqslant nm, \tag{6.5}$$

$$\text{where } v_j(w) = v_{a_j,s_j} \text{ and } p_{j,l}(w) = \frac{p_{q-nm+l,a_j,s_j}(wz_{s_j})}{(q-mn+l-a_j)!} \tag{6.6}$$

with a_j and s_j defined in (A.18). Thus (6.5) can be written as $A_w\bar{y} = v$, where $A_w = \{p_{j,l}(w)\}_{j,l=1}^{nm}$, $\bar{y} = (y_{q-nm+1}, \dots, y_q)$ and $v = (v_1, \dots, v_{nm})$. According to (6.3) and (6.6), each $p_{j,l}(w)$ is a polynomial in w of degree $k+n+l-a_j$ with the leading coefficient $\frac{z_{s_j}^{k+n+l-a_j}}{(k+n+l-a_j)!}$. By Corollary A.11, $\det A_w$ is a polynomial in w of degree exactly $\mu = \frac{n^2m^2+n^2m+2nkm}{2}$, where $k = q - n(m+1)$. Thus we can find $b > 0$ and $c_0 > 0$ such that

$$|\det A_w| \geqslant b|w|^\mu \quad \text{for } |w| \geqslant c_0. \tag{6.7}$$

In particular, $\det A_w \neq 0$ for $|w| \geqslant c_0$ and therefore the system (6.5) has a unique solution for any $w \in \mathbb{K}$ with $|w| \geqslant c_0$. Thus for such w, there exists a unique $y = y^w \in X$ satisfying (6.2). Since $v_j(w)$ is a polynomial in w of degree at most $n - a_j$, whose coefficients are bounded when $u_0, \dots, u_m \in B$, there is $c_1 > 0$ such that

$$\bigl|v_j(w)\bigr| \leqslant c_1|w|^{n-a_j} \quad \text{for } |w| \geqslant c_0 \text{ and } 1 \leqslant j \leqslant nm. \tag{6.8}$$

We use the Cramer's formula for the solution of a uniquely solvable system of linear equations. To this end, for $1 \leqslant l \leqslant nm$, we consider the matrix $A_{w,l}$, being A_w with the lth column replaced by the vector $v(w)$ defined in (6.6). Note that all the entries of A_w are polynomials in w. Taking into account the degrees of these polynomials together with (6.8), we see that $\det A_{w,l}$ is a polynomial in w of degree at most $\mu - q + (m+1)n - l$, whose coefficients are bounded when $u_0, \dots, u_m \in B$. Hence there is $c_2 > 0$ such that

$$|\det A_{w,j}| \leqslant c_2|w|^{\mu-q+(m+1)n-l} \quad \text{for } 1 \leqslant j \leqslant nm \text{ and } |w| \geqslant c_0. \tag{6.9}$$

By the Cramer's formula, $y^w_{q-nm+l} = \frac{\det A_{w,l}}{\det A_w}$ for $1 \leqslant l \leqslant mn$. According to (6.7) and (6.9), we have

$$\bigl|y^w_{q-nm+l}\bigr| \leqslant \frac{c_2}{b}|w|^{(m+1)n-q-l} \quad \text{for } 1 \leqslant l \leqslant nm \text{ and } |w| \geqslant c_0. \tag{6.10}$$

Since $y^w_l = 0$ for $1 \leqslant l \leqslant q - nm$, (6.10) implies that there is $c_3 > 0$ for which $\|y^w\| \leqslant c_3|w|^{(m+1)n-q-1}$ whenever $|w| \geqslant c_0$. By (6.3) and (6.10), there is $c_4 > 0$ such that

$$\bigl|\bigl(e^{wz_rS_r}\bigl(u_0 + y^w\bigr)\bigr)_j\bigr| \leqslant c_4|w|^{n-j} \quad \text{for } 1 \leqslant r \leqslant m,\ n < j \leqslant q \text{ and } |w| \geqslant c_0. \tag{6.11}$$

By (6.2), $(e^{wz_rS_r}(u_0 + y^w))_j = (u_r)_j$ for $1 \leqslant j \leqslant n$ and $1 \leqslant r \leqslant m$. Hence (6.11) ensures the existence of $c_5 > 0$ for which $\|u_r - e^{wz_rS_r}(u_0 + y^w)\| \leqslant c_5|w|^{-1}$ whenever $|w| \geqslant c_0$. Denoting

$x = u_0 + y^w$, we get $\|u_0 - x\| \leqslant c_3|w|^{(m+1)n-q-1}$ and $\|u_r - e^{wz_r S_r}x\| \leqslant c_5|w|^{-1}$ for $w \in \mathbb{K}$, $|w| \geqslant c_0$. Thus

$$\sup_{w\in\mathbb{K},\,|w|\geqslant c_0}\ \sup_{u_0,\dots,u_m\in B}\ \inf_{x\in X} M(x,w,u_0,\dots,u_m) \leqslant \max\{c_3,c_5\}, \quad \text{where}$$

$$M(x,w,u_0,\dots,u_m) = |w|\cdot\max\bigl\{|w|^{q-n(m+1)}\|u_0-x\|,\ \bigl\|u_1-e^{wz_1S_1}x\bigr\|,\dots,\ \bigl\|u_m-e^{wz_mS_m}x\bigr\|\bigr\}.$$

For $1 \leqslant |w| \leqslant c_0$, we have $M(0,w,u_0,\dots,u_m) \leqslant cc_0^{q+1-n(m+1)}$, where $c \geqslant 1$ is such that $\|u\| \leqslant c$ for any $u \in B$. Hence

$$\sup_{w\in\mathbb{K},\,1\leqslant|w|\leqslant c_0}\ \sup_{u_0,\dots,u_m\in B}\ \inf_{x\in X} M(x,w,u_0,\dots,u_m) \leqslant cc_0^{q+1-n(m+1)}.$$

By the last two displays, the left-hand side in (6.1) does not exceed $\max\{c_3, c_5, cc_0^{q+1-n(m+1)}\}$ and therefore it is finite. □

Introducing multiplicity into Lemma 6.2 does not change anything, but it will allow us to obtain a coordinate-free version of the previous lemma that will be needed in the sequel.

Lemma 6.3. *Let $n,m,r \in \mathbb{N}$, $\{e_{j,l}\colon 1 \leqslant j \leqslant r,\ 1 \leqslant l \leqslant q\}$ a basis in a qr-dimensional Banach space X with $q \geqslant n(m+1)$, $z \in \mathbb{K}^{[m]}$ and $S_1,\dots,S_m$ linear operators on X such that each $X_j = \operatorname{span}\{e_{j,l}\colon 1 \leqslant l \leqslant q\}$ is invariant for each S_t and each $S_t|_{X_j}$ has a nicely upper triangular matrix in the basis $\{e_{j,1},\dots,e_{j,q}\}$. Then (6.1) holds for any bounded subset B of $E = \operatorname{span}\{e_{j,l}\colon 1 \leqslant j \leqslant r,\ 1 \leqslant l \leqslant n\}$.*

Proof. Let B be a bounded subset of E and $E_j = \operatorname{span}\{e_{j,1},\dots,e_{j,n}\}$ for $1 \leqslant j \leqslant r$. Clearly, $E = E_1 \oplus \dots \oplus E_r$. Since E is finite-dimensional, we can find bounded subsets B_j in E_j for $1 \leqslant j \leqslant r$ for which $B \subseteq B_1 + \dots + B_r$. By Lemma 6.2 applied to the restrictions of S_t to X_j,

$$\sup_{\substack{w\in\mathbb{K},\,|w|\geqslant 1\\ u_0,\dots,u_m\in B_j}}\ \inf_{x\in X_j}\ |w|\cdot\max\bigl\{|w|^{q-n(m+1)}\|u_0-x\|,\ \bigl\|u_1-e^{wz_1S_1}x\bigr\|,\dots,\ \bigl\|u_m-e^{wz_mS_m}x\bigr\|\bigr\} = c_j < \infty$$

for $1 \leqslant j \leqslant r$. Using the facts that $B \subseteq B_1 + \dots + B_r$, the triangle inequality and the above display, we see that the left-hand side in (6.1) does not exceed $c_1 + \dots + c_s$ and therefore is finite. □

As we announced, the following is a coordinate-free version of Lemma 6.2.

Lemma 6.4. *Let $m,n,q \in \mathbb{N}$, $q \geqslant n(m+1)$, $z \in \mathbb{K}^{[m]}$, X a finite-dimensional Banach space, $T \in L(X)$ and E a subspace of X such that $T^n(E) = \{0\}$ and $E \subseteq T^{q-n}(X)$. Assume also that $T_1,\dots,T_m \in L(X)$ are given by $T_j = z_jT + a_{j,2}T^2 + a_{j,3}T^3 + \cdots$, where $a_{j,s} \in \mathbb{K}$ and the series converges pointwise. Then for any bounded subset B of E,*

$$\sup_{\substack{w\in\mathbb{K},\,|w|\geqslant 1\\ u_0,\dots,u_m\in B}}\ \inf_{x\in X}\ |w|\cdot\max\bigl\{|w|^{q-n(m+1)}\|u_0-x\|,\ \bigl\|u_1-e^{wT_1}x\bigr\|,\dots,\bigl\|u_m-e^{wT_m}x\bigr\|\bigr\}$$
$$<\infty. \tag{6.12}$$

Proof. Fix a bounded subset B of E and let $r=\dim E$. Since $E\subseteq T^{q-n}(X)$, we can pick an r-dimensional subspace F of X such that $T^{q-n}(F)=E$. Let $g_1,\dots,g_r$ be a basis in F. Consider a qr-dimensional Banach space Z with a basis $\{e_{j,l}:\ 1\leqslant j\leqslant r,\ 1\leqslant l\leqslant q\}$ and let $S\in L(Z)$ be defined as $Se_{j,1}=0$ and $Se_{j,l}=e_{j,l-1}$ if $l\geqslant 2$. We also consider a linear map $C:Z\to X$ defined by $Ce_{j,l}=T^{q-l}g_j$. Since $T^q(F)=T^n(E)=\{0\}$, we have $T^qg_j=0$ and therefore $X_0=C(Z)$ is invariant for T. Moreover, it is easy to see that $RC=CS$, where $R\in L(X_0)$ is the restriction of T to X_0. Since $E=\operatorname{span}\{T^{q-n}g_1,\dots,T^{q-n}g_r\}$, we see that $E\subseteq C(G)$, where $G=\operatorname{span}\{e_{j,l}:\ 1\leqslant j\leqslant r,\ 1\leqslant l\leqslant n\}$. Since G is finite-dimensional and B is a bounded subset of E, there is a bounded subset B_1 of G such that $B\subseteq C(B_1)$. Now let $S_j=z_jS+a_{j,2}S^2+a_{j,3}S^3+\cdots$ (convergence is not an issue since $S^q=0$). Clearly, the operators $z_j^{-1}S_j$ satisfy all the conditions of Lemma 6.3, which implies that

$$\sup_{\substack{w\in\mathbb{K},\,|w|\geqslant 1\\ v_0,\dots,v_m\in B_1}}\ \inf_{y\in Z}\ |w|\cdot\max\bigl\{|w|^{q-n(m+1)}\|v_0-y\|,\ \bigl\|v_1-e^{wS_1}y\bigr\|,\dots,\bigl\|v_m-e^{wS_m}y\bigr\|\bigr\}=c<\infty.$$

Applying the operator C, we get

$$\sup_{\substack{w\in\mathbb{K},\,|w|\geqslant 1\\ v_0,\dots,v_m\in B_1}}\ \inf_{y\in Z}|w|\cdot\max\bigl\{|w|^{q-n(m+1)}\|Cv_0-Cy\|,\ \bigl\|Cv_1-Ce^{wS_1}y\bigr\|,\dots,\bigl\|Cv_m-Ce^{wS_m}y\bigr\|\bigr\}$$
$$\leqslant c\|C\|<\infty.$$

Using the definitions of S_j and T_j together with the equality $RC=CS$ with R being the restriction of T to X_0, we see that $e^{wR_j}C=Ce^{wS_j}$, where $R_j\in L(X_0)$ is the restriction of T_j to the invariant subspace X_0. This observation together with the inclusion $B\subseteq C(B_1)$ and the last display show that

$$\sup_{\substack{w\in\mathbb{K},\,|w|\geqslant 1\\ u_0,\dots,u_m\in B}}\ \inf_{y\in Z}|w|\cdot\max\bigl\{|w|^{q-n(m+1)}\|u_0-Cy\|,\ \bigl\|u_1-e^{wT_1}Cy\bigr\|,\dots,\bigl\|u_m-e^{wT_m}Cy\bigr\|\bigr\}$$
$$<\infty.$$

Since C takes values in X, the above display implies (6.12). $\square$

Corollary 6.5. *Let $m\in\mathbb{N}$, $z\in\mathbb{K}^{[m]}$, X a topological vector space, $T\in L(X)$, $T_1,\dots,T_m\in L(X)$ are given by $T_j=I+z_jT+a_{j,2}T^2+a_{j,3}T^3+\cdots$, where $a_{j,s}\in\mathbb{K}$ and the series converges pointwise and*

$$\Lambda_m(T)=\operatorname{span}\biggl(\bigcup_{n\in\mathbb{N}}T^{mn}\bigl(\ker T^{(m+1)n}\bigr)\biggr). \tag{6.13}$$

Then for each $u_0,\dots,u_m\in\Lambda_m(T)$, there is a sequence $\{x_k\}_{k\in\mathbb{N}}$ in X such that $x_k\to u_0$ and $T_j^kx_k\to u_j$ for $1\leqslant j\leqslant m$. If T is exponentiable and $\{w_k\}_{k\in\mathbb{Z}_+}$ is a sequence in $\mathbb{K}$ satisfying

$|w_k| \to \infty$, then there is a sequence $\{y_k\}_{k\in\mathbb{Z}_+}$ in X such that $y_k \to u_0$ and $e^{w_k z_j T} y_k \to u_j$ for $1 \leqslant j \leqslant m$.

Proof. The set Σ of $(u_0,\dots,u_m) \in X^{m+1}$ for which there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ in X such that $x_k \to u_0$ and $T_j^k x_k \to u_j$ for $1 \leqslant j \leqslant m$ is clearly a linear subspace of X^{m+1}. If T is exponentiable and $\{w_n\}_{n\in\mathbb{Z}_+}$ is a sequence in $\mathbb{K}$ satisfying $|w_k| \to \infty$, then the set Σ_0 of $(u_0,\dots,u_m) \in X^{m+1}$ for which there exists a sequence $\{y_k\}_{k\in\mathbb{Z}_+}$ in X such that $y_k \to u_0$ and $e^{w_k z_j T} y_k \to u_j$ for $1 \leqslant j \leqslant m$ is also a linear subspace of X^{m+1}. We have to show that $\Lambda_m(T)^{m+1} \subseteq \Sigma$ and $\Lambda_m(T)^{m+1} \subseteq \Sigma_0$. Since Σ and Σ_0 are linear subspaces of X^{m+1}, it is enough to verify that

$$\bigl(T^{mn}\bigl(\ker T^{(m+1)n}\bigr)\bigr)^{m+1} \subseteq \Sigma \quad \text{and} \quad \bigl(T^{mn}\bigl(\ker T^{(m+1)n}\bigr)\bigr)^{m+1} \subseteq \Sigma_0 \quad \text{for each } n \in \mathbb{N}. \tag{6.14}$$

Let $n \in \mathbb{N}$ and $u_0,\dots,u_m \in T^{mn}(\ker T^{(m+1)n})$. Pick $v_0,\dots,v_m \in \ker T^{(m+1)n}$ such that $T^{mn} v_j = u_j$ for $0 \leqslant j \leqslant m$. Clearly, $X_0 = \operatorname{span}\{T^l v_j\colon\ 0 \leqslant j \leqslant m,\ 0 \leqslant l \leqslant (m+1)n\}$ is invariant for T and for all T_j, and that the restriction $R \in L(X_0)$ of T to X_0 is nilpotent. Moreover, $T^n(E) = \{0\}$, where $E = \operatorname{span}\{u_0,\dots,u_m\}$ and $T^{nm}(X_0) \supseteq E$. The nilpotency of R implies that the restrictions $R_j \in L(X_0)$ of T_j to X_0 can be written as $R_j = e^{S_j}$, where $S_j = z_j R + b_{j,2} R^2 + b_{j,3} R^3 + \cdots$. Now we equip X_0 with any norm $\|\cdot\|$. By Lemma 6.4 with $q = n(m+1)$,

$$\sup_{w\in\mathbb{K},\ |w|\geqslant 1}\ \inf_{x\in X_0} |w| \max\bigl\{\|u_0 - x\|, \bigl\|u_1 - e^{wS_1}x\bigr\|, \dots, \bigl\|u_m - e^{wS_m}x\bigr\|\bigr\} < \infty \quad \text{and}$$

$$\sup_{w\in\mathbb{K},\ |w|\geqslant 1}\ \inf_{x\in X_0} |w| \max\bigl\{\|u_0 - x\|, \bigl\|u_1 - e^{wz_1R}x\bigr\|, \dots, \bigl\|u_m - e^{wz_mR}x\bigr\|\bigr\} < \infty.$$

Since $R_j^k = e^{kS_j}$ and R_j is the restriction of T_j to X_0, the first equality in the above display implies that there is a sequence $\{x_k\}_{k\in\mathbb{N}}$ in X_0 such that $\|u_0 - x_k\| = O(k^{-1})$ and $\|u_j - T_j^k x_k\| = O(k^{-1})$ as $k \to \infty$. Hence $x_k \to u_0$ and $T_j^k x_k \to u_j$ in X.

Since R is the restriction of T to X_0, the second equality in the above display implies that there is a sequence $\{y_k\}_{k\in\mathbb{Z}_+}$ in X_0 for which $\|u_0 - y_k\| = O(|w_k|^{-1})$ and $\|u_j - e^{w_k z_j T} y_k\| = O(|w_k|^{-1})$ as $k \to \infty$. Since $|w_k| \to \infty$, $y_k \to u_0$ and $T_j^k y_k \to u_j$ in X. That is, (6.14) is satisfied. $\square$

Lemma 6.4 in the case $m = 1$ and $z = 1$ immediately implies the following corollary.

Corollary 6.6. *Let $n, q \in \mathbb{N}$, $q \geqslant 2n$, X a finite-dimensional Banach space, $T \in L(X)$ and E a subspace of X such that $T^n(E) = \{0\}$ and $E \subseteq T^{q-n}(X)$. Then for any bounded subset B of E,*

$$\sup_{\substack{w\in\mathbb{K},\ |w|\geqslant 1\\ u_0,u_1\in B}}\ \inf_{x\in X} \max\bigl\{|w|^{q+1-2n}\|u_0 - x\|, |w|\bigl\|u_1 - e^{wT}x\bigr\|\bigr\} < \infty. \tag{6.15}$$

We need the following elementary lemma.

Lemma 6.7. *Let X be a Banach space, $n \in \mathbb{N}$, $S \in L(X)$ and $x \in X$ such that $S^n x = 0$. Then for each $z \in \mathbb{K}$ with $|z| \geqslant 1$, $\|e^{zS}x\| \leqslant |z|^{n-1}\|x\|e^{\|S\|}$.*

Proof. Let $z \in \mathbb{K}$, $|z| \geqslant 1$. Since $S^n x = 0$, $e^{zS}x = x + zSx + \dots + \frac{z^{n-1}}{(n-1)!}S^{n-1}x$. Hence

$$\left\|e^{zS}x\right\| \leqslant \sum_{j=0}^{n-1} \frac{|z|^j}{j!}\|S\|^j\|x\| \leqslant |z|^{n-1}\|x\|\sum_{j=0}^{n-1}\frac{\|S\|^j}{j!} \leqslant |z|^{n-1}\|x\|e^{\|S\|}. \qquad \square$$

The next application of Lemma 6.2 deals with backward shift tuples of linear operators. It will be key to derive Theorems 1.5, 5.7 and Corollary 1.6. Note that if $(T_1, \dots, T_m) \in L(X)^m$ is a backward shift m-tuple with respect to a filtration $\{X_a\}_{a \in \mathbb{N}_q^m}$, then $T_j^q = 0$ for $1 \leqslant j \leqslant m$. In particular, each T_j is nilpotent and therefore exponentiable.

Lemma 6.8. *Let $n, m \in \mathbb{N}$, $q = 2^m n$, X a topological vector space, $(T_1, \dots, T_m) \in L(X)^m$ a backward shift m-tuple with respect to a filtration $\{X_a\}_{a \in \mathbb{N}_q^m}$ of X. Assume also that $\|\cdot\|$ is a norm on X and E is a finite-dimensional subspace of $X_{(n,\dots,n)}$. Then there exists a finite-dimensional subspace Y of X such that $E \subseteq Y$ and, for any bounded subset B of E,*

$$\sup_{\substack{u_0,\dots,u_m \in B \\ z \in \mathbb{K}^{[m]},\ 1 \leqslant |z_1| \leqslant \dots \leqslant |z_m|}} \inf_{x \in Y} |z_1| \cdot \max\left\{\|u_0 - x\|, \left\|u_1 - e^{z_1 T_1}x\right\|, \dots, \left\|u_m - e^{z_m T_m}x\right\|\right\} < \infty. \tag{6.16}$$

Proof. We use induction with respect to m. In the case $m = 1$ we have $T_1^n(E) = \{0\}$ and $E \subseteq T_1^n(X)$. Pick a finite-dimensional subspace G of X such that $E = T_1^n(G)$ and let $Y = G + T_1(G) + \dots + T_1^{q-1}(G)$. Applying Lemma 6.4 with $m = 1$ and $q = 2n$ to the restriction of T_1 to Y, we get the required estimate.

Assume now that $m \geqslant 2$ and the required estimate is correct for any smaller m. Consider the elements $b = (n, \dots, n)$, $d = (q/2, \dots, q/2, n)$ and $g = (q/2, \dots, q/2, q)$ of $\mathbb{N}^m$. Fix a finite-dimensional linear subspace E of X_b. It is straightforward to see that $(T_1|_{X_d}, \dots, T_{m-1}|_{X_d}) \in L(X_d)^{m-1}$ is a backward shift $m-1$-tuple on X_d with respect to $\{X_{a,n}\}_{a \in \mathbb{N}_{q/2}^{m-1}}$. By the induction hypothesis there exists a finite-dimensional subspace V of X_d such that $E \subseteq V$ and for any bounded subset B of E,

$$\begin{aligned}\alpha(B) &= \sup_{\substack{u_0,\dots,u_{m-1} \in B \\ z \in \mathbb{K}^{[m-1]},\ 1 \leqslant |z_1| \leqslant \dots \leqslant |z_{m-1}|}} \inf_{y \in V} |z_1| \max\left\{\|u_0 - y\|, \left\|u_1 - e^{z_1 T_1}y\right\|, \dots, \left\|u_{m-1} - e^{z_{m-1}T_{m-1}}y\right\|\right\} \\ &< \infty. \end{aligned} \tag{6.17}$$

Next, since $V \subseteq X_d$ is finite-dimensional, $T_m^n(X_d) = \{0\}$ and $X_d \subseteq T_m^{q-n}(X_g)$, we can find a finite-dimensional subspace Y of X_g such that $T_m(Y) \subseteq Y$ and $V \subseteq T_m^{q-n}(Y)$. Now we can apply Corollary 6.6 to the restriction of T_m to the invariant subspace Y to ensure that for any $\gamma > 0$,

$$\beta(\gamma)=\sup_{\substack{z\in\mathbb{K},\ |z|\geqslant 1\\ y,v\in V,\ \|y\|\leqslant\gamma,\ \|v\|\leqslant\gamma}}\ \inf_{f\in Y}\max\bigl\{|z|^{q+1-2n}\|y-f\|,\ |z|\bigl\|v-e^{zT_m}f\bigr\|\bigr\}<\infty. \tag{6.18}$$

Let now B be a bounded subset of E, $z\in\mathbb{K}^{[m]}$, $1\leqslant|z_1|\leqslant\cdots\leqslant|z_m|$ and let $\alpha=\alpha(B)>0$ be the number defined by (6.17). By (6.17), there is $y\in V$ such that

$$\|u_0-y\|\leqslant 2\alpha|z_1|^{-1}\quad\text{and}\quad\bigl\|u_j-e^{z_jT_j}y\bigr\|\leqslant 2\alpha|z_1|^{-1}\quad\text{for } 1\leqslant j\leqslant m-1. \tag{6.19}$$

In particular, $\|y\|\leqslant\|u_0\|+\|u_0-y\|\leqslant c+2\alpha$, where $c=\sup\{\|u\|:\ u\in B\}$. Since $u_m\in B$, we have $u_m\in E\subseteq V$ and $\|u_m\|\leqslant c\leqslant c+2\alpha$. Thus we can use (6.18) with $\gamma=c+2\alpha$ to find $h\in Y$ such that

$$\|h\|\leqslant 2\beta|z_m|^{2n-q-1}\quad\text{and}\quad\bigl\|u_m-e^{z_mT_m}(y+h)\bigr\|\leqslant 2\beta|z_m|^{-1}, \tag{6.20}$$

where $\beta=\beta(c+2\alpha)$. Now let $x=y+h$. According to (6.19) and (6.20),

$$\|u_0-x\|\leqslant\|u_0-y\|+\|h\|\leqslant 2\alpha|z_1|^{-1}+2\beta|z_m|^{2n-q-1}\leqslant 2(\alpha+\beta)|z_1|^{-1}. \tag{6.21}$$

From (6.20) it immediately follows that

$$\bigl\|u_m-e^{z_mT_m}x\bigr\|\leqslant 2\beta|z_m|^{-1}\leqslant 2\beta|z_1|^{-1}. \tag{6.22}$$

Now let $1\leqslant j\leqslant m-1$. Using (6.19), we get

$$\bigl\|u_j-e^{z_jT_j}x\bigr\|\leqslant\bigl\|u_j-e^{z_jT_j}y\bigr\|+\bigl\|e^{z_jT_j}h\bigr\|\leqslant 2\alpha|z_1|^{-1}+\bigl\|e^{z_jT_j}h\bigr\|.$$

Since $h\in X_g$, $T_j^{q/2}h=0$. Since $T_j^q=0$, $Y_j=Y+T_j(Y)+\cdots+T_j^{q-1}(Y)$ is a finite-dimensional subspace of X invariant for T_j. Let c_j be the norm of the restriction of T to Y_j. By Lemma 6.7, $\|e^{z_jT_j}h\|\leqslant|z_j|^{q/2-1}\|h\|e^{c_j}$. Since $\|h\|\leqslant 2\beta|z_m|^{2n-q-1}$ and $|z_j|\leqslant|z_m|$, we obtain

$$\bigl\|e^{z_jT_j}h\bigr\|\leqslant|z_j|^{q/2-1}\|h\|e^{c_j}\leqslant 2\beta e^{c_j}|z_m|^{q/2-1}|z_m|^{2n-q-1}=2\beta e^{c_j}|z_m|^{2n-2^{m-1}n-2}.$$

Since $m\geqslant 2$, $2n-2^{m-1}n-2\leqslant -2\leqslant -1$ and therefore $|z_m|^{2n-2^{m-1}n-2}\leqslant|z_m|^{-1}\leqslant|z_1|^{-1}$. Thus the last two displays imply that $\|u_j-e^{z_jT_j}x\|\leqslant 2(\alpha+\beta e^{c_j})|z_1|^{-1}$ for $1\leqslant j\leqslant m-1$. Combining this estimate with (6.21) and (6.22), we see that

$$|z_1|\max\bigl\{\|u_0-x\|,\ \bigl\|u_1-e^{z_1T_1}x\bigr\|,\ldots,\ \bigl\|u_m-e^{z_mT_m}x\bigr\|\bigr\}\leqslant\delta=2\alpha+2\beta\max_{1\leqslant j\leqslant m-1}e^{c_j}.$$

Hence the left-hand side in (6.16) does not exceed δ, which proves (6.16). □

We would like to get rid of the condition $|z_1|\leqslant\cdots\leqslant|z_m|$ in Lemma 6.8. For $z\in\mathbb{K}^m$, we denote

$$\nu(z)=\min_{1\leqslant j\leqslant m}|z_j|. \tag{6.23}$$

Lemma 6.9. *Let $n, m \in \mathbb{N}$, $q = 2^m n$, X a topological vector space, $(T_1, \dots, T_m) \in L(X)^m$ a backward shift m-tuple with respect to a filtration $\{X_a\}_{a \in \mathbb{N}_q^m}$ of X. Assume also $\|\cdot\|$ is a norm on X and E is a finite-dimensional subspace of $X_{(n,\dots,n)}$. Then there exists a finite-dimensional subspace Y of X such that $E \subseteq Y$ and for any bounded subset B of E,*

$$\sup_{\substack{u_0,\dots,u_m \in B \\ z \in \mathbb{K}^{[m]},\, \nu(z) \geqslant 1}} \inf_{x \in Y} \nu(z) \cdot \max\bigl\{\|u_0 - x\|, \bigl\|u_1 - e^{z_1 T_1} x\bigr\|, \dots, \bigl\|u_m - e^{z_m T_m} x\bigr\|\bigr\} < \infty. \tag{6.24}$$

Proof. As usual, $\mathcal{S}_m$ is the group of permutations of $\mathbb{N}_m$. Direct application of Lemma 6.8 ensures that for any $\sigma \in \mathcal{S}_m$, there exists a finite-dimensional subspace Y_σ of X such that $E \subseteq Y_\sigma$ and for any bounded subset B of E,

$$\begin{aligned} &a_\sigma(B) \\ &\quad= \sup_{\substack{u_0,\dots,u_m \in B \\ z \in \mathbb{K}^{[m]},\, 1 \leqslant |z_{\sigma(1)}| \leqslant \dots \leqslant |z_{\sigma(m)}|}} \inf_{x \in Y_\sigma} |z_{\sigma(1)}| \cdot \max\bigl\{\|u_0 - x\|, \bigl\|u_1 - e^{z_1 T_1} x\bigr\|, \dots, \bigl\|u_m - e^{z_m T_m} x\bigr\|\bigr\} \\ &\quad< \infty. \end{aligned}$$

Now, the left-hand side in (6.24) with $Y = \sum_{\sigma \in \mathcal{S}_m} Y_\sigma$ does not exceed $\max_{\sigma \in \mathcal{S}_m} a_\sigma(B)$, which proves (6.24). $\square$

Corollary 6.10. *Let $n, m \in \mathbb{N}$, $q = 2^m n$, X a topological vector space, $(T_1, \dots, T_m) \in L(X)^m$ a backward shift m-tuple with respect to a filtration $\{X_a\}_{a \in \mathbb{N}_q^m}$ of X. Then for any $u_0, \dots, u_m \in X_{(n,\dots,n)}$ and any sequence $\{z_k = (z_{k,1}, \dots, z_{k,m})\}_{k \in \mathbb{Z}_+}$ in $\mathbb{K}^{[m]}$ satisfying $\nu(z_k) \to \infty$, there is a sequence $\{x_k\}_{k \in \mathbb{Z}_+}$ in X such that $x_k \to u_0$ and $e^{z_{k,j} T_j} x_k \to u_j$ for $1 \leqslant j \leqslant m$.*

Proof. According to the hypothesis, $E = \operatorname{span}\{u_0, \dots, u_m\}$ is a finite-dimensional subspace of $X_{(n,\dots,n)}$. Consider any norm $\|\cdot\|$ on X. By Lemma 6.9, there is a finite-dimensional subspace Y of X such that

$$\sup_{z \in \mathbb{K}^{[m]},\, \nu(z) \geqslant 1} \inf_{x \in Y} \nu(z) \cdot \max\bigl\{\|u_0 - x\|, \bigl\|u_1 - e^{z_1 T_1} x\bigr\|, \dots, \bigl\|u_r - e^{z_m T_m} x\bigr\|\bigr\} < \infty.$$

Hence we can pick a sequence $\{x_k\}_{k \in \mathbb{Z}_+}$ in Y such that $\|u_0 - x_k\| = O(\nu(z_k)^{-1})$ and $\|u_j - e^{z_{k,j} T_j} x_k\| = O(\nu(z_k)^{-1})$ for $1 \leqslant j \leqslant m$ as $k \to \infty$. Since $\nu(z_k) \to \infty$, we have $\|x_k - u_0\| \to 0$ and $\|u_j - e^{z_{k,j} T_j} x_k\| \to 0$. For $1 \leqslant j \leqslant m$ let $Y_j = Y + T_j(Y) + \dots + T_j^{q-1}(Y)$. Clearly $Y \subseteq Y_j$ and Y_j are finite-dimensional. Since $T_j^q = 0$, we have $T_j(Y_j) \subseteq Y_j$. Since Y_j is finite-dimensional, the norm topology on Y_j coincides with the topology inherited from X. Since $u_0, x_k \in Y$ and $u_j, e^{z_{k,j} T_j} x_k \in Y_j$, $x_k \to u_0$ and $e^{z_{k,j} T_j} x_k \to u_j$ in X. $\square$

7. Proofs of Theorems 1.3 and 5.7

Proposition 7.1. *Let $m \in \mathbb{N}$, $z \in \mathbb{K}^{[m]}$, X a topological vector space and $T \in L(X)$ such that $\Lambda_m(T)$, defined in (6.13), is dense in X. Assume also that $T_1, \dots, T_m \in L(X)$ are given by the formulae $T_j = I + z_j T + \sum_{l=2}^{\infty} a_{j,l} T^l$ with $a_{j,l} \in \mathbb{K}$, where the series in the right-hand side converges pointwise. Then $(T_1, \dots, T_m)$ is d-mixing. If additionally, T is exponentiable, then the m-tuple $\{e^{w z_1 T}, \dots, e^{w z_m T}\}_{w \in \mathbb{K}}$ of operator groups is d-mixing.*

Proof. By Corollary 6.5, $\Sigma(T_1,\dots,T_m)$ is dense in X^{m+1} and $\Sigma(\{e^{w_k z_1 T},\dots,e^{w_k z_m T}\}_{k\in\mathbb{Z}_+})$ is dense in X^{m+1} whenever $|w_k|\to\infty$, provided T is exponentiable. The conclusion now follows by Lemma 3.3. $\square$

Proof of Theorem 1.3. By [28, Proposition 4.9], we may assume without loss of generality that $a_{0,j}=1$ for each $1\leqslant j\leqslant m$. Let $(z_1,\dots,z_m):=(a_{1,1},\dots,a_{1,m})$. By Proposition 7.1, it suffices to demonstrate that the space $\Lambda_m(T)$ defined in (6.13) is dense in X. Since T is a backward space shift, there is a sequence $\{X_n\}_{n\in\mathbb{N}}$ of linear subspaces of X such that the sum of X_n is dense, $T(X_1)=\{0\}$ and $T(X_{n+1})$ is a dense subspace of X_n for each $n\in\mathbb{N}$. Let $Y_n=T^{nm}(X_{n(m+1)})$ for $n\in\mathbb{N}$. Since $T(X_{k+1})$ is a dense subspace of X_k for each $k\in\mathbb{N}$, Y_n is a dense subspace of X_n for each $n\in\mathbb{N}$. On the other hand, since $X_{n(m+1)}\subseteq\ker T^{n(m+1)}$, $Y_n\subseteq\Lambda_m(T)$ for each $n\in\mathbb{N}$. Since $\Lambda_m(T)$ is a linear space, it contains the sum Z of Y_n. Since the sum of X_n is dense in X and each Y_n is dense in X_n, Z is dense in X. Hence $\Lambda_m(T)$ is dense in X. $\square$

Proof of Theorem 5.7. Let $m\in\mathbb{N}$, X be a topological vector space and let $(T_1,\dots,T_m)\in L(X)^m$ be a backward space shift m-tuple. By definition, there is a filtration $\{X_a\}_{a\in\mathbb{N}^m}$ of X and for any $n\in\mathbb{N}$, there is a filtration $\{X_a^{(n)}\}_{a\in\mathbb{N}_n^m}$ of $X_{(n,\dots,n)}$ such that $Y_n=X^{(n)}_{(n,\dots,n)}$ is invariant for each T_j and $(T_1|_{Y_n},\dots,T_m|_{Y_n})\in L(Y_n)^m$ is a backward shift m-tuple with respect to $\{X_a^{(n)}\}_{a\in\mathbb{N}_n^m}$.

Let $\{r_k=(r_{k,1},\dots,r_{k,m})\}_{k\in\mathbb{Z}_+}$ be a sequence in $\mathbb{N}^m$ and $\{z_k=(z_{k,1},\dots,z_{k,m})\}_{k\in\mathbb{Z}_+}$ a sequence in $\mathbb{K}^{[m]}$ such that $\nu(r_k)\to\infty$ and $\nu(z_k)\to\infty$, where ν is defined in (6.23). In order to show that $(I+T_1,\dots,I+T_m)$ is strongly d-mixing, it suffices to verify that the sequence $\{((I+T_1)^{r_{k,1}},\dots,(I+T_m)^{r_{k,m}})\}_{k\in\mathbb{Z}_+}$ is d-mixing. By Lemma 3.3, it is enough to demonstrate that

$$\Sigma=\Sigma\big(\big\{(I+T_1)^{r_{k,1}},\dots,(I+T_m)^{r_{k,m}}\big\}_{k\in\mathbb{Z}_+}\big)\quad\text{is dense in } X^{m+1}. \tag{7.1}$$

Similarly, in order to prove that $\{e^{zT_1},\dots,e^{zT_m}\}_{z\in\mathbb{K}}$ is a strongly d-mixing m-tuple of semigroups provided T_j are exponentiable, it suffices to verify that the sequence $\{(e^{z_{k,1}S_1},\dots,e^{z_{k,m}S_m})\}_{k\in\mathbb{Z}_+}$ is d-mixing. By Lemma 3.3, it is enough to show that

$$\Sigma_0=\Sigma\big(\big\{e^{z_{k,1}T_1},\dots,e^{z_{k,m}T_m}\big\}_{k\in\mathbb{Z}_+}\big)\quad\text{is dense in } X^{m+1}. \tag{7.2}$$

Since $\{X_{(n,\dots,n)}\}_{n\in\mathbb{N}}$ is an increasing sequence of subspaces of X with dense union, in order to prove (7.1) and (7.2), it suffices to demonstrate that $\Sigma\cap X^{m+1}_{(n,\dots,n)}$ is dense in $X^{m+1}_{(n,\dots,n)}$ for each $n\in\mathbb{N}$ and that $\Sigma_0\cap X^{m+1}_{(n,\dots,n)}$ is dense in $X^{m+1}_{(n,\dots,n)}$ for each $n\in\mathbb{N}$ provided T_j are exponentiable. Now let $n\in\mathbb{N}$ and $q=2^m n$. Then $X^{(q)}_{(n,\dots,n)}$ is dense in $X_{(n,\dots,n)}$. Moreover, $(T_1|_{Y_q},\dots,T_m|_{Y_q})$ is a backward shift m-tuple on Y_q with respect to $\{X_a^{(q)}\}_{a\in\mathbb{N}_q^m}$. By Corollary 6.10, $(X^{(q)}_{(n,\dots,n)})^{m+1}\subseteq\Sigma_0$. Since $X^{(q)}_{(n,\dots,n)}$ is dense in $X_{(n,\dots,n)}$, it follows that $\Sigma_0\cap X^{m+1}_{(n,\dots,n)}$ is dense in $X^{m+1}_{(n,\dots,n)}$ for each $n\in\mathbb{N}$ provided T_j are exponentiable, which proves (7.2). Now, if $R_j=T_j|_{Y_q}\in L(Y_q)$, then the operators

$$S_j=\ln(I+R_j)=\sum_{l=1}^{\infty}\frac{(-1)^l}{l}R_j^l=\sum_{l=1}^{q-1}\frac{(-1)^l}{l}R_j^l\quad\text{for } 1\leqslant j\leqslant m$$

(we use the equalities $R_j^q = 0$) also form a backward shift m-tuple on Y_q with respect to $\{X_a^{(q)}\}_{a\in\mathbb{N}_q^m}$. By Corollary 6.10, for any $u_0, \dots, u_m \in X_{(n,\dots,n)}^{(q)}$ there is a sequence $\{x_k\}_{k\in\mathbb{N}}$ in Y_q such that $x_k \to u_0$ and $e^{r_{k,j}S_j}x_k \to u_j$. Since $e^{r_{k,j}S_j}$ is exactly the restriction of $(I+T_j)^{r_{k,j}}$ to the invariant subspace Y_q, we have $(u_0, \dots, u_m) \in \Sigma$. Thus $(X_{(n,\dots,n)}^{(q)})^{m+1} \subseteq \Sigma$. Since $X_{(n,\dots,n)}^{(q)}$ is dense in $X_{(n,\dots,n)}$, it follows that $\Sigma \cap X_{(n,\dots,n)}^{m+1}$ is dense in $X_{(n,\dots,n)}^{m+1}$ for each $n \in \mathbb{N}$, which proves (7.1). □

8. Proofs of Theorem 1.5 and Corollary 1.6

According to Theorem 5.7, Theorem 1.5 will be proved if we verify the following proposition.

Proposition 8.1. *Let X be a separable infinite-dimensional Fréchet space non-isomorphic to ω and $r \in \mathbb{N}$. Then there exists a linearly independent sequence $\{e_a\}_{a\in\mathbb{N}^r}$ in X with dense linear span and $S_1, \dots, S_r \in L(X)$ such that the operators S_j are exponentiable, pairwise commuting and $(S_1, \dots, S_r)$ is a backward shift r-tuple with respect to $\{e_a\}_{a\in\mathbb{N}^r}$ (see Remark 5.5 for the definition).*

Proof. The main lemma in [11] ensures the existence of sequences $\{x_n\}_{n\in\mathbb{N}}$ in X and $\{f_n\}_{n\in\mathbb{N}}$ in X^* such that $x_n \to 0$, $E = \operatorname{span}\{x_n\colon n \in \mathbb{N}\}$ is dense in X, the set $\{f_n\colon n \in \mathbb{N}\}$ is uniformly equicontinuous, $f_n(x_m) = 0$ if $n \neq m$ and $f_n(x_n) \neq 0$ for any $n \in \mathbb{N}$. The latter condition implies the linear independence of $\{x_n\}_{n\in\mathbb{N}}$. Fix a bijection $\gamma : \mathbb{N}^r \to \mathbb{N}$ and let $e_a = x_{\gamma(a)}$ and $g_a = f_{\gamma(a)}$ for $a \in \mathbb{N}^r$. Then e_a are linearly independent and $\operatorname{span}\{e_a\colon a \in \mathbb{N}^r\} = E$ is dense in X. Moreover, $g_a(e_a) \neq 0$ for $a \in \mathbb{N}^r$ and $g_a(e_b) = 0$ when $a \neq b$. For $n \in \mathbb{N}$, $n \geqslant r$, we denote

$$\varepsilon_n = \min\bigl\{\bigl|g_a(e_a)\bigr|\colon a \in \mathbb{N}^r,\ |a| = n+1\bigr\}, \quad \text{where } |a| = a_1 + \cdots + a_r \text{ for } a \in \mathbb{N}^r.$$

Since $g_a(e_a) \neq 0$, $\varepsilon_n > 0$ for $n \geqslant r$. Pick any sequence $\{\alpha_n\}_{n\geqslant r}$ of positive numbers satisfying

$$\alpha_{n+1} \geqslant 2^n \alpha_n \varepsilon_n^{-1} \quad \text{for any } n \geqslant r \tag{8.1}$$

and consider the operators $S_j : X \to X$ for $1 \leqslant j \leqslant r$ defined by the formula

$$S_j x = \sum_{a\in\mathbb{N}^r} \frac{\alpha_{|a|} g_{a+\mathrm{e}_j}(x)}{\alpha_{|a|+1} g_{a+\mathrm{e}_j}(e_{a+\mathrm{e}_j})} e_a. \tag{8.2}$$

Since $\{f_n\colon n \in \mathbb{N}\} = \{g_a\colon a \in \mathbb{N}^r\}$ is uniformly equicontinuous, there exists a continuous seminorm p on X such that each $|g_a(x)| \leqslant p(x)$ for each $x \in X$ and $a \in \mathbb{N}^r$. Since $x_n \to 0$, the closed balanced convex hull K of $\{x_n\colon n \in \mathbb{N}\}$ is a compact subset of X. Hence the Minkowskii functional q of the set K is a norm on $X_K = \operatorname{span}(K)$, defining a topology stronger than the one inherited from X. It is also well known that (X_K, q) is a Banach space since K is compact. Clearly $q(e_a) \leqslant 1$ for each $a \in \mathbb{N}_a$. From (8.1) and the definition of ε_n it follows that the series defining S_j can be written as

$$S_j x = \sum_{a\in\mathbb{N}^r} c_{j,a} g_{a+\mathrm{e}_j}(x) e_a \quad \text{with } 0 < |c_{j,a}| < 2^{-|a|} \quad \text{and therefore}$$

$$\sum_{a\in\mathbb{N}^r} |c_{j,n}| \leqslant C = \sum_{a\in\mathbb{N}^r} 2^{-|a|}.$$

Thus the series defining S_j converges absolutely in the Banach space (X_K, q) and therefore converges in X. Moreover, $q(S_j x) \leqslant Cp(x)$ for each $x \in X$. Thus S_j are continuous as linear operators from X to the Banach space X_K and therefore $S_j \in L(X)$. From the above display it also follows that $S_j e_a = 0$ if $a_j = 1$ and $S_j e_a = c_{j,a-\mathrm{e}_j} g_a(\mathrm{e}_a) e_{a-\mathrm{e}_j}$ otherwise. Hence $(S_1, \dots, S_r)$ is a backward shift r-tuple with respect to $\{e_a\}_{a\in\mathbb{N}^r}$. Using the definition of S_j and the equalities $g_a(e_b) = 0$ for $a \neq b$, it is easy to verify that $S_j S_l e_a = S_l S_j e_a$ for $1 \leqslant j < l \leqslant r$ and $a \in \mathbb{N}^r$. Indeed, if either $a_j = 1$ or $a_l = 1$, we have $S_j S_l e_a = S_l S_j e_a = 0$. If $a_j \geqslant 2$ and $a_l \geqslant 2$, then $S_j S_l e_a = S_l S_j e_a = \frac{\alpha_{|a|-2}}{\alpha_{|a|}} e_{a-\mathrm{e}_j-\mathrm{e}_l}$. Since E is dense in X, then $S_1, \dots, S_r$ are pairwise commuting. It remains to show that each S_j is exponentiable. Let $1 \leqslant j \leqslant r$. As we have already shown $q(S_j x) \leqslant Cp(x)$ for each $x \in X$. Since K is compact and p is continuous, there is $c > 0$ such that $p(x) \leqslant c$ for each $x \in K$. Hence $p(x) \leqslant cq(x)$ for any $x \in X_K$. Thus $q(S_j^2 x) \leqslant Cp(S_j x) \leqslant Ccq(S_j x) \leqslant C^2 cp(x)$. Iterating this argument, we get $q(S_j^n x) \leqslant C^n c^{n-1} p(x)$ for each $n \in \mathbb{N}$. Hence for any $x \in X$ and $z \in \mathbb{K}$, the series $\sum_{n=1}^{\infty} \frac{z^n}{n!} S_j^n x$ converges absolutely in the Banach space X_K. Thus we can define a linear operator $e^{zS_j} : X \to X$ by the formula $e^{zS_j} x = x + \sum_{n=1}^{\infty} \frac{z^n}{n!} S_j^n x$. Moreover,

$$q\big(e^{zS_j} x - x\big) \leqslant p(x) \sum_{n=1}^{\infty} \frac{|z|^n C^n c^{n-1}}{n!} = c^{-1}\big(e^{cC|z|} - 1\big) p(x).$$

Thus each e^{zS_j} is a continuous linear operator from X to X_K and therefore $e^{zS_j} \in L(X)$. The above display also implies that $e^{zS_j} x \to x$ as $z \to 0$ for any $x \in X$, which ensures strong continuity of the operator group $\{e^{zS_j}\}_{z\in\mathbb{K}}$. □

As already mentioned, Proposition 8.1 and Theorem 5.7 imply Theorem 1.5. It remains to prove Corollary 1.6.

Proof of Corollary 1.6. Let X be a separable infinite-dimensional Fréchet space. The case when X is non-isomorphic to ω follows immediately from Theorem 1.5. It remains to consider the case when X is isomorphic to ω. In this case we can interpret X as the space $\mathbb{K}^{\mathbb{N}^r}$ with the coordinate-wise convergence topology. Consider $S_j \in L(X)$ defined by the formula $(S_j x)_a = x_{a+\mathrm{e}_j}$. It is straightforward to see that $(S_1, \dots, S_r)$ is a backward shift r-tuple with respect to the canonical basis of $\mathbb{K}^{\mathbb{N}^r}$. By Theorem 5.7, $(I + S_1, \dots, I + S_r)$ is strongly d-mixing. □

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Appendix A

A.1. Sobolev spaces

Lemma A.1. *Let $f \in W^{2,2}[-\pi,\pi]$, $f(-\pi)=f(\pi)$, $f'(-\pi)=f'(\pi)$, $c_0=\|f\|_{L^\infty[-\pi,\pi]}$ and $c_1=\|f''\|_{L^2[-\pi,\pi]}$. Then $\|f\|_{W^{2,2}[-\pi,\pi]}\leqslant\sqrt{3c_1^2+c_0^2}$.*

Proof. For $n\in\mathbb{Z}$, let $\hat f_n=\int_{-\pi}^{\pi}f(x)e^{-inx}\,dx$ be the nth Fourier coefficient of f. Using the Parseval identity and the equalities $f(-\pi)=f(\pi)$ and $f'(-\pi)=f'(\pi)$, we get

$$\|f\|^2_{L^2[-\pi,\pi]}=\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}|\hat f_n|^2,\qquad \|f'\|^2_{L^2[-\pi,\pi]}=\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}|n\hat f_n|^2,$$
$$\|f''\|^2_{L^2[-\pi,\pi]}=\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\bigl|n^2\hat f_n\bigr|^2.$$

Hence $\|f\|^2_{L^2[-\pi,\pi]}\leqslant\|f'\|^2_{L^2[-\pi,\pi]}+|\hat f_0|^2$ and $\|f'\|_{L^2[-\pi,\pi]}\leqslant\|f''\|_{L^2[-\pi,\pi]}$. That is, $\|f'\|_{L^2[-\pi,\pi]}\leqslant c_1$ and $\|f\|^2_{L^2[-\pi,\pi]}\leqslant c_0^2+c_1^2$. Thus

$$\|f\|^2_{W^{2,2}[-\pi,\pi]}=\|f\|^2_{L^2[-\pi,\pi]}+\bigl\|f'\bigr\|^2_{L^2[-\pi,\pi]}+\bigl\|f''\bigr\|^2_{L^2[-\pi,\pi]}\leqslant 3c_1^2+c_0^2.\qquad\square$$

Lemma A.2. *Let $-\infty<\alpha<\beta<\infty$ and $a_0,a_1,b_0,b_1\in\mathbb{C}$. Then there exists $f\in C^2[\alpha,\beta]$ such that*

$$f(\alpha)=a_0,\qquad f'(\alpha)=a_1,\qquad f(\beta)=b_0,\qquad f'(\beta)=b_1,\tag{A.1}$$
$$\|f\|_{L^\infty[\alpha,\beta]}\leqslant\frac{|a_0+b_0|}{2}+\frac{|a_0-b_0|}{2}+\frac{(\beta-\alpha)(|a_1|+|b_1|)}{5},\tag{A.2}$$
$$\bigl\|f''\bigr\|^2_{L^2[\alpha,\beta]}\leqslant\frac{24|a_0-b_0|^2}{(\beta-\alpha)^3}+\frac{12}{\beta-\alpha}\bigl(|a_1|^2+|b_1|^2\bigr).\tag{A.3}$$

Proof. For brevity, we denote $\tau=\beta-\alpha$. Consider the following polynomial of degree at most 3:

$$q(t)=\frac{a_0+b_0}{2}+\frac{\tau(a_1-b_1)}{8}\bigl(1-t^2\bigr)+\frac{b_0-a_0}{4}\bigl(3t-t^3\bigr)+\frac{\tau(a_1+b_1)}{8}\bigl(t^3-t\bigr).\tag{A.4}$$

The reason for considering q is that it is the unique polynomial of degree at most 3 satisfying

$$q(-1)=a_0,\qquad q(1)=b_0,\qquad q'(-1)=\frac{\tau a_1}{2}\quad\text{and}\quad q'(1)=\frac{\tau b_1}{2}.\tag{A.5}$$

Using (A.4), we immediately see that for each $t\in[-1,1]$,

$$\bigl|q(t)\bigr|\leqslant\frac{|a_0+b_0|}{2}+\frac{\tau|a_1-b_1|}{8}\bigl(1-t^2\bigr)+\frac{|a_0-b_0|}{4}\bigl|3t-t^3\bigr|+\frac{\tau|a_1+b_1|}{8}\bigl|t-t^3\bigr|.$$

Taking into account that for each $t \in [-1, 1]$, $1 - t^2 \leqslant 1$, $|3t - t^3| \leqslant 2$ and $|t - t^3| \leqslant \frac{2}{3\sqrt{3}} < \frac{2}{5}$, we obtain

$$\begin{aligned}\|q\|_{L^\infty[-1,1]} &\leqslant \frac{|a_0+b_0|}{2} + \frac{\tau|a_1-b_1|}{8} + \frac{|a_0-b_0|}{2} + \frac{\tau|a_1+b_1|}{20} \\ &\leqslant \frac{|a_0+b_0|}{2} + \frac{|a_0-b_0|}{2} + \frac{\tau(|a_1|+|b_1|)}{5}. \end{aligned} \tag{A.6}$$

Differentiating q twice, we get $q''(t) = \frac{\tau(b_1-a_1)}{4} + (\frac{3(a_0-b_0)}{2} + \frac{3\tau(a_1+b_1)}{4})t$. Integrating $|q''(t)|^2 = q''(t)\overline{q''(t)}$ from -1 to 1, we get

$$\begin{aligned}\|q''\|^2_{L^2[-1,1]} &= \frac{\tau^2|b_1-a_1|^2 + 3|2(a_0-b_0) + \tau(a_1+b_1)|^2}{8} \\ &\leqslant \frac{\tau^2|b_1-a_1|^2}{8} + 3|a_0-b_0|^2 + \frac{3}{4}\tau^2|a_1+b_1|^2. \end{aligned}$$

Using the easy inequality $\frac{|b_1-a_1|^2}{8} + \frac{3|b_1+a_1|^2}{4} \leqslant \frac{3(|a_1|^2+|b_1|^2)}{2}$, we get

$$\|q''\|_{L^2[-1,1]} \leqslant 3|a_0-b_0|^2 + \frac{3\tau^2}{2}\big(|a_1|^2 + |b_1|^2\big). \tag{A.7}$$

Now we consider the polynomial f defined by the formula $f(x) = q(\frac{2}{\tau}x - \frac{\alpha+\beta}{\tau})$. It is straightforward to see that $f(\alpha) = q(-1)$, $f(\beta) = q(1)$, $f'(\alpha) = \frac{2}{\tau}q'(-1)$ and $f'(\beta) = \frac{2}{\tau}q'(1)$. These equalities together with (A.5) imply (A.1). Clearly $\|f\|_{L^\infty[\alpha,\beta]} = \|q\|_{L^\infty[-1,1]}$. Hence (A.6) implies (A.2). Finally, $f''(x) = \frac{4}{\tau^2}q''(\frac{2}{\tau}x - \frac{\alpha+\beta}{\tau})$. Making the linear change of variables $t = \frac{2}{\tau}x - \frac{\alpha+\beta}{\tau}$ in the integral defining $\|f''\|^2_{L^2[\alpha,\beta]}$, we have $\|f''\|^2_{L^2[\alpha,\beta]} = \frac{8}{\tau^3}\|q''\|^2_{L^2[-1,1]}$. This equality together with (A.7) gives (A.3). □

Lemma A.3. *There exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of 2π-periodic functions on $\mathbb{R}$ such that $f_n|_{[-\pi,\pi]} \in W^{2,2}[-\pi,\pi]$, the sequence $\{\|f_n\|_{W^{2,2}[-\pi,\pi]}\}_{n\in\mathbb{N}}$ is bounded and $f_n(x) = 2e^{inx} - e^{2inx}$ whenever $|x - \frac{2\pi k}{n}| \leqslant \frac{2}{n^5}$ for some $k \in \mathbb{Z}$.*

Proof. For $n \in \mathbb{N}$, let $h_n : \mathbb{R} \to \mathbb{C}$, $h_n(x) = 2e^{inx} - e^{2inx}$. Clearly h_n is periodic with the period $\frac{2\pi}{n}$. Let also $\alpha_n = \frac{2}{n^5}$ and $\beta_n = \frac{2\pi}{n} - \frac{2}{n^5}$. By Lemma A.2, there is $g_n \in C^2[\alpha_n, \beta_n]$ such that

$$g_n(\alpha_n) = h_n(\alpha_n), \qquad g_n(\beta_n) = h_n(\beta_n), \qquad g_n'(\alpha_n) = h_n'(\alpha_n), \qquad g_n'(\beta_n) = h_n'(\beta_n), \tag{A.8}$$

$$\|g_n\|_{L^\infty[\alpha_n,\beta_n]} \leqslant \max\big\{\big|h_n(\alpha_n)\big|, \big|h_n(\beta_n)\big|\big\} + \frac{(\beta_n-\alpha_n)(|h_n'(\alpha_n)| + |h_n'(\beta_n)|)}{5}, \tag{A.9}$$

$$\|g_n''\|^2_{L^2[\alpha_n,\beta_n]} \leqslant \frac{24|h_n(\alpha_n) - h_n(\beta_n)|^2}{(\beta_n-\alpha_n)^3} + \frac{12}{\beta_n-\alpha_n}\big(\big|h_n'(\alpha_n)\big|^2 + \big|h_n'(\beta_n)\big|^2\big). \tag{A.10}$$

Periodicity of h_n with the period $\frac{2\pi}{n}$ and the equalities (A.8) imply that there is a unique $f_n \in C^1(\mathbb{R})$ such that f_n is periodic with the period $\frac{2\pi}{n}$, $f_n|_{[\alpha_n,\beta_n]} = g_n$ and $f_n|_{[\beta_n, \frac{2\pi}{n}+\alpha_n]} =$

$h_n|_{[\beta_n, \frac{2\pi}{n}+\alpha_n]}$. Indeed, the last two equalities define f_n on $[\alpha_n, \frac{2\pi}{n}+\alpha_n]$, while (A.8) ensures C^1-gluing at the point β_n as well as the boundary condition $f_n(\alpha_n) = f_n(\frac{2\pi}{n}+\alpha_n)$, $f_n'(\alpha_n) = f_n'(\frac{2\pi}{n}+\alpha_n)$, which makes a periodic C^1-extension possible. Periodicity of f_n with the period $\frac{2\pi}{n}$ and the equality $f_n|_{[\beta_n, \frac{2\pi}{n}+\alpha_n]} = h_n|_{[\beta_n, \frac{2\pi}{n}+\alpha_n]}$ imply that $f_n(x) = 2e^{inx} - e^{2inx}$ whenever $|x - \frac{2\pi k}{n}| \leqslant \frac{2}{n^5}$ for some $k \in \mathbb{Z}$. Since f_n is piecewise C^2, $f_n|_{[-\pi,\pi]} \in W^{2,2}[-\pi,\pi]$. It remains to estimate $\|f_n\|_{W^{2,2}[-\pi,\pi]}$. Obviously, $|h_n(x)| \leqslant 3$ for each $x \in \mathbb{R}$. Since $f_n|_{[\beta_n, \frac{2\pi}{n}+\alpha_n]} = h_n|_{[\beta_n, \frac{2\pi}{n}+\alpha_n]}$, we get

$$\left|h_n(\alpha_n)\right| = \left|h_n(\beta_n)\right| \leqslant 3 \quad \text{and} \quad \|f_n\|_{L^\infty[\beta_n, \frac{2\pi}{n}+\alpha_n]} \leqslant 3. \tag{A.11}$$

Using the obvious inequality $|e^{it} - e^{is}| \leqslant |t-s|$ for $t, s \in \mathbb{R}$, we obtain

$$\left|h_n'(\alpha_n)\right| = \left|h_n'(\beta_n)\right| = \left|2in\left(2e^{2in^{-4}} - e^{4in^{-4}}\right)\right| \leqslant 2n \cdot 2n^{-4} = 4n^{-3}. \tag{A.12}$$

Next,

$$\begin{aligned}\left|h_n(\alpha_n) - h_n(\beta_n)\right| &= \left|2\left(e^{2in^{-4}} - e^{-2in^{-4}}\right) - \left(e^{4in^{-4}} - e^{-4in^{-4}}\right)\right| = \left|4\sin\left(2n^{-4}\right) - 2\sin\left(4n^{-4}\right)\right| \\ &= 4\sin\left(2n^{-4}\right)\left(1 - \cos\left(2n^{-4}\right)\right) = 16\sin^3\left(n^{-4}\right)\cos\left(n^{-4}\right) \\ &\leqslant 16n^{-12}.\end{aligned} \tag{A.13}$$

Using (A.9), (A.11)–(A.13) and the equality $\beta_n - \alpha_n = 2\pi n^{-1} - 4n^{-5}$, we obtain

$$\|f_n\|_{L^\infty[\alpha_n,\beta_n]} \leqslant 3 + 5^{-1}\left(2\pi n^{-1} - 4n^{-5}\right)8n^{-3} < 9 \quad \text{for each } n \in \mathbb{N}.$$

From the above display, the second inequality in (A.11) and $\frac{2\pi}{n}$-periodicity of f_n, we obtain

$$\|f_n\|_{L^\infty[-\pi,\pi]} \leqslant \max\{3, 9\} = 9. \tag{A.14}$$

Direct computation shows that $f_n''(x) = h_n''(x) = -2n^2e^{inx} + 4n^2e^{2inx}$ and therefore $|f_n''(x)| \leqslant 6n^2$ for $x \in [\beta_n, \frac{2\pi}{n}+\alpha_n]$. Hence

$$\left\|f_n''\right\|^2_{L^2[\beta_n, \frac{2\pi}{n}+\alpha_n]} \leqslant 4n^{-5} \cdot 36n^4 = 144n^{-1}.$$

Using (A.10), (A.12), (A.13) and the equality $\beta_n - \alpha_n = 2\pi n^{-1} - 4n^{-5}$, we get

$$\left\|f_n''\right\|^2_{L^2[\alpha_n,\beta_n]} \leqslant \frac{24 \cdot 16^2 n^{-24}}{(2\pi n^{-1} - 4n^{-5})^3} + \frac{24 \cdot 16n^{-6}}{2\pi n^{-1} - 4n^{-5}} \leqslant 24 \cdot 32n^{-21} + 24 \cdot 8n^{-5} < 960n^{-5}.$$

By the last two displays

$$\left\|f_n''\right\|^2_{L^2[\alpha_n, \frac{2\pi}{n}+\alpha_n]} = \left\|f_n''\right\|^2_{L^2[\beta_n, \frac{2\pi}{n}+\alpha_n]} + \left\|f_n''\right\|^2_{L^2[\alpha_n,\beta_n]} \leqslant 144n^{-1} + 960n^{-5} \leqslant 1104n^{-1}.$$

Since f'' is periodic with the period $\frac{2\pi}{n}$, from the above display it follows that

$$\|f_n''\|^2_{L^2[-\pi,\pi]} = n\|f_n''\|^2_{L^2[\alpha_n,\frac{2\pi}{n}+\alpha_n]} \leqslant n \cdot 1104n^{-1} \leqslant 1104. \tag{A.15}$$

According to Lemma A.1, inequalities (A.14) and (A.15) imply that

$$\|f_n\|_{W^{2,2}[-\pi,\pi]} \leqslant \sqrt{9^2 + 3 \cdot 1104} < 64 \quad \text{for each } n \in \mathbb{N}.$$

Thus the sequence $\{f_n\}$ satisfies all desired conditions. □

A.2. Determinants

For $a, k \in \mathbb{N}$, $a \geqslant k$, we consider the $k \times k$ Töplitz matrix

$$C_{a,k} = \left\{\frac{1}{(a+l-j)!}\right\}_{j,l=1}^{k}. \tag{A.16}$$

Lemma A.4. *For any $a, k \in \mathbb{N}$, $a \geqslant k$, the matrix $C_{a,k}$ is invertible. Moreover,*

$$\det C_{a,1} = \frac{1}{a!}, \qquad \det C_{a,2} = \frac{1}{a!(a+1)!} \quad \textit{and} \quad \det C_{a,k} = \frac{1!2! \cdot \ldots \cdot (k-1)!}{a!(a+1)! \cdot \ldots \cdot (a+k-1)!} \quad \textit{for } k \geqslant 3. \tag{A.17}$$

Proof. The case $k = 1$ is obvious. Indeed, $\det C_{b,1} = \frac{1}{b!}$ for any $b \in \mathbb{N}$, which agrees with (A.17). It remains to consider the case $k \geqslant 2$. For $2 \leqslant j \leqslant k$ we subtract the $(j-1)$th row of $C_{a,k}$ multiplied by $a - j + 2$ from the jth row of $C_{a,k}$. The resulting $k \times k$ matrix N must have the same determinant as $C_{a,k}$. On the other hand, the first column of N has shape $(\frac{1}{a!}, 0, \ldots, 0)$. Thus, $\det C_{a,k} = \det N = \frac{1}{a!} \det K$, where K is the $(k-1) \times (k-1)$ matrix obtained from N by eliminating the first row and the first column. From the way the matrix N was constructed it follows that $K = \{\frac{l}{(a+1+l-j)!}\}_{j,l=1}^{k-1}$. Dividing the lth column of K by l, we obtain the matrix $C_{a+1,k-1}$. Hence

$$\det C_{a,k} = \frac{1}{a!} \det K = \frac{(k-1)!}{a!} \det C_{a+1,k-1}.$$

Applying the recurrent formula in the above display $k-1$ times and the equality $\det C_{b,1} = \frac{1}{b!}$, we get the required formula for $\det C_{a,k}$. Since $\det C_{a,k} \neq 0$, $C_{a,k}$ is invertible. □

We also need to compute the determinants of the following Vandermonde-like matrices. Let $n, m, k \in \mathbb{N}$ and $z_1, \ldots, z_m \in \mathbb{C}$. For $1 \leqslant j \leqslant nm$, we consider positive integers s_j and a_j such that

$$j = (s_j - 1)n + a_j, \quad \text{where } 1 \leqslant s_j \leqslant m \text{ and } 1 \leqslant a_j \leqslant n. \tag{A.18}$$

It is easy to see that the map $j \mapsto (s_j, a_j)$ is a bijection from $\mathbb{N}_{nm}$ to $\mathbb{N}_m \times \mathbb{N}_n$. Another way to view this map is to say that it enumerates $\mathbb{N}_m \times \mathbb{N}_n$ in lexicographical ordering. We consider the $nm \times nm$ matrix

$$A_{n,m,k} = A_{n,m,k}(z_1, \dots, z_m) = \left\{ \frac{z_{s_j}^{k+n+l-a_j}}{(k+n+l-a_j)!} \right\}_{j,l=1}^{mn}. \tag{A.19}$$

Another way to introduce the matrix $A_{n,m,k}(z_1, \dots, z_m)$ is by listing its rows. For $a, b \in \mathbb{N}$ and $z \in \mathbb{C}$ we consider the vector

$$A_{a,b}^z = \left(\frac{z^a}{a!}, \frac{z^{a+1}}{(a+1)!}, \dots, \frac{z^{a+b-1}}{(a+b-1)!} \right) \in \mathbb{C}^b. \tag{A.20}$$

If $A_1, \dots, A_r$ are r vectors in $\mathbb{C}^r$, then $A = [A_1, \dots, A_r]$ stands for the $r \times r$ matrix in which A_j occupies the jth row. Then

$$\begin{aligned} A_{n,m,k} &= A_{n,m,k}(z_1, \dots, z_m) \\ &= \big[A_{n+k,nm}^{z_1}, A_{n+k-1,nm}^{z_1}, \dots, A_{k+1,nm}^{z_1}, \\ &\quad A_{n+k,nm}^{z_2}, A_{n+k-1,nm}^{z_2}, \dots, A_{k+1,nm}^{z_2}, \dots, \\ &\quad A_{n+k,nm}^{z_m}, A_{n+k-1,nm}^{z_1}, \dots, A_{k+1,nm}^{z_m}\big]. \end{aligned}$$

Proposition A.5. *The determinant of $A_{n,m,k}(z_1, \dots, z_m)$ has the shape*

$$\det A_{n,m,k}(z_1, \dots, z_m) = a_{n,m,k} \prod_{j=1}^{m} z_j^{n(n+k)} \prod_{1 \leqslant j < l \leqslant m} (z_l - z_j)^{n^2}, \tag{A.21}$$

where

$$a_{n,m,k} = \frac{(1!2! \cdot \ldots \cdot (n-1)!)^n}{(n+k)!(n+k+1)! \cdot \ldots \cdot (nm+k-1)!}. \tag{A.22}$$

Most of this section is devoted to the proof of Proposition A.5. We start with several observations. Obviously, $\det A_{n,m,k}(z_1, \dots, z_m)$ is a polynomial in $z_1, \dots, z_m$ with rational coefficients and therefore can be considered as an element of the ring $\mathcal{P} = \mathbb{C}[z_1, \dots, z_m]$ of polynomials in $z_1, \dots, z_m$ with complex coefficients. It is straightforward to see that $A_{l,nm}^z = z^l B_{l,nm}^z$, where $B_{l,j}^z = (\frac{1}{l!}, \frac{z}{(l+1)!}, \dots, \frac{z^{j-1}}{(l+j-1)!})$. It follows that

$$\det A_{n,m,k} = \det B_{n,m,k} \cdot \prod_{j=1}^{m} \prod_{l=1}^{n} z_j^{k+l} = (z_1 \cdots z_m)^{n(n+2k+1)/2} \det B_{n,m,k}, \tag{A.23}$$

where the matrix $B_{n,m,k}$ is obtained from $A_{n,m,k}$ by replacing the rows $A_{l,nm}^{z_j}$ with $B_{l,nm}^{z_j}$. The matrix $B_{n,m,k}$ has the property that any entry in its jth column is a monomial in $z_1, \dots, z_m$ of degree $j-1$. By definition of the determinant, $\det B_{n,m,k}$ is a homogeneous polynomial in $z_1, \dots, z_m$

of degree $1 + 2 + \cdots + (nm - 1) = \frac{nm(nm-1)}{2}$ (zero polynomial is considered as a homogeneous polynomial of any degree we like). Combining this observation with (A.23), we see that $\det A_{n,m,k}$ is a homogeneous polynomial of degree $\frac{nm(nm-1)}{2} + \frac{mn(n+2k+1)}{2} = \frac{n^2m^2+n^2m+2nkm}{2}$. Note also that if we swap z_j and z_l with $j \neq l$, the corresponding $A_{n,m,k}$ matrices can be obtained from one another by n transpositions of rows. Since a transposition of rows of a matrix multiplies its determinant by -1, we see that the polynomial $\det A_{n,m,k}$ is symmetric if n is even and is antisymmetric if n is odd. These observations are summarized in the following lemma.

Lemma A.6. $\det A_{n,m,k}$ *is a homogeneous polynomial of degree* $\frac{n^2m^2+n^2m+2nkm}{2}$ *and* $\det A_{n,m,k}$ *is symmetric if n is even and is antisymmetric if n is odd.*

We also need a couple of general algebraic results. One is the existence and uniqueness of prime factorization of polynomials of several variables over any field [16, Theorem 5, p. 149]. This result combined with the obvious observation that a polynomial of degree (exactly) one is always prime gives the following lemma.

Lemma A.7. *Let $Q \in \mathcal{P}$, $k_1, \dots, k_s \in \mathbb{N}$ and let $u_1, \dots, u_s \in \mathcal{P}$ be pairwise linearly independent polynomials of degree 1 such that $u_j^{k_j}$ is a divisor of Q for $1 \leqslant j \leqslant s$. Then $u_1^{k_1} \cdots u_s^{k_s}$ is a divisor of Q.*

In order to formulate the next lemma, we introduce the following notation. For $Q \in \mathcal{P}$, $1 \leqslant j \leqslant m$ and $l \in \mathbb{N}$, we denote the lth derivative of Q with respect to z_j by $Q^{(l)}_{z_j}$. That is, $Q^{(l)}_{z_j} = \frac{\partial^l Q}{\partial z_j^l}$.

Lemma A.8. *Let $Q \in \mathcal{P}$, $k \in \mathbb{N}$ and $u \in \mathcal{P}$ a polynomial of degree 1: $u = a_0 + a_1 z_1 + \cdots + a_m z_m$. Fix $1 \leqslant j \leqslant m$ such that $a_j \neq 0$. Then u^k is a divisor of Q if and only if $Q(w) = Q'_{z_j}(w) = \cdots = Q^{(k-1)}_{z_j}(w) = 0$ for any $w = (w_1, \dots, w_m) \in \mathbb{C}^m$ such that $u(w) = 0$.*

Proof. Case $k = 1$ is elementary. Indeed, it means that a polynomial vanishing on a hyperplane must be a multiple of the degree 1 polynomial defining this hyperplane. The rest is a straightforward induction with respect to k. $\square$

Remark A.9. Lemma A.7 holds true for polynomials over any field. Lemma A.8 holds for polynomials over any infinite field and fails over finite fields.

The following lemma is the key ingredient of the proof of Proposition A.5.

Lemma A.10. *The polynomials $z_1^{n(n+k)}$ and $(z_2 - z_1)^{n^2}$ are divisors of* $\det A_{n,m,k}(z_1, \dots, z_m)$.

Proof. According to Lemma A.8, Lemma A.10 will be proved if we verify that

$$\frac{\partial^j}{\partial z_1^j} \det A_{n,m,k}(z_1, \dots, z_m)\bigg|_{z_1=0} = 0 \quad \text{for } j < n(n+k) \quad \text{and} \tag{A.24}$$

$$\frac{\partial^j}{\partial z_1^j} \det A_{n,m,k}(z_1, \dots, z_m)\bigg|_{z_1=z_2} = 0 \quad \text{for } j < n^2. \tag{A.25}$$

Thus it remains to prove (A.24) and (A.25). In order to do that, we have to remind how one differentiates the determinant. Let $s \in \mathbb{N}$ and let $A = \{a_{j,l}\}_{j,l=1}^{s}$ be an $s \times s$ matrix such that each $a_{j,l}$ is a differentiable function of a variable t. Let also $A_1, \dots, A_s$ be the rows of the matrix A. In our notation $A = [A_1, \dots, A_s]$. Then

$$\frac{d}{dt} \det A = \sum_{j=1}^{s} \det\left[A_1, \dots, A_{j-1}, \frac{d}{dt} A_j, A_{j+1}, \dots, A_s\right].$$

The above formula allows us to find higher derivatives of $\det A$ provided the matrix entries are differentiable the appropriate number of times,

$$\frac{d^a}{dt^a} \det A = \sum_{\nu \in \mathbb{Z}_+^s,\, \nu_1 + \dots + \nu_s = a} \frac{a!}{\nu_1! \cdot \ldots \cdot \nu_s!} \det\left[\frac{d^{\nu_1}}{dt^{\nu_1}} A_1, \dots, \frac{d^{\nu_s}}{dt^{\nu_s}} A_s\right] \quad \text{for each } a \in \mathbb{N}.$$

Thus the function $\frac{d^a}{dt^a} \det A$ is a linear combination with positive integer coefficients of the functions

$$\det\left[\frac{d^{\nu_1}}{dt^{\nu_1}} A_1, \dots, \frac{d^{\nu_s}}{dt^{\nu_s}} A_s\right], \quad \text{where } \nu_l \in \mathbb{Z}_+ \text{ and } \nu_1 + \dots + \nu_s = a.$$

Since the rows of $A_{n,m,k}(z_1, \dots, z_m)$ starting from the $(n+1)$th row do not depend on z_1, the polynomial $(\det A_{n,m,k})_{z_1}^{(j)}$ is a linear combination of $\det A_{n,m,k}^{\nu}$, where $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}_+^n$, $\nu_1 + \dots + \nu_n = j$ and $A_{n,m,k}^{\nu}$ is obtained from $A_{n,m,k}(z_1, \dots, z_m)$ by differentiating with respect to z_1 the first row ν_1 times, the second row ν_2 times, etc. Thus (A.24) and (A.25) will be proved if we verify that

$$\det A_{n,m,k}^{\nu}(z_1, \dots, z_m)\big|_{z_1=0} = 0 \quad \text{for } \nu \in \mathbb{Z}_+^n,\ \nu_1 + \dots + \nu_n < n(n+k) \quad \text{and} \tag{A.26}$$

$$\det A_{n,m,k}^{\nu}(z_1, \dots, z_m)\big|_{z_1=z_2} = 0 \quad \text{for } \nu \in \mathbb{Z}_+^n,\ \nu_1 + \dots + \nu_n < n^2. \tag{A.27}$$

In order to prove (A.26) and (A.27), we need to differentiate the rows of $A_{n,m,k}(z_1, \dots, z_m)$. That is, we differentiate the vectors $A_{a,b}^{z}$ defined in (A.20). The following is easily verified:

$$\frac{d^\nu}{dz^\nu} A_{a,b}^{z} = A_{a-\nu,b}^{z} \quad \text{if } 0 \leqslant \nu \leqslant a \quad \text{and} \quad \frac{d^\nu}{dz^\nu} A_{a,b}^{z} = 0 \quad \text{if } \nu \geqslant a+b, \tag{A.28}$$

$$\frac{d^\nu}{dz^\nu} A_{a,b}^{z} = \left(0, \dots, 0, 1, z, \frac{z^2}{2!}, \dots, \frac{z^{a+b-1-\nu}}{(a+b-1-\nu)!}\right) \quad \text{if } a < \nu < a+b. \tag{A.29}$$

Using (A.28), we also see that

$$\frac{d^{\nu_1}}{dz^{\nu_1}} A_{a_1,b}^{z} = \frac{d^{\nu_2}}{dz^{\nu_2}} A_{a_2,b}^{z} \quad \text{if } a_1 - \nu_1 = a_2 - \nu_2. \tag{A.30}$$

From (A.28) and (A.29) it immediately follows that

$$\frac{d^\nu}{dz^\nu} A^z_{a,b}\bigg|_{z=0} = (0,\dots,0) \quad \text{if } 0 \leqslant \nu \leqslant a-1 \text{ or } \nu \geqslant a+b, \tag{A.31}$$

$$\frac{d^\nu}{dz^\nu} A^z_{a,b}\bigg|_{z=0} = (0,\dots,0,1,0,\dots,0) \quad \text{with 1 in } (\nu-a+1)\text{th place if } a \leqslant \nu < a+b. \tag{A.32}$$

Recall that for $1 \leqslant j \leqslant n$, the jth row of $A_{n,m,k}$ is $A^{z_1}_{k_j,nm}$, where $k_j = n+k-j+1$. By (A.31) and (A.32), the jth row of $A^\nu_{n,m,k}|_{z_1=0}$ for $1 \leqslant j \leqslant n$ is zero if $0 \leqslant \nu_j \leqslant n+k-j$ or $\nu > nm+n+k-j$ and is the $(\nu_j+j-n-k)$th basic vector otherwise. The only way for $\det A^\nu_{n,m,k}|_{z_1=0}$ to be non-zero is for its first n rows to be pairwise different basic vectors. Thus $\det A^\nu_{n,m,k}|_{z_1=0} = 0$ unless $m_j = \nu_j+j-n-k$ for $1 \leqslant j \leqslant n$ are pairwise different numbers from $\mathbb{N}_{nm}$. If m_j are pairwise different positive integers, then $m_1+\dots+m_n \geqslant 1+\dots+n$. Using the definition of m_j, we see that the latter inequality is equivalent to $\nu_1+\dots+\nu_n \geqslant n(n+k)$. Thus $\det A^\nu_{n,m,k}|_{z_1=0} = 0$ unless $\nu_1+\dots+\nu_n \geqslant n(n+k)$, which proves (A.26) and therefore (A.24).

In order to complete the proof it suffices to verify (A.27). Let $\nu \in \mathbb{Z}^n_+$. Since the jth row of $A_{n,m,k}$ is $A^{z_1}_{n+k-j+1,nm}$ for $1 \leqslant j \leqslant n$, the jth row of $A^\nu_{n,m,k}$ is $\frac{d^{\nu_j}}{dz_1^{\nu_j}} A^{z_1}_{n+k-j+1,nm}$. If $\nu_j \leqslant n+k-j+1$, formula (A.28) implies that the jth row of $A^\nu_{n,m,k}(z_1,\dots,z_m)$ is exactly $A^{z_1}_{n+k-j+1-\nu_j,nm}$. If $n+k-j+1-\nu_j \geqslant k+1$, the matrix $A^\nu_{n,m,k}|_{z_1=z_2}$ has exactly the same row as the jth row appear among the rows with numbers from $n+1$ to $2n$. In the latter case $\det A^\nu_{n,m,k}|_{z_1=z_2} = 0$ as required. Thus it remains to consider the case $n+k-j+1-\nu_j \leqslant k$ or equivalently $\nu_j \geqslant n-j+1$ for $1 \leqslant j \leqslant n$. If $1 \leqslant j < l \leqslant n$, then, according to (A.30), the jth and the lth rows of $A^\nu_{n,m,k}$ coincide if $j+\nu_j = l+\nu_l$. Hence $\det A^\nu_{n,m,k} = 0$ unless the numbers $1+\nu_1, 2+\nu_2, \dots, n+\nu_n$ are pairwise different. Thus it remains to consider the case when $1+\nu_1, 2+\nu_2, \dots, n+\nu_n$ are pairwise different and $\nu_j + j \geqslant n+1$ for each j. The sum of n pairwise different integers $\geqslant n+1$ is at least $(n+1)+(n+2)+\dots+2n$. Hence $\sum_{j=1}^n (\nu_j+j) \geqslant \sum_{j=1}^n n+j$. It follows that $\nu_1+\dots+\nu_n \geqslant n^2$. Thus $\det A^\nu_{n,m,k}|_{z_1=z_2} = 0$ unless $\nu_1+\dots+\nu_n \geqslant n^2$, which proves (A.27) and therefore (A.25). $\square$

Proof of Proposition A.5. By Lemma A.6, $\det A_{n,m,k}$ is either symmetric or antisymmetric. Hence Lemma A.10 implies that $z_j^{n(n+k)}$ and $(z_l - z_j)^{n^2}$ are divisors of $\det A_{n,m,k}$ whenever $l \neq j$. By Lemma A.7,

$$Q_{n,m,k} = Q_{n,m,k}(z_1,\dots,z_m) = \prod_{j=1}^m z_j^{n(n+k)} \prod_{1 \leqslant j<l \leqslant m} (z_l - z_j)^{n^2}$$

is a divisor of $\det A_{n,m,k}$. Now, $\deg Q_{n,m,k} = \frac{n^2m^2+n^2m+2nkm}{2}$. By Lemma A.6, $\det A_{n,m,k}$ is a homogeneous polynomial of degree $\deg Q_{n,m,k}$. Since $Q_{n,m,k}$ is a divisor of $\det A_{n,m,k}$, there is a constant $a_{n,m,k} \in \mathbb{C}$ such that $\det A_{n,m,k} = a_{n,m,k} Q_{n,m,k}$. It remains to compute $a_{n,m,k}$.

We start with the case $m=1$. In this case $A_{n,1,k}(z_1) = \{\frac{z_1^{n+k+l-j}}{(n+k+l-j)!}\}^n_{j,l=1}$. It immediately follows that $\det A_{n,1,k}(z_1) = z_1^{n(n+k)} \det C_{n+k,n}$, where the matrix $C_{n+k,n}$ is defined in (A.16). By Lemma A.4, we have $\det A_{n,1,k}(z_1) = \frac{1!2!\cdot\ldots\cdot(n-1)!\, z_1^{n(n+k)}}{(n+k)!(n+k+1)!\cdot\ldots\cdot(2n+k-1)!}$. Hence

$$a_{n,1,k} = \frac{1!2!\cdot\ldots\cdot(n-1)!}{(n+k)!(n+k+1)!\cdot\ldots\cdot(2n+k-1)!}. \tag{A.33}$$

Now we consider the case $m \geqslant 2$. It is straightforward to see that the z_m-degree of $Q_{n,m,k}$ is exactly $\mu = mn^2 + nk$. Moreover,

$$Q_{n,m,k}(z_1, \dots, z_m) = \sum_{j=0}^{\mu} R_j(z_1, \dots, z_{m-1}) z_m^j, \quad \text{where } R_\mu = Q_{n,m-1,k},$$

and each R_j is a polynomial in the variables $z_1, \dots, z_{m-1}$. According to the above display,

$$\det A_{n,m,k} = \sum_{j=0}^{\mu} a_{n,m,k} R_j(z_1, \dots, z_{m-1}) z_m^j \quad \text{with } R_\mu = Q_{n,m-1,k}. \tag{A.34}$$

Denoting the (j,l)th entry of $A_{n,m,k}$ by $\alpha_{j,l} = \alpha_{j,l}(z_1, \dots, z_m)$, we see that

$$\det A_{n,m,k} = \sum_{\sigma \in \mathcal{S}_{nm}} s(\sigma) A_\sigma, \quad \text{where } A_\sigma = \prod_{j=1}^{nm} \alpha_{j,\sigma(j)},$$

$\mathcal{S}_{nm}$ is the group of bijections of $\mathbb{N}_{nm}$ and $s(\sigma) = 1$ if the permutation σ is even, $s(\sigma) = -1$ if σ is odd. It is straightforward to see that the z_m-degree of the monomial A_σ is $\mu = mn^2 + nk$ if the set $\{n(m-1)+1, \dots, nm\}$ is invariant for σ and is less than μ otherwise. Thus according to (A.34) and the above display,

$$a_{n,m,k} z_m^\mu Q_{n,m-1,k} = \sum_{\sigma \in \mathcal{S}_{n(m-1)}} \sum_{\pi \in \mathcal{S}_n} s(\sigma)s(\pi) B_\sigma D_\pi = \left(\sum_{\sigma \in \mathcal{S}_{n(m-1)}} s(\sigma) B_\sigma \right) \cdot \left(\sum_{\pi \in \mathcal{S}_n} s(\pi) D_\pi \right),$$
$$\text{where } B_\sigma = \prod_{j=1}^{n(m-1)} \alpha_{j,\sigma(j)} \text{ and } D_\pi = \prod_{j=1}^{n} \alpha_{n(m-1)+j,n(m-1)+\pi(j)}. \tag{A.35}$$

The first factor in the right-hand side of (A.35) is $\det A_{n,m-1,k}(z_1, \dots, z_{m-1})$, while the second factor is $z_m^\mu \det C_{nm+k,n}$. Hence

$$a_{n,m,k} Q_{n,m-1,k}(z_1, \dots, z_{m-1}) = \det A_{n,m-1,k}(z_1, \dots, z_{m-1}) \det C_{nm+k,n}.$$

Using Lemma A.4 and the equality $\det A_{n,m-1,k} = a_{n,m-1,k} Q_{n,m-1,k}$, we can rewrite the above display:

$$a_{n,m,k} = a_{n,m-1,k} \frac{1!2! \cdot \ldots \cdot (n-1)!}{(nm+k)!(nm+k+1)! \cdot \ldots \cdot (n(m+1)+k-1)!}. \tag{A.36}$$

From (A.33) and (A.36) we immediately obtain the explicit formula (A.22) for $a_{n,m,k}$. $\square$

We conclude this section by deriving the following corollary of Proposition A.5.

Corollary A.11. *Let $m, n, k \in \mathbb{N}$, $z = (z_1, \dots, z_m) \in \mathbb{K}^{[m]}$ and $p_{j,l}(w)$ for $1 \leqslant j, l \leqslant nm$ be polynomials in one variable w such that the leading term $\hat{p}_{j,l}$ of $p_{j,l}$ is*

$$\hat{p}_{j,l}(w) = \frac{z_{s_j}^{k+n+l-a_j}}{(k+n+l-a_j)!} w^{k+n+l-a_j}, \tag{A.37}$$

where s_j and a_j are defined in (A.18). Then the determinant of the matrix $A_w = \{p_{j,l}(w)\}_{j,l=1}^{nm}$ is a polynomial in w of degree exactly $\frac{n^2m^2+n^2m+2nkm}{2}$.

Proof. Let $\widehat{A}_w$ be the matrix composed of leading terms of the polynomials $p_{j,l}$: $\widehat{A}_w = \{\hat{p}_{j,l}(w)\}_{j,l=1}^{nm}$. Using the standard formula for the determinant, we see that

$$\det A_w = \sum_{\sigma \in \mathcal{S}_{nm}} s(\sigma) P_\sigma(w) \quad \text{and} \quad \det \widehat{A}_w = \sum_{\sigma \in \mathcal{S}_{nm}} s(\sigma) \widehat{P}_\sigma(w),$$

$$\text{where } P_\sigma(w) = \prod_{j=1}^{nm} p_{j,\sigma(j)}(w) \text{ and } \widehat{P}_\sigma(w) = \prod_{j=1}^{nm} \hat{p}_{j,\sigma(j)}(w).$$

Just considering the degrees of the polynomials involved, we see that each $\widehat{P}_\sigma(w)$ is a monomial in w of degree exactly $\mu = \frac{n^2m^2+n^2m+2nkm}{2}$ and is the leading term of the polynomial $P_\sigma(w)$. Hence $\deg(P_\sigma - \widehat{P}_\sigma) < \mu$ for each $\sigma \in \mathcal{S}_{nm}$. Thus, according to the last display, $Q(w) = \det A_w - \det \widehat{A}_w$ is a polynomial of degree strictly less than μ. Next, according to (A.19) and (A.37), $\widehat{A}_w = A_{n,m,k}(wz_1, \dots, wz_m)$. By Proposition A.5, $\det \widehat{A}_w = bw^\mu$, where $b = b(z_1, \dots, z_m) \neq 0$ since $z \in \mathbb{K}^{[m]}$. Thus $\det A_w = bw^\mu + Q(w)$ with $b \neq 0$ and $\deg Q < \mu$. It follows that $\deg \det A_w = \mu$. □

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