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A CHARACTERISATION OF PHASE TYPE DISTRIBUTIONS

by

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ABTRACT

This paper examines the moments of the distribution of a sum of independent, but non-identical exponential variates. Certain algebraically tractable mixes of exponential parameters are considered with a view to identifying how different is the distribution from the gamma distribution, which arises in the identical exponential variates case. It is shown that the distribution is often close to gamma, but also that a lognormal distribution is often a close approximation. This has relevance to the modelling of maintenance times.

1. INTRODUCTION

The motivation for this analysis is the observation that in repairable and maintained systems the time to restore a system to working order rarely conforms to the conventional assumption of an exponential distribution. However it is possible that the times taken to execute the sub-tasks making up many maintenance actions could be each approximately exponential, but with varying values of the rate parameter. One particular point of interest is whether the distribution of the sum of non-identical exponential variates is close to a lognormal distribution, which is a favoured model for maintenance times. There may also be similarities with the Weibull model or the gamma distribution, the latter being the exact model for the sum of identical exponential variates. The gamma distribution in that case is often referred to as the Erlang distribution, as in the classic Cox (1965) book on renewal theory where mention is also made of the possible generalisation examined here. In Cox it is put in the context of times spent in the different stages of a process leading to failure, which is analogous to the situation highlighted here.

A phase type distribution is defined by Neuts(1981) as the probability distribution of the time until absorption in a Markov process with a finite number of transient states and one absorbing state. If the transition rate from state i is λ_i, the distribution of Y, the time from entering the initial state to absortion has probability density function

$$f(y) = \sum A_i \lambda_i e^{-\lambda_i y}$$

assuming all λ_i to be distinct. It can be shown via Laplace transforms that

$$A_i = \prod_{j \neq i} \frac{\lambda_i}{(\lambda_j - \lambda_i)}.$$

Phase type distributions have finite central moments and are dense in the class of distribution functions. As a result any repair time distribution could be approximated (with varying precision) by a phase type distribution. Johnson and Taaffe(1990), for example, take certain distributions and via the moments, produce an approximately equivalent phase type distribution.

This paper looks at certain phase type distributions and quantifies the closeness, or othersise, to other models, the lognormal in particular.

2. STRUCTURING THE RATE PARAMETERS

One of the standard ways to identify distributions is via moments and a simple way to obtain expressions for these in the case of variable sums is by using the moment generating function, provided the variables are independent of each other. It will be assumed here that times for the maintenance sub-tasks are independent.

Let the sub-task times be $\{X_i : i = 1, 2, \ , \ , n\}$ and the moment generating function of $Y = \sum_{i=1}^{n} X_i$ be $M(s)$. By definition $M(s)$ is the expected value of e^{sY} and by the usual rules of expectation

$$M(s) \quad = \quad \prod_{i=1}^{n} g_i(s) \tag{1}$$

where $g_i(s)$ is the moment generating function for X_i. If X_i is exponentially distributed with rate parameter λ_i then

$$M_i(s) \quad = \quad \frac{\lambda_i}{\lambda_i - s}.$$

Expressions for the moments $E(Y^r)$ are given by $\frac{d^r}{ds^r}[M(s)]$ at $s = 0$.

For certain families of distributions there is a particular relationship between the defining moments. The relationship between skewness (the standardised third moment) and the coefficient of variation is demonstrated graphically for the gamma, Weibull, lognormal and loglogistic families of distributions in Cox and Oakes (1984). It is possible under certain conditions to obtain points on this graph representing the distribution of Y and of interest here is where these lie in relation to the well-known families.

Obtaining the $E(Y^r)$ for the relevant r is in principle straight forward but these will be multivariate functions of the $\{\lambda_i\}$ and hence identification of the distribution is unclear unless some structure is placed on the $\{\lambda_i\}$ in order to reduce the number of unknown quantities in the moments.

The variables contributing to Y may be taken in any order and a useful ordering is one where the rate parameter increases with i, that is,

$$\lambda_{i+1} \ < \ \lambda_i \, .$$

Let $\lambda_1 = \lambda$. Various approximations to a set of increasing λ_i may be proposed, based only on λ and perhaps one or two constants. For example, mean sub-task times decreasing in geometric progression would be given by

$$\lambda_i = \lambda / c^{i-1}, \qquad c < 1.$$

3. GENERALISING THE MOMENTS

Differentiating (1) with respect to s,

$$\begin{aligned} M'(s) &= \sum_i [\, g_i'(s) \prod_{j:\, j \neq i} g_j(s) \,] \\ &= M(s) \sum_i \frac{g_i'(s)}{g_i(s)} . \end{aligned}$$

The mean of Y is given by $M'(0) = \sum_i g_i'(0)$, the sum of the means of the X_i, as expected. Differentiating again,

$$\begin{aligned} M''(s) &= M'(s) \sum_i \frac{g_i'(s)}{g_i(s)} + M(s) \, [\sum_i \frac{g_i(s)\, g_i''(s) - \{g_i'(s)\}^2}{\{g_i(s)\}^2}] \\ M''(0) &= [M'(0)]^2 + \sum_i [g_i''(0) - \{g_i'(0)\}^2] . \end{aligned}$$

So the variance of Y,

$$V(Y) = \sum_i [g_i''(0) - \{g_i'(0)\}^2]$$

which again is as expected, the sum of the variances of the X_i, and hence the coefficient of variation is given by

$$\gamma = \frac{\sqrt{V(Y)}}{E(Y)} = \frac{\sqrt{\sum_i [g_i''(0) - \{g_i'(0)\}^2]}}{\sum_i g_i'(0)} . \tag{2}$$

The skewness of Y, γ_3, is

$$E[(\frac{Y - E(Y)}{\sqrt{V(Y)}})^3] = \frac{E(Y^3) - 3E(Y^2)\, E(Y) + 2[E(Y)]^2}{[V(Y)]^{3/2}} \tag{3}$$

The third derivative of (1) with respect to s is

$$M'''(s) = M''(s)\sum_i \frac{g_i'(s)}{g_i(s)} + 2\,M'(s)\,[\sum_i \frac{g_i(s)\,g_i''(s) - \{g_i'(s)\}^2}{\{g_i(s)\}^2}]$$

$$+ \; M(s)\,[\sum_i \frac{\{g_i(s)\}^2 g_i'''(s) - 3g_i(s)\,g_i'(s)\,g_i''(s) + 2\{g_i'(s)\}^3}{\{g_i(s)\}^3}]\,.$$

$$E(Y^3) = M'''(0) = M''(0)M'(0) + 2\,M'(0)\;V(Y)\; + \sum_i[\,g_i'''(0) - 3\,g_i'(0)\,g_i''(0) + 2\{g_i'(0)\}^3]$$

$$= \; M''(0)\,M'(0) \; + \; 2\,M'(0)\,[M''(0) \; - \; \{M'(0)\}^2] \; + \; \sum_i \sigma_i^{3/2}\eta_i$$

where σ_i and η_i are respectively the standard deviation and skewness of X_i.

Substituting for $E(Y^3)$ in (2) yields

$$\gamma_3 \quad = \quad \frac{\sum \sigma_i^{3/2}\eta_i}{[\sum \sigma_i^2]^{3/2}} \qquad (4)$$

Equations (2) and (4) can also be obtained via the result given in Kendall(1994), that the rth cumulant of a sum of independent random variables is the sum of the rth cumulants of its components.

4. EXPONENTIAL $\{X_i\}$

Given $\quad g_i(s) \; = \; \dfrac{\lambda_i}{\lambda_i - s}\,,$

$$g_i'(s) \; = \; \frac{\lambda_i}{(\lambda_i - s)^2}\,; \qquad g_i'(0) \; = \; \frac{1}{\lambda_i}\,,$$

$$g_i''(s) = \; \frac{2\lambda_i}{(\lambda_i - s)^3}\,; \qquad g_i''(0) \; = \frac{2}{\lambda_i^2}\,,$$

$$g_i'''(s) \; = \; \frac{6\lambda_i}{(\lambda_i - s)^4}\,; \qquad g_i'''(0) \; = \frac{6}{\lambda_i^3}\,.$$

So, $\quad \sigma_i^2 \; = \; \dfrac{2}{\lambda_i^2} - \dfrac{1}{\lambda_i^2} \; = \; \dfrac{1}{\lambda_i^2}$

and from (1) $\quad \gamma \; = \; \dfrac{\sqrt{\sum \frac{1}{\lambda_i^2}}}{\sum \frac{1}{\lambda_i}}\,. \qquad (5)$

Now $\sigma_i^{3/2}\eta_i = \dfrac{6}{\lambda_i^3} - 3\dfrac{2}{\lambda_i^2}\dfrac{1}{\lambda_i} + 2\dfrac{1}{\lambda_i^3} = \dfrac{2}{\lambda_i^3}$

and from (3)

$$\gamma_3 = \frac{\sum \frac{2}{\lambda_i^3}}{\left(\sum \frac{1}{\lambda_i^2}\right)^{3/2}} . \tag{6}$$

Relations (5) and (6) may be combined to give γ_3 as a function of γ,

$$\gamma_3 = \frac{2}{\gamma^3}\left[\frac{\sum \frac{1}{\lambda_i^3}}{\left(\sum \frac{1}{\lambda_i}\right)^3} \right] . \tag{7}$$

When the X_i are i.i.d, $\lambda_i = \lambda$ and $\gamma = \dfrac{\sqrt{n\frac{1}{\lambda^2}}}{n\frac{1}{\lambda}} = \dfrac{1}{\sqrt{n}}$

and

$$\gamma_3 = \frac{2}{1/n^{3/2}} \frac{n\frac{1}{\lambda^3}}{\left(n\frac{1}{\lambda}\right)^3} = \frac{2}{\sqrt{n}}$$

which yields the curve $\gamma_3 = 2\gamma$ shown on the Cox & Oakes graph.

Now suppose that the λ_i are not all equal. Two cases will be considered.

(i) $\dfrac{1}{\lambda_i} = \dfrac{c^{i-1}}{\lambda}$, that is, the means in geometric progression

(ii) $\dfrac{1}{\lambda_i} = \dfrac{1 + (i-1)a}{\lambda}$, that is, the means in arithmetic progression,

and without loss of generality we can take $\lambda = 1$.

Case (i) : For convenience the value of c will be taken to be less than 1.

From (5)

$$\gamma = \frac{\sqrt{\sum c^{2(i-1)}}}{\sum c^{i-1}} = \frac{(1-c^{2n})^{1/2}(1-c)}{(1-c^n)\sqrt{1-c^2}} = \sqrt{\frac{(1+c^n)(1-c)}{(1-c^n)(1+c)}} . \tag{8}$$

From (6) $$\gamma_3 = \frac{2\sum c^{3(i-1)}}{(\sum c^{2(i-1)})^{3/2}} = 2\,\frac{(1 - c^{3n})\,(1 - c^2)^{3/2}}{(1 - c^3)\,(1 - c^{2n})^{3/2}}\,. \tag{9}$$

Figure 1 shows a plot of γ_3 against γ for c in the range [0,1] and for n = 2, 4, 6, 10, 15. Functions of the form $\gamma_3 = f(\gamma)$ are also shown for the lognormal and Weibull families. In the case of the lognormal $\gamma_3 = 3\gamma + \gamma^3$. Tabulated moment properties of the Weibull distribution can be found in Cox(1962).

The case $c = 0$ is equivalent to $n = 1$ (the exponential), and the case $c = 1$ is the i.i.d case yielding the gamma distribution. Clearly a range of general cases give a distribution which in practice is close to lognormal, or at least close enough for the nearest alternative, the gamma distribution, not to be distinctly preferable.

Case (ii) : Here we again substitute into the formulae of (5) and (6) and effectively replace c^{i-1} by $1+(i-1)a$ in the derivation of (8) and (9).

$$\gamma = \frac{\sqrt{\sum[1 + (i-1)a]^2}}{\sum[1 + (i-1)a]} = \frac{[n + an(n\text{-}1) + \frac{a^2(n\text{-}1)n(2n\text{-}1)}{6}]^{1/2}}{n + \frac{a(n\text{-}1)n}{2}}$$

$$= \frac{2\,[6 + 6a(n\text{-}1) + a^2(n\text{-}1)(2n\text{-}1)]^{1/2}}{\sqrt{6n}[2 + a(n\text{-}1)]} \tag{10}$$

$$\gamma_3 = \frac{2\sum[1 + (i-1)a]^3}{\{\sum[1 + (i-1)a]^2\}^{3/2}} = \frac{2[n + \frac{3an(n\text{-}1)}{2} + \frac{a^2(n\text{-}1)n(2n\text{-}1)}{2} + \frac{a^3(n\text{-}1)^2n^2}{4}]}{[n + an(n-1) + \frac{a^2(n-1)n(2n-1)}{6}]^{3/2}}$$

$$= \frac{3\sqrt{6}[4 + 6a(n\text{-}1) + 2a^2(n\text{-}1)(2n\text{-}1) + a^3(n\text{-}1)^2 n]}{\sqrt{n}[6 + 6a(n-1) + a^2(n-1)(2n-1)]^{3/2}} \tag{11}$$

[Using $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$; $\sum_{i=1}^{n} i^2 = \frac{n(n+1)\,(2n+1)}{6}$; $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$]

When $a = 0$, Y will have a gamma distribution for all n. When $a = 1$, the exponential means are evenly spaced. Other values of a will only yield distinct cases when n is small as when n

becomes large any value for a is very close to the case $a = 1$.

Figure 2 shows a plot of γ_3 against γ for $a \in [0, \infty)$ and $n = 2, 4, 6, 10, 15$. The case $n = 2$ is the same as in Figure 1 since means in this case can be classed as in both arithmetic and geometric progression. In the arithmetic case the distributions stay closer to gamma than lognormal compared to the geometric case for all values of n larger than 2. As stated above varying a makes a reduced difference the larger n becomes. For a given n there are limiting values for γ and γ_3.

As $a \to \infty$,

$$\gamma \to 2\sqrt{\frac{2n-1}{6n(n-1)}} \quad \text{and} \quad \gamma_3 \to \frac{3}{(2n-1)}\sqrt{\frac{6n(n-1)}{2n-1}} \ .$$

5. DISCUSSION

The results shown here confine themselves to sums of exponential variables which have a selection of means with two well defined patterns. In practice mean times for sub-tasks will not conform to this ideal. It is clear though that many sets of means will be similar enough to the cases analysed here to suggest that a lognormal distribution is a reasonable model for total time though for small values of n the gamma distribution may be a better choice, at least theoretically.

The analysis is intended to give support to the lognormal assumption for maintenance times, and importantly to allow Markovian analysis of repairable systems by dividing repair times into stages of exponential duration.

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Figure 1 - geometric means, n = 2, 4, 6, 10, 15

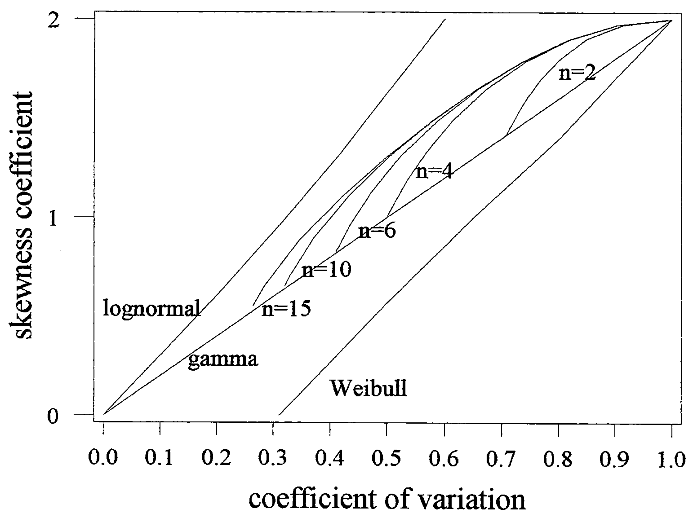
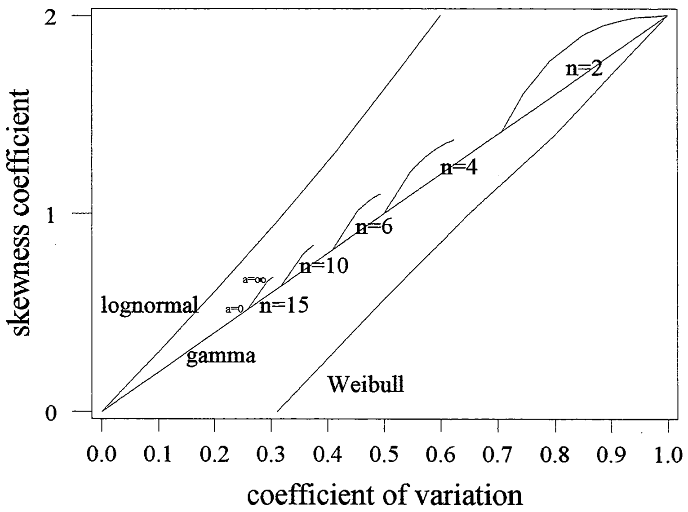


Figure 2 - arithmetic means, n = 2, 4, 6, 10, 15



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