Banjo, Elizabeth (2013). Representation theory of algebras related to the partition algebra. (Unpublished Doctoral thesis, City University London)

CITY UNIVERSITY LONDON

Representation Theory Of Algebras Related To The Partition Algebra

by

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A thesis submitted in partial fulfillment for the degree of Doctor of Philosophy

in the
Department of Mathematics
School of Engineering and Mathematical Sciences

April 2013

Declaration of Authorship

I, Elizabeth O. Banjo, declare that this thesis titled, Representation Theory Of Algebras Related To The Partition Algebra and the work presented in it are my own. I confirm that:

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- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
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"Do not think that what is hard for thee to master is impossible for man; but if a thing is possible and proper to man, deem it attainable by thee."

Marcus Aurelius

Abstract

The main objective of this thesis is to determine the complex generic representation theory of the Juyumaya algebra. We do this by showing that a certain specialization of this algebra is isomorphic to the small ramified partition algebra, introduced by Martin (the representation theory of which is computable by a combination of classical and category theoretic techniques). We then use this result and general arguments of Cline, Parshall and Scott to prove that the Juyumaya algebra $\mathcal{E}_n(x)$ over the complex field is generically semisimple for all $n \in \mathbb{N}$. The theoretical background which will facilitate an understanding of the construction process is developed in suitable detail. We also review a result of Martin on the representation theory of the small ramified partition algebra, and fill in some gaps in the proof of this result by providing proofs to results leading to it.

Acknowledgements

The successful completion of this thesis was made possible through the invaluable contribution of a number of people. To say “thank you” to everyone is not even enough to express my gratitude.

I am deeply grateful to Prof. Paul Martin for making the experience of research an enjoyable one by giving me an interesting problem to work on, for generously sharing his knowledge and ideas, for his patience and tremendous support, and for believing in me. I feel very fortunate to have been one of his students. My debt is indeed greater because he has been with and for me ever since my days as an undergraduate student.

I am indebted to Prof. Robert Marsh for many invaluable discussions, for useful material, for proofreading this thesis, and willingness to spend time and share his knowledge in order to make this thesis possible.

To Dr. Marcos Alvarez for his support during the early part of my PhD.

I want to thank Dr. Alison Parker, and Prof. Andreas Fring for agreeing to be part of my examining committee and for their time and useful comments.

The stimulating environment at City University has allowed me to learn from many people. It is a pleasure to thank the entire Mathematics department for their help and support throughout this journey. I mention especially Prof. Andreas Fring for his help in finding funding for the last year of my PhD, and Dr. Maud De Visscher for proofreading part of this thesis.

I wish to thank all the members of staff at University of Leeds (where this thesis was completed) especially the following people: Paul Martin, Robert Marsh, Alison Parker, William Crawley-Boevey, Andrew Hubery, Yann Palu, Charles Harris, Jonathan Partington, Jessica Brennan, for their kind hospitality.

My time in Leeds has been one of the best periods of my life, and I owe it to other exceptional people that I have met there. To all my irreplaceable friends especially Raquel Simoes, Liliana Badillo Sanchez, and Mayra Montalvo Ballesterous, I would like to express my appreciation for their constant encouragement and support. To Mrs Martin and family for warmly welcoming me into their family; I am happy and honoured to have met them.

My sincere thanks to Fidel Ighile for being such a supportive friend and for always being there for me. Special thanks also to Zoe Gumm for her kindness, friendship, support, and for all those memorable times we spent together as office mates at City.

I would also like to thank my family for their unconditional love and support.

Financial support for my research program from City University London is gratefully acknowledged.

Above all, I thank God for giving me the intellect to comprehend the subject, and for giving me the strength and endurance to complete the program. Most specially thankful for sending beautiful people into my life when I needed them the most.

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To Fidel Ighile

Chapter 1

Introduction

Representation theory is concerned with the study of how various algebraic structures (such as groups, monoids, algebras) act on vector spaces while respecting the operations on these algebraic structures. In group theory, the idea of representation is to find a group of permutations or linear transformations with the same structure as a given, abstract, group (see, for example, [1]). Formally, a representation of a group is a homomorphism $G \to GL_n(F)$ for a field F, giving an invertible $n \times n$ matrix for each element of G. More abstractly, representations of a group G may be defined in terms of modules over the group algebra over F.

One of the most fundamental problems in representation theory is to construct and classify irreducible representations of a given algebraic structure, up to isomorphism. This problem is usually difficult and often can be solved only partially [2]. The problem has been solved for some algebras such as the partition algebras over $\mathbb{C}$ [3], and for some groups such as the symmetric groups [1] and the wreath product groups over $\mathbb{C}$ [4].

For F a field and $\delta' \in F$ the partition algebras $P_n(\delta')$ $(n = 1, 2, \ldots)$ are a tower of finite dimensional unital algebras over F each with a basis of set partitions. These algebras appeared independently in the work of Martin [3, 5, 6] and Jones [7]. Their work on the partition algebra stemmed from studies of the Temperley-Lieb algebra and the Potts model in statistical mechanics. The partition algebras have a rich representation theory. For example, Martin [3, 6], Martin and Saleur [8], Doran and Wales [9], Halverson and Ram [10], Martin and Woodcock [11] have extensively studied the structure and the representation theory of the partition algebra $P_n(\delta')$, with $\delta' \in \mathbb{C}$. They revealed that $P_n(\delta')$ is semisimple whenever δ'

is not an integer in $[0, 2n-1]$, and they analyzed the irreducible representations in both the semisimple and non-semisimple cases.

Martin and Elgamal have studied a certain generalization [12] of the partition algebra called the *ramified* partition algebra. For each natural number n, poset T, and any $|T|$-tuple of scalars $\delta = (\delta_1, \ldots, \delta_d) \in F^d$, the ramified partition algebra $P_n^{(T)}(\delta)$ is a certain subalgebra of the tensor product of partition algebras $\bigotimes_{t \in T} P_n(\delta_t)$. The partition algebra coincides with the case $|T| = 1$. For fixed n and T the ramified partition algebra, like the partition algebra, has a basis independent of δ. In case $T = \underline{2} := (\{1,2\}, \leq)$, it was shown in [12] that there are unboundedly many choices of δ such that $P_n^{(\underline{2})}(\delta)$ is not semisimple for sufficiently large n, but that it is generically semisimple for all n.

Some years later, while working on a different problem (restriction rules for wreaths), Martin discovered another algebra called the *small* ramified partition algebra [13]. The small ramified partition algebras P_n^{ς} are subalgebras of the ramified partition algebras. They are also subalgebras of the tensor product of the symmetric group algebra FS_n and the partition algebra $P_n(\delta')$. Unlike the partition algebras and the ramified partition algebras, the small ramified partition algebras are independent of parameters. As shall become clear as we proceed, we have algebra inclusions

$$\begin{array}{ccccc} & \subset & P_n^{(\underline{2})}(\delta) & \subset & \\ P_n^{\varsigma} & & & & P_n(\delta_1) \otimes_F P_n(\delta_2) \\ & \subset & FS_n \otimes_F P_n(\delta') & \subset & \end{array}$$

where $\delta = (\delta_1, \delta_2)$, and $\delta' = \delta_2$ in the inclusion $FS_n \otimes_F P_n(\delta') \subset P_n(\delta_1) \otimes_F P_n(\delta_2)$.

Like some algebras such as the Temperley-Lieb algebras (see, for example, [3, 14]) and the Brauer algebras [15–17], the algebras $P_n^{(T)}(\delta)$, P_n^{ς}, and $P_n(\delta')$ are examples of "diagram algebras". A diagram algebra is a finite dimensional algebra with a basis given by a collection of certain diagrams and multiplication described combinatorially by diagram concatenation.

This thesis is concerned with the representation theory of a certain algebra which we shall call the Juyumaya algebra of braids and ties (or simply the Juyumaya algebra). The Juyumaya algebras are a family of finite dimensional $\mathbb{C}$-algebras $\{\mathcal{E}_n(x) \colon n \in \mathbb{N},\ x \in \mathbb{C}\}$. These algebras were introduced by Juyumaya in [18] and studied further by Aicardi and Juyumaya [19] and by Ryom-Hansen [20].

Motivation for investigating the representation theory of this algebra comes from observations on the representation theory of the small ramified partition algebra. Our first discovery of the connection between these two algebras was that they have equal dimension. (This intriguing result was hinted at [20].)

The Juyumaya algebras $\mathcal{E}_n(x)$ are a generalisation of the Iwahori-Hecke algebras [21]. The complex generic representation theory of the Iwahori-Hecke algebras is reasonably well known (see, for example [21] for a review). Like the Iwahori-Hecke algebras it turns out, as we shall show, that the Juyumaya algebras are generically semisimple. In contrast to the Iwahori-Hecke case however, the generic representation theory of the Juyumaya algebras over the field of complex numbers was only known for the cases $n = 1, 2, 3$ [19], [20]. Here we determine the result for all n.

Our method is to establish, for each n, an isomorphism between $\mathcal{E}_n(1)$ (over $\mathbb{C}$) and the small ramified partition algebra P_n^{ς}, of known complex representation theory and then to use general arguments of Cline, Parshall and Scott [22].

In dealing with the study of algebraic structures and their respective representations, it is common to take a category-theoretic approach to modules. This is the way we shall proceed in this thesis. Our approach for finding the simple $P_n^{\varsigma}-$modules is motivated by the work of Cox et.al. on "towers of recollement" [23] and some results of Green [24]. Towers of recollement are used in algebraic representation theory, for example, [3, 25, 26]. The tower of recollement is somewhat connected in the semisimple case to the Jones basic construction [27]. In fact, the idea behind the approach is roughly the following: If A is an algebra, and $e \in A$ an idempotent, then the category eAe-mod of left eAe-modules embeds in A-mod. More simply, the idea is that if eAe-mod may be relatively simply analysed, the embedding then gives partial knowledge of A-mod [28].

Once we view an algebraic structure in terms of its category of modules, it is natural to compare such categories. This leads to the notion of "Morita equivalence". Two rings R and S are said to be *Morita equivalent* if their respective categories R-mod and S-mod of (left) modules are equivalent. Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *equivalent* if there exists functors $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$, $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ satisfying $\mathcal{F} \circ \mathcal{G} \cong \mathcal{I}_{\mathcal{D}}$ and $\mathcal{G} \circ \mathcal{F} \cong \mathcal{I}_{\mathcal{C}}$, where $\cong$ denotes isomorphism of functors and $\mathcal{I}$ is the identity functor.

1.1 Structure of the thesis

We will adopt the convention of placing a QED box at the end of some results to imply that we will not provide the proof of that result but interested reader can find the proof in the reference provided in the header of the result. We will begin each chapter with a brief summary of what that chapter contains. However, for convenience, here is an overview of the arrangement and content of this thesis.

In Chapter 2, we begin with a brief tour through representation theory of partition algebras, with emphasis determined by what is useful for the later chapters. The goal of the thesis is to present the connection of the small ramified partition algebra to the Juyumaya algebra.

In Chapter 3, we recall the definition of the small ramified partition algebra after reviewing the ramified partition algebra, a generalisation of the partition algebra.

The focus then turns to working out the irreducible representations of the the small ramified partition algebra in Chapter 4. We look at an illustrative example. Before describing explicitly the structure of P_n^{ς}, we give an indexing set for the irreducible representations of P_n^{ς}. Since we are taking a category theoretic approach, it is natural to ask about the category of P_n^{ς}-modules, and we do so here. After setting the scene with the category of P_n^{ς}-modules, we then exploit some properties of this category to construct the irreducible representations of P_n^{ς}. An observation reveals that the basis elements of the small ramified partition algebra is somewhat related to some wreath products of symmetric groups. As an alternative method to construct the irreducible representations of P_n^{ς} we consider the wreath product groups and describe its representation theory.

In Chapter 5, we recall the definition of the Juyumaya algebra. We also present the main results of the thesis and prove them.

Unfortunately, constraints of time prevent the development of a wider investigation of the representation theory within the thesis. Chapter 6 considers the progress made so far and looks at some aspects of the theory that we have not yet had time to develop, but which would be interesting subjects for further research.

Appendix A contains some representation theory of the symmetric groups over $\mathbb{C}$. This is useful for Chapter 4 but is removed to the appendix to facilitate the flow of narrative.

Appendix B contains an account of some necessary preliminaries for our studies - review on algebras, modules, and the core classical representation theory of algebras. We begin by giving the definitions of a matrix representation of a group and that of a module. We continue with the analysis of the relationship between simple modules and semisimple modules. Then we consider for which algebras we can reduce the study of their representation theory to the study of their simple modules. Such algebras are called semisimple, and the Artin-Wedderburn Theorem will give a complete classification in this case. If an algebra is not semisimple, then the Jacobson radical of the algebra can be regarded as a measure of its non-semisimplicity. The Krull-Schmidt Theorem then tells us that it is enough to determine the indecomposable modules.

This thesis is based on a published article of the author, titled *The generic representation theory of the Juyumaya algebra of braids and ties* [29]. However, some of the notation has been improved and some of the arguments have been discussed comprehensively here.

Chapter 2

A review of the partition algebra

One of the main algebras of interest in this thesis is the small ramified partition algebra. It is an algebra with a diagrammatic formulation akin to the partition algebra. It will be convenient, therefore, to recall this familiar example in a suitable formalism and then generalise to the small ramified partition algebra. We also briefly summarise the basic representation theory of the partition algebras that will be useful later on. Details can be found in [3]. Much of the standard terms and notation in representation theory we use here are reviewed for reference in Appendix B.

2.1 Partition monoid

For $n \in \mathbb{N}$, we define $\underline{n} = \{1, 2, \ldots, n\}$ and $\underline{n}' = \{1', 2', \ldots, n'\}$.

Definition 2.1. A *(set) partition of a set* X is a collection $\{S_1, S_2, S_3, \ldots\}$ of non-empty subsets of X such that

$$S_1 \cup S_2 \cup S_3 \cup \ldots = X \text{ and } S_i \cap S_j = \emptyset \text{ whenever } i \neq j.$$

We denote the set of all partitions of X by $\mathcal{P}_X$.

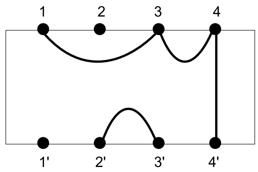


FIGURE 2.1: A diagrammatic representation of a partition from $\mathcal{P}_{\underline{4}\cup\underline{4}'}$.

Example 2.2.

$$\begin{aligned}\mathcal{P}_{\underline{2}\cup\underline{2}'} \;=\; & \{\{\{1\},\{2\},\{1'\},\{2'\}\},\{\{1,2,1',2'\}\},\{\{1,2,1'\},\{2'\}\},\\ & \{\{1,2,2'\},\{1'\}\},\{\{1,1',2'\},\{2\}\},\{\{2,1',2'\},\{1\}\},\{\{1,2\},\{1',2'\}\},\\ & \{\{1,1'\},\{2,2'\}\},\{\{1,2'\},\{1',2\}\},\{\{1,2\},\{1'\},\{2'\}\},\\ & \{\{1,1'\},\{2\},\{2'\}\},\{\{1,2'\},\{1'\},\{2\}\},\{\{1',2\},\{1\},\{2'\}\},\\ & \{\{2,2'\},\{1\},\{1'\}\},\{\{1',2'\},\{1\},\{2\}\}\}.\end{aligned}$$

We call the individual subsets in a partition of X *parts*. For instance, $\{1,2\}$ is a part of the partition $\{\{1,2\},\{1'\},\{2'\}\} \in \mathcal{P}_{\underline{2}\cup\underline{2}'}$ in Example 2.2.

We shall see in Theorem 2.4 that the set $\mathcal{P}_{\underline{n}\cup\underline{n}'}$ of all partitions on $\underline{n} \cup \underline{n}'$ forms a monoid, the so-called *partition monoid* (see, for example, [10], [6], or [12]), under an associative binary operation we describe shortly.

A set partition $p \in \mathcal{P}_{\underline{n}\cup\underline{n}'}$ may be represented by a diagram on the vertex set $\underline{n} \cup \underline{n}'$ as follows. In a rectangular frame, we arrange vertices labelled $1,\dots,n$ in a row (increasing from left to right) and vertices labelled $1',\dots,n'$ in a parallel row directly below. When such a diagram is arranged in this way we may talk about the top and bottom rows of p. We then add edges in such a way that two vertices are connected by a path if and only if they belong to the same part of p. For example, the partition

$$\{\{1,3,4,4'\},\{2',3'\},\{1'\},\{2\}\} \in \mathcal{P}_{\underline{4}\cup\underline{4}'}$$

is represented by the diagram pictured in Figure 2.1.

The diagram representing a set partition is not unique. Two such diagrams are regarded as equivalent if they have the same connected components. We will thus identify diagrams on the vertex set $\underline{n} \cup \underline{n}'$ if they are equivalent and the term *partition diagram* will be used to mean the equivalence class of the given diagram.

In the same way, we will not distinguish between a set partition and a diagram which represents it.

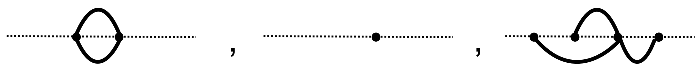
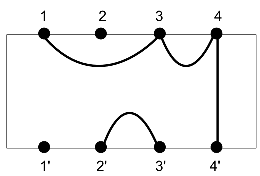
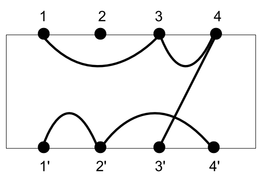


FIGURE 2.2: A closed loop, an isolated vertex, and an open string, respectively that may appear in the middle row during the composition of two partition diagrams. The dotted line here just indicates the middle row

In order to describe the product of these partition diagrams, let $p, q \in \mathcal{P}_{\underline{n} \cup \underline{n}'}$. We first place the partition diagram representing p above the partition diagram representing q so that vertices $1', \ldots, n'$ of p are identified with vertices $1, \ldots, n$ of q. This new diagram consists of a top row, bottom row, and the part where the vertices coincide which we will call the "middle row". In this middle row, there are three topologically different connected components that are isolated from the boundaries in composition that may appear, namely closed loops, isolated vertices, and open strings. (These connected components are illustrated respectively in Figure 2.2). We finally remove this middle row as well as any connected components; the resulting diagram is the product pq. An example is given as follows.

If

$p \mapsto$ 

$q \mapsto$ 

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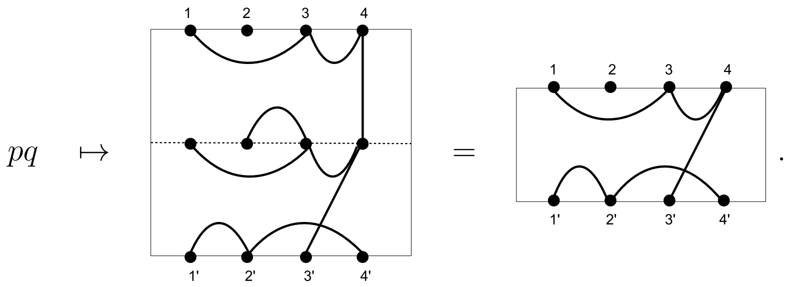
then the product of the partition diagrams of p and q in $\mathcal{P}_{\underline{n}\cup\underline{n}'}$ is



Lemma 2.3 (See [3, Prop. 1])**.** *The product on $\mathcal{P}_{\underline{n}\cup\underline{n}'}$ defined above is associative and well-defined up to equivalence.* □

Theorem 2.4. *The set $\mathcal{P}_{\underline{n}\cup\underline{n}'}$ forms a monoid.*

Proof. It is easy to verify that the identity element is the partition

$$\{\{1,1'\},\{2,2'\},\{3,3'\},\ldots,\{n,n'\}\}.$$

Associativity follows from Lemma 2.3. □

The submonoids of the partition monoid $\mathcal{P}_{\underline{n}\cup\underline{n}'}$ include the following.

Definition 2.5.

(1) The Brauer monoid $\mathcal{B}_{\underline{n}\cup\underline{n}'} = \{x \in \mathcal{P}_{\underline{n}\cup\underline{n}'}:$ each part of x contains exactly two elements of $\underline{n}\cup\underline{n}\}$. (See, for example, [30], [15]).

(2) The Temperley-Lieb monoid $T_{\underline{n}\cup\underline{n}'} = \{x \in \mathcal{B}_{\underline{n}\cup\underline{n}'}:\ x$ is planar$\}$. The word *planar* here means that if we consider the basis elements as diagrams, then there are no edge crossings in the diagram. (See, for example, [31]).

(3) The symmetric group $S_{\underline{n}\cup\underline{n}'} = \{x \in \mathcal{P}_{\underline{n}\cup\underline{n}'}:$ each part of x has exactly two elements (one primed and the other unprimed) of $\underline{n}\cup\underline{n}\}$. (See, for example, [10]).

2.2 The partition algebra

A convenient situation occurs when we use an algebraic structure such as a group or a monoid as a basis for an *algebra* (see section B.1.3 for a definition) over a field or a ring. Since we already know how elements multiply in these algebraic structures, we can use the product operation on them to define the product in their algebra. In the case of a monoid, this construction is known as the *monoid algebra* and will be denoted by FG where F is the field and G is the monoid.

Definition 2.6. Let $P_n(\delta') = \mathbb{C}\mathcal{P}_{\underline{n}\cup\underline{n}'}$ be the $\mathbb{C}$-vector space with basis $\mathcal{P}_{\underline{n}\cup\underline{n}'}$. We define a product on $P_n(\delta')$ as follows. Given $p, q \in \mathcal{P}_{\underline{n}\cup\underline{n}'}$, define

$$p \circ q = \delta'^{l}(pq),$$

the scalar multiple of (the monoid product) pq by the scalar $\delta'^{l} \in \mathbb{C}$ where l is the number of connected components removed from the middle row when constructing the product pq. The linear extension of the product $\circ$ gives $P_n(\delta')$ the structure of an associative $\mathbb{C}$-algebra which is known as the *partition algebra*.

The dimension of $P_n(\delta')$ is the Bell number B_{2n} (see, for example [32]), the number of ways to partition a set of $2n$ elements. The sequence is A020557 in the Sloane's On-line Encyclopedia of Integer Sequences [33].

The partition algebra is an example of a *monoid algebra* (see, for example [34, p. 106], [35, §5.1, Ex. 4]). More examples of a monoid algebra are as follows:

For each monoid defined in Definition 2.5, we can construct an associative algebra in the same way that we construct the partition algebra $P_n(\delta')$ from the partition monoid $\mathcal{P}_{\underline{n}\cup\underline{n}'}$. For example, we obtain the Brauer algebra $B_n(\delta')$ from $\mathcal{B}_{\underline{n}\cup\underline{n}'}$, the Temperley-Lieb algebra $TL_n(\delta)$ from $T_{\underline{n}\cup\underline{n}'}$, and the group algebra of the symmetric group $\mathbb{C}S_n$ from $S_{\underline{n}\cup\underline{n}'}$ in this way.

We now briefly summarise a category-theoretical approach to the representation theory of $P_n(\delta')$. This approach was introduced by J.A. Green in the Schur algebra setting, [24], but has turned out to be useful in the context of diagram algebras, see for example, [23], [36], [25], and [26]. In the case of the partition algebra $P_n(\delta')$, good references to the formalism are [3, 6].

Definition 2.7.

(1) Let A be an algebra over a field F. An element $e \in A$ is an *idempotent* in case $e^2 = e$.

(2) An idempotent e of A is a *central idempotent* in case it is in the centre of A.

Given an algebra A over a field and an idempotent $e \in A$ then e determines a second algebra, namely

$$eAe = \{eae \colon a \in A\}$$

with binary operation given by that of A restricted to eAe and with identity $e = e1e$. (If $e \neq 1$, then the algebra eAe is not a subalgebra of A. Although in the thesis, for convenience, we shall refer to such an algebra as an *idempotent subalgebra* of A.) Thus, we may define functors between A-mod (the category of left A-modules) and eAe-mod:

$$\begin{aligned} F \colon A\text{-mod} &\to eAe\text{-mod} \\ M &\mapsto eM \end{aligned} \tag{2.1}$$

$$\begin{aligned} G \colon eAe\text{-mod} &\to A\text{-mod} \\ N &\mapsto {}_A Ae \otimes_{eAe} N \end{aligned} \tag{2.2}$$

The functor F is called *localisation*, and G is called *globalisation*, with respect to e. We shall return to consider such functors for the small ramified partition algebras (since algebras are rings) in Section 4.6. As a first illustration of how these functors may be applied to algebras we consider the partition algebra $P_n(\delta')$.

There is a natural inclusion

$$P_{n-1}(\delta') \subsetneq P_n(\delta') \tag{2.3}$$

given by adding vertices labelled n and n' with a vertical edge connecting them in the rightmost part of an arbitrary partition diagram $q \in P_{n-1}(\delta')$.

For $n \geq 1, \delta' \neq 0$, consider the idempotent e_n in $P_n(\delta')$ defined by $1/\delta'$ times the partition diagram where i is joined (by an edge) to i' for $i = 1, \ldots, n-1$, and there is no edge joining n to n'. This is illustrated in Figure 2.3.

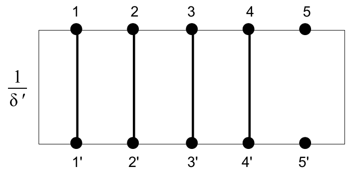


FIGURE 2.3: The idempotent e_5 in $P_5(\delta')$

Theorem 2.8 (See [3, Theorem 1]). *For each $n \in \mathbb{N}$, $\delta' \neq 0$ and idempotent $e_n \in P_n(\delta')$ as defined above, there is an isomorphism of algebras*

$$e_n P_n(\delta') e_n \cong P_{n-1}(\delta'). \qquad \square$$

Thus, according to Green [24], there are associated functors

$$\begin{aligned} F\colon P_n\text{-mod} &\to P_{n-1}\text{-mod} \\ M &\mapsto e_n M \end{aligned}$$

and

$$\begin{aligned} G\colon P_{n-1}\text{-mod} &\to P_n\text{-mod} \\ N &\mapsto P_n e_n \otimes_{P_{n-1}} N \end{aligned}$$

Following Martin [6], we define the propagating number for a partition diagram q, denoted by $\#(q)$, to be the number of distinct parts of q containing elements from both the top and bottom row of q. The product of partition diagrams has the property that if q_1, q_2 are partition diagrams, we have

$$\#(q_1 q_2) \leq \min(\#(q_1), \#(q_2)).$$

The ideal $P_n(\delta')e_nP_n(\delta')$ is spanned by all diagrams having a propagating number strictly less than n. Furthermore, we have

Lemma 2.9 (See [3]). *For each n, and $\delta' \neq 0$, the following is an isomorphism of algebras*

$$P_n(\delta')/P_n(\delta')e_nP_n(\delta') \cong \mathbb{C}S_n. \qquad \square$$

Let $\widehat{S_n}$ denote any index set for the irreducible representations of the symmetric group S_n. (See Appendix A for a good choice.)

It follows, by [24], Theorem 2.8, and Lemma 2.9, that

Theorem 2.10 (See [3, p. 72–73])**.** *Let $\widehat{\Lambda_n}$ denote an index set for the irreducible representations of $P_n(\delta')$. Then $\widehat{\Lambda_n}$ is the disjoint union*

$$\widehat{\Lambda_n} = \widehat{\Lambda_{n-1}} \,\dot{\cup}\, \widehat{S_n}. \tag{2.4}$$

$\square$

Theorem 2.11 (See [8, Coro. 10.3, §6])**.** *For each integer $n \geq 0$, the algebra $P_n(\delta')$ is semisimple over $\mathbb{C}$ whenever δ' is not an integer in the range [0, 2n-1].* $\square$

Chapter 3

The Small Ramified Partition Algebras

In order to define the *small* ramified partition algebra, it will be helpful to recall the definition of the ramified partition algebra, given in [12], from which this algebra can be constructed. We shall mainly base our exposition on the notations and terminology of [12], as well as key results from that paper.

The purpose of Section 3.1 is to lay out some notation and terminology which will be used later. In Section 3.2, we review the definition of the ramified partition algebra from which the small ramified partition algebra can be constructed. We then recall the definition of the small ramified partition algebra.

3.1 Some definitions and notation

Given $n \in \mathbb{N}$, we let S_n denote the symmetric group on $\underline{n}$. The group algebra FS_n of S_n is embedded in $P_n(\delta')$ as the span of the partitions with every part having exactly two elements, one primed and the other unprimed, of $\underline{n} \cup \underline{n}'$. When we write $\underline{d}$ for a *poset*, we mean $\{1, 2, \ldots, d\}$ equipped with the natural partial order $\leq$ (although we will often concentrate on the case $\underline{2} = (\{1, 2\},\ 1 \leq 2)$ in this thesis). Throughout this chapter, F will denote a field. For $X' \subset X$ and $c \in P_X$ we define $c|_{X'}$ as the collection of the sets of the form $c_i \cap X'$ where the c_i are the parts of the partition c.

Definition 3.1. For a set X, we define the *refinement* partial order on $\mathcal{P}_X$ as follows. For $p,\ q \in \mathcal{P}_X$, we say p is a refinement of q, denoted $p \leq q$, if each part of q is a union of one or more parts of p.

For example, the set partition $p = \{\{1,2\},\{3,4,5\},\{6\}\}$ is a refinement of the set partition $q = \{\{1,2,3,4,5\},\{6\}\}$ since $\{1,2,3,4,5\} \in q$ is the union of the parts $\{1,2\}$ and $\{3,4,5\}$ of p, and $\{6\}$ in q is a part of p.

Proposition 3.2 (See [12, Prop. 1])**.** *Let $p,\ q \in \mathcal{P}_X$ and $Y \subseteq X$. Then $p \leq q$ implies $p|_Y \leq q|_Y$.* □

3.2 Ramified partition algebra

The ramified partition algebra was introduced by Martin and Elgamal [12] as a generalisation of the ordinary partition algebra $P_n(\delta')$ (see Chapter 2 for a review).

3.2.1 The ramified partition monoid

Definition 3.3. Let $(T, \leq)$ be a finite poset. For a set X, we define $\mathbf{P}_X^T$ to be the subset of the Cartesian product $\prod_T \mathcal{P}_X$ consisting of those elements $q = (q_i \colon i \in T)$ such that $q_i \leq q_j$ whenever $i \leq j$. Any such element $q \in \mathbf{P}_X^T$ will be referred to as a *T-ramified partition.*

For example, some elements of $\mathbf{P}^{\underline{2}}_{\underline{2}\cup\underline{2}'}$ are listed below:

$$\begin{aligned}
\pi_1 &= (\{\{1\},\{2\},\{1'\},\{2'\}\},\{\{1\},\{2\},\{1'\},\{2'\}\}) \\
\pi_2 &= (\{\{1,2\},\{1'\},\{2'\}\},\{\{1,2,1'\},\{2'\}\}) \\
\pi_3 &= (\{\{1,2'\},\{2,1'\}\},\{\{1,2,1',2'\}\}),
\end{aligned}$$

and so on.

We now recall from [12] the diagrammatic realization of an element of $\mathbf{P}^T_{\underline{n}\cup\underline{n}'}$. We shall only need the case $T = \underline{2}$ here. We first look at an example from [12]. The diagram in Figure 3.1 represents

$$(\{\{1,2,3\},\{1',2'\},\{3'\},\{4,5'\},\{5,4'\}\},\{\{1,2,3,1',2'\},\{3'\},\{4,5'\},\{5,4'\}\}).$$

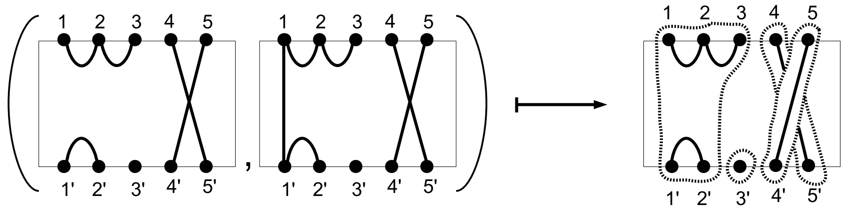


FIGURE 3.1: A diagram representing a $\underline{2}$-ramified partition

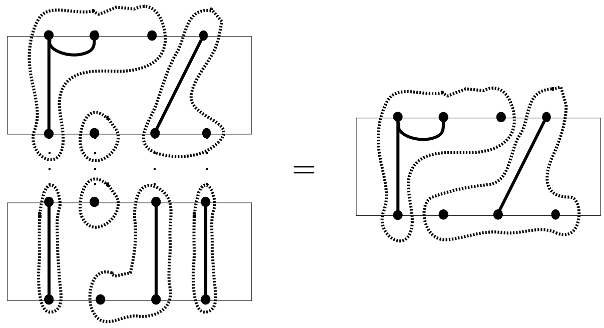


FIGURE 3.2: The composition of ramified $2n$- partition diagrams

Consider an element (p, q) in $\mathbf{P}^{\mathbf{2}}_{\underline{n}\cup\underline{n}'}$. Then p and q can be thought of as partition algebra diagrams in which the connected components (the parts of p) are grouped into disjoint sets or "islands". The islands are the parts of q. Note that islands can cross (as illustrated in Figure 3.1), but it is not hard to draw them unambiguously.

Similarly, a diagram representing a T-ramified partition is not unique. We say two diagrams are equivalent if they give rise to the same T-ramified partition.

The term *ramified partition diagram* (or sometimes *ramified* $2n$*-partition diagram* to indicate the number of vertices) will be used to mean the equivalence class of the given diagram.

We refer to the edges in the underlying partition algebra diagram of a ramified partition diagram as *bones*.

The composition of ramified $2n$-partition diagrams in $\mathbf{P}^{T}_{\underline{n}\cup\underline{n}'}$ is as follows. First identify the bottom of one ramified $2n$-partition diagram with the top of the other, composing the underlying partition algebra diagrams as in Section 2.1. The islands in the composition are the connected components of the union of the islands in each of the diagrams. Then discard any island connected components that are isolated from the boundaries in composition as shown in Figure 3.2.

Throughout, we shall identify a ramified partition with its ramified partition diagram and speak of them interchangeably.

Proposition 3.4 (See [12, Prop. 2]). *For any d-tuple $\delta = (\delta_1, \dots, \delta_d) \in F^d$, the set $\mathbf{P}^T_{\underline{n}\cup\underline{n}'}$ forms a basis for a subalgebra of $\bigotimes_{t\in T} P_n(\delta_t)$.*

Proof. The proof can be found in [12]. □

Proposition 3.5. *For each $n \in \mathbb{N}$ the set of ramified $2n$-partition diagrams, $\mathbf{P}^T_{\underline{n}\cup\underline{n}'}$, with multiplication defined by composition of diagrams (as defined above), is a monoid.*

Proof. Clearly, the identity element in $\mathbf{P}^T_{\underline{n}\cup\underline{n}'}$ is

$$(\{\{1,1'\},\{2,2'\},\dots,\{n,n'\}\},\{\{1,1'\},\{2,2'\},\dots,\{n,n'\}\}).$$

It remains to check that the multiplication operation is associative, but this is easy to verify. □

3.2.2 The ramified partition algebras

For $\delta = (\delta_1, \delta_2, \dots, \delta_d) \in F^d$, the *$T$-ramified partition algebra* $P^{(T)}_n(\delta)$ over F is the finite dimensional algebra with basis $\mathbf{P}^T_{\underline{n}\cup\underline{n}'}$ and product induced by the product of $\mathbf{P}^T_{\underline{n}\cup\underline{n}'}$ in a way made precise as follows.

Let $r, s \in \mathbf{P}^T_{\underline{n}\cup\underline{n}'}$. When forming the product rs in $\mathbf{P}^T_{\underline{n}\cup\underline{n}'}$ a crucial step involved the removal of connected components that are isolated from the boundaries after the composition of diagrams of r and s. Instead, replace any bone (resp. island) connected components that are isolated from the boundaries in composition by a factor δ_1 (resp. δ_2) as shown in Figure 3.3. In [12] it is shown that this operation (extended linearly over F) gives $P^{(T)}_n(\delta)$ the structure of an associative F-algebra.

A line joining the top part of a diagram and the bottom part will be called a *propagating line* (but note that in general, equivalent diagrams might have a different number of propagating lines). The propagating number (see Section 2.2) of a partition diagram is the same as the smallest number of propagating lines in a diagram representing it. A propagating line with an island around it will be called a *propagating stick*.

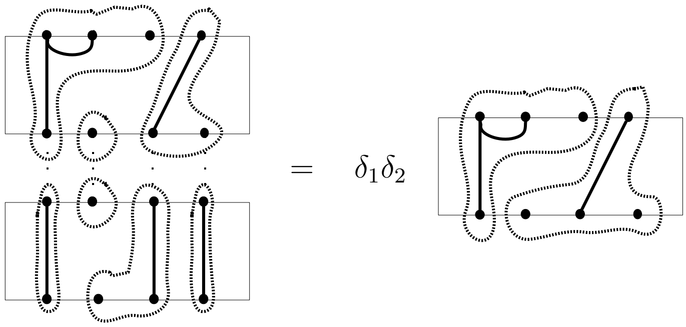


FIGURE 3.3: The composition of diagrams in $P_4^{(\underline{2})}(\delta)$

The complex generic representation theory of $P_n^{(T)}(\delta)$ has been determined in the case $T = \underline{2}$ in [12]. It was shown that there are infinitely many choices of δ such that $P_n^{(\underline{2})}(\delta)$ is not semisimple for sufficiently large n, but that it is generically semisimple for all n.

3.2.3 Small Ramified Partition Algebra P_n^{ς}

In this section we recall the definition of the small ramified partition algebra. To define this algebra we require the following definitions.

Definition 3.6. We define diag-$\mathcal{P}_n$ to be the subset of $\mathcal{P}_{\underline{n}\cup\underline{n}'}$ such that i, i' are in the same part for all $i \in \mathbb{N}$.

For example, recall from [3] the special elements in $\mathcal{P}_{\underline{n}\cup\underline{n}'}$ as follows.

$$\begin{aligned}
1 &= \{\{1,1'\},\ \{2,2'\},\ \ldots\{i,i'\},\ \ldots\{n,n'\}\} \\
A^{i,j} &= \{\{1,1'\},\ \{2,2'\},\ \ldots\{i,i',j,j'\},\ \ldots\{n,n'\}\} \quad i,j = 1,2,\ldots,n \\
\sigma_{i,j} &= \{\{1,1'\},\ \{2,2'\},\ \ldots\{i,j'\},\{j,i'\}\ \ldots\{n,n'\}\} \quad i,j = 1,2,\ldots,n \\
e_i &= \{\{1,1'\},\ \{2,2'\},\ \ldots\{i\},\{i'\},\ \ldots\{n,n'\}\} \quad i = 1,2,\ldots,n.
\end{aligned}$$

Here, 1 and $A^{i,j}$ are in diag-$\mathcal{P}_n$ while $\sigma_{i,j}$ and e_i are not. Note that $\sigma_{i,i+1}$ corresponds to the simple transposition $(i, i+1) \in S_n$, and the elements of the set $\{\sigma_{i,i+1} \colon 1 \le i \le n-1\}$ generate FS_n.

Definition 3.7. For any $\delta' \in F$, we define Δ_n to be the subalgebra of $P_n(\delta')$ generated by the elements of S_n and the $A^{i,j}$, $i, j = 1, 2, \ldots, n$.

Proposition 3.8. *The map*

$$\varsigma \colon S_n \times \textit{diag-}\mathcal{P}_n \to S_n \times \mathcal{P}_{\underline{n}\cup\underline{n}'}$$

given by

$$(a, b) \mapsto (a, ba)$$

defines an injective map.

Proof. The well-definedness of ς is clear. To prove that ς is an injective map, it suffices to show that if (a, ba) is equal to (c, dc) in $S_n \times \mathcal{P}_{\underline{n}\cup\underline{n}'}$, then (a, b) is equal to (c, d) in $S_n \times \text{diag-}\mathcal{P}_n$. Assume that $(a, ba) = (c, dc)$. Since $a = c$, then $bc = dc$. But c is invertible, thus, $b = d$. $\square$

Note that ς is not a surjective map as there are some elements in $S_n \times \mathcal{P}_{\underline{n}\cup\underline{n}'}$ that are not images of elements in $S_n \times \text{diag-}\mathcal{P}_n$ under ς. For example, although any non-identical pair of permutations is an element in $S_n \times \mathcal{P}_{\underline{n}\cup\underline{n}'}$, it is not an image of any element in $S_n \times \text{diag-}\mathcal{P}_n$ under the map ς.

Definition 3.9. We define $\mathbb{P}_{\underline{n}\cup\underline{n}'}$ to be the subset of the Cartesian product $S_n \times \mathcal{P}_{\underline{n}\cup\underline{n}'}$ given by the elements $q = (q_1, q_2)$ such that q_1 is a refinement of q_2.

Proposition 3.10 (See [13, p. 5])**.** *The set $B_n^{\varsigma} := \varsigma(S_n \times \textit{diag-}\mathcal{P}_n)$ lies in $\mathbb{P}_{\underline{n}\cup\underline{n}'}$ and forms a basis for a subalgebra of $FS_n \otimes_F \Delta_n$.* $\square$

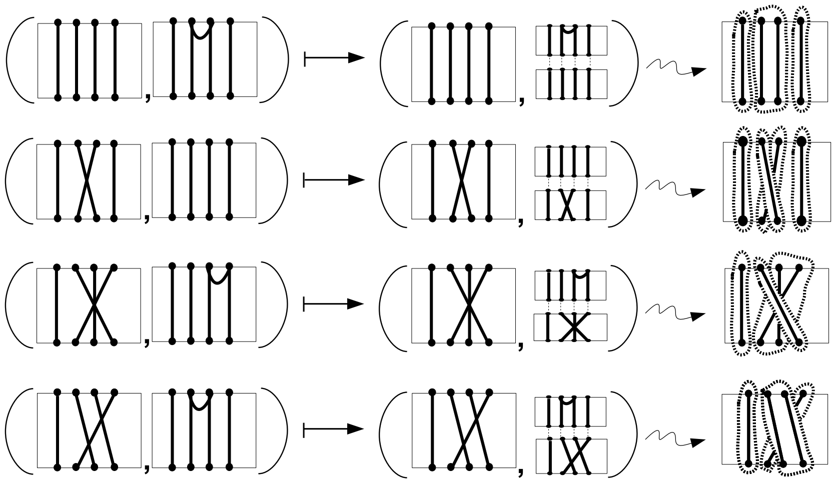
Definition 3.11. The associative algebra P_n^{ς} over F is the free F-module with B_n^{ς} as basis and multiplication inherited from the multiplication on $P_n^{(\underline{2})}(\delta)$. We call P_n^{ς} the *small ramified partition algebra.*

It is easy to check that

Lemma 3.12. *The multiplication on P_n^{ς} is well-defined up to equivalence.*

There is a diagram representation of B_n^{ς} since its elements are $\underline{2}$-ramified partitions (see [13, p. 6]). Each element of the basis $\varsigma(S_n \times \text{diag-}\mathcal{P}_n)$ of B_n^{ς} is obtained by taking a permutation in S_n and partitioning its parts (propagating lines) into disjoint islands.

Example 3.13. The map defined in Proposition 3.8 is illustrated by the following pictures.



In particular, these pictures describe the diagrammatic realization of some basis elements in P_4^ς.

Lemma 3.14 (See [13, §3.4]). *The dimension of P_n^ς is given by $n!B_n$, where B_n is the nth Bell number.* □

Remark 3.15. Notice that, P_n^ς is spanned by diagrams with propagating number n (See Example 3.13). This means that, unlike the ramified partition algebras, the small ramified partition algebras do not depend on parameter δ.

Definition 3.16. For any $\delta' \in F$, we define Γ_n as the subalgebra of $P_n(\delta')$ generated by the elements of $A^{i,j}$, $i, j = 1, 2, \ldots, n$.

Note that the natural injection of Γ_n into P_n^ς is given by

$$A^{i,j} \mapsto (1, A^{i,j})$$

and there exists a natural injection of FS_n into P_n^ς given by

$$\sigma_{i,i+1} \mapsto (\sigma_{i,i+1}, \sigma_{i,i+1}) = \varsigma(\sigma_{i,i+1}, 1).$$

Proposition 3.17 (See [13, Prop. 2]). *The algebra P_n^ς is generated by $(1, A^{i,i+1})$ and $(\sigma_{i,i+1}, \sigma_{i,i+1})$ ($i = 1, 2, \ldots, n-1$).* □

Chapter 4

The Representation Theory of the small ramified partition algebra P_n^{ς}

In this chapter we study the representations of the small ramified partition algebras of Section 3. Our aim is to classify their finite dimensional representations over an algebraically closed field of characteristic zero. The final Theorem (Theorem 4.57) in this chapter is due to Martin [13]. However, the proof in [13] is very terse. Here we present an explicit proof of the Theorem by providing the proofs (which we have not found in the literature) of the results leading to it. We follow closely the notation of [13].

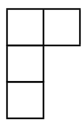
In Section 4.1 and Section 4.2, we recall some relevant definitions that will be needed later. In Section 4.3, we give a concise exposition of the representation theory of the wreath product $G \wr S_n$ with G a finite group (see, for example [4, Chapter 4], [37, Chapter 5], [38, Section 3.1], [39], [40, Appendix A]) over an algebraically closed field F of characteristic zero. The focus then turns to working out the irreducible representations of P_n^{ς}. The case P_2^{ς} is worked out as an illustrative example in Section 4.4. Before describing explicitly the structure of P_n^{ς}, we describe an indexing set for the irreducible representations of P_n^{ς} in Section 4.5. In Section 4.6, we begin our study of the category of P_n^{ς}-modules. In what follows, in Section 4.7, we recall the definition of Morita equivalence (see, for example [41, p. 325]) and a result about Morita equivalence of F-algebras. This result is then applied to certain algebras related to the small ramified partition algebra. We give an explicit construction of the simple P_n^{ς}-modules in Section 4.8.

4.1 Set partition shapes and combinatorics

Definition 4.1. We define the *shape* of a set partition b to be the list of sizes of parts of b in non-increasing order.

It is clear that the shape of a partition of $\underline{n}$ is an integer partition of n. We write $b \Vdash \mu$ to denote that b has shape μ.

Remark 4.2. We can think of the shape of b as a *Young diagram*. For example, a Young diagram with shape



corresponds to $(2, 1, 1)$.

See Appendix A for more details.

The following power notation is useful in the case when several parts of b are of the same length:

$$\mu = (\underbrace{\lambda_1, \lambda_1, \ldots, \lambda_1}_{p_1}, \underbrace{\lambda_2, \lambda_2, \ldots, \lambda_2}_{p_2}, \ldots) \qquad \rightsquigarrow \qquad \lambda^p = (\lambda_1^{p_1}, \lambda_2^{p_2}, \ldots).$$

Exponents equal to unity are omitted.

Example 4.3. The set partition $b = \{\{1, 2\}, \{3, 5, 7\}, \{4, 6\}\} \in \mathcal{P}_{\underline{7}}$ has shape $(3, 2^2)$.

For the following, we adopt the convention of multiplying permutations right to left.

The symmetric group S_n acts on $\mathcal{P}_{\underline{n}}$ from the left via the map

$$\begin{aligned} S_n \times \mathcal{P}_{\underline{n}} &\to \mathcal{P}_{\underline{n}} \\ (\pi, a) &\mapsto \pi a := \{\pi a' \colon a' \in a\}, \end{aligned}$$

where $\pi a' := \{\pi i \colon i \in a'\}$. Thus, for each $a \in \mathcal{P}_{\underline{n}}$ and any $\pi,\ \pi' \in S_n$ the following holds:

$$\pi(\pi' a) = (\pi\pi')a.$$

Example 4.4. Consider $\pi = (12),\ \pi' = (132) \in S_3$. If $a = \{\{1,2\},\{3\}\} \in \mathcal{P}_{\underline{3}}$ then the action described above gives the following.

$$(12)\left((132)\{\{1,2\},\{3\}\}\right) \;=\; ((12)(132))\{\{1,2\},\{3\}\} \;=\; \{\{3,2\},\{1\}\}.$$

We next introduce a partial order on partitions of n.

Definition 4.5. Suppose λ and λ' are two partitions of n. We say that λ is a *refinement* of λ', denoted $\lambda \leq \lambda'$, if the parts of λ' are unions of parts of λ.

(We write $\lambda < \lambda'$ if $\lambda \leq \lambda'$ and $\lambda \neq \lambda'$.)

For example, the diagram of partitions of 4 ordered by refinement is shown in Figure 4.1.

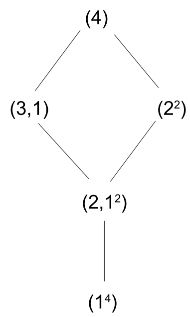


FIGURE 4.1: Diagram of partitions of 4 ordered by refinement

To specify a function μ from a set S to a set T, given an ordered list, $\underline{x}$, of the elements of S, we may write $\mu\colon \underline{x} \mapsto \underline{y}$, meaning $\mu(x_i) = y_i$ for all i. But if almost all $\mu(x_i) = t_0$, with t_0 some given element of T then it is convenient, following [13], to write

$$\mu = \frac{x_{i_1}}{\mu(x_{i_1})}\frac{x_{i_2}}{\mu(x_{i_2})}\cdots \tag{4.1}$$

where $\{i_1, i_2, \ldots\}$ is the set of i such that $\mu(x_i) \neq t_0$.

For example, $\mu\colon (1,2,3,4,\ldots) \mapsto (3,4,5,0,0,0,\ldots)$ becomes $\frac{1}{3}\frac{2}{4}\frac{3}{5}$ (with $t_0 = 0$)

4.2 The stabilizer of a set partition

We now introduce the so-called *stabilizer* [42, p. 144] of a set partition in S_n. The representations induced from stabilizers play a vital role in the theory of P_n^{ς} (as we shall see shortly).

Definition 4.6. Let $b \in \mathcal{P}_{\underline{n}}$ be a set partition. The *stabilizer* $S(b)$ of b in S_n is the group of all permutations $\sigma \in S_n$ such that $\sigma b = b$.

Theorem 4.7 (See, for example [42, p. 144])**.** *Let $b \in \mathcal{P}_{\underline{n}}$ be a partition of shape $\lambda^p = (\lambda_1^{p_1}, \lambda_2^{p_2}, \ldots)$. The group $S(b)$ is isomorphic to the* direct product

$$\prod_i (S_{\lambda_i} \wr S_{p_i}) \tag{4.2}$$

of wreath products *of symmetric groups.* □

(The wreath product is discussed in Section 4.3.)

The subgroup $S(b)$ contains all permutations $\sigma \in S_n$ which preserve the parts of the partition b, or that permute parts of the same size. Thus, we mention two subgroups in $S(b)$ for $b \Vdash \lambda^p$:

Let $S^0(b)$ denote the group that permutes within parts: $S^0(b) \cong (S_{\lambda_1})^{\times p_1} \times (S_{\lambda_2})^{\times p_2} \times \ldots \subset S_n$; and let $S^1(b)$ denote the group that permutes parts of equal size: $S^1(b) \cong S_{p_1} \times S_{p_2} \times \ldots \subset S_n$.

Example 4.8. Consider $b \in \mathcal{P}_{\underline{3}}$. If

$$\begin{aligned}
b = \{\{1\},\{2\},\{3\}\} &\text{ then } S(b) = \{S_1 \wr S_3 \cong S_3\},\\
b = \{\{1,2\},\{3\}\} &\text{ then } S(b) = \{(S_2 \wr S_1) \times (S_1 \wr S_1) \cong S_2\},\\
b = \{\{1,3\},\{2\}\} &\text{ then } S(b) = \{(S_2 \wr S_1) \times (S_1 \wr S_1) \cong S_2\},\\
b = \{\{2,3\},\{1\}\} &\text{ then } S(b) = \{(S_2 \wr S_1) \times (S_1 \wr S_1) \cong S_2\},\\
b = \{\{1,2,3\}\} &\text{ then } S(b) = \{S_3 \wr S_1 \cong S_3\},
\end{aligned}$$

Definition 4.9. Let G be a group acting on a set X, and let $x \in X$. Then the *orbit of x under G* is the subset of X defined by

$$Gx = \{gx \colon g \in G\}.$$

It is easy to verify that

Lemma 4.10. *The orbit of $b \in \mathcal{P}_{\underline{n}}$ under S_n consists of those set partitions of the same shape as b.*

As Example 4.4 illustrates, S_n acts transitively on set partitions of a fixed shape, i.e. the action has exactly one orbit. Thus the number of set partitions of a given shape λ^p, using the orbit-stabilizer theorem (see [43, Theorem 3.9.2]), is

$$\mathcal{D}_{\lambda^p} = \frac{n!}{\prod_i((\lambda_i!)^{p_i}p_i!)} = \frac{n!}{|S(b)|} \tag{4.3}$$

(where b is any set partition of shape λ^p).

We shall establish later a construction of irreducible representations of our algebra P_n^ς directly in terms of representations of $S(b)$. Since $S(b)$ is the direct product of wreath products of symmetric groups, one is led to study wreath product groups.

4.3 Representations of wreath products

In this section, our attention is restricted to wreath products $G \wr H$ with $H = S_n$ and G any finite group. We recall the classification of irreducible representations of the group $G \wr S_n$ for any finite group G over the complex fields $\mathbb{C}$. For such groups, the representation theory is closely related to that of G and of the symmetric groups (see [1] for a review). For a comprehensive treatment of this topic refer e.g. to [4, Chapter 4], [37, Chapter 5], [44, §2]. However, the exposition given in [4] and [37] is quite lengthy while that given in [44] is brief and somewhat abstract. Here we discuss the subject in a concise and lucid manner. Good references for applications of wreath product groups and its representations are [38, Section 3.1], [39], [40, Appendix A].

It is enough for us to study the wreath factors of $S(b)$ since the field we are working over is the complex field.

4.3.1 Wreath product definition

Notation: Let $|G|$ denote the order of a group G.

Recall the *direct product* of two groups G and H

$$G \times H = \{(g, h) \colon g \in G, h \in H\}$$

with identity element $1_{G\times H} = (1_G, 1_H)$ and group operations

$$\begin{aligned}(g_1, h_1)(g_2, h_2) &= (g_1 g_2, h_1 h_2)\\ (g, h)^{-1} &= (g^{-1}, h^{-1}).\end{aligned}$$

We denote by $\mathrm{Aut}(G)$ the automorphism group of G. (Recall that $\mathrm{Aut}(G) = \{f \colon G \to G \colon f \text{ is an isomorphism}\}$ and that $\mathrm{Aut}(G)$ is a group under function composition.)

The notion of *semidirect product* of two groups generalises the idea of a direct product.

Definition 4.11. Suppose that X is a group with a normal subgroup G and a subgroup H such that

$$X = GH \quad \textit{and} \quad G \cap H = \{e\}.$$

Then X is said to be the *internal semidirect product* of G and H.

Since G is normal in X, for each $h \in H$ we have an automorphism of G given by $\psi \colon g \mapsto hgh^{-1}$. It is easy to verify that $\psi(h_1 h_2) = \psi(h_1)\psi(h_2)$; thus $\psi \colon H \to \mathrm{Aut}(G)$ is a homomorphism.

Definition 4.12. Let G and H be groups. Let $\psi \colon H \to \mathrm{Aut}(G)$ be a homomorphism. We define a binary operation $\cdot$ on $G \times H$ by

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \psi(h_1)(g_2), h_1 h_2).$$

The set $G \times H$, equipped with the operation $\cdot$ forms a group, called the *external semidirect product* of G and H with respect to ψ and is denoted $G \rtimes_\psi H$.

For simplicity's sake, we frequently omit the ψ and simply write $G \rtimes H$ instead. Often we write $(g_1, h_1)(g_2, h_2)$ instead of $(g_1, h_1) \cdot (g_2, h_2)$.

The identity element of $G \rtimes H$ is $(1_G, 1_H)$.

Let $G_1 = \{(g, 1_H) \colon g \in G\}$ and $H_1 = \{(1_G, h) \colon h \in H\}$. It is straightforward to show that these are subgroups of $G \rtimes H$ and that they are isomorphic to G and H respectively. The group operation $\cdot$ shows that

$$(g, 1_H)(1_G, h) = (g\psi(1_H)(1_G), h) = (g, h) \in G_1H_1.$$

In fact, $G \rtimes H$ is the internal semidirect product of G_1 and H_1.

We now define a special semidirect product that will be of particular interest to us, namely, the *wreath product.*

Definition 4.13. Suppose H is a subgroup of S_n acting on the set $\underline{n} = \{1, \ldots, n\}$. Define

$$G^n = \{f | f \colon \underline{n} \to G\}$$

to be the set of all mappings from $\underline{n}$ into a group G.

The *wreath product* of G and H, denoted by $G \wr H$, is, as a set, the cartesian product

$$G^n \times H = \{(f; \pi) | f \colon \underline{n} \to G, \pi \in H\}$$

with multiplication given by

$$(f; \pi)(f'; \pi') = (f f'_\pi; \pi\pi')$$

where $f_\pi \in G^n$ is the mapping $f_\pi \colon \underline{n} \to G$, defined by

$$f_\pi(i) = f(\pi^{-1}(i)), \qquad \text{for all } i \in \underline{n};$$

and for two maps f and $f' \colon \underline{n} \to G$,

$$f f'(i) := f(i) f'(i), \qquad \text{for all } i \in \underline{n}.$$

Its order (if G is finite) is $|G|^n|H|$.

It is easy to check that G^n is a normal subgroup of $G \wr H$ and that $G \wr H$ is a semidirect product of G^n and H.

Theorem 4.14. *Let G, H be groups as defined in Definition 4.13. Then $G \wr H$ is a group.* □

Proof. The identity element in $G \wr H$ is $(e; 1_H)$, where e is defined by

$$e(i) = 1_G \text{ for all } i \in \underline{n}$$

and 1_H is the identity of H. The inverse of an element $(f;\pi)$ in $G \wr H$ is $(f^{-1}_{\pi^{-1}}; \pi^{-1})$. The associativity is verified as follows: consider any three elements $(f^1;\pi_1)$, $(f^2;\pi_2)$, and $(f^3;\pi_3)$ in $G \wr H$. Then,

$$\begin{aligned}\left((f^1;\pi_1)(f^2;\pi_2)\right)(f^3;\pi_3) &= (f^1 f^2_{\pi_1}; \pi_1\pi_2)(f^3;\pi_3)\\ &= ((f^1 f^2_{\pi_1}) f^3_{\pi_1\pi_2}; \pi_1\pi_2\pi_3).\end{aligned}$$

However, by definition,

$$(f^1 f^2_{\pi_1}) f^3_{\pi_1\pi_2}(i) = f^1(i) f^2(\pi_1^{-1}(i)) f^3(\pi_2^{-1}\pi_1^{-1}(i)), \text{ for all } i \in \underline{n}. \tag{4.4}$$

Consider

$$\begin{aligned}(f^1;\pi_1)\left((f^2;\pi_2)(f^3;\pi_3)\right) &= (f^1;\pi_1)(f^2 f^3_{\pi_2}; \pi_2\pi_3)\\ &= (f^1(f^2 f^3_{\pi_2})_{\pi_1}; \pi_1\pi_2\pi_3).\end{aligned}$$

Again by definition,

$$\begin{aligned}f^1(f^2 f^3_{\pi_2})_{\pi_1}(i) &= f^1(i)(f^2 f^3_{\pi_2})(\pi_1^{-1}(i))\\ &= f^1(i) f^2(\pi_1^{-1}(i)) f^3(\pi_2^{-1}\pi_1^{-1}(i)), \text{ for all } i \in \underline{n}.\end{aligned} \tag{4.5}$$

Since the right hand sides of Equations (4.4) and (4.5) are equal, we have the associative law of multiplication of the elements of $G \wr H$. Thus, $G \wr H$ is a group. □

4.3.2 Conjugacy classes of wreath product groups

We shall describe the conjugacy classes of $G \wr S_n$. In order to do this, we introduce a notation. The use of this notation facilitates the calculation of the order of the set of conjugacy classes.

We write Λ for the set of all integer partitions including the empty partition $\emptyset$, and Λ_n for the subset consisting of partitions of n. For example,

$$\Lambda_3 = \{(1^3), (2,1), (3)\}.$$

For G a group, we write $\Lambda_{\mathbb{C}}(G)$ for an index set for ordinary irreducible representations (together, in principle, with a map to explicit representations) of G. Thus, $\Lambda_{\mathbb{C}}(S_n) = \Lambda_n$ (see for example, [45]). We shall use the analogous notation, $\Lambda_{\mathbb{C}}(A)$, for any algebra A over $\mathbb{C}$.

We set $r = |\Lambda_{\mathbb{C}}(G)|$. For X, Y any sets, we write $\mathrm{Mor}(X, Y)$ for the set of maps $f \colon X \to Y$. Thus an element V of $\mathrm{Mor}(\Lambda_{\mathbb{C}}(G), \Lambda)$ may be expressed as an ordered r-tuple $(V_1, V_2, \ldots, V_r)$ of integer partitions (a *multipartition*). For any finite set S, we write $\mathrm{Mor}(S, \Lambda)_n$ for the subset of $\mathrm{Mor}(S, \Lambda)$ consisting of multipartitions of the form $(V_1, V_2, \ldots)$ such that $\sum_i |V_i| = n$.

Theorem 4.15 (See [4, Corollary 4.4.4])**.** *There exists a bijection*

$$\begin{aligned} \Lambda_{\mathbb{C}}(G \wr S_n) &\to \mathrm{Mor}(\Lambda_{\mathbb{C}}(G), \Lambda)_n \\ L_V &\mapsto V \end{aligned}$$

Note that, by Proposition B.31, we can deduce that the number of conjugacy classes of $G \wr S_n$ is

$$|\mathrm{Mor}(\Lambda_{\mathbb{C}}(G), \Lambda)_n|.$$

As an example, we consider $\mathrm{Mor}(\Lambda_{\mathbb{C}}(S_3), \Lambda)_2$. Using the above ordering on $\Lambda_3 = \Lambda_{\mathbb{C}}(S_3)$, the elements are:

$$\begin{array}{ccc} ((2), \emptyset, \emptyset), & (\emptyset, (2), \emptyset), & (\emptyset, \emptyset, (2)), \\ ((1^2), \emptyset, \emptyset), & (\emptyset, (1^2), \emptyset), & (\emptyset, \emptyset, (1^2)), \\ ((1), (1), \emptyset), & ((1), \emptyset, (1)), & (\emptyset, (1), (1)). \end{array}$$

Hence, there are 9 conjugacy classes in the group $S_3 \wr S_2$. The order of this group is $6^2 \times 2 = 72$.

4.3.3 Induced representations

If G is a group and H is a subgroup of G, then a representation of G can be constructed from a representation of H by *induction* (see, for example [46, §4.1]).

This technique is of particular relevance for Section 4.3.4 as all irreducible representations of wreath products are obtained as induced representations.

Definition 4.16. Let $\rho\colon H \to GL(W)$ be a representation of H in a complex vector space W. Let $\mathrm{Ind}_H^G(W)$ be a vector space defined by

$$\mathrm{Ind}_H^G(W) = \mathbb{C}G \otimes_{\mathbb{C}H} W.$$

Then G acts on $\mathrm{Ind}_H^G(W)$ as follows:

$$s(g \otimes w) = sg \otimes w \qquad s \in G,\ g \in \mathbb{C}G,\ w \in W.$$

This action of G on $\mathrm{Ind}_H^G(W)$ is the *representation of G induced by ρ* and is denoted by $\rho \uparrow G$.

Dually, the *restriction* (see, for example [46, §4.1]) of a representation $\psi\colon G \to GL(V)$ defines a representation of a subgroup H. In this case, the representation is denoted by $\psi \downarrow H$ and the vector space $\mathrm{Res}_H^G(V) = V$.

4.3.4 Ordinary irreducible representations of wreath product groups

Let F denote an algebraically closed field, say $F = \mathbb{C}$, G a finite group and H a subgroup of S_n. We define a group

$$G^* := G_1 \times G_2 \times \ldots \times G_n$$

which is the direct product of n copies G_i of G, where

$$G_i = \{(f; 1_H)\colon f(j) = 1_G \text{ for all } j \neq i\} \cong G.$$

(G^* is often called the *base group* of the wreath product.) Let H' be the group

$$H' := \{(e; \pi)\colon \pi \in H\}.$$

Note that H' is a complement of G^* and isomorphic to H.

Since F is an algebraically closed field, the irreducible representations of G^* over F are the *outer tensor products* (see Appendix B for definition)

$$T^* := T_1 \# T_2 \# \cdots \# T_n$$

of irreducible representations T_i of G over F, where $\#$ denotes the outer tensor product.

The representing matrices of the outer tensor products can be obtained as the Kronecker product

$$\begin{aligned} T^*(f;1_H) &:= T_1(f(1)) \times T_2(f(2)) \times \cdots \times T_n(f(n)) \\ &= t^1_{a_1b_1}(f(1)) t^2_{a_2b_2}(f(2)) \dots t^n_{a_nb_n}(f(n)). \end{aligned} \tag{4.6}$$

(where the $t^i_{a_ib_i}(f(i))$ are the matrix entries of $T_i(f(i))$).

To obtain the irreducible representations of wreath product groups, first we derive the *inertia group* $G \wr H_{T^*}$ of this representation T^*, which is defined by

$$G \wr H_{T^*} = \{(f;\pi) \in G \wr H \colon T^{*(f;\pi)} \sim T^*\}$$

where $\sim$ denotes equivalence of representations and $T^{*(f;\pi)}$ is the representation conjugate to T^* defined as follows:

$$\begin{aligned} T^{*(f;\pi)}((f';1_H)) &:= T^*\left((f;\pi)^{-1}(f';1_H)(f;\pi)\right) \\ &= T^*\left((f^{-1}_{\pi^{-1}}, \pi^{-1})(f';1_H)(f;\pi)\right) \\ &= T^*\left((f^{-1}_{\pi^{-1}} f'_{\pi^{-1}}, \pi^{-1})(f;\pi)\right) \\ &= T^*\left((f^{-1}_{\pi^{-1}} f'_{\pi^{-1}} f_{\pi^{-1}}, 1_H)\right) \\ &= T^*\left((f^{-1}f'f)_{\pi^{-1}}; 1_H\right). \end{aligned} \tag{4.7}$$

The group $G \wr H_{T^*}$ by definition is a product

$$G \wr H_{T^*} = G^* H'_{T^*}$$

of G^* with a subgroup H'_{T^*} of the complement H' of G^*. The group H'_{T^*} will be called the *inertia factor* of T^* :

$$H'_{T^*} = \{(e;\pi)\colon T^{*(e;\pi)} \sim T^*\}.$$

We notice that, by substituting e for f into Equation (4.7),

$$T^{*(e;\pi)} = T^*(f_{\pi^{-1}}; 1_H).$$

To describe the inertia factor explicitly, we distinguish the irreducible representations (over F) of T^* with respect to their type. That is,

Definition 4.17. Let $F^1, F^2, \ldots, F^r$ be a fixed listing of the r pairwise inequivalent representations of G over F. T^* is said to be of *type* $(n) = (n_1, n_2, \ldots, n_r)$ with respect to the above listing if n_j is the number of factors T_i of T^* equivalent to F^j.

Let S_{n_j} be the subgroup of S_n consisting of the elements permuting exactly the n_j indices of the n_j factors T_i of T^* which are equivalent to F^j.

Define

$$S'_{(n)} = S'_{n_1} \times S'_{n_2} \times \cdots \times S'_{n_r}$$

with

$$S'_{n_j} = \{(e;\pi)\colon \pi \in S_{n_j}\}.$$

In this setup, it was proved in [4] that

$$H'_{T^*} = H' \cap S'_{(n)}$$

so that for the inertia group of T^* the following holds:

$$G \wr H_{T^*} = G^*(H \cap S_{(n)})' = G \wr (H \cap S_{(n)}).$$

The representations $\widehat{T}^*$ whose matrices are defined as follows form the irreducible representations of $G \wr H_{T^*}$:

$$\widehat{T}^*(f;\pi) = t^1_{a_1 b_{\pi^{-1}(1)}}(f(1)) t^2_{a_2 b_{\pi^{-1}(2)}}(f(2)) \dots t^n_{a_n b_{\pi^{-1}(n)}}(f(n)).$$

Let T' be an irreducible representation of the inertia factor H'. Let $\widehat{T}^*$ be determined using the method outlined above. Then the representation $\widehat{T}^* \otimes T'$ is an irreducible representation of $G \wr H_{T^*}$.

Proposition 4.18 (See [4, 4.3.33])**.** *The induced representation* $(\widehat{T}^* \otimes T') \uparrow (G \wr H)$ *is irreducible and every irreducible representation of* $G \wr H$ *over* F *is of this form.* □

The dimension of the irreducible representation $(\widehat{T}^* \otimes T') \uparrow (G \wr H)$ of $G \wr H$ is given by

$$\dim\Big((\widehat{T}^* \otimes T') \uparrow (G \wr H)\Big) = \dim(\widehat{T}^* \otimes T') \frac{|G \wr H|}{|G \wr H_{T^*}|}.$$

Example 4.19. As an example, we derive the ordinary irreducible representations of $S_2 \wr S_2$. We will denote an irreducible representation T of a group G by $[\lambda]$ where λ is the partition associated with T. For example, the irreducible representations of the group S_2 are $[1^2]$ and $[2]$.

(1) The irreducible representations of the basis group S_2^* are

$$[2]\#[2], \quad [2]\#[1^2], \quad [1^2]\#[1^2], \quad [1^2]\#[2].$$

With respect to the listing $[2]$, $[1^2]$ of the irreducible representations of S_2, the types of these representations are:

$$(2,0) \quad (1,1) \quad (0,2) \quad (1,1).$$

Hence a complete system of irreducible representations of S_2^* with pairwise different types is

$$[2]\#[2], \quad [2]\#[1^2], \quad [1^2]\#[1^2].$$

(2) The corresponding inertia groups are: $S_2 \wr S_2$, $S_2 \times S_2$, $S_2 \wr S_2$; the inertia factors are: S_2', S_1', S_2'.

(3) Consequently, the irreducible ordinary representations of $S_2 \wr S_2$ are:

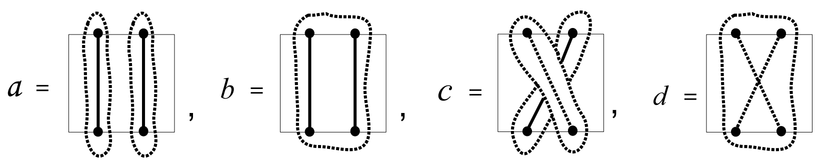
$$\begin{aligned}
\widehat{[2]\#[2]} \otimes [2]' &= \widehat{[2]\#[2]}, \\
\widehat{[2]\#[2]} \otimes [1^2]', & \\
\widehat{[2]\#[1^2]} \otimes [1]' \uparrow (S_2 \wr S_2) &= [2]\#[1^2] \uparrow (S_2 \wr S_2), \\
\widehat{[1^2]\#[1^2]} \otimes [2]' &= \widehat{[1^2]\#[1^2]}, \\
\widehat{[1^2]\#[1^2]} \otimes [1^2]'. &
\end{aligned}$$

Their dimensions are $1, 1, 2, 1, 1$, respectively, satisfying Theorem B.3. Here, we have $1^2 + 1^2 + 2^2 + 1^2 + 1^2 = 8 = |S_2 \wr S_2|$.

4.4 The regular P_n^ς-module

Recall that the set of all ramified partition diagrams on $\underline{n} \cup \underline{n}'$ forms a monoid, written as B_n^ς.

For example, $B_2^\varsigma = \{a, b, c, d\}$ where a, b, c, d are:



Consider the free $\mathbb{C}$-module $\mathbb{C}B_n^\varsigma = P_n^\varsigma$ with basis B_n^ς. This module is a monoid algebra over $\mathbb{C}$ by virtue of the monoid multiplication. We now describe the *regular representation* of the monoid B_n^ς.

Set $r = |B_n^\varsigma|$. The action of B_n^ς on the monoid algebra $P_n^\varsigma = \{\lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_r g_r \colon \lambda_i \in \mathbb{C},\ g_i \in B_n^\varsigma$ can be expressed as

$$g(\lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_r g_r) = (\lambda_1 g g_1 + \lambda_2 g g_2 + \cdots + \lambda_r g g_r)$$

for all $g \in B_n^\varsigma$. We obtain a *left regular representation* of B_n^ς in this fashion.

Example 4.20. Let $B_2^\varsigma = \{a, b, c, d\}$ as described above. The elements of the algebra $\mathbb{C}B_2^\varsigma = P_2^\varsigma$ have the form

$$\lambda_1 a + \lambda_2 b + \lambda_3 c + \lambda_4 d \quad (\lambda_i \in \mathbb{C}).$$

We have

$$\begin{aligned}
a(\lambda_1 a + \lambda_2 b + \lambda_3 c + \lambda_4 d) &= \lambda_1 a + \lambda_2 b + \lambda_3 c + \lambda_4 d, \\
b(\lambda_1 a + \lambda_2 b + \lambda_3 c + \lambda_4 d) &= \lambda_1 b + \lambda_2 b + \lambda_3 d + \lambda_4 d, \\
c(\lambda_1 a + \lambda_2 b + \lambda_3 c + \lambda_4 d) &= \lambda_1 c + \lambda_2 d + \lambda_3 a + \lambda_4 b, \\
d(\lambda_1 a + \lambda_2 b + \lambda_3 c + \lambda_4 d) &= \lambda_1 d + \lambda_2 d + \lambda_3 b + \lambda_4 b.
\end{aligned}$$

By taking matrices relative to the basis a, b, c, d of $\mathbb{C}B_2^\varsigma$ we obtain the regular representation of B_2^ς :

$$a \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$c \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad d \rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

We do not yet have the tools for a systematic analysis of the representation theory of a monoid, in particular B_n^ς, but a couple of observations are in order. Suppose a representation R of an algebraic structure has been found which consists of matrices each being an $n \times n$ matrix. We can form another representation R' by a *similarity transformation* (see, for example [47, §5.2])

$$R'(g) = S^{-1}R(g)S,$$

S being a nonsingular matrix. Thus, R and R' are equivalent representations (see Definition B.2). Using similarity transformations, it is often possible to bring each matrix in the representation monoid (or group) into a diagonal form of (B.3).

Example 4.21. Consider the regular representation of the monoid B_2^ς in Example 4.20. Over $\mathbb{C}$, we choose the basis $\{-b-d, b-a+c-d, -b+d, b-a+d-c\}$.

Then

$$a \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$c \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad d \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

These are indeed all direct matrix sums of the form (B.3). Thus we have decomposed the regular representation of B_2^ς (over $\mathbb{C}$) into its irreducible parts. In particular, there are exactly 4 1-dimensional inequivalent representations in the regular representation above. By the Artin-Wedderburn theorem (see Theorem B.40), this decomposition can only happen for a semisimple algebra and the regular representations obtained are the only ones. For a general field, we have

Proposition 4.22. *Let F be an arbitrary field. Then the algebra P_2^ς is semisimple over F provided 2 is invertible in F.*

Proof. It is easy to see that the elements $-b-d, b-a+c-d, -b+d, b-a+d-c$ form a basis of P_2^ς over a field F whenever 2 is invertible in F. Thus, the proposition follows from the above argument. □

Note that the sum of the squares of the dimensions of these inequivalent irreducible represention is $|B_2^\varsigma|$.

We shall show in Section 4.6 that, for each $n \in \mathbb{N}$, P_n^ς is semisimple over $F = \mathbb{C}$. Before turning our attention to the decomposition of the regular P_n^ς-module (for $n \geq 2$) into simple modules, we describe an indexing set for the simple P_n^ς-modules.

4.5 Indexing set for the simple P_n^ς-modules

In this section, we describe an indexing set for the simple modules of P_n^ς. This will be useful later.

Let Λ (resp. Λ^*) be the set of all finite Young diagrams including (resp. excluding) the empty diagram. We write $\mathrm{Mor}^f(\Lambda^*, \Lambda)$ for the set of functions

$$\mu\colon \Lambda^* \to \Lambda$$

with only finitely many $\lambda \in \Lambda^*$ such that $\mu(\lambda) \neq \emptyset$. This condition means that the *degree* of $\mu \in \mathrm{Mor}^f(\Lambda^*, \Lambda)$

$$|\mu| = \sum_{\lambda} |\lambda||\mu(\lambda)|$$

is well defined.

We denote the subset of $\mathrm{Mor}^f(\Lambda^*, \Lambda)$ of functions of degree $N \in \mathbb{N}$ by $\mathrm{Mor}_N(\Lambda^*, \Lambda)$.

For example, using notation (4.1) we have,

(i) $\mathrm{Mor}_1(\Lambda^*, \Lambda) = \left\{ \frac{(1)}{(1)} \right\}$.

(ii) $\mathrm{Mor}_2(\Lambda^*, \Lambda) = \left\{ \frac{(2)}{(1)}, \frac{(1^2)}{(1)}, \frac{(1)}{(2)}, \frac{(1)}{(1^2)} \right\}$.

(iii) $\mathrm{Mor}_3(\Lambda^*, \Lambda) = \left\{ \frac{(3)}{(1)}, \frac{(21)}{(1)}, \frac{(1^3)}{(1)}, \frac{(2)}{(1)}\frac{(1)}{(1)}, \frac{(1^2)}{(1)}\frac{(1)}{(1)}, \frac{(1)}{(3)}, \frac{(1)}{(21)}, \frac{(1)}{(1^3)} \right\}$.

The *shape* of a function $\mu \in \mathrm{Mor}^f(\Lambda^*, \Lambda)$ is an integer partition $\kappa(\mu)$ defined as follows. We define it using ascending power notation (see Section 4.1), in terms of which $\kappa(\mu)$ is given by the function

$$\alpha(i) = \sum_{\lambda \vdash i} |\mu(\lambda)|.$$

This can then be recast in ordinary power notation as described above.

For example, consider $\mu : ((3), (1^3), (2), (1^2), \ldots) \mapsto ((1), (1^2), (1), \emptyset, \ldots)$. Then
$\alpha(2) = \sum_{\lambda \vdash 2} |\mu(\lambda)| = |\mu((2))| + |\mu((1^2))| = |(1)| + |\emptyset| = 1$,
$\alpha(3) = \sum_{\lambda \vdash 3} |\mu(\lambda)| = |\mu((3))| + |\mu((1^3))| = |(1)| + |(1^2)| = 3$.

Therefore, μ has shape $\kappa(\mu) = (3^3, 2)$

Let $\mathrm{Mor}_{\lambda^p}(\Lambda^*, \Lambda)$ denote the subset of $\mathrm{Mor}^f(\Lambda^*, \Lambda)$ consisting of maps of shape λ^p. We have

$$\mathrm{Mor}_N(\Lambda^*, \Lambda) = \bigcup_{\lambda^p \vdash N} \mathrm{Mor}_{\lambda^p}(\Lambda^*, \Lambda).$$

For example,

$$\mathrm{Mor}_3(\Lambda^*, \Lambda) = \mathrm{Mor}_{(3)}(\Lambda^*, \Lambda) \cup \mathrm{Mor}_{(2,1)}(\Lambda^*, \Lambda) \cup \mathrm{Mor}_{(1^3)}(\Lambda^*, \Lambda)$$

where

$$\mathrm{Mor}_{(3)}(\Lambda^*, \Lambda) = \left\{ \frac{(3)}{(1)}, \frac{(21)}{(1)}, \frac{(1^3)}{(1)} \right\},$$

$$\mathrm{Mor}_{(2,1)}(\Lambda^*, \Lambda) = \left\{ \frac{(2)}{(1)}\frac{(1)}{(1)}, \frac{(1^2)}{(1)}\frac{(1)}{(1)} \right\},$$

and

$$\mathrm{Mor}_{(1^3)}(\Lambda^*, \Lambda) = \left\{ \frac{(1)}{(3)}, \frac{(1)}{(21)}, \frac{(1)}{(1^3)} \right\}.$$

If κ has just a single 'factor' i^m then $\mathrm{Mor}_{i^m}(\Lambda^*, \Lambda)$ is just the set of maps from Λ_i to Λ such that $\alpha(i) = m$ (and $\alpha(j) = 0,$ for all $j \neq i$).

Lemma 4.23. *The map*

$$\mathrm{Mor}_{i^m}(\Lambda^*, \Lambda) \to \mathrm{Mor}_m(\Lambda_i, \Lambda)$$
$$\mu \mapsto \mu|_{\Lambda_i}$$

is a bijection. □

By Theorem 4.15 and Lemma 4.23,

$$\Lambda_{\mathbb{C}}(S_n \wr S_m) = \mathrm{Mor}_{(n^m)}(\Lambda^*, \Lambda).$$

Thus with $b \vdash \lambda^p$

$$\begin{aligned} \Lambda_{\mathbb{C}}(S(b)) &= \Lambda_{\mathbb{C}}(\times_i(S_{\lambda_i} \wr S_{p_i})) \\ &= \times_i(\Lambda_{\mathbb{C}}(S_{\lambda_i} \wr S_{p_i})) \\ &= \times_i \mathrm{Mor}_{(\lambda_i^{p_i})}(\Lambda^*, \Lambda) \\ &= \mathrm{Mor}_{(\lambda^p)}(\Lambda^*, \Lambda). \end{aligned} \tag{4.8}$$

We have (as we shall show in Theorem 4.57)

$$\Lambda_{\mathbb{C}}(P_n^{\varsigma}) = \mathrm{Mor}_n(\Lambda^*, \Lambda) = \bigcup_{\lambda^p \vdash n} \mathrm{Mor}_{\lambda^p}(\Lambda^*, \Lambda).$$

4.6 Decomposition of the regular P_n^ς-module

To find all the irreducible modules in P_n^ς, we rely on some results of [24] as well as [23].

We recall from Section 2.2 the following. Given an algebra A and an idempotent $e \in A$ we may define functors

$$\begin{aligned} F\colon A\text{-mod} &\to eAe\text{-mod} \\ M &\mapsto eM \end{aligned} \tag{4.9}$$

$$\begin{aligned} G\colon eAe\text{-mod} &\to A\text{-mod} \\ N &\mapsto {}_A Ae \otimes_{eAe} N \end{aligned} \tag{4.10}$$

The functor G usually takes an irreducible module N to a module $G(N)$ which is not irreducible. We want to define another functor G'. This functor takes irreducibles to irreducibles [24].

If M is an A-module and M_0 an A-submodule of M, define

$$M_{(e)} = \sum_{\substack{M_0 \subseteq M \\ eM_0 = 0}} M_0.$$

Then

$$\begin{aligned} G'\colon eAe\text{-mod} &\to A\text{-mod} \\ R &\mapsto (Ae \otimes_{eAe} R)/(Ae \otimes_{eAe} R)_{(e)} \end{aligned} \tag{4.11}$$

By [24, §6.2], every simple eAe-module arises in the following way:

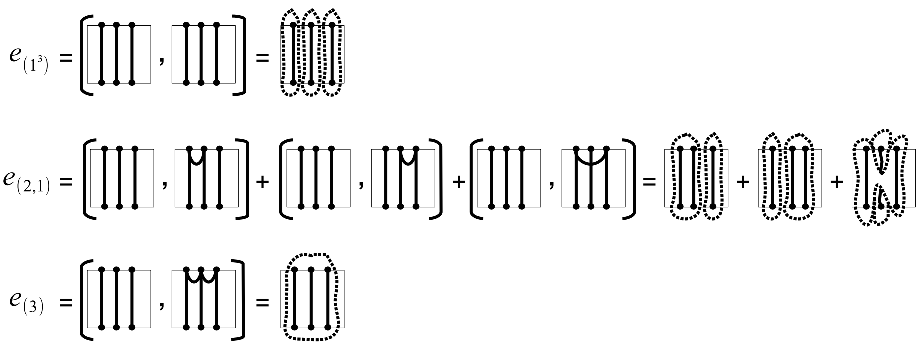
Theorem 4.24 (Green [24])**.** *Let $\{L(\lambda), \lambda \in \Lambda\}$ be a full set of irreducible A-modules, indexed by a set Λ. Set $\Lambda^e = \{\lambda \in \Lambda\colon eL(\lambda) \neq 0\}$. Then $\{eL(\lambda)\colon \lambda \in \Lambda^e\}$ is a full set of irreducible eAe-modules. The remaining irreducible modules $L(\lambda)$ (with $\lambda \in \Lambda \backslash \Lambda^e$) are a full set of irreducible A/AeA-modules.* □

From now on, we will write $[a, b]$ for $\varsigma(a, b)$.

Consider the element e_{λ^p} of P_n^{ς} defined by

$$e_{\lambda^p} := \sum_{b \vdash \lambda^p} [1, b].$$

Example 4.25. For $n = 3$,



Lemma 4.26. *The element e_{λ^p} is central in P_n^{ς}.*

Proof. For e_{λ^p} to be central in P_n^{ς}, it is enough to show that $e_{\lambda^p}[a, b'] = [a, b']e_{\lambda^p}$ for all $[a, b'] \in P_n^{\varsigma}$.

$$\begin{aligned} e_{\lambda^p}[a, b'] &= \sum_{b \Vdash \lambda^p} [1, b][a, b'] \\ &= \sum_{b \Vdash \lambda^p} [a, bb'] \end{aligned}$$

On the other hand,

$$\begin{aligned} [a, b']e_{\lambda^p} &= \sum_{b \Vdash \lambda^p} [a, b'][1, b] \\ &= \sum_{b \Vdash \lambda^p} [a, b'aba^{-1}] \\ &= \sum_{b \Vdash \lambda^p} [a, b'b] && \text{since, as } b \text{ runs over elements of shape } \lambda^p \text{, so does } aba^{-1}. \\ &= \sum_{b \Vdash \lambda^p} [a, bb'] && \text{as elements in diag-}\mathcal{P}_n \text{ commute.} \end{aligned}$$

Therefore e_{λ^p} is central in P_n^{ς}. $\square$

We shall often say that the shape of $[a, b] = (a, ba)$ is the shape of b. Recall from Definition 4.5 a partial order on integer partition which we shall be utilizing hereafter.

Proposition 4.27. *For each shape λ^p, the ideal $P_n^\varsigma e_{\lambda^p}$ has basis* $\{[a, b]\colon$ *shape of* $[a, b] \geq \lambda^p\}$.

Proof. We first need to show that if $[a, b_0]$ has shape $\geq \lambda^p$ then $[a, b_0] \in P_n^\varsigma e_{\lambda^p}$. We prove this by induction on the shape of $[a, b_0]$. Suppose $[a, b_0]$ has shape (n). Then

$$[a, b_0]\left(\sum_{b \Vdash \lambda^p}[1, b]\right) = (\star)[a, b_0] \quad \in \quad P_n^\varsigma e_{\lambda^p} \tag{4.12}$$

where $\star$ is the number of elements b in diag-$\mathcal{P}_n$ of shape λ^p.

Now let $[a, b_0]$ have shape $\lambda^{p'}$ (where $\lambda^{p'} \geq \lambda^p$). Suppose $[a, b_0'] \in P_n^\varsigma e_{\lambda^p}$ for $[a, b_0']$ of larger shape than $[a, b_0]$. Then

$$[a, b_0]\left(\sum_{b \Vdash \lambda^p}[1, b]\right) = (\star)[a, b_0] + r \quad \in \quad P_n^\varsigma e_{\lambda^p} \tag{4.13}$$

where $\star$ is the number of elements $[a, b]$ of shape λ^p such that b is a refinement of b_0 and where r is a sum of terms of form $[a', b']$ with the shape of b' greater than $\lambda^{p'}$. Therefore, $[a, b_0] \in P_n^\varsigma e_{\lambda^p}$.

We also need to show that for any shape λ^p, basis elements in P_n^ς whose shapes are greater than or equal to λ^p form a basis of $P_n^\varsigma e_{\lambda^p}$.

Suppose $[a, b_0]$ is an arbitrary element of P_n^ς. Then $[a, b_0]\left(\sum_{b \Vdash \lambda^p}[1, b]\right)$ is a sum of elements of the form $[a, b_0 aba^{-1}]$ (with $b \Vdash \lambda^p$). Note that the shape of an element $[a, b_0 aba^{-1}]$ is greater than or equal to λ^p. This implies that

$$x\left(\sum_{b \Vdash \lambda^p}[1, b]\right) \in \text{ span } \{[a, b]\colon \text{ shape of } [a, b] \geq \lambda^p\} \quad \forall x \in P_n^\varsigma.$$

Thus, the elements $[a, b]$ of shape greater than or equal to λ^p span $P_n^\varsigma e_{\lambda^p}$. ☐

Proposition 4.28. $P_n^\varsigma e_{\lambda^{p'}} \subsetneq P_n^\varsigma e_{\lambda^p}$ *if and only if* $\lambda^{p'} > \lambda^p$.

Proof. Assume $\lambda^{p'} > \lambda^p$. Take any $[a, b] \in P_n^\varsigma e_{\lambda^p}$, for b of shape greater than or equal to $\lambda^{p'}$. Then $[a, b]$ has shape greater than or equal to λ^p. Therefore, every basis element of $P_n^\varsigma e_{\lambda^{p'}}$ is a basis element of $P_n^\varsigma e_{\lambda^p}$ which implies that $P_n^\varsigma e_{\lambda^{p'}} \subsetneqq P_n^\varsigma e_{\lambda^p}$. On the other hand, assume $P_n^\varsigma e_{\lambda^{p'}} \subsetneqq P_n^\varsigma e_{\lambda^p}$. Take $[a,b]$ of shape $\lambda^{p'}$, so $[a,b] \in P_n^\varsigma e_{\lambda^{p'}}$ which implies $[a,b] \in P_n^\varsigma e_{\lambda^p}$. Therefore, $[a,b]$ has shape greater than or equal to λ^p. This means $\lambda^{p'} \geq \lambda^p$ but if $\lambda^{p'} = \lambda^p$ then $P_n^\varsigma e_{\lambda^{p'}} = P_n^\varsigma e_{\lambda^p}$, a contradiction. Thus, $\lambda^{p'} > \lambda^p$ □

As a concrete explanation of Proposition 4.28 we look at the following example.

Example 4.29. We use the elements constructed in Example 4.25 here.

$$\begin{aligned}
P_3^\varsigma e_{(1^3)} &= P_3^\varsigma \\
P_3^\varsigma e_{(2,1)} &= \mathbb{C}\text{-span}\ \{[a,b]\colon a \in S_3,\ b \text{ is of shape } (3) \text{ or } (2,1)\} \\
P_3^\varsigma e_{(3)} &= \mathbb{C}\text{-span}\ \{[a,b]\colon a \in S_3,\ b \text{ is of shape } (3)\}
\end{aligned}$$

Therefore, $P_3^\varsigma e_{(3)} \subsetneqq P_3^\varsigma e_{(2,1)} \subsetneqq P_3^\varsigma e_{(1^3)}$, corresponding to $(3) > (2,1) > (1^3)$. We see that the assertion in Proposition 4.28 holds.

Set

$$I_{>\,\lambda^p} = \sum_{\lambda^{p'} > \lambda^p} P_n^\varsigma e_{\lambda^{p'}}.$$

Each section

$$M_{\lambda^p} := P_n^\varsigma e_{\lambda^p} / I_{>\,\lambda^p}$$

in the filtration stated in Proposition 4.28 has basis parameterized by elements $[a,b] \in \varsigma(S_n \times \text{diag-}\mathcal{P}_n)$ of shape λ^p.

Example 4.30. The elements constructed in Example 4.25 induce a filtration for P_3^ς by ideals (see Example 4.29)

$$P_3^\varsigma e_{(3)} \subsetneqq P_3^\varsigma e_{(2,1)} \subsetneqq P_3^\varsigma e_{(1^3)}.$$

Then the sections in the filtration above are as follows.

$$\begin{aligned}
M_{(1^3)} &= \frac{P_3^\varsigma e_{(1^3)}}{P_3^\varsigma e_{(2,1)} + P_3^\varsigma e_{(3)}} = \mathbb{C}\text{-span}\{[a,b]\colon a \in S_3,\ b \text{ is of shape } (1^3)\} \\
M_{(2,1)} &= \frac{P_3^\varsigma e_{(2,1)}}{P_3^\varsigma e_{(3)}} = \mathbb{C}\text{-span}\{[a,b]\colon a \in S_3,\ b \text{ is of shape } (2,1)\} \\
M_{(3)} &= P_3^\varsigma e_{(3)} = \mathbb{C}\text{-span}\{[a,b]\colon a \in S_3,\ b \text{ is of shape } (3)\}
\end{aligned}$$

Note that for each λ^p, the dimension of M_{λ^p} is $n!\mathcal{D}_{\lambda^p}$ (where $\mathcal{D}_{\lambda^p}$ is defined in Equation (4.3)).

Next, we decompose the sections as far as possible.

As a vector space we have [13]

$$M_{\lambda^p} = \bigoplus_{b \Vdash \lambda^p} \mathbb{C}[S_n, b].$$

Note that $\mathbb{C}[S_n, b]$ is an $S(b)$-module via the embedding $g \mapsto [g, 1]$ of $S(b)$ into B_n^ς.

Proposition 4.31. *The map*

$$\begin{aligned} \vartheta \colon \mathbb{C}S_n &\to \mathbb{C}[S_n, b] \\ a &\mapsto [a, b] \end{aligned}$$

is an $S(b)$-module isomorphism.

Proof. The module $\mathbb{C}[S_n, b]$ is generated by elements $\langle \{[a, b] \colon a \in S_n\} \rangle$. Assume $g \in S(b)$. The element g acts on $a \in \mathbb{C}S_n$ as follows:

$$g \cdot a = ga \tag{4.14}$$

The action of $S(b)$ on $\mathbb{C}[S_n, b]$ is as follows.

$$\begin{aligned} S(b) \times [S_n, b] &\to [S_n, b] \\ (g, [a, b]) &\mapsto [g, 1][a, b] = [ga, gbg^{-1}] \\ &= [ga, b] \qquad \text{since } g \in S(b). \end{aligned} \tag{4.15}$$

Comparing (4.14) and (4.15) we see that $\mathbb{C}[S_n, b]$ is an $S(b)$-module isomorphic to $\mathbb{C}S_n$ as an $S(b)$-module. $\square$

Corollary 4.32. *The modules $M_{(1^n)}$ and $M_{(n)}$ are isomorphic to $\mathbb{C}S_n$ as $\mathbb{C}S_n$-modules.*

Recall that the number of set partitions of shape λ^p is $\mathcal{D}_{\lambda^p} = \frac{|S_n|}{S(b)}$. It follows from Proposition 4.31 that $\mathbb{C}[S_n, b]$ decomposes into $\mathcal{D}_{\lambda^p}$ copies of the regular $S(b)$-module.

Remark 4.33. In P_n^{ς}, multiplication is given by

$$[a,b][a',b'] = [aa', bab'a^{-1}]. \tag{4.16}$$

Assuming b and $ab'a^{-1}$ have the same shape, say λ^p, if $b \neq ab'a^{-1}$ then $b(ab'a^{-1})$ has shape $> \lambda^p$. Therefore, the multiplication in Equation 4.16 is zero in M_{λ^p}. Precisely, multiplication of two elements $[a,b]$, $[a',b'] \in M_{\lambda^p}$ is

$$[a,b][a',b'] = \begin{cases} 0, & \text{if } b \neq aba^{-1} \\ [aa',b'], & \text{if } b' = aba^{-1}. \end{cases} \tag{4.17}$$

Lemma 4.34. *The element $e_{\lambda^p} + I_{>\lambda^p} := \overline{e_{\lambda^p}}$ is central and idempotent in M_{λ^p}.*

Proof. The proof that $\overline{e_{\lambda^p}}$ is central in M_{λ^p} follows a similar argument to the proof of Lemma 4.26.

To show that $\overline{e_{\lambda^p}}$ is idempotent in M_{λ^p}, we have to show that $(\overline{e_{\lambda^p}})^2 = \overline{e_{\lambda^p}}$.

$$\begin{aligned} (\overline{e_{\lambda^p}})^2 &= \left(\sum_{b \Vdash \lambda^p} \overline{[1,b]} \right)^2 \\ &= \sum_{b \Vdash \lambda^p} \overline{[1,b]} + \star \end{aligned}$$

where $\star$ denotes the sum of elements whose shapes are greater than λ^p. But $\star$ is zero using Equation (4.17). Therefore $\overline{e_{\lambda^p}}$ is an idempotent element in M_{λ^p}. □

Proposition 4.35. *The section*

$$M_{\lambda^p} = (e_{\lambda^p} + I_{>\lambda^p}) \left(\frac{P_n^{\varsigma}}{I_{>\lambda^p}} \right) (e_{\lambda^p} + I_{>\lambda^p}).$$

Proof.

$$\begin{aligned} (e_{\lambda^p} + I_{>\lambda^p}) \left(\frac{P_n^{\varsigma}}{I_{>\lambda^p}} \right) (e_{\lambda^p} + I_{>\lambda^p}) &= \frac{e_{\lambda^p} P_n^{\varsigma} e_{\lambda^p} + I_{>\lambda^p}}{I_{>\lambda^p}} \\ &= \frac{P_n^{\varsigma} e_{\lambda^p} + I_{>\lambda^p}}{I_{>\lambda^p}} \quad \text{since } e_{\lambda^p} \text{ is central in } P_n^{\varsigma} \\ &= \frac{P_n^{\varsigma} e_{\lambda^p}}{I_{>\lambda^p}} \\ &= M_{\lambda^p} \end{aligned}$$

□

Proposition 4.35 says, in other words, that M_{λ^p} is an idempotent subalgebra of the quotient algebra of P_n^{ς} by $I_{>\lambda^p}$. The identity element is $e_{\lambda^p} + I_{>\lambda^p}$.

By Equation (2.2) and Equation (4.11), there exists a functor

$$\begin{aligned} F_1 \colon M_{\lambda^p}\text{-mod} &\to P_n^{\varsigma}/I_{>\lambda^p}\text{-mod} \\ N &\mapsto \frac{\frac{P_n^{\varsigma}}{I_{>\lambda^p}}(e_{\lambda^p} + I_{>\lambda^p}) \otimes_{M_{\lambda^p}} N}{(\frac{P_n^{\varsigma}}{I_{>\lambda^p}}(e_{\lambda^p} + I_{>\lambda^p}) \otimes_{M_{\lambda^p}} N)_{(e_{\lambda^p}+I_{>\lambda^p})}}. \end{aligned} \tag{4.18}$$

Proposition 4.36 ([24, 6.2e])**.** *If V is irreducible over M_{λ^p} then $F_1(V)$ is irreducible over $P_n^{\varsigma}/I_{>\lambda^p}$.* □

Thus F_1 induces a map

$$\mathfrak{F}_1 \colon \{\text{simple } M_{\lambda^p}\text{-modules}\} \to \{\text{simple } P_n^{\varsigma}/I_{>\lambda^p}\text{-modules}\}.$$

By Theorem 4.24, we deduce that

Proposition 4.37. *Let V be an irreducible $P_n^{\varsigma}/I_{>\lambda^p}$-module. Then $V \in Im(\mathfrak{F}_1)$ if and only if $(e_{\lambda^p} + I_{>\lambda^p})V \neq 0$.*

Those simple modules not hit by $\mathfrak{F}_1$ correspond to simple modules over $\frac{\frac{P_n^{\varsigma}}{I_{>\lambda^p}}}{\frac{P_n^{\varsigma}}{I_{>\lambda^p}}(e_{\lambda^p}+I_{>\lambda^p})\frac{P_n^{\varsigma}}{I_{>\lambda^p}}} \cong P_n^{\varsigma}/(P_n^{\varsigma}e_{\lambda^p})$.

We have

Proposition 4.38. *Let V be an irreducible $P_n^{\varsigma}/I_{>\lambda^p}$-module. Then $(e_{\lambda^p}+I_{>\lambda^p})V \neq 0$ if and only if $(P_n^{\varsigma}e_{\lambda^p} + I_{>\lambda^p})V \neq 0$.*

Proof. Note that $(P_n^{\varsigma}e_{\lambda^p}+I_{>\lambda^p})V = P_n^{\varsigma}/I_{>\lambda^p}(e_{\lambda^p}+I_{>\lambda^p})V$. Now if $(e_{\lambda^p}+I_{>\lambda^p})V \neq 0$ then $P_n^{\varsigma}/I_{>\lambda^p}(e_{\lambda^p}+I_{>\lambda^p})V \neq \{0\}$ (as $1+I_{>\lambda^p} \in P_n^{\varsigma}/I_{>\lambda^p}$). If $(e_{\lambda^p}+I_{>\lambda^p})V = 0$ then $P_n^{\varsigma}/I_{>\lambda^p}(e_{\lambda^p}+I_{>\lambda^p})V = \{0\}$. That is, $P_n^{\varsigma}/I_{>\lambda^p}(e_{\lambda^p}+I_{>\lambda^p})V \neq \{0\}$ implies $(e_{\lambda^p}+I_{>\lambda^p})V \neq 0$. □

There is a functor induced by the natural epimorphism from P_n^{ς} to $P_n^{\varsigma}/I_{>\lambda^p}$

$$F_2 \colon P_n^{\varsigma}/I_{>\lambda^p}\text{-mod} \to P_n^{\varsigma}\text{-mod}. \tag{4.19}$$

Suppose V is a $P_n^{\varsigma}/I_{>\lambda^p}$-module. Define a map

$$\begin{aligned}\phi\colon P_n^{\varsigma} &\to P_n^{\varsigma}/I_{>\lambda^p}\\ a &\mapsto a + I_{>\lambda^p}\end{aligned}$$

If $a \in P_n^{\varsigma}$, define

$$av := \phi(a)v \quad \text{for all } a \in P_n^{\varsigma}, v \in V \tag{4.20}$$

Then V becomes a P_n^{ς}-module via this action. Thus $F_2(V) = V$ is now regarded as P_n^{ς}-module. Define F_2 to be the identity on morphisms. Then F_2 is a functor.

Proposition 4.39. *Let V be a $P_n^{\varsigma}/I_{>\lambda^p}$-module. If V' is a P_n^{ς}-submodule of $F_2(V)$, then V' is a $P_n^{\varsigma}/I_{>\lambda^p}$-submodule of V.*

Proof. Assume $V' \subseteq F_2(V)$ is a P_n^{ς}-submodule. If $b \in P_n^{\varsigma}/I_{>\lambda^p}$, $v \in V'$, then $b = \phi(a)$, for some $a \in P_n^{\varsigma}$. Thus $bv = \phi(a)v = av \in V'$ by Equation (4.20). Therefore, V' is a $P_n^{\varsigma}/I_{>\lambda^p}$-submodule of V. □

The converse of Proposition 4.39 is also true. That is,

Proposition 4.40. *Let V be a $P_n^{\varsigma}/I_{>\lambda^p}$-module. If V' is a $P_n^{\varsigma}/I_{>\lambda^p}$-submodule of V then V' is a P_n^{ς}-submodule of $F_2(V)$.*

Proof. Let $a \in P_n^{\varsigma}, v \in V'$. Then $av = \phi(a)v$. But $\phi(a)v \in V'$ (by Equation (4.20)) as V' is a $P_n^{\varsigma}/I_{>\lambda^p}$-submodule of V. Therefore, V' is a P_n^{ς}-submodule of $F_2(V)$. □

Proposition 4.41. *A $P_n^{\varsigma}/I_{>\lambda^p}$-module V is irreducible if and only if $F_2(V)$ is irreducible over P_n^{ς}.*

Proof. Suppose V is an irreducible $P_n^{\varsigma}/I_{>\lambda^p}$-module, and assume $V' \subseteq F_2(V)$ is a P_n^{ς}-submodule with $V' \neq 0$. By Proposition 4.39, V' is a $P_n^{\varsigma}/I_{>\lambda^p}$-submodule of V. Since V is an irreducible $P_n^{\varsigma}/I_{>\lambda^p}$-module this implies that $V' = V$. Hence $F_2(V)$ is an irreducible P_n^{ς}-module. The converse is similar, using Proposition 4.40. □

Thus, the functor F_2 induces a map

$$\mathfrak{F}_2\colon \{\text{simple } P_n^{\varsigma}/I_{>\lambda^p}\text{-modules}\} \to \{\text{simple } P_n^{\varsigma}\text{-modules}\}.$$

Proposition 4.42. *Let W be an irreducible P_n^ς-module. Then $W \in Im(\mathfrak{F}_2)$, if and only if $I_{>\lambda^p}W = 0$.*

Proof. Let $W \in \mathrm{Im}(\mathfrak{F}_2)$. That is, $W = \mathfrak{F}_2(V) = V$ for V an irreducible $P_n^\varsigma/I_{>\lambda^p}$-module. Let $a \in I_{>\lambda^p}$, $v \in V$. Then by Equation (4.20), $av = \phi(a)v = 0v = 0$. Therefore, $I_{>\lambda^p}W = I_{>\lambda^p}V = 0$. Conversely, suppose W is an irreducible P_n^ς-module and $I_{>\lambda^p}W = 0$. Let $V = W$ regarded as a $P_n^\varsigma/I_{>\lambda^p}$-module as follows:

$$\text{For} \quad a + I_{>\lambda^p} \in P_n^\varsigma/I_{>\lambda^p}, \qquad (a + I_{>\lambda^p})v := av \quad \text{for any } v \in V. \tag{4.21}$$

This action is well-defined, as $I_{>\lambda^p}V = 0$. Then $W = F_2(V)$. Hence, $W \in \mathrm{Im}(\mathfrak{F}_2)$ by Proposition 4.41. □

Thus, the simple modules F_2 misses are the simple P_n^ς-module W for which $I_{>\lambda^p}W \neq 0$. That is, $P_n^\varsigma e_{\lambda^{p'}}W \neq 0$ for some $\lambda^{p'} > \lambda^p$.

Proposition 4.43. *An irreducible P_n^ς-module W lies in $Im(\mathfrak{F}_2\mathfrak{F}_1)$ if and only if*
(a) $I_{>\lambda^p}W = 0$

(b) $P_n^\varsigma e_{\lambda^p}W \neq 0$

Proof. Assume $W \in \mathrm{Im}(\mathfrak{F}_2\mathfrak{F}_1)$. Then $W \in \mathrm{Im}(\mathfrak{F}_2)$, so $I_{>\lambda^p}W = 0$. Write $W = F_2(F_1(M))$ with $F_1(M)$ an irreducible $P_n^\varsigma/I_{>\lambda^p}$-module and M an irreducible M_{λ^p}-module. By Propositions 4.37 and 4.38, $F_1(M) \in \mathrm{Im}(\mathfrak{F}_1)$ implies

$$\begin{aligned} &((P_n^\varsigma e_{\lambda^p}) + I_{>\lambda^p})F_1(M) \neq 0 \\ \Rightarrow &(P_n^\varsigma e_{\lambda^p})W \neq 0 \qquad \text{as required.} \end{aligned}$$

Conversely, suppose (a) and (b) hold. By (a) and Proposition 4.42, $W \in \mathrm{Im}(\mathfrak{F}_2)$, i.e. $W = F_2(V)$ where V is an irreducible $P_n^\varsigma/I_{>\lambda^p}$-module. By (b) $P_n^\varsigma e_{\lambda^p}W \neq 0$, which implies $(P_n^\varsigma e_{\lambda^p} + I_{>\lambda^p})V = (P_n^\varsigma e_{\lambda^p}/I_{>\lambda^p})V \neq 0$. Therefore $V \in \mathrm{Im}(\mathfrak{F}_1)$ by Proposition 4.37 and Proposition 4.38. Hence $W \in \mathrm{Im}(\mathfrak{F}_2\mathfrak{F}_1)$ as required. □

Suppose W is an irreducible P_n^ς-module. Consider $P_n^\varsigma e_{\lambda^p}W$ for all λ^p. Take λ^p maximal such that $P_n^\varsigma e_{\lambda^p}W \neq 0$. If $\lambda^p = (1^n)$ then $P_n^\varsigma e_{\lambda^p} = P_n^\varsigma$. Then $P_n^\varsigma e_{\lambda^p}W = P_n^\varsigma W = W \neq 0$. So a maximal such λ^p exists. Then if $\lambda^{p'} > \lambda^p$, $P_n^\varsigma e_{\lambda^{p'}}W = 0$. This implies, $I_{>\lambda^p}W = 0$. Therefore (a) and (b) hold or in other words, $W \in \mathrm{Im}(\mathfrak{F}_2\mathfrak{F}_1)$ for λ^p.

Proposition 4.44. *Let* W *be an irreducible* P_n^{ς}*-module. Then* $W \in Im(\mathfrak{F}_2\mathfrak{F}_1)$ *for a unique* λ^p*.*

Proof. We argue by contradiction. Set $I_{>\lambda^p} = \sum_{\lambda^{p''}>\lambda^p} P_n^{\varsigma} e_{\lambda^{p''}}$. Suppose W arises in two ways. That is,

$$\begin{aligned} I_{>\lambda^p} W &= 0 \\ P_n^{\varsigma} e_{\lambda^p} W &\neq 0 \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} I_{>\lambda^{p'}} W &= 0 \\ P_n^{\varsigma} e_{\lambda^{p'}} W &\neq 0 \quad \text{ and } \lambda^p \neq \lambda^{p'}. \end{aligned} \tag{4.23}$$

Then for all $\lambda^{p''} > \lambda^p$, $P_n^{\varsigma} e_{\lambda^{p'}} W = 0$. Therefore, $\lambda^{p'} \not> \lambda^p$. Similarly, $\lambda^p \not> \lambda^{p'}$. So λ^p, $\lambda^{p'}$ are incomparable. Now $W = e_{\lambda^p} P_n^{\varsigma} W = e_{\lambda^p} W$. Then

$$\begin{aligned} e_{\lambda^{p'}} W &= e_{\lambda^{p'}} e_{\lambda^p} W \\ &= \left(\sum_{b' \Vdash \lambda^{p'}} [1, b'] \sum_{b \Vdash \lambda^p} [1, b] \right) W \end{aligned}$$

The element

$$\sum_{b' \Vdash \lambda^{p'}} [1, b'] \sum_{b \Vdash \lambda^p} [1, b]$$

is a sum of basis elements of shape greater than λ^p, since λ^p and $\lambda^{p'}$ are incomparable. Hence

$$\sum_{b' \Vdash \lambda^{p'}} [1, b'] \sum_{b \Vdash \lambda^p} [1, b] W = 0,$$

so $e_{\lambda^{p'}} W = 0$, and therefore, $P_n^{\varsigma} e_{\lambda^{p'}} W = 0$. This contradicts Equation (4.23). Thus, $W \in \text{Im}(\mathfrak{F}_2\mathfrak{F}_1)$ for a unique λ^p. □

It also follows that

$$\Lambda_{\mathbb{C}}(P_n^{\varsigma}) = \dot{\bigcup}_{\lambda^p \vdash n} \Lambda_{\mathbb{C}}(\mathcal{M}_{\lambda^p})$$

We shall discuss the details shortly.

4.7 Morita theory

This section contains a brief account of the theory of Morita equivalence. Morita theory addresses the question of when two algebras have equivalent categories of modules.

Definition 4.45. Let A and B be algebras over a field F. Then A and B are said to be *Morita equivalent* (see, for example, [41, p. 325]) if there is an equivalence from the category of left A-modules to the category of left B-modules.

Theorem 4.46 (See, for example [48])**.** *Two algebras A, B are Morita equivalent if and only if there exists an idempotent $e \in A$ such that $A \cong AeA$ and $B \cong eAe$.* $\square$

Consider the idempotent $[1, b_0]$, $b_0 \Vdash \lambda^p$, and recall from Equation (4.17) the multiplication of elements in M_{λ^p}. We have

$$[1,b_0]M_{\lambda^p} = [1,b_0] \bigoplus_{w\in S_n; b \Vdash \lambda^p} \mathbb{C}[w,b] = \bigoplus_{w\in S_n; b \Vdash \lambda^p} \mathbb{C}[1,b_0][w,b] = \bigoplus_{w\in S_n} \mathbb{C}[w,b_0].$$

Thus,

$$[1,b_0]M_{\lambda^p}[1,b_0] = \bigoplus_{w\in S_n} \mathbb{C}[w,b_0][1,b_0] = \bigoplus_{w\in S_n} [w, b_0 w b_0 w^{-1}]$$

$$= \bigoplus_{w\in S(b_0)} [w,b_0] \cong \mathbb{C}S(b_0)$$

and

$$M_{\lambda^p}[1,b_0]M_{\lambda^p} = M_{\lambda^p} \bigoplus_{w\in S_n} \mathbb{C}[w,b_0] = \left(\bigoplus_{x\in S_n; b\Vdash \lambda^p} \mathbb{C}[x,b] \right) \bigoplus_{w\in S_n} \mathbb{C}[w,b_0]$$

$$= \bigoplus_{x\in S_n; b\Vdash \lambda^p} \bigoplus_{w\in S_n} \mathbb{C}[x,b][w,b_0] = \bigoplus_{x,\, w\in S_n; b\Vdash\lambda^p} [xw, bxb_0x^{-1}] = M_{\lambda^p}$$

Thus by Theorem 4.46

Theorem 4.47. *The algebras M_{λ^p} and $\mathbb{C}S(b_0)$ (with $b_0 \Vdash \lambda^p$) are Morita equivalent.* $\square$

Properties of a ring which are preserved under Morita equivalence are called *Morita invariants* (See, for example [49, p. 243]). Examples of such properties include a ring being semisimple, artinian, noetherian, e.t.c.. By Maschke's theorem (see Proposition B.26 and Corollary B.34) and Theorem B.38, $\mathbb{C}S(b_0)$ is split semisimple for every shape. Consequently, we have

Corollary 4.48. *For each λ^p, M_{λ^p} is semisimple over $\mathbb{C}$.* $\square$

We notice that dimensionality is not a Morita invariant property, and in fact differs in our case. We illustrate this point with an example.

Example 4.49. Recall from Example 4.30 the sections $M_{(1^3)}, M_{(2,1)}, M_{(3)}$ in the filtration for P_3^ς. Their dimensions are $6, 18, 6$ respectively. Also recall from Example 4.8 that for each $b \in \mathcal{P}_{\underline{3}}$

$$S(b) \cong S_3 \text{ if } b \Vdash (1^3) \text{ or } (3)$$
$$S(b) \cong S_2 \text{ if } b \Vdash (2,1)$$

Although $M_{(2,1)}$ and $S(b)$ with $b \Vdash (2,1)$ are Morita equivalent by Theorem 4.47, their dimensions are not equal since $S(b)$ has dimension 2 and $M_{(2,1)}$ has dimension 18.

Next we construct explicitly the simple modules of P_n^ς and compute their dimensions.

4.8 Explicit construction of simple modules of P_n^ς (illustrated by an example)

Recall that the complex representation theory of the symmetric groups is known (see Appendix A). In particular, for a partition λ of an integer n there are standard constructions for *primitive idempotents* which generate irreducible representations of $\mathbb{C}S_n$.

Definition 4.50.

(1) Let A be an algebra over a field F. Two idempotents e and f are *orthogonal* if $ef = fe = 0$.

(2) An idempotent e of A is *primitive* if it is not possible to write e as the sum of two orthogonal idempotents.

We proceed to construct the irreducible representations of P_n^{ς}, using information from Appendix A. We relegate the details of the representation theory of the symmetric group, that is useful for this section, to the Appendix to avoid obscuring the computation by too many details.

We have seen in Section 4.6 that by decomposing M_{λ^p} for all λ^p we get the simple P_n^{ς}-modules. It is convenient to illustrate the decomposition of M_{λ^p} with an example. We work out all the simple modules of P_3^{ς}.

Example 4.51. Consider each section M_{λ^p} in the filtration for P_3^{ς} constructed in Example 4.30. We proceed by examining these cases one at a time.

It is easy to see that the map

$$\begin{aligned} \phi\colon M_{(3)} &\to \mathbb{C}S_3 \\ [a,b] &\mapsto a \end{aligned}$$

defines a $\mathbb{C}$-algebra isomorphism.

As seen in Appendix A, $\mathbb{C}S_3$ regarded as a $\mathbb{C}S_3$-module decomposes as a direct sum

$$\mathbb{C}S_3 = \mathbb{C}S_3 y_{t_1} \oplus \mathbb{C}S_3 y_{t_2} \oplus \mathbb{C}S_3 y_{t_3} \oplus \mathbb{C}S_3 y_{t_4} \tag{4.24}$$

where $t_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$, $t_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$, $t_3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$, $t_4 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$, and

$$\begin{aligned} y_{t_1} &= e + (12) + (23) + (13) + (123) + (132) \\ y_{t_2} &= e + (12) - (13) - (123) \\ y_{t_3} &= e - (12) + (13) - (132) \\ y_{t_4} &= e - (12) - (23) - (13) + (123) + (132) \end{aligned}$$

respectively, their Young symmetrizers.

Via ϕ we get

$$M_{(3)} = \phi^{-1}(\mathbb{C}S_3 y_{t_1}) \oplus \phi^{-1}(\mathbb{C}S_3 y_{t_2}) \oplus \phi^{-1}(\mathbb{C}S_3 y_{t_3}) \oplus \phi^{-1}(\mathbb{C}S_3 y_{t_4})$$

as an $M_{(3)}$-module. For simplicity, set $V_{t_i} := \phi^{-1}(\mathbb{C}S_3 y_{t_i})$ so that

$$M_{(3)} = V_{t_1} \oplus V_{t_2} \oplus V_{t_3} \oplus V_{t_4}.$$

Via ϕ $M_{(3)}$ can be regarded as a $\mathbb{C}S_3$-module. Now applying ϕ^{-1} to (4.24), we have

$$M_{(3)} = M_{(3)}\phi^{-1}(y_{t_1}) \oplus M_{(3)}\phi^{-1}(y_{t_2}) \oplus M_{(3)}\phi^{-1}(y_{t_3}) \oplus M_{(3)}\phi^{-1}(y_{t_4}).$$

Let $\hat{y_{t_i}} := \phi^{-1}(y_{t_i})$. Then

$$\begin{aligned} M_{(3)} &= M_{(3)}\hat{y_{t_1}} \oplus M_{(3)}\hat{y_{t_2}} \oplus M_{(3)}\hat{y_{t_3}} \oplus M_{(3)}\hat{y_{t_4}} \\ &= V_{t_1} \oplus V_{t_2} \oplus V_{t_3} \oplus V_{t_4}. \end{aligned}$$

It is easy to see that these summands are P_3^{ς}-submodules of $M_{(3)}$. We show that they are irreducible P_3^{ς}-modules.

If $N \subset V_{t_i}$ is a nonzero P_3^{ς}-submodule, then N is an $M_{(3)}$-submodule of V_{t_i}. This imply that $N = V_{t_i}$. Therefore V_{t_i} is an irreducible P_3^{ς}-module.

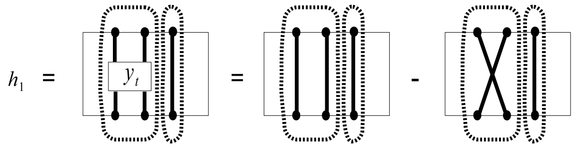
The decomposition of $M_{(1^3)}$ follows a similar argument to that of $M_{(3)}$. By Corollary 4.32, $M_{(1^3)}$ as a vector space decomposes as

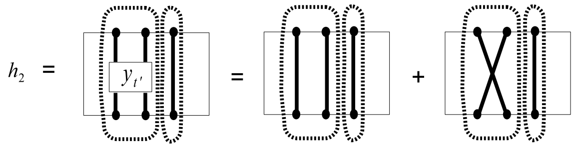
$$M_{(1^3)} = M_{(1^3)}\check{y_{t_1}} \oplus M_{(1^3)}\check{y_{t_2}} \oplus M_{(1^3)}\check{y_{t_3}} \oplus M_{(1^3)}\check{y_{t_4}}$$

where $\check{y_{t_i}}$ denotes the Young symmetrizer associated with a Young tableau t_i.

We already know, by Theorem 4.47, that $M_{(2,1)}$ has two simple modules. Since $M_{(2,1)}$ has dimension 18 then these simple modules must each be of dimension 3. By the Artin-Wedderburn theorem, we expect 3 copies each of the 3-dimensional simple modules in the regular representation.

The condition for idempotence, $f^2 = f$, leads to two elements h_1, h_2 in $M_{(2,1)}$. We can represent them in terms of diagrams with the following convention: Given a diagram d in P_n^{ς} with an island $\mathfrak{I}$ containing r noncrossing bones, there is a corresponding natural embedding ψ of $\mathbb{C}S_r$ into P_n^{ς} mapping a permutation σ to d with σ in the island. Given $y \in \mathbb{C}S_r$ we denote the image of y under ψ by drawing d with a box covering $\mathfrak{I}$ labelled with y. For example, we represent h_1 and h_2 as





where t, t' are standard Young tableaux $\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 2\\ \hline\end{array}$ respectively on the number of bones lying in that island. The young symmetrizer associated with t (resp. t') is denoted by y_t (resp. $y_{t'}$).

We check readily that h_1, h_2 are orthogonal idempotents. Thus, we have a decomposition of $M_{(2,1)}$ into subspaces:

$$M_{(2,1)}h_1 \oplus M_{(2,1)}h_2.$$

It is easy to verify that left-multiplication of elements of B_n^{ς} on $M_{(2,1)}h_1$ and $M_{(2,1)}h_2$ give 3-dimensional representation each of P_n^{ς} (of course, elements that are not of shape $(2,1)$ are sent to zero). Simple calculations show that these representations are irreducible. Figure 4.2 gives the decomposition of $M_{(2,1)}$ into summands.

In Figure 4.2, the simple modules (a), (b), (c) are equivalent to each other. Also the simple modules (d), (e), (f) are equivalent but are inequivalent to (d), (e), (f).

Lemma 4.52 (See [13]). *Any submodule of* M_{λ^p} *contains an element of form* $q = \sum_i c_i[x_i, b]$, *with* $b \Vdash \lambda^p$.

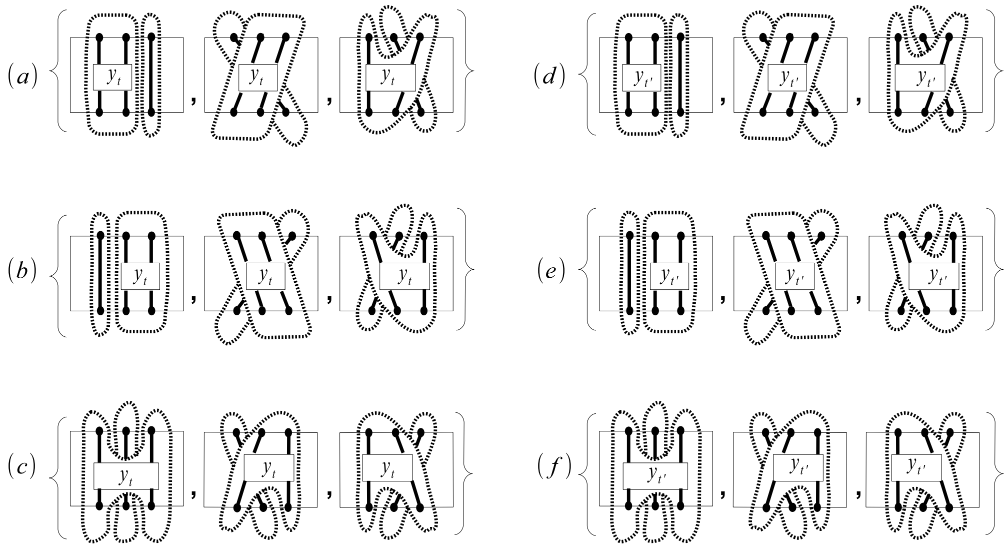


FIGURE 4.2: Some simple modules of P_3^ς.

Proof. Let $U \neq 0$ be a submodule of M_{λ^p}. Let $m := \sum_{ij} c_{ij}[x_i, y_j] \in U$ with $m \neq 0$. Then $V := P_n^\varsigma m$ is a submodule of U. Choose l so that some scalar $c_{il} \neq 0$. Then

$$\begin{aligned}[1, y_l] \sum_{ij} c_{ij}[x_i, y_j] &= \sum_{ij} c_{ij}[1, y_l][x_i, y_j] \\ &= \sum_{ij} c_{ij}[x_i, y_l y_j] \in V \\ &= \sum_{i} c_{il}[x_i, y_l] \in U. \end{aligned}$$

□

We write T_b^L (resp. T_b^R) for a traversal of the left (resp. right) cosets of $S(b)$ in S_n.

Recall that P_n^ς is generated by $[1, A^{1,2}]$ and $[S_n, 1]$. The element $[1, A^{1,2}]$ acts on $q = \sum_i c_i[x_i, b]$ as 1 or takes the shape of q up in the order described in Section 4.1 which is regarded as 0 in M_{λ^p}. We consider the action of $[S_n, 1]$ in two parts:

(a) $[S(b), 1]$: The subspace $\mathbb{C}[S(b), 1]q$ gives a $\mathbb{C}S(b)$-submodule consisting of elements which are of the form $\sum_i \tilde{c}_i[\tilde{x}_i, b]$ (with $\tilde{x}_i$ in S_n). So there exists an element of the same form as q generating an irreducible $S(b)$-submodule. (We assume that q is in fact such an element).

(b) a traversal: take an element w of a traversal of $S(b)$ in S_n. Then the action of w is

$$wq = [w, 1] \sum_i c_i[x_i, b] = \sum_i c_i[wx_i, wbw^{-1}]. \tag{4.25}$$

Set $b^w = wbw^{-1}$. Then wq generates an irreducible $S(b^w)$-module.

We observe that

Proposition 4.53. *The map*

$$\begin{aligned} \phi\colon S(b) &\to S(b^w) \\ g &\mapsto wgw^{-1} \end{aligned}$$

is a group isomorphism. □

and hence

Proposition 4.54. *The map*

$$\begin{aligned} \psi\colon \mathbb{C}S(b)q &\to \mathbb{C}S(b^w)wq \\ gq &\mapsto \phi(g)wq = wgq \end{aligned}$$

is an isomorphism of $\mathbb{C}S(b)$-modules (via the group isomorphism ϕ).

Corollary 4.55. *The module $\mathbb{C}S(b^w)wq$ is an irreducible $\mathbb{C}S(b^w)$-module.*

Proof. Let $V \subseteq \mathbb{C}S(b^w)wq$ be a non-zero $\mathbb{C}S(b^w)$-submodule. Then V is a non-zero $\mathbb{C}S(b)$-submodule of $\mathbb{C}S(b^w)wq$ (via ϕ). This implies that ψ^{-1} is a non-zero $\mathbb{C}S(b)$-submodule of $\mathbb{C}S(b)q$ (as ψ is an isomorphism of $\mathbb{C}S(b)$-modules). Thus $\psi^{-1} = \mathbb{C}S(b)q$ by the argument in (a) above. Since ψ is also an isomorphism of vector spaces, we have that $V = \mathbb{C}S(b^w)wq$ and the result holds. □

Let L_μ be the irreducible $S(b)$-submodule $\mathbb{C}S(b)q$ of U. Then, as $\mathbb{C}S(b)$-module, L_μ is isomorphic to $L_\mu(w)$, where $L_\mu(w) = \mathbb{C}S(b^w)wq$, for all $w \in T_b^L$. Since these subspaces involve different basis elements, we get

$$\oplus_{w \in T_b^L} L_\mu(w) \subseteq U.$$

Thus, every irreducible P_n^{ς}-submodule is at least a sum (as a vector space) of $\mathcal{D}_{\lambda^p}$ spaces each of which is an (isomorphic) simple module for $S(b)$ for the appropriate b. In particular,

Proposition 4.56 (See [13]). *For each inequivalent simple $S(b)$-module L_μ (i.e. with $\mu \in \mathrm{Mor}_{\lambda^p}(\Lambda^*, \Lambda)$ and $\lambda^p \Vdash b$) of dimension m_μ and basis $\{v_i | i = 1, \ldots, m_\mu\}$, say, there is a simple P_n^{ς}-module of L_μ^{ς} of dimension*

$$dim\ L_\mu^{\varsigma} = m_\mu \mathcal{D}_{\lambda^p} \tag{4.26}$$

and basis $\{[wv_i, b^w] | i = 1, \ldots, m_\mu,\ \ w \in T_b^L\}$. The modules $\{L_\mu^{\varsigma}\}$ are pairwise inequivalent. □

Since $\mathbb{C}S(b)$ is split semisimple (over $\mathbb{C}$) for every shape, we have that the multiplicity of L_μ in the b-th summand is $m_\mu \mathcal{D}_{\lambda^p}$, since the summand is $\mathcal{D}_{\lambda^p}$ copies of the regular $S(b)$-module. Thus each $\mathcal{M}_{\lambda^p}$ is semisimple (see also Proposition 4.48), and hence

Theorem 4.57 (See [13]). *Let $n \in \mathbb{N}$. Then the algebra P_n^{ς} is split semisimple over $\mathbb{C}$. The simple modules may be indexed by the set $\mathrm{Mor}_n(\Lambda^*, \Lambda)$. The dimensions of the simple modules are given by $m_\mu \mathcal{D}_{\lambda^p}$.*

Proof. Immediate from previous results. □

Chapter 5

The Juyumaya algebra of braids and ties - Connection to P_n^{ς}

This chapter introduces the main object of the thesis, the Juyumaya algebra. In Section 5.1, we recall the definition of the Juyumaya algebra. We prove the main results of the thesis in Section 5.2 and Section 5.3.

5.1 The Juyumaya algebra of braids and ties

Following [20], we recall the Juyumaya algebra over the ring $\mathbb{C}[u, u^{-1}]$.

Definition 5.1 (See [20, §2])**.** Let u be an indeterminate over $\mathbb{C}$ and $\mathcal{A}$ be the principal ideal domain $\mathbb{C}[u, u^{-1}]$. The algebra $\mathcal{E}_n^{\mathcal{A}}(u)$ over $\mathcal{A}$ is the unital associative $\mathcal{A}$-algebra generated by the elements $T_1,\ T_2, \ldots, T_{n-1}$ and $E_1,\ E_2, \ldots, E_{n-1}$, which satisfy the defining relations

$$
\begin{array}{lll}
(A1) & T_iT_j = T_jT_i & \text{if } \; |i-j|>1 \\
(A2) & E_iE_j = E_jE_i & \\
(A3) & E_i^2 = E_i & \\
(A4) & E_iT_i = T_iE_i & \\
(A5) & E_iT_j = T_jE_i & \text{if } \; |i-j|>1 \\
(A6) & T_iT_jT_i = T_jT_iT_j & \text{if } \; |i-j|=1 \\
(A7) & E_jT_iT_j = T_iT_jE_i & \text{if } \; |i-j|=1 \\
(A8) & E_iE_jT_j = E_iT_jE_i = T_jE_iE_j & \text{if } \; |i-j|=1 \\
(A9) & T_i^2 = 1+(u-1)E_i(1-T_i) &
\end{array}
$$

for all i, j.

Let $\mathbb{C}(u)$ be the field of rational functions. We define $\mathcal{E}_n^0(u)$ as

$$\mathcal{E}_n^0(u) := \mathcal{E}_n^{\mathcal{A}}(u) \otimes_{\mathcal{A}} \mathbb{C}(u)$$

where $\mathbb{C}(u)$ is made into an $\mathcal{A}$-module through inclusion.

Corollary 5.2 (See [20, Corollary 3])**.** *The dimension of $\mathcal{E}_n^0(u)$ is given by $n!B_n$, where B_n is the nth Bell number.* □

The Bell number making appearance in Corollary 5.2 indicates that there might be a connection between the Juyumaya algebra and the (small ramified) partition algebra. In section 5.2 we present this connection.

From the presentation of $\mathcal{E}_n^{\mathcal{A}}(u)$, relations $(A1)$, $(A6)$, $(A9)$ form a deformation of the defining Coxeter relations (see [21, §1]) of the symmetric group S_n. It is straightforward to verify the following result.

Proposition 5.3. *There exists a homomorphism from $\mathcal{E}_n^{\mathcal{A}}(u)$ to the group ring $\mathcal{A}S_n$ of the symmetric group given by*

$$
\begin{aligned}
X \colon \mathcal{E}_n^{\mathcal{A}}(u) &\to \mathcal{A}S_n \\
T_i &\mapsto (i, i+1) \\
E_i &\mapsto 0.
\end{aligned}
$$

□

In particular, $\mathcal{A}S_n$ is isomorphic to a quotient of $\mathcal{E}_n^{\mathcal{A}}(u)$.

5.2 Relationship of the Juyumaya algebra to P_n^ς

In response to a remark by Ryom-Hansen in [20] regarding the dimension of $\mathcal{E}_n^0(u)$: "The appearance of the Bell number is somewhat intriguing and may indicate a connection to the partition algebra ..., we do not think at present that the connection can be very direct", we present new results that establish a connection between the Juyumaya algebra and the partition algebra, via the small ramified partition algebra.

Let $\mathbb{C}$ be the field of complex numbers which is a $\mathbb{C}[u, u^{-1}]$-algebra (that is, with u acting as a complex number x). Denote the $\mathbb{C}$-algebra $\mathcal{E}_n^{\mathcal{A}}(u) \otimes_{\mathcal{A}} \mathbb{C}$ by $\mathcal{E}_n(x)$. Here, we shall only need the case $x = 1$.

Recall from Definition 3.7 and Section 3.2.3 the definitions of Δ_n and $A^{i,i+1}$ respectively.

Proposition 5.4. *The map* $\rho\colon \mathcal{E}_n(1) \to \mathbb{C}S_n \otimes_{\mathbb{C}} \Delta_n$ *given by*

$$E_i \mapsto (1, A^{i,i+1})$$
$$T_i \mapsto (\sigma_{i,i+1}, \sigma_{i,i+1})$$

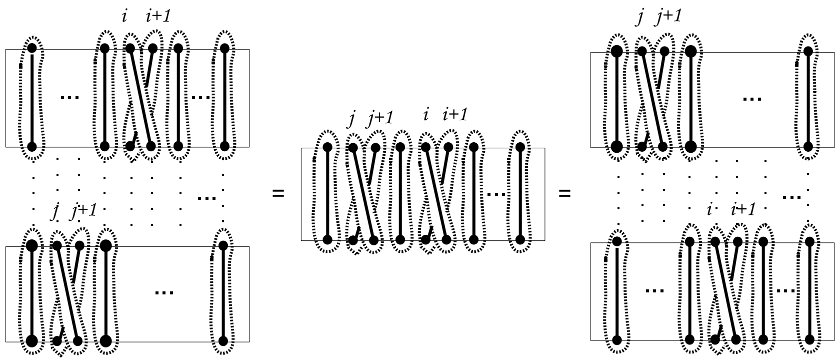
defines a $\mathbb{C}$*-algebra homomorphism.*

Proof. To show that this map is an algebra homomorphism we check that the relations (A1)–(A9) hold when $(1, A^{i,i+1})$ is put in place of E_i and $(\sigma_{i,i+1}, \sigma_{i,i+1})$ is put in place of T_i as follows.

Assume $|i - j| > 1$. Then

(A1) $\rho(T_iT_j) = (\sigma_{i,i+1}, \sigma_{i,i+1})\,(\sigma_{j,j+1}, \sigma_{j,j+1}) = (\sigma_{i,i+1}\sigma_{j,j+1},\ \sigma_{i,i+1}\sigma_{j,j+1})$ and $\rho(T_jT_i) = (\sigma_{j,j+1}, \sigma_{j,j+1})\,(\sigma_{i,i+1}, \sigma_{i,i+1}) = (\sigma_{j,j+1}\sigma_{i,i+1},\ \sigma_{j,j+1}\sigma_{i,i+1})$. Since $|i - j| > 1$, $\sigma_{i,i+1}\sigma_{j,j+1} = \sigma_{j,j+1}\sigma_{i,i+1}$. Thus, $(\sigma_{i,i+1}\sigma_{j,j+1},\ \sigma_{i,i+1}\sigma_{j,j+1}) = (\sigma_{j,j+1}\sigma_{i,i+1},\ \sigma_{j,j+1}\sigma_{i,i+1})$ as required.

Diagrammatically, this may be represented as follows.



(A2) $\rho(E_iE_j) = (1, A^{i,i+1})\,(1, A^{j,j+1}) = (1,\ A^{i,i+1}A^{j,j+1}) = (1,\ A^{j,j+1}A^{i,i+1})$
$= (1, A^{j,j+1})\,(1, A^{i,i+1}) = \rho(E_jE_i)$.

The second equality follows from the definition of the tensor product of algebras, the third equality is a consequence of the commutativity of the elements $A^{k,k+1}$ and the fourth equality follows again from the definition of the tensor product of algebras.

(A3) $\rho(E_i^2) = (1, A^{i,i+1})\,(1, A^{i,i+1}) = (1,\ A^{i,i+1}A^{i,i+1}) = (1,\ A^{i,i+1}) = \rho(E_i)$.

(A4) Similar to the proof of relation (A2), $\rho(E_iT_i) = (1, A^{i,i+1})\,(\sigma_{i,i+1},\ \sigma_{i,i+1})$
$= (1\sigma_{i,i+1},\ A^{i,i+1}\sigma_{i,i+1}) = (\sigma_{i,i+1}1,\ \sigma_{i,i+1}A^{i,i+1})$
$= (\sigma_{i,i+1},\ \sigma_{i,i+1})\,(1, A^{i,i+1}) = \rho(T_iE_i)$.

(A5) $\rho(E_iT_j) = (1, A^{i,i+1})\,(\sigma_{j,j+1},\ \sigma_{j,j+1}) = (1\sigma_{j,j+1},\ A^{i,i+1}\sigma_{j,j+1})$ and
$\rho(T_jE_i) = (\sigma_{j,j+1},\ \sigma_{j,j+1})\,(1, A^{i,i+1}) = (\sigma_{j,j+1}1,\ \sigma_{j,j+1}A^{i,i+1})$.
Since $|i-j| > 1,\ A^{i,i+1}\sigma_{j,j+1} = \sigma_{j,j+1}A^{i,i+1}$. Thus,
$(\sigma_{j,j+1},\ A^{i,i+1}\sigma_{j,j+1}) = (\sigma_{j,j+1},\ \sigma_{j,j+1}A^{i,i+1})$ as required.

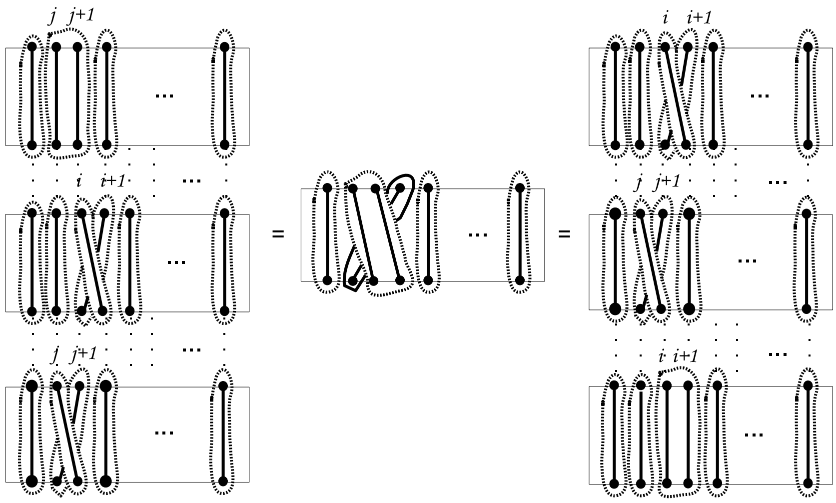
(A6) $\rho(T_iT_jT_i) = (\sigma_{i,i+1}, \sigma_{i,i+1})\,(\sigma_{j,j+1},\ \sigma_{j,j+1})\,(\sigma_{i,i+1}, \sigma_{i,i+1})$
$= (\sigma_{i,i+1}\sigma_{j,j+1}\sigma_{i,i+1},\ \sigma_{i,i+1}\sigma_{j,j+1}\sigma_{i,i+1})$ and
$T_jT_iT_j$ corresponds to $(\sigma_{j,j+1}, \sigma_{j,j+1})\,(\sigma_{i,i+1},\ \sigma_{i,i+1})\,(\sigma_{j,j+1}, \sigma_{j,j+1})$
$= (\sigma_{j,j+1}\sigma_{i,i+1}\sigma_{j,j+1}\ \sigma_{j,j+1}\sigma_{i,i+1}\sigma_{j,j+1})$.
Since $|i-j| = 1,\ \sigma_{i,i+1}\sigma_{j,j+1}\sigma_{i,i+1} =\ \sigma_{j,j+1}\sigma_{i,i+1}\sigma_{j,j+1}$.
Thus, $(\sigma_{i,i+1}\sigma_{j,j+1}\sigma_{i,i+1}\ \sigma_{i,i+1}\sigma_{j,j+1}\sigma_{i,i+1}) =$
$(\sigma_{j,j+1}\sigma_{i,i+1}\sigma_{j,j+1}, \sigma_{j,j+1}\sigma_{i,i+1}\sigma_{j,j+1})$ as required.

(A7) The element $E_jT_iT_j$ is mapped to $(1, A^{j,j+1})(\sigma_{i,i+1}, \sigma_{i,i+1})(\sigma_{j,j+1}, \sigma_{j,j+1})$
$= (\sigma_{i,i+1}\sigma_{j,j+1},\ A^{j,j+1}\sigma_{i,i+1}\sigma_{j,j+1})$ and
the element $T_iT_jE_i$ is mapped to $(\sigma_{i,i+1}, \sigma_{i,i+1})(\sigma_{j,j+1}, \sigma_{j,j+1})(1, A^{i,i+1})$

$= (\sigma_{i,i+1}\sigma_{j,j+1},\ \sigma_{i,i+1}\sigma_{j,j+1}A^{i,i+1})$.

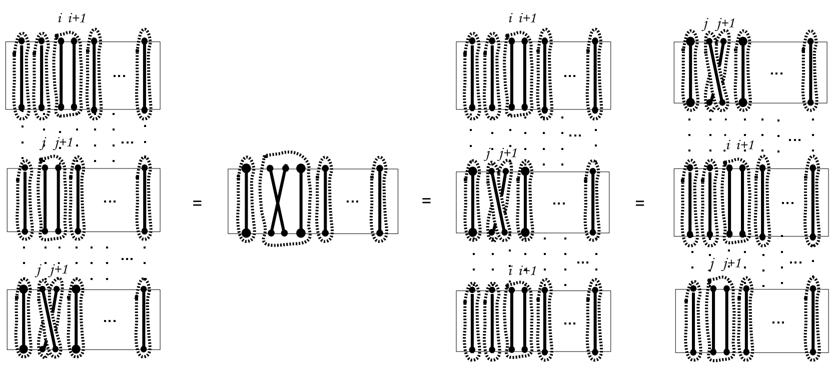
Since $|i-j| = 1,\ A^{j,j+1}\sigma_{i,i+1}\sigma_{j,j+1} = \sigma_{i,i+1}\sigma_{j,j+1}A^{i,i+1}$ and the result follows.

Proving relation (A7) using diagrams:



(A8) The element $E_iE_jT_j$ corresponds to $(1, A^{i,i+1})(1, A^{j,j+1})(\sigma_{j,j+1}, \sigma_{j,j+1})$
$= (\sigma_{j,j+1}, A^{i,i+1}A^{j,j+1}\sigma_{j,j+1})$,
the element $E_iT_jE_i$ corresponds to $(1, A^{i,i+1})(\sigma_{j,j+1}, \sigma_{j,j+1})(1, A^{i,i+1})$
$= (\sigma_{j,j+1}, A^{i,i+1}\sigma_{j,j+1}A^{i,i+1})$, and
the element $T_jE_iE_j$ corresponds to $(\sigma_{j,j+1}, \sigma_{j,j+1})(1, A^{i,i+1})(1, A^{j,j+1})$
$= (\sigma_{j,j+1}, \sigma_{j,j+1}A^{i,i+1}A^{j,j+1})$.
We have $A^{i,i+1}A^{j,j+1}\sigma_{j,j+1} = A^{i,i+1}\sigma_{j,j+1}A^{i,i+1} = \sigma_{j,j+1}A^{i,i+1}A^{j,j+1}$ since $|i-j| = 1$ as required.

Relation (A8) may be described using diagrams as follows.



(A9) Since u is specialised to 1, relation A9 states that $T_i^2 = 1$ and the relation corresponds to $(\sigma_{i,i+1}, \sigma_{i,i+1})\ (\sigma_{i,i+1}, \sigma_{i,i+1}) = (\sigma_{i,i+1}\sigma_{i,i+1}, \sigma_{i,i+1}\sigma_{i,i+1}) = (1, 1)$ as required.

$\square$

We leave it as an exercise to use diagrams to check relations (A2) – (A6) and (A9). Next we show that

Theorem 5.5. *The map $\phi\colon \mathcal{E}_n(1) \to P_n^\varsigma$ given by*

$$E_i \mapsto (1, A^{i,i+1})$$
$$T_i \mapsto (\sigma_{i,i+1}, \sigma_{i,i+1})$$

defines a $\mathbb{C}$-algebra isomorphism.

Proof. The map ϕ is well-defined since by Proposition 3.17 $(1, A^{i,i+1})$ and $(\sigma_{i,i+1}, \sigma_{i,i+1})$ generates precisely P_n^ς.

In order to check that ϕ is an algebra homomorphism, we need to verify that the defining relations of $\mathcal{E}_n(1)$ are satisfied in P_n^ς and this has already been shown in Proposition 5.4. All that remains is to show that the map is an isomorphism. By Corollary 5.2 and by Corollary 3.14, the dimensions of $\mathcal{E}_n(1)$ and P_n^ς are equal. Moreover, the map ϕ is surjective since the images of the generators E_i and T_i of $\mathcal{E}_n(1)$ generate P_n^ς.

Thus, the preceding facts together imply that ϕ is an isomorphism. $\square$

5.3 Representation theory

Generic irreducible representations of the Juyumaya algebra are constructed for the cases $n = 2, 3$ in [19], [20]. Here we provide a proof of generic semisimplicity of the Juyumaya algebra for all n, by reference to Chapter 4.

In the previous section we established, for each $n \in \mathbb{N}$, an isomorphism between the algebras $\mathcal{E}_n(1)$ and P_n^ς. With this result we have implicitly determined the complex representation theory of $\mathcal{E}_n(1)$ since the representation theory of P_n^ς over $\mathbb{C}$ is already known (See Chapter 4). With the knowledge that the algebra P_n^ς over $\mathbb{C}$ is split semisimple, we can now prove that

Theorem 5.6. *For all n, the algebra $\mathcal{E}_n(x)$ is generically semisimple.*

Before we provide the proof of Theorem 5.6, it is worth clarifying our notion of generic. Our notion is essentially the same as that of Cline, Parshall, and Scott [22, §1]. Precisely,

Definition 5.7. Let P be a property of finite dimensional algebras over fields. Given a commutative, Noetherian domain $\mathcal{O}$ with quotient field $K = k(0)$, and a finite dimensional algebra A over $\mathcal{O}$ such that P holds for the K-algebra A_K. Then P is said to hold *generically* for A if there exists a non-empty open subset $\Omega \subseteq \text{Spec } \mathcal{O}$ such that P holds for the residual algebras $A_{k(p)}$ for all $p \in \Omega$.

For example, the property that an algebra be split semisimple is a generic property but in our case it holds on a Zariski non-empty open subset of the complex space.

Proof of Theorem 5.6.
By Theorem 5.5, $\mathcal{E}_n(1)$ is isomorphic to the algebra P_n^ς and by Theorem 4.57 P_n^ς is split semisimple over $\mathbb{C}$. This implies that $\mathcal{E}_n(1)$ is split semisimple over $\mathbb{C}$. Since split semisimplicity is a generic property therefore, $\mathcal{E}_n(x)$ is split semisimple for generic choices of $x \in \mathbb{C}$. But $\mathcal{E}_n(x)$ is semisimple if and only if it is split semisimple (since we are working over an algebraically closed field of characteristic zero). $\square$

Chapter 6

Conclusion

In this chapter, we summarise what has been achieved so far and make some suggestions as to possible further directions in which this research could continue.

6.1 Summary

We have determined the generic representation theory of the Juyumaya algebra beyond the cases $n = 2, 3$ over the field of complex numbers. In order to understand the representation theory of the Juyumaya algebra over the field of complex numbers, it was crucial for us to study the representation theory of the small ramified partition algebra since these algebras are isomorphic as $\mathbb{C}-$algebras. Thus, to begin our study of the small ramified partition algebra, it was helpful to, first of all, familiarise ourselves with both the partition algebra and the ramified partition algebra.

It was worth studying the representation theory of wreath products as it is closely tied to the combinatorial representation theory of the small ramified partion algebra. While the representation theory of wreath products are by now reasonably known, there is a lack of concise presentations suitable for readers seeking a fast read on this topic. Chapter 4, therefore, has a considerable emphasis on the exposition of this material as a way of bridging the gap. On the other hand, to some extent the exposition of the representation theory of the small ramified partition algebra is not new material but is presented in a terse manner in the only reference [13] found. We have tried to improve this by giving a detailed description and providing the results with proofs which are not found in that paper.

6.2 Discussion

The subject of representation theory of algebras is a vast one. As such, there are a number of interesting open problems on this subject.

The small ramified partition algebra P_{n-1}^{ς} can be embedded in P_n^{ς} by adding vertices labelled n and n' with a propagating stick (see Section 3.2.2 for definition) connecting them in the rightmost part of an arbitrary diagram $p \in P_{n-1}^{\varsigma}$. We have the following tower of algebras

$$P_{n-1}^{\varsigma} \subset P_n^{\varsigma} \subset \dots .$$

The Bratteli diagram for the tower of small ramified partition algebras is a graph with vertices organised into levels indexed by $n \in \mathbb{N}$ such that the vertices on level n are labelled by the index set $\mathrm{Mor}_n(\Lambda^*, \Lambda)$ (described in Section 4.5) corresponding to the irreducible representations of P_n^{ς}.

The Bratteli diagram for the inclusion $P_{n-1}^{\varsigma} \subset P_n^{\varsigma}$ for $n \leq 4$ is shown in Figure 6.1.

The number of paths from the top of the Bratteli diagram (in Figure 6.1) to $\mu \in \mathrm{Mor}_n(\Lambda^*, \Lambda)$ is the label (in bold) on vertex μ and thus is the dimension of μ. In row $n = 4$, the dimensions of the irreducible modules are $1, 3, 2, 3,$
$1, 4, 8, 4, 3, 3, 6, 3, 3, 6, 6$,$6, 6, 1, 3, 2, 3, 1$ (reading from left to right). Furthermore, $1^2 + 3^2 + 2^2 + 3^2 + 1^2$+$4^2 + 8^2 + 4^2 + 3^2$+$3^2 + 6^2 + 3^2 + 3^2$+$6^2 + 6^2 + 6^2 + 6^2 +$ 1^2+$3^2 + 2^2 + 3^2 + 1^2 = 360$ which is $4!15$ (from Corollary 3.14), the dimension of P_4^{ς}.

We have not described the branching rule for $P_{n-1}^{\varsigma} \subset P_n^{\varsigma}$ for all n. A good starting point might be to prove the conjecture in Martin's paper [13], relating to an algorithm for describing a restriction rule for P_n^{ς} to P_{n-1}^{ς}.

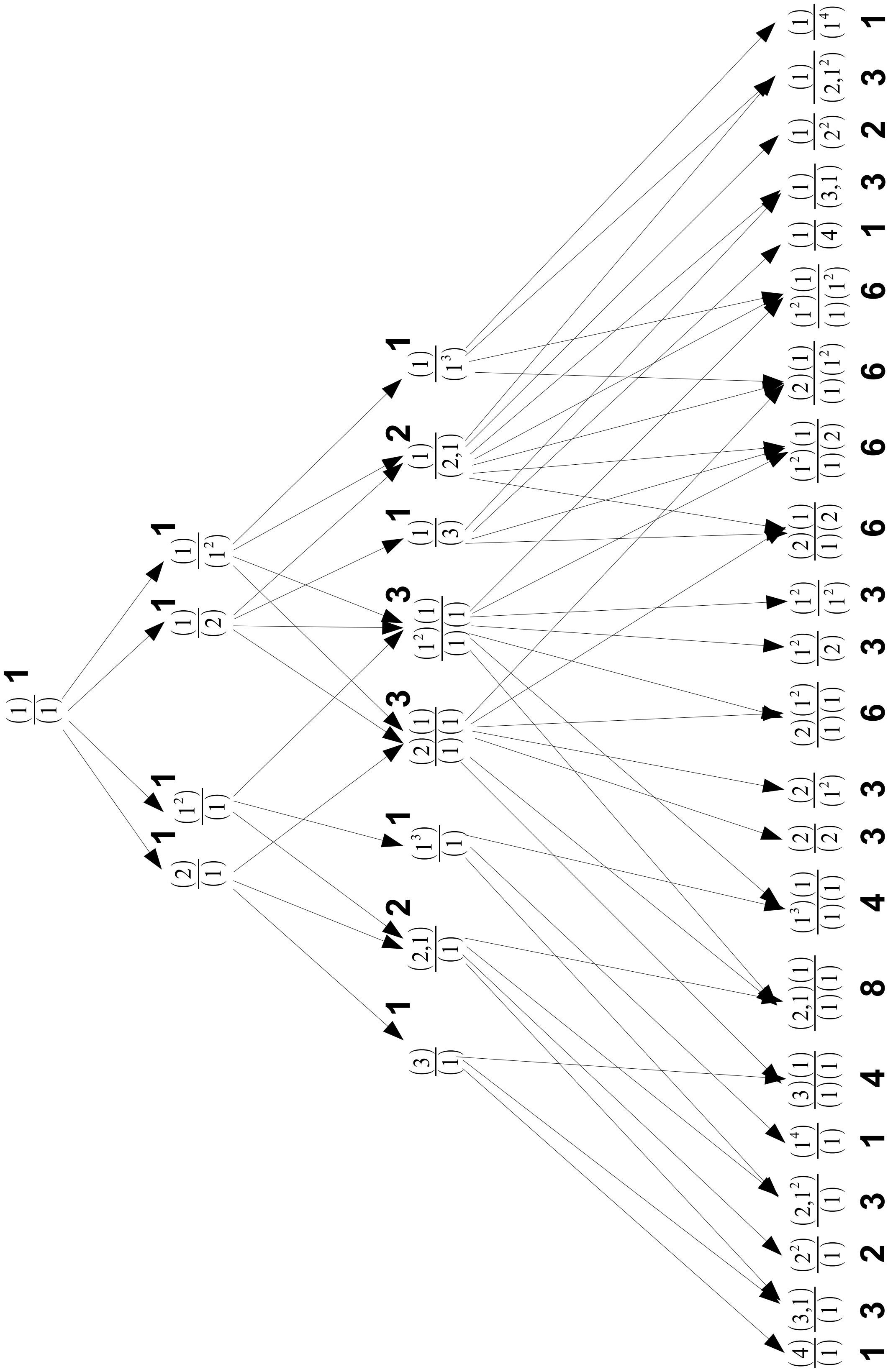


FIGURE 6.1: Bratteli diagram for P_n^{ς}.

Appendix A

The Representation Theory of the Symmetric Group

Here we briefly review some combinatorial notions from the representation theory of the symmetric groups. This appendix provides some necessary tools for the construction of the irreducible representations of the small ramified partition algebra studied in Chapter 4.

A.1 Partitions and Young tableaux

Definition A.1. A *partition* λ of a nonnegative integer $n \in \mathbb{N}$ is a finite sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$ such that $n = \sum_{i=1}^{l} \lambda_i$.

For example, the partitions of the integer 4 are:

$$(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1).$$

We write $\lambda \vdash n$ to denote that λ is a partition of n. By definition, $n = 0$ has a unique partition, namely the empty sequence $\emptyset$. A partition λ is represented graphically by a Young diagram.

Definition A.2. A *Young diagram* of $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_l)$ is a left-justified array of boxes with l rows, and λ_i boxes on the ith row.

For example, the Young diagrams of the partitions of 4 are

$$\ydiagram{1,1,1,1}\,,\ \ydiagram{2,1,1}\,,\ \ydiagram{2,2}\,,\ \ydiagram{3,1}\,,\ \ydiagram{4}\,.$$

Definition A.3. A *(Young) tableau of shape* λ is obtained by filling in the boxes of a Young diagram of λ with non-repeated entries $t_{i,j} \in \{1, 2, \dots, n\}$.

For example, let $\lambda = (2, 1) \vdash 3$. Then the Young tableaux of shape $(2, 1)$ are

$$\begin{ytableau} 1 & 2 \\ 3 \end{ytableau}\,,\ \begin{ytableau} 2 & 1 \\ 3 \end{ytableau}\,,\ \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}\,,\ \begin{ytableau} 3 & 1 \\ 2 \end{ytableau}\,,\ \begin{ytableau} 2 & 3 \\ 1 \end{ytableau}\,,\ \begin{ytableau} 3 & 2 \\ 1 \end{ytableau}\,.$$

Definition A.4. A *standard (Young) tableau* of shape λ is a Young tableau of shape λ such that the entries of each row are in increasing order from left to right and the entries of each column are in increasing order from top to bottom.

For example, the following are all standard tableaux of shape $(2, 2, 1)$.

$$\begin{ytableau} 1 & 2 \\ 3 & 4 \\ 5 \end{ytableau}\,,\ \begin{ytableau} 1 & 2 \\ 3 & 5 \\ 4 \end{ytableau}\,,\ \begin{ytableau} 1 & 3 \\ 2 & 4 \\ 5 \end{ytableau}\,,\ \begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 \end{ytableau}\,,\ \begin{ytableau} 1 & 4 \\ 2 & 5 \\ 3 \end{ytableau}\,.$$

A.1.1 Specht Modules for S_n

To each Young tableau, we construct a primitive idempotent (see Definition 4.50 for a definition) which generates a simple module of $\mathbb{C}S_n$. This simple module is known as the *Specht module* (See, for example, [1, Section 2.3]). The primitive idempotents are constructed from corresponding "symmetrizers" and "antisymmetrizers". We briefly discuss these tools and a construction here. A technique for explicitly constructing the irreducible representations of $P_n^{\ltimes}$ is considered in Section 4.8 of Chapter 4, based on the concepts of Young tableaux, symmetrizers, and antisymmetrizers.

Definition A.5. Let t be a Young tableau. Then the *row stabilizer* of t, denoted R_t, is the subgroup of S_n which permutes elements within each row of t.

Definition A.6. Let t be a Young tableau. Then the *column stabilizer* of t, denoted C_t, is the subgroup of S_n which permutes elements within each column of t.

Example A.7. Let $t = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$. Then

$$\begin{aligned} R_t = \{&(123), (132), (12), (13), (23), (1), (123)(45), \\ &(132)(45), (12)(45), (13)(45), (23)(45), (45)\} \\ C_t = \{&(146), (164), (14), (16), (46), (1), (146)(25), \\ &(164)(25), (14)(25), (16)(25), (46)(25), (25)\} \end{aligned}$$

To each Young tableau, we associate a primitive idempotent.

Definition A.8. Let t be a Young tableau. The *symmetrizer* s_t, the *antisymmetrizer* a_t, and the *Young symmetrizer* y_t associated with t are defined as

$$\begin{aligned} s_t &= \sum_{r \in R_t} r; \\ a_t &= \sum_{c \in C_t} sgn(c)c; \\ y_t &= \sum_{c,r} sgn(c)rc. \end{aligned}$$

where *sgn* stands for the sign of the permutation c.

The symmetrizer, antisymmetrizer, and the Young symmetrizer, generate left ideals that provide the irreducible representations of the symmetric group [50, Theorem 5.12.2]. In particular,

Theorem A.9 (See, e.g.,[51, Theorem 5.4])**.** *The Young symmetrizer* y_t *associated to the Young tableau* t *is a primitive idempotent, and the invariant subspace* $S^\lambda := \mathbb{C}S_n y_t$*, for each* $\lambda \vdash n$*, of* $\mathbb{C}S_n$ *yields an irreducible representation of* S_n*.* □

The module S^λ are called the *Specht module* [50].

The irreducible representations for different Young diagrams are inequivalent, but for different Young tableaux of the same shape they are equivalent [52, Lemma 4.7]. Moreover the complete decomposition of the regular representations of S_n is governed by the following theorem.

Theorem A.10 (See, e.g.,[53, Proposition 7.2.2] and [50, Theorem 5.12.2])**.** *Every irreducible representation of* S_n *is isomorphic to* S^λ *for a unique* λ*. Furthermore,*

S^λ satisfy

$$n! = \sum_{\lambda \vdash n} (dim\ S^\lambda)^2,$$

so the Specht modules give a complete set of inequivalent irreducible modules.

As an example, we construct all the irreducible representations of S_3.

Example A.11. There are four standard Young tableaux for $n = 3$.

$$t_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array} : \quad \begin{aligned} s_{t_1} &= e + (12) + (23) + (13) + (123) + (132) \\ a_{t_1} &= e \\ y_{t_1} &= a_{t_1} s_{t_1} = e + (12) + (23) + (13) + (123) + (132) \end{aligned}$$

$$t_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} : \quad \begin{aligned} s_{t_2} &= e + (12) \\ a_{t_2} &= e - (13) \\ y_{t_2} &= a_{t_2} s_{t_2} = e + (12) - (13) - (123) \end{aligned}$$

$$t_3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} : \quad \begin{aligned} s_{t_3} &= e + (13) \\ a_{t_3} &= e - (12) \\ y_{t_3} &= a_{t_3} s_{t_3} = e - (12) + (13) - (132) \end{aligned}$$

$$t_4 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} : \quad \begin{aligned} s_{t_4} &= e \\ a_{t_4} &= e - (12) - (23) - (13) + (123) + (132) \\ y_{t_4} &= a_{t_4} s_{t_4} = e - (12) - (23) - (13) + (123) + (132) \end{aligned}$$

In the example above, it is easy to see that y_{t_1} and y_{t_4} each generates an inequivalent one-dimensional representations. On can show directly that $y_{t_4} y_{t_1} = 0$. In fact, one can prove that if tableaux t and t' are not equal then $y_t y_{t'} = 0$ for general S_n (see [51, lemma IV.6]). To construct a basis for the representations of S_3

generated by y_{t_2}, we multiply e_2 on the left by elements of S_3.

$$\begin{aligned}
ey_{t_2} &= e + (12) - (13) - (123) = y_{t_2} \\
(12)y_{t_2} &= (12) + e - (132) - (23) := q_2 \\
(13)y_{t_2} &= (13) + (123) - e - (12) = -y_{t_2} \\
(23)y_{t_2} &= (23) + (132) - (123) - (13) = y_{t_2} - q_2 \\
(123)y_{t_2} &= (123) + (13) - (23) - (132) = -y_{t_2} + q_2 \\
(132)y_{t_2} &= (132) + (23) - (12) - e = -q_2
\end{aligned}$$

We see that $\mathbb{C}S_3 y_{t_2}$ is spanned by y_{t_2} and q_2. Since these elements are linearly independent, they form a basis for $\mathbb{C}S_3 y_{t_2}$. It is straightforward to verify that y_{t_3} also generates a two dimensional irreducible representation. Since S_3 has only one two-dimensional representation it is necessary that this representation generated by y_{t_3} is isomorphic to the one described above. However, the invariant subspace generated by y_{t_3} is different from that generated by y_{t_2}. We note that the invariant subspaces generated by the idempotents $y_{t_1}, y_{t_2}, y_{t_3}, y_{t_4}$ of the four standard Young tableaux together span the whole of $\mathbb{C}S_n$. We conclude that the regular representation of S_3 is fully decomposed into irreducible representations by using Young symmetrizers associated with the standard Young tableaux, as was claimed above.

Appendix B

Representation theory of finite groups

We assemble an arsenal of basic tools to use for the study of the representation theory of the small ramified partition algebra. We assume basic knowledge of groups, rings, fields, and vector spaces. A complete exposition of group representations can be found in [1], [45], or [54]. This chapter contains no new material but it is intended to keep this thesis reasonably self-contained. We shall omit most proofs on the assumption that the reader will have seen this material before.

B.1 Group representations and modules

Let F be a field. Unless stated otherwise, we will always assume that F is algebraically closed. We shall use the notation $\dim V$ to denote the dimension of a vector space V over F. We recall that $GL(V)$ denotes the group of all invertible linear transformations of a vector space V onto itself over F. We write $GL(n, F)$ for the group of invertible $n \times n$ matrices over F.

B.1.1 Matrix representations

Definition B.1. A *matrix representation* of a finite group G over F is a homomorphism

$$\rho : G \to GL(n, F).$$

Thus, if ρ is a function from G to $GL(n, F)$, then ρ is a representation if and only if

$$\rho(e) = I \text{ the identity matrix, and} \tag{B.1}$$

$$\rho(gh) = \rho(g)\rho(h), \qquad \text{for all } g, h \in G. \tag{B.2}$$

The conditions (B.1) and (B.2), applied with $h = g^{-1}$, imply that each $\rho(g)$ is invertible and

$$\rho(g^{-1}) = \rho(g)^{-1} \qquad \text{for all } g \in G.$$

The *dimension*, or *degree*, of ρ is the integer n.

Definition B.2. Two representations $\rho, \rho' \colon G \to GL(n, F)$ of a group G are said to be *equivalent* if there exists a fixed invertible matrix T such that

$$\rho'(g) = T\rho(g)T^{-1} \text{ for all } g \in G.$$

Otherwise, ρ and ρ' are said to be *inequivalent*.

We write $\rho \sim \rho'$ to imply that ρ and ρ' are equivalent representations.

Theorem B.3 (See, e.g.,[1, Proposition 1.10.1])**.** *Let $\rho_1, \ldots, \rho_l$ be a complete set of inequivalent irreducible representations of a group G. Then*

$$\sum_{i=1}^{l} (dim\ \rho_i)^2 = |G|. \qquad \square$$

An approach to the representation theory of finite groups involves yet another equivalent concept, that of finitely generated modules over the group algebra. Much of the material in the remainder of the thesis shall be presented in terms of modules. It is therefore necessary at this juncture to review some elementary module theory.

B.1.2 Modules and Algebras

Definition B.4. Let R be a ring with unit, meaning R has a multiplicative identity 1, and let M be an abelian group written additively. We say that M is a *left R-module* if there is a map from $R \times M \to M$ such that for all $r, s, \in R$, and $m, n \in M$, the following conditions are satisfied:

(1) $rm \in M$;

(2) $(r+s)m = rm + sm$;

(3) $r(m+n) = rm + rn$;

(4) $(rs)m = r(sm)$;

(5) $1m = m$.

If F is a field, then the definition of an F-module is precisely that of an F-vector space. Thus a module is the natural generalization of a vector space when working over an arbitrary ring instead of a field [55].

A *right R-module M* is defined similarly, with the exception that the ring acts on the right. If R is commutative, then every left R-module can, in an obvious way, be given a right R-module structure, and hence it is not necessary to distinguish between left and right R-modules. In this thesis all modules will be left modules, unless stated otherwise. An example of a module is a vector space V over F, together with a multiplication $(v, g) \mapsto vg$ for $v \in V$ and $g \in G$ (and the multiplication satisfies the above axioms). Then V, following [56], is referred to as an FG-module.

Matrix representation lies at the concrete end of the spectrum of representation theory. At the abstract, theoretic end of the spectrum is found the module theoretic approach. A result that enables this approach is the bijection between FG-modules and matrix representations of G over F, which we reveal in the following result.

Theorem B.5 (See, e.g.,[56, Theorem 4.4])**.**
(1) If $\rho\colon G \to GL(n, F)$ is a representation of G, then the vector space F^n of column vectors becomes an FG-module with the action of G given by $gv = \rho(g)v$.

(2) Conversely, if V is a finite dimensional FG-module, we can choose a basis $\{v_1, \ldots, v_n\}$ of V and let $\rho(g)$ be the matrix describing the action of g on V with respect to this basis. Then $g \mapsto \rho(g)$ is a representation of G. □

Thus representations of a group G over a field F can be identified with its FG-modules. Our viewpoint will primarily be that of modules over an algebra, although on occasion it will be convenient to work with the matrix representation ρ

arising from a given FG-module, where $\rho\colon G \to GL(n,F)$ is defined by $\rho(g)v = gv$ for $v \in F^n$ and $g \in G$.

Definition B.6. Let R, S be rings. We say that an abelian group M is an (R,S)-*bimodule* over R and S if M is a left R-module and a right S-module and if we have

$$r(ms) = (rm)s \qquad \text{for all } r \in R, m \in M, \text{ and } s \in S.$$

We shall sometimes write ${}_RM,\ M_R,\ {}_RM_S$ for a left R-module, a right R-module, and a (R,S)-bimodule respectively.

Example B.7.

(1) Every left R-module is an $(R,\mathbb{Z})$-bimodule, and every right R-module is a $(\mathbb{Z},R)$-bimodule.

(2) If R is commutative, then any R-module is an (R,R)-bimodule.

Definition B.8. Let M be an R-module, and let N be a subgroup of M. Then N is a R-*submodule* (or *submodule*) of M if $rn \in N$ for all $r \in R$ and $n \in N$.

Example B.9.

(1) The (left) R-submodules of R are exactly the left ideals of R.

(2) For every R-module M, the zero subspace $\{0\}$, and M itself, are R-submodules of M.

Definition B.10. A non-zero R-module M is *irreducible* if the only submodules of M are $\{0\}$ and M; otherwise M is called *reducible*.

Remark B.11. Irreducible modules are also called *simple* modules.

Definition B.12. Let M be an R-module. If M' is a submodule of M, then the *quotient module* M/M' of M by M' is the quotient group M/M' considered as an R-module by defining $r(m+M') = rm + M'$ for $r \in R$, and $m + M' \in M/M'$.

B.1.2.1 Operations on Modules

Definition B.13. Let N_1 and N_2 be submodules of an R-module M. Then the *sum* of the submodules is defined to be

$$N_1 + N_2 = \{x + y\colon x \in N_1, y \in N_2\}.$$

The sum of two submodules N_1 and N_2 of a module M is also a submodule of M, as is $N_1 \cap N_2$. If $N_1 \cap N_2 = \{0\}$, then the sum of N_1 and N_2 is said to be *direct*, and we denote it by $N_1 \oplus N_2$. Let $\{u_1, u_2, \ldots, u_m\}$ be a basis of N_1, and $\{v_1', v_2', \ldots, v_n\}$ be a basis of N_2. The resulting representation matrices relative to the basis $\{u_1, u_2, \ldots, u_m, v_1', v_2', \ldots, v_n\}$ have the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{B.3}$$

where A and B are of dimensions m and n respectively.

Definition B.14. A submodule N of a module M is a *direct summand* of M if there is some other submodule N' of M such that $M = N \oplus N'$.

If M and N are R-modules, then we denote the set of all R-module homomorphisms from M to N by $\hom_R(M, N)$. We will sometimes write $\mathrm{End}_R(M)$ for $\hom_R(M, M)$.

Let R, S be rings with units, let M be an (R, S)-bimodule, and let N be an R-module. Then $\hom_R(M, N)$ becomes an S-module in the following way: For $s \in S$ and $\rho \in \hom_R(M, N)$, we define $s\rho \in \hom_R(M, N)$ by

$$(s\rho)(m) = \rho(ms).$$

Definition B.15. Let M be an (R, S)-bimodule and let N be an S-module. The *tensor product* of M and N over S is an R-module, denoted $M \otimes_S N$, with generating set $\{m \otimes n \colon m \in M, n \in N\}$ and defining relations:

- $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$ for all $m_1, m_2 \in M$ and $n \in N$.
- $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$ for all $m \in M$ and $n_1, n_2 \in N$.
- $(ms) \otimes n = m \otimes (sn)$ for all $m \in M,\ n \in N$, and $s \in S$.
- $(rm) \otimes n = r(m \otimes n)$ for all $m \in M,\ n \in N$, and $r \in R$.

Definition B.16. Suppose G_1 and G_2 are groups and that $M_1,\ M_2$ are RG_1-modules and RG_2-modules respectively. Then the *outer tensor product* (see, for example, [57]) of M_1 and M_2, denoted $M_1 \# M_2$, is defined as the $R(G_1 \times G_2)$-module obtained by defining the action of any $(g_1, g_2) \in G_1 \times G_2$ on an element

$m_1 \otimes m_2$ of the R-module $M_1 \otimes_R M_2$, by

$$(g_1, g_2)(m_1 \otimes m_2) = g_1 m_1 \otimes g_2 m_2.$$

B.1.2.2 Finiteness conditions

Definition B.17. Let M be an R-module. The submodules of M are said to satisfy the *ascending chain condition (A.C.C.)* if every chain of submodules of M

$$M_1 \subset M_2 \subset \cdots$$

terminates, that is, if there exists an index j such that $M_j = M_{j+1} = \cdots$.

Analogously,

Definition B.18. Let M be an R-module. The submodules of M are said to satisfy the *descending chain condition (D.C.C.)* if every chain of submodules of M

$$M_1 \supset M_2 \supset \cdots$$

terminates.

An R-module M whose submodules satisfy the A.C.C. (resp. D.C.C.) is termed *Noetherian* (resp. *Artinian*).

Definition B.19. Let M be a left R-module. A *composition series* for M is a sequence of submodules of M

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_l = \{0\}$$

such that all quotient modules M_i/M_{i+1} are simple ($i = 0, 1, \ldots, n-1$). The quotient modules M_i/M_{i+1} are called the *composition factors* of this series and the number n is the *length of the series*.

Two composition series are said to be *equivalent* if they have the same number of factors and if the factors can be paired off in such a way that corresponding factors are isomorphic over R [58].

Evidently, not every module has a composition series. For example, the $\mathbb{Z}$-module has no composition series see, for example, [59, Prop. 7.11]. Some criteria for the existence of composition series for a module are stated in the following theorem.

Theorem B.20 (Jordan-Hölder. See, e.g., [60, Theorem 3.2.1] or [50, Theorem 3.7.1])**.** *The following statements about an R-module M are equivalent:*

(i) M has a composition series;

(ii) M satisfies the ascending chain condition (A.C.C.) and descending chain condition (D.C.C.);

(iii) every sequence of submodules of M can be refined (that is, submodules can be inserted) to yield a composition series. □

Definition B.21. Let M and N be R-modules. An *R-module homomorphism* $\alpha\colon M \to N$ is a linear map such that $\alpha(rm) = r\alpha(m)$ for all $r \in R$ $m \in M$.

B.1.3 Group algebras

Definition B.22. Let R be a ring and let G be a group. The *group ring* of G over R, denoted by RG, consists of all finite formal R-linear combinations of elements of G, i.e.:

$$RG = \left\{ \sum_i r_i g_i \colon r_i \in R, g_i \in G \right\},$$

whose multiplication operation is defined by R-linearly extending the group multiplication operation of G. Explicitly, we define the multiplication in RG as follows:

$$\left(\sum_i r_i g_i \right) \left(\sum_j s_j g_j \right) = \sum_{i,j} (r_i s_j)(g_i g_j)$$

for all $r_i, s_j \in R$.

In the case where $R = F$ is a field, the group ring is an F-vector space with G as a basis and hence having finite dimension $|G|$. In this case, FG is called the *group algebra* instead since it satisfies a mathematical structure we now define.

Definition B.23. An *(associative) algebra* A over a field F, or an *$F-$algebra*, is a nonempty set A, together with three operations, called *addition* (denoted by $+$), *multiplication* (denoted by juxtaposition) and *scalar multiplication* (also denoted by juxtaposition), for which the following properties must be satisfied:

- A is a vector space over F under addition and scalar multiplication.
- A is a ring with identity under addition and multiplication.
- If $r \in F$ and $a, b \in A$, then

$$r(ab) = (ra)b = a(rb).$$

Definition B.24.

(1) An algebra is *finite-dimensional* if its vector space is finite-dimensional.

(2) An algebra is *commutative* if A is a commutative ring.

(3) An element $a \in A$ is *invertible* if there is $b \in A$ for which $ab = ba = 1$.

(4) The *centre* of an $F-$algebra A is the set

$$Z(A) = \{a \in A \colon ax = xa \text{ for all } x \in A\}$$

of all elements of A that commute with every element of A.

Example B.25.

- Any ring is a $\mathbb{Z}$-algebra.
- The matrix ring $M_n(F)$ is a finite-dimensional F-algebra.

We now state the fundamental theorem on decomposition of modules (or representations).

Theorem B.26 (Maschke. See, e.g., [56, Theorem 8.1])**.** *Let F be a field of characteristic zero and G a finite group. Let V be a finite-dimensional FG-module with a submodule $U \subseteq V$. Then there exists a subspace $W \subseteq V$ such that $V = U \oplus W$.* □

A useful notion, and one which is somewhat easy for a module to satisfy, is indecomposability.

Definition B.27. An FG-module V is *indecomposable* if it cannot be written as a direct sum of two non-trivial submodules. Otherwise, V is *decomposable.*

Clearly, if V is an irreducible module it has no proper submodules and hence cannot be written as a direct sum of non-trivial submodules. Therefore, any irreducible module is automatically indecomposable. But the converse is not true in general. That is, there exist indecomposable modules which have proper submodules.

Theorem B.28 (See, e.g.,[58, Theorem 14.2])**.** *If the submodules of V satisfy the D.C.C., then V can be expressed as a direct sum of a finite number of indecomposable modules.* □

Definition B.29. An FG-module is said to be *completely reducible* if it is a direct sum of irreducible FG-modules.

Note that by Definition B.29, every irreducible FG-module V is completely reducible. A module which is both reducible, and completely reducible is decomposable. However, a decomposable module need not be completely reducible.

Remark B.30. A completely reducible module is also called *semisimple*.

A natural question to ask is: given G, how many irreducible $\mathbb{C}G$-modules are there? The following result reveals the answer.

Proposition B.31 (See, e.g.,[61, Prop. 6.3])**.** *If G is finite, then the number of inequivalent irreducible G-modules is equal to the number of conjugacy classes of G.* □

It is worth mentioning that, in general, there is no natural one-to-one correspondence between the conjugacy classes of G and the irreducible $\mathbb{C}G$-modules [55]. However, if $G = S_n$, then a conjugacy class consists of all permutations of a given cycle-type (hence there is such a correspondence). But a cycle-type is just a partition of n. Thus,

Corollary B.32. *The number of inequivalent irreducible S_n-modules is the number of partitions of n.*

We state another useful result in representation theory.

Lemma B.33 (Schur. See, e.g., [56, Lemma 9.1])**.** *Let F be the field of complex numbers, G be a group, and U, V be irreducible FG-modules.*

(1) Every FG-homomorphism $U \to V$ is either zero or an isomorphism.

(2) Every FG-isomorphism $U \to U$ is a scalar multiple of the identitiy map 1_U. □

A consequence of Maschke's theorem states that every FG-module is a direct sum of irreducible FG-submodules, where F is a field of characteristic zero (such as $\mathbb{R}$ or $\mathbb{C}$). In essence, this reduces representation theory to the study of irreducible FG-modules [56].

Corollary B.34 (See, e.g., [56, Theorem 8.7])**.** *Suppose that G is a finite group and that F is a field of characteristic zero. Then every non-zero FG-module is completely reducible.*

In the thesis, we shall be concerned only with the case $F = \mathbb{C}$, which is called *ordinary (or complex)* representation theory. Since $\mathbb{C}$ has characteristic zero, we see from Corollary B.34 that every $\mathbb{C}G$-module is semisimple for any finite group G. The remainder of this section concentrates on algebras that have this property.

Let A be an algebra. Our interest is in A-modules which are semisimple and in determining conditions on A under which each A-module will satisfy the property of semisimplicity. Thus, the following theorem reveals the connection between simple modules and semisimple modules.

Theorem B.35 (See, e.g., [62, Prop. 4.28])**.** *The following statements about an A-module M are equivalent:*

(1) Any submodule of M is a direct summand of M.

(2) M is semisimple.

(3) M is a sum of simple submodules. □

Lemma B.36 (See, e.g., [63, Lemma 6.4.4])**.** *If M is a module satisfying condition (1) of the above theorem, then any submodule of M also satisfies that condition.* □

The following results follows immediately from Theorem B.35 and Lemma B.36.

Corollary B.37 (See, e.g., [64, Lemma. 3.3])**.**

(1) A submodule of a semisimple module is again semisimple. The direct sum of any set of semisimple modules is again semisimple.

(2) A quotient of a semisimple module is again semisimple. □

Theorem B.38 (See, e.g., [58, Theorem 25.2])**.** *An algebra A is semisimple if and only if A is semisimple as an A-module.* □

Thus, if G is a finite group and F is a field of characteristic zero, then FG is semisimple by Corollary B.34.

Definition B.39. An algebra D is said to be a *division algebra* if the non-zero elements of D form a group under multiplication.

We now give a complete classification of the finite dimensional semisimple algebras. The following astounding result forms the foundation for our approach in decomposing the small ramified partition algebra in Chapter 4.

Theorem B.40 (Artin-Wedderburn. See, e.g., [65, Theorem 3.3.2])**.** *An algebra A is semisimple over F if and only if it is isomorphic to a direct sum of matrix algebras*

$$A \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_l}(D_l)$$

where $n_1, \ldots, n_l \in \mathbb{N}$ and $D_1, \ldots, D_l$ division algebras. □

A corollary of the Artin-Wedderburn theorem states that if an algebra A satisfies the assumptions of Theorem B.40, then A has exactly l isomorphism classes of simple modules. If S_i is the simple module corresponding to $M_{n_i}(D_i)$, then dim $S_i = n_i$ and S_i occurs precisely n_i times in a decomposition of A into simple modules.

In the case where an algebra A is not semisimple, one can measure how far from semisimple it is by finding the smallest ideal I in A such that A/I is semisimple. This ideal I of A is called the (Jacobson) radical of A.

Definition B.41. The *Jacobson radical* of an algebra A, denoted $\mathcal{J}(A)$, is the intersection of all the maximal ideals of A.

Theorem B.42 (Jacobson. See, e.g., [66, Prop. 2] and [55, Theorem 23])**.** *Let A be a finite dimensional algebra. The ideal $\mathcal{J}(A)$ is*

(i) the intersection of all maximal submodules of A,

(ii) the smallest submodule I of A such that A/I is semisimple. □

Thus an algebra can also be said to be semisimple if its Jacobson radical is the null ideal [67]. Given a finite dimensional A-module M, it is clear that we can decompose M as a direct sum of indecomposable modules. The Krull-Schmidt theorem says that this decomposition is essentially unique, and so it is enough to classify the indecomposable modules of an algebra.

Theorem B.43 (Krull-Schmidt. See, e.g., [68, Coro. 19.22] or [50, Theorem 3.8.1])**.** *Let A be a finite dimensional algebra. Then any finite dimensional representation of A can be uniquely (up to an isomorphism and the order of summands) decomposed into a direct sum of indecomposable representations.* □

B.1.3.1 The regular FG-module

Definition B.44. Let G be a finite group and F be $\mathbb{C}$. The vector space FG, with the natural multiplication $(g \times v) \mapsto gv \quad (v \in FG, g \in G)$, is called the *(left) regular* FG-module.

The *right* regular FG-module is defined similarly but with G acting on the right of FG. Henceforth, we shall use the term "regular FG-module" always to mean "left regular FG-module"; it will be clear, however, that the subsequent discussion applies equally well to the right regular FG-module. Note that the dimension of the regular FG-module is equal to $|G|$.

Definition B.45. An FG-module V is *faithful* if the identity element of G is the only element g for which

$$gv = v \qquad \text{for all } v \in V.$$

Proposition B.46 (See, e.g., [56, Prop. 6.6])**.** *The regular FG-module is faithful.* □

Example B.47. Let $G = S_3 = \{e, g_1, g_2, g_3, g_4, g_5\}$ where $e = (1)(2)(3),\ g_1 = (12),\ g_2 = (13),\ g_3 = (23),\ g_4 = (123),\ g_5 = (132)$. The elements of FG have the form

$$\lambda_1 e + \lambda_2 g_1 + \lambda_3 g_2 + \lambda_4 g_3 + \lambda_5 g_4 + \lambda_6 g_5 \quad (\lambda_i \in F).$$

We find the matrices of $e,\ g_1, g_4$: (here we multiply permutations from left to right)

$$\begin{aligned}
e(\lambda_1 e + \lambda_2 g_1 + \lambda_3 g_2 + \lambda_4 g_3 + \lambda_5 g_4 + \lambda_6 g_5) &= \lambda_1 e + \lambda_2 g_1 + \lambda_3 g_2 + \lambda_4 g_3 + \lambda_5 g_4 + \lambda_6 g_5,\\
g_1(\lambda_1 e + \lambda_2 g_1 + \lambda_3 g_2 + \lambda_4 g_3 + \lambda_5 g_4 + \lambda_6 g_5) &= \lambda_1 g_1 + \lambda_2 e + \lambda_3 g_4 + \lambda_4 g_5 + \lambda_5 g_2 + \lambda_6 g_3,\\
g_4(\lambda_1 e + \lambda_2 g_1 + \lambda_3 g_2 + \lambda_4 g_3 + \lambda_5 g_4 + \lambda_6 g_5) &= \lambda_1 g_4 + \lambda_2 g_3 + \lambda_3 g_1 + \lambda_4 g_2 + \lambda_5 g_5 + \lambda_6 e.
\end{aligned}$$

By taking matrices relative to the basis $e, g_1, g_2, g_3, g_4, g_5$ of FG, we obtain the following matrix representation of G :

$$e \rightarrow \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, \quad g_1 \rightarrow \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix},$$

$$g_4 \rightarrow \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0 \end{pmatrix}.$$

We leave computing the rest of the matrices as an exercise.

As a consequence of Maschke's theorem, we have:

Corollary B.48 (See, e.g., [69, Theorem 2.31])**.** *Every irreducible representation of the group algebra $\mathbb{C}G$ occurs in the regular representation of $\mathbb{C}G$.* □

B.1.3.2 Projective modules

Let A be a finite dimensional algebra.

Definition B.49. A sequence of A-modules

$$L \xrightarrow{\psi} M \xrightarrow{\phi} N \tag{B.4}$$

is *exact* at M if im $\psi =$ ker ϕ.

If a sequence $L_0 \to L_1 \to \ldots \to L_k \to L_{k+1}$ is exact at every module $L_i \colon 1 \leq i \leq k$, then it is called an *exact sequence.* An exact sequence of the form

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\phi} N \to 0 \tag{B.5}$$

is called a *short exact sequence.*

Note that in a short exact sequence as above we have that

$$M/L \cong N$$

by the isomorphism theorem, and dim $M =$ dim $L +$ dim N. In other words, all short exact sequences can essentially be written in the form

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\phi} M/L \to 0$$

where ψ is an inclusion map of a submodule L of M and ϕ is the natural epimorphism. The module M in the short exact sequence (B.5) is an *extension* of L by N.

Lemma B.50. *Given a short exact sequence* (B.5) *the following are equivalent:*

(i) *There exists a homomorphism* $\mu \colon N \to M$ *such that* $\phi\mu = 1_N$.

(ii) *There exists a homomorphism* $\tau \colon M \to L$ *such that* $\tau\psi = 1_L$.

(iii) *There exists a module* Y *with* $M = Y \oplus$ *ker* ϕ.

Definition B.51. A short exact sequence (B.5) is *split* if it satisfies one of the three equivalent conditions in Lemma B.50.

Proposition B.52 (See, e.g., [59, Prop. 6.34])**.** *If an exact sequence* (B.5) *splits, then* $M \cong L \oplus N$. □

Definition B.53. Let R be a ring. A set $\{m_i\}$ of elements of an R-module M is called *R-free* if the only solution to

$$\sum_i r_i m_i = 0, \qquad r_i \in R$$

is $r_i = 0$ for all i.

Definition B.54. A subset $\{m_i\}$ of an R-module M is called a *set of generators* of M if every $m \in M$ can be expressed in the form

$$m = \sum_i r_i(m) m_i \qquad r_i(m) \in R$$

an R-linear combination of a finite number of the $\{m_i\}$.

Definition B.55. A R-free set of generators of M is called a *basis* of M. A (left) R-module with a basis is called a *free (left) R-module.*

Definition B.56. An R-module P is said to be *projective* if P is a direct summand of a free module, i.e. if there exists an R-module Q such that $P \oplus Q$ is a free R-module.

Every free module is projective, but not vice-versa: a projective module which is not free, for example, is $\mathbb{Z}$ regarded as a $\mathbb{Z} \oplus \mathbb{Z}$-module.

Proposition B.57 (See, e.g., [59, Prop. 6.73 and Prop. 6.76])**.** *The following are equivalent for an R-module P:*

(1) P is projective;

(2) if $P \xrightarrow{g'} E'$ *is an R-module homomorphism and* $E \xrightarrow{f} E'$ *is a surjective R-module homomorphism, then there exists an R-module homomorphism* $P \xrightarrow{g} E$ *such that* $g' = f \circ g$*. That is, the following diagram can be completed such that it is commutative:*

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists g"'] \arrow[d, "g'"] \\ E \arrow[r, two heads, "f"'] & E' \end{tikzcd}$$

(3) P is a direct summand of a free module;

(4) Every short exact sequence of the form

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\phi} P \to 0$$

splits. □

B.2 Character Theory of Groups

In the case of large groups the explicit construction of irreducible representations can be difficult [70]. It will become clear that the *character* of a representation encapsulates a great deal of information about the representation such as determining whether or not a representation is irreducible. We assume thoughout this section that $F = \mathbb{C}$.

Definition B.58. The *trace* of an $n \times n$ matrix $A = (a_{ij})$, written Tr A, is given by

$$\text{Tr } A = \sum_{i=1}^{n} a_{ii}.$$

That is, the trace of A is the sum of the diagonal entries of A.

Definition B.59. The *character* of a representation ρ of a group G is the function $\chi\colon G \to \mathbb{C}$ defined by

$$\chi(g) = \text{Tr } \rho(g) \text{ for all } g \in G.$$

Naturally enough, we define the character of an FG-module with basis $\mathbb{B}$ to be the character χ of the corresponding representation, namely

$$\chi(g) = TrA_g$$

where A_g is the matrix of g relative to $\mathbb{B}$.

Theorem B.60 (See, e.g., [71, Theorem I])**.** *Equivalent representations of a group have the same character.* □

The converse of Proposition (B.60) is also true. That is, if two representations have the same character, then they must be equivalent. The result corresponding to Proposition (B.60) for modules is that isomorphic FG-modules have the same character.

Definition B.61. Let G be a group. We say that χ is a *character* of G if χ is the character of some representation of G. Moreover, we say that a character is *irreducible* if it is the character of an irreducible representation; and it is *reducible* if it is the character of a reducible representation. A *complex character* is the character of a complex representation.

Note that characters are invariant under conjugation and so χ takes a constant value on any conjugacy class C. Such functions are called *class functions*.

Definition B.62. A *class function* on a group G is a function $f\colon G \to \mathbb{C}$ such that $f(g) = f(h)$ whenever g and h are in the same conjugacy class.

The sums and scalar multiples of class functions are again class functions, so the set $R(G)$ of all class functions on G forms a subspace of the vector space of all functions from G to $\mathbb{C}$. Also, $R(G)$ has a basis consisting of those functions that have the value 1 on precisely one conjugacy class and 0 elsewhere. Thus

$$\dim\, R(G) = \text{ number of conjugacy classes of } G.$$

Theorem B.63 (See, e.g.,[72, Theorem 12.2.23])**.** *The number of irreducible characters of G is equal to the number of conjugacy classes of G.* □

The irreducible characters of a finite group G are class functions, and the number of them is equal to the number of conjugacy classes of G by Theorem B.63. Therefore, it is convenient to record all the values of all the irreducible characters of G in an array. This array is known as the character table of G.

Definition B.64. The *character table* of a group G is an array whose rows are indexed by the irreducible characters of G and whose columns are indexed by the conjugacy classes (or, in practice, by conjugacy class representatives).

Thus a character table is a concise way to describe all irreducible characters of a group G. From this table, characters of G can be written as sums of irreducible characters and, as we shall see later on, there are many more facts about the structure of G that can be read from its character table.

Definition B.65. Let G be a group with FG-module. The *regular* character, denoted χ^{reg}, is the character of the regular FG-module.

The values of χ^{reg} on the elements of a group G are easily described and given in the next result.

Proposition B.66 (See, e.g.,[56, Prop. 13.20])**.** *Let χ^{reg} be the regular character of G. Then*

$$\chi^{reg}(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

□

Another result which is helpful for computing characters over the ground field $\mathbb{C}$ is the following.

Proposition B.67 (See, e.g.,[56, Prop. 13.20]). *Let χ be a character of G. Then $\chi(g^{-1}) = \overline{\chi(g)}$ for all $g \in G$, where $\overline{\chi(g)}$ denotes the complex conjugate of $\chi(g)$.*

B.2.1 Inner products of characters

A method for determining whether a representation is irreducible is by using inner products.

The characters of a finite group G are functions from G to C. The set of all such functions form a vector space over $\mathbb{C}$, if we adopt the natural rules for adding functions and multiplying functions by complex numbers. It is easy to see that the following definition satisfies the conditions of an inner product on the vector space of all functions from G to $\mathbb{C}$, [56].

Definition B.68. Let χ and ψ be characters of G. The *inner product* of χ and ψ is

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}. \tag{B.6}$$

By Proposition B.67, Definition B.68 becomes

Corollary B.69. *Let χ and ψ be characters; then*

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$

Next, we state an important theorem for irreducible characters.

Theorem B.70 (See, e.g.,[1, Theorem 1.9.3]). *Let χ and ψ be characters of two non-isomorphic irreducible FG-modules. Then we have*

(i) $\langle \chi, \psi \rangle = 0$,

(ii) $\langle \chi, \chi \rangle = 1$. □

Theorem B.70 has many interesting consequences.

Corollary B.71. *Let $\chi_1, \chi_2, \ldots, \chi_k$ be the irreducible characters of G. If ψ is any character of G, then*

(i) $\psi = d_1\chi_1 + d_2\chi_2 + \ldots + d_k\chi_k$ *for some non-negative integer* $d_1, d_2, \ldots, d_k$.

(ii) $\langle \psi, \chi_i \rangle = d_i$ *for all* i.

(iii) $\langle \psi, \psi \rangle = d_1^2 + d_2^2 + \ldots + d_k^2$.

(iv) Let V be a $\mathbb{C}G$-module with character ψ. Then V is irreducible if and only if $\langle \psi, \psi \rangle = 1$.

(v) Let U and V be $\mathbb{C}G$-modules, with characters χ and ψ, respectively. Then $U \cong W$ if and only if $\chi = \psi$.

(vi) Any distinct of irreducible characters $\chi_1, \chi_2, \ldots, \chi_k$ of G are linearly independent vectors in the vector space of all functions from G to $\mathbb{C}$.

B.3 Category Theory

We assume familiarity with some category theory basics. See, for example, [73], [74], or [75]. However, in this section we recall a few points in order to establish some general notation.

B.3.1 Definition of a Category

Definition B.72. A category $\mathcal{C}$ consists of the following data:

1. a collection $\mathrm{Ob}(\mathcal{C})$ of *objects*

2. a collection of *arrows* (often called *morphisms*) $\hom(A, B)$ for each pair A, B of objects where each morphism $f \in \hom(A, B)$ has a domain A and codomain B so that $f : A \to B$.

3. a binary operation $\circ$ known as *composition* of morphisms such that for each ordered triple A, B, C of objects we have

$$\begin{aligned} hom(A,B) \times hom(B,C) &\to hom(A,C) \\ (f,g) &\mapsto g \circ f \end{aligned}$$

satisfying the following laws:

- Associative : for all $f\colon A \to B,\ g\colon B \to C,\ h\colon C \to D,$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \arrow[dr, "h\circ g"] & \\
& C \arrow[r, "h"'] & D
\end{tikzcd}$$

- Identity: for each object A there is given a morphism:

$$1_A\colon A \to A$$

  called the **identity** morphism satisfying the following:
  for all $f\colon A \to B$,
$$f \circ 1_A = f = 1_B \circ f$$

  .

Example B.73.

- **Set**: The objects are sets, morphisms are functions, and composition is the usual composition of functions.
- **Grp**: The objects in this category are groups, morphisms are homomorphisms, and composition is the usual composition.
- **Ab**: The category of abelian groups - the objects are abelian groups, the morphisms are group homomorphisms.
- R-mod: Given a ring, R-mod is the category of all left R-modules. Thus, Ob(R-mod) is the collection of all left R-modules and the set of morphisms from M to N (where M, N are objects of R-mod) is the set of all R-module homomorphisms from M to N. We write $Hom(M, N)$ rather than $hom(M, N)$.

Definition B.74. A *subcategory* $\mathcal{B}$ of a category $\mathcal{C}$ is a category for which:

- each object of $\mathcal{B}$ is an object of $\mathcal{C}$
- for all objects B, B' in $\mathcal{B}$, $\mathcal{B}(B, B') \subseteq \mathcal{C}(B, B')$; and

- compositions and identity arrows are the same in $\mathcal{B}$ as in $\mathcal{C}$.

Definition B.75. The *product of two categories* $\mathcal{C}$ and $\mathcal{D}$, denoted $\mathcal{C} \times \mathcal{D}$ has as objects pairs (C, D) of objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$; and arrows $(f, g)\colon (C, D) \to (C', D')$ for $f\colon C \to C' \in \mathcal{C}$ and $g\colon D \to D' \in \mathcal{D}$. The composition and identity are defined componentwise; that is,

$$(C, D) \xrightarrow{(f,g)} (C', D') \xrightarrow{(f',g')} (C'', D'')$$

is defined in terms of the compositions in C and D by

$$\begin{aligned} (f', g') \circ (f, g) &= (f' \circ f, g' \circ g). \\ 1_{(C,D)} &= (1_C, 1_D). \end{aligned}$$

B.3.2 Monomorphisms, Epimorphisms, and Isomorphisms

When we think about sets, groups and functions, we are often interested in functions with special properties such as being injective (one-to-one), surjective (onto), or bijective (defining an isomorphism). Appropriate analogues of these concepts also play an important role in categorical reasoning.

Definition B.76. An arrow $f\colon B \to C$ in a category $\mathcal{C}$ is a *monomorphism* (or *"is monic"*) if, for any pair of arrows $g\colon A \to B$ and $h\colon A \to B$, the equality $f \circ g = f \circ h$ implies that $g = h$.

Definition B.77. An arrow $f\colon B \to C$ in a category $\mathcal{C}$ is an *epimorphism* (or *"is epic"*) if, for any pair of arrow $g\colon B \to C$ and $h\colon B \to C$, the equality $g \circ f = h \circ f$ implies that $g = h$.

Definition B.78. An arrow $f\colon B \to C$ in a category $\mathcal{C}$ is an *isomorphism* if there is an arrow $f'\colon B \to A$, called the *inverse* of f, such that $f' \circ f = 1_B$. The objects A and B are said to be *isomorphic* if there is an isomorphism between them.

An important concept in category theory is the concept of category of categories where the mappings or arrows between categories (the categories are objects in this context) are functors, which we shall now define.

B.3.3 Functor

Definition B.79. A *(covariant) functor*

$$F\colon C \to D$$

between categories C and D is a mapping of objects to objects and arrows to arrows, in such a way that:

- $F(f\colon A \to B) = F(f)\colon F(A) \to F(B)$
- $F(g \circ f) = F(g) \circ F(f)$
- $F(1_A) = 1_{F(A)}$

Example B.80. The *forgetful functor* $U\colon \mathbf{Ab} \to \mathbf{Set}$ from the category of abelian groups to the category of sets is the functor that forgets the abelian group structure on the objects of **Ab**.

Definition B.81. A *contravariant functor* F is one that maps objects to objects as before, but that maps arrows to arrows going the opposite direction, that is, F is a functor from C^{op} to D.

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