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## Relativistic Landau resonances

E. A. Evangelidis

Laboratory of Non-Conventional Sources of Energy, Demokritos University of Thrace, Xanthi, Greece

G. J. J. Botha

Department of Applied Mathematics, University of Leeds, Leeds, UK

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[1] The possible interactions between plasma waves and relativistic charged particles are considered. An electromagnetic perturbation in the plasma is formulated as an elliptically polarized wave, and the collisionless plasma is described by a distribution in phase space, which is realized in cylindrical coordinates. The linearized Vlasov equation is solved in the semi-relativistic limit, to obtain the distribution function in the rest frame of the observer. The perturbed currents supported by the ionized medium are then calculated, so that an expression can be written for the total amount of energy available for transfer through the Landau mechanism. It is found that only certain modes of the perturbed current are available for this energy transfer. The final expressions are presented in terms of Stokes parameters, and applied to the special cases of a thermal as well as a nonthermal plasma. The thermal plasma is described by a Maxwellian distribution, while two nonthermal distributions are considered: the kappa distribution and a generalized Weibull distribution.

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### 1. Introduction

[2] Highly energetic particles are prevalent in space and laboratory plasmas. Relativistic particles flowing from pulsars [Melrose, 2003] have been studied through extensive observations of the Crab nebula [Oort and Walraven, 1956; Bogovalov and Khangoulyan, 2002; Mori et al., 2004]. Relativistic protons and electrons accelerated by solar flares and coronal mass ejections [Klein et al., 2001; Maia et al., 2001; Reames, 2002], as well as high-energy electrons in Jupiter's radiation belt [Bolton et al., 2002] and in the Earth's inner magnetosphere [Friedel et al., 2002] and radiation belt [Li and Temerin, 2001; Lorentzen et al., 2001], have been observed in the solar system and near-Earth environment. Together with the presence of high-energy alpha particles in fusion plasmas [Hawryluk, 1998; Perkins et al., 1999; Aymar et al., 2002; Testa et al., 2004], these occurrences of highly energetic particles add to the interest in the interaction of the particles with their plasma environment.

[3] The same theoretical framework is used in the laboratory and in space to describe the particle interaction with waves in the surrounding plasma [Stix, 1962]. The idea of slowing down highly energetic particles by means of Alfvén waves was proposed by, among others, Kahn [1971] in connection with the Crab nebula. In the Earth's magnetosphere, Horne and Thorne [1998] identified electromagnetic ion cyclotron (EMIC) waves, oblique magnetosonic waves,

and whistler waves as possible wave modes capable of resonating with electrons. EMIC waves were identified as a possible cause of pitch angle scatter [Summers et al., 1998], while whistler waves may provide a mechanism to accelerate electrons to relativistic energies as well as to affect the electron scatter rates [Horne and Thorne, 2003; Summers et al., 2004]. Diffusion rates of radiation belt electrons are known to be a function of the electron energy [Horne et al., 2003], and as such are also influenced by the resonances between electrons and the plasma waves. As whistler waves appear to be highly effective in resonating with radiation belt electrons [Meredith et al., 2003], these wave-particle interactions have been modeled numerically in various studies [Summers and Ma, 2000; Summers et al., 2004; Omura and Summers, 2004].

[4] In this paper we would like to re-examine the role of electromagnetic waves in exchanging energy with near-relativistic charged particles (e.g., protons, electrons, and alpha particles) in tenuous media, where collisions are an unlikely mechanism for the transfer of energy and momentum. In the literature, Landau resonances are described analytically by explicitly using the dielectric tensor, which quickly becomes a monumental task [Stix, 1962; Miyamoto, 1989]. Rather than follow this route, we use a known identity for Bessel functions (see equation (42)), to simplify the expressions. The final answer, written in terms of Stokes parameters, shows that only certain modes of the perturbed currents supported by the ionized medium are available for energy transfer through the Landau mechanism. The analytical results presented in this paper will provide valuable insight when applied to (among various applications) the

study of resonant interactions between waves and energetic particles in the Earth's radiation belt.

[5] In the next section the required expressions for such an energy transfer are derived. This analysis is performed in the semi-relativistic limit, which is then compared to the classical case in section 3. The general expressions are then applied to the special cases of a thermalized plasma (section 4) as well as a nonthermalized plasma (section 5). Two nonthermal distributions are considered: the kappa distribution and a generalization of the Weibull distribution. Finally, a comparison is made between positive and negative charged particles.

## 2. Relativistic Plasma

[6] A relativistic plasma distribution in a static, uniform magnetic field  $\mathbf{H}_0 = (0, 0, H_0)$  without an externally applied electric field can be described by the collisionless Boltzmann (or Vlasov) equation,

$$\frac{\partial f}{\partial t} + \frac{c^2}{\mathcal{E}} \mathbf{p} \cdot \frac{\partial f}{\partial \mathbf{r}} + \dot{\mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0, \quad (1)$$

where  $f$  is the distribution of particles in phase space,  $\mathbf{p}$  is the relativistic momentum,  $c$  is the speed of light, and with the relativistic energy given by

$$\mathcal{E} = \frac{mc^2}{\sqrt{1 - (v/c)^2}}, \quad (2)$$

where  $m$  is the rest mass of the particles moving at velocity  $v$ . Equation (1) is expressed in cylindrical coordinates; that is, the position vector becomes

$$\mathbf{r} = r \cos \phi \mathbf{e}_1 + r \sin \phi \mathbf{e}_2 + z \mathbf{e}_3, \quad (3)$$

and the momentum becomes

$$\mathbf{p} = p_{\perp} \cos \phi \mathbf{e}_1 + p_{\perp} \sin \phi \mathbf{e}_2 + p_{\parallel} \mathbf{e}_3, \quad (4)$$

where the perpendicular and parallel components refer to the direction relative to the magnetic field  $\mathbf{H}_0$ . An electromagnetic perturbation will be described by the vector potential

$$\mathbf{A} = \mathbf{A}_k e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \quad \mathbf{A}_k = (a_1, ia_2, 0). \quad (5)$$

This is the expression for a polarized monochromatic plane wave, with wave vector  $\mathbf{k} = (k_1, 0, k_3)$  and the frequency  $\omega$  determined by the dispersion relation of the type of wave under consideration. The Lorentz force can be written as

$$\begin{aligned} \dot{\mathbf{p}}_0 &= \frac{qc}{\mathcal{E}} \mathbf{p}_0 \times \mathbf{H}_0 \\ &= \frac{qcH_0}{\mathcal{E}} (p_{\perp} \sin \phi, -p_{\perp} \cos \phi, 0), \end{aligned} \quad (6)$$

$$\dot{\mathbf{p}}_1 = q \left[ -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} + \frac{c}{\mathcal{E}} \mathbf{p}_0 \times (\nabla \times \mathbf{A}) \right], \quad (7)$$

in the zero (unperturbed) and the first-order (perturbed) approximation. The notation  $q$  contains the sign and size of a particle's charge. The velocity in the unperturbed state is given by

$$\mathbf{v}_0 = \frac{c^2}{\mathcal{E}} \mathbf{p}_0. \quad (8)$$

The steady state Vlasov equation (1)

$$\mathbf{v}_0 \cdot \frac{\partial f_0}{\partial \mathbf{r}} + \dot{\mathbf{p}}_0 \cdot \frac{\partial f_0}{\partial \mathbf{p}} = 0 \quad (9)$$

is written, with the help of equations (4) and (6), in the form

$$\mathbf{v}_0 \cdot \frac{\partial f_0}{\partial \mathbf{r}} - \Omega \frac{\partial f_0}{\partial \phi} = 0, \quad (10)$$

where the definition

$$\Omega = \frac{qcH_0}{\mathcal{E}} \quad (11)$$

has been used. From this it is concluded that a coordinate independent distribution function, which is also axisymmetric in momentum space, is a solution. Let  $f[(p'_{\perp})^2, (p'_{\parallel})^2]$  be such a function in a frame of reference moving with velocity  $v_R$  with respect to an observer considered at rest. In the frame of the observer this becomes

$$f_0(p^2) = f_0 \left[ p_{\perp}^2, \frac{(p_{\parallel} - \mathcal{E}v_R/c^2)^2}{1 - v_R^2/c^2} \right] \quad (12)$$

for motion parallel to the magnetic field [Kahn, 1971; Landau and Lifshitz, 1975]. Since  $v_R/c \ll 1$ , a Taylor expansion gives

$$\begin{aligned} f_0(p^2) &= f_0(p_{\perp}^2, p_{\parallel}^2) - \frac{2p_{\parallel} \mathcal{E} v_R}{c^2} f_0'(p_{\perp}^2, p_{\parallel}^2) \\ &+ \left[ \left( p_{\parallel}^2 + \frac{\mathcal{E}^2}{c^2} \right) f_0'(p_{\perp}^2, p_{\parallel}^2) + \frac{2p_{\parallel}^2 \mathcal{E}^2}{c^2} f_0''(p_{\perp}^2, p_{\parallel}^2) \right] \left( \frac{v_R}{c} \right)^2 \\ &+ \mathcal{O} \left( \frac{v_R}{c} \right)^3, \end{aligned} \quad (13)$$

where we have used a binomial expansion for the denominator, and

$$f_0' = \frac{\partial f_0}{\partial p^2}. \quad (14)$$

Only first-order accuracy is maintained in this analysis; that is, only the first two terms of expansion (13) are used. With this ground state, the Vlasov equation (1) is linearized as

$$\frac{\partial f_1}{\partial t} + \frac{c^2}{\mathcal{E}} \mathbf{p}_0 \cdot \frac{\partial f_1}{\partial \mathbf{r}} + \dot{\mathbf{p}}_0 \cdot \frac{\partial f_1}{\partial \mathbf{p}} + \dot{\mathbf{p}}_1 \cdot \frac{\partial f_0}{\partial \mathbf{p}} = 0, \quad (15)$$

where  $f_1$  is the perturbed distribution. Assume that  $f_1$  takes the form

$$f_1 = f(\phi) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \quad (16)$$

where  $\mathbf{k}$  is the wave vector used in perturbation (5). Substituting all the above expressions in the linearized Vlasov equation (15), the differential equation for the perturbed distribution function

$$\frac{df'(\phi)}{d\phi} - i(\chi \cos \phi + \mu)f'(\phi) - Q(\phi) = 0 \quad (17)$$

is obtained, where

$$\chi = \left(\frac{ck_1}{qH_0}\right)p_{\perp} = \beta p_{\perp}, \quad (18)$$

$$\mu = \frac{ck_3}{qH_0} \left(p_{\parallel} - \frac{\omega \mathcal{E}}{c^2 k_3}\right), \quad (19)$$

$$Q(\phi) = i \frac{2qk_3}{c\Omega} \left(\frac{\omega}{k_3} - v_R\right) (a_1 \cos \phi + ia_2 \sin \phi) p_{\perp} f'_0. \quad (20)$$

By solving this first-order linear ODE in the standard way, i.e.,

$$f(\phi) = e^{g(\phi)} \left[ C_1 + \int Q(\phi) e^{-g(\phi)} d\phi \right], \quad (21)$$

where  $C_1$  is the integration constant and

$$g(\phi) = i(\chi \sin \phi + \mu\phi), \quad (22)$$

as well as employing the expansions

$$e^{\pm i\chi \sin \phi} = \sum_{m=-\infty}^{\infty} J_m(\chi) e^{\pm im\phi}, \quad (23)$$

the final result

$$f(\phi) = \frac{i2qk_3}{c\Omega} \left(\frac{\omega}{k_3} - v_R\right) f'_0 p_{\perp} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{J_m(\chi) J_n(\chi)}{1 - (m + \mu)^2} e^{i(n-m)\phi} \cdot (A_m \sin \phi - iB_m \cos \phi) \quad (24)$$

is obtained, where the definitions

$$A_m = a_1 + a_2(m + \mu) \quad (25)$$

$$B_m = a_1(m + \mu) + a_2 \quad (26)$$

have been used. The constant of integration  $C_1$  was found to be zero under the boundary condition  $f(\phi + 2\pi) = f(\phi)$ . As an aside, it should be noted that *Lifshitz and Pitaevskii* [1981, equation (53.11)] solved ODE (17) by placing the integration constant in the integration limits of (21). This leads to unnecessary complications where Landau's causality argument has to be invoked, together with the periodicity of  $f(\phi)$ , in order to determine  $C_1$ . Solution (24) is the expression for the perturbed distribution function, to be used further in the evaluation of the currents supported

by the ionized medium, the statistical properties of which are determined by the function  $f_0$ .

[7] The current produced by charge  $q$  moving with velocity  $\mathbf{v}$  is

$$q\mathbf{v} = \frac{qc^2}{\mathcal{E}} \mathbf{p}. \quad (27)$$

Integrating the product of this and the perturbed distribution (16) over momentum space, we obtain the associated current distribution

$$\mathbf{j} = \frac{i2qk_3}{H_0} \left(\frac{\omega}{k_3} - v_R\right) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \int_0^{\infty} dp_{\perp} \int_{-\infty}^{\infty} dp_{\parallel} \int_0^{2\pi} d\phi \cdot \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f'_0 \frac{J_n(\chi) J_m(\chi)}{1 - (m + \mu)^2} \cdot e^{i(n-m)\phi} (A_m \sin \phi - iB_m \cos \phi) \begin{bmatrix} p_{\perp}^3 \cos \phi \\ p_{\perp}^3 \sin \phi \\ p_{\perp}^2 p_{\parallel} \end{bmatrix}, \quad (28)$$

where (4) and (24) have been used. The orthogonality of the trigonometric functions shows that the only surviving currents are those for the following combinations of the indices  $m$  and  $n$ :

$$n - m = 0, \pm 1, \pm 2. \quad (29)$$

The corresponding currents are indicated with respective superscripts in brackets. Thus they are

$$\mathbf{j}^{(0)} = 2\kappa \int_0^{\infty} dp_{\perp} \int_{-\infty}^{\infty} dp_{\parallel} f'_0 p_{\perp}^3 \sum_{m=-\infty}^{\infty} \frac{J_m^2(\chi)}{1 - (m + \mu)^2} \begin{bmatrix} B_m \\ iA_m \\ 0 \end{bmatrix}, \quad (30)$$

$$\mathbf{j}^{(+1)} = -2\kappa(a_1 - a_2) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \int_0^{\infty} dp_{\perp} \int_{-\infty}^{\infty} dp_{\parallel} \cdot f'_0 p_{\perp}^2 p_{\parallel} \sum_{m=-\infty}^{\infty} \frac{J_m(\chi) J_{m+1}(\chi)}{m + \mu + 1}, \quad (31)$$

$$\mathbf{j}^{(-1)} = -2\kappa(a_1 + a_2) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \int_0^{\infty} dp_{\perp} \int_{-\infty}^{\infty} dp_{\parallel} \cdot f'_0 p_{\perp}^2 p_{\parallel} \sum_{m=-\infty}^{\infty} \frac{J_{m-1}(\chi) J_m(\chi)}{m + \mu - 1}, \quad (32)$$

$$\mathbf{j}^{(+2)} = -\kappa(a_1 - a_2) \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \int_0^{\infty} dp_{\perp} \int_{-\infty}^{\infty} dp_{\parallel} \cdot f'_0 p_{\perp}^3 \sum_{m=-\infty}^{\infty} \frac{J_m(\chi) J_{m+2}(\chi)}{m + \mu + 1}, \quad (33)$$

$$\mathbf{j}^{(-2)} = -\kappa(a_1 + a_2) \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} \int_0^\infty dp_\perp \int_{-\infty}^\infty dp_\parallel$$

$$\cdot f_0' p_\perp^3 \sum_{m=-\infty}^\infty \frac{J_{m-2}(\chi) J_m(\chi)}{m + \mu - 1}, \quad (34)$$

where the abbreviation

$$\kappa = \frac{\pi q k_3}{H_0} \left( \frac{\omega}{k_3} - v_R \right) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \quad (35)$$

has been used. The electric field consistent with the vector potential (5) is

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = \frac{\omega}{c} (ia_1, -a_2, 0) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \quad (36)$$

and its complex conjugate

$$\mathbf{E}^* = -\frac{\omega}{c} (ia_1, a_2, 0) e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \quad (37)$$

which is to be used below. Notice that the electric field is perpendicular to the  $\pm 1$  currents. Therefore only the 0 and  $\pm 2$  modes are available to do work. In order to calculate the work available in each channel, we form the scalar product of the currents with the complex conjugate electric field. The results are summarized below:

$$\mathbf{j}^{(0)} \cdot \mathbf{E}^* = \kappa_E \int_0^\infty dp_\perp \int_{-\infty}^\infty dp_\parallel f_0' p_\perp^3$$

$$\cdot \sum_{m=-\infty}^\infty J_m^2(\chi) \left[ \frac{(a_1 + a_2)^2}{m + \mu - 1} + \frac{(a_1 - a_2)^2}{m + \mu + 1} \right], \quad (38)$$

$$\mathbf{j}^{(+2)} \cdot \mathbf{E}^* = (a_1^2 - a_2^2) \kappa_E \int_0^\infty dp_\perp \int_{-\infty}^\infty dp_\parallel$$

$$\cdot f_0' p_\perp^3 \sum_{m=-\infty}^\infty \frac{J_m(\chi) J_{m+2}(\chi)}{m + \mu + 1}, \quad (39)$$

$$\mathbf{j}^{(-2)} \cdot \mathbf{E}^* = (a_1^2 - a_2^2) \kappa_E \int_0^\infty dp_\perp \int_{-\infty}^\infty dp_\parallel$$

$$\cdot f_0' p_\perp^3 \sum_{m=-\infty}^\infty \frac{J_{m-2}(\chi) J_m(\chi)}{m + \mu - 1}, \quad (40)$$

where we have used the abbreviation

$$\kappa_E = \frac{i\pi q k_3 \omega}{c H_0} \left( \frac{\omega}{k_3} - v_R \right). \quad (41)$$

Using the identity [Evangelidis, 1973, 1977a, 1977b, 1978, 1984]

$$\sum_{m=-\infty}^\infty \frac{J_{m+p}(\chi) J_m(\chi)}{m - \mu - q} = \frac{-\pi J_{p+\mu+q}(\chi) J_{-(\mu+q)}(\chi)}{\sin[(\mu+q)\pi]} \quad (42)$$

in the expressions above, they simplify to

$$\mathbf{j}^{(0)} \cdot \mathbf{E}^* = \pi \kappa_E \int_0^\infty dp_\perp \int_{-\infty}^\infty dp_\parallel f_0' p_\perp^3$$

$$\cdot \left[ (a_1 + a_2)^2 \frac{J_{-(\mu-1)}(\chi) J_{(\mu-1)}(\chi)}{\sin[(\mu-1)\pi]} \right.$$

$$\left. + (a_1 - a_2)^2 \frac{J_{-(\mu+1)}(\chi) J_{(\mu+1)}(\chi)}{\sin[(\mu+1)\pi]} \right], \quad (43)$$

$$\mathbf{j}^{(+2)} \cdot \mathbf{E}^* = \pi \kappa_E (a_1^2 - a_2^2) \int_0^\infty dp_\perp \int_{-\infty}^\infty dp_\parallel$$

$$\cdot f_0' p_\perp^3 \frac{J_{-(\mu-1)}(\chi) J_{(\mu+1)}(\chi)}{\sin[(\mu+1)\pi]}, \quad (44)$$

$$\mathbf{j}^{(-2)} \cdot \mathbf{E}^* = \pi \kappa_E (a_1^2 - a_2^2) \int_0^\infty dp_\perp \int_{-\infty}^\infty dp_\parallel$$

$$\cdot f_0' p_\perp^3 \frac{J_{-(\mu+1)}(\chi) J_{(\mu-1)}(\chi)}{\sin[(\mu-1)\pi]}. \quad (45)$$

These integrals have singularities at

$$\mu + 1 = M, \quad \mu - 1 = N, \quad (46)$$

where  $M$  and  $N$  are integers. Bessel functions with lowest order have the highest amplitude and will dominate the results. Thus only  $M = N = 0$  will be used in the present analysis.

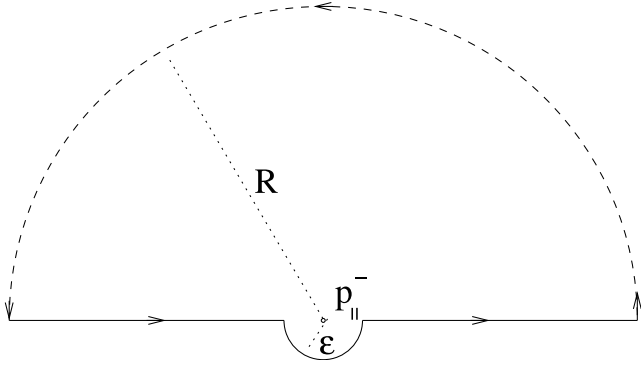
[8] Expressions (44) and (45) give the contribution of the poles  $\mu \pm 1 \rightarrow 0$  as the integral goes from  $-\infty$  to  $\infty$ , in terms of the same Stokes parameter. See *Befeki* [1966] for a definition of Stokes parameters. In contrast, in expression (43) the contribution of each pole is associated with a different Stokes parameter. Expressions (43) to (45) are quite general and can be used in the computation of the transfer of energy through the Landau mechanism. They are particularly suited to the calculation of the integrals over the poles for contours satisfying the causality requirement, that is, going around the poles in a counterclockwise sense in the lower imaginary half plane (Figure 1). To avoid unnecessary repetitions, the idea is presented in the evaluation of the integral in expression (44) only. First it is noticed that as the points on the real axis approach the pole, the sine terms can be represented by their arguments, so that

$$\lim_{\mu+1 \rightarrow 0} \int_{-\infty}^\infty \frac{\pi f(p_\perp, p_\parallel)}{\sin[(\mu+1)\pi]} dp_\parallel = \lim_{\mu \rightarrow -1} \int_{-\infty}^\infty \frac{f(p_\perp, p_\parallel)}{\mu+1} dp_\parallel$$

$$= \lim_{p_\parallel \rightarrow p_\parallel^-} \frac{q H_0}{c k_3} \int_{-\infty}^\infty \frac{f(p_\perp, p_\parallel)}{p_\parallel - p_\parallel^-} dp_\parallel \quad (47)$$

upon the use of definition (19). The poles will be signified by  $p_\parallel^+$  and  $p_\parallel^-$ , whereby

$$p_\parallel^\pm = \frac{\mathcal{E}}{c^2 k_3} (\omega \pm \Omega). \quad (48)$$



**Figure 1.** Integration path of  $\int_{-\infty}^{\infty} dp_{\parallel}$  in expression (44) around the singularity  $p_{\parallel}^{-}$ , with  $\varepsilon \rightarrow 0$ . The contribution from the dashed semicircle is zero when  $R \rightarrow \infty$ . For further reading, see *Boyd and Sanderson* [2003].

Thus integration along the path of Figure 1 gives

$$PV \int_{-\infty}^{\infty} \frac{f(p_{\perp}, p_{\parallel})}{\sin[(\mu+1)\pi]} dp_{\parallel} + \frac{qH_0}{ck_3} \left[ i\pi f(p_{\perp}, p_{\parallel}^{-}) \right],$$

where  $PV$  indicates the Cauchy principal value. The principal value represents the irreversible Joule heating, i.e., the energy transfer from the wave (field) to the particles (current). The second term represents the energy transferred through the Landau mechanism, which can flow in either direction between wave and particles. The total amount of energy available for transfer is thus

$$\begin{aligned} W &= \frac{1}{2} [\mathbf{j}^{(0)} + \mathbf{j}^{(-2)} + \mathbf{j}^{(+2)}] \cdot \mathbf{E}^* \\ &= W_J + W_L, \end{aligned} \quad (49)$$

where  $W_J$  is the energy going into the Joule heating, and where the amount of energy available for transfer through the Landau mechanism is

$$\begin{aligned} W_L &= -\frac{\pi^2 q^2 \omega}{2c^2} \left( \frac{\omega}{k_3} - v_R \right) \int_0^{\infty} dp_{\perp} p_{\perp}^3 \\ &\quad \cdot \left[ (a_1 + a_2)^2 J_0^2(\chi) f_0' \left[ p_{\perp}^2, (p_{\parallel}^+)^2 \right] \right. \\ &\quad + (a_1 - a_2)^2 J_0^2(\chi) f_0' \left[ p_{\perp}^2, (p_{\parallel}^-)^2 \right] \\ &\quad + (a_1^2 - a_2^2) J_0(\chi) J_2(\chi) \\ &\quad \left. \cdot \left\{ f_0' \left[ p_{\perp}^2, (p_{\parallel}^+)^2 \right] + f_0' \left[ p_{\perp}^2, (p_{\parallel}^-)^2 \right] \right\} \right]. \end{aligned} \quad (50)$$

This is the sought for expression under quite general assumptions.

### 3. Nonrelativistic Plasma

[9] For a nonrelativistic plasma, equation (1) is valid but with the energy given by  $E = mc^2$  instead of (2), and with a distribution  $f_0(p_{\perp}^2, p_{\parallel}^2)$  instead of (12). Assuming the same

perturbation as (5) and (16), we obtain ODE (17) with  $Q$  defined as

$$Q(\phi) = i \frac{2q\omega}{c\Omega} (a_1 \cos \phi + ia_2 \sin \phi) p_{\perp} f_0'. \quad (51)$$

From this it is clear that the rest of the analysis can be obtained by setting  $v_R = 0$ , so that the final answer is given by

$$\begin{aligned} W_L &= -\frac{\pi^2 q^2 \omega^2}{2c^2 k_3} \int_0^{\infty} dp_{\perp} p_{\perp}^3 \left[ (a_1 + a_2)^2 J_0^2(\chi) f_0' \left[ p_{\perp}^2, (p_{\parallel}^+)^2 \right] \right. \\ &\quad + (a_1 - a_2)^2 J_0^2(\chi) f_0' \left[ p_{\perp}^2, (p_{\parallel}^-)^2 \right] \\ &\quad + (a_1^2 - a_2^2) J_0(\chi) J_2(\chi) \\ &\quad \left. \cdot \left\{ f_0' \left[ p_{\perp}^2, (p_{\parallel}^+)^2 \right] + f_0' \left[ p_{\perp}^2, (p_{\parallel}^-)^2 \right] \right\} \right]. \end{aligned} \quad (52)$$

### 4. Thermal Plasma

[10] To employ the derived expressions in the calculation of the energy transferable between particles and waves, one has to specify the thermal state of the plasma under consideration. For an axisymmetric plasma in thermal equilibrium, the distribution function in the rest frame of the observer is a Maxwellian

$$f_0 = \frac{N}{(2\pi m k_B T)^{3/2}} \exp\left(-\frac{p^2}{2m k_B T}\right), \quad (53)$$

where  $N$  is the number of particles per unit volume,  $k_B$  is the Boltzmann constant, and  $T$  is the absolute temperature. Its derivative, as defined by (14), is

$$f_0' = -\frac{N}{\sqrt{\pi^3 (2m k_B T)^5}} \exp\left(-\frac{p_{\perp}^2 + p_{\parallel}^2}{2m k_B T}\right). \quad (54)$$

This becomes

$$f_0' |_{p_{\parallel}^{\pm}} = -\frac{N}{\sqrt{\pi^3 (2m k_B T)^5}} \Theta^{\pm} \exp\left(-\frac{p_{\perp}^2}{2m k_B T}\right) \quad (55)$$

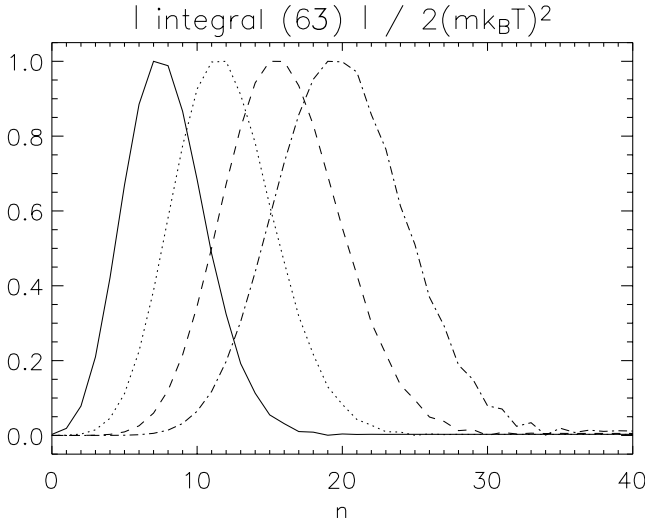
at resonances  $p_{\parallel}^+$  and  $p_{\parallel}^-$ , where

$$\Theta^{\pm} = \exp\left(-\frac{(p_{\parallel}^{\pm})^2}{2m k_B T}\right). \quad (56)$$

A substitution into (50) gives the available energy for transfer through the Landau mechanism as

$$\begin{aligned} W_L &= \frac{\pi N q^2 \omega}{2c^2 \sqrt{\pi (2m k_B T)^5}} \left( \frac{\omega}{k_3} - v_R \right) \int_0^{\infty} dp_{\perp} p_{\perp}^3 e^{-p_{\perp}^2/2m k_B T} \\ &\quad \cdot \left\{ (a_1^2 - a_2^2) (\Theta^+ + \Theta^-) J_0(\chi) J_2(\chi) \right. \\ &\quad \left. + [(a_1^2 + a_2^2) (\Theta^+ + \Theta^-) + 2a_1 a_2 (\Theta^+ - \Theta^-)] J_0^2(\chi) \right\}. \end{aligned} \quad (57)$$





**Figure 2.** Absolute convergence of integral (63) for different values of  $y$ : solid line is  $y = 2$ ; dotted line is  $y = 3$ ; dashed line is  $y = 4$ ; dash-dotted line is  $y = 5$ . The different curves have been normalized to fit the same scale, and  $n$  is the number of terms included in the series.

Using the values of the momentum at the two poles as given by (48), we obtain

$$\Theta^+ + \Theta^- = 2e^{-\gamma(\omega^2 + \Omega^2)} \cosh(2\gamma\omega\Omega), \quad (58)$$

$$\Theta^+ - \Theta^- = -2e^{-\gamma(\omega^2 + \Omega^2)} \sinh(2\gamma\omega\Omega), \quad (59)$$

where the definition

$$\gamma = \frac{1}{2mk_B T} \left( \frac{\mathcal{E}}{c^2 k_3} \right)^2 \quad (60)$$

has been employed. Consequently, expression (57) becomes

$$\begin{aligned} W_L = & \frac{\pi N q^2 \omega}{c^2 \sqrt{\pi(2mk_B T)^5}} \left( \frac{\omega}{k_3} - v_R \right) e^{-\gamma(\omega^2 + \Omega^2)} \\ & \cdot \int_0^\infty dp_\perp p_\perp^3 e^{-p_\perp^2/2mk_B T} \\ & \cdot \{ (a_1^2 - a_2^2) \cosh(2\gamma\omega\Omega) J_0(\chi) J_2(\chi) \\ & + [(a_1^2 + a_2^2) \cosh(2\gamma\omega\Omega) - 2a_1 a_2 \sinh(2\gamma\omega\Omega)] J_0^2(\chi) \}. \end{aligned} \quad (61)$$

The integral over the perpendicular component of the momentum is evaluated using result (A3) in Appendix A. With the abbreviation

$$\alpha = \frac{1}{2mk_B T} \quad (62)$$

and with  $\beta$  defined as in (18), the results

$$\int_0^\infty p_\perp^3 e^{-\alpha p_\perp^2} J_0^2(\beta p_\perp) dp_\perp = 2(mk_B T)^2 \sum_{n=0}^\infty \frac{(n+1)(2n)!}{(n!)^3} (-y)^n \quad (63)$$

$$\begin{aligned} & \int_0^\infty p_\perp^3 e^{-\alpha p_\perp^2} J_0(\beta p_\perp) J_2(\beta p_\perp) dp_\perp \\ & = 2(mk_B T)^2 y \sum_{n=0}^\infty \frac{(2n+2)!}{(n!)^2 (n+2)!} (-y)^n \end{aligned} \quad (64)$$

follow, where

$$y = \frac{\beta^2}{4\alpha} = \frac{k_B T / 2mc^2}{\Omega_0^2 / (ck_1)^2} = \left( \frac{ck_1}{\Omega_0} \right)^2 \frac{k_B T}{2mc^2} \quad (65)$$

$$\Omega_0 = \frac{qH_0}{mc}. \quad (66)$$

The frequency  $\Omega_0$  is the Larmor frequency in the nonrelativistic approximation. Keeping only the first two terms in the summation series, equation (61) becomes

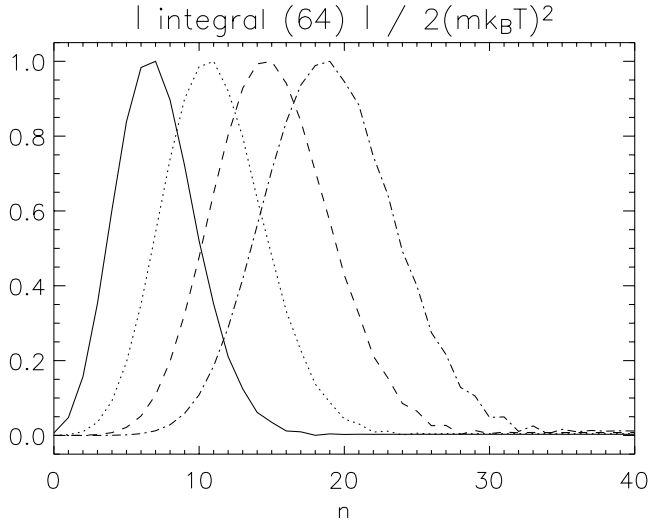
$$\begin{aligned} W_L = & \frac{\pi N q^2 \omega}{2c^2 \sqrt{2\pi} mk_B T} \left( \frac{\omega}{k_3} - v_R \right) e^{-\gamma(\omega^2 + \Omega^2)} \\ & \cdot \{ [(a_1^2 + a_2^2) + (a_1^2 - a_2^2)y] \cosh(2\gamma\omega\Omega) \\ & - 2a_1 a_2 \sinh(2\gamma\omega\Omega) \} (1 - 4y). \end{aligned} \quad (67)$$

The number of terms to be included is determined by the value of  $y$ . For  $y = 0.01$  we need five terms in integral (63) and four terms in integral (64) for the numerical results to change less than one unit at the sixth decimal place. For  $y = 0.1$  the same accuracy requires seven terms in both integrals, while  $y = 1$  requires 21 terms in (63) and 20 terms in (64). More terms are required to achieve the same accuracy as the value of  $y$  increases, as indicated by Figures 2 and 3. The physics behind the numerical values reflects the interplay of the thermal perturbations against the perturbations of the wave mode propagating in the plasma, as shown in (65).

## 5. Nonthermal Plasma

[11] It is well established that electron and ion distribution functions in numerous astrophysical situations are not Maxwellian. High-energy electron distribution tails with strong density gradients appear throughout the solar system [see *Owocki and Scudder*, 1983, and references therein], as well as in the distribution of fusion alpha particles in JET and the envisaged ITER [*Pam ela et al.*, 2003]. These nonthermal observations and measurements are often fitted by the so-called kappa distribution [*Owocki and Scudder*, 1983; *Hasegawa et al.*, 1985; *Scudder*, 1992; *Summers and Thorne*, 1992; *Saito et al.*, 2000; *Leubner and Schupfer*, 2002; *Leubner*, 2004], which resembles a Maxwellian at low-energy ranges but varies as a power law in its high-energy tail.

[12] Another versatile distribution that is widely used is the Weibull distribution. By changing the values of its parameters, one can make it assume the characteristics of other types of distributions, like the Rayleigh and exponential distributions. The Weibull and Rayleigh distributions are widely used in wind energy analysis [*Spera*, 1994; *Şahin and Aksakal*, 1999; *Ulgen and Hepbasli*, 2002], as well as in the study of waves and fluid flows [*Fonseca and Soares*, 2004; *Xu et al.*, 2004; *Mahdi and Ashkar*, 2004], the

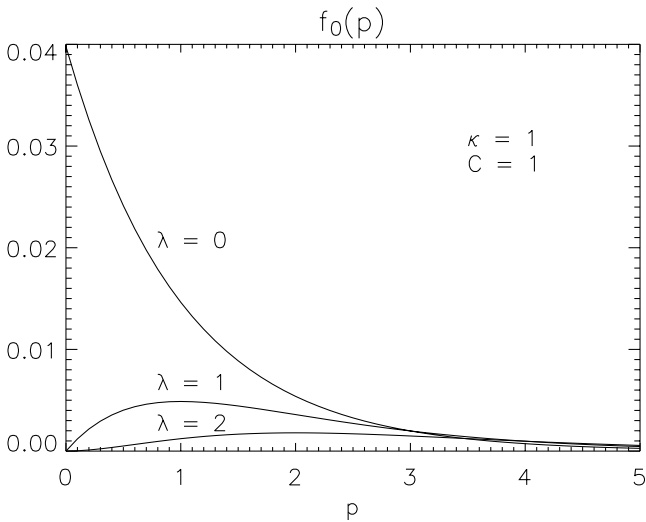


**Figure 3.** Absolute convergence of integral (64). The curves are normalized, and the notation of Figure 2 is used.

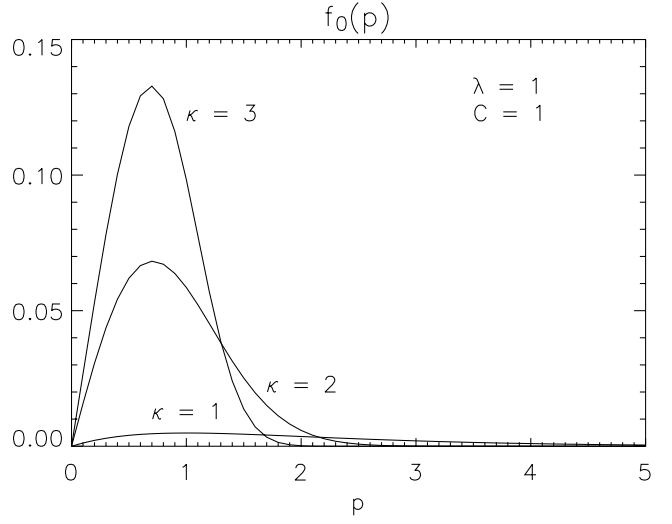
modeling of particle behavior in various fields [Gille, 1999; Ohtsuki and Emori, 2000; Cheng et al., 2003], and in the study of mesospheric [Meek et al., 2004] and material strength models [Zhao, 2004]. Given the considerable literature on Weibull distributions, we would like to introduce the more general class of functions

$$F(p) = (p/C)^\lambda \exp[-(p/C)^\kappa], \quad (68)$$

where  $\lambda$ ,  $C$ , and  $\kappa$  are constants and the momentum is given by  $p$ . For Weibull distributions,  $\lambda = \kappa - 1$ , for Rayleigh distributions,  $\lambda = 1$  and  $\kappa = 2$ , and for exponential distributions,  $\lambda = 0$  and  $\kappa = 1$ . The constant  $C$  is called the scale factor, whereas  $\kappa$  is the shape factor. In Figures 4–6 we explore the influence of  $\lambda$ ,  $\kappa$ , and  $C$  on distribution (68). In the analysis below, the kappa distribution and distribution



**Figure 4.** Influence of parameter  $\lambda$  on the generalized Weibull distribution (81). The curves are produced with  $\kappa = 1$  and  $C = 1$ .



**Figure 5.** Influence of parameter  $\kappa$  on the generalized Weibull distribution (81). The curves are produced with  $\lambda = 1$  and  $C = 1$ .

(68) will each be evaluated in the transfer of energy through Landau resonances.

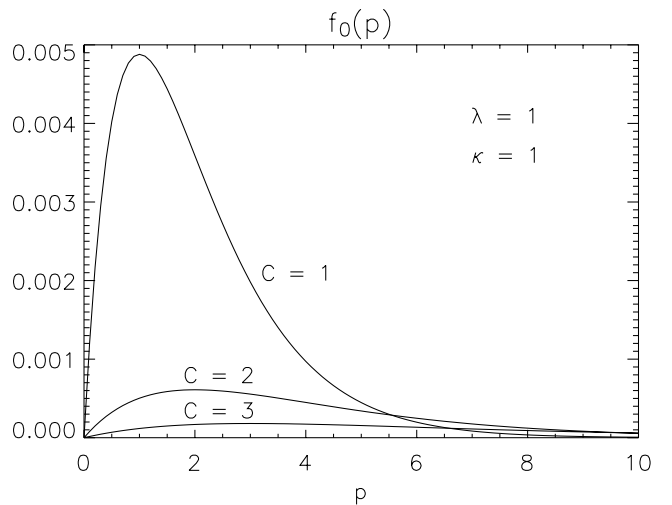
### 5.1. Kappa Distribution

[13] When normalized to the total number of particles  $N$  over the momentum volume, the kappa distribution takes on the form

$$f_0 = \frac{1}{\sqrt{\pi^3}} \frac{N}{C^3} \frac{1}{\sqrt{\kappa}} \frac{\Gamma(\kappa)}{\Gamma(\kappa - \frac{1}{2})} \left(1 + \frac{p^2}{\kappa C^2}\right)^{-(\kappa+1)}, \quad (69)$$

where  $C$  and  $\kappa$  are both constants and where the momentum is defined by  $p^2 = p_\perp^2 + p_\parallel^2$ . Its derivative, as defined by (14), is

$$f'_0 = -\frac{1}{\sqrt{\pi^3}} \frac{N}{C^5} \frac{\kappa+1}{\kappa^{3/2}} \frac{\Gamma(\kappa)}{\Gamma(\kappa - \frac{1}{2})} \left(1 + \frac{p^2}{\kappa C^2}\right)^{-(\kappa+2)}, \quad (70)$$



**Figure 6.** Influence of parameter  $C$  on the generalized Weibull distribution (81). The curves are produced with  $\lambda = 1$  and  $\kappa = 1$ .



which is evaluated at resonances  $p_{\parallel}^+$  and  $p_{\parallel}^-$ . The energy available for transfer through the Landau mechanism is given by (50), in which there are two types of integrals to consider,

$$I_A = \int_0^{\infty} J_0^2(\beta p_{\perp}) f_0' \left[ p_{\perp}^2, \left( p_{\parallel}^{\pm} \right)^2 \right] p_{\perp}^3 dp_{\perp}, \quad (71)$$

$$I_B = \int_0^{\infty} J_0(\beta p_{\perp}) J_2(\beta p_{\perp}) f_0' \left[ p_{\perp}^2, \left( p_{\parallel}^{\pm} \right)^2 \right] p_{\perp}^3 dp_{\perp}. \quad (72)$$

In order to evaluate these integrals, we integrate over  $p$  using the substitution  $p_{\perp}^2 = p^2 - (p_{\parallel}^{\pm})^2$ , where  $p_{\parallel}^{\pm}$  are treated as constants. This makes it necessary to evaluate the Bessel functions as well as  $f_0'$  at the resonance points  $p_{\parallel}^+$  and  $p_{\parallel}^-$ , which means that (50) has to be rewritten as

$$W_L = -\frac{\pi^2 q^2 \omega}{2c^2} \left( \frac{\omega}{k_3} - v_R \right) \cdot \left\{ (a_1 + a_2)^2 I_A|_{p_{\parallel}^+} + (a_1 - a_2)^2 I_A|_{p_{\parallel}^-} + (a_1^2 - a_2^2) [I_B|_{p_{\parallel}^+} + I_B|_{p_{\parallel}^-}] \right\}. \quad (73)$$

By making a series expansion of the Bessel functions, the Bessel products in integrals (71) and (72) become

$$J_0^2(\xi) = 1 - \frac{1}{2} \xi^2 + \frac{3}{2^5} \xi^4 - \frac{5}{3^2 \cdot 2^6} \xi^6 + \dots \quad (74)$$

$$J_0(\xi) J_2(\xi) = \frac{\xi^2}{8} \left[ 1 - \frac{1}{3} \xi^2 + \frac{5}{2^7} \xi^4 - \frac{7}{5 \cdot 3^2 \cdot 2^6} \xi^6 + \dots \right], \quad (75)$$

where  $\xi = \beta C \sqrt{\zeta^2 - (\zeta_{\pm})^2}$ . In order to simplify the notation, we use  $\zeta = p/C$  and  $\zeta_{\pm} = p_{\parallel}^{\pm}/C$ . In the analysis that follows, only the first two terms in series (74) and (75) will be retained. Using the integral definition of the beta function in terms of the trigonometric functions, as well as the relation between the beta and gamma functions, the results

$$I_A = \frac{1}{2\sqrt{\pi^3}} \frac{N}{C} \frac{1}{\sqrt{\kappa}} \frac{\Gamma(\kappa)}{\Gamma(\kappa - 1/2)} \cdot \left\{ \zeta_{\pm}^2 - 1 + \frac{(\beta C)^2}{2} \left[ \frac{2\kappa}{\kappa - 1} + \zeta_{\pm}^2 (\zeta_{\pm}^2 - 2) \right] \right\} \quad (76)$$

$$I_B = \frac{1}{16\sqrt{\pi^3}} \frac{N}{C} \frac{1}{\sqrt{\kappa}} \frac{\Gamma(\kappa)}{\Gamma(\kappa - 1/2)} \left\{ (\beta C)^2 \left[ \frac{2\kappa}{\kappa - 1} + \zeta_{\pm}^2 (\zeta_{\pm}^2 - 2) \right] - \frac{(\beta C)^4}{3} \left[ \frac{6\kappa}{\kappa - 1} \left( \zeta_{\pm}^2 - \frac{\kappa}{\kappa - 2} \right) + \zeta_{\pm}^4 (\zeta_{\pm}^2 - 3) \right] \right\} \quad (77)$$

are obtained. Substituting these results into (73), the final expression becomes

$$W_L = -\frac{\sqrt{\pi} q^2 \omega}{4c^2} \left( \frac{\omega}{k_3} - v_R \right) \frac{N}{C} \frac{1}{\sqrt{\kappa}} \frac{\Gamma(\kappa)}{\Gamma(\kappa - 1/2)} \cdot \left\{ (a_1 + a_2)^2 R_+ + (a_1 - a_2)^2 R_- + \frac{(\beta C)^2}{2} \frac{\kappa}{\kappa - 1} (5a_1^2 + 3a_2^2) + S \right\}, \quad (78)$$

where  $R_{\pm}$  and  $S$  are defined as

$$R_{\pm} = (\zeta_{\pm}^2 - 1) + \frac{(\beta C)^2}{2} \zeta_{\pm}^2 (\zeta_{\pm}^2 - 2), \quad (79)$$

$$S = \frac{(\beta C)^2}{8} (a_1^2 - a_2^2) \left\{ \zeta_+^2 (\zeta_+^2 - 2) + \zeta_-^2 (\zeta_-^2 - 2) - \frac{(\beta C)^2}{3} \left[ \frac{6\kappa}{\kappa - 1} \left( \zeta_+^2 + \zeta_-^2 - \frac{2\kappa}{\kappa - 2} \right) + \zeta_+^4 (\zeta_+^2 - 3) + \zeta_-^4 (\zeta_-^2 - 3) \right] \right\}. \quad (80)$$

## 5.2. Generalized Weibull Distribution

[14] When normalized to the total number of particles  $N$  over the momentum volume, distribution (68) takes on the form

$$f_0 = \frac{1}{4\pi} \left[ \Gamma \left( \frac{\lambda + 3}{\kappa} \right) \right]^{-1} \frac{\kappa N}{C^3} \left( \frac{p}{C} \right)^{\lambda} \exp \left[ - \left( \frac{p}{C} \right)^{\kappa} \right], \quad (81)$$

where  $p = \sqrt{p_{\perp}^2 + p_{\parallel}^2}$ . Its derivative, defined by (14), is

$$f_0' = \frac{1}{8\pi} \frac{N}{C^5} \frac{\kappa}{\Gamma \left( \frac{\lambda + 3}{\kappa} \right)} (\lambda - \kappa \zeta^{\kappa}) \zeta^{(\lambda - 2)} e^{-\zeta^{\kappa}}, \quad (82)$$

where we have used  $\zeta = p/C$  as before. This is evaluated at resonances  $\zeta_+$  and  $\zeta_-$ , with  $\zeta_{\pm} = p_{\parallel}^{\pm}/C$ . The energy available for transfer through the Landau mechanism is given by (73), which contains integrals (71) and (72). Using only the first two terms of series (74) and (75), as well as the integral definition of the gamma function, the results

$$I_A = -\frac{1}{4\pi} \frac{N}{C} \left[ \Gamma \left( \frac{\lambda + 3}{\kappa} \right) \right]^{-1} \left\{ \Gamma \left( \frac{\lambda + 2}{\kappa} \right) - (\beta C)^2 \cdot \left[ \Gamma \left( \frac{\lambda + 4}{\kappa} \right) - \zeta_{\pm}^2 \Gamma \left( \frac{\lambda + 2}{\kappa} \right) \right] \right\} \quad (83)$$

$$I_B = -\frac{1}{16\pi} \frac{N}{C} \left[ \Gamma \left( \frac{\lambda + 3}{\kappa} \right) \right]^{-1} \left\{ (\beta C)^2 \cdot \left[ \Gamma \left( \frac{\lambda + 4}{\kappa} \right) - \zeta_{\pm}^2 \Gamma \left( \frac{\lambda + 2}{\kappa} \right) \right] - \frac{(\beta C)^4}{2} \cdot \left[ \Gamma \left( \frac{\lambda + 6}{\kappa} \right) - 2\zeta_{\pm}^2 \Gamma \left( \frac{\lambda + 4}{\kappa} \right) + \zeta_{\pm}^4 \Gamma \left( \frac{\lambda + 2}{\kappa} \right) \right] \right\} \quad (84)$$

follow. Substituting these results into (73), the final expression becomes

$$W_L = \frac{\pi q^2 \omega}{8c^2} \left( \frac{\omega}{k_3} - v_R \right) \frac{N}{C} \left[ \Gamma \left( \frac{\lambda + 3}{\kappa} \right) \right]^{-1} \cdot \left\{ 2(a_1^2 + a_2^2) \left[ \Gamma \left( \frac{\lambda + 2}{\kappa} \right) - (\beta C)^2 \Gamma \left( \frac{\lambda + 4}{\kappa} \right) \right] + (\beta C)^2 \Gamma \left( \frac{\lambda + 2}{\kappa} \right) U - V \right\}, \quad (85)$$

where the definitions

$$U = (a_1 + a_2)^2 \zeta_+^2 + (a_1 - a_2)^2 \zeta_-^2 \quad (86)$$

$$\begin{aligned}
V &= \frac{(\beta C)^2}{4} (a_1^2 - a_2^2) \left\{ (\zeta_+^2 + \zeta_-^2) \Gamma\left(\frac{\lambda+2}{\kappa}\right) \right. \\
&\quad - 2\Gamma\left(\frac{\lambda+4}{\kappa}\right) + \frac{(\beta C)^2}{2} \left[ (\zeta_+^4 + \zeta_-^4) \Gamma\left(\frac{\lambda+2}{\kappa}\right) \right. \\
&\quad \left. \left. - 2(\zeta_+^2 + \zeta_-^2) \Gamma\left(\frac{\lambda+4}{\kappa}\right) + 2\Gamma\left(\frac{\lambda+6}{\kappa}\right) \right] \right\} \quad (87)
\end{aligned}$$

have been used.

## 6. Negative Charge Carriers

[15] In the previous sections the symbol  $q$  includes the sign and magnitude of the charge. If  $q$  is taken to be only the magnitude, then the analysis up until now concerns the resonances of positive charged particles with the wave. By writing explicitly  $-q$  for negative charges, the previous analysis can be repeated with the same definitions of  $\Omega$ ,  $\chi$ , and  $\mu$ , as given by (11), (18), and (19). Instead of ODE (17), we now solve for

$$\frac{df(\phi)}{d\phi} + i(\chi \cos \phi + \mu)f(\phi) - Q(\phi) = 0, \quad (88)$$

with  $Q$  defined by (20). This has the solution

$$\begin{aligned}
f(\phi) &= \frac{i2qk_3}{c\Omega} \left(\frac{\omega}{k_3} - v_R\right) f'_0 p_\perp \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{J_m(\chi) J_n(\chi)}{1 - (m + \mu)^2} \\
&\quad \cdot e^{i(m-n)\phi} (F_m \sin \phi + iG_m \cos \phi), \quad (89)
\end{aligned}$$

where the definitions

$$F_m = a_1 - a_2(m + \mu) \quad (90)$$

$$G_m = a_1(m + \mu) - a_2 \quad (91)$$

have been used. By following the previous analysis closely, we obtain the intermediate results

$$\begin{aligned}
\mathbf{j}^{(0)} \cdot \mathbf{E}^* &= \pi \kappa_E \int_0^\infty dp_\perp \int_{-\infty}^\infty dp_\parallel f'_0 p_\perp^3 \\
&\quad \cdot \left[ (a_1 + a_2)^2 \frac{J_{-(\mu+1)}(\chi) J_{(\mu+1)}(\chi)}{\sin[(\mu+1)\pi]} \right. \\
&\quad \left. + (a_1 - a_2)^2 \frac{J_{-(\mu-1)}(\chi) J_{(\mu-1)}(\chi)}{\sin[(\mu-1)\pi]} \right], \quad (92)
\end{aligned}$$

$$\begin{aligned}
\mathbf{j}^{(+2)} \cdot \mathbf{E}^* &= \pi \kappa_E (a_1^2 - a_2^2) \int_0^\infty dp_\perp \int_{-\infty}^\infty dp_\parallel \\
&\quad \cdot f'_0 p_\perp^3 \frac{J_{-(\mu+1)}(\chi) J_{(\mu-1)}(\chi)}{\sin[(\mu-1)\pi]}, \quad (93)
\end{aligned}$$

$$\begin{aligned}
\mathbf{j}^{(-2)} \cdot \mathbf{E}^* &= \pi \kappa_E (a_1^2 - a_2^2) \int_0^\infty dp_\perp \int_{-\infty}^\infty dp_\parallel \\
&\quad \cdot f'_0 p_\perp^3 \frac{J_{-(\mu-1)}(\chi) J_{(\mu+1)}(\chi)}{\sin[(\mu+1)\pi]}, \quad (94)
\end{aligned}$$

with  $\kappa_E$  defined by (41). Integrating over the same singularities as before, the total amount of energy available for transfer through the Landau mechanism is

$$\begin{aligned}
W_L &= -\frac{\pi^2 q^2 \omega}{2c^2} \left(\frac{\omega}{k_3} - v_R\right) \int_0^\infty dp_\perp p_\perp^3 \\
&\quad \cdot \left[ (a_1 + a_2)^2 J_0^2(\chi) f'_0 \left[ p_\perp^2, (p_\parallel^-) \right]^2 \right. \\
&\quad + (a_1 - a_2)^2 J_0^2(\chi) f'_0 \left[ p_\perp^2, (p_\parallel^+) \right]^2 \\
&\quad + (a_1^2 - a_2^2) J_0(\chi) J_2(\chi) \\
&\quad \left. \cdot \left\{ f'_0 \left[ p_\perp^2, (p_\parallel^+) \right]^2 + f'_0 \left[ p_\perp^2, (p_\parallel^-) \right]^2 \right\} \right], \quad (95)
\end{aligned}$$

where the poles  $p_\parallel^\pm$  are defined as in (48). The difference between (50) and (95) is that the poles are associated with different Stokes parameters. Indeed, (95) can be obtained by the substitution of  $q \rightarrow -q$  in (50).

[16] For a thermalized plasma, the Maxwellian (53) is used in (95). Repeating the analysis in section 4,

$$\begin{aligned}
W_L &= \frac{\pi N q^2 \omega}{2c^2 \sqrt{2\pi m k_B T}} \left(\frac{\omega}{k_3} - v_R\right) e^{-\gamma(\omega^2 + \Omega^2)} \\
&\quad \cdot \left\{ [(a_1^2 + a_2^2) + (a_1^2 - a_2^2)\gamma] \cosh(2\gamma\omega\Omega) \right. \\
&\quad \left. + 2a_1 a_2 \sinh(2\gamma\omega\Omega) \right\} (1 - 4\gamma) \quad (96)
\end{aligned}$$

is obtained for negative charged particles.

[17] For a nonthermal plasma described by the kappa distribution (69), the analysis of section 5.1 is repeated. From (95) it follows that the energy available for transfer through the Landau mechanism takes the form

$$\begin{aligned}
W_L &= -\frac{\pi^2 q^2 \omega}{2c^2} \left(\frac{\omega}{k_3} - v_R\right) \\
&\quad \cdot \left\{ (a_1 + a_2)^2 I_A|_{p_\parallel^-} + (a_1 - a_2)^2 I_A|_{p_\parallel^+} \right. \\
&\quad \left. + (a_1^2 - a_2^2) [I_B|_{p_\parallel^+} + I_B|_{p_\parallel^-}] \right\} \quad (97)
\end{aligned}$$

for negative charged particles, where the integrals  $I_A$  and  $I_B$  are defined by (76) and (77). The final expression for  $W_L$  becomes

$$\begin{aligned}
W_L &= -\frac{\sqrt{\pi} q^2 \omega}{4c^2} \left(\frac{\omega}{k_3} - v_R\right) \frac{N}{C} \frac{1}{\sqrt{\kappa} \Gamma(\kappa - 1/2)} \\
&\quad \cdot \left\{ (a_1 + a_2)^2 R_- + (a_1 - a_2)^2 R_+ \right. \\
&\quad \left. + \frac{(\beta C)^2}{2} \frac{\kappa}{\kappa - 1} (5a_1^2 + 3a_2^2) + S \right\}, \quad (98)
\end{aligned}$$

where  $R_\pm$  and  $S$  are defined by (79) and (80), respectively.

[18] If the nonthermal plasma is described by the generalized Weibull distribution (81), the analysis in section 5.2 is followed using (97) for the energy available for transfer through the Landau mechanism, where the integrals  $I_A$  and

$I_B$  are defined by (83) and (84). The final expression for  $W_L$  takes the form

$$W_L = \frac{\pi q^2 \omega}{8c^2} \left( \frac{\omega}{k_3} - v_R \right) \frac{N}{C} \left[ \Gamma \left( \frac{\lambda+3}{\kappa} \right) \right]^{-1} \cdot \left\{ 2(a_1^2 + a_2^2) \left[ \Gamma \left( \frac{\lambda+2}{\kappa} \right) - (\beta C)^2 \Gamma \left( \frac{\lambda+4}{\kappa} \right) \right] + (\beta C)^2 \Gamma \left( \frac{\lambda+2}{\kappa} \right) U' - V \right\}, \quad (99)$$

with

$$U' = (a_1 + a_2)^2 \zeta_-^2 + (a_1 - a_2)^2 \zeta_+^2 \quad (100)$$

and with  $V$  defined by (87).

## 7. Summary

[19] An analysis has been presented of the possible role of the Landau mechanism in the transfer of energy to or from a collisionless plasma by means of waves. The Vlasov equation (1) was perturbed by a polarized monochromatic plane wave (5) to produce a plasma state that is axisymmetric in momentum space. In the semi-relativistic limit (13) this plasma state, as seen in the rest frame of the observer, was used as the ground state for the linearized Vlasov equation (15). With the plane wave perturbation, the linearized Vlasov equation reduced to a first-order linear ODE (17), which was solved in the standard way. This allowed us to calculate the current (28) associated with the perturbed distribution. It became possible to identify the spectral composition of the wave modes, and also to isolate the contribution of the various components of the wave spectrum, as shown by (29). Using the electric field associated with the plane wave (37), only certain modes survived, (38) to (40), that are capable of performing any work. The known identity (42) allowed the simplification of the integration along paths according to Landau's causality argument, so that an expression of the energy available for transfer between the plasma and the wave through the Landau mechanism could be written down in (50). For waves to play a role, their frequencies have to be comparable to that of the cyclotron frequency of the corresponding charge carrier. It is remarkable that for a polarized wave interacting with a plasma, each mode appears in some combination of the Stokes parameters, which differs for the different channels of transfer.

[20] This result was then applied to the specific case of a thermal plasma in section 4, by assuming a Maxwellian distribution in the rest frame of the observer. At the same time, we have explored a nonthermal distribution of electrons and ions which is becoming a dominant feature in fusion plasmas and in a host of astrophysical plasmas. This was done by considering the kappa distribution (section 5.1) as well as a generalized form of the Weibull distribution (section 5.2). Finally, we provided explicit expressions for all the main results in terms of negative charge carriers in section 6.

[21] It should be noted that the evaluation of integrals involving Bessel function products was greatly simplified using the results in Appendix A. The application of the main

result (50) of this paper to a Maxwellian, kappa, as well as a generalized Weibull distribution should facilitate the direct application of this analysis to experimental and observational data.

## Appendix A: Evaluation of Integrals (63) and (64)

[22] From *Gradshteyn and Ryzhik's* [1980] equation (6.633.1) it is known that

$$\int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_\mu(\beta x) J_\nu(\gamma x) dx = \frac{\beta^\mu \gamma^\nu}{2^{\mu+\nu+1}} \frac{\alpha^{-(\mu+\nu+\lambda+2)/2}}{\Gamma(\nu+1)} \cdot \sum_{n=0}^\infty \frac{\Gamma[n+1+(\mu+\nu+\lambda)/2]}{n! \Gamma(n+\mu+1)} \left( -\frac{\beta^2}{4\alpha} \right)^n \cdot F \left( -n, -n-\mu; \nu+1; \frac{\gamma^2}{\beta^2} \right), \quad (A1)$$

where  $F$  is a hypergeometric function with the parameter constraints  $\Re(\alpha) > 0$  and  $\Re(\mu+\nu+\lambda) > -2$ , as well as  $\beta > 0$  and  $\gamma > 0$ . In thermonuclear and space plasmas the Bessel function arguments  $\beta$  and  $\gamma$  are equal. It is therefore appropriate to make use of Gauss' identity [*Gradshteyn and Ryzhik*, 1980, equation (9.122.1)]

$$F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} \quad (A2)$$

with  $\gamma \neq 0, -1, -2, \dots$ . Thus, it follows that

$$\int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_\mu(\beta x) J_\nu(\beta x) dx = \frac{1}{2\alpha^{1+\lambda/2}} \left( \frac{\beta^2}{4\alpha} \right)^{(\mu+\nu)/2} \sum_{n=0}^\infty \left( -\frac{\beta^2}{4\alpha} \right)^n \cdot \frac{\Gamma[n+1+(\mu+\nu+\lambda)/2] \Gamma(2n+\mu+\nu+1)}{n! \Gamma(n+\mu+1) \Gamma(n+\nu+1) \Gamma(n+\mu+\nu+1)} \quad (A3)$$

with the constraint  $\nu \neq -1, -2, -3, \dots$ . This expression is used consistently in the evaluation of the integrals involving Bessel function products. It is observed that the evaluation of the integrals involved in plasma physics are reduced to the evaluation of mere gamma functions, certainly a trivial task. [*Stix*, 1962; *Miyamoto*, 1989].

[23] The series in (A3) is absolutely convergent, as can be seen from the following considerations. If  $u_n$  and  $u_{n+1}$  are the  $n$ th and  $(n+1)$ th terms, then

$$\left| \frac{u_{n+1}}{u_n} \right| = \frac{n+1+(\mu+\nu+\lambda)/2}{n+1} \left| -\frac{\beta^2}{4\alpha} \right| \cdot \frac{(2n+\mu+\nu+2)(2n+\mu+\nu+1)}{(n+\mu+1)(n+\nu+1)(n+\mu+\nu+1)}, \quad (A4)$$

which becomes

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| \approx \lim_{n \rightarrow \infty} \frac{1}{n+2(\mu+\nu+2)} \left( \frac{\beta^2}{\alpha} \right), \quad (A5)$$

where only terms not going to zero in the limit are kept in the numerator and denominator. The limit (A5) is zero,

which means the series converges absolutely for all values of  $\beta^2/\alpha$ .

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G. J. J. Botha, Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK. (gert@maths.leeds.ac.uk)

E. A. Evangelidis, Laboratory of Non-Conventional Sources of Energy, Demokritos University of Thrace, Kimeria, Xanthi, Greece. (eevangel@env.duth.gr)