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Faculty of Actuarial Science and Statistics

Types of Dependence and Time-dependent Association between Two Lifetimes in Single Parameter Copula Models

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Abstract

Most publications on modeling insurance contracts on two lives, assuming dependence of the two lifetimes involved, focus on the time of inception of the contract. The dependence between the lifetimes is usually modeled through a copula and the effect of this dependence on the pricing of a joint life policy is measured. This paper investigates the effect of association at the outset on the mortality in the future. The conditional law of mortality of an individual, given his survival and given the life status of the partner is derived. The conditional joint survival distribution of a couple at any duration, given that the two lives are then alive, is also derived. We analyze how the degree of dependence between the two members of a couple varies throughout the duration of a contract. We will do that for (mainly Archimedean) copula models, with one parameter for the degree of dependence. The conditional distributions hence derived provide the basis for the calculation of prospective provisions. Key words: Archimedean copulas, multiple life contingencies, aging, valuing policies.

1 Introduction

Traditionally, it has been presumed in multiple life contingencies that the remaining lifetimes of the lives involved are mutually independent. As some empirical investigations have shown, for a married couple, this standard assumption is based on computational convenience rather than realism.

Over the past few years, several papers about insurance contracts on two lives have been published which allow for dependence between the two lifetimes. Frees et al. (1996) and Carriere (2000) present alternative ways of modeling dependence of times of death of coupled lives and applying them to a dataset. In both papers, a significant degree of positive correlation between lifetimes was observed. This implies, for instance, that joint life annuities are underpriced while last survivor annuities are overpriced. Carriere and Chan (1986) present boundaries of single premiums for last survivor annuities. All three papers adopt a methodology based on copulas, an approach which is popular nowadays. Other papers studying bounds of single premiums are Dhaene et al. (2000) and Denuit et al. (2001).

All these papers study the impact of dependency of two remaining lifetimes on the pricing of life insurance products on the lives concerned. Dependency, however, also affects the valuation of such contracts over time. Prospective provisions (also known as reserves) are based on laws of mortality which apply to the policy valuation date. If the remaining lifetimes of a couple are dependent at the outset of a policy, then any of the two lives' survival probabilities may depend on the life status of the partner. Moreover, the joint distribution of remaining lifetimes, given the survival of both partners to a certain date, is affected as well. Using copula models to modify a formula used in the valuation of survivorship life insurance policies, Margus (2002) has pointed out that mortality rates of a life whose spouse is still alive differs from the mortality rates of a life whose spouse already died before the time of valuation. Youn et al. (2002) show that a lot of well-known relationships between probabilities and single premiums in multiple life contingencies are not valid in case of dependent lifetimes. They establish, however, that the validity of those relationships can be restored if the definition of individual survival probabilities allows for the life status of the partner. Carriere (2000), stating that the copula should affect the lives from the time of marriage, rather than from the time of birth, uses similar arguments.

Moreover, it is not sufficient just to conclude that there is dependence between remaining lifetimes. It is also essential to state what type of dependence applies. Hougaard (2000) points out that within a time framework, one can basically distinguish between three different categories:

1. Instantaneous dependence: dependence caused by common events affecting both lives at the same time.

2. Long-term dependence: dependence which is caused by a common risk environment, affecting the surviving partner for their remaining lifetime.

3. Short-term dependence: the event of death of one life changes the mortality of the other life immediately, but this effect diminishes over time.

Instantaneous dependence is implied due to the fact that both members of a couple are prone to common events such as accidents. Dependence is assumed to be of a long-term nature if the force of mortality of the survivor is a constant or increasing function of the time of death of the partner. On the other hand, dependence is taken to be short-term if this force of mortality is a decreasing function of the time-of-death of the partner. An example of long-term dependence is the mechanism of matching couples ("birds of a feather flock together"). For instance, two partners often come from the same neighborhood which determines their common risks. The

"broken heart syndrome" (as researched in Parkes et al., 1969, and Jagger and Sutton, 1991) is the most well-known example of short-term dependence.

It is a question which type of dependence prevails within the framework of multiple life contingencies. Hougaard (2000) suggests that in case of a married couple, short-term dependence is perhaps more relevant than long-term dependence. This assertion is underpinned by one of the main results from the empirical work in Parkes et al. (1969) and Jagger and Sutton (1991). Both studies show that within about 6 months after death of the partner, the mortality of widowers is comparable with that of married men. Youn and Shemyakin (1999, 2001) show that, when implementing a copula model, ignoring the difference between the physical ages of the two partners can lead to an underestimation of the instantaneous dependence and short-term dependence.

As long as both lives of a couple are alive, the degree of association between the respective remaining lifetimes depends on time. This gives rise to another question, namely which patterns of association over time between members of a couple are realistic. Moreno (1994) discusses this issue, also known as aging, in the context of frailty models, which are an important subclass of Archimedean copula models.

This paper analyzes the time-dependent association implied by copula models, and identifies their type of dependence. We assess the impact of the life history of one life on the mortality of the other life. If the one life has died before the other life, we will also distinguish between the different times of death. In this respect, our approach differs from the one in Margus (2002) who considers only one state "Dead" of the partner without specifying when the partner died. We also investigate how the dependence between the two lives will change over time, as long as they are both still alive. We consider one and the same couple at the outset and restrict our study in this paper to models with one parameter for dependence only.

We have developed a methodology which is based on analyzing the force of mortality of one life. Our paper sheds new light on theoretical properties of copulas. Besides, it gives an answer to the important question which copula models may be suitable for modelling dependence, which exhibits itself in the practice of the insurance of couples. Some of the copulas discussed in this contribution have not been widely studied before, and this has led to some interesting findings. Our approach helps the actuary to choose an appropriate copula and provides a framework for the calculation of provisions of contracts on two lives.

The organization of this paper is as follows. In Section 2, we specify the conditional marginal and joint survival functions which we want to analyze. Section 3 gives an introduction to copula models and outlines the structure for the next two sections. Section 4 concerns the previous death of one life and analyses the impact on the death of the other life, while Section 5 deals with the case where both lives are still alive. A numerical example about provisions of an insurance contract on two lives is given in Section 6. A discussion in Section 7 concludes the paper.

2 Conditional marginal and joint survival functions

We consider a contract effected on two lives (x) and (y), aged x and y, respectively at duration 0. The complete remaining lifetimes of (x) and (y) are denoted by T_x and T_y, respectively, with marginal survival functions $S_1(s_1)$ and $S_2(s_2)$. We assume that T_x and T_y are continuously distributed, with upper bounds $\omega_x - x$ and $\omega_y - y$, respectively. The variables ω_x and ω_y denote the limiting ages of (x) and (y). The joint survival function is denoted by $S(s_1, s_2)$.

We want to calculate the prospective provision at duration $t \geq 0$. We assume that the policy is in force if at least one of both lives is alive.

Consider first of all the case where life (x) is still alive and (y) passed away between duration

t_y and $t_y + dt$, with $t_y \in [0, t]$. Then the survival function of remaining lifetime of (x) at duration t, given death of (y) at time t_y, is given by:

$$S_{1;t}\left(s \,|\, T_y = t_y\right) = P\left[T_x > t + s \,|\, T_x > t, T_y = t_y\right] = \frac{-\frac{d}{dt_y}P\left[T_x > t + s, T_y > t_y\right]}{-\frac{d}{dt_y}P\left[T_x > t, T_y > t_y\right]}$$

$$= \frac{\frac{d}{dt_y}S\left(t + s, t_y\right)}{\frac{d}{dt_y}S\left(t, t_y\right)}. \tag{1}$$

Next, consider the case where both are still alive at duration $t \geq 0$. Then the survival function of remaining lifetime of (x) at t, given survival of (y) to time t, is given by

$$S_{1;t}\left(s \,|\, T_y > t\right) = P\left[T_x > t + s \,|\, T_x > t, T_y > t\right] = \frac{P\left[T_x > t + s, T_y > t\right]}{P\left[T_x > t, T_y > t\right]} = \frac{S\left(t + s, t\right)}{S\left(t, t\right)}. \tag{2}$$

The expressions for $S_{2;t}\left(s \,|\, T_x = t_x\right)$ and $S_{2;t}\left(s \,|\, T_x > t\right)$ are similar to (1) and (2), respectively, with x and y interchanged. The joint survival function, given survival of both to t, is defined as

$$S_t\left(s_1, s_2\right) = P\left[T_x > t + s_1, T_y > t + s_2 \,|\, T_x > t, T_y > t\right] = \frac{S\left(t + s_1, t + s_2\right)}{S\left(t, t\right)}. \tag{3}$$

3   Copula models

We start this section by giving a general introduction to copulas in Subsection 3.1. Then, we show how the equations established in Section 2 are specified if a copula model applies. We make a distinction between previous death of one life (Subsection 3.2) and survival of both (Subsection 3.3). Finally, in Subsection 3.4, we will give an overview of the copula families to be considered in this paper.

3.1   Introduction to general copulas

If we apply a copula model, the joint distribution is determined by the marginals and a copula function of two arguments, denoted by $C\left[\cdot, \cdot\right]$. Assuming that $0 \leq u \leq 1$ and $0 \leq v \leq 1$, this function has the following properties:

1. $C\left[0, v\right] = C\left[u, 0\right] = 0$;

2. $C\left[1, v\right] = v$ and $C\left[u, 1\right] = u$;

3. $C\left[\cdot, \cdot\right]$ is nondecreasing in each argument.

For an overview of applications of copulas in actuarial science, see Frees and Valdez (1998). In the sequel, we will use survival copulas only. Then the joint survival function is a survival copula function, with the marginal survival functions as its arguments:

$$S\left(s_1, s_2\right) = C\left[S_1\left(s_1\right), S_2\left(s_2\right)\right].$$

The functions introduced in Section 2 will be re-expressed in the next two subsections.

3.2 Previous death of one partner

For (1), we get:

$$S_{1;t}\left(s\left|T_y=t_y\right.\right) = P\left[T_x > t+s\left|T_x > t, T_y = t_y\right.\right] = \frac{\left(C_2\left[S_1\left(t+s\right),v\right]\right)_{v=S_2(t_y)}}{\left(C_2\left[S_1\left(t\right),v\right]\right)_{v=S_2(t_y)}}, \qquad (4)$$

with $C_2\left[\cdot,\cdot\right]$ denoting the partial derivative of $C\left[\cdot,\cdot\right]$ with respect to its second argument. Obviously, this definition only makes sense if $C_2\left[S_1\left(t\right),v\right] \neq 0$ for $v = S_2\left(t_y\right)$.

We prefer to specify the mortality in terms of the forces of mortality, rather than the survival function, if possible. The advantage of this is the multiplicative relationship between this quantity and the related quantity applying to the case of independence. Provided that $\left(C_2\left[S_1\left(t+s\right),v\right]\right)_{v=S_2(t_y)} \neq 0$, the forces of mortality are given as:

$$\begin{aligned} \mu_1\left(x+t+s\left|T_y=t_y\right.\right) &= -\frac{d}{ds}\ln\left[\left(C_2\left[S_1\left(t+s\right),v\right]\right)_{v=S_2(t_y)}\right] \\ &= \mu_1\left(x+t+s\right)\frac{S_1\left(t+s\right)\left(C_{21}\left[u,v\right]\right)_{u=S_1(t+s);v=S_2(t_y)}}{\left(C_2\left[S_1\left(t+s\right),v\right]\right)_{v=S_2(t_y)}}, \qquad (5) \end{aligned}$$

with $\mu_1\left(x+t+s\right)$ denoting the force of mortality at age $x+t+s$ corresponding to the distribution of T_x. Furthermore $C_{21}\left[\cdot,\cdot\right]$ is the second derivative with respect to its second and first argument.

Another advantage of the force of mortality representation is that it allows us to adopt the concepts in Hougaard (2000). We will focus on the specification of conditional laws of mortality through forces of mortality and analyze their behavior as a function of t_y, the time of death of the partner. For several copulas, we will work out whether there is <u>long-term</u> or <u>short-term</u> <u>dependence</u> between the lives. Our method is strongly based on Hougaard's definition, which follows below.

Definition 1 *The remaining lifetimes T_x and T_y exhibit <u>short-term</u> <u>dependence</u> if $\mu_1\left(x+t+s\left|T_y=t_y\right.\right)$ is an increasing function of $t_y \in [0,t]$ (or alternatively, if $\mu_2\left(y+t+s\left|T_x=t_x\right.\right)$ is an increasing function of $t_x \in [0,t]$). On the other hand, there is <u>long-term</u> <u>dependence</u> between T_x and T_y if $\mu_1\left(x+t+s\left|T_y=t_y\right.\right)$ is constant or decreasing as a function of $t_y \in [0,t]$ (or equivalently, if $\mu_2\left(y+t+s\left|T_x=t_x\right.\right)$ is constant or decreasing as a function of $t_x \in [0,t]$).*

As we will see below now, the case of a force of mortality which is constant as a function of the time of death does not exist in copula models. It follows from (5) that if the force of mortality of (x) is constant, then

$$\frac{C_{21}\left[u,v\right]}{C_2\left[u,v\right]} = d\frac{\ln\left[C_2\left[u,v\right]\right]}{du}, \quad u,v \in [0,1],$$

is independent of v. This implies:

$$\ln\left[C_2\left[u,v\right]\right] = K_1\left(u\right) + K_2, \qquad (6)$$

with $K_1\left(u\right)$ denoting a real valued differentiable function, depending only on u and the parameters of dependence, and not on v. In (6), K_2 is a real valued constant. This leads to

$$C\left[u,v\right] = v\exp\left[K_1\left(u\right) + K_2\right] + K_3,$$

with K_3 denoting another real valued constant. The condition $C[u, 0] = 0$ for $u \in [0, 1]$ implies $K_3 = 0$. Then, the condition $C[u, 1] = u$ for all $u \in [0, 1]$ implies $K_1(u) = \ln[u] - K_2$. Hence,

$$C[u, v] = uv,$$

being the independence copula. In other words, if two lifetimes are dependent and a copula model applies, then the force of mortality of one life always depends on the time of death of the other life.

We will identify the type of dependence for some copula families, which is the topic of Section 4.

3.3 Both lives survive

Next, we consider the case of survival of both. Then, equation (2) becomes:

$$S_{1;t}(s|T_y > t) = \Pr[T_x > t + s|T_x > t, T_y > t] = \frac{\Pr[T_x > t + s, T_y > t]}{\Pr[T_x > t, T_y > t]} = \frac{C(S_1(t + s), S_2(t))}{C(S_1(t), S_2(t))}. \tag{7}$$

A similar relationship holds for $S_{2;t}(s|T_y > t) = \Pr[T_y > t + s|T_x > t, T_y > t]$:

$$S_{2;t}(s|T_x > t) = \Pr[T_y > t + s|T_x > t, T_y > t] = \frac{\Pr[T_x > t, T_y > t + s]}{\Pr[T_x > t, T_y > t]} = \frac{C(S_1(t), S_2(t + s))}{C(S_1(t), S_2(t))}. \tag{8}$$

Once again, whenever possible, we prefer to express the effect of survivorship of the partner through a force of mortality function. The force of mortality corresponding to survival function (7) is given by:

$$\begin{aligned}
\mu_1(x + t + s|T_y > t) &= -\frac{d}{ds}\ln[C(S_1(t + s), S_2(t))] \\
&= \mu_1(x + t + s)\frac{S_1(t + s)(C_1[u, v])_{u=S_1(t+s);v=S_2(t)}}{C(S_1(t + s), S_2(t))}. \tag{9}
\end{aligned}$$

Equation (3) becomes:

$$S_t(s_1, s_2) = C_t[S_{1;t}(s_1|T_y > t), S_{2;t}(s_2|T_x > t)] = \frac{C[S_1(t + s_1), S_2(t + s_2)]}{S(t, t)}, \tag{10}$$

determined by the marginals $S_{1;t}(s|T_y > t)$ and $S_{2;t}(s|T_x > t)$ as well as a new copula C_t.

In Section 5, we derive the form of the copula C_t for some copula families. We assume that t is such that $S(t, t) > 0$.

3.4 Copulas to be considered

In this paper, in both Sections 4 and 5, we will first of all discuss the three special cases of independence, comonotonicity (Fréchet upper bound) and countermonotonicity (Fréchet lower bound). This is followed by Archimedean copulas.

If the two lifetimes are independent, the copula is specified as $C[u, v] = uv$. The Fréchet upper bound gives maximal positive dependence, with $C[u, v] = \min[u, v]$. On the other hand, the Fréchet lower bound represents maximal negative dependence, and we have $C[u, v] = \max[u + v - 1, 0]$.

The Fréchet lower bound will only be discussed for illustrative purposes. In general, for the sake of practical relevance, we assume that the two lifetimes have a type of positive dependence.

More formally, we assume that the two lifetimes are Positive Quadrant Dependent (PQD in short). According to Lehmann (1966), this is the case if

$$\Pr\left[T_x > s_1, T_y > s_2\right] \geq \Pr\left[T_x > s_1\right]\Pr\left[T_y > s_2\right] \;\; \forall s_1, s_2 \in \mathbb{R}.$$

In general, it may not be easy to obtain straightforward expressions for the equations in the previous subsection. Archimedean copulas have been well known and widely applied for their mathematical tractability, as well as their flexibility, and will therefore be considered. An Archimedean copula is constructed by a generator, being a function $\phi\left(\cdot\right) : [0,1] \to \mathbb{R}^+$ with a continuous first and second derivative, denoting $\phi'\left(\tau\right)$ and $\phi''\left(\tau\right)$ respectively, satisfying

$$\phi\left(1\right) = 0, \;\; \phi'\left(\tau\right) < 0 \text{ and } \phi''\left(\tau\right) > 0, \quad 0 \leq \tau \leq 1,$$

for all $\tau \in (0,1)$. Then the copula generated by the function $\phi\left(\cdot\right)$ is expressed as

$$C\left[u,v\right] = \phi^{[-1]}\left(\phi\left(u\right) + \phi\left(v\right)\right), \quad 0 \leq u, v \leq 1,$$

with $\phi^{[-1]}\left(\cdot\right)$, being the pseudo-inverse function of $\phi\left(\cdot\right)$:

$$\phi^{[-1]}\left(\tau\right) = \begin{cases} \phi^{-1}\left(\tau\right), & \text{for} \quad 0 \leq \tau \leq \phi\left(0\right) \\ 0, & \text{for} \quad \phi\left(0\right) \leq \tau \leq \infty. \end{cases}$$

In the above equation, $\phi^{-1}\left(\cdot\right)$ is the inverse function of $\phi\left(\cdot\right)$. The pseudo-inverse and the inverse function coincide completely if $\lim_{\tau \downarrow 0}\phi\left(\tau\right) = \infty$. If this property holds, the generator is said to be strict. For the sake of mathematical convenience (to avoid technical complications with the specification of the force of mortality), we will only discuss Archimedean copulas with strict generators, hence

$$C\left[u,v\right] = \phi^{-1}\left(\phi\left(u\right) + \phi\left(v\right)\right), \quad 0 \leq u, v \leq 1. \tag{11}$$

Archimedean copulas have been introduced in Genest and MacKay (1986a,1986b). For more details about Archimedean copulas and their properties, see Chapter 4 of Nelsen (1999). We will use some of the families described therein as illustrations. We only consider Archimedean copulas which allow for at least a range of Positive Quadrant Dependence and which explicitly contain the important special case of independence. In this paper, nine families are object of study. They are tabulated in Table 1.

Some of the copula types in this table have well known names. Families $1, 2, 3, 5$ and 6 are known as the Clayton (or Cook-Johnson or Oakes), the Gumbel-Hougaard, the Frank, the Ali-Mikhail-Haq and the Joe family, respectively. The last column displays the maximal value of the population version of Kendall's coefficient of concordance, which in the remainder of this paper we shall refer to as Kendall's τ. The higher the value of this quantity, the stronger the association between the random variables. The maximal value is equal to 1, corresponding to Fréchet's upper bound. All families except one are positively ordered, which implies that the population version of Kendall's τ increases as the value of the parameter increases. Only Frank's family is negatively ordered. One can read from the table that all families except one cover the entire range of Positive Quadrant Dependence. The exception is Ali-Mikhail-Haq. The limiting case for $\theta \uparrow 1$ gives the copula $C\left[u,v\right] = uv/\left(u + v - uv\right)$, which has a value of Kendall's τ equal to $1/3$.

Number	Generator $\phi(\tau)$	PQD for	Independence for	Maximum τ
1	$\tau^{-\theta}-1$	$\theta>0$	$\theta\downarrow 0$	1 $(\theta\to\infty)$
2	$(-\ln\tau)^{\theta}$	$\theta\geq 1$	$\theta=1$	1 $(\theta\to\infty)$
3	$-\ln\frac{e^{\theta\tau}-1}{e^{\theta}-1}$	$\theta<0$	$\theta\uparrow 0$	1 $(\theta\to-\infty)$
4	$(1-\ln t)^{\theta}-1$	$\theta\geq 1$	$\theta=1$	1 $(\theta\to\infty)$
5	$\ln\frac{1-\theta(1-t)}{t}$	$\theta\in[0,1)$	$\theta=0$	$\frac{1}{3}$ $(\theta\uparrow 1)$
6	$-\ln\left[1-(1-t)^{\theta}\right]$	$\theta\geq 1$	$\theta=1$	1 $(\theta\to\infty)$
7	$-\ln\frac{(1+t)^{-\theta}-1}{2^{-\theta}-1}$	$\theta\in[-1,\infty)\setminus\{0\}$	$\theta=-1$	1 $(\theta\to\infty)$
8	$\exp\left[t^{-\theta}\right]-e$	$\theta>0$	$\theta\downarrow 0$	1 $(\theta\to\infty)$
9	$\frac{1}{t^{\theta}}-t^{\theta}$	$\theta>0$	$\theta\downarrow 0$	1 $(\theta\to\infty)$

Table 1: Archimedean copula families

4 Copula models, one life died

4.1 Special cases

4.1.1 Independence

For $C[u,v]=u\,v$, (4) leads to

$$S_{1;t}\left(s\left|T_y=t_y\right.\right)=\frac{S_1(t+s)}{S_1(t)},$$

while (5) yields:

$$\mu_1\left(x+t+s\left|T_y=t_y\right.\right)=\mu_1\left(x+t+s\right)\frac{S_1(t+s)}{S_1(t+s)}=\mu_1\left(x+t+s\right),$$

which is as expected: the force of mortality of (x) is not affected at all by the time of death of (y).

4.1.2 Fréchet upper bound

For $C[u,v]=\min[u,v]$, we have $C_2[u,v]=I_{\{u>v\}}$, so $S_{1;t}\left(s\left|T_y=t_y\right.\right)$ only exists if $S_2(t_y)<S_1(t)$, in which case we get:

$$S_{1;t}\left(s\left|T_y=t_y\right.\right)=\begin{cases}1 & \text{for } s<S_1^{[-1]}(S_2(t_y))-t \\ 0 & \text{for } s>S_1^{[-1]}(S_2(t_y))-t\end{cases}, \tag{12}$$

defining

$$S_1^{[-1]}(\kappa)=\begin{cases}S_1^{-1}(\kappa) & \text{for } 0<\kappa\leq 1 \\ \omega_x & \text{for } \kappa=0\end{cases}. \tag{13}$$

In words, given death of (y) at t_y, (x) dies with certainty at age $x+ S_1^{[-1]}(S_2(t_y))$. Although a force of mortality cannot be specified (since (12) is degenerate), this can be considered as an example of long term dependence: $S_1^{[-1]}(S_2(t_y))$ increases in t_y: the sooner (y) dies, the sooner (x) dies. Note that if (y) dies at 0, then (x) dies at 0.

4.1.3 Fréchet lower bound

For $C[u,v] = \max[u+v-1,0]$, we have $C_2[u,v] = I_{\{v>u-1\}}$, so $S_{1;t}(s\,|T_y = t_y)$ only exists if $S_2(t_y) > 1 - S_1(t)$, which is always satisfied. In this case we get:

$$S_{1;t}(s\,|T_y = t_y) = \begin{cases} 1 & \text{for} \quad s < S_1^{[-1]}(1 - S_2(t_y)) - t \\ 0 & \text{for} \quad s > S_1^{[-1]}(1 - S_2(t_y)) - t \end{cases},$$

with $S_1^{[-1]}(\cdot)$ as defined in (13). In other words, given death of (y) at t_y, (x) dies with certainty at age $x+ S_1^{-1}(1 - S_2(t_y))$. The sooner (y) dies, the later (x) dies. Note that $S_{1;t}(s\,|T_y = t_y) = \omega_x$ for $t_y = 0$: death of (y) at 0 implies death of (x) not before attaining the limiting age. As stated in Margus (2002), the death of one life prevents the death of the other life.

4.2 Archimedean copulas

Substituting (11) into (5) leads to:

$$\mu_1(x+t+s\,|T_y = t_y)$$
$$= \mu_1(x+t+s) S_1(t+s) \left(-\phi'(S_1(t+s))\right) \left(-\frac{\left(\phi^{-1}\right)''(v)}{\left(\phi^{-1}\right)'(v)}\right)_{(v=\phi(S(t+s,t_y)))}. \quad (14)$$

Note that, in this expression only the function

$$-\left(\frac{\left(\phi^{-1}\right)''(v)}{\left(\phi^{-1}\right)'(v)}\right)_{(v=\phi(S(t+s,t_y)))} = -d\frac{\ln\left[-\left(\left(\phi^{-1}\right)'(v)\right)_{(v=\phi(S(t+s,t_y)))}\right]}{dv}, \quad (15)$$

depends on t_y. If this function decreases (increases) in t_y, then the dependence is of a long-term (short-term) nature.

Our analyses establish that almost all of the copulas exhibit long term dependence. The first four families listed in the table have a frailty specification. Frailty is a basic example of long term dependence. This has been pointed out in Hougaard (2000), and will be demonstrated here again.

Case 2 (Frailty) *If the inverse of the generator is a Laplace transform, we have the general expression*

$$\phi^{-1}(v) = \int_{z=0}^{\infty} e^{-zv}dF(z),$$

with $F(z)$ denoting the c.d.f. of frailty. Then we have

$$\left(\phi^{-1}\right)'(v) = \int ze^{-zv}dF(z),$$

and

$$\left(\phi^{-1}\right)''(v) = \int z^2 e^{-zv}dF(z).$$

So

$$\frac{d}{dt_y}\left(-\frac{\left(\phi^{-1}\right)''(v)}{\left(\phi^{-1}\right)'(v)}\right)_{(v=\phi(S(t+s,t_y)))}$$

$$= \phi'\left(S_2\left(t_y\right)\right)S_2\left(t_y\right)\mu\left(y+t_y\right)\frac{\int z^3 e^{-zv}dF\left(z\right)\int z e^{-zv}dF\left(z\right)-\left(\int z^2 e^{-zv}dF\left(z\right)\right)^2}{\left(\int z e^{-zv}dF\left(z\right)\right)^2}$$

$$\leq 0.$$

As the first illustration, we discuss the Clayton family. This type has some special properties, as we will see in this, as well as in next section. This is followed by the Gumbel-Hougaard copula (applied in Youn and Shemyakin (1999, 2001), Youn et al., 2002, and Denuit et al., 2001) and Frank's copula (applied in Frees et al., 1996, Carriere, 2000, and Margus, 2002).

Example 3 (Clayton) *The inverse of the generator, $\phi^{-1}\left(\tau\right)=\left(\tau+1\right)^{-\frac{1}{\theta}}$ is the Laplace transform of the Gamma(α,β) distribution with parameters $\alpha=\theta^{-1}$ and $\beta=1$. In general the Laplace transform of a Gamma(α,β) distribution is $\phi^{-1}\left(\tau\right)=\left(\frac{\tau}{\beta}+1\right)^{-\frac{1}{\theta}}$, leading to a generator $\phi\left(\tau\right)=\beta\left(t^{-\theta}-1\right)$. From Genest and MacKay (1986b), we know that a generator is determined up to a positive multiplier. Hence, the value of β does not affect the joint distribution. Equation (14) gives:*

$$\mu_1\left(x+t+s\left|T_y=t_y\right.\right)=\mu_1\left(x+t+s\right)\left(\theta+1\right)\frac{\left(S_1\left(t+s\right)\right)^{-\theta}}{\left(S_1\left(t+s\right)\right)^{-\theta}+\left(S_2\left(t_y\right)\right)^{-\theta}-1},$$

which is decreasing in t_y. Note the special case $t_y=0$ (death immediately after issue), in which case the above expression leads to:

$$\mu_1\left(x+t+s\left|T_y=t_y\right.\right)=\left(\theta+1\right)\mu_1\left(x+t+s\right).$$

In other words, if (y) dies immediately after issue of a contract, the force of mortality (x) is $(\theta+1)$ times the marginal force of mortality for any time in the future. We will discuss this feature further in Section 5 where Clayton's copula is considered again.

Example 4 (Gumbel-Hougaard) *The inverse of the generator, $\phi^{-1}\left(\tau\right)=\exp\left(-\tau^{\frac{1}{\theta}}\right)$ is the Laplace transform of the positive stable distribution, as pointed out in Frees and Valdez (1998). Equation (14) gives:*

$$\mu_1\left(x+t+s\left|T_y=t_y\right.\right) = \mu_1\left(x+t+s\right)\left(\frac{\left(-\ln S_1\left(t+s\right)\right)^{\theta}}{\left(-\ln S_1\left(t+s\right)\right)^{\theta}+\left(-\ln S_2\left(t_y\right)\right)^{\theta}}\right)^{1-\frac{1}{\theta}}$$

$$\left(1+\left(\theta-1\right)\left(\left(-\ln S_1\left(t+s\right)\right)^{\theta}+\left(-\ln S_2\left(t_y\right)\right)^{\theta}\right)^{-\frac{1}{\theta}}\right),$$

which is decreasing as a value of t_y.

Example 5 (Frank) *The inverse of the generator, $\phi^{-1}\left(\tau\right)=\ln\left[1+\left(e^{\theta}-1\right)e^{-\tau}\right]/\theta$ is the Laplace transform of the logarithmic series distribution on the positive integers, as pointed out in Frees and Valdez (1998). Equation (14) gives:*

$$\mu_1\left(x+t+s\left|T_y=t_y\right.\right) = \mu_1\left(x+t+s\right)S_1\left(t+s\right)\left(-\theta\right)\frac{e^{\theta S_1(t+s)}}{1-e^{\theta S_1(t+s)}}$$

$$\left(\frac{1-e^{\theta}}{\left(1-e^{\theta}\right)-\left(1-e^{\theta S_1(t+s)}\right)\left(1-e^{\theta S_2(t_y)}\right)}\right),$$

which is decreasing as a value of t_y.

A simple check with <u>Mathematica</u> shows that Family 4 in Table 1 has a frailty specification as well, implying that dependence is also of a long term type. The frailty distribution involves a Bessel function. Families 5 to 8 in Table 1 feature long term dependence as well. Only Family 9, has short term dependence in some cases, as will be shown below.

Example 6 (Family 9) *The inverse of the generator is* $\phi^{-1}(\tau) = 2^{-\frac{1}{\theta}} \left(-\tau + \sqrt{4 + \tau^2} \right)^{\frac{1}{\theta}}$. *Equation (14) gives:*

$$
\mu_1 \left(x + t + s \left| T_y = t_y \right. \right) = \mu_1 \left(x + t + s \right) \theta \left(\frac{1}{S_1 \left(t + s \right)^\theta} - S_1 \left(t + s \right)^\theta \right)
$$

$$
\left(\frac{w \left(t + s, t_y \right) + \frac{1}{\theta} \sqrt{4 + \left(w \left(t + s, t_y \right) \right)^2}}{4 + \left(w \left(t + s, t_y \right) \right)^2} \right), \tag{16}
$$

with

$$
w \left(t + s, t_y \right) = \left(\frac{1}{S_1 \left(t + s \right)^\theta} - S_1 \left(t + s \right)^\theta \right) + \left(\frac{1}{S_2 \left(t_y \right)^\theta} - S_2 \left(t_y \right)^\theta \right).
$$

For small t_y and $t + s$, i.e. young ages and or short durations, (16) is increasing as a function of t_y. On the other hand, for larger values of t_y and/or $t + s$, it is decreasing as a function of t_y.

5 Copula models, both survive

5.1 Special cases

5.1.1 Independence

For $C[u, v] = u\,v$, (9) leads to

$$
\mu_1 \left(x + t + s \left| T_y > t \right. \right) = \mu_1 \left(x + t + s \right),
$$

which is as expected: in case of independence, the mortality of one life does not depend on the life history of the other life.

Furthermore, (10) leads to

$$
S_t \left(s_1, s_2 \right) = \frac{S_1 \left(t + s \right) S_2 \left(t + s \right)}{S \left(t, t \right)} = S_{1;t} \left(s \left| T_y > t \right. \right) S_{1;t} \left(s \left| T_y > t \right. \right).
$$

Hence, $C_t \left(u, v \right) = uv$: the joint distribution of remaining lifetime, given survival of both is the independence copula again (as expected).

5.1.2 Fréchet upper bound

For $C[u, v] = \min[u, v]$, we have

$$
S_{1;t} \left(s \left| T_y > t \right. \right) = \frac{\min \left[S_1 \left(t + s \right), S_2 \left(t \right) \right]}{\min \left[S_1 \left(t \right), S_2 \left(t \right) \right]},
$$

11

and a similar expression for $S_{2;t}\left(s\,|T_y>t\right)$. In this case, (10) leads to

$$
\begin{aligned}
C_t\left[S_{1;t}\left(s\,|T_y>t\right),S_{2;t}\left(s\,|T_y>t\right)\right] & = \frac{\min\left[S_1\left(t+s\right),S_2\left(t+s\right)\right]}{S\left(t,t\right)} \\
& = \min\left[S_{1;t}\left(s\,|T_y>t\right),S_{2;t}\left(s\,|T_y>t\right)\right].
\end{aligned}
$$

In words, if the joint distribution has the Fréchet upper bound as copula at the outset, it will continue to have the Fréchet upper bound as copula in the future.

5.1.3 Fréchet lower bound

For $C\left[u,v\right]=\max\left[u+v-1,0\right]$, we have

$$
S_{1;t}\left(s\,|T_y>t\right)=\frac{\max\left[S_1\left(t+s\right)+S_2\left(t\right)-1,0\right]}{S\left(t,t\right)},
$$

and a similar expression for $S_{2;t}\left(s\,|T_y>t\right)$. In this case, (10) leads to

$$
\begin{aligned}
C_t\left[S_{1;t}\left(s\,|T_y>t\right),S_{2;t}\left(s\,|T_y>t\right)\right] & = \frac{\max\left[S_1\left(t+s\right)+S_2\left(t+s\right)-1,0\right]}{S\left(t,t\right)} \\
& = \max\left[S_{1;t}\left(s\,|T_y>t\right)+S_{2;t}\left(s\,|T_y>t\right)-1,0\right].
\end{aligned}
$$

In words, if the joint distribution has the Fréchet lower bound as copula at the outset, it will continue to have the Fréchet lower bound as copula in the future.

5.2 Archimedean copulas

We start this subsection by deriving the copula of the conditional joint survival function, given survival of both lives to a certain time in Subsubsection 5.2.1. Thereafter, in Subsubsection 5.2.2, we derive some time-dependent measures of association. Finally, in Subsubsection 5.2.3, we give some examples, extracted from Table 1.

5.2.1 Updated joint distribution

For Archimedean copulas, with the generator as defined in Subsection 3.4, the conditional marginal survival function of remaining lifetime of (x), given survival of (x) and (y) to t, has the following expression:

$$
S_{1;t}\left(s\,|T_y>t\right)=\frac{\phi^{-1}\left(\phi\left(S_1\left(t+s\right)\right)+\phi\left(S_2\left(t\right)\right)\right)}{S\left(t,t\right)}, \tag{17}
$$

with a similar expression for $S_{2;t}\left(s\,|T_x>t\right)$. The force of mortality as defined in (9), is:

$$
\begin{aligned}
& \mu_1\left(x+t+s\,|T_y>t\right) \\
= & \ \mu_1\left(x+t+s\right)\,S_1\left(t+s\right)\left(-\phi'\left(S_1\left(t+s\right)\right)\right)\left(-\frac{\left(\phi^{-1}\right)'\left(v\right)}{\left(\phi^{-1}\right)\left(v\right)}\right)_{\left(v=\phi(S(t+s,t))\right)}. \tag{18}
\end{aligned}
$$

The next theorem demonstrates that, if the copula underlying the joint survival function at the outset is Archimedean, then the copula of the conditional joint survival function, given survival of both, is Archimedean as well.

Theorem 7 *If the copula of a joint survival function copula at time* 0 *is Archimedean, then the copula of the conditional joint survival function, given survival of both to* t, *is also Archimedean. Let* $\phi(\cdot)$ *denote the generator of the copula at time* 0. *Then* $\phi_t(\cdot)$, *the generator of the Archimedean copula at time* t, *is*

$$\phi_t(\tau) = \phi(\tau \cdot S(t,t)) - \phi(S(t,t)), \quad \tau \in [0,1]. \tag{19}$$

Proof. We show that the joint distribution of remaining lifetime, given survival of both to t, comprises the copula generated by $\phi_t(\cdot)$ in (19). Note, first of all that $\phi_t(\cdot)$ has all the properties of a generator for an Archimedean copula, namely $\phi_t(1) = 0$, $\phi_t'(\tau) < 0$, and $\phi_t''(\tau) > 0$. The inverse of the generator $\phi_t(\cdot)$, denoted by $\phi_t^{-1}(\cdot)$, is

$$\phi_t^{-1}(\tau) = \frac{\phi^{-1}(\tau + \phi(S(t,t)))}{S(t,t)}.$$

Applying (11), this leads to the updated copula, denoted by $C_t[\cdot, \cdot]$:

$$C_t[u,v] = \frac{\phi^{-1}(\phi(u \cdot S(t,t)) + \phi(v \cdot S(t,t)) - \phi(S(t,t)))}{S(t,t)}.$$

Substituting $u = S_{1;t}(s | T_y > t)$ and $v = S_{2;t}(s | T_y > t)$ gives, using expression (17)

$$
\begin{aligned}
C_t[S_{1;t}(s | T_y > t), S_{2;t}(s | T_x > t)] &= \frac{\phi^{-1}(\phi(S_1(t+s)) + \phi(S_2(t+s)))}{S(t,t)} \\
&= \Pr[T_x > t+s, T_y > t+s | T_x > t, T_y > t].
\end{aligned}
$$

∎

A similar result has been derived in Manatunga and Oakes (1996) in the context of the important subclass of frailty models.

Next, we consider the case of copula families which remains constant in time. They can be derived by solving the equation

$$\phi_t(\tau) = \alpha(S(t,t))\phi(\tau), \tag{21}$$

for $\tau \in [0,1]$. The function $\alpha(S(t,t))$ depends on $S(t,t)$ only, and not on τ. In the Appendix, we prove that the solution of this equation is:

$$\phi(v) = \frac{K}{\theta}\left(v^{-\theta} - 1\right), \quad \theta \in \mathbb{R} \setminus \{0\}, K \geq 0.$$

which is the generator of Clayton's copula.

It is well known that Clayton's copula implies association which is independent of time. Note, furthermore, that the Clayton family is the only type which satisfies this property.

In this subsection we use the notion of concordance ordering of copulas. A copula $C_{(1)}$ is smaller than $C_{(2)}$ if $C_{(1)}(u,v) \leq C_{(2)}(u,v)$ for all $u, v \in [0,1]$. For our analysis of the copulas introduced in Subsection 3.4, we make use of some of the following theorems. The first one is from Nelsen (1999). The second one is due to Nelsen (1999) and is an extension of a result in Genest and MacKay (1986a). The last two have been derived in Genest and MacKay (1986a) and can also be found in Nelsen (1999). We define $\omega_{xy} = \min[t | S(t,t) = 0]$ as the limiting age which the joint-life status can obtain.

Theorem 8 *Let* C_{t_1} *and* C_{t_2} *be Archimedean copulas, generated, respectively, by* $\phi_{t_1}(\cdot)$ *and* $\phi_{t_2}(\cdot)$. *Then, if* $\phi_{t_1} \circ \phi_{t_2}^{[-1]}$ *is concave, then* $C_{t_1} < C_{t_2}$.

Theorem 9 *Let C_{t_1} and C_{t_2} be Archimedean copulas, generated, respectively, by $\phi_{t_1}(\cdot)$ and $\phi_{t_2}(\cdot)$. Assume both generators to be continuously differentiable on $(0,1)$. Then if $\phi'_{t_1}(\cdot)/\phi'_{t_2}(\cdot)$ is nondecreasing on $(0,1)$, then $C_{t_1} < C_{t_2}$.*

Theorem 10 *Let $\{C_t \,|\, t \in [0, \omega_{xy}]\}$, be a family of copulas with continuously differentiable generators $\phi_t(\cdot)$. Then $C = \lim_{t \to \omega_{xy}} C_t$ is an Archimedean copula if and only if there exists a function in Ω such that for all s, τ in $(0,1)$,*

$$\lim_{t \to \omega_{xy}} = \frac{\phi_t(s)}{\phi'_t(\tau)} = \frac{\phi(s)}{\phi'(\tau)}. \tag{22}$$

Theorem 11 *Let $\{C_t \,|\, t \in [0, \omega_{xy}]\}$, be a family of copulas with differentiable generators $\phi_t(\cdot)$ in Ω. Then $\lim_{t \to \omega_{xy}} C_t(u,v) = \min[u,v]$ if and only if,*

$$\lim_{t \to \omega_{xy}} = \frac{\phi_t(\tau)}{\phi'_t(\tau)} = 0. \tag{23}$$

Theorems 8 and 9 are used to investigate whether dependence has a monotone development (increasing or decreasing) over time. If dependence is decreasing, Theorem 10 can be used to find the limiting form of dependence. For instance, if $\phi(s)/\phi'(\tau) = t \ln s$, then the limiting form is independence. On the other hand, if dependence is increasing, Theorem 11 can be used to check if maximal dependence is attained in the limit. Should this not be the case, then Theorem 10 could be applied to look for another limiting form. Note that we are actually dealing with two parameters in the copula, namely θ and $S(t,t)$, and we fix θ.

In each of the illustrating examples in Subsubsection 5.2.3 we derive the generator as a function of time.

5.2.2 Time-dependent measures of association

Several time-dependent measures of association have been developed in the literature. We discuss two which are related to Kendall's tau, denoted by $\tau(T_x, T_y)$. It is defined as:

$$\tau(T_x, T_y) = 4 \int_{u=0}^{1} \int_{v=0}^{1} C[u,v] \, dC[u,v] + 1. \tag{24}$$

Kendall's tau does not depend on the distribution of the marginals and this explains its popularity. Independence, comonotonicity and countermonotonicity imply values for Kendall's tau of 0, 1 and -1, respectively.

From Genest and MacKay (1986a, 1986b), we know that for Archimedean copulas, (24) reduces to

$$\tau(X_1, X_2) = 4 \int_{u=0}^{1} \frac{\phi(v)}{\phi'(v)} dv + 1. \tag{25}$$

One measure of time-dependent association is Kendall's tau pertaining to the copula constructed by the generator $\phi_t(\tau)$, defined in (19). We will denote it by $\widetilde{\tau}_t(T_1, T_2)$. Hence

$$\widetilde{\tau}_t(T_x, T_y) = 4 \int_{u=0}^{1} \frac{\phi_t(v)}{\phi'_t(v)} dv + 1 = \frac{4}{S(t,t)} \int_{u=0}^{1} \frac{\phi(u \cdot S(t,t)) - \phi(S(t,t))}{\phi'_t(u \cdot S(t,t))} dv + 1. \tag{26}$$

Manatunga and Oakes (1996) define this version of Kendall's tau as the *truncated* tau. They present alternative definitions of $\widetilde{\tau}_t(T_x, T_y)$.

The second measure to be discussed is the cross-ratio function $CR\left(S\left(t,t\right)\right)$, which has been introduced by Clayton (1978). Its characteristics are discussed in Oakes (1989). It is defined by Oakes as:

$$CR\left(S\left(t_1,t_2\right)\right) = \frac{S\left(t_1,t_2\right)\frac{d}{dt_1}\frac{d}{dt_2}S\left(t_1,t_2\right)}{\frac{d}{dt_1}S\left(t_1,t_2\right)\frac{d}{dt_2}S\left(t_1,t_2\right)}. \tag{27}$$

An interpretation of this quantity as an odds-ratio is given in Anderson et al. (1992). Some properties of the cross-ratio function are derived in Gupta (2003). Oakes also points out that

$$\overline{\tau}_t\left(T_x,T_y\right) = \frac{CR\left(S\left(t_1,t_2\right)\right)-1}{CR\left(S\left(t_1,t_2\right)\right)+1}, \tag{28}$$

is a conditional version of Kendall's tau.

For two reasons, we prefer the cross-ratio function to the truncated tau, as the degree of time-dependent association. First of all, an alternative definition of $CR\left(\cdot\right)$ is

$$CR\left(S\left(t_1,t_2\right)\right) = \frac{\mu_1\left(x+t_1\left|T_y=t_2\right.\right)}{\mu_1\left(x+t_1\left|T_y>t_2\right.\right)}. \tag{29}$$

The interpretation is clear: it indicates the relative rate of increase of the force of mortality of the survivor at t_1 upon death of the partner at t_2.

Secondly, the cross-ratio function is easier to evaluate than the truncated tau, as no integration is required. This is shown by dividing (14) by (18), using $t_2=t_1=t=t_y$. This leads to:

$$CR\left(S\left(t,t\right)\right) = \left(\frac{\phi^{-1}\left(v\right)\left(\phi^{-1}\right)''\left(v\right)}{\left(\left(\phi^{-1}\right)'\left(v\right)\right)^2}\right)_{v=\phi\left(S\left(t,t\right)\right)}. \tag{30}$$

In other words, the cross ratio function only depends on the inverse of the generator. This result has been derived in Oakes (1994).

Hougaard (2000) and Hougaard et al. (1992) have applied the cross-ratio function in the statistical analysis of twin data.

In each of the illustrating examples in Subsubsection 5.2.3, we will derive the cross-ratio function. Note that the papers introduced in this subsubsection focus on frailty models (being a subclass of Archimedean copulas) while this contribution deals with Archimedean copulas in general.

Some other measures of time-dependent association between two lives have appeared in the literature. Anderson et al. (1992) introduce the "Conditional expected residual life" and the "Conditional probability of survival". Some of their properties have been derived in Gupta (2003). Bassan and Spizzichino (2001) introduce and discuss a bivariate aging function which can be used in the special case of exchangeable lifetimes (i.e. (x) and (y) have the same law of mortality).

5.2.3 Examples

First of all, we discuss the Clayton copula, followed by Gumbel-Hougaard and Frank.

Example 12 (Clayton) *For $\phi\left(\tau\right) = \delta\left(\tau^{-\theta}-1\right)$, $\theta>0$ and any $\delta>0$, we have for (18):*

$$\mu_1\left(x+t+s\left|T_y>t\right.\right) = \mu_1\left(x+t+s\right)\frac{\left(S_1\left(t+s\right)\right)^{-\theta}}{\left(S_1\left(t+s\right)\right)^{-\theta}+\left(S_2\left(t\right)\right)^{-\theta}-1}.$$

For (19), we have

$$\phi_t(\tau) = \delta \left(S(t,t) \right)^{-\theta} \left(\tau^{-\theta} - 1 \right) = \delta \left(S(t,t) \right)^{-\theta} \phi(\tau), \tag{31}$$

so the copula essentially remains the same throughout time. It follows that the association remains the same as well. This is a confirmation of a previous result. The cross-ratio function is therefore constant in time, and equal to $\theta + 1$.

Example 13 (Gumbel-Hougaard) *For $\phi(\tau) = (-\ln[\tau])^{\theta}$, $\theta \geq 1$, we have for (18):*

$$\mu_1(x+t+s \,|T_y > t) = \mu_1(x+t+s) \left(\frac{(-\ln S_1(t+s))^{\theta}}{(-\ln S_1(t+s))^{\theta} + (-\ln S_2(t))^{\theta}} \right)^{1-\frac{1}{\theta}}. \tag{32}$$

For (19), we have

$$\phi_t(\tau) = (-\ln[\tau S(t,t)])^{\theta} - (-\ln[S(t,t)])^{\theta}. \tag{33}$$

Using Theorem 9, we obtain that $C_t < C$ and using Theorem 10, we find that the association between the two lifetimes reduces to zero. In other words, the two lives become less dependent as they age. The cross-ratio function is:

$$CR\left((S(t,t)) \right) = 1 + \frac{\theta - 1}{-\ln S(t,t)}. \tag{34}$$

Example 14 (Frank) *For $\phi(\tau) = -\ln\left[\left(e^{\theta\tau} - 1 \right) / \left(e^{\theta} - 1 \right) \right]$, we have for (18):*

$$\begin{aligned}
&\mu_1(x+t+s \,|T_y > t) \\
&= \mu_1(x+t+s) \\
&\quad \frac{\left(1 - e^{\theta S_2(t)} \right) e^{\theta S_1(t+s)} (-\theta) S_1(t+s)}{\left((e^{\theta} - 1) + \left(e^{\theta S_1(t+s)} - 1 \right) \left(e^{\theta S_2(t)} - 1 \right) \right) \ln\left[1 + \frac{\left(e^{\theta S_1(t+s)} - 1 \right) \left(e^{\theta S_2(t)} - 1 \right)}{e^{\theta} - 1} \right]}.
\end{aligned} \tag{35}$$

For (19), we have

$$\phi_t(\tau) = -\ln \frac{e^{\theta S(t,t)\tau} - 1}{e^{\theta S(t,t)} - 1}. \tag{36}$$

One can see from this expression that the copula pertaining to time belongs to the Frank family as well with parameter θ updated to $\theta \cdot S(t,t)$. As time proceeds, the parameter approaches zero, implying independence. The cross-ratio function is:

$$CR\left(S(t,t) \right) = -\frac{\theta S(t,t)}{1 - e^{\theta S(t,t)}}. \tag{37}$$

Most copulas give a decreasing dependence over time leading to independence as t approaches ω_{xy}. Families 8 and 9 of Table 1 are the only types where dependence is increasing over time. Family 8 will be treated below.

Example 15 (Family 8 of Table 1) *If $\phi(\tau) = \exp\left(t^{-\theta} \right) - e$, $\theta \geq 1$, then the inverse is $\phi^{-1}(\tau) = (\ln[\tau + e])^{-\frac{1}{\theta}}$. we have for (18):*

$$\begin{aligned}
&\mu_1(x+t+s \,|T_y > t) \\
&= \mu_1(x+t+s) \frac{S_1(t+s)^{-\theta}}{\ln\left[\exp\left[S_1(t+s)^{-\theta} \right] + \exp\left[S_2(t_y)^{-\theta} \right] - e \right]} \\
&\quad \frac{\exp\left[S_1(t+s)^{-\theta} \right]}{\exp\left[S_1(t+s)^{-\theta} \right] + \exp\left[S_2(t_y)^{-\theta} \right] - e}.
\end{aligned} \tag{38}$$

Copula	Duration 10	Duration 20	Duration 30
Clayton	$117, 158$	$79, 471.1$	$48, 556.4$
Gumbel-Hougaard	$116, 682$	$77, 329.8$	$47, 973.5$
Family 8	$117, 024$	$79, 822.6$	$48, 650$
Family 9	$116, 949$	$80, 458.3$	$47, 998.9$
Independence	$115, 605$	$75, 038.7$	$45, 170.5$

Table 2: Provisions if (y) is still alive

For (19), we have

$$\phi_t\left(\tau\right) = \frac{\exp\left[\left(\tau \cdot S\left(t, t\right)\right)^{-\theta}\right] - \exp\left[\left(S\left(t, t\right)\right)^{-\theta}\right]}{S\left(t, t\right)}. \tag{39}$$

Using Theorem 9, we obtain that $C_t > C$ and using Theorem 11, we find that the association between the two lifetimes increases to comonotonicity. In other words, the two lives become more dependent as they age. The cross-ratio function is

$$CR\left(S\left(t, t\right)\right) = 1 + \theta\left(1 + \left[S\left(t, t\right)\right]^{-\theta}\right). \tag{40}$$

6 Numerical example

We consider a policy taken out by a couple where $(x) = (y) = 60$. The policy secures various benefits, one of which is a whole-life annuity due of $10, 000$ p.a., payable on life (x) while alive, independent of the life status of (y). These benefits are payable by single premium, which is $156, 309$ (irrespective of the degree of association, as (x) and (y) are coupled by the copula upon issue). The marginals $S_1\left(\cdot\right)$ and $S_2\left(\cdot\right)$ are specified by the British life tables PMA92C20 and PFA92C20, respectively. We assume that deaths are uniformly distributed between consecutive integer ages. Interest is at 4% p.a..

As copulas, we choose:

1. Clayton (long-term dependence, association constant over time);

2. Gumbel-Hougaard (long-term dependence, association decreasing in time);

3. Family 8 (long-term dependence, association increasing in time);

4. Family 9 (partially short-term dependence, association increasing in time);

For each copula, the parameter corresponds to a value of Kendall's τ equal to 0.5 at the outset. We calculate the provisions at durations 10, 20 and 30. Table 2 shows the provisions that apply if (y) is still alive on the valuation date. Obviously these provisions are higher than in the case of independence.

Figure 1 displays the curves for duration 10. For Clayton, Gumbel-Hougaard, and Family 8, the provisions are increasing as a function of t_y, the time-of-death of (y). Since all these copulas feature long-term dependence, this is not a very surprising result. The more distant the time of death of the partner, the higher the mortality of the remaining life, and hence the smaller the provisions. The provisions relating to Family 9 are slightly decreasing in t_y (from about $109, 150$ for $t_y = 0$ to $108, 900$ for $t_y = 10$), reflecting the short-term dependence for short durations. The provisions are maximal for this Family 9, while they are minimal for Gumbel-Hougaard, which varies between $46, 000$ and $52, 000$. Figure 2 gives the provision at duration 20. Again, Family
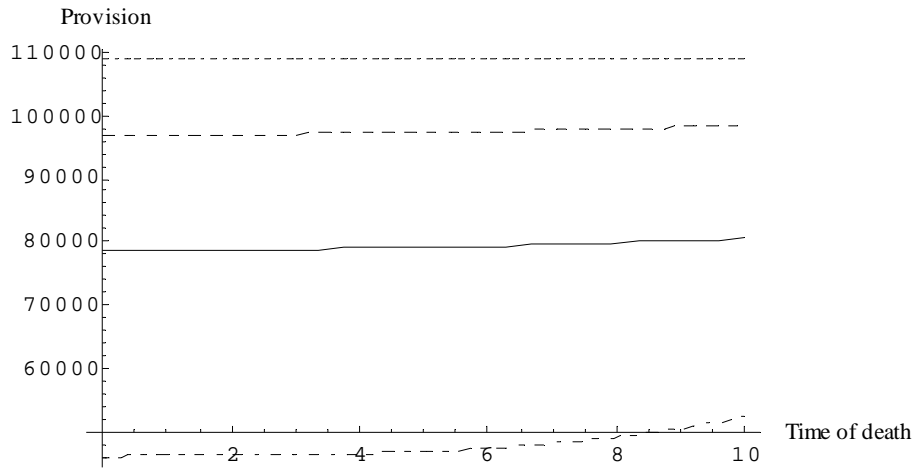
Figure 1: Provision at duration 10 as a function of the time of death of (y). (Gumbel-Hougaard: dashed-dotted; Clayton: solid; Family 8: dashed; Family 9: dotted).
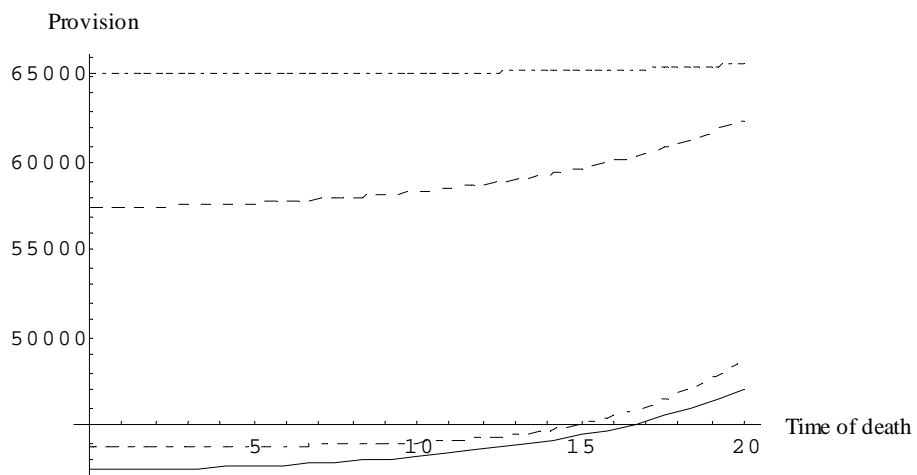

Figure 2: Provision at duration 20 as a function of the time of death of (y). (Gumbel-Hougaard: dashed-dotted; Clayton: solid; Family 8: dashed; Family 9: dotted).
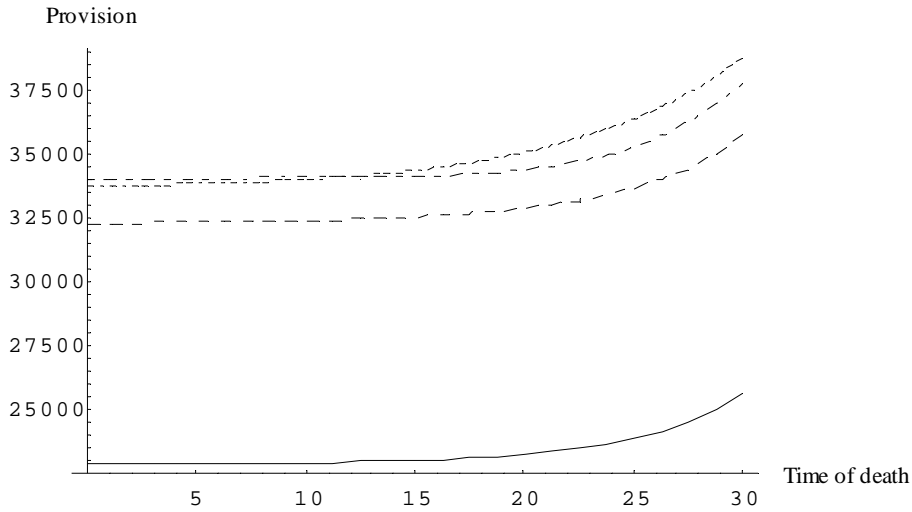
Figure 3: Provision at duration 30 as a function of the time of death of (y). (Clayton: solid; Family 8: dashed; Family 9: dotted; Gumbel-Hougaard: dashed-dotted).

9 (now slightly increasing as a function of t_y) gives the largest provisions, followed by Family 8. Now it is Clayton giving minimal values. Figure 3 gives the provisions at duration 30. Now Gumbel-Hougaard and Family 9 display the largest values, while the provisions corresponding to Clayton's copula are still minimal. We have done a similar exercise regarding provisions to be kept if the female annuitant is alive and the male has died, again for Kendall's τ equal to 0.5. In that case the graphs look a bit different. We have also done investigations based on smaller and larger values of Kendall's coefficient of concordance. Different patterns arise for different cases, and the sizes of the provisions are by no means ordered, in the sense that e.g. Family 9 and Gumbel-Hougaard always give maximal values and minimal values for the provisions. The provisions produced from Gumbel-Hougaard's copula become relatively larger compared to other copulas as the duration goes up, while relatively low values result from Clayton's copula.

It is difficult to explain all the features arising. This example, however, mainly serves to illustrate that provisions can differ substantially for different copulas, even if the association at the outset is the same. This implies that a copula should be chosen with care. Probably the results will differ for different combinations of ages at issue.

7 Discussion

In this paper, we have derived the type of association, as well as the aging properties, of some single parameter dependence models.

If two lifetimes are independent at the outset, then the time of death of one life has no impact at all on the mortality of the other life. Moreover, the conditional remaining lifetimes, given survival of both to a certain time, will remain independent. These results are easily understood by intuitive reasoning.

In case of a copula model, the mortality of one life always depends on the time of death of the other life, unless the lifetimes are independent. We have discussed several Archimedean copulas which allow for Positive Quadrant Dependence and incorporated the special case of

independence. We conclude that for all the copulas studied, there is no one which incorporates pure short-term dependence.

The question is whether this is realistic within the framework of married couples. Jagger & Sutton (1991) and Parkes et al. (1969) have shown that in most empirical studies the mortality of widowers increases sharply after death of their spouse, but returns to normal levels within a period of about six months. Hougaard (2000) suggests that for married couples, short-term dependence is probably more relevant than long-term dependence, and that for twins, probably the opposite holds. Firstly, contrary to twins, married couples do not have common genes, but only share their living environment. Secondly, the broken heart syndrome is a more significant feature among married couples than it is among twins. The applications carried out in Hougaard (2000), Hougaard et al. (1992) and Anderson et al. (1992) are all based on twin data.

Probably both the long term and the short term dependence effects merit separate parameters, so single parameter models may be too limited to capture all the types of dependence featured. This is a topic for future research.

Furthermore, we have investigated the aging properties of some Archimedean copula models. We have established that in most cases the association between lifetimes decreases over time. In other words, individuals become less dependent as they age. Moreno (1994), devoting a discussion to aging in some frailty models, suggests that this pattern is intuitively desirable. His reasoning is that "if two lives have died, their survival experiences become independent of each other". He uses the frailty distributions of Inverse Gaussian and a multiple point distribution as illustrations. Moreno argues that increasing dependence can occur if the mortality of some couples is extremely low, which is the case for Poisson distributed frailty.

These arguments may make sense if dependence is of a distinct long term nature, as specified by frailty models. For married couples, the assumption that dependence will diminish over time, seems stark. A young individual who is still engaged in paid employment will usually have a more extensive social network than a retiree, and hence better able to handle the shock of death of the partner. Moreover, youngsters have more opportunities to remarry. If an individual has been married for a long while, the shock of death of the spouse may have more dramatic (emotional) effects than for a life who has only got a partner for a couple of years.

This all gives arguments to use more sophisticated models to describe dependence, by allowing for short term dependence, as well as instantaneous dependence. Youn and Shemyakin (1999, 2001) have captured these two types of dependence by classifying individuals according to the age difference between the female and the male member of each couple. However, given any age difference, they use the Gumbel-Hougaard copula to specify the dependence between the lifetimes. As we have seen in this paper, using this copula implies assuming that there is long-term dependence and association is decreasing as both lives age.

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A   Proof in Subsubsection 5.2.1

In this Appendix, we solve the equation (21), namely

$$\phi_t(\tau) = \alpha(S(t,t))\phi(\tau), \tag{41}$$

for $\phi(\tau)$. Taking derivatives of both sides with respect to τ gives

$$\alpha(S(t,t)) = \frac{S(t,t)\phi'(S(t,t)\cdot\tau)}{\phi'(\tau)}. \tag{42}$$

We take the derivative of both sides with respect to τ again, and establish that the following relationship must hold:

$$\frac{S(t,t)\phi''(S(t,t)\cdot\tau)}{\phi'(S(t,t)\cdot\tau)} = \frac{\phi''(\tau)}{\phi'(\tau)}. \tag{43}$$

Hence, the right hand side of the above equation does not depend on $S(t,t)$. Taking the derivative of both with respect to $S(t,t)$ gives the following differential equation for $v = S(t,t)\cdot\tau$:

$$\left(v\cdot\phi'''(v) + \phi''(v)\right)\phi'(v) - v\left[\phi''(v)\right]^2 = 0. \tag{44}$$

This differential equation is solved by

$$\phi(v) = \frac{K_2}{1+K_1}v^{1+K_1} + K_3, \tag{45}$$

for some real valued constants K_1, K_2, and K_3. The conditions $\phi'(v) < 0$ and $\phi''(v) > 0$ give $K_2 < 0$ and $K_1 < 0$. Furthermore, the condition $\phi(1) = 0$ gives $K_3 = -\frac{K_2}{1+K_1}$. The final condition $\lim_{v\downarrow 0}\phi(v) = \infty$ leads to $K_1 < -1$. Substituting $\theta = -(1+K_1)$, we get

$$\phi(v) = -\frac{K_2}{\theta}\left(v^{-\theta} - 1\right), \tag{46}$$

which is the generator of Clayton's copula.

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