

Northumbria Research Link

Citation: Conte, Robert, Grundland, A. Michel and Huard, Benoit (2009) Elliptic solutions of isentropic ideal compressible fluid flow in (3+1) dimensions. *Journal of Physics A: Mathematical and Theoretical*, 42 (13). p. 135203. ISSN 1751-8113

Published by: IOP Publishing

URL: <http://dx.doi.org/10.1088/1751-8113/42/13/135203>

This version was downloaded from Northumbria Research Link:
<http://nrl.northumbria.ac.uk/11571/>

Northumbria University has developed Northumbria Research Link (NRL) to enable users to access the University's research output. Copyright © and moral rights for items on NRL are retained by the individual author(s) and/or other copyright owners. Single copies of full items can be reproduced, displayed or performed, and given to third parties in any format or medium for personal research or study, educational, or not-for-profit purposes without prior permission or charge, provided the authors, title and full bibliographic details are given, as well as a hyperlink and/or URL to the original metadata page. The content must not be changed in any way. Full items must not be sold commercially in any format or medium without formal permission of the copyright holder. The full policy is available online: <http://nrl.northumbria.ac.uk/policies.html>

This document may differ from the final, published version of the research and has been made available online in accordance with publisher policies. To read and/or cite from the published version of the research, please visit the publisher's website (a subscription may be required.)

www.northumbria.ac.uk/nrl



Elliptic solutions of isentropic ideal compressible fluid flow in (3+1) dimensions

R. Conte^{1,2}, A.M. Grundland^{3,4}, B. Huard⁵

1. LRC MESO, École normale supérieure de Cachan (CMLA) et CEA-DAM, 61, avenue du Président Wilson, F-94235 Cachan Cedex, France.
 2. Service de physique de l'état condensé (URA 2464), CEA-Saclay, F-91191 Gif-sur-Yvette Cedex, France
 3. Centre de Recherches Mathématiques, Université de Montréal, C.P. 6128, Succ. Centre-ville, Montréal, (QC) H3C 3J7, Canada,
 4. Université du Québec, Trois-Rivières CP500 (QC) G9A 5H7, Canada
 5. Département de mathématiques et de statistique, Université de Montréal C.P. 6128, Succ. Centre-ville, Montréal, (QC) H3C 3J7, Canada
- Email addresses : Robert.Conte@cea.fr, grundlan@crm.umontreal.ca, huard@dms.umontreal.ca

Abstract

A modified version of the conditional symmetry method, together with the classical method, is used to obtain new classes of elliptic solutions of the isentropic ideal compressible fluid flow in (3+1) dimensions. We focus on those types of solutions which are expressed in terms of the Weierstrass \wp -functions of Riemann invariants. These solutions are of special interest since we show that they remain bounded even when these invariants admit the gradient catastrophe. We describe in detail a procedure for constructing such classes of solutions. Finally, we present several examples of an application of our approach which includes bumps, kinks and multi-wave solutions.

1 Introduction

The purpose of this paper is to construct bounded elliptic solutions of a compressible isentropic ideal flow in (3 + 1) dimensions. Such solutions exist even in the case where the Riemann invariants admit the gradient catastrophe.

Let us first present a brief description of a procedure detailed in [10] for constructing rank- k solutions in terms of Riemann invariants for the case of an isentropic compressible ideal fluid in (3 + 1) dimensions. Such a model is governed by the equations

$$u_t^\alpha + \sum_{\beta=1}^4 \sum_{j=1}^3 \mathcal{A}_\beta^{j\alpha}(u) u_j^\beta = 0, \quad \alpha = 1, 2, 3, 4, \quad (1.1)$$

where $\mathcal{A}^1, \mathcal{A}^2$ and \mathcal{A}^3 are 4×4 real-valued matrix functions of the form

$$\mathcal{A}^j = \begin{pmatrix} u^i & \delta_{i1}\kappa^{-1}a & \delta_{i2}\kappa^{-1}a & \delta_{i3}\kappa^{-1}a \\ \delta_{i1}\kappa a & u^i & 0 & 0 \\ \delta_{i2}\kappa a & 0 & u^i & 0 \\ \delta_{i3}\kappa a & 0 & 0 & u^i \end{pmatrix}, \quad j = 1, 2, 3,$$

$\kappa = 2(\gamma - 1)^{-1}$ and γ is the adiabatic exponent of the medium under consideration. The independent and dependent variables are denoted by $x = (t = x^0, x^1, x^2, x^3) \in X \subset \mathbb{R}^4$ and $u = (a, \vec{u}) \in U \subset \mathbb{R}^4$, respectively, and u_i stands for the first order partial derivatives of u , i.e. $u_i^\alpha \equiv \partial u^\alpha / \partial x^i$, $\alpha = 1, \dots, 4$, $i = 0, 1, 2, 3$. Here, the quantity a stands for the velocity of sound in the medium and \vec{u} is the velocity vector field of the flow. Throughout this paper, we adopt the summation convention over repeated lower and upper indices.

The purpose of this article is to obtain rank- k solutions of system (1.1) expressible in terms of Riemann invariants. To this end, we seek solutions $u(x)$ of (1.1) defined implicitly by the following set of relations between the variables u^α, r^A and x^i ,

$$u = f(r^1(x, u), \dots, r^k(x, u)), \quad r^A(x, u) = \lambda_i^A(u)x^i, \quad \ker(\lambda_0^A \mathcal{I}_4 + \mathcal{A}^i(u)\lambda_i^A) \neq 0, \quad (1.2)$$

for some function $f : \mathbb{R}^k \rightarrow \mathbb{R}^4$ and $A = 1, \dots, k \leq 3$. Such a solution is called a rank- k solution if $\text{rank}(u_i^\alpha) = k$. The functions $r^A(x, u)$ are called the Riemann invariants associated with the wave vectors $\lambda^A = (\lambda_0^A, \vec{\lambda}^A) \in \mathbb{R}^4$ of the system (1.1). Here, $\vec{\lambda}^A = (\lambda_1^A, \lambda_2^A, \lambda_3^A)$ denotes a direction of wave propagation and the eigenvalue λ_0^A is a phase velocity of the considered wave. Two types of admissible wave vectors for the isentropic equations (1.1) are obtained by solving the dispersion relation

$$\det(\lambda_0(u)\mathcal{I}_4 + \lambda_i(u)\mathcal{A}^i(u)) = [(\lambda_0 + \vec{u} \cdot \vec{\lambda})^2 - a^2\vec{\lambda}^2](\lambda_0 + \vec{u} \cdot \vec{\lambda})^2 = 0. \quad (1.3)$$

They are called the entropic (E) and acoustic (S) wave vectors and are defined by

$$\text{i) } \lambda^E = (\varepsilon a + \vec{u} \cdot \vec{e}, -\vec{e}), \quad \varepsilon = \pm 1, \quad \text{ii) } \lambda^S = (\det(\vec{u}, \vec{e}, \vec{m}), -\vec{e} \times \vec{m}), \quad |\vec{e}|^2 = 1, \quad (1.4)$$

where \vec{e} and \vec{m} are unit and arbitrary vectors, respectively.

The construction of rank- k solutions through the conditional symmetry method (CSM) is achieved by considering an overdetermined system, consisting of the original system (1.1) in 4 independent variables together with a set of compatible first order differential constraints (DCs),

$$\xi_a^i(u)u_i^\alpha = 0, \quad \lambda_i^A \xi_a^i = 0, \quad a = 1, \dots, 4 - k, \quad (1.5)$$

for which a symmetry criterion is automatically satisfied. Such notions as conditional symmetry, conditional symmetry algebra and conditionally invariant solution for the original system (1.1) we use in accordance with definitions given in [10]. Under the above circumstances, the following result holds :

The isentropic compressible ideal fluid equations (1.1) admit a $(4 - k)$ -dimensional conditional symmetry algebra L if and only if there exists a set of $(4 - k)$ linearly independent vector fields

$$X_a = \xi_a^i(u) \frac{\partial}{\partial x^i}, \quad a = 1, \dots, 4-k, \quad \ker(\mathcal{A}^i(u)\lambda_i^A) \neq 0, \quad \lambda_i^A \xi_a^i = 0, \quad A = 1, \dots, k \leq 3, \quad (1.6)$$

which satisfy on some neighborhood of $(x_0, u_0) \in X \times U$ the trace conditions

$$\text{i) } \operatorname{tr}\left(\mathcal{A}^\mu \frac{\partial f}{\partial r} \lambda\right) = 0, \quad \text{ii) } \operatorname{tr}\left(\mathcal{A}^\mu \frac{\partial f}{\partial r} \eta_{(a_1} \frac{\partial f}{\partial r} \dots \eta_{a_s)} \frac{\partial f}{\partial r} \lambda\right) = 0, \quad \mu = 1, \dots, 4, \quad (1.7)$$

where

$$\lambda = (\lambda_i^A) \in \mathbb{R}^{k \times 4}, \quad r = (r^1, \dots, r^k) \in \mathbb{R}^k, \quad \frac{\partial f}{\partial r} = \left(\frac{\partial f^\alpha}{\partial r^A}\right) \in \mathbb{R}^{4 \times k},$$

$$\eta_{a_s} = \left(\frac{\partial \lambda_{a_s}^A}{\partial u^\alpha}\right) \in \mathbb{R}^{k \times 4}, \quad s = 1, \dots, k-1,$$

and (a_1, \dots, a_s) denotes the symmetrization over all indices in the bracket. Solutions of the system which are invariant under the Lie algebra L are precisely rank- k solutions of the form (1.2).

This result is a special case of the proposition in [10]. Note that these symmetries are not symmetries of the original system, but they can be used to construct solutions of the overdetermined system composed of (1.1) and (1.5).

For the case of rank-1 entropic solution E , the wave vector λ^E is a non-zero multiple of (1.4 i). Therefore, the corresponding vector fields X_i and Riemann invariant r become

$$X_i = -(a + \vec{e} \cdot \vec{u})^{-1} e_i \frac{\partial}{\partial t} + \frac{\partial}{\partial x^i}, \quad i = 1, 2, 3, \quad (1.8)$$

$$r(x, u) = (a + \vec{u} \cdot \vec{e})t - \vec{e} \cdot \vec{x}, \quad |\vec{e}|^2 = 1,$$

where we chose $\varepsilon = 1$ in (1.4 i). Rank-1 solutions invariant under the vector fields $\{X_1, X_2, X_3\}$ are obtained through the change of coordinates

$$\bar{t} = t, \quad \bar{x}^1 = r(x, u), \quad \bar{x}^2 = x^2, \quad \bar{x}^3 = x^3, \quad \bar{a} = a, \quad \bar{u}^1 = u^1, \quad \bar{u}^2 = u^2, \quad \bar{u}^3 = u^3, \quad (1.9)$$

on $\mathbb{R}^4 \times \mathbb{R}^4$. Assuming that the direction of the wave vector \vec{e} is constant, the fluid dynamics equations (1.1) transform into the system

$$\frac{\partial \bar{a}}{\partial \bar{x}^1} = \kappa^{-1} e_i \frac{\partial \bar{u}^i}{\partial \bar{x}^1}, \quad \frac{\partial \bar{u}^i}{\partial \bar{x}^1} = \kappa e_i \frac{\partial \bar{a}}{\partial \bar{x}^1}, \quad i = 1, 2, 3, \quad (1.10)$$

with the invariance conditions

$$\bar{a}_{\bar{t}} = \bar{a}_{\bar{x}^j} = 0, \quad \bar{u}_{\bar{t}}^\alpha = \bar{u}_{\bar{x}^j}^\alpha = 0, \quad j = 2, 3, \quad \alpha = 1, 2, 3. \quad (1.11)$$

The general rank-1 entropic E solution takes the form

$$\bar{a}(\bar{t}, \bar{x}) = \bar{a}(\bar{x}^1), \quad \bar{u}^i(\bar{t}, \bar{x}) = \kappa e_i \bar{a}(\bar{x}^1) + C_i, \quad C_i \in \mathbb{R}, \quad i = 1, 2, 3, \quad (1.12)$$

where the Riemann invariant $\bar{x}^1 = r(x, u)$ is given by

$$r(x, u) = [(1 + \kappa)a + \vec{e} \cdot \vec{C}]t - \vec{e} \cdot \vec{x}, \quad \vec{C} = (C_1, C_2, C_3) \in \mathbb{R}^3.$$

A similar procedure can be applied to the rank-1 acoustic solution S . Here, the wave vector λ^S is a non-zero multiple of (1.4 ii) and the corresponding vector fields X_i and Riemann invariant are

$$r(x, u) = \det(\vec{u}, \vec{e}, \vec{m})t - (\vec{e} \times \vec{m}) \cdot \vec{x}, \quad X_i = \frac{(\vec{e} \times \vec{m})_i}{\det(\vec{u}, \vec{e}, \vec{m})} \frac{\partial}{\partial t} + \frac{\partial}{\partial x^i}, \quad i = 1, 2, 3. \quad (1.13)$$

Again, the change of variables (1.9) leads to transformed dynamical equations, which we integrate in order to find rank-1 acoustic solution of the form

$$\begin{aligned} \bar{a}(\bar{t}, \bar{x}) &= a_0, \quad \bar{u}^1(\bar{t}, \bar{x}) = \bar{u}^1(\bar{x}^1), \quad \bar{u}^2(\bar{t}, \bar{x}) = \bar{u}^2(\bar{x}^1), \quad C \in \mathbb{R}, \\ \bar{u}^3(\bar{t}, \bar{x}) &= (e_1 m_2 - e_2 m_1)^{-1} [C - (e_2 m_3 - e_3 m_2) \bar{u}^1(\bar{x}^1) - (e_3 m_1 - e_1 m_3) \bar{u}^2(\bar{x}^1)]. \end{aligned} \quad (1.14)$$

Here \bar{u}^1 et \bar{u}^2 are arbitrary functions of the Riemann invariant $\bar{x}^1 = r(x, u)$ which has the explicit form

$$r(x, u) = Ct - \det(\vec{x}, \vec{e}, \vec{m}). \quad (1.15)$$

In general, the overdetermined system composed of (1.7 i) and (1.7 ii) is nonlinear and cannot always be solved in a closed form. Nevertheless, particular rank- k solutions for many physically interesting systems of PDEs are well worth pursuing. These particular solutions of (1.7 i) and (1.7 ii) can be obtained by assuming that the function f is in the form of a rational function, which may also be interpreted as a truncated Laurent series in the variables r^A . This method can work only for equations having the Painlevé property [4]. Consequently, these equations can be very often integrated in terms of known functions.

Applying a version of the conditional symmetry method to the isentropic model (1.1), several new classes of solutions have been constructed in a closed form [9, 10]. Comparing these results with the ones obtained via the generalized method of characteristics (GMC) [20], it was shown that more diverse classes of solutions are involved in superpositions (i.e. rank- k solutions) than in the case of the GMC [10].

This paper is a continuation of the papers [9, 10]. The objective is to construct bounded elliptic solutions of the isentropic system (1.1) using the version of the CSM proposed in [10]. These types of solutions are obtained through a proper selection of differential constraints (DCs) compatible with the initial system of equations (1.1). That is, the solution should satisfy both the initial system (1.1) and the differential constraints (1.5). Among the new results obtained, we have rank-2 and rank-3 periodic bounded solutions expressed in terms of Weierstrass \wp -functions. They represent

bumps, kinks and multiple waves, all of which depend on Riemann invariants. These solutions remain bounded even when the invariants admit a gradient catastrophe.

The paper is organized as follows. In Section 2 we construct rank-2 and rank-3 elliptic solutions of the system, among which multiple waves and doubly periodic solutions are included, and we show that they remain bounded everywhere. Section 3 summarizes the results obtained and contains some suggestions for future developments.

2 Rank-2 and rank-3 solutions

The construction approach outlined in Section 1 has been applied to the isentropic flow equations (1.1) in order to obtain rank-2 and rank-3 solutions. The results of our analysis are summarized in Tables 1 and 2. Several of them possess a certain amount of freedom. They depend on one or two arbitrary functions of one or two Riemann invariants, depending on the case. The range of the types of solutions obtained depends on different combinations of the vector fields X_a as given in (1.6). For convenience, we denote by $E_i E_j$, $E_i S_j$, $S_i S_j$, $E_i E_j E_k$, etc, $i, j, k = 1, 2, 3$, the solutions which are the result of nonlinear superpositions of rank-1 solutions associated with different types of wave vectors (1.4 i) and (1.4 ii). By r^1, r^2 and r^3 we denote the Riemann invariants which coincide with the group invariants of the differential operators X_a of the solution under consideration.

The arbitrary functions appearing in the solutions listed in Tables 1 and 2 allow us to change the geometrical properties of the governed fluid flow in such a way as to exclude the presence of singularities. This fact is of special significance since, as is well known [16, 21], in most cases, even for arbitrary smooth and sufficiently small initial data at $t = t_0$ the magnitude of the first derivatives of Riemann invariants becomes unbounded in some finite time T . Thus, solutions expressible in terms of Riemann invariants usually admit a gradient catastrophe. Nevertheless, we have been able to show that it is still possible in these cases to construct bounded solutions expressed in terms of elliptic functions, through the proper selection of the arbitrary functions appearing in the general solution. For this purpose it is useful to select DCs corresponding to a certain class of the nonlinear Klein-Gordon equation which is known to possess rich families of bounded solutions [1]. We choose elliptic solutions of the Klein-Gordon equation because a group theoretical analysis has already been performed [22]. The obtained results can be adapted to the isentropic ideal compressible fluid flow in $(3 + 1)$ dimensions. Thus, we specify the arbitrary function(s) appearing in the general solutions listed in Tables 1 and 2, say ϕ , to the differential constraint in the form of the Klein-Gordon ϕ^6 -field equation in three independent variables r^1, r^2 and r^3 which form the coordinates of the Minkowski space $M(1, 2)$

$$\phi_{r^1 r^1} - \phi_{r^2 r^2} - \phi_{r^3 r^3} = c\phi^5, \quad c \in \mathbb{R}. \quad (2.1)$$

Here, we choose r^1 to be timelike and r^2, r^3 to be spacelike coordinates. It is well known (see e.g. [22]) that equation (2.1) is invariant with respect to the similitude Lie algebra

$sim(1, 2)$ involving the following generators

$$\begin{aligned}
D &= r^i \partial_{r^i} - \frac{1}{2} \phi \partial_\phi, & P_i &= \partial_{r^i}, & i &= 1, 2, 3, \\
L_{ab} &= r^a \partial_{r^b} - r^b \partial_{r^a}, & a \neq b &= 2, 3, \\
K_{1a} &= -(r^1 \partial_{r^a} - r^a \partial_{r^1}), & a &= 2, 3,
\end{aligned} \tag{2.2}$$

where D denotes a dilation, P_i represents translations, L_{ab} stands for rotations and K_{1a} for Lorentz boosts. A systematic use of the subgroup structure [22] of the invariance group of (2.1) allows us to generate all symmetry variables ξ in terms of the Riemann invariants r^1, r^2, r^3 . We concentrate here only on the case when symmetry variables are invariants of the assumed subgroups G_i of $Sim(1, 2)$ having generic orbits of codimension one. For illustration purposes, we perform a symmetry reduction analysis on four selected members of the list of subalgebras given in ([22], Table IV) which involve dilations. For each selected subalgebra in Minkowski space $M(1, 2)$, we compute the group invariants ξ of the corresponding Lie subgroup and reduce the equation (2.1) to a second order ODE. The application of the symmetry reduction method to equation (2.1) leads to solutions of the form

$$\phi(r) = \alpha(r)F(\xi(r)), \quad r = (r^1, r^2, r^3), \tag{2.3}$$

where the multiplier $\alpha(r)$ and the symmetry variable $\xi(r)$ are given explicitly by group theoretical considerations and $F(\xi)$ satisfies an ODE obtained by substituting (2.3) into equation (2.1). The results of our computation are listed below.

1. $\{D, P_1\}$: $\alpha = \{4c[(r^2)^2 + (r^3)^2]\}^{-1/4}$, $\xi = \frac{1}{2} \arctan \frac{r^3}{r^2}$, $F'' + F + F^5 = 0$,
2. $\{D, L_{31}\}$: $\alpha = \{-c(r^1)^2/4\}^{-1/4}$, $\xi = \frac{(r^2)^2 + (r^3)^2}{(r^1)^2}$,
 $\xi(1 + \xi)F'' + \left(2\xi + \frac{3}{2}\right)F' + \frac{3}{16}F + F^5 = 0$,
3. $\{D + \frac{1+q}{q}K_{12}, L_{23}\}$: $\alpha = \left\{-\frac{(2q+1)}{c}\right\}^{1/4}(r^1 + r^2)^{q/2}$,
 $\xi = [(r^1)^2 - (r^2)^2 - (r^3)^2](r^1 + r^2)^q$,
 $F'' + \frac{3q+l}{2q+1}\frac{1}{\xi}F' + F^5 = 0$, $q = -l/3, l = 2, 4 - 3l$, $l \in \mathbb{Z}^+$,
4. $\{D + \frac{1}{2}K_{12}, L_1 - K_{13}\}$: $\alpha = (9/4C)^{1/4}\{r^2 - (r^1 + r^3)^2/4\}^{-1/2}$,
 $\xi = \frac{6(r^3 - r^1) + 6r^2(r^1 + r^3) - (r^1 + r^3)^3}{8(r^2 - (r^1 + r^3)^2/4)^{3/2}}$, $(1 + \xi^2)F'' + \frac{7}{3}F' + \frac{1}{3}F + F^5 = 0$.

The parity invariance of (2.1) suggests the substitution

$$F(\xi) = [H(\xi)]^{1/2}$$

which transforms the equations listed in (2.4) to

$$\{D, P_1\} \quad H'' = \frac{H'^2}{2H} - 2(H + H^3), \quad (2.5)$$

$$\{D, L_{31}\} \quad H'' = \frac{H'^2}{2H} - \frac{1}{\xi(1+\xi)} \left[\left(2\xi + \frac{3}{2} \right) H' + \frac{3}{8}H + 2H^3 \right], \quad (2.6)$$

$$\left\{ D + \frac{1+q}{q} K_{12}, L_{23} \right\} \quad H'' = \frac{H'^2}{2H} - \left[\frac{m}{\xi} H' + 2H^3 \right], m = \frac{3q+l}{2q+1} = (0, 4/3, 2), \quad (2.7)$$

$$\left\{ D + \frac{1}{2} K_{12}, L_1 - K_{13} \right\} \quad H'' = \frac{H'^2}{2H} - \frac{1}{1+\xi^2} \left[\frac{7}{3} \xi H' + \frac{2}{3} H + 2H^3 \right], \quad (2.8)$$

where the three admissible values for the scalar m come from group theoretical considerations [22]. Each of these four equations possesses a first integral

$$K' = \frac{1}{4} G g^2 \frac{(gH)^2}{gH} - \frac{c_0}{4} (gH)^3 - 3e_0 gH, \quad (2.9)$$

in which the four sets of functions G, g and constants e_0, c_0 obey the respective conditions

$$G = -\frac{3c_0}{4}, \quad g^2 = \frac{4e_0}{c_0},$$

$$G = -\frac{3c_0}{4} \xi(\xi+1), \quad g^2 = -\frac{64e_0}{c_0} \xi,$$

$$G = -\frac{3c_0}{4}, \quad (a, e_0, g^2) = (0, 0, k_1), \quad (4/3, 0, k_1 \xi^{4/3}), \quad (2, e_0, -\frac{16e_0}{c_0} \xi^2),$$

$$G = -\frac{3c_0}{4} (\xi^2 + 1), \quad g = k_1 (1 + \xi^2)^{1/3}, \quad e_0 = 0,$$

(k_1 denotes an arbitrary nonzero real constant). Under a transformation $(H, \xi) \rightarrow (U, \zeta)$ which preserves the Painlevé property,

$$H(\xi) = U(\zeta)/g(\xi), \quad \left(\frac{d\zeta}{d\xi} \right)^2 = \frac{1}{Gg^2},$$

the equation (2.9) becomes autonomous

$$U'^2 - c_0 U^4 - 12e_0 U^2 - 4K'U = 0, \quad c_0 \neq 0.$$

When $K' = 0$, U^{-1} is either a sine, cosine, hyperbolic sine or a hyperbolic cosine function, depending on the signs of the constants, therefore bounded solutions are easily characterized.

When $K' \neq 0$, it is convenient to first integrate this elliptic equation in terms of the Weierstrass function $\wp(\zeta, g_2, g_3)$,

$$U(\zeta) = \frac{K'}{\wp(\zeta) - e_0}, \quad g_2 = 12e_0^2, \quad g_3 = -8e_0^3 - c_0 K'^2,$$

$$\wp'^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) = 4\wp^3 - g_2\wp - g_3,$$

(where we abbreviate $\wp(\zeta, g_2, g_3)$ by $\wp(\zeta)$) then to use the classical formulae which connect \wp and various bounded Jacobi functions.

This correspondence is quite easy to write down if one uses the symmetric notation of Halphen [14] to represent the Jacobi functions. Halphen introduces three basis functions

$$h_\alpha(u) = \sqrt{\wp(u) - e_\alpha}, \quad \alpha = 1, 2, 3,$$

and the connection between the Weierstrass \wp function and the Jacobi copolar trio $\text{cs}, \text{ds}, \text{ns}$ is given by [14, p. 46]

$$\frac{\text{cs}(z|k)}{h_1(u)} = \frac{\text{ds}(z|k)}{h_2(u)} = \frac{\text{ns}(z|k)}{h_3(u)} = \frac{u}{z} = \frac{1}{\sqrt{e_1 - e_3}}, \quad k^2 = \frac{e_2 - e_3}{e_1 - e_3},$$

where k is the modulus of the Jacobi elliptic functions. For full details on Halphen's symmetric notation, see [15]. We give here explicit solutions in terms of the Weierstrass function, leaving the conversion to Jacobi's notation to the reader. These solutions are obtained by convenient choices of the normalization constants e_0, c_0, K', k_1 .

With the normalization $e_0 = -1/3$, $c_0 = -4/3$, $K' = C$, the general solution of (2.5) is

$$F^2(\xi) = \frac{C}{\wp(\xi) + 1/3}, \quad \zeta = \xi, \quad g_2 = \frac{4}{3}, \quad g_3 = \frac{8}{27} + \frac{4}{3}C^2, \quad C \in \mathbb{R}. \quad (2.10)$$

With the normalization $e_0 = k_0^{-2}/48$, $c_0 = -(4/3)k_0^{-2}$, $K' = C$, the solution of (2.6) has the form

$$F^2(\xi) = \frac{C\xi^{-1/2}}{\wp(\zeta) - \frac{1}{48k_0^2}}, \quad \zeta = -2k_0 \operatorname{argth} \sqrt{\xi + 1}, \quad g_2 = \frac{1}{192k_0^4}, \quad g_3 = -\frac{1}{13824k_0^6} + \frac{4C^2}{3k_0^2},$$

with $k_0, C \in \mathbb{R}$.

The three cases for equation (2.7) associated with the subalgebra $\{D + \frac{1+q}{q}K_{12}, L_{23}\}$ yield the respective solutions

$$\begin{aligned} q = -k/3 : F^2(\xi) &= \frac{C}{\wp(\xi)}, \quad \zeta = \xi, \quad g_2 = 0, \quad g_3 = \frac{4C^2}{3}, \\ q = 4 - 3k : F^2(\xi) &= \frac{C\xi^{-2/3}}{\wp(\zeta)}, \quad \zeta = 3k_0\xi^{1/3}, \quad g_2 = 0, \quad g_3 = \frac{4C^2}{3k_0^2}, \\ q = k - 2 : F^2(\xi) &= \frac{C\xi^{-1}}{\wp(\zeta) - \frac{1}{12k_0^2}}, \quad \zeta = k_0 \log \xi, \quad g_2 = \frac{1}{12k_0^4}, \quad g_3 = -\frac{1}{216k_0^6} + \frac{4C^2}{3k_0^2}. \end{aligned}$$

Finally, equation (2.8) integrates as (equation no 4 in (2.4))

$$F^2(\xi) = \frac{C(\xi^2 + 1)^{-1/3}}{\wp(\zeta)}, \quad \zeta = \xi \quad {}_2F_1\left(\frac{1}{2}, \frac{5}{6}; \frac{3}{2}; -\xi^2\right), \quad g_2 = 0, \quad g_3 = \frac{4C^2}{3k_0^2},$$

where ${}_2F_1$ denotes the hypergeometric function.

Using these results, we construct bounded rank-3 solutions of the equations (1.1). For this purpose, for each general solution appearing in Tables 2 and 3, we introduce the arbitrary functions into the Klein-Gordon equation (2.1) and select only the solutions expressed in terms of the Weierstrass \wp -function.

For illustration, let us now discuss the case of the rank-3 entropic solution $E_1 E_2 E_3$ which represents a superposition of three rank-1 entropic solutions E_i given by (1.12). We assume that the entropic wave vectors λ^{E_1} , λ^{E_2} and λ^{E_3} are linearly independent and take the form

$$\lambda^{E_i} = (a + \vec{e}^i \cdot \vec{u}, -\vec{e}^i), \quad |\vec{e}^i|^2 = 1, \quad i = 1, 2, 3.$$

Hence the corresponding vector fields X_i and Riemann invariants are given by

$$X = \frac{\partial}{\partial x^3} - \frac{\det(\vec{e}^1, \vec{e}^2, \vec{e}^3)}{\beta_3} \frac{\partial}{\partial t} + \frac{\beta_1}{\beta_3} \frac{\partial}{\partial x^1} + \frac{\beta_2}{\beta_3} \frac{\partial}{\partial x^2}, \quad r^i(x, u) = (a + \vec{e}^i \cdot \vec{u})t - \vec{e}^i \cdot \vec{x}, \quad (2.11)$$

where $\beta_i = (\vec{e}^2 \times \vec{e}^3)_i (a + \vec{e}^1 \cdot \vec{u}) + (\vec{e}^1 \times \vec{e}^3)_i (a + \vec{e}^2 \cdot \vec{u}) + (\vec{e}^1 \times \vec{e}^2)_i (a + \vec{e}^3 \cdot \vec{u})$. The rank-3 entropic solutions invariant under the vector field X are obtained through the change of coordinates

$$\bar{t} = t, \bar{x}^1 = r^1(x, u), \bar{x}^2 = r^2(x, u), \bar{x}^3 = r^3(x, u), \bar{a} = a, \bar{u}^1 = u^1, \bar{u}^2 = u^2, \bar{u}^3 = u^3, \quad (2.12)$$

on $\mathbb{R}^4 \times \mathbb{R}^4$. Specifying the form of the solution as a linear superposition of rank-1 solutions (1.12),

$$\bar{a} = \bar{a}_1(r^1) + \bar{a}_2(r^2) + \bar{a}_3(r^3), \quad \vec{u} = \kappa(\vec{e}^1 \bar{a}_1(r^1) + \vec{e}^2 \bar{a}_2(r^2) + \vec{e}^3 \bar{a}_3(r^3)) \quad (2.13)$$

the fluid dynamics equations (1.1) transform to

$$\sum_{i=1}^2 \sum_{j=i+1}^3 [\kappa(\vec{e}^i \cdot \vec{e}^j)^2 + (1 - \kappa)(\vec{e}^i \cdot \vec{e}^j) - 1] \bar{a}'_i(r^i) \bar{a}'_j(r^j) = 0, \quad (2.14)$$

while the invariance conditions have the form

$$\bar{a}_{\bar{t}} = \bar{u}_{\bar{t}}^1 = \bar{u}_{\bar{t}}^2 = \bar{u}_{\bar{t}}^3 = 0. \quad (2.15)$$

This solution exists if and only if the three entropic wave vectors $\vec{e}^1, \vec{e}^2, \vec{e}^3$ intersect at a certain specific angle given by

$$\cos \phi_{ij} = -\frac{1}{\kappa}, \quad i \neq j = 1, 2, 3,$$

where ϕ_{ij} denotes the angle between the wave vectors \vec{e}^i and \vec{e}^j [10, 20]. Imposing the condition that each of the functions $\bar{a}_i(r^i)$, $i = 1, 2, 3$ obeys the ODE

$$F'' + F + F^5 = 0,$$

then according to (2.10), the rank-3 entropic solution takes the form

$$a = \sum_{i=1}^3 \frac{C_i}{\left(\wp(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}C_i^4) + \frac{1}{3}\right)^{1/2}}, \quad \vec{u} = \kappa \sum_{i=1}^3 \frac{C_i \vec{\lambda}^i}{\left(\wp(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}C_i^4) + \frac{1}{3}\right)^{1/2}}, \quad (2.16)$$

$$r^i = -(1 + \kappa) \frac{C_i}{\left(\wp(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}C_i^4) + \frac{1}{3}\right)^{1/2}} t + \vec{\lambda}^i \cdot \vec{x}, \quad i = 1, 2, 3.$$

Making use of an explicit expression for the zeros of the \wp -function, we show that the values for which the denominator in solution (2.16) vanish are not located on the real axis for a specific choice of the constants of integration C_i . Then we have the following result.

If the constants of integration C_i are equal to $\sqrt{19}/6$, then the elliptic rank-3 entropic solution (2.16) of the isentropic ideal compressible fluid flow equations (1.1) is bounded.

Indeed, according to recent results obtained by Duke and Imamoglu in [6], the location of the zeros of the \wp -function can be given explicitly in terms of generalized hypergeometric functions.

Considering a lattice $\mathcal{L} = \mathbb{Z} + \tau\mathbb{Z}$, $\text{Im } \tau > 0$, the doubly periodic Weierstrass \wp -function is defined by

$$\wp(z; \tau) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right),$$

where the sum ranges over all $\omega \in \mathcal{L}$. Note that the \wp -function assumes every value of the extended complex plane exactly twice in \mathcal{L} and since it is even, its zeros are of the form $\pm z_0$. The value of z_0 can be determined from the following theorem [6].

The zeros $\pm z_0$ of the \wp -function are given by

$$z_0 = \frac{1 + \tau}{2} + \frac{c_2 s^{1/4} {}_3F_2\left(\frac{1}{3}, \frac{2}{3}, 1; \frac{3}{4}, \frac{5}{4}; s\right)}{{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; 1 - s\right)}, \quad c_2 = -\frac{i\sqrt{6}}{3\pi}, \quad |s| < 1, \quad |1 - s| < 1, \quad (2.17)$$

where ${}_pF_q$ denotes the generalized hypergeometric functions and s is a function of the modular discriminant Δ and the Eisenstein series E_4 ,

$$s = 1 - \frac{1728\Delta}{E_4^3}, \quad \Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24}, \quad E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, \quad q = e^{2\pi i \tau}. \quad (2.18)$$

It is understood that the principal branch is to be taken in the radical expression $s^{1/4}$.

We now illustrate the application of this theorem by showing that the function $\wp(z, 4/3, 1)$ is always strictly positive for real z . This case is the specific case of solution (2.10) for which $C = \sqrt{19}/6$. From given invariants g_2 and g_3 , by finding the roots e_1, e_2, e_3 of the cubic polynomial $4t^3 - g_2t - g_3$, one can determine the values of the periods ω_1 and ω_2 . In the case when $g_2 = 4/3$ and $g_3 = 1$, we obtain the periods $\omega_1 = 2.81$ and $\omega_2 = 1.405 + i2.902$, hence $\tau = \omega_2/\omega_1 = 0.5 + i1.033$.

For any pair of periods ω_1, ω_2 , the lattice $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ can be rescaled in such a way that ω_1 is normalized to 1 by using the well-known formula for the \wp -function

$$\wp(z, \omega_1, \omega_2) = \wp(z/\omega_1, \omega_2/\omega_1)/\omega_1^2. \quad (2.19)$$

Introducing the numeric value of τ into (2.18), we can evaluate from (2.17) and (2.19) the zeros of the Weierstrass function $\wp(z, 4/3, 1)$, $z_0 = \pm(1.405 + i0.929)$, which are indeed complex. Note that the \wp -function always possesses a double pole at $z = 0$ and that it tends to $+\infty$ at this point. Since it is continuous for all real $z \in (0, \omega_1)$, this implies that it must always be strictly positive. Hence, $\wp(z, 4/3, 1) + 1/3 > 0$ on the real interval $(0, \omega_1)$ and the function $(\wp(z, 4/3, 1) + 1/3)^{-1}$ is therefore bounded for all real z . Choosing the constants C_i as values of the initial constant C for which solution (2.10) is bounded then guarantees the boundedness of functions a and \vec{u} in solution (2.16). Therefore, solution (2.16) is bounded everywhere, even when the Riemann invariants r^i admit the gradient catastrophe. A similar analysis can be applied to every solution presented in Table 3 to show that they are bounded. This can be accomplished by the same procedure as presented above through the use of the theorem from [6].

This solution is physically interesting since it remains bounded for every value of the Riemann invariants r^i . Thus, it represents a bounded solution with periodic flow velocities (see figure 1). Similarly, it is possible to submit the arbitrary functions of the differential equations no 2, 3, 4 listed in (2.4) to obtain other types of bounded solutions. Table 3 presents these various types of solutions with their corresponding Riemann invariants. They are all bounded solutions of periodic, bump or kink type. Note that these solutions of (1.1) admit gradient catastrophes at some finite time. Hence, some discontinuities can occur like shock waves [5, 13]. Note also that the solutions remain bounded even when the first derivatives of r^i tend to infinity after some finite time T . However, after time T , the solution cannot be represented in parametric form by the Riemann invariants and ceases to exist.

3 Concluding remarks

The methods presented in this paper can be applied quite broadly and can usually provide at least certain particular solutions of hydrodynamic type equations. The conditional symmetries refer to the symmetries of the overdetermined system obtained by subjecting the original system (1.1) to certain differential constraints defined by setting the characteristics of the vector fields X_a to zero. The conditional symmetries are not symmetries of the original system (1.1). However, they are used to construct classes of rank-3 solutions of this system which are not obtainable by the classical symmetry approach. Among the new results obtained, we have rank-2 and rank-3 periodic solutions expressed in terms of the Weierstrass \wp -function that we have shown to be bounded over the real axis. They represent bumps, kinks and multiple-wave solutions, all of which depend on Riemann invariants. These solutions remain bounded even when the invariants admit the gradient catastrophe.

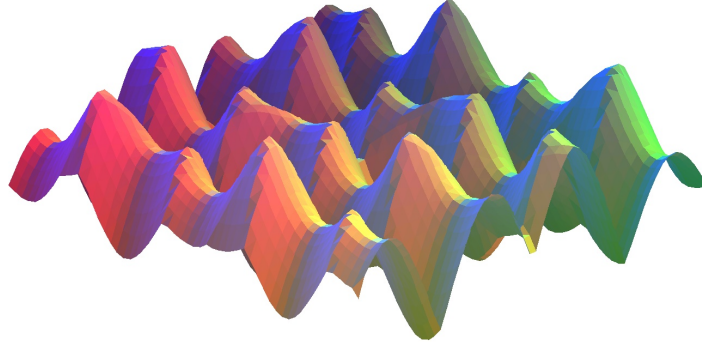


Figure 1: Bounded entropic rank-3 solution

Among the questions that one may ask is what role do exact analytical solutions play in the physical interpretation. One possible response is that such solutions may display qualitative behaviour which would otherwise be difficult to detect numerically or by approximations. For example, the doubly periodic properties of certain solutions expressed in terms of the Weierstrass \wp -function would not be very easily seen numerically.

One could also inquire about the stability property of the obtained solutions. Indeed, solutions which possess the property of stability should be observable physically and such analysis could be the starting point for perturbative computations. This task will be undertaken in a future work.

Acknowledgement

The authors would like to thank William Duke (University of California, Los Angeles) for helpful and interesting discussion on the topic of this paper. This work has been partially supported by research grants from NSERC of Canada. RC is pleased to thank the Laboratoire de physique mathématique du Centre de Recherches Mathématiques at the Université de Montréal for its continuous support.

References

- ¹ Ablowitz M.J., Clarkson P.A., *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, London Math. Soc., Cambridge Univ. Press, London, 1991.
- ² Chandrasekhar S., *Hydrodynamic and hydromagnetic stability*, Dover, New York, 1981.

- ³ Clarkson P.A. and Winternitz P., Symmetry reduction and exact solutions of nonlinear partial differential equations, *The Painlevé property, one century later*, 591–660, ed. R. Conte, CRM series in mathematical physics (Springer, New York, 1999).
- ⁴ R. Conte, M. Musette, *The Painlevé Handbook*, Springer Verlag, New York, 2008.
- ⁵ Courant R., Friedrichs K.O., *Supersonic flow and shock waves*, Interscience Publ., New York, 1948.
- ⁶ Duke W. and Imamoglu O., The zeros of the Weierstrass \wp -function and hypergeometric series, *Math. Ann.* 340 (2008), 4, 897-905.
- ⁷ Faddeev D.K., *Computational Methods of Linear Algebra*, W H Freeman & Co. (1963).
- ⁸ E. Goursat, *Leçons sur l'intégration des équations aux dérivées partielles du second ordre à deux variables indépendantes*, Gauthier-Villars, Paris, 1890.
- ⁹ Grundland A.M. and Huard B., Riemann invariants and rank- k solutions of hyperbolic systems, *J. Nonlin. Math. Phys.*, 13, 3 (2006), 393–419.
- ¹⁰ Grundland, A. M., Huard, B., Conditional symmetries and Riemann invariants for hyperbolic systems of PDEs. *J. Phys. A* 40 (2007), no. 15, 4093–4123.
- ¹¹ Grundland A.M. and Lalague L., Lie subgroups of the symmetry group of equations describing a nonstationary and isentropic flow, *Can. J. Phys.* 72, 9, 362–374 (1994).
- ¹² Grundland A.M. and Lalague L., Invariant and partially-invariant solutions of the equations describing a non-stationary and isentropic flow for an ideal and compressible fluid in (3+1) dimensions, *J. Phys. A : Math. Gen.* 29 (1996), 1723–1739.
- ¹³ John F., Formation of singularities in one-dimensional nonlinear wave propagation, *Comm. Pure Appl. Math.* 27, 377–405 (1974).
- ¹⁴ G.-H. Halphen, *Traité des fonctions elliptiques et de leurs applications*, Gauthier-Villars, Paris. Partie 1, Théorie des fonctions elliptiques et de leurs développements en série, 492p (1886).
- ¹⁵ W. Magnus, F. Oberhettinger and R.P. Soni, *Formulas and theorems for the special functions of mathematical physics*, third enlarged ed., Springer, Berlin, 1966, Orme ZK111
- ¹⁶ von Mises R., *Mathematical theory of compressible fluid flow*, Academic Press, New York, 1958.
- ¹⁷ Olver P.J., *Applications of Lie groups to differential equations*, Graduate Texts in Math. 107, Springer-Verlag, New York, 1986.

- ¹⁸ Olver P. J. and Vorobev E. M., *Nonclassical and Conditional Symmetries*, in *CRC Handbook of Lie Group Analysis*, Editor: N H Ibragimov, CRC press, London, 1995, Vol. 3, Chapt. XI.
- ¹⁹ Ovsiannikov L.V., *Group analysis of differential equations*, Academic Press, New York, 1982.
- ²⁰ Peradzynski Z., On certain classes of exact solutions for gasdynamics equations, *Archives of Mechanics*, 9, 2 (1972), 287–303.
- ²¹ Rozdestvenskii B., Janenko N., *Systems of quasilinear equations and their applications to gas dynamics*, A.M.S., Vol 55, Providence, 1983.
- ²² Winternitz P., Grundland A.M. and Tuszynski J.A., Exact solutions of the multi-dimensional classical ϕ^6 field equations obtained by symmetry reduction, *J. Math. Phys.* 28, 9, 2194–2212 (1987)

Table 1. Rank-2 solutions with the freedom of one, two or three arbitrary functions of one or two variables. Unassigned unknown functions $a(\cdot), u(\cdot), \dots$ are arbitrary functions of their respective arguments.

No	Type	Vector Fields	Riemann Invariants	Solutions
1	$E_1 S_1$	$X_1 = \frac{\partial}{\partial x^2} - \frac{\sigma_2}{\beta_1} \frac{\partial}{\partial t} - \frac{\beta_2}{\beta_1} \frac{\partial}{\partial x^1}$ $X_2 = \frac{\partial}{\partial x^3} - \frac{\sigma_3}{\beta_1} \frac{\partial}{\partial t} - \frac{\beta_3}{\beta_1} \frac{\partial}{\partial x^1}$ $\beta_i = -(\bar{e}^2 \times \bar{m}^2)_i (a + \bar{e}^1 \cdot \bar{u}) + e_i^1 [\bar{u}, \bar{e}^2, \bar{m}^2]$ $\sigma_j = -e_1^1 (\bar{e}^2 \times \bar{m}^2)_j + e_j^1 (\bar{e}^2 \times \bar{m}^2)_1, j = 2, 3$	$r^1 = ((1+k)\bar{a}_1(r^1) + C_2)t - \bar{e}^1 \cdot \bar{x}$ $r^2 = Ct - [\bar{x}, \bar{e}^2, \bar{m}^2], \quad [\bar{e}^1, \bar{e}^2, \bar{m}^2] = 0$ $C_2 = (C_1 e_1^1 - e_3^1)^{-1}$	$\bar{a} = \bar{a}_1(r^1) + a_0, \quad [\bar{u}_2, \bar{e}^2, \bar{m}^2] = C$ $\bar{u} = k\bar{a}_1(r^1) + \bar{u}_2(r^2), \quad \bar{u}_2^3(r^2) = C_1 \bar{u}_2^1(r^2)$ $a_0, C, C_1, C_2 \in \mathbb{R}$
2a	$S_1 S_2$	$X_1 = \frac{\partial}{\partial t} + u^1 \frac{\partial}{\partial x^1} + u^2 \frac{\partial}{\partial x^2}$ $X_2 = \frac{\partial}{\partial x^3}$	$r^1 = x^1 - u^1 t$ $r^2 = x^2 - u^2 t$	$\bar{a} = a_0, \quad \bar{u}^1 = -\phi_{r^2}, \quad \bar{u}^2 = \phi_{r^1},$ $\phi = \varphi(\alpha_1 r^1 + \alpha_2 r^2) + \beta_1 r^1 + \beta_2 r^2 + \gamma,$ $\bar{u}^3 = \bar{u}^3(r^1, r^2), \quad a_0, \alpha_i, \beta_i, \gamma \in \mathbb{R}, i = 1, 2,$
2b	$S_1 S_2$	$X_1 = \frac{\partial}{\partial t} + u^1 \frac{\partial}{\partial x^1} + u^2 \frac{\partial}{\partial x^2}$ $X_2 = \frac{\partial}{\partial x^3}$	$r^1 = x^1 - u^1 t$ $r^2 = x^2 - u^2 t$	$\bar{a} = a_0, \quad \bar{u}^2 = \bar{u}^3 = g(x^1 - x^2), \quad a_0 \in \mathbb{R},$ $\bar{u}^1 = b(x^1 - tg(x^1 - x^2), x^2 - tg(x^1 - x^2))$
2c	$S_1 S_2$	$X_2 = \frac{\partial}{\partial x^2} - \frac{\sigma_2}{\beta_1} \frac{\partial}{\partial t} - \frac{\beta_2}{\beta_1} \frac{\partial}{\partial x^1}$ $X_3 = \frac{\partial}{\partial x^3} - \frac{\sigma_3}{\beta_1} \frac{\partial}{\partial t} - \frac{\beta_3}{\beta_1} \frac{\partial}{\partial x^1}$ $\beta_j = \lambda_j^2 [\bar{u}, \bar{e}^1, \bar{m}^1] - \lambda_j^1 [\bar{u}, \bar{e}^2, \bar{m}^2]$ $\sigma_i = \lambda_1^1 \lambda_i^2 - \lambda_i^1 \lambda_1^2$	$r^1 = \left(C_1 + \frac{\lambda_1^1 C_2}{\lambda_1^2} \right) t - \bar{\lambda}^1 \cdot \bar{x}$ $r^2 = \left(C_2 + \frac{\lambda_2^1}{\lambda_1^1} C_1 + G(r^1) \right) t - \bar{\lambda}^2 \cdot \bar{x}$ $\lambda_i^j = -(\bar{e}^j \times \bar{m}^j)_i$ $G(r^1) = \frac{1}{\lambda_1^1} ((\lambda_1^1 \lambda_2^2 - \lambda_2^1 \lambda_1^2) \bar{u}_1^2(r^1) + (\lambda_1^1 \lambda_3^2 - \lambda_1^2 \lambda_3^1) \bar{u}_1^3(r^1))$	$\bar{a} = a_0, \quad a_0, C_1, C_2 \in \mathbb{R}$ $\bar{u}^1 = \frac{1}{\lambda_1^1} (C_1 - \lambda_2^1 \bar{u}_1^2(r^1) - \lambda_3^1 \bar{u}_1^3(r^1))$ $- \left(\frac{\lambda_2^2}{\lambda_1^2} \eta + \frac{\lambda_2^2}{\lambda_1^2} \right) \bar{u}_2^2(r^2) + \frac{C_2}{\lambda_1^2}$ $\bar{u}^2 = \bar{u}_1^2(r^1) + \bar{u}_2^2(r^2)$ $\bar{u}^3 = \bar{u}_1^3(r^1) + \eta \bar{u}_2^2(r^2), \quad \eta = \frac{\lambda_2^2 \lambda_1^1 - \lambda_1^2 \lambda_2^1}{\lambda_1^1 \lambda_3^2 - \lambda_1^2 \lambda_3^1}$
3	$E_1 E_2 S_1$	$X = \frac{\partial}{\partial x^3} - \frac{\sigma_1}{\beta_{12}} \frac{\partial}{\partial t} + \frac{\beta_{23}}{\beta_{12}} \frac{\partial}{\partial x^1} + \frac{\beta_{31}}{\beta_{12}} \frac{\partial}{\partial x^2}$ $\sigma_1 = \varepsilon_{ijk} e_i^1 e_j^2 (\bar{e}^3 \times \bar{m})_k$ $\beta_{ij} = (e_j^1 e_i^2 - e_i^1 e_j^2) [\bar{u}, \bar{e}^3, \bar{m}^3]$ $+ (e_j^2 (\bar{e}^3 \times \bar{m}^3)_i - e_i^2 (\bar{e}^3 \times \bar{m}^3)_j) (a + \bar{e}^1 \cdot \bar{u})$ $+ (e_i^1 (\bar{e}^3 \times \bar{m}^3)_j - e_j^1 (\bar{e}^3 \times \bar{m}^3)_i) (a + \bar{e}^2 \cdot \bar{u})$	$r^1 = \frac{\beta \bar{u}_3^1(r^3) t - e_1^1 x^1 - e_2^2 x^2}{1 - \alpha(1+\kappa)t}$ $r^2 = \frac{-\beta \bar{u}_3^1(r^3) t - e_1^2 x^1 - e_2^2 x^2}{1 - \alpha(1+\kappa)t}$ $r^3 = x^3 - u_0^3 t$	$\bar{a} = \frac{\alpha((e_1^1 + e_1^2)x^1 + (e_2^1 + e_2^2)x^2)}{1 - \alpha(1+\kappa)t}, \quad \bar{u}^3 = u_0^3$ $\bar{u}^1 = \frac{-\kappa \alpha ((e_1^1)^2 + (e_1^2)^2)x^1 + (e_1^1 e_2^1 + e_1^2 e_2^2)x^2 - \bar{u}_3^1(r^3)}{1 - \alpha(1+\kappa)t}$ $\bar{u}^2 = \kappa \alpha \left(\frac{e_2^1 (\beta \bar{u}_3^1(r^3) t - e_1^1 x^1 - e_2^2 x^2)}{1 - \alpha(1+\kappa)t} + \frac{e_2^2 (-\beta \bar{u}_3^1(r^3) t - e_1^2 x^1 - e_2^2 x^2)}{1 - \alpha(1+\kappa)t} \right) + \frac{e_2^2 - e_1^1}{e_1^1 - e_1^2} \bar{u}_3^1(r^3)$ $\alpha, u_0^3 \in \mathbb{R}$

Table 2. Rank-3 solutions. Unassigned unknown functions $a(\cdot), u(\cdot), \dots$ are arbitrary functions of their respective arguments.

N_0	Type	Vector Fields	Riemann Invariants	Solutions
1	$E_1 E_2 E_3$	$X_1 = \frac{\partial}{\partial x^3} + \frac{\sigma_1}{\beta_3} \frac{\partial}{\partial t} + \frac{\beta_1}{\beta_3} \frac{\partial}{\partial x^1} + \frac{\beta_2}{\beta_3} \frac{\partial}{\partial x^2}$ $\sigma_1 = -[\bar{e}^1, \bar{e}^2, \bar{e}^3]$ $\beta_i = (\bar{e}^2 \times \bar{e}^3)_i (a + \bar{e}^1 \cdot \bar{u})$ $+ (\bar{e}^1 \times \bar{e}^3)_i (a + \bar{e}^2 \cdot \bar{u})$ $+ (\bar{e}^1 \times \bar{e}^2)_i (a + \bar{e}^3 \cdot \bar{u})$	$r^i = (1 + \kappa) a_i(r^3) t - \bar{e}^i \cdot \bar{x}, i = 1, 2, 3$ $\bar{e}^i \cdot \bar{e}^j = -1/\kappa, i \neq j = 1, 2, 3$	$\bar{a} = \bar{a}_1(r^1) + \bar{a}_2(r^2) + \bar{a}_3(r^3)$ $\bar{u} = \kappa(\bar{e}^1 \bar{a}_1(r^1) + \bar{e}^2 \bar{a}_2(r^2) + \bar{e}^3 \bar{a}_3(r^3))$
2a	$E_1 S_1 S_2$	$X = e_1^2 \frac{\partial}{\partial x^1} + e_2^2 \frac{\partial}{\partial x^2}$	$r^1 = ((1 + k^{-1})f(r^1) + a_0 + u_0^3)t - x^3$ $r^2 = t - x^1 \sin g(r^2, r^3) + x^2 \cos g(r^2, r^3)$ $\frac{\partial r^3}{\partial t} + (f(r^1) + u_0^3) \frac{\partial r^3}{\partial x^3} = 0$	$\bar{a} = k^{-1}f(r^1) + a_0, \quad \bar{u}^1 = \sin g(r^2, r^3)$ $\bar{u}^2 = -\cos g(r^2, r^3), \quad \bar{u}^3 = f(r^1) + u_0^3$ $a_0, u_0^3 \in \mathbb{R}$
2b	$E_1 S_1 S_2$	$X = e_1^2 \frac{\partial}{\partial x^1} + e_2^2 \frac{\partial}{\partial x^2}$	$r^1 = \frac{1 - (1+k^{-1})At}{(1+k^{-1})B + a_0 + u_0^3} t - x^3$ $r^2 = t - x^1 \sin g(r^2, r^3) + x^2 \cos g(r^2, r^3)$ $r^3 = \Psi \left[\frac{1}{A} (A(ka_0 - u_0^3)t + x^3 - ka_0 - B) \right] ((1+k)At - k)^{-k/k+1}$	$\bar{a} = k^{-1}(Ar^1 + B) + a_0,$ $\bar{u}^1 = \sin g(r^2, r^3), \bar{u}^2 = -\cos g(r^2, r^3)$ $\bar{u}^3 = Ar^1 + B + u_0^3, \quad a_0, u_0^3 \in \mathbb{R}$
2c	$E_1 S_1 S_2$	$X = \frac{\partial}{\partial x^3}$	$r^1 = (k^{-1}f(r^1) + a_0)t - x^1 \cos f(r^1) - x^2 \sin f(r^1)$ $r^2 = -t \cos f(r^1) - x^2$ $r^3 = -t \sin f(r^1) + x^1$	$\bar{a} = k^{-1}f(r^1) + a_0, \quad \bar{u}^1 = \sin f(r^1)$ $\bar{u}^2 = -\cos f(r^1), \quad a_0 \in \mathbb{R}$ $\bar{u}^3 = g(r^2 \cos f(r^1) + r^3 \sin f(r^1))$

Table 3 : Bounded real solutions for the nonscattering solution $E_1 E_2 E_3$ obtained by submitting the arbitrary functions to the various reductions (2.5)-(2.8) of the Klein-Gordon equation (2.1).

no	Riemann invariants	Solution	Type and comments
1	$r^i = -(1 + \kappa)$	$a = \sum_{i=1}^3 \frac{C_i}{\left(\wp\left(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}C_i^4\right) + \frac{1}{3}\right)^{1/2}} t + \bar{\lambda}^i \cdot \vec{x}$ $\vec{u} = \kappa \sum_{i=1}^3 \frac{C_i \bar{\lambda}^i}{\left(\wp\left(r^i, \frac{4}{3}, \frac{8}{27} + \frac{4}{3}C_i^4\right) + \frac{1}{3}\right)^{1/2}}$	Periodic solution $C_i \in \mathbb{R}$
2a	$r^i = -(1 + \kappa)$	$a = \sum_{i=1}^3 \left(\frac{C_i}{\wp\left(r^i, 0, \frac{4C_i^2}{3}\right)} \right)^{1/2} t + \bar{\lambda}^i \cdot \vec{x}$ $\vec{u} = \kappa \sum_{i=1}^3 \left(\frac{C_i}{\wp\left(r^i, 0, \frac{4C_i^2}{3}\right)} \right)^{1/2}, \quad \vec{u} = \kappa \sum_{i=1}^3 \left(\frac{C_i}{\wp\left(r^i, 0, \frac{4C_i^2}{3}\right)} \right)^{1/2} \bar{\lambda}^i$	Periodic Solution $C_i > 0$
2b	$r^i = -(1 + \kappa)$ $\zeta_i = 3k_0(r^i)^{1/3}$	$a = \sum_{i=1}^3 \left(\frac{C_i (r^i)^{-2/3}}{\wp(\zeta_i, 0, \frac{4C_i^2}{3k_0^2})} \right)^{1/2} t + \bar{\lambda}^i \cdot \vec{x}$ $\vec{u} = \kappa \sum_{i=1}^3 \left(\frac{C_i (r^i)^{-2/3}}{\wp(\zeta_i, 0, \frac{4C_i^2}{3k_0^2})} \right)^{1/2}, \quad \vec{u} = \kappa \sum_{i=1}^3 \left(\frac{C_i (r^i)^{-2/3}}{\wp(\zeta_i, 0, \frac{4C_i^2}{3k_0^2})} \right)^{1/2} \bar{\lambda}^i$	Bump $k_0 \in \mathbb{R}, C_i > 0$
2c	$r^i = -(1 + \kappa)$	$a = \sum_{i=1}^3 \left(\frac{C_i (r^i)^{-1}}{\left(\wp(\zeta_i, 12e_0^2, -8e_0^3 + 16C_i^2 e_0) - e_0\right)} \right)^{1/2} t + \bar{\lambda}^i \cdot \vec{x}$ $\vec{u} = k_0 \ln r^i$	Bump $e_0 \in \mathbb{R}, C_i > 0$
3	$r^i = -(1 + \kappa)$ $\zeta_i = r^i {}_2F_1\left(\frac{1}{2}, \frac{5}{6}; \frac{3}{2}; -(r^i)^2\right)$	$a = \sum_{i=1}^3 \left(\frac{C_i ((r^i)^2 + 1)^{-1/3}}{\wp\left(\zeta_i, 0, \frac{4C_i^2}{3k_0^2}\right)} \right)^{1/2} t + \bar{\lambda}^i \cdot \vec{x}$ $\vec{u} = \kappa \sum_{i=1}^3 \left(\frac{C_i ((r^i)^2 + 1)^{-1/3}}{\wp\left(\zeta_i, 0, \frac{4C_i^2}{3k_0^2}\right)} \right)^{1/2} \bar{\lambda}^i$	Kink $k_0 \in \mathbb{R}, C_i > 0$