ISSN 1755-5361

# University of Essex

Department of Economics

# Discussion Paper Series

No. 722 November 2012

# Universal interactive preferences

Jayant V. Ganguli and Aviad Heifetz

# Universal interactive preferences

Jayant V. Ganguli and Aviad Heifetz*

November 11, 2012

**Abstract**

We prove that a universal preference type space exists under much more general conditions than those postulated by Epstein and Wang (1996). To wit, it is enough that preferences can be encoded by a countable collection of continuous functionals, while the preferences themselves need not necessarily be continuous or regular, like, e.g., in the case of lexicographic preferences. The proof relies on a far-reaching generalization of a method developed by Heifetz and Samet (1998).

## 1 Introduction

Classical game theory has largely been developed under the assumption that players have Savage (1954) preferences, and can hence be modeled as maximizing subjective expected utilities. In single-person decision problems, in contrast, a voluminous literature axiomatizes and analyzes many additional classes of preference relations, which are obviously relevant in strategic interactions as well. How should games with incomplete information be modeled and handled with such more general preferences?

With Savage (1954) preference relations, games with incomplete information are modeled by probabilistic type spaces (Harsanyi, 1967-68). Each type of each player is associated with a probabilistic belief over the space of states of 'nature' – the players' von Neumann and

*We thank Simone Cerria-Vioglio, Yi Chun-Chen, Sander Heinsalu, Fabio Maccheroni, Miklos Pinter, and Amanda Friedenberg for helpful comments and discussions. The usual disclaimer applies. Ganguli: Dept of Economics, University of Essex, <jayantvivek@gmail.com>. Heifetz: Dept of Economics and Management, Open University of Israel, <aviadhe@openu.ac.il>.

Morgenstern (1944) utility indices from their action profiles, the external signals they get, etc. – and the other players' types. A strategy of each player is a measurable mapping from her types to her actions. Thus, from each player's perspective, her own actions coupled with the strategy profiles of the other players constitute *acts* from their types and nature into the space of everybody's action profiles; integration with respect to the probabilistic belief of each of the player's types of the payoffs associated by nature to each action profile defines a Savage (1954) preference relation over these acts. Moreover, by considering the type's marginal belief over nature, over nature and the other players' marginal beliefs on nature, etc., we see how each type's belief encapsulates an infinite hierarchy of *mutual* beliefs of all orders.

Type spaces can be readily extended to more general classes of preferences, by endowing each type directly with a preference relation over acts which are measurable functions from nature and the others' types into everybody's action profiles. The type's marginal preference over constant acts, over acts which are measurable with respect to the other players' marginal preferences over constant acts, etc., form a hierarchy of mutual preferences. In the particular case in which the preference relations satisfy Savage (1954) axioms and for each player states of nature associate real-valued von Neumann and Morgenstern (1944) payoffs to the players' action profiles, each of these preference relations can be represented by a probability measure over nature and the other players' types, as in Harsanyi's formulation.

Given a class of preference relations over acts, does the corresponding class of type spaces contain a *universal* space, i.e. one which 'embeds' all others in the sense of containing all preference hierarchies which appear in some type space? This is a pertinent question since, in applications, 'small' type spaces are tailored to the problem at hand, and it is important to know whether any generality is lost by this restriction or rather the same analysis could, in principle, be carried out in a *universal* space and deliver the same result. Furthermore, robustness results are most relevant if they obtain in a universal space, which allows for all possible perturbations, rather than within any particular, restricted type space.

For the case of preferences based on probabilistic beliefs, Mertens and Zamir (1985), followed by Brandenburger and Dekel (1993), Heifetz (1993), and Mertens, Sorin, and Zamir (1994) showed that under suitable topological or regularity assumptions, the set of *all* hierarchies of probabilistic beliefs constitutes a type space, which is hence universal.[1] In the absence of regularity, however, Heifetz and Samet (1999) showed that there exist hierarchies of beliefs which are not types in any type space. Nevertheless, Heifetz and Samet (1998) showed that even in the absence of regularity, the set of all profiles of belief hierarchies appearing in type spaces is itself a type space, which is universal.[2]

What happens with more general classes of preferences? Epstein and Wang (1996) showed that when preferences are regular in the appropriate sense, the set of all preference hierarchies forms a type space and Chen (2010) proved that it is universal. Alternatively, if one restricts attention to algebras of events then Di Tillio (2008) showed that a universal space exists under very mild conditions. However, what happens in the absence of regularity and when the pertinent class of events forms a $\sigma$-algebra?

In this paper, we show that a universal space exists under milder and more general conditions on preferences than those postulated by Epstein and Wang (1996). To wit, it suffices that preferences can be encoded monotonically by some countable collection of continuous real-valued functionals over acts. As long as such a representation exists, it is immaterial whether or not the preference relations themselves are continuous on acts. For example, a lexicographic preference represented by a finite sequence of $\ell$ continuous functionals is not itself continuous, since an act may be superior to all acts in some increasing sequence, but inferior to their limit. Nevertheless, there does exist a universal space in the category of type spaces where each type is associated with a lexicographic preference representable by a collection of $\ell$ continuous functionals over acts.

The method of proof is a far-reaching generalization of the one employed by Heifetz and

[1] Other developments under regularity assumptions include Battigalli and Siniscalchi (1999) for conditional beliefs in dynamic games, Mariotti, Meier, and Piccione (2005) for compact possibility models, Ahn (2007) for compact sets of probabilistic beliefs, Gul and Pesendorfer (2010) to study interdependent preferences that accommodate reciprocity, Bergemann, Morris, and Takahashi (2011) to study strategic distinguishability of types, Heifetz, Meier, and Schipper (2012) to study unawareness, and Heifetz and Kets (2012) to study bounded reasoning.

[2] Meier (2008), Pinter and Udvari (2011), Heinsalu (2012), Kets (2012), and Pinter (2012) provide recent developments of more general type spaces using the Heifetz and Samet (1998) approach, while Moss and Viglizzo (2004) formulate type spaces as coalgebras and show the existence of a final coalgebra which provides the universal type space.

Samet (1998).[3] It proceeds by 'collecting' all preference hierarchies that appear in types spaces in the category, and showing that the resulting collection is a universal space within that category. Each object in this collection is a hierarchy of preference relations extended over ever-richer sets of acts. A crucial point of the argument made in proposition 4 is that even if the ever-extended preference relation is not itself continuous, the fact that it is encoded by continuous functionals is sufficient to imply that the limit preference is uniquely defined. One must then furthermore show that this limit preference varies in a measurable way with the hierarchy. This follows from a functional monotone class theorem employed in lemma 1.

The paper is organized as follows. Sections 2 and 3 introduce the notation and definitions for our study of type spaces as well as the main result. Section 4 contains the statements of our results including the main measure-theoretic lemma and the main theorem while section 5 provides examples of the kinds of preferences accommodated by our construction and concludes. Proofs appear in the appendix.

## 2 Preliminaries

For any measurable space $Y$ with an associated $\sigma$-algebra $\Sigma_Y$, let $\mathcal{F}(Y)$ denote the set of all real-valued bounded acts, i.e. bounded measurable functions from $Y$ to the set of outcomes $\mathbb{R}$. The set $\mathbb{R}$ can be identified with the subset of constant acts in $\mathcal{F}(Y)$ with slight abuse of notation, i.e. for any $c \in \mathbb{R}$, $c \in \mathcal{F}(Y)$ is the constant act such that $c(y) = c$ for all $y \in Y$.

Let $L$ be a countable index set. We say that a binary relation $\succsim$ over $\mathcal{F}(Y)$ admits a **monotone continuous $L$ representation** if there exists a function

$$U : \mathcal{F}(Y) \to \mathbb{R}^L$$

and a preorder – a transitive and reflexive binary relation – $\trianglerighteq$ on $\mathbb{R}^L$ such that

$$g \succsim f \quad \text{iff} \quad U(g) \trianglerighteq U(f), \tag{1}$$

$$\text{if } g_n(y) \to g(y) \ \ \forall y \in Y \text{ then } U(g_n) \to U(g), \tag{2}$$

[3] It is an interesting question for future research whether the results of Moss and Viglizzo (2006) could be generalized in order to study the existence of the universal type space for a general class of preferences.

where convergence is coordinate-wise, i.e. $U_\ell(g_n) \to U_\ell(g)$ with $U_\ell$ denoting the $\ell$-th coordinate of $U$, $\ell \in L$, and

$$\text{if } f \geq g \text{ then } U_\ell(f) \geq U_\ell(g), \ell \in L. \tag{3}$$

For a given index set $L$ and an order $\trianglerighteq$ on $\mathbb{R}^L$ denote by $\mathcal{P}(Y)$ the set of preference relations on $\mathcal{F}(Y)$ that admit a monotone continuous $L$ representation. For each $\succsim\in \mathcal{P}(Y)$, we fix a standard representation $U$ and abusing notation, denote by $\mathcal{P}(Y)$ the set of these representations. We refer to the representation $U$ as the $L$-utility.

Denote by $\Sigma_{\mathcal{P}(Y)}$the $\sigma$-algebra on $\mathcal{P}(Y)$ generated by the sets of the form

$$\beta_\ell^{r_\ell}(f) = \{U \in \mathcal{P}(Y) | U_\ell(f) \geq r_\ell\} \text{ and } \beta_\ell^f(r_\ell) = \{U \in \mathcal{P}(Y) | r_\ell \geq U_\ell(f)\}$$

for $r_\ell \in \mathbb{R}, \ell\in \mathbb{L}$ and acts $f \in \mathcal{F}(Y)$. Then for $f \in \mathcal{F}(Y)$ and $r = (r_\ell)_{\ell\in L} \in \mathbb{R}^L$

$$\beta^r(f) = \{U \in \mathcal{P}(Y) | U(f) \trianglerighteq r\} = \cap_{\ell\in L} \{U \in \mathcal{P}(Y) | U_\ell(f) \geq r_\ell\} = \cap_{\ell\in L}\beta_\ell^{r_\ell}(f)$$

and

$$\beta^f(r) = \{U \in \mathcal{P}(Y) | r \trianglerighteq U(f)\} = \cap_{\ell\in L} \{U \in \mathcal{P}(Y) | r_\ell \geq U_\ell(f)\} = \cap_{\ell\in L}\beta_\ell^f(r_\ell)$$

which are hence measurable events in $\Sigma_{\mathcal{P}(Y)}$ since $L$ is countable.

**Remark 1.** *This is the only place where we use the assumption that the index set $L$ is countable. We could more generally allow the index set $L$ to be of arbitrary cardinality and assume additionally that $\beta^r(f), \beta^f(r) \in \Sigma_{\mathcal{P}(Y)}$ for every $f \in \mathcal{F}(Y)$ and $r = (r_\ell)_{\ell\in L} \in \mathbb{R}^L$.*

For measurable spaces $Y$ and $Z$ and a measurable function $\phi : Y \to Z$, define the preference mapping $\hat{\phi} : \mathcal{P}(Y) \to \mathcal{P}(Z)$ where for any $f \in \mathcal{F}(Z)$,

$$\hat{\phi}(U_Y)(f) = U_Y(f \circ \phi). \tag{4}$$

The set of players is $I$ and $I_0 = I \cup \{0\}$ denotes the set of players and "nature" (player 0). As usual, for any collection $\{Y_i\}_{i\in I_0}$, $Y_{-i} = \times_{i'\in I_0\setminus\{i\}}Y_{i'}$. We consider the product, finite or infinite, of measurable spaces as a measurable space with the product $\sigma$-algebra.

## 3 Type spaces

Let $S$ be a measurable space whose elements are the states of nature.

**Definition 1** (Type space)**.** *A type space on $S$ is a tuple $<(T_i)_{I \in I_0}, (m_i)_{i \in I}> = < T, m >$, where*

1. *$T_0 = S$ and $T_i$, for $i \in I$ is a measurable space and*

2. *for each $i \in I$, $m_i : T_i \to \mathcal{P}(T_{-i})$ is measurable.*

The elements of $T$ are called states of the world and an element of $T_i$ is called an $i$-type. For any $f \in \mathcal{F}(T_{-i})$, $r \in \mathbb{R}^L$, the belief operators $B_i^r(f)$ and $B_i^f(r)$ are defined by

$$B_i^r(f) = \{t \in T \mid m_i(t_i)(f) \trianglerighteq r\} \text{ and } B_i^f(r) = \{t \in T \mid r \trianglerighteq m_i(t_i)(f)\} \tag{5}$$

Then, recalling that $\beta_i^r(f) = \{U \in \mathcal{P}(T_{-i}) \,|U(f) \trianglerighteq r\}$ and $\beta_i^f(r) = \{U \in \mathcal{P}(T_{-i}) \,|r \trianglerighteq U(f)\}$ we have that $B_i^r(f) = m_i^{-1}(\beta_i^r(f)) \times T_{-i}$ and $B_i^f(r) = m_i^{-1}\left(\beta_i^f(r)\right) \times T_{-i}$ are measurable events.

Let $< T, m >$ and $< T', m' >$ be type spaces on $S$. Type morphisms defined below are mappings that preserve the preference structures, as given by $m$ and $m'$.

**Definition 2** (Type morphisms)**.** *Let $\phi_i : T_i \to T'_i$ , $i \in I_0$ be measurable functions. Then, $\phi = (\phi_i)_{i \in I_0} : T \to T'$ is a type morphism if*

1. *$\phi_0$ is the identity on $S$ and*

2. *for each $i \in I$ and $t_i \in T_i$, $m'_i(\phi_i(t_i)) = \hat{\phi}_i(m_i(t_i))$, i.e. for every $f \in \mathcal{F}(T'_{-i})$*

$$m'_i(\phi_i(t_i))(\ f) \ = m_i(t_i)(f \circ \phi)\,. \tag{6}$$

Then, it can be verified that a type morphism $\phi$ preserves belief operators, i.e. for each $i \in I$, $f \in \mathcal{F}(T'_{-i})$

$$\phi_i^{-1}(B_i^r(f \circ \phi_i)) = B_i^r(f) \text{ and } \phi_i^{-1}\left(B_i^{f \circ \phi_i}(r)\right) = B_i^f(r). \tag{7}$$

# 4 The universal type space

**Definition 3** (Universal type space)**.** *A type space* $< T^*, m^* >$ *on* $S$ *is universal if for every type space* $< T, m >$ *on* $S$*, there exists a unique type morphism from* $< T, m >$ *to* $< T^*, m^* >$*.*

Our main result is the following.

**Main Theorem.** *For any measurable space* $S$ *there exists a universal type space on* $S$*.*

Before proceeding to the proof, we note the following result.

**Proposition 1.** *There is at most one universal type space on* $S$ *up to a type isomorphism.*

## 4.1 Main measure-theoretic lemma

The main measure-theoretic lemma needed for the construction of the universal type space is the following.

**Lemma 1.** *Let* $(Y, \Sigma_Y)$ *be a measurable space. Let* $\mathcal{G} \subseteq \mathcal{F}(Y)$ *be such that the* $\sigma$*-algebra* $\Sigma_Y$ *is generated by*

$$\boldsymbol{A}_{\mathcal{G}} = \left\{ f^{-1}(E) : f \in \mathcal{G}, E \subseteq \mathbb{R} \text{ } \textit{Borel measurable} \right\}$$

*and such that* $G$ *satisfies the following properties.*

*(i) The constant function* $1 \in \mathcal{G}$

*(ii) For any* $f, f' \in \mathcal{G}$ *and* $\alpha, \alpha' \in \mathbb{R}$*,* $\alpha f + \alpha' f' \in \mathcal{G}$*.*

*(iii) For any* $f, f' \in \mathcal{G}$*,* $\min\{f, f'\} \in \mathcal{G}$*.*

*Let* $\Sigma_{\mathcal{G}}$ *be the* $\sigma$*-algebra on* $\mathcal{P}(Y)$ *generated by sets of the form*

$$\left\{ \beta_\ell^{r_\ell}(f) \,|\, f \in \mathcal{G}, r_\ell \in \mathbb{R}, \ell \in L \right\}$$

*and*

$$\left\{ \beta_\ell^{f}(r_\ell) \,|\, f \in \mathcal{G}, r_\ell \in \mathbb{R}, \ell \in L \right\}.$$

*Then* $\Sigma_{\mathcal{P}(Y)} = \Sigma_{\mathcal{G}}$*.*

### 4.2 Hierarchies of preferences

We now define spaces of hierarchies of preferences $H_i^k$ for each $k \geq 0$ and $i \in I_0$. For every $k \geq 0$, $H_0^k = S$ and for every $i \in I$, $H_i^0$ is a singleton. As usual $H^k = \times_{i \in I_0} H_i^k$. We define inductively

$$H_i^{k+1} = H_i^k \times \mathcal{P}(H_{-i}^k) = H_i^0 \times \left( \times_{k'=0}^{k} \mathcal{P}(H_{-i}^{k'}) \right). \tag{8}$$

The space of $i$-hierarchies for player $i \in I$ is

$$H_i = H_i^0 \times \left( \times_{k'=0}^{\infty} \mathcal{P}(H_{-i}^{k'}) \right) \tag{9}$$

and the projection from $H_i$ to $H_i^k$ is denoted $\pi_i^k$.

Given a type space $T$, we can define an $i$-description map $h_i : T_i \to H_i$ for each $i \in I_0$ as follows. For all $k \geq 0$, let $h_0^k$ be the identity on $S$. For $i \in I$, $h_i^0 : T_i \to H_i^0$ is uniquely defined since $H_i^0$ is a singleton. Inductively, define $h_i^{k+1} : T_i \to H_i^{k+1}$ for $k \geq 0$ by

$$h_i^{k+1}(t_i) = \left( h_i^k(t_i), \hat{h}_{-i}^k(m_i(t_i)) \right) = \left( h_i^0(t_i), \hat{h}_{-i}^0(m_i(t_i)), \ldots, \hat{h}_{-i}^k(m_i(t_i)) \right) \tag{10}$$

where $\hat{h}_{-i}^k : \mathcal{P}(T_{-i}) \to \mathcal{P}(H_{-i}^k)$ is the mapping between the sets of preferences as defined in (4) in section 2. Now define $h_i : T_i \to H_i$, $i \in I$ as the unique function that satisfies for all $k \geq 0$, $h_i^k = \pi_i^k(h_i)$, i.e.

$$h_i(t_i) = \left( h_i^0(t_i), \hat{h}_{-i}^0(m_i(t_i)), \ldots, \hat{h}_{-i}^k(m_i(t_i)), \ldots \right) \tag{11}$$

and define $h_0$ to be the identity on $S$. The first result is as follows.

**Proposition 2.** *Type morphisms preserve i-descriptions.*

We can now define the universal type space by setting $T_0^* = S$ and $T_i^*$ to be the set of all $i$-descriptions in $H_i$, i.e., all hierarchies $t_i^* \in H_i$ for which $t_i^* = h_i(t_i)$ for some $t_i \in T_i$ in some type space $< T, m >$ over $S$. The $\sigma$-algebra of $T_i^*$ is the one inherited from $H_i$. We define $m_i^* : T_i^* \to \mathcal{P}(T_{-i}^*)$ by

$$m_i^*(t_i) = \hat{h}_{-i}(m_i(t_i)). \tag{12}$$

The next result establishes that $< T^*, m^* >$ thus defined is a type space.

**Proposition 3.** $< (T_i^*)_{i \in I_0}, (m_i^*)_{i \in I} >$ *is a type space on S.*

**Proposition 4.** *For every type space $< T, m >$, the description map $h : T \to T^*$ is a type morphism.*

**Lemma 2.** *The hierarchy description maps $h_i : T_i^* \to T_i^*$ are the identity maps.*

We now re-state and prove the main result.

**Theorem 1.** $< (T_i^*)_{i \in I_0}, (m_i^*)_{i \in I} >$ *is the universal type space.*

## 5 Examples

Some examples of preferences where $\succsim$ admits a continuous $L$-representation and are accommodated in our construction of the universal type space are as follows.

1. If $|L| = 1$ and $\trianglerighteq$ is the usual order $\geq$ on $\mathbb{R}$, $\succsim$ admits a continuous representation such as those for preferences under risk and ambiguity including Choquet expected utility, multiple-prior maxmin expected utility, invariant biseparable preferences, 'smooth' ambiguity preferences, variational preferences, uncertainty-averse preferences, vector expected utility preferences as discussed in Gilboa and Marinacci (2011) and rank-dependent and prospect theory preferences as discussed in Wakker (2010) and for social preferences as discussed in Marinacci, Maccheroni, and Rustichini (2011).

2. If $1 < |L| < \infty$, respectively $L = \mathbb{N}$, and $\trianglerighteq$ is the lexicographic order on $\mathbb{R}^{|L|}$, respectively $\mathbb{R}^{\mathbb{N}}$, then $\succsim$ is lexicographic, i.e. $g \succ f$ iff for some $\bar{\ell} \in L$ it is the case $U_\ell(g) = U_\ell(f)\ \forall \ell < \bar{\ell}$ and $U_{\bar{\ell}}(g) > U_{\bar{\ell}}(f)$. In the first case $\succsim$ is of order (at most) $|L|$ while in the second case there is no a priori bound on $\bar{\ell}$. Moreover, when $U_\ell$ is linear for every $\ell \in L$, by the Riesz representation theorem $\succsim$ has a unique representation by a Lexicographic Probability System (LPS) whose order may be finite or infinite (Blume, Brandenburger, and Dekel, 1991).

3. If $1 < |L|$ and $\trianglerighteq$ is the partial order on $\mathbb{R}^L$ defined by $(r_\ell)_{\ell \in L} \trianglerighteq (r'_\ell)_{\ell \in L}$ iff $r_\ell \geq r'_\ell$ $\forall \ell \in L$, $\succsim$ may be incomplete and admit continuous versions of representations in Galaabaatar and Karni (2012), Ok, Ortoleva, and Riella (2012), and Ok (2012) among others.

4. $U$ can represent some instances of preferences over menus that feature behavior such as self-control and temptation, or non-Bayesian updating, or self-deception as described in Lipman and Pesendorfer (2011). For example, when $|L| = 2$, $U = (U_1, U_2)$ represents self-control preferences over menus similarly to the representation axiomatized in Gul and Pesendorfer (2001) (Theorem 3) with $U_1$ representing the commitment utility and $U_2$ representing temptation utility. The utility $V(\cdot)$ of compact non-empty menu $F \subseteq \mathcal{F}(Y)$ is either

$$V(F) = \max_{f \in F} \{U_1(f) + U_2(f)\} - \max_{f \in F} U_2(f) \text{ (self control)}$$

or

$$V(F) = \max_{f \in F} U_1(f) \text{ subject to } U_2(f) \geq U_2(f') \text{ for all } f' \in F \text{ (no self control).}$$

## References


Ahn, D. (2007): "Hierarchies of ambiguous beliefs," *Journal of Economic Theory*, 136, 286–301.

Battigalli, P., and M. Siniscalchi (1999): "Hierarchies of conditional beliefs and interactive epistomology for dynamic games," *Journal of Economic Theory*, 88, 188–230.

Bergemann, D., S. Morris, and S. Takahashi (2011): "Interdependent Preferences and Strategic Distinguishability," mimeo, Princeton University Economic Theory Center Research Paper No. 008-2011.

Blume, L., A. Brandenburger, and E. Dekel (1991): "Lexicographic Probabilities and Choice Under Uncertainty," *Econometrica*, 59(1), pp. 61–79.

Brandenburger, A., and E. Dekel (1993): "Hierarchies of belief and common knowledge," *Journal of Economic Theory*, 59, 189–198.

Chen, Y.-C. (2010): "Universality of the Epstein-Wang type structure," *Games and Economic Behavior*, 68(1), 389 – 402.

Dellacherie, C., and P.-A. Meyer (1978): *Probabilities and Potential A*, North Holland Mathematics Studies 29. North Holland, New York.

Di Tillio, A. (2008): "Subjective expected utility in games," *Theoretical Economics*, 3, 287–323.

Epstein, L., and T. Wang (1996): "'Beliefs about beliefs' without probabilities," *Econometrica*, 64, 1343–1373.

Galaabaatar, T., and E. Karni (2012): "Subjective expected utility with incomplete preferences," mimeo, Johns Hopkins University, forthcoming in Econometrica.

Gilboa, I., and M. Marinacci (2011): "Ambiguity and the Bayesian Paradigm," mimeo, Bocconi University, forthcoming in 'Advances in Economics and Econometrics: Theory and Applications,Tenth World Congress of the Econometric Society', Cambridge University Press.

Gul, F., and W. Pesendorfer (2001): "Temptation and self-control," *Econometrica*, 69, 1403–1435.

——— (2010): "Interdependent Preference Models as a Theory of Intentions," mimeo, Princeton University.

Harsanyi, J. C. (1967-68): "Games with incomplete information played by 'Bayesian' players: Parts I-III," *Management Science*, 14, 159–182, 320–334, and 486–502.

Heifetz, A. (1993): "The Bayesian formulation of incomplete information–the non-compact case," *International Journal of Game Theory*, 21, 329–338.

Heifetz, A., and W. Kets (2012): "All types naive and canny," mimeo, Northwestern University.

Heifetz, A., M. Meier, and B. Schipper (2012): "Unawareness, beliefs, and speculative trade," mimeo, UC Davis.

Heifetz, A., and D. Samet (1998): "Topology-free typology of beliefs," *Journal of Economic Theory*, 82, 324–341.

——— (1999): "Coherent beliefs are not always types," *Journal of Mathematical Economics*, 32, 475–488.

Heinsalu, S. (2012): "Universal type space with unawareness," mimeo, Yale University.

Kets, W. (2012): "Bounded reasoning and higher-order uncertainty," mimeo, Northwestern University.

Lipman, B., and W. Pesendorfer (2011): "Temptation," mimeo, Boston University, forthcoming in 'Advances in Economics and Econometrics: Theory and Applications,Tenth World Congress of the Econometric Society', Cambridge University Press.

Marinacci, M., F. Maccheroni, and A. Rustichini (2011): "Social Decision Theory: Choosing Within and Between Groups," forthcoming in the Review of Economic Studies, doi: 10.1093/restud/rds006.

Mariotti, T., M. Meier, and M. Piccione (2005): "Hierarchies of beliefs for compact possibility models," *Journal of Mathematical Economics*, 21, 303–324.

Meier, M. (2008): "Universal knowledge-belief structures," *Games and Economic Behavior*, 62, 53–66.

Mertens, J.-F., S. Sorin, and S. Zamir (1994): "Repeated Games: Part A: Background Material," CORE Discussion Paper 9420, Universite Catholique de Louvain.

Mertens, J.-F., and S. Zamir (1985): "Formulation of Bayesian analysis for games with incomplete information," *International Journal of Game Theory*, 14, 1–29.

Moss, L. S., and I. D. Viglizzo (2004): "Harsanyi Type Spaces and Final Coalgebras Constructed from Satisfied Theories," *Electronic Notes in Theoretical Computer Science*, 106, 279–295.

——— (2006): "Final Coalgebras for Functors on Measurable Spaces," *Information and Computation*, 204, 610–636.

Ok, E. (2012): "Functional (Utility) Representation of Preorders," Chapter 9 in Elements of Order Theory, mimeo, New York University.

Ok, E., P. Ortoleva, and G. Riella (2012): "Incomplete Preferences under Uncertainty: Indecisiveness in Beliefs vs. Tastes," *Econometrica*, 80, 1791–1808.

Pinter, M. (2012): "Type spaces with non-additive beliefs," mimeo, Corvinus University of Budapest.

Pinter, M., and Z. Udvari (2011): "Generalized type spaces," mimeo, MPRA Paper No. 34107.

Savage, L. J. (1954): *The foundations of statistics.* John Wiley and Sons.

von Neumann, J., and O. Morgenstern (1944): *Theory of Games and Economic Behavior.* Princeton University Press.

Wakker, P. (2010): *Prospect theory for risk and ambiguity.* Cambridge University Press.

# 6 Appendix: proofs

**Proof of proposition 1.** Let $< T, m >$ and $< T', m' >$ be universal type spaces on $S$. Then there are type morphisms $\phi : T \to T'$ and $\phi' : T' \to T$. Thus $\phi'(\phi) : T \to T$ is a type morphism. However, the identity mapping from $T$ to $T$ is also a type morphism and so by the uniqueness of type morphisms to universal type spaces, it follows that $\phi'(\phi)$ is the identity on $T$ which proves that $\phi$ is a type isomorphism. ■

**Proof of lemma 1.** Clearly $\Sigma_{\mathcal{P}(Y)} \supseteq \Sigma_{\mathcal{G}}$, since $\Sigma_{\mathcal{P}(Y)}$ is generated by the sets of the form

$$\{\beta_\ell^{r_\ell}(f) \,|\, f \in \mathcal{F}(Y), r_\ell \in \mathbb{R}, \ell \in L\}$$

and

$$\left\{\beta_\ell^{f}(r_\ell) \,|\, f \in \mathcal{F}(Y), r_\ell \in \mathbb{R}, \ell \in L\right\}.$$

We establish that $\Sigma_{\mathcal{P}(Y)} \subseteq \Sigma_{\mathcal{G}}$. Let $\mathcal{F}' \subseteq \mathcal{F}(Y)$ be the collection of acts $f$ such that $\beta_\ell^{r_\ell}(f), \beta_\ell^{f}(r_\ell) \in \Sigma_{\mathcal{G}}$ for all $r_\ell \in \mathbb{R}$ and $\ell \in L$. We show that $\mathcal{F}' \supseteq \mathcal{F}(Y)$ which establishes the result, since $\Sigma_{\mathcal{G}}$ then contains all the generators of $\Sigma_{\mathcal{P}(Y)}$.

We prove that $\mathcal{F}' \supseteq \mathcal{F}(Y)$ by employing the functional monotone class theorem (Dellacherie and Meyer (1978) theorem 22.3, p. 15-1).[4] Given assumptions (i)-(iii) on G, and the fact that $\mathcal{F}(Y)$ is the set of $\Sigma_Y$-measurable acts while $\Sigma_Y$ is generated by $\boldsymbol{A}_{\mathcal{G}}$, it remains to show that $\mathcal{F}'$ is closed under bounded monotone convergence. Indeed, let $\{f_n\}_{n=1}^{\infty}$ be a bounded monotone sequence of functions in $\mathcal{F}'$ converging to $f \in \mathcal{F}(Y)$.[5] Then for all $r_\ell \in \mathbb{R}, \ell \in L$, by the continuity(2) and monotonicity (3) of $L$-utility,

$$\beta_\ell^{r_\ell}(f) = \cap_{k=1}^{\infty} \cup_{m=1}^{\infty} \cap_{n \geq m} \beta_\ell^{r_\ell - \frac{1}{k}}(f_n)$$

and

$$\beta_\ell^{f}(r_\ell) = \cap_{n \geq 1} \beta_\ell^{f_n}(r_\ell).$$

Since the assumption that $f_n \in \mathcal{F}'$ for every $n \geq 1$ means, in particular, that $\beta_\ell^{r_\ell - \frac{1}{k}}(f_n), \beta_\ell^{f_n}(r_\ell) \in \Sigma_{\mathcal{G}}$ for every $k \geq 1$, it thus follows that also $\beta_\ell^{r_\ell}(f), \beta_\ell^{f}(r_\ell) \in \Sigma_{\mathcal{G}}$. As this holds for all $r_\ell \in \mathbb{R}, \ell \in L$, we conclude that $f \in \mathcal{F}'$, as required.■

[4] The corresponding notation there has $\mathcal{H} = \mathcal{F}'$ and $\mathcal{C} = \mathcal{G}$.

[5] $\{f_n\}_{n=1}^{\infty}$ is a sequence for which (i) there exists $M < \infty$ such that $0 \leq f_n(y) \leq M$ for all $y \in Y$ and $n = 1, 2, \ldots$ and (ii) $f_n(y)$ is increasing in $n$ for all $y \in Y$.

**Proof of proposition 2.** Let $\phi : T \to T'$ be a type morphism. We have to show that $h_i'(\phi_i(t_i)) = h_i(t_i)$ for all $t_i \in T_i$ and $i \in I_0$. For $i = 0$, this follows immediately since $\phi_0, h_0^k, h_0, h_0'^k, h_0'$ are all the identity map on $S$. For $i \in I$, $h_i^0(t_i) = h_k'^0(\phi_i(t_i))$ since $H_i^0$ is a singleton.

Suppose, inductively, that we have already proved that $h_i^k(t_i) = h_i'^k(\phi_i(t_i))$ for every $t_i \in T_i$ and every $i \in I_0$. In the following sequence of equalities, the second equality stems from the fact that type morphisms preserve preferences (6) and the induction hypothesis is used in the third equality. For any $f \in \mathcal{F}(H_{-i}^k)$

$$\begin{aligned}\hat{h}_{-i}'^k(m_i'(\phi_i(t_i)))(f) &= m_i'(\phi_i(t_i))\left(f \circ h_{-i}'^k\right) \\ &= m_i(t_i)\left(f \circ h_{-i}'^k \circ \phi_{-i}\right) = m_i(t_i)\left(f \circ h_{-i}^k\right) = \hat{h}_{-i}^k(m_i(t_i))(f)\end{aligned}$$

It then follows that

$$\begin{aligned}h_i'^{k+1}(\phi_i(t_i)) = & \left(h_i'^k(\phi_i(t_i)), \hat{h}_{-i}'^k\left(m_i'(\phi_i(t_i))\right)\right) \\ = & \left(h_i^k(t_i), \hat{h}_{-i}^k\left(m_i(t_i)\right)\right) \\ = & h_i^{k+1}(t_i)\end{aligned}$$

as needed. ■

**Proof of proposition 3.** To show that $< T^*, m^* >$ is a type space on $S$, we have to show that $m_i^*$ is a measurable mapping for each $i \in I$. For $t_i^*$, let $t_i$ be the $i$-type used to define $m_i^*(t_i^*) \in \mathcal{P}(T_{-i}^*) \subseteq \mathcal{P}(H_{-i})$, i.e. $m_i^*(t_i^*) = \hat{h}_{-i}(m_i(t_i))$.

Consider the preference relation on $\mathcal{F}(H_{-i}^k)$ induced by $m^*(t_i^*)$, i.e. $\hat{\pi}_{-i}^k(m_i^*(t_i^*))$ where $\hat{\pi}_{-i}^k : \mathcal{P}(H_{-i}) \to \mathcal{P}(H_{-i}^k)$ is the preference mapping defined in (4) corresponding to the projection $\pi_i^k : H_i \to H_i^k$. Then,

$$\begin{aligned}\hat{\pi}_{-i}^k\left(m_i^*(t_i^*)\right) = & \hat{\pi}_{-i}^k\left(\hat{h}_{-i}\left(m_i(t_i)\right)\right) \\ = & \hat{h}_{-i}^k\left(m_i(t_i)\right) \text{ since } h_i^k(t_i) = \pi_i^k\left(h_i(t_i)\right) \\ = & (k+1)^{th} \text{ coordinate of } h_i(t_i) \\ = & (k+1)^{th} \text{ coordinate of the hierarchy } t_i^* \\ \equiv & \left(t_i^*\right)^{k+1}\end{aligned} \tag{13}$$

Let $\mathcal{G}_k \subseteq \mathcal{F}(H_{-i})$ be the set of acts that are measurable with respect to $H_{-i}^k$, i.e. $\mathcal{G}_k$ is the set of acts $f_k$ such that for every Borel measurable $E \subseteq \mathbb{R}$ there exists some measurable $E_k \subseteq H_{-i}^k$ for which $f_k^{-1}(E) = \left(\pi_{-i}^k\right)^{-1}(E_k)$. Let

$$\mathcal{G} = \cup_{k=0}^{\infty} \mathcal{G}_k$$

Then $\boldsymbol{A}_{\mathcal{G}} = \left\{f^{-1}(E) : f \in \mathcal{G}, E \subseteq \mathbb{R} \text{ Borel measurable}\right\}$ is the collection of all cylinders with finite-dimensional bases, which generates the $\sigma$-algebra on $H_{-i}$. Moreover, (i) the constant act 1 is in $\mathcal{G}_0$

and hence in $\mathcal{G}$; furthermore, if $f, f' \in \mathcal{G}$ then $f \in \mathcal{G}_k$ and $f' \in \mathcal{G}_{k'}$ for some $k, k'$, and if, without loss of generality $k \geq k'$ then $f' \in \mathcal{G}_k$. It thus follows that (ii) $\alpha f + \alpha' f' \in \mathcal{G}_k \subset \mathcal{G}$ for every $\alpha, \alpha' \in \mathbb{R}$, and (iii) $\min\{f, f'\} \in \mathcal{G}_k \subset \mathcal{G}$.

Lemma 1 then implies that $\Sigma_{\mathcal{P}(H_{-i})} = \Sigma_{\mathcal{G}}$, i.e. that $\Sigma_{\mathcal{P}(H_{-i})}$ is generated by the sets of the form

$$\{\beta_\ell^{r_\ell}(f) \,|\, f \in \mathcal{G}, r_\ell \in \mathbb{R}, \ell \in L\} = \cup_{k=0}^{\infty} \{\beta_\ell^{r_\ell}(f_k) \,|\, f_k \in \mathcal{G}_k, r_\ell \in \mathbb{R}, \ell \in L\}$$

and

$$\left\{\beta_\ell^{f}(r_\ell) \,|\, f \in \mathcal{G}, r_\ell \in \mathbb{R}, \ell \in L\right\} = \cup_{k=0}^{\infty} \left\{\beta_\ell^{f_k}(r_\ell) \,|\, f_k \in \mathcal{G}_k, r_\ell \in \mathbb{R}, \ell \in L\right\}.$$

But if $f_k \in \mathcal{G}_k, r_\ell \in \mathbb{R}, \ell \in L$, then denoting by $f^k \in F\left(H_{-i}^k\right)$ the act on $H_{-i}^k$ for which $f_k = f^k \circ \pi_{-i}^k$, from (13) we get that

$$(m_i^*)^{-1}(\beta_\ell^{r_\ell}(f_k)) = \{t_i^* \,|(m_i^*(t_i^*))_\ell(f_k) \geq r_\ell\} = \left\{t_i^* \,\middle|\left((t_i^*)^{k+1}\right)_\ell (f^k) \geq r_\ell\right\} \quad (14)$$

$$(m_i^*)^{-1}\left(\beta_\ell^{f_k}(r_\ell)\right) = \{t_i^* \,|r_\ell \geq (m_i^*(t_i^*))_\ell(f_k)\} = \left\{t_i^* \,\middle|\, r_\ell \geq \left((t_i^*)^{k+1}\right)_\ell (f^k)\right\} \quad (15)$$

which are hence measurable subsets in $H_i$. This proves that $m_i^*$ is a measurable mapping, as required. ■

**Proof of proposition 4.** The functions $h_i, i \in I$, are measurable and $h_0$ is the identity. Since the range of $h_i$ is $T_i^*$, it is also measurable as a function to $T_i^*$. Also, from (13), it follows that for acts $f_k$ in $\mathcal{F}(H_{-i})$ that are measurable with respect to the $\sigma$-algebra on $H_{-i}^k$, $(m_i^*(t_i^*))_\ell(f_k)$ does not depend on the specific type $t_i$ chosen to define $m_i^*(t_i^*)$, since there exists $f^k \in \mathcal{F}\left(H_{-i}^k\right)$ such that $f_k = f^k \circ \pi_{-i}^k$ and so

$$(m_i^*(t_i^*))_\ell(f_k) = \left((t_i^*)^{k+1}\right)_\ell (f^k) = \left(\hat{h}_{-i}(m_i(t_i))\right)_\ell (f_k) = (m_i(t_i))_\ell(f_k \circ h_{-i}) \quad (16)$$

for any $t_i$ such that $h_i(t_i) = t_i^*$ and every $\ell \in L$.

Now, every measurable act $f \in F(H_{-i})$ is a pointwise limit of a sequence of functions $f_k \in F(H_{-i})$ which are, respectively, measurable with respect to the $\sigma$-algebra on $H_{-i}^k$. The continuity of $(m_i^*(t_i^*))_\ell$ and $(m_i(t_i))_\ell$ in (2) then implies that

$$(m_i^*(t_i^*))_\ell(f) = \lim_{k\to\infty}(m_i^*(t_i^*))_\ell(f_k) = \lim_{k\to\infty}(m_i(t_i))_\ell(f_k \circ h_{-i}) = (m_i(t_i))_\ell(f \circ h_{-i}) \quad (17)$$

for every $\ell \in L$ and $i \in I$, which proves that $h$ is a type morphism. ■

**Proof of lemma 2.** It suffices to show that for each $k$ and $i \in I$, the function $h_i^k$ on $T^*$ is the projection on $H_i^k$. We show this by induction on $k$. It is clearly true for $k = 0$. Suppose that $h^k = \pi^k$. By definition, $(h_i(t^*))^{k+1} = \hat{h}_{-i}^k(m_i^*(t_i^*))$. Using the induction hypothesis we get $\hat{h}_{-i}^k(m_i^*(t_i^*)) = \hat{\pi}_{-i}^k(m_i^*(t_i^*))$, implying that $(h_i(t^*))^{k+1} = \hat{\pi}_{-i}^k(m_i^*(t_i^*)) = (t_i^*)^{k+1}$.■

**Proof of theorem 1 [Main Theorem].** For any type space $< T, m >$, the description map $h : T \to T^*$ is a type morphism by proposition 4. We need to show that it is unique. Suppose $\phi : T \to T^*$ is a type morphism. Then for each $i \in I$ and $t_i \in T_i$, $h_i(t_i) = h_i(\phi_i(t_i))$ by proposition 2. However, from lemma 2, we get $h_i(\phi_i(t_i)) = \phi_i(t_i)$. Hence, $\phi_i = h_i$ and the result follows.■