Iyer, S. (2003). Application of stochastic methods in the valuation of social security pension schemes (Report No. Actuarial Research Paper No. 151). London, UK: Faculty of Actuarial Science & Insurance, City University London.

Cass Business School
City of London

Cass means business

Faculty of Actuarial Science and Statistics

Application of Stochastic Methods in the Valuation of Social Security Pension Schemes.

Subramaniam Iyer

Actuarial Research Paper No. 151

August 2003

ISBN 1-901615-72-3

Cass Business School
106 Bunhill Row
London EC1Y 8TZ
T +44 (0)20 7040 8470
www.cass.city.ac.uk

APPLICATION OF STOCHASTIC METHODS IN THE VALUATION OF SOCIAL SECURITY PENSION SCHEMES

Subramaniam Iyer

1. INTRODUCTION

The actuarial valuation of social security pension schemes has invariably been based on the deterministic approach. Although it is recognized that the determining parameters are subject to stochastic variation, the valuation is based on constant values of the parameters, chosen so as to represent the "average" that is expected each year over the time period covered by the valuation (Iyer, 1999). Sometimes results are also presented for alternative sets of parameters (typically, for two additional sets on either side, termed "conservative" and "optimistic"), or the sensitivity of the results to individual key parameters are illustrated, but apart from having some recourse to the simulation technique, stochastic methods have not seen much application (Plamondon et.al., 2002).

This contrasts with the case of other areas of insurance, including that of occupational pensions, where stochastic methods are increasingly being applied (Daykin et.al., 1994; Booth et.al., 1999).

This paper attempts to develop a stochastic valuation method for a newly introduced social security pension scheme. The approach is not based on the simulation technique. Rather, the purpose is to derive algebraic expressions for the variances of, and the co-variances among, the important aggregates that characterize the development of the scheme, i.e. the financial projections of insured salaries and benefit expenditures and their discounted values, taking into account stochastic variation in the following factors:

- The age-specific cohort survival
- The time-specific intake of new entrants
- The time-specific real rate of return on the Fund (i.e. the return net of salary escalation).

On this basis, a method of determining premiums is developed which recognizes explicitly and quantifies the underlying stochastic variations.

In order to keep the mathematical manipulations within reasonable limits, however, the following simplifying assumptions are made in regard to the pension model:

- The scheme provides pensions only on retirement (invalidity and survivors' pensions are not considered). There are no pensioners at the outset.
- The ages of entry and retirement are each unique, and constant over time
- The age-specific mortality rates are constant over time
- The pension is based on the final salary and is fully indexed to salary escalation.

Another important set of assumptions concerns independence. Each individual member of the scheme is assumed to be independent of every other member – whether in a different cohort or even in the same cohort - in regard to the mortality factor. Further, the investment return factor is assumed to be independent of the mortality factor and the new entrant factor.

For the development of the theory a continuous formulation is adopted. The paper does not enter into the details of practical application. However, numerical illustrations of the main results are provided in Appendix 8.

2. DEMOGRAPHIC PROJECTIONS, ALLOWING FOR STOCHASTIC VARIATION IN COHORT SURVIVAL

2.1 Notation and assumptions

In classical demographic projections, the number of survivors to various ages from a given group of individuals is projected exactly in line with the (assumed) underlying life table function. In this paper, it is recognized that cohort survival as above represents only the average progression of the cohort. The actual outcome is subject to stochastic variation around the average. The aim is to quantify the extent of the variation.

For the present purposes, the age *x* and time *t* are treated as continuous variables. The population element aged *x*, at time *t*, is, in general, denoted by *P(x, t)*. That is to say, the population aged between *x* and *x+dx* at time *t* is *P(x, t)dx*. When it is necessary to distinguish between the Active Population and the Retired Population, they are indicated, respectively, by the symbols *Ac(x, t)* and *Re(x, t)*. The Total Active and Retired Populations at time t are denoted by the symbols *A(t)* and *R(t)*.

The entry age is denoted by *b* and the retirement age by *r*. The initial active population, *Ac(x, 0)*, *b = x < r*, is assumed to be given, with *Ac(b, 0)dt* representing the number of new entrants at *t = 0* – more precisely, the number of new entrants in the interval *(-dt, 0)*. The number of new entrants at time *t* - or rather, in the interval *(t-dt, t)* - is given by $Ac(b,0)e^{rt}\,dt$, where ? is an assumed constant. **The "dt" is hereafter omitted in the text for brevity, but should be understood as implied.**

Stochastic variation in survival is assumed to apply to each population element from the time it enters the purview of the scheme. The applicable mortality table, representing the "average" survival pattern, is denoted by $\{l_x\}$, *b = x < ?*, where *?* is the limiting age, with ***? -r < r-b*** (i.e. the potential active lifetime exceeds the potential retirement lifetime).

2.2 Initial Population cohorts

If *P(x, t)* derives from an initial population cohort, the condition *t-x+b < 0* would apply, and the corresponding initial population element would be *Ac(x-t, 0)*. The number surviving to age *x* out of this number has the Binomial distribution with parameter ${}_t p_{x-t}$. Therefore (see Appendix 1, Note 1), the expected value and variance of *P(x, t)* are given by

$$EP(x,t) = Ac(x-t,0)\,{}_t p_{x-t} \qquad (1)$$

$$VP(x,t) = Ac(x-t,0)\,{}_t p_{x-t}(1-{}_t p_{x-t}) \qquad (2)$$

If *P(x, t)* and *P(y, u)*, both deriving from initial population cohorts, are from different cohorts, i.e. *x-t ? y-u*, they will be independent and therefore uncorrelated. However, if they derive from the same cohort (*x-t = y-u*), they will be correlated. Assuming ***x > y***, the covariance is given by (see Appendix 1, Note 1),

$$COV(P(x,t),P(y,u)) = Ac(x-t,0)\,{}_t p_{x-t}(1-{}_u p_{y-u}) \qquad (3)$$

2.3 New Entrant cohorts

We shall first consider the case corresponding to a unit new entrant at *t=0*. If *P(x, t)* derives from a new entrant cohort (*t-x+b > 0*), the size of the cohort at entry would be $e^{(t-x+b)r}$, and the expected value and variance of *P(x, t)* would be given by

$$EP(x,t) = e^{(t-x+b)r}\ {}_{x-b}p_b = e^{rt}\frac{D_x^{(r)}}{D_b^{(r)}} \tag{4}$$

$$VP(x,t) = e^{(t-x+b)r}\ {}_{x-b}p_b(1-{}_{x-b}p_b) = e^{rt}[\frac{D_x^{(r)}}{D_b^{(r)}} - \frac{D_{xx}^{(r)}}{D_{bb}^{(r)}}] \tag{5}$$

where $D_x^{(r)} = l_x e^{-rx}, D_{xx}^{(r)} = l_x l_x e^{-rx}$.

The above formulae correspond to a unit new entrant at *t = 0*. The actual number of new entrants at *t = 0* being *Ac(b, 0)*, it is necessary to adjust formulae (4) and (5). The adjustment is obtained by multiplying both the formulae by *Ac(b, 0)*. It might be thought that formula (5) should be multiplied by $Ac(b,0)^2$, but this would be incorrect, in view of the assumed independence of individual members within a cohort (see Appendix 1, Note 2). The adjusted formulae are given by

$$EP(x,t) = Ac(b,0)e^{rt}\frac{D_x^{(r)}}{D_b^{(r)}} \tag{6}$$

$$VP(x,t) = Ac(b,0)e^{rt}[\frac{D_x^{(r)}}{D_b^{(r)}} - \frac{D_{xx}^{(r)}}{D_{bb}^{(r)}}] \tag{7}$$

P(x, t) and *P(y, u)* will be independent and uncorrelated if one is from an initial population cohort and the other is from a new entrant cohort. Similarly, if *P(x, t)* and *P(y, u)*, both deriving from new entrant cohorts, are from different cohorts, i.e. *x-t ? y-u*, they will be independent and therefore uncorrelated. However, if they derive from the same cohort (*x-t = y-u*), they will be correlated. Assuming *x > y*, the covariance is given by the following formula (see Appendix 1, Note 1), including the adjustment required to take into account the actual number of new entrants at *t = 0*, namely *Ac(b, 0)*,

$$COV(P(x,t),P(y,u)) = Ac(b,0)e^{(t-x+b)r}\ {}_{x-b}p_b(1-{}_{y-b}p_b) = Ac(b,0)e^{rt}\frac{D_x^{(r)}}{D_b^{(r)}}(1-{}_{y-b}p_b) \tag{8}$$

2.4 Total Active and Retired Population projections: Expected values and variances

Since the different cohorts are considered to be evolving independently of each other, the variances are additive across cohorts at each point in time. When the expected values of the different cohorts surviving to time *t* are added to produce the expected total active and retired populations, the corresponding variances can be added to obtain the variance of the total active and retired populations. Due to the differences in the formulae, the elements from the initial population cohorts and those from new entrant cohorts should be added separately, over the appropriate age ranges, $a_1(t), a_2(t); a_0(t), a_1(t)$. This will give rise to the following formulae for the expected value and variance of *A(t)* or *R(t)*:

$$EA(t) or ER(t) = Ac(b,0)[\int_{a_1(t)}^{a_2(t)} \frac{Ac(x-t,0)}{Ac(b,0)} \, {}_t p_{x-t} dx + e^{rt}\int_{a_0(t)}^{a_1(t)} \frac{D_x^{(r)}}{D_b^{(r)}} dx] \qquad (9)$$

$$VA(t) or VR(t) = Ac(b,0)[\int_{a_1(t)}^{a_2(t)} \frac{Ac(x-t,0)}{Ac(b,0)} \, {}_t p_{x-t}(1- {}_t p_{x-t}) dx + e^{rt}\int_{a_0(t)}^{a_1(t)} (\frac{D_x^{(r)}}{D_b^{(r)}} - \frac{D_{xx}^{(r)}}{D_{bb}^{(r)}}) dx] \qquad (10)$$

For the simplified pension model assumed in this paper, with constant entry and retirement ages *b* and *r*, constant mortality table and with no pensioners at the outset of the scheme, the limits $a_0(t), a_1(t), a_2(t)$ will be as follows (Note: *? -r < r-b*):

Active/Retired population	**Time range**	**Age range for IP** $(a_1(t), a_2(t))$	**Age range for NE** $(a_0(t), a_1(t))$
Active population *A(t)*	*0 = t < r-b*	*b+t, r*	*b, b+t*
	t = r-b	*nil*	*b, r*
Retired population *R(t)*	*0 = t < ? -r*	*r, r+t*	*nil*
	? -r = t < r-b	*r, ?*	*nil*
	r-b = t < ? -b	*b+t, ?*	*r, b+t*
	t = ? -b	*nil*	*r, ?*

2.5 Total Active and Retired Population projections: Co-variances

A(t) will be correlated with *A(u)*, and *R(t)* with *R(u)*, to the extent that there are common cohorts between either pair. Similarly *R(t)* will be correlated with *A(u), u < t*, if there are common cohorts. However, *R(t)* will not be correlated with *A(u)* if *u > t*, since there cannot be common cohorts between them.

Consider, for example, *A(t)* and *A(u)*, with ***u < t*** If *Ac(x, t)* and *Ac(y, u)* are from the same cohort, then *x-t = y-u*, or *x = y+t-u*. If common cohorts exist over the range $(c_0(t,u), c_2(t,u))$ of *y*, of which $(c_0(t,u), c_1(t,u))$ concerns new entrant cohorts and $(c_1(t,u), c_2(t,u))$ relates to initial population cohorts, using (3) and (8) above, the co-variance between *A(t)* and *A(u)* can be expressed as

$$Ac(b,0)[e^{rt}\int_{c_0(t,u)}^{c_1(t,u)} \frac{D_{y+t-u}^{(r)}}{D_b^{(r)}}(1- {}_{y-b}p_b) dy + \int_{c_1(t,u)}^{c_2(t,u)} \frac{Ac(y-u,0)}{Ac(b,0)} \, {}_t p_{y-u}(1- {}_u p_{y-u}) dy] \qquad (11)$$

Similar formulae apply to the covariance between *R(t)* and *R(u)* and between *R(t)* and *A(u)*, ***u < t***.
The appropriate limits of integration are obtained according to the following rules:
If either $(a_0(t), a_1(t))$ or $(a_0(u), a_1(u))$ = nil, then $(c_0(t,u), c_1(t,u))$ = nil.
If $a_0(t) - t + u > a_1(u)$ or $a_1(t) - t + u < a_0(u)$ then $(c_0(t,u), c_1(t,u))$ = nil.
Otherwise, if $a_0(t) - t + u < a_0(u)$ then $c_0(t,u) = a_0(u)$;
if $a_0(t) - t + u > a_0(u)$ then $c_0(t,u) = a_0(t) - t + u$;
if $a_1(t) - t + u < a_1(u)$ then $c_1(t,u) = a_1(t) - t + u$;
if $a_1(t) - t + u > a_1(u)$ then $c_1(t,u) = a_1(u)$.
If either $(a_1(t), a_2(t))$ or $(a_1(u), a_2(u))$ = nil, then $(c_1(t,u), c_2(t,u))$ = nil.
If $a_1(t) - t + u > a_2(u)$ or $a_2(t) - t + u < a_1(u)$ then $(c_1(t,u), c_2(t,u))$ = nil.

Otherwise, if $a_1(t) - t + u < a_1(u)$ then $c_1(t,u) = a_1(u)$;

if $a_1(t) - t + u > a_1(u)$ then $c_1(t,u) = a_1(t) - t + u$;

if $a_2(t) - t + u < a_2(u)$ then $c_2(t,u) = a_2(t) - t + u$;

if $a_2(t) - t + u > a_2(u)$ then $c_2(t,u) = a_2(u)$.

The limits of integration for the simplified pension model are given in Appendix 2.

2.6 The long-term situation

In the long-term (*t* = ? -*b*) both the active and the retired populations would consist entirely of new entrant cohorts, and the above formulae would simplify as follows:

$$EA(t) = Ac(b,0)e^{rt}\frac{\overline{N}_b^{(r)} - \overline{N}_r^{(r)}}{D_b^{(r)}}; ER(t) = Ac(b,0)e^{rt}\frac{\overline{N}_r^{(r)}}{D_b^{(r)}} \quad (12)$$

$$VA(t) = Ac(b,0)e^{rt}\left[\frac{\overline{N}_b^{(r)} - \overline{N}_r^{(r)}}{D_b^{(r)}} - \frac{\overline{N}_{bb}^{(r)} - \overline{N}_{rr}^{(r)}}{D_{bb}^{(r)}}\right]; VR(t) = Ac(b,0)e^{rt}\left[\frac{\overline{N}_r^{(r)}}{D_b^{(r)}} - \frac{\overline{N}_{rr}^{(r)}}{D_{bb}^{(r)}}\right] \quad (13)$$

where, $\overline{N}_{xx}^{(r)} = \int_x^v D_{yy}^{(r)} dy$

It will be noted that although *EA(t)* and *ER(t)* and *VA(t)* and *VR(t)* increase continuously and tend to infinity as *t* tends to infinity (unless ? = 0), the respective coefficients of variation, *CVA(t)* and *CVR(t)*, are proportional to $Ac(b,0)^{-\frac{1}{2}}e^{-\frac{1}{2}rt}$. Therefore, they are inversely related to the square root of the population factor *Ac(b, 0)* and tend to zero as t tends to infinity, provided ? > 0. This is in conformity with the law of large numbers.

2.7 The Demographic Ratio

Consider the Demographic Ratio, defined as

$$DR(t) = \frac{R(t)}{A(t)} \quad (14)$$

R(t) can be regarded as independent of *A(t)* as they are constituted by different cohorts. Therefore *R(t)* and *A(t)* are uncorrelated, and if the third and higher central moments of *A(t)* as well as fourth and higher powers of *CVA(t)* are ignored, the following approximate expressions will be obtained for the expected value and the variance of the Demographic Ratio (see Appendix 1, Note 3):

$$EDR(t) = \frac{ER(t)}{EA(t)}[1 + CVA^2(t)] \quad (15)$$

$$VDR(t) = \frac{ER^2(t)}{EA^2(t)}[CVR^2(t) + CVA^2(t)] \quad (16)$$

Equation (15) indicates that the ratio *ER(t)/EA(t)* is a biased estimator of the Demographic Ratio, although the bias declines with time if ? > 0. Equation (16) shows that the variance of this ratio also tends to decline with time.

Further, it can be shown that the coefficient of variation, *CVDR(t)* decreases with *t*, tending to *0* as *t? 8*, so that the precision of the estimated Demographic Ratio increases with time. The initial population size has also a similar effect; for a given *t*, the larger the value of *Ac(b,0)*, the smaller will be *CVA(t)* and *CVR(t)* and the larger will be the precision of the estimated Demographic Ratio.

3. DEMOGRAPHIC PROJECTIONS, ALLOWING ALSO FOR STOCHASTIC INTAKE OF NEW ENTRANTS

3.1 Notation and assumptions

In classical demographic projections, the annual number of new entrants is assumed deterministically. For example, the number of new entrants in the interval (*t-dt, t*) may be taken as $Ac(b,0)e^{rt}dt$, where ? is an assumed parameter (see section 2.1).

In this section, the number of new entrants in the interval (*t-dt, t*) is assumed to be $Ac(b,0)e^{\int_0^t r(z)dz}$ dt, where $e^{r(z)}$ for different values of z are independent identically distributed (IID) variables, with expected value and second raw moment given by

$$E(e^{r(z)}) = e^{r};E(e^{2r(z)}) = e^{p} \qquad (17)$$

so that the variance is given by

$$V(e^{r(z)}) = e^{p} - e^{2r} \qquad (18)$$

For the variance to be positive, we should have p > 2?. When p = 2?, the variance is zero, meaning that the new entrant factor is deterministic. The symbol f is used to denote the difference **p – ?**.

As regards the initial population cohorts, the formulae developed in section 2.2 above are also valid in the present case. It is therefore necessary to discuss only the new entrant cohorts.

3.2 New entrant cohorts: Expected value and variance of *P(x, t)*

As in section 2, we shall first develop the formulae for a unit new entrant at time *t = 0.* If *P(x, t)* derives from a new entrant cohort, this cohort would have entered at time *t-x+b*, and would have numbered $L = e^{\int_0^{t-x+b} r(z)dz}$ at entry. In view of the IID property of $e^{r(z)}$ we have

$$E(L) = e^{(t-x+b)r};V(L) = e^{(t-x+b)p} - e^{2(t-x+b)r} \qquad (19)$$

The conditional expectation and variance of *P(x, t)*, given *L*, are given by

$$E(P(x,t)\,|\,L) = L\,{}_{x-b}p_b;V(P(x,t)\,|\,L) = L\,{}_{x-b}p_b(1-{}_{x-b}p_b) \qquad (20)$$

The unconditional expectation of *P(x, t)* is therefore

$$E(P(x,t)) = EE(P(x,t)\,|\,L) = {}_{x-b}p_b\,E(L) = {}_{x-b}p_b\,e^{(t-x+b)r} = e^{rt}\frac{D_x^{(r)}}{D_b^{(r)}} \qquad (21)$$

The unconditional variance of *P(x, t)* is obtained as *VP(x, t) = EV(P(x,t)|L)+VE(P(x,t)|L)*

$$EV(P(x,t)\,|\,L) = {}_{x-b}p_b(1-{}_{x-b}p_b)E(L) = e^{(t-x+b)r}\,{}_{x-b}p_b(1-{}_{x-b}p_b) \qquad (22)$$

$$VE(P(x,t)\,|\,L) = {}_{x-b}p_b^2V(L) = {}_{x-b}p_b^2(e^{(t-x+b)p} - e^{2(t-x+b)r}) \qquad (23)$$

Adding (22) and (23) and simplifying, we get

$$VP(x,t) = e^{pt}\frac{D_{xx}^{(p)}}{D_{bb}^{(p)}} - e^{2rt}\left(\frac{D_x^{(r)}}{D_b^{(r)}}\right)^2 + e^{rt}\left(\frac{D_x^{(r)}}{D_b^{(r)}} - \frac{D_{xx}^{(r)}}{D_{bb}^{(r)}}\right) \qquad (24)$$

The formulae (21) and (24), which correspond to a unit new entrant at *t = 0,* should be adjusted to correspond to the actual number of new entrants at *t = 0*, namely *Ac(b, 0)*. As in the case of section 2 above, for the same reasons, the adjustment factor would, in either case, be *Ac(b, 0).* The final expressions, therefore, are

$$EP(x,t) = Ac(b,0)e^{rt}\frac{D_x^{(r)}}{D_b^{(r)}} \tag{25}$$

$$VP(x,t) = Ac(b,0)[e^{pt}\frac{D_{xx}^{(p)}}{D_{bb}^{(p)}} - e^{2rt}(\frac{D_x^{(r)}}{D_b^{(r)}})^2 + e^{rt}(\frac{D_x^{(r)}}{D_b^{(r)}} - \frac{D_{xx}^{(r)}}{D_{bb}^{(r)}})] \tag{26}$$

It will be noted that (25) is identical to (6), and that (26) will reduce to (7) if p = 2?

3.3 New entrant cohorts: Covariance between *P(x, t)* and *P(y, u),* from the same cohort

Unlike the case of section 2, *P(x, t)* and *P(y, u)* will be correlated, whether or not they are from the same cohort. As before, the formulae are first developed for a unit new entrant at *t = 0*, and then adjusted to correspond to *Ac(b, 0)* new entrants at *t = 0.*

Since they are from the same cohort, *x-t = y-u*. Assume ***x > y***. *L* has the same definition as in section 3.2. We use the formula (see Appendix 1, Note 4):

$$COV(P(x,t),P(y,u)) = ECOV(P(x,t),P(y,u) \mid L) + COV(E(P(x,t) \mid L), E(P(y,u) \mid L)) \tag{27}$$

The development of the formula proceeds as follows:

$$COV(P(x,t),P(y,u) \mid L) = L\,{}_{x-b}p_b(1 - {}_{y-b}p_b) \tag{28}$$

$$ECOV(P(x,t),P(y,u) \mid L) = {}_{x-b}p_b(1 - {}_{y-b}p_b)E(L) = e^{(t-x+b)r}\,{}_{x-b}p_b(1 - {}_{x-b}p_b) \tag{29}$$

$$E(P(x,t) \mid L) = L\,{}_{x-b}p_b; E(P(y,u) \mid L) = L\,{}_{y-b}p_b \tag{30}$$

$$E[E(P(x,t) \mid L)E(P(y,u) \mid L)] = {}_{x-b}p_b\,{}_{y-b}p_b E(L^2) \tag{31}$$

$$EE(P(x,t) \mid L) * EE(P(y,u) \mid L) = {}_{x-b}p_b\,{}_{y-b}p_b E^2(L) \tag{32}$$

$$COV[E(P(x,t) \mid L), E(P(y,u) \mid L)] = {}_{x-b}p_b\,{}_{y-b}p_b(E(L^2) - E^2(L)) \tag{33}$$

Substituting for $E(L^2)$ and *E(L)* in (33), combining with (29) and simplifying, and incorporating the adjustment for correspondence with *Ac(b, 0)* new entrants at t = 0,

$$COV(P(x,t),P(y,u)) = Ac(b,0)[(e^{pt}\frac{D_x^{(p)}}{D_b^{(p)}} - e^{2rt}\frac{D_x^{(2r)}}{D_b^{(2r)}})\,{}_{y-b}p_b + e^{rt}\frac{D_x^{(r)}}{D_b^{(r)}}(1 - {}_{y-b}p_b)] \tag{34}$$

It will be noted that (34) will reduce to (26) when $x = y$, and to (8) when p = 2?.

3.4 New Entrant cohorts: Covariance between *P(x, t)* and *P(y, u),* from different cohorts

Since they are not from the same cohort, *x-t ? y-u*. Let us assume that ***x-t < y-u***, which means that *P(x, t)* is from a later cohort than *P(y, u).* Let $L_1 = e^{\int_0^{t-x+b} r(z)dz}$ and $L_2 = e^{\int_0^{u-y+b} r(z)dz}$. We use the following formula (see Appendix 1, Note 4)

$$ECOV(P(x,t),P(y,u) \mid L_1, L_2) + COV(E(P(x,t) \mid L_1), E(P(y,u) \mid L_2)) \tag{35}$$

The first term of (35) is zero, because *P(x, t)* and *P(y, u)* are uncorrelated if the respective cohorts are non-stochastic. Therefore only the second term is relevant. The derivation proceeds as follows, per unit new entrant at *t = 0*:

$$E(L_1) = e^{(t-x+b)r}; E(L_2) = e^{(u-y+b)r} \tag{36}$$

$$E(L_1L_2) = E(e^{\int_0^{t-x+b} r(z)dz + \int_0^{u-y+b} r(z)dz}) = E(e^{2\int_0^{u-y+b} r(z)dz + \int_{u-y+b}^{t-x+b} r(z)dz}) = e^{(u-y+b)p + (t-x-u+y)r} \tag{37}$$

$$EE(P(x,t)\,|\,L_1) = E(L_1\,{}_{x-b}p_b) = {}_{x-b}p_b\,E(L_1) \qquad (38)$$

$$EE(P(y,u)\,|\,L_2) = E(L_2\,{}_{y-b}p_b) = {}_{y-b}p_b\,E(L_2) \qquad (39)$$

$$E[E(P(x,t)\,|\,L_1) * E(P(y,u)\,|\,L_2)] = {}_{x-b}p_b\,{}_{y-b}p_b\,E(L_1L_2) \qquad (40)$$

Substituting from (36) and (37) in (38), (39) and (40), the required co-variance is obtained as (40) – (38)*(39) which, after allowing for the actual number of new entrants at *t = 0*, namely *Ac(b, 0)* and simplification, gives the following result:

$$COV(P(x,t), P(y,u)) = Ac(b,0)e^{rt}\frac{D_x^{(r)}}{D_b^{(r)}}[e^{fu}\frac{D_y^{(f)}}{D_b^{(f)}} - e^{ru}\frac{D_y^{(r)}}{D_b^{(r)}}] \qquad (41)$$

If ***x-t > y-u***, i.e. *P(x,t)* is from an earlier cohort than *P(y,u)*, the expression for the co-variance would become

$$Ac(b,0)e^{ru}\frac{D_y^{(r)}}{D_b^{(r)}}[e^{ft}\frac{D_x^{(f)}}{D_b^{(f)}} - e^{rt}\frac{D_x^{(r)}}{D_b^{(r)}}]$$

It will be noted that if p = 2?, the expression within the square brackets will, in either case, disappear, making the covariance zero.

3.5 Total Active and Retired Population projections: Expected values, variances and co-variances

The expected values of *A(t)* and *R(t)* are not affected by the stochastic nature of new entrants. The formulae already derived in section 2 will therefore hold. However, the expressions for the variances and co-variances will be considerably more complicated. Consider the covariance between *A(t)* and *A(u)*. The general expression for this covariance would be

$$COV(A(t), A(u)) = \int_{a_0(t)}^{a_2(t)}\int_{a_0(u)}^{a_2(u)} COV(Ac(x,t), Ac(y,u))dydx = K + L + M \qquad (42)$$

where *K, L* and *M* are components which are discussed below.

If *Ac(x, t)* derives from an initial population cohort, then it will not be correlated with any other *Ac(y, u)* except if they are both from the same cohort. The component *K* refers to the integral of the co-variances among initial population cohorts, and will be the same as the second member on the right-hand-side of (11). If *Ac(x, t)* is from a new entrant cohort, it will be correlated with any *Ac(y, u)* which is from the same or another new entrant cohort, although the expression for their covariance would depend upon whether they are from the same cohort (equation (34)) or from different cohorts (equation (41)). The component *L* refers to the integral obtained by applying formula (41) over the whole range of integration for new entrant cohorts. The component *M* is a correction to *L* to allow for the correct formula, i.e. (34), in those cases where *Ac(x, t)* and *Ac(y, u)* are from the same cohort. The expressions for *K, L* and *M*, after simplification, allowing for *Ac(b,0)* new entrants at *t = 0*, are as follows (note: $L = L_1 + L_2 - L_3$):

$$K = Ac(b,0)[\int_{c_1(t,u)}^{c_2(t,u)}\frac{Ac(y-u,0)}{Ac(b,0)}\,{}_tp_{y-u}(1 - {}_up_{y-u})dy] \qquad (43)$$

$$L_1 = Ac(b,0)e^{rt}e^{fu}\int_{a_0(t)}^{a_1(t)}\frac{\bar{N}_{y_0}^{(f)} - \bar{N}_{a_1(u)}^{(f)}}{D_b^{(f)}}\frac{D_x^{(r)}}{D_b^{(r)}}dx \qquad (44)$$

$$L_2 = Ac(b,0)e^{ru}e^{ft}\int_{a_0(t)}^{a_1(t)} \frac{\overline{N}^{(r)}_{a_0(u)} - \overline{N}^{(r)}_{y_0}}{D_b^{(r)}} \frac{D_x^{(f)}}{D_b^{(f)}} dx \qquad (45)$$

$$L_3 = Ac(b,0)e^{r(t+u)} \frac{\overline{N}^{(r)}_{a_0(t)} - \overline{N}^{(r)}_{a_1(t)}}{D_b^{(r)}} \frac{\overline{N}^{(r)}_{a_0(u)} - \overline{N}^{(r)}_{a_1(u)}}{D_b^{(r)}} \qquad (46)$$

$$M = Ac(b,0)e^{rt}\int_{c_0(t,u)}^{c_1(t,u)} \frac{D^{(r)}_{y+t-u}}{D_b^{(r)}}(1 - {}_{y-b}p_b)dy \qquad (47)$$

where $\overline{N}_x^{(r)} = \int_x^w D_y^{(r)} dy$. The values of $a_0(t), a_1(t)$ were defined in section 2.4 and the values of $c_0(t,u), c_1(t,u), c_2(t,u)$ were defined in section 2.5. The value of y_0 is x-t+u, subject to the minimum of $a_0(u)$ and the maximum of $a_1(u)$. Demographic projections are illustrated in Appendix 8, Table 1.

K (43) is not affected by ? or p, but it can be shown that L_1, L_2, L_3 and *M* ((44)-(47)) will all increase if either ? or p is increased. Therefore, an increase in ? or p will lead to an increase in *COV(A(t),A(u))*.

It will be noted that when p = 2?, $L_1 + L_2 = L_3$, so that the expression for the co-variance in (42) will reduce to *K*+*M*, which is identical to (11). Eventually, when the initial population has disappeared from the scene (t > ? – b), expression (42) for the co-variance would reduce to *L*+*M*.

The expression (42) is also valid for the variance of *A(t)*, which is obtained by putting $u = t$. Similar expressions also apply to the co-variance between *R(t)* and *R(u)* and that between *R(t)* and *A(u)*. Provided the initial relative age-distribution remains invariant, it will be evident that the coefficients of variation of *A(t)* and *R(t)* will be inversely proportional to the square root of *Ac(b,0)* which means that the lower the initial population, the higher the coefficients of variation.

3.6 Expressions for co-variances in the long-term

For t exceeding specified values, general expressions can be derived for *COV(A(t),A(u))*, *COV(R(t),R(u))*, *COV(R(t),A(u))* and *COV(A(t),R(u))*, where ***u = t*** (see Appendices 4, 5 and 6). In particular, the following expressions, for the variance of *A(t)* and *R(t)* and for the covariance between *R(t)* and *A(t)*, are valid for ***t > ?-b***:

$$VA(t) = Ac(b,0)[P_1e^{pt} - P_2e^{2rt} + P_3e^{rt}] \qquad (48)$$

where

$$P_1 = \int_b^r \frac{D_x^{(f)}}{D_b^{(f)}} \frac{\overline{N}_b^{(r)} - \overline{N}_x^{(r)}}{D_b^{(r)}} dx + \int_b^r \frac{D_x^{(r)}}{D_b^{(r)}} \frac{\overline{N}_x^{(f)} - \overline{N}_r^{(f)}}{D_b^{(f)}} dx$$

$$P_2 = \left[\frac{\overline{N}_b^{(r)} - \overline{N}_r^{(r)}}{D_b^{(r)}}\right]^2$$

$$P_3 = \frac{\overline{N}_b^{(r)} - \overline{N}_r^{(r)}}{D_b^{(r)}} - \frac{\overline{N}_{bb}^{(r)} - \overline{N}_{rr}^{(r)}}{D_{bb}^{(r)}}$$

$$VR(t) = Ac(b,0)[Q_1e^{pt} - Q_2e^{2rt} + Q_3e^{rt}] \qquad (49)$$

where

$$Q_1 = \int_r^v \frac{D_x^{(f)}}{D_b^{(f)}} \frac{\bar{N}_r^{(r)} - \bar{N}_x^{(r)}}{D_b^{(r)}} dx + \int_r^v \frac{D_x^{(r)}}{D_b^{(r)}} \frac{\bar{N}_x^{(f)}}{D_b^{(f)}} dx$$

$$Q_2 = [\frac{\bar{N}_r^{(r)}}{D_b^{(r)}}]^2$$

$$Q_3 = \frac{\bar{N}_r^{(r)}}{D_b^{(r)}} - \frac{\bar{N}_{rr}^{(r)}}{D_{bb}^{(r)}}$$

$$COV(R(t), A(t)) = Ac(b,0)[R_1 e^{pt} - R_2 e^{2rt}] \quad (50)$$

where

$$R_1 = \frac{\bar{N}_b^{(r)} - \bar{N}_r^{(r)}}{D_b^{(r)}} \frac{N_r^{(f)}}{D_b^{(f)}}$$

$$R_2 = \frac{\bar{N}_b^{(r)} - \bar{N}_r^{(r)}}{D_b^{(r)}} \frac{\bar{N}_r^{(r)}}{D_b^{(r)}}$$

It will be noted that if p = 2?, the first two terms of (48) will cancel out and (48) will reduce to the expression for *VA(t)* in (13). Similarly, (49) will reduce to the expression for *VR(t)* in (13), and the expression for the covariance in (50) will reduce to zero.

The expected values *EA(t)* and *ER(t)* being the same as those in (12), it will be seen the coefficients of variation of *A(t)* and *R(t)* are given by the expressions

$$CVA^2(t) = \frac{1}{Ac(b,0)} [\frac{P_1}{P_2} e^{(p-2r)t} - 1 + \frac{P_3}{P_2} e^{-rt}] \quad (51)$$

$$CVR^2(t) = \frac{1}{Ac(b,0)} [\frac{Q_1}{Q_2} e^{(p-2r)t} - 1 + \frac{Q_3}{Q_1} e^{-rt}] \quad (52)$$

In addition, if *Corr(R(t),A(t))* denotes the correlation coefficient between *A(t)* and *R(t)*, the ratio of *COV(R(t),A(t))* to the product *EA(t)ER(t)* will be given by

$$Corr(R(t), A(t)) CVA(t) CVR(t) = \frac{1}{Ac(b,0)} [\frac{R_1}{R_2} e^{(p-2r)t} - 1] \quad (53)$$

It will be noted that when p > 2?, all the three expressions will diverge to infinity as *t* tends to infinity. For a given *t*, a decrease in the initial population size will lead to an increase in all the three coefficients of variation. It can be shown that an increase in p will have a similar effect.

3.7 The Demographic Ratio

Considering the Demographic Ratio *DR(t)*, unlike the case in section 2.7, *R(t)* and *A(t)* will now be correlated. If, in addition to what was ignored in section 2.7, the second and higher powers of *Corr(R(t),A(t))CVA(t)CVR(t)* and correlations between *R(t)* and *A(t)* of order higher than one are ignored, the following approximate expressions can be derived (see Appendix 1, Note 3):

$$EDR(t) = \frac{ER(t)}{EA(t)} [1 + CVA^2(t) - Corr(R(t), A(t)) CVA(t) CVR(t)] \quad (54)$$

$$VDR(t) = \frac{ER^2(t)}{EA^2(t)} [CVR^2(t) + CVA^2(t) - 2Corr(R(t), A(t)) CVA(t) CVR(t)] \quad (55)$$

Unlike section 2.7, these two expressions, individually, will diverge to infinity as *t? 8*. However, the coefficient of variation *CVDR(t)* can be shown to tend to zero as *t? 8*. For a given *t, CVDR(t)* will increase with decreasing initial population size and with increasing p. These effects are illustrated numerically in Appendix 8 (Tables 6 and 7).

4. FINANCIAL PROJECTIONS (STATIC SALARY BASIS)

4.1 Notation and assumptions

As in classic projections, it is assumed that the growth of an individual's salary is governed by a salary escalation due to the increase in the general level of earnings, on top of the progression due to age/seniority along a salary scale. By static salary basis, it is meant that salary escalation is not taken into account directly in the projections (although the salary scale effect is). The assumption is also made that pensions are fully adjusted to salary escalation. In this circumstance, the analysis of ratios between benefit and salary projections (e.g. pay-as-you-go premiums) will be equivalent to that based on projections allowing directly for salary escalation. In fact, salary escalation is allowed for, indirectly, in the assumption concerning the investment return (see section 5, below).

The average annual salary of the *Ac(x, t)* persons aged *x*, at time *t*, is denoted by *sa(x, t)* and the annual pension amount of the *Re(x, t)* pensioners is denoted by *pe(x, t)*. *sa(x, 0)* at the start of the scheme is assumed to be given. The average annual salary at entry of the *Ac(b, 0)* persons entering at *t = 0* is *sa(b, 0)*. The total annual salary of the *Ac(x,t)* persons is denoted by *Sa(x, t)* and the total annual pensions of the *Re(x, t)* retired persons is denoted by *Be(x, t)*. Pensions are assumed to be based on the final salary at retirement and the proportionate pension rate for entry age *x* is denoted by *k(x)*.

Individual salaries progress according to the salary scale *s(x)*, *b = x = r*, which is assumed to remain constant over time. Since salary escalation over time is not directly taken into account, the salary at entry for all new entrant cohorts would be *sa(b, 0)*.

4.2 Expected values, variances and co-variances

The following relationships hold between the financial and demographic projection elements:

$$Sa(x,t) = Ac(x,t)sa(x,t); Be(x,t) = \mathrm{Re}(x,t)\,pe(x,t) \qquad (56)$$

where, for the initial population cohorts

$$sa(x,t) = sa(x-t,0)\frac{s(x)}{s(x-t)}; pe(x,t) = sa(x-t,0)\frac{s(r)}{s(x-t)}k(x-t) \qquad (57)$$

and for the new entrant cohorts

$$sa(x,t) = sa(b,0)\frac{s(x)}{s(b)}; pe(x,t) = sa(b,0)\frac{s(r)}{s(b)}k(b) \qquad (58)$$

The expected values of the financial projection elements are therefore related to those of the corresponding demographic projection elements as follows:

$$ESa(x,t) = sa(x,t)EAc(x,t); EBe(x,t) = pe(x,t)E\,\mathrm{Re}(x,t) \qquad (59)$$

Based on (9), this leads to the following expressions for *ES(t)* and *EB(t)*:

$$ES(t) = Sa(b,0)[\int_{a_1(t)}^{a_2(t)} \frac{Sa(x-t,0)}{Sa(b,0)} \frac{s(x)}{s(x-t)}\, {}_t p_{x-t}\, dx + e^{rt} \int_{a_0(t)}^{a_1(t)} \frac{D_x^{s(r)}}{D_b^{s(r)}} dx] \qquad (60)$$

$$EB(t) = Sa(b,0)[\int_{a_1(t)}^{a_2(t)} \frac{Sa(x-t,0)}{Sa(b,0)} k(x-t) \frac{s(r)}{s(x-t)}\, {}_t p_{x-t}\, dx + k(b)e^{rt} \int_{a_0(t)}^{a_1(t)} \frac{s(r)D_x^{(r)}}{D_b^{s(r)}} dx] \qquad (61)$$

where $D_x^{s(r)} = s(x)D_x^{(r)}$.

The co-variances of the financial projection elements are related to those of the corresponding demographic projection elements as follows:

$$COV(Sa(x,t), Sa(y,u)) = sa(x,t)sa(y,u)COV(Ac(x,t), Ac(y,u)) \quad (62)$$

$$COV(Be(x,t), Be(y,u)) = pe(x,t)pe(y,u)COV(\mathrm{Re}(x,t), \mathrm{Re}(y,u)) \quad (63)$$

$$COV(Be(x,t), Sa(y,u)) = pe(x,t)sa(y,u)COV(\mathrm{Re}(x,t), Ac(y,u)) \quad (64)$$

These adjustments should be incorporated in (42) to (47), as appropriate, to obtain the expressions for the various co-variances of the financial projections. In order to illustrate this, the expression for *COV*(S(*t*),*S*(*u*)), *u* = *t*, is indicated below:

$$COV(S(t), S(u)) = K + L_1 + L_2 - L_3 + M \quad (65)$$

$$K = Sa(b,0)[\int_{c_1(t,u)}^{c_2(t,u)} \frac{Sa(y-u,0)}{Sa(b,0)} sa(y-u,0) \frac{s(y+t-u)s(y)}{s(y-u)^2} {}_t p_{y-u}(1 - {}_u p_{y-u})dy] \quad (66)$$

$$L_1 = Sa(b,0)sa(b,0)e^{rt}e^{fu} \int_{a_0(t)}^{a_1(t)} \frac{\bar{N}_{y_0}^{s(f)} - \bar{N}_{a_1(u)}^{s(f)}}{D_b^{s(f)}} \frac{s(r)D_x^{(r)}}{D_b^{s(r)}} dx \quad (67)$$

$$L_2 = Sa(b,0)sa(b,0)e^{ru}e^{ft} \int_{a_0(t)}^{a_1(t)} \frac{\bar{N}_{a_0(u)}^{s(r)} - \bar{N}_{y_0}^{s(r)}}{D_b^{s(r)}} \frac{s(r)D_x^{(f)}}{D_b^{s(f)}} dx \quad (68)$$

$$L_3 = Sa(b,0)sa(b,0)e^{r(t+u)} \frac{\bar{N}_{a_0(t)}^{s(r)} - \bar{N}_{a_1(t)}^{s(r)}}{D_b^{s(r)}} \frac{\bar{N}_{a_0(u)}^{s(r)} - N_{a_1(u)}^{s(r)}}{D_b^{s(r)}} \quad (69)$$

$$M = Sa(b,0)sa(b,0)e^{rt} \int_{c_0(t,u)}^{c_1(t,u)} \frac{D_{y+t-u}^{(r)}}{D_b^{(r)}} \frac{s(y+t-u)s(y)}{s(b)^2} (1 - {}_{y-b}p_b)dy \quad (70)$$

where $\bar{N}_x^{s(r)} = \int_x^w D_y^{s(r)} dy$. Expressions for *COV(B(t),B(u))* and *COV(B(t),S(u))* and for *COV(S(t),B(u)*, *u* = *t*, can be developed on similar lines (see Appendix 3). The formula for *COV(S(t),S(u))* is also valid for *VS(t)*, with *t* = *u*, and the formula for *COV(B(t),B(u))* is valid for *VB(t).* Financial projections are illustrated in Appendix 8, Tables 2 and 3.

Expressions (48) to (53) concerning the demographic projections in the mature situation can be adapted to the financial projections and will lead to similar expressions. The analysis of the effects of the parameters ? and p and of the size of the initial population will be similar to that in sections 3.5 and 3.6 and will lead to the same conclusions.

4.3 The capitalized value of pension awards

Let *Bc(t)* denote the capitalized value of the pensions awarded to those retiring at time t, and *a** the capitalized value of an indexed unit pension at age *r*. Then,

$$Bc(t) = c(t)Ac(r,t)a*$$

where $c(t) = sa(r-t,0)\frac{s(r)}{s(r-t)}k(r-t)$ if $t < r-b$; $c(t) = sa(b,0)\frac{s(r)}{s(b)}k(b)$ if $t = r-b$.

$$Ea* = \bar{a}_r^{(d)}; \; Va* = 2\frac{\bar{a}_r^{(d)} - \bar{a}_r^{(a)}}{(a-d)} - (\bar{a}_r^{(d)})^2 \text{ (see Appendix 1, Note 5)}$$

c(t) is deterministic and *a** and *Ac(r, t)* can be regarded as independent. Therefore,

$$EBc(t) = c(t)EAc(r,t)Ea* \quad (71)$$

From Appendix 1, Note 6, it follows that

$$VBc(t) = c^2(t)[VAc(r,t)(Va*+E^2a*) + Va*E^2 Ac(r,t)] \qquad (72)$$

EAc(r, t) given by (1) or (4), and *VAc(r, t)* by (2) or (26), depending upon whether t < or > $r - b$, can be substituted in the above expression..
The co-variance between *Bc(t)* and *S(t)* will be zero if $t < r - b$. If $t > r - b$,

$$COV(Bc(t), S(t)) = Ea*k(b)\int_b^r sa(b,0)^2 \frac{s(r)s(x)}{s(b)^2} COV(Ac(r,t), Ac(x,t))dx$$

Substituting for *COV(Ac(r, t), Ac(x, t))* from the expression following (41) and simplifying, for $t > r\text{-}b$,

$$COV(Bc(t), S(t)) = Ac(b,0)Ea*sa(b,0)^2 k(b)[e^{pt}\frac{D_r^{s(f)}}{D_b^{s(f)}} - e^{2rt}\frac{D_r^{s(r)}}{D_b^{s(r)}}]\frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}} \qquad (73)$$

5. DISCOUNTED VALUES OF SALARY AND BENEFIT PROJECTIONS

Discounting of projections is relevant in the context of funded financial systems. The following development is based on the **force of "real interest"**, defined here as nominal interest less salary escalation. By using this force of interest to discount the projections on the **static salary basis,** salary escalation is effectively taken into account.

5.1 Notation and assumptions

In the classical approach, the force of real interest, denoted by **d**, would be assumed to be a constant throughout the period of projections (or taking different constant values over successive sub-intervals). In this paper, however, the force of real interest *d(t)* is assumed to be distributed such that $e^{-d(z)}$ for different values of z are independent, identically distributed (IID) variables with expected value and second raw moment given by

$$E(e^{-d(z)}) = e^{-d}; E(e^{-2d(z)}) = e^{-a} \qquad (74)$$

so that the variance is given by

$$V(e^{-d(z)}) = e^{-a} - e^{-2d} \qquad (75)$$

For the variance to be positive, we should have **a < 2d**. If a = 2d the variance is zero, meaning that the interest factor is deterministic. The symbol **?** is used to denote the difference **a – d**.

5.2 Case of deterministic projections

We shall first consider the case where the projections themselves are deterministic, so that only the interest element is stochastic. Let *S(t)* and *B(t)* denote the salary and benefit projections on a **static salary basis**, ignoring salary escalation. It is assumed that pensions are fully indexed to salary escalation. Let *DS(t)* and *DB(t)* denote the values of *S(t)* and *B(t)* discounted to time $t = 0$, and *TDS(t)* and *TDB(t)* the totals of the discounted values over $(0, t)$.

In view of the IID properties described in section 5.1,

$$EDS(t) = E(S(t)e^{-\int_0^t d(z)dz}) = S(t)e^{-dt}; EDS^2(t) = E(S(t)^2 e^{-2\int_0^t d(z)dz}) = S(t)^2 e^{-at} \qquad (76)$$

$$ETDS(t) = E[\int_0^t S(u)e^{-\int_0^u d(z)dz} du] = \int_0^t S(u)e^{-du}du \qquad (77)$$

$$VDS(t) = EDS^2(t) - [EDS(t)]^2 = S(t)^2[e^{-at} - e^{-2dt}] \qquad (78)$$

$$COV(DS(t), DB(t)) = S(t)B(t)[e^{-at} - e^{-2dt}] \qquad (79)$$

Squaring *TDS(t)* we have

$$TDS^2(t) = \int_0^t S(u)e^{-\int_0^u d(z)dz} du \int_0^t S(v)e^{-\int_0^v d(z)dz} dv = 2\int_0^t S(u)e^{-\int_0^u d(z)dz} du \int_0^u S(v)e^{-\int_0^v d(z)dz} dv$$

Noting that $u > v$, the above expression can be simplified as

$$TDS^2(t) = 2\int_0^t S(u)\int_0^u S(v)e^{-2\int_0^v d(z)dz - \int_v^u d(z)dz} dvdu$$

Taking expectations we have

$$ETDS^2(t) = 2\int_0^t S(u)e^{-du} du \int_0^u S(v)e^{-(a-d)v} dv \qquad (80)$$

Therefore the variance of *TDS(t)* is given by

$$VTDS(t) = 2\int_0^t S(u)e^{-du}\int_0^u S(v)e^{-qv} dvdu - [\int_0^t S(u)e^{-du} du]^2 \qquad (81)$$

The above formula can also be expressed as

$$VTDS(t) = 2\int_0^t S(u)e^{-du}\int_0^u S(v)[e^{-qv} - e^{-dv}]dvdu \qquad (82)$$

It can similarly be shown that the covariance between *TDS(t)* and *TDB(t)* is given by

$$\int_0^t S(u)e^{-du}\int_0^u B(v)[e^{-qv} - e^{-dv}]dvdu + \int_0^t B(u)e^{-du}\int_0^u S(v)[e^{-qv} - e^{-dv}]dvdu \qquad (83)$$

It can be shown that when a = 2d, i.e. the interest element is deterministic, the expression for *VTDS(t)* in (82) will vanish. Two further useful results are the following:

$$COV(TDB(t), DB(t)) = B(t)e^{-dt}\int_0^t B(u)[e^{-qu} - e^{-du}]du \qquad (84)$$

$$COV(TDS(t), DB(t)) = B(t)e^{-dt}\int_0^t S(u)[e^{-qu} - e^{-du}]du \qquad (85)$$

5.3 Case of stochastic projections

In developing the formulae for this case, the assumption is made that the interest element is independent of the mortality and new entrant elements. Consider first, *DS(t).*

$$DS(t) = S(t)e^{-\int_0^t d(z)dz} = S(t)\Phi(t) \qquad (86)$$

S(t) depends on the mortality and new entrant elements, while F(t) depends on the interest element. Therefore they are independent of each other.

$$EDS(t) = ES(t)E\Phi(t) = e^{-dt}ES(t) \qquad (87)$$

where *ES(t)* is given by (60). *VDS(t)* is derived as follows:

$$VDS(t) = ES^2(t)E\Phi^2(t) - (ES(t))^2(E\Phi(t))^2 = ES^2(t)e^{-at} - E(S(t))^2 e^{-2dt}$$

$$VDS(t) = VS(t)e^{-at} + (ES(t))^2[e^{-at} - e^{-2dt}] \qquad (88)$$

ETDS(t) and *VTDS(t)* are derived as follows:

$$ETDS(t) = E\int_0^t DS(u)du = \int_0^t EDS(u)du = \int_0^t ES(u)e^{-du}du \qquad (89)$$

$$VTDS(t) = \int_0^t\int_0^t COV(DS(u), DS(v))dvdu = 2\int_0^t\int_0^u COV(DS(u), DS(v))dvdu$$

$$= 2[\int_0^t\int_0^u E[DS(u)DS(v)]dvdu - \int_0^t\int_0^u EDS(u)EDS(v)dvdu] = 2[A - B]$$

$$E[DS(u)DS(v)] = E[\Phi(u)S(u)\Phi(v)S(v)] = E[\Phi(u)\Phi(v)]E[S(u)S(v)]$$

$$E[\Phi(u)\Phi(v)] = E[e^{-\int_0^u d(z)dz}e^{-\int_0^v d(z)dz}] = E[e^{-2\int_0^v d(z)dz - \int_v^u d(z)dz}] = e^{-av - d(u-v)} = e^{-du}e^{-(a-d)v}$$

$$A = \int_0^t e^{-du} \int_0^u e^{-(a-d)v} E(S(u)S(v))dvdu$$

$$B = \int_0^t e^{-du} \int_0^u e^{-dv} ES(u)ES(v)dvdu$$

$$VTDS(t) = 2[\int_0^t e^{-du} \int_0^u e^{-qv} COV(S(u), S(v))dvdu + \int_0^t e^{-du} ES(u)du \int_0^u (e^{-qv} - e^{-dv}) ES(v)dv] \quad (90)$$

When *S(t)* is deterministic, *COV(S(u),S(v))* = 0 and the first member of (90) will vanish so that (90) reduces to (82). When a = 2d, i.e. the interest element is deterministic, the second member of (90) will vanish and (90) will reduce to

$$VTDS(t) = 2\int_0^t e^{-du} \int_0^u e^{-dv} COV(S(u), S(v))dvdu$$

The expression for *VTDB(t)* will be similar to (90). It can be shown on the same lines that the covariance between *TDS(t)* and *TDB(t)* will be given by

$$\int_0^t e^{-du} \int_0^u e^{-qv} COV(S(u), B(v))dvdu + \int_0^t e^{-du} ES(u)du \int_0^u (e^{-qv} - e^{-dv}) EB(v)dv$$

$$+ \int_0^t e^{-du} \int_0^u e^{-qv} COV(B(u), S(v))dvdu + \int_0^t e^{-du} EB(u)du \int_0^u (e^{-qv} - e^{-dv}) ES(v)dv \quad (91)$$

Two further useful results are:

$$COV(DB(t), TDB(t)) = e^{-dt} \int_0^t [e^{-qu} COV(B(t), B(u)) + (e^{-qu} - e^{-du}) EB(t)EB(u)]du \quad (92)$$

$$COV(DB(t), TDS(t)) = e^{-dt} \int_0^t [e^{-qu} COV(B(t), S(u)) + (e^{-qu} - e^{-du}) EB(t)ES(u)]du \quad (93)$$

Discounted financial projections are illustrated in Appendix 8, Table 4.

Let the two terms of (90) be indicated by *VTDS1(t)* and *VTDS2(t)*. An increase in either d or a will lead to a decrease in either term and therefore in *VTDS(t)*. *VTDS1(t)* involves the factor *Ac(b,0)* whereas *VTDS2(t)* involves $Ac(b,0)^2$ and *VTDS2* will predominate, so that the co-variances between *S(t)* and *S(u)* will not have much effect on *VTDS(t)* unless the initial population is relatively small. The same conclusions apply to *VTDB(t)* and *COV(TDS(t),TDB(t))*. For a given *t*, the coefficients of variation *CVTDS(t)* and *CVTDB(t)* will increase with decreasing initial population size and with increasing p or decreasing a. These effects will be reflected in the stochastic premiums (see section 6).

5.4 Convergence of the expressions in the long-term

Based on the expressions for *COV(S(t),S(u))* etc. – see section 4.2 and Appendix 3 – long-term expressions for these co-variances can be derived and further, long-term expressions for the variances of and the co-variances between totals of discounted values can be developed (see Appendices 4, 5 and 6).

The development in Appendix 4 shows that, for any *t* exceeding a sufficiently large value t_0 (> *2(r-b)*)

$$VTDS(t) - VTDS(t_0) = F(t_0) - F(t) \quad (94)$$

F(t) is given by an expression of the form,

$$F(t) = F_1 e^{-(d-r)t} + F_2 e^{-(a-p)t} + F_3 e^{-(a-2r)t} + F_4 e^{-(a-r)t} + F_5 e^{-2(d-r)t} \quad (95)$$

Given the conditions (a) p > 2? and (b) a < 2d, the further conditions required for the expression in (95) to tend to 0 as *t* tends to 8 are:

(c) d > ? and (d) a > p.

For, (a) and (d) imply that a > 2?. The limiting value of *VTDS(t)* will then be given by,

$$VTDS(\infty) = VTDS(t_0) + F(t_0) \quad (96)$$

It turns out that $VTDB(t) - VTDB(t_0)$ (for $t_0 > (r-b)+2(v-r)$) and $Cov(TDB(t),TDS(t)) - Cov(TDB(t_0),TDS(t_0))$ (for $t_0 > 2(r-b)+(v-r)$) too have a structure similar to (94) and (95) (see Appendices 5 and 6), and will converge as in (96) under the same conditions. Further, based on (89), it can be shown that, *ETDS(t) – ETDS(* t_0 *)* (for $t_0 > r-b$), and *ETDB(t) – ETDB(* t_0 *)* (for $t_0 > v-b$), respectively take the forms *P(* t_0 *) – P(t)* and *Q(* t_0 *) – Q(t)*, where *P(t)* and *Q(t)* are given by,

$$P(t) = \frac{Sa(b,0)}{d-r}\frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}}e^{-t(d-r)}; Q(t) = \frac{Sa(b,0)}{d-r}\frac{\overline{N}_r^{s(r)}}{D_b^{s(r)}}k(b)e^{-t(d-r)} \quad (97)$$

It follows that *ETDS(t)* and *ETDB(t)* will converge as *t ? 8* subject to condition © above (d > ?).

6 APPLICATION OF STOCHASTIC METHODS IN THE FINANCING OF SOCIAL SECURITY PENSION SCHEMES

6.1 General principles

The classical method of determining premiums for any given financial system (see Iyer, 1999, chapter 1) is on the basis of deterministic projections, which is effectively in terms of expected values. Any margins which are then added to absorb the effect of chance variations are necessarily arbitrary. However, since expressions for the variances and co-variances are now available, an improved method can be applied incorporating the appropriate adjustment for chance variations.

Any financial system essentially seeks a level premium, which, when applied to a function *f(S)* of projected salaries, will balance another function *g(B)* of projected benefits. The classical method determines the premium *pr* by the formula

$$pr = \frac{Eg(B)}{Ef(S)} \quad (98)$$

Following the method outlined in Booth et. al., 1999 (p. 640) an alternative approach (based on the so-called *percentile* principle) is to consider a variable *u* defined by

$$u = kf(S) - g(B) \quad (99)$$

An assumption is required concerning the statistical distribution of *u*. The normal distribution could possibly be justified on the basis of the central limit theorem, when the population size is sufficiently large. The premium *pr** (referred to in the sequel as the stochastic premium) is determined such that the probability of failure of the financial system (i.e. that *u* will be negative) is less than a pre-determined e. *pr** will be given by the larger root of the equation

$$E^2(u) - y_e^2 V(u) = 0 \quad (100)$$

where, y_e is the $(1-e)^{th}$ percentile of the appropriate distribution.
Alternatively, given any premium *pr***, the probability that its application will lead to the failure of the financial system can be determined.

6.2 The pay-as-you-go system

In the framework of the continuous formulation in this paper, the pay-as-you-go system aims at balancing the income and expenditure at a given point in time. Given projections established at *t=0*, the relevant functions for the pay-as-you-go premium at time *t=n* are

$$f(S) = S(n); g(B) = B(n)$$

The classical premium is given by

$$PAYG(n) = \frac{ES(n)}{EB(n)}$$

The stochastic premium *PAYG*(n)* is given by the larger root of (100), where

$$E(u) = kES(n) - EB(n); V(u) = k^2VS(n) + VB(n) - 2kCOV(S(n), B(n)) \qquad (101)$$

6.3 The Average Premium system

The Average Premium system aims to balance expenditure over successive intervals of several years through the application of a level premium over each interval, allowing for any accumulated fund at the start of the interval. The relevant functions corresponding to the interval (*0, n*), starting with a fund *F(0)*, are

$$f(S) = TDS(n); g(B) = TDB(n) - F(0)$$

The classical premium is given by

$$AVP(0,n) = \frac{ETDB(n) - F(0)}{ETDS(n)}$$

The stochastic premium is the larger root of (100), where

$$E(u) = F(0) + kETDS(n) - ETDB(n)$$

$$V(u) = k^2VTDS(n) + VTDB(n) - 2kCOV(TDS(n), TDB(n)) \qquad (102)$$

6.4 The Reserve Ratio system

The Reserve Ratio system applies a level premium over an interval of years such that a Fund is accumulated bearing a specified ratio to the level of the expenditure at the end of the period. For the interval (*0, n*), the relevant functions, allowing for a reserve ratio ? and an initial fund *F(0)* are

$$f(S) = TDS(n); g(B) = TDB(n) + \Lambda DB(n) - F(0)$$

The classical premium is given by

$$RRP(0, n, \Lambda) = \frac{ETDB(n) + \Lambda EDB(n) - F(0)}{ETDS(n)}$$

The stochastic premium is the larger root of (100), where

$$E(u) = F(0) + kETDS(n) - ETDB(n) - \Lambda EDB(n)$$

$$V(u) = k^2VTDS(n) + VTDB(n) + \Lambda^2VDB(n) - 2kCOV(TDS(n), TDB(n)) \qquad (103)$$

$$-2k\Lambda COV(TDS(n), DB(n)) + 2\Lambda COV(TDB(n), DB(n))$$

6.5 The Terminal Funding system

In the framework of the continuous formulation in this paper, the Terminal Funding system fully capitalizes the pensions awarded at any time *t* by the corresponding contribution income. For determining the Terminal Funding premium at *t = n*, based on projections established at *t = 0*, the relevant functions are

$$f(S) = S(n); g(B) = Bc(n)$$

The classical method is based on the formula

$$TFP(n) = \frac{EBc(n)}{ES(n)} \quad (104)$$

The stochastic premium would be the larger root of (100), where

$$E(u) = kES(n) - EBc(n) \quad (105)$$

$$V(u) = k^2 VS(n) + VBc(n) - 2kCOV(Bc(n), S(n)) \quad (106)$$

6.6 The General Average Premium system

The General Average Premium system is based on the concept of a constant premium that will ensure the financial equilibrium of a pension scheme throughout the infinite lifetime of the scheme following an actuarial valuation (at *t = 0*), taking credit for the accumulated fund. The General Average Premium system can be regarded as the limiting form of the Average Premium system, when *n ? 8* . Provided the necessary conditions for convergence are met (i.e. d > ? and a > p – see section 5.4, above), the formulae and the procedure of section 6.3 above can be applied, by substituting *ETDS(8)* for *ETDS(n)*, *VTDS(8)* for *VTDS(n)*, etc.

6.7 Sensitivity of the premiums to population size and parameters

In contrast to the classical premiums, the stochastic premiums are affected by the size of the initial population, being inversely related to the population size. However, the sensitivity to a change in the population size is considerably higher for the Pay-as-you-go and Terminal Funding systems, as compared to the other systems. This is to be explained by the fact that the projected values (on which the *PAYG* and *TF* systems depend directly) are much more sensitive to population size changes than the totals of discounted projected values (to which the other systems relate). In all cases, the stochastic premium tends to the classical premium as the population size tends to infinity.

As regards the effect of the parameters, the classical premiums are affected by ? and d but are independent of the secondary parameters p and a. Stochastic *PAYG* and *TFP* premiums are affected by p, with an increase in p leading to an increase in the premiums, but are not affected by a. Stochastic *AVP* and *RRP* premiums are affected by both p and a, increasing with increasing p and decreasing with increasing a, although the effect of changing p is relatively marginal. These effects of changing population size and changing parameters on the stochastic premiums are illustrated in Appendix 8, Tables 5 to 8.

7. CONCLUSION

The traditional, deterministic, approach to the actuarial valuation of social security pension schemes has continued practically unchanged to this day. This paper has explored the possibility of applying stochastic methods, drawing on the techniques being applied in other areas of insurance but tailoring them to the special characteristics of social security pensions. Although it is based on certain simplifying assumptions, the paper has shown that it is feasible to apply stochastic methods in this area as well.

The paper has brought out several interesting features of the stochastic aspects of social security pension projections. It has demonstrated in particular that deterministic projections can be highly misleading when the population size is small. Fortunately most social security pension schemes, being national in scope, have high coverage in terms of numbers of individuals. Nevertheless, the paper has shown that stochastic methods can throw considerably more light on the evolution of a social security pension scheme and therefore enable more informed policy decisions to be taken.

The paper has been largely theoretical and further work is certainly required, in particular, to facilitate practical application. It is hoped that the paper will succeed in drawing attention to this potential new area of application of stochastic methods, stimulate interest in further research and eventually, lead to the adoption of stochastic methods in social security pension scheme valuations.

REFERENCES

(1) Booth, P.; Chadburn, R.; Cooper, D.; Haberman, S.; James, D. 1999. *Modern actuarial theory and practice* (London, C.R.C. Press).
(2) Daykin, C. D.; Pentikainen, T.; Pesonen, M. 1994. *Practical risk theory for actuaries* (London, Chapman and Hall).
(3) Iyer, S. 1999. *Actuarial mathematics of social security pensions* (Geneva, International Labour Office).
(4) Plamondon, P.; Drouin, A.; Binet, G.; Cichon, M.; McGillivray, W. R.; Bedard, M.; Perez-Montas, H. 2002. *Actuarial practice in social security* (Geneva, International Labour Office).

APPENDIX 1
MATHEMATICAL NOTES

Note 1 (sections 2.2, 2.3)

Consider a cohort of size N at age z, which is subject to a given life table $\{l_x\}$. Let *N(y)* and *N(x)* denote the survivors to ages y and x ($z < y < x$), where y and x are not necessarily integral. Then *N(y)* has the binomial distribution with parameters N and $p = \frac{l_y}{l_z}$ and we have

$$E(N(y)) = Np; V(N(y)) = Np(1-p)$$

Let $p' = \frac{l_x}{l_z}$ and $p'' = \frac{l_x}{l_y}$. Then $p' = pp''$.

$$E(N(x)) = Np'; V(N(x)) = Np'(1-p')$$

$$E(N(x) \mid N(y)) = N(y)p''$$

$$E(N(x) * N(y)) = E(N(y) * E(N(x) \mid N(y)) = E(N^2(y)p'')$$

$$= p''[V(N(y)) + E^2(N(y))] = p''[Np(1-p) + N^2p^2] = Npp''(1-p) + N^2p^2p''$$

$$= Np'(1-p) + Np * Np' = Np'(1-p) + E(N(x))E(N(y))$$

Therefore the covariance between *N(x)* and *N(y)* is given by

$$COV(N(x), N(y)) = Np'(1-p) = N\frac{l_x}{l_z}(1-\frac{l_y}{l_z})$$

Note 2 (section 2.3)

A parallel can be drawn with the distinction between u and v, defined by

$$u = nx; v = \sum_{i=1}^{n} x_i$$

where the x_i's are independent variables having the same distribution as x. We have

$$E(u) = E(v) = nE(x)$$

But the variances are not equal. They are given by

$$V(u) = n^2V(x); V(v) = nV(x)$$

Note 3 (sections 2.7 and 3.7)

Let x and y be two variables and let $u = \frac{x}{y}$. Let the main parameters of x and y be:

$$E(x) = m; V(x) = s^2; E(y) = n; V(y) = q^2; COV(x, y) = k$$

Approximate expressions for *E(u)* and *V(u)* can be derived as follows:

$$E(u) = E(\frac{x}{y}) = E(\frac{x-m+m}{y-n+n}) = \frac{m}{n}E(1+\frac{x-m}{m})(1+\frac{y-n}{n})^{-1} = \frac{m}{n}(1-\frac{k}{mn}+\frac{q^2}{n^2})$$

$$E(u^2) = \frac{m^2}{n^2}E(1+\frac{x-m}{m})^2(1+\frac{y-n}{n})^{-2} = \frac{m^2}{n^2}(1+\frac{s^2}{m^2}+3\frac{q^2}{n^2}-4\frac{k}{mn})$$

$$V(u) = E(u^2) - E^2(u) = \frac{m^2}{n^2}(\frac{s^2}{m^2}+\frac{q^2}{n^2}-2\frac{k}{mn})$$

Note 4 (sections 3.3 and 3.4)

Let *x*, *y* be two variables, either being dependant on *z*. Then, the conditional co-variance between *x* and *y* is given by,

$$Cov(x, y| z) = E(xy | z) - E(x | z) E(y | z)$$

Taking the expectation over the range of *z*,

$$ECov(x, y| z) = EE(xy | z) - E[E(x| z) E(y| z)]$$

The co-variance between the conditional expectations is given by

$$Cov(E(x | z), E(y | z)) = E[E(x| z) E(y| z)] - EE(x | z) EE(y| z)$$

By adding the two results, and noting that *EE* signifies unconditional expectation, it follows that

$$Cov(x, y) = ECov(x, y| z) + Cov(E(x | z), E(y | z))$$

The above result is valid even if *z* represents a group of variables. Therefore, if *x* depends on *z* and *y* depends on *w*,

$$Cov(x, y) = ECov(x, y| z, w) + Cov(E(x| z), E(y | w))$$

Note 5 (section 4.3)

These formulae for *Ea** and *Va** correspond to formulae (24.60) and (24.62) on pp. 644-645 of Booth et. al., 1999, but relate to the continuous formulation adopted in this paper. When a = 2d, the formula for *Va** reduces to formula (24.32) on p.629 of Booth et. al.

Note 6 (section 4.3)

Let *x* and *y* be two independent variables, and let *u* = *xy*.

$$E(xy) = E(x)E(y)$$

$$V(u) = E(u^2) - E^2(u) = E(x^2 y^2) - E^2(x)E^2(y)$$

Since *x* and *y* are independent, so are their squares. Therefore,

$$E(x^2 y^2) = E(x^2) E(y^2) = (V(x) + E^2(x))(V(y) + E^2(y))$$

It follows that

$$V(u) = V(x)V(y) + V(x)E^2(y) + V(y)E^2(x)$$

APPENDIX 2
LIMITS OF INTEGRATION $c_0(t,u)$, $c_1(t,u)$ AND $c_2(t,u)$ FOR SIMPLIFIED PENSION SCHEME

Co-variants	Range of t	Range of u	Range NE $c_0, c_1(t,u)$	Range IP $c_1, c_2(t,u)$
A(t) and *A(u)*	*t < r-b*	*u < t*	*b, b+u*	*b+u, r-t+u*
	t = r-b	*u < t-r+b*	*nil*	*nil*
		t > u = t-r+b	*b, r-t+u*	*nil*
R(t) and *R(u)*	*t < ?-r*	*u < t*	*nil*	*r, r+u*
	?-r = t < r-b	*t > u = t-?+r*	*nil*	*r, ?-t+u*
		u < t-? +r	*nil*	*nil*
	r-b = t < ?-b	*t > u = r-b*	*r, b+u*	*b+u, ?-t+u*
		r-b > u > t-? +r	*nil*	*r, ?-t+u*
		u < t-? +r	*nil*	*nil*
	t = ?-b	*t > u = t-?+r*	*r, ?-t+u*	*nil*
		u < t-? +r	*nil*	*nil*
R(t) and *A(u)*	*t < ?-r*	*u < t*	*nil*	*r-t+u, r*
	?-r = t < r-b	*t > u = t-?+r*	*nil*	*r-t+u, r*
		u < t-? +r	*nil*	*r-t+u, ?-t+u*
	r-b = t < ?-b	*t > u =r-b*	*r-t+u, r*	*nil*
		r-b > u > t-? +r	*r-t+u, b+u*	*b+u, r*
		t-r+b = u < t-?+r	*r-t+u, b+u*	*b+u, ?-t+u*
		u < t-r+b	*b, b+u*	*b+u, ?-t+u*
	t = ?-b	*t > u = t-?+r*	*r-t+u, r*	*nil*
		t-r+b = u < t-?+r	*r-t+u, ?-t+u*	*nil*
		t-? +b = u < t-r+b	*b, ?-t+u*	*nil*
		u < t-? +b	*nil*	*nil*

APPENDIX 3
EXPRESSIONS FOR *COV(B(t), B(u))* AND *COV(S(t),B(u))*

Expression for *COV(B(t),B(u)), u = t*

$$COV(B(t),B(u)) = K + L_1 + L_2 - L_3 + M$$

$$K = Sa(b,0)[\int_{c_1(t,u)}^{c_2(t,u)} \frac{Sa(y-u,0)}{Sa(b,0)} sa(y-u,0) \frac{s(r)^2}{s(y-u)^2} \, {}_t p_{y-u} (1 - {}_u p_{y-u}) k(y-u)^2 dy]$$

$$L_1 = Sa(b,0) sa(b,0) e^{rt} e^{fu} \int_{a_0(t)}^{a_1(t)} \frac{\bar{N}_{y_0}^{(f)} - \bar{N}_{a_1(u)}^{(f)}}{D_b^{s(f)}} \frac{s(r)^2 D_x^{(r)}}{D_b^{s(r)}} k(b)^2 dx$$

$$L_2 = Sa(b,0) sa(b,0) e^{ru} e^{ft} \int_{a_0(t)}^{a_1(t)} \frac{\bar{N}_{a_0(u)}^{(r)} - \bar{N}_{y_0}^{(r)}}{D_b^{s(r)}} \frac{s(r)^2 D_x^{(f)}}{D_b^{s(f)}} k(b)^2 dx$$

$$L_3 = Sa(b,0) sa(b,0) e^{r(t+u)} s(r)^2 \frac{\bar{N}_{a_0(t)}^{(r)} - \bar{N}_{a_1(t)}^{(r)}}{D_b^{s(r)}} \frac{\bar{N}_{a_0(u)}^{(r)} - N_{a_1(u)}^{(r)}}{D_b^{s(r)}} k(b)^2$$

$$M = Sa(b,0) sa(b,0) e^{rt} \int_{c_0(t,u)}^{c_1(t,u)} \frac{D_{y+t-u}^{(r)}}{D_b^{(r)}} \frac{s(r)^2}{s(b)^2} (1 - {}_{y-b} p_b) k(b)^2 dy$$

Expression for *COV(S(t),B(u)), u = t*

$$COV(S(t),B(u)) = K + L_1 + L_2 - L_3 + M$$

$$K = Sa(b,0)[\int_{c_1(t,u)}^{c_2(t,u)} \frac{Sa(y-u,0)}{Sa(b,0)} sa(y-u,0) \frac{s(y+t-u)s(r)}{s(y-u)^2} \, {}_t p_{y-u} (1 - {}_u p_{y-u}) k(y-u) dy]$$

$$L_1 = Sa(b,0) sa(b,0) e^{rt} e^{fu} \int_{a_0(t)}^{a_1(t)} \frac{\bar{N}_{y_0}^{(f)} - \bar{N}_{a_1(u)}^{(f)}}{D_b^{s(f)}} \frac{s(r) D_x^{s(r)}}{D_b^{s(r)}} k(b) dx$$

$$L_2 = Sa(b,0) sa(b,0) e^{ru} e^{ft} \int_{a_0(t)}^{a_1(t)} \frac{\bar{N}_{a_0(u)}^{(r)} - \bar{N}_{y_0}^{(r)}}{D_b^{s(r)}} \frac{s(r) D_x^{s(f)}}{D_b^{s(f)}} k(b) dx$$

$$L_3 = Sa(b,0) sa(b,0) e^{r(t+u)} s(r) \frac{\bar{N}_{a_0(t)}^{s(r)} - \bar{N}_{a_1(t)}^{s(r)}}{D_b^{s(r)}} \frac{\bar{N}_{a_0(u)}^{(r)} - N_{a_1(u)}^{(r)}}{D_b^{s(r)}} k(b)$$

$$M = Sa(b,0) sa(b,0) e^{rt} \int_{c_0(t,u)}^{c_1(t,u)} \frac{D_{y+t-u}^{(r)}}{D_b^{(r)}} \frac{s(y+t-u)s(r)}{s(b)^2} (1 - {}_{y-b} p_b) k(b) dy$$

Expression for *COV(S(u),B(t)), u = t*

$$COV(B(t),S(u)) = K + L_1 + L_2 - L_3 + M$$

$$K = Sa(b,0)[\int_{c_1(t,u)}^{c_2(t,u)} \frac{Sa(y-u,0)}{Sa(b,0} sa(y-u,0) \frac{s(r)s(y)}{s(y-u)^2} \, {}_t p_{y-u} (1 - {}_u p_{y-u}) k(y-u) dy]$$

$$L_1 = Sa(b,0) sa(b,0) e^{rt} e^{fu} \int_{a_0(t)}^{a_1(t)} \frac{\bar{N}_{y_0}^{s(f)} - \bar{N}_{a_1(u)}^{s(f)}}{D_b^{s(f)}} \frac{s(r) D_x^{(r)}}{D_b^{s(r)}} k(b) dx$$

$$L_2 = Sa(b,0)sa(b,0)e^{ru}e^{ft}\int_{a_0(t)}^{a_1(t)} \frac{\overline{N}^{s(r)}_{a_0(u)} - \overline{N}^{s(r)}_{y_0}}{D_b^{s(r)}} \frac{s(r)D_x^{(f)}}{D_b^{s(f)}} k(b)dx$$

$$L_3 = Sa(b,0)sa(b,0)e^{r(t+u)}s(r)\frac{\overline{N}^{(r)}_{a_0(t)} - \overline{N}^{(r)}_{a_1(t)}}{D_b^{s(r)}} \frac{\overline{N}^{s(r)}_{a_0(u)} - N^{s(r)}_{a_1(u)}}{D_b^{s(r)}} k(b)$$

$$M = Sa(b,0)sa(b,0)e^{rt}\int_{c_0(t,u)}^{c_1(t,u)} \frac{D^{(r)}_{y+t-u}}{D_b^{(r)}} \frac{s(r)s(y)}{s(b)^2}(1- {}_{y-b}p_b)k(b)dy$$

APPENDIX 4
LONG-TERM EXPRESSIONS FOR *COV(A(t),A(u)), COV(S(t),S(u))* and *VTDS(t)*

Co-variance between *A(t)* and *A(u)*, (*t = 2(r–b), u = t*)

The formulae and expressions will be identical to those for *COV(S(t),S(u))* below, except that the salary scale function will be eliminated throughout. Thus, all commutation functions will be replaced by the corresponding functions without the salary scale being incorporated (replace $D_x^{s(r)} by D_x^{(r)}, \overline{N}_x^{s(r)} by \overline{N}_x^{(r)}$ etc). Moreover, the *adjustment factor should be Ac(b,0) (instead of Sa(b,0)sa(b,0))*.

Co-variance between *S(t)* and *S(u)*, (*t = 2(r–b), u = t*) , according to range of *u*

The adjustment factor Sa(b,0)sa(b,0) should apply to all the following formulae.

***u* = *r* - *b*:** $COV(S(t),S(u)) = e^{rt} f(u)$, where

$$f(u) = \frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}}[e^{fu}\frac{\overline{N}_b^{s(f)} - \overline{N}_{b+u}^{s(f)}}{D_b^{s(f)}} - e^{ru}\frac{\overline{N}_b^{s(r)} - \overline{N}_{b+u}^{s(r)}}{D_b^{s(r)}}]$$

where ? = p – ?.

***r* - *b* < *u* = *t* – (*r* – *b*):** $COV(S(t),S(u)) = k_1 e^{pt}e^{-fz} - k_2 e^{2rt}e^{-rz}$

where $z = t - u$;

$$k_1 = \frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}} \frac{\overline{N}_b^{s(f)} - \overline{N}_r^{s(f)}}{D_b^{s(f)}}$$

$$k_2 = [\frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}}]^2$$

***u* > *t* – (*r* – *b*):** $COV(S(t),S(u)) = e^{pt}f_1(z) - e^{2rt} f_2(z) + e^{rt} f_3(z)$

where $z = t - u$;

$$f_1(z) = e^{-fz}[\frac{\overline{N}_b^{s(r)} - \overline{N}_{b+z}^{s(r)}}{D_b^{s(r)}} \frac{\overline{N}_b^{s(f)} - \overline{N}_r^{s(f)}}{D_b^{s(f)}} + \int_{b+z}^{r} \frac{\overline{N}_{x-z}^{s(f)} - \overline{N}_r^{s(f)}}{D_b^{s(f)}} \frac{D_x^{s(r)}}{D_b^{s(r)}} dx] + e^{-rz}\int_{b+z}^{r} \frac{\overline{N}_b^{s(r)} - \overline{N}_{x-z}^{s(r)}}{D_b^{s(r)}} \frac{D_x^{s(f)}}{D_b^{s(f)}} dx$$

$$f_2(z) = e^{-rz}[\frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}}]^2$$

$$f_3(z) = \int_b^{r-z} \frac{D_{y+z}^{s(r)}}{D_b^{s(r)}} \frac{s(y)}{s(b)}(1 - \frac{l_y}{l_b})dy$$

Variance of *TDS(t)*

In the following, $f(u), k_1, k_2, f_1(z), f_2(z), f_3(z)$ denote the functions occurring in the expressions relating to *COV(S(t),S(u)), after incorporating the adjustment factor*.

Let the two components of (90) be denoted by *VTDS1(t)* and *VTDS2(t)*.
For *t =2(r – b),*

$$VTDS1(t) - VTDS1(2(r-b)) = H(2(r-b)) - H(t)$$

where,

$$H(t) = H_1 e^{-t(d-r)} + H_2 e^{-t(a-p)} + H_3 e^{-t(a-2r)} + H_4 e^{-t(a-r)}$$

$$H_1 = \frac{2}{d-r}[\int_0^{r-b} e^{-qu} f(u)du + k_1 \frac{e^{-(r-b)(q-f)}}{q-f} - k_2 \frac{e^{-(r-b)(q-r)}}{q-r}]$$

$$H_2 = \frac{2}{a-p}[\int_0^{r-b} e^{qz} f_1(z)dz - k_1 \frac{e^{(r-b)(q-f)}}{q-f}]$$

$$H_3 = \frac{2}{a-2r}[k_2 \frac{e^{(r-b)(q-r)}}{q-r} - \int_0^{r-b} e^{qz} f_2(z)dz]$$

$$H_4 = \frac{2}{a-r}\int_0^{r-b} e^{qz} f_3(z)dz$$

For *t = r – b,*

$$VTDS2(t) - VTDS2(r-b) = G(r-b) - G(t)$$

$$G(t) = G_1 e^{-t(d-r)} + G_2 e^{-t(a-2r)} + G_3 e^{-2t(d-r)}$$

$$k = Sa(b,0)\frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}}$$

$$G_1 = \frac{2}{d-r}[k\int_0^{r-b} (e^{-qu} - e^{-du})ES(u)du + k^2\{\frac{e^{-(r-b)(q-r)}}{q-r} - \frac{e^{-(r-b)(d-r)}}{d-r}\}]$$

$$G_2 = \frac{-2k^2}{(q-r)(a-2r)}$$

$$G_3 = \frac{k^2}{(d-r)^2}$$

It follows that for $t_0 \geq 2(r-b)$,

$$VTDS(t) - VTDS(t_0) = F(t_0) - F(t)$$

$$F(t) = F_1 e^{-(d-r)t} + F_2 e^{-(a-p)t} + F_3 e^{-(a-2r)t} + F_4 e^{-(a-r)t} + F_5 e^{-2(d-r)t}$$

where $F_1 = H_1 + G_1; F_2 = H_2; F_3 = H_3 + G_2; F_4 = H_4; F_5 = G_3$.

Note 1. The expressions for *Cov(S(t),S(u))* are obtained by simplifying the respective integrals, within the appropriate integration limits (section 2.4 and Appendix 2).

Note 2. The expression for *VTDS1(t) – VTDS1(2(r-b))* is obtained by expressing the relevant integral as the sum of sub-integrals, as follows (and then simplifying each):

$$\int_{2(r-b)}^{t} e^{-du}\int_{0}^{u} e^{-qv} Cov(S(u),S(v))dvdu = \int_{2(r-b)}^{t} e^{-du}[\int_{0}^{r-b} + \int_{r-b}^{u-r+b} + \int_{u-r+b}^{u}]e^{-qv} Cov(S(u),S(v))dvdu$$

A similar procedure applies to the expression for *VTDS2(t) – VTDS2(r-b).*

APPENDIX 5
LONG-TERM EXPRESSIONS FOR *COV(R(t),R(u)), COV(B(t),B(u))* and *VTDBt)*

Co-variance between *R(t)* and *R(u),* (*t = (r–b)+2(? -r), u = t*)

The formulae and expressions will be identical to those for *COV(B(t),B(u)* below, except that the salary scale functions (*s(b), s(r)*) and the pension rate function *k(b)* will be suppressed throughout. Moreover, the *adjustment factor should be Ac(b,0) (instead of Sa(b,0)sa(b,0)).*

Co-variance between *B(t)* and *B(u),* (*t = (r–b)+2(?-r), u = t*), according to range of *u*

The adjustment factor Sa(b,0)sa(b,0) should apply to all the following formulae.

***r-b = u = ? - b*:** $COV(B(t),B(u)) = e^{rt} f(u)$, where

$$f(u) = \frac{\overline{N}_r^{(r)}}{D_b^{(r)}}[e^{fu}\frac{\overline{N}_r^{(f)} - \overline{N}_{b+u}^{(f)}}{D_b^{(f)}} - e^{ru}\frac{\overline{N}_r^{(r)} - \overline{N}_{b+u}^{(r)}}{D_b^{(r)}}]\frac{s(r)^2}{s(b)^2}k(b)^2$$

where ? = p – ?.

***? - b < u = t – (? - r)*:** $COV(B(t),B(u)) = k_1 e^{pt} e^{-fz} - k_2 e^{2rt} e^{-rz}$

where $z = t - u$;

$$k_1 = \frac{\overline{N}_r^{(r)}}{D_b^{(r)}}\frac{\overline{N}_r^{(f)}}{D_b^{(f)}}\frac{s(r)^2}{s(b)^2}k(b)^2$$

$$k_2 = [\frac{\overline{N}_r^{(r)}}{D_b^{(r)}}]^2\frac{s(r)^2}{s(b)^2}k(b)^2$$

***u > t – (? - r)*:** $COV(B(t),B(u)) = e^{pt} f_1(z) - e^{2rt} f_2(z) + e^{rt} f_3(z)$

where $z = t - u$;

$$f_1(z) = [e^{-fz}[\frac{\overline{N}_r^{(r)} - \overline{N}_{r+z}^{(r)}}{D_b^{(r)}}\frac{\overline{N}_r^{(f)}}{D_b^{(f)}} + \int_{r+z}^{v}\frac{\overline{N}_{x-z}^{(f)}}{D_b^{(f)}}\frac{D_x^{(r)}}{D_b^{(r)}}dx] + e^{-rz}\int_{r+z}^{v}\frac{\overline{N}_r^{(r)} - \overline{N}_{x-z}^{(r)}}{D_b^{(r)}}\frac{D_x^{(f)}}{D_b^{(f)}}dx]\frac{s(r)^2}{s(b)^2}k(b)^2$$

$$f_2(z) = e^{-rz}[\frac{\overline{N}_r^{(r)}}{D_b^{(r)}}]^2\frac{s(r)^2}{s(b)^2}k(b)^2$$

$$f_3(z) = [\int_{r}^{v-z}\frac{D_{y+z}^{(r)}}{D_b^{(r)}}(1-\frac{l_y}{l_b})dy]\frac{s(r)^2}{s(b)^2}k(b)^2$$

Expression for the variance of *TDB(t)*

In what follows, the functions $f(u), k_1, k_2, f_1(z), f_2(z), f_3(z)$ refer to those corresponding to *COV(B(t),B(u)), after incorporating the adjustment factor.*

As in the case of *VTDS(t)* (see (90)), let *VTDB(t) = VTDB1(t) + VTDB2(t).*

For *t = (r – b) + 2(?– r),*

$$VTDB1(t) - VTDB1((r-b)+2(v-r)) = H(2(r-b)) - H(t)$$

where,

$$H(t) = H_1 e^{-t(d-r)} + H_2 e^{-t(a-p)} + H_3 e^{-t(a-2r)} + H_4 e^{-t(a-r)}$$

$$H_1 = \frac{2}{d-r}\left[\int_{r-b}^{v-b} e^{-qu} f(u)du + k_1 \frac{e^{-(v-b)(q-f)}}{q-f} - k_2 \frac{e^{-(v-b)(q-r)}}{q-r}\right]$$

$$H_2 = \frac{2}{a-p}\left[\int_0^{v-r} e^{qz} f_1(z)dz - k_1 \frac{e^{(v-r)(q-f)}}{q-f}\right]$$

$$H_3 = \frac{2}{a-2r}\left[k_2 \frac{e^{(v-r)(q-r)}}{q-r} - \int_0^{v-r} e^{qz} f_2(z)dz\right]$$

$$H_4 = \frac{2}{a-r}\int_0^{v-r} e^{qz} f_3(z)dz$$

For *t = ? – b*,

$$VTDB2(t) - VTDB2(v-b) = G(r-b) - G(t)$$

$$G(t) = G_1 e^{-t(d-r)} + G_2 e^{-t(a-2r)} + G_3 e^{-2t(d-r)}$$

$$k^* = Sa(b,0)\frac{\overline{N}_r^{s(r)}}{D_b^{s(r)}}$$

$$G_1 = \frac{2}{d-r}\left[k^*\int_0^{v-b}(e^{-qu} - e^{-du})EB(u)du + k^{*2}\left\{\frac{e^{-(v-b)(q-r)}}{q-r} - \frac{e^{-(r-b)(d-r)}}{d-r}\right\}\right]$$

$$G_2 = \frac{-2k^{*2}}{(q-r)(a-2r)}$$

$$G_3 = \frac{k^{*2}}{(d-r)^2}$$

It follows that for $t_0 \geq (r-b) + 2(v-r)$,

$$VTDB(t) - VTDB(t_0) = F(t_0) - F(t)$$

$$F(t) = F_1 e^{-(d-r)t} + F_2 e^{-(a-p)t} + F_3 e^{-(a-2r)t} + F_4 e^{-(a-r)t} + F_5 e^{-2(d-r)t}$$

where $F_1 = H_1 + G_1; F_2 = H_2; F_3 = H_3 + G_2; F_4 = H_4; F_5 = G_3$.

Note. The procedures for the development and simplification of the above expressions are similar to those adopted in Appendix 4 (see Notes 1 and 2, Appendix 4).

APPENDIX 6
LONG-TERM EXPRESSIONS FOR *COV(R(t),A(u)), COV(B(t),S(u))* and *Cov(TDB(t),TDS(t))*

Co-variance between *R(t)* and *A(u)*, (*t = 2(r–b)+(? -r), u = t*)

The formulae and expressions will be identical to those for *COV(B(t),S(u))* below, except that the pension rate function (*k(b)*) and the salary scale function (*s(x)*) will be suppressed throughout. Thus, all commutation functions will be replaced by the corresponding functions without the salary scale being incorporated (replace $D_x^{s(r)} by D_x^{(r)}, \overline{N}_x^{s(r)} by \overline{N}_x^{(r)}$ etc). Moreover, the *adjustment factor should be Ac(b,0) (instead of Sa(b,0)sa(b,0)).*

Co-variance between *B(t)* and *S(u)*, (*t = 2(r–b)+(? -r), u = t*) , according to range of *u*

The adjustment factor Sa(b,0)sa(b,0) should apply to all the following formulae.

u = r - b: $COV(B(t),S(u)) = e^{rt} f(u)$, where

$$f(u) = \frac{\overline{N}_r^{(r)}}{D_b^{(r)}}[e^{fu}\frac{\overline{N}_b^{s(f)} - \overline{N}_{b+u}^{s(f)}}{D_b^{s(f)}} - e^{ru}\frac{\overline{N}_b^{s(r)} - \overline{N}_{b+u}^{s(r)}}{D_b^{s(r)}}]\frac{s(r)}{s(b)}k(b)$$

where ? = p – ?.

r - b < u = t – (? – b): $COV(B(t),S(u)) = k_1 e^{pt} e^{-fz} - k_2 e^{2rt} e^{-rz}$

where $z = t - u$;

$$k_1 = \frac{\overline{N}_r^{(r)}}{D_b^{(r)}}\frac{\overline{N}_b^{s(f)} - \overline{N}_r^{s(f)}}{D_b^{s(f)}}\frac{s(r)}{s(b)}k(b)$$

$$k_2 = \frac{\overline{N}_r^{(r)}}{D_b^{(r)}}\frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}}\frac{s(r)}{s(b)}k(b)$$

t – (? -b) < u = t – (r – b): $COV(B(t),S(u)) = e^{pt} f_1(z) - e^{2rt} f_2(z) + e^{rt} f_3(z)$

where $z = t - u$; $f_1(z) =$

$$[e^{-fz}\{\frac{\overline{N}_r^{(r)} - \overline{N}_{b+z}^{(r)}}{D_b^{(r)}}\frac{\overline{N}_b^{s(f)} - \overline{N}_r^{s(f)}}{D_b^{s(f)}} + \int_{b+z}^{v}\frac{\overline{N}_{x-z}^{s(f)} - \overline{N}_r^{s(f)}}{D_b^{s(f)}}\frac{D_x^{(r)}}{D_b^{(r)}}dx\} + e^{-rz}\int_{b+z}^{v}\frac{\overline{N}_b^{s(r)} - \overline{N}_{x-z}^{s(r)}}{D_b^{s(r)}}\frac{D_x^{(f)}}{D_b^{(f)}}dx]\frac{s(r)}{s(b)}k(b)$$

$$f_2(z) = e^{-rz}\frac{\overline{N}_r^{(r)}}{D_b^{(r)}}\frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}}\frac{s(r)}{s(b)}k(b)$$

$$f_3(z) = [\int_b^{v-z}\frac{D_{y+z}^{(r)}}{D_b^{(r)}}\frac{s(y)}{s(b)}(1 - \frac{l_y}{l_b})dy]\frac{s(r)}{s(b)}k(b)$$

t – (r-b) < u = t – (? -r): $COV(B(t),S(u)) = e^{pt} f_4(z) - e^{2rt} f_2(z) + e^{rt} f_5(z)$

where $z = t - u$;

$$f_4(z) = [e^{-fz}\int_r^{v}\frac{\overline{N}_{x-z}^{s(f)} - \overline{N}_r^{s(f)}}{D_b^{s(f)}}\frac{D_x^{(r)}}{D_b^{(r)}}dx + e^{-rz}\int_r^{v}\frac{\overline{N}_b^{s(r)} - \overline{N}_{x-z}^{s(r)}}{D_b^{s(r)}}\frac{D_x^{(f)}}{D_b^{(f)}}dx]\frac{s(r)}{s(b)}k(b)$$

$$f_5(z) = [\int_{r-z}^{v-z}\frac{D_{y+z}^{(r)}}{D_b^{(r)}}\frac{s(y)}{s(b)}(1 - \frac{l_y}{l_b})dy]\frac{s(r)}{s(b)}k(b)$$

u > t – (? -r): $COV(B(t),S(u)) = e^{pt} f_6(z) - e^{2rt} f_2(z) + e^{rt} f_7(z)$

where $z = t - u$;

$$f_6(z) = [e^{-fz}\int_r^{r+z}\frac{\overline{N}_{x-z}^{s(f)} - \overline{N}_r^{s(f)}}{D_b^{s(f)}}\frac{D_x^{(r)}}{D_b^{(r)}}dx + e^{-rz}\{\int_r^{r+z}\frac{\overline{N}_b^{s(r)} - \overline{N}_{x-z}^{s(r)}}{D_b^{s(r)}}\frac{D_x^{(f)}}{D_b^{(f)}}dx + \frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}}\frac{D_{r+z}^{(f)}}{D_b^{(f)}}\}]\frac{s(r)}{s(b)}k(b)$$

$$f_7(z) = [\int_{r-z}^{r}\frac{D_{y+z}^{(r)}}{D_b^{(r)}}\frac{s(y)}{s(b)}(1-\frac{l_y}{l_b})dy]\frac{s(r)}{s(b)}k(b)$$

Co-variance between *A(t)* and *R(u)*, (*t = ? -b, u = t*)

The formulae and expressions will be identical to those for *COV(S(t),B(u))* below, except that the pension rate function (*k(b)*) and the salary scale function (*s(x)*) will be suppressed throughout. Thus, all commutation functions will be replaced by the corresponding functions without the salary scale being incorporated (replace $D_x^{s(r)} by D_x^{(r)}, \overline{N}_x^{s(r)} by \overline{N}_x^{(r)}$ etc). Moreover, *the adjustment factor should be Ac(b,0) (instead of Sa(b,0)sa(b,0)).*

Co-variance between *S(t)* and *B(u)*, (*t = ? -b, u = t*), according to range of *u*

The adjustment factor Sa(b,0)sa(b,0) should apply to all the following formulae.

***r –b = u < ? - b*:** $COV(S(t), B(u)) = e^{rt} g(u)$, where

$$g(u) = \frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}}[\frac{\overline{N}_r^{(f)} - \overline{N}_{b+u}^{(f)}}{D_b^{(f)}} - \frac{\overline{N}_r^{(r)} - \overline{N}_{b+u}^{(r)}}{D_b^{(r)}}]\frac{s(r)}{s(b)}k(b)$$

where ? = p – ?.

***u = ? – b*:** $COV(S(t), B(u)) = h_1 e^{pt} e^{-fz} - h_2 e^{2rt} e^{-rz}$

where $z = t - u$;

$$h_1 = \frac{\overline{N}_r^{(f)}}{D_b^{(f)}}\frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}}\frac{s(r)}{s(b)}k(b)$$

$$h_2 = \frac{\overline{N}_r^{(r)}}{D_b^{(r)}}\frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}}\frac{s(r)}{s(b)}k(b)$$

Co-variance between *TDS(t)* and *TDB(t)*

In the following, $f(u), k_1, k_2, f_1(z), f_2(z), f_3(z), f_4(z), f_5(z), f_6(z), f_7(z)$ denote the functions occurring in the expressions relating to *COV(B(t),S(u))* and $g(u), h_1, h_2$ denote the functions occurring in the expressions for *COV(S(t),B(u)), after incorporating the respective adjustment factors.*

Let the four components of (91) be denoted by *CovTDBTDS1(t), CovTDBTDS2(t), CovTDSTDB1(t)* and *CovTDSTDB2(t).*

.

For *t =2(r – b) + (? – r),*

$$CovTDBTDS1(t) - CovTDBTDS1(2(r-b) + (v - r)) = H(2(r-b) + (v - r)) - H(t)$$

where,

$$H(t) = H_1 e^{-t(d-r)} + H_2 e^{-t(a-p)} + H_3 e^{-t(a-2r)} + H_4 e^{-t(a-r)}$$

$$H_1 = \frac{1}{d-r}[\int_0^{r-b} e^{-qu} f(u)du + k_1 \frac{e^{-(r-b)(q-f)}}{q-f} - k_2 \frac{e^{-(r-b)(q-r)}}{q-r}]$$

$$H_2 = \frac{1}{a-p}[\int_0^{v-r} e^{qz} f_6(z)dz + \int_{v-r}^{r-b} e^{qz} f_4(z)dz + \int_{r-b}^{v-b} e^{qz} f_1(z)dz - k_1 \frac{e^{(v-b)(q-f)}}{q-f}]$$

$$H_3 = \frac{1}{a-2r}[k_2 \frac{e^{(v-b)(q-r)}}{q-r} - \int_0^{v-b} e^{qz} f_2(z)dz]$$

$$H_4 = \frac{1}{a-r}[\int_0^{v-r} e^{qz} f_7(z)dz + \int_{v-r}^{r-b} e^{qz} f_5(z)dz + \int_{r-b}^{v-b} e^{qz} f_3(z)dz]$$

For $t = ? - b$,

$$CovTDBTDS2(t) - CovTDBTDS2(v-b) = G(v-b) - G(t)$$

$$G(t) = G_1 e^{-t(d-r)} + G_2 e^{-t(a-2r)} + G_3 e^{-2t(d-r)}$$

$$k = Sa(b,0)\frac{\overline{N}_b^{s(r)} - \overline{N}_r^{s(r)}}{D_b^{s(r)}}; k^* = Sa(b,0)\frac{\overline{N}_r^{s(r)}}{D_b^{s(r)}}$$

$$G_1 = \frac{1}{d-r}[k^* \int_0^{r-b} (e^{-qu} - e^{-du})ES(u)du + kk^*\{\frac{e^{-(r-b)(q-r)}}{q-r} - \frac{e^{-(r-b)(d-r)}}{d-r}\}]$$

$$G_2 = \frac{-kk^*}{(q-r)(a-2r)}$$

$$G_3 = \frac{kk^*}{2(d-r)^2}$$

For $t = ? - b$,

$$CovTDSTDB1(t) - CovTDSTDB1(v-b) = L(v-b) - L(t)$$

$$L(t) = L_1 e^{-t(d-r)} + L_2 e^{-t(a-p)} + L_3 e^{-t(a-2r)}$$

$$L_1 = \frac{1}{d-r}[\int_{r-b}^{v-b} e^{-qu} g(u)du + h_1 \frac{e^{-(v-b)(q-f)}}{q-f} - h_2 \frac{e^{-(v-b)(q-r)}}{q-r}]$$

$$L_2 = \frac{-h_1}{(a-p)(q-f)}$$

$$L_3 = \frac{h_2}{(a-2r)(q-r)}$$

For $t = ? - b$,

$$CovTDSTDB2(t) - CovTDSTDB2(v-b) = G(v-b) - G(t)$$

$$J(t) = J_1 e^{-t(d-r)} + J_2 e^{-t(a-2r)} + J_3 e^{-2t(d-r)}$$

$$J_1 = \frac{1}{d-r}[k \int_0^{v-b} (e^{-qu} - e^{-du})EB(u)du + kk^*\{\frac{e^{-(v-b)(q-r)}}{q-r} - \frac{e^{-(v-b)(d-r)}}{d-r}\}]$$

$$J_2 = \frac{-kk^*}{(q-r)(a-2r)}$$

$$J_3 = \frac{kk^*}{2(d-r)^2}$$

It follows that for $t_0 \geq 2(r-b)$ +(? - r),

$$Cov(TDS(t),TDB(t)) - Cov(TDS(t_0),TDB(t_0)) = F(t_0) - F(t)$$

$$F(t) = F_1 e^{-(d-r)t} + F_2 e^{-(a-p)t} + F_3 e^{-(a-2r)t} + F_4 e^{-(a-r)t} + F_5 e^{-2(d-r)t}$$

where $F_1 = H_1 + G_1 + L_1 + J_1; F_2 = H_2 + L_2; F_3 = H_3 + G_2 + L_3 + J_2; F_4 = H_4; F_5 = G_3 + J_3$.

Note. The procedures for the development and simplification of the above expressions are similar to those adopted in Appendix 4 (see Notes 1 and 2, Appendix 4).

APPENDIX 7
LIST OF SYMBOLS

(in the approximate order in which they first appear in the text)

P(x, t)	Population aged *x* at time *t* (active or retired)
?	New entrant growth parameter (new entrant intake deterministic)
b	Entry age of new entrants
r	Retirement age
?	Limiting age of the mortality table
Ac(x, t)	Active Population, aged *x* at time *t*
Re(x, t)	Retired Population, aged *x* at time *t*
EF	Expected value of (any stochastic function) *F*
VF	Variance of *F*
COV(F,G)	Co-variance between (stochastic functions) *F* and *G* (also *CovFG*)
$D_x^{(r)}, \overline{N}_x^{(r)}$	Commutation functions based on force of interest ?
$D_{xx}^{(r)}, \overline{N}_{xx}^{(r)}$	Special commutation functions ($D_{xx}^{(r)} = l_x l_x e^{-rx}$)
$(a_0(t), a_1(t))$	Age-limits of the surviving new entrant population at time *t*
$(a_1(t), a_2(t))$	Age limits of the surviving initial population at time *t*
P(t)	Total population at time *t* (active or retired)
A(t)	Total Active Population at time *t*
R(t)	Total Retired Population at time *t*
$c_0(t,u), c_1(t,u)$	Age-limits (at *u*) for common new entrant cohorts between *P(t)* and *P(u)*
$c_1(t,u), c_2(t,u)$	Age-limits (at u) for common initial population cohorts between *P(t),P(u)*
DR(t)	Demographic ratio
CVF	Coefficient of variation of (a stochastic function) *F*
Corr(F,G)	Correlation coefficient between stochastic functions *F* and *G* (also *CorrFG*)
?	Primary new entrant growth parameter (new entrant intake stochastic)
p	Secondary new entrant growth parameter (new entrant intake stochastic)
f	p - ?
s(x)	Salary scale function
$D_x^{s(r)}, \overline{N}_x^{s(r)}$	Commutation functions based on force of interest ?, incorporating the salary scale function
sa(x, t)	Average annual salary of the active population aged *x* at time *t*
pe(x, t)	Average annual pension of the retired population aged *x* at time *t*
Sa(x, t)	Annual salary bill of the active population aged *x* at time *t*
Be(x, t)	Annual pension bill of the retired population aged *x* at time *t*

S(t)	Total salary bill at time *t*
B(t)	Total pension bill at time *t*
Bc(t)	Capitalized value of the pensions of those retiring at time *t*
*a**	Capitalized value of an indexed unit pension at age *r*
k(x)	Proportionate rate of pension for entry age *x*
d	Force of real interest (deterministic) or primary real interest parameter (force of interest stochastic)
a	Secondary real interest parameter (force of interest stochastic)
?	a – d
DS(t)	Discounted value of *S(t)*
DB(t)	Discounted value of *B(t)*
TDS(t)	Total over (*0, t*) of discounted salaries
TDB(t)	Total over (*0, t*) of discounted pension expenditures
pr	Classical (deterministic) premium
*pr**	Stochastic premium
PAYG(n)	Pay-as-you-go premium at $t = n$
AVP(0, n)	Average premium over (*0, n*)
$RRP(0, n : \Lambda)$	Reserve Ratio system premium over (*0, n*), with ratio = ?
TFP(n)	Terminal Funding premium at $t = n$
GAP	General Average Premium

APPENDIX 8
NUMERICAL ILLUSTRATIONS

A hypothetical retirement pension scheme

The illustrations relate to a newly introduced pension scheme that provides retirement pensions, for life, from age 65. The pension rate is 1 per cent of the final salary per year of service. Pre-scheme service is not taken into account. The pension is fully adjusted, in line with the salary escalation of active persons.

The age-distribution of the initial insured active population (per 10,000), as well as of the corresponding initial salary bill, is shown in the table below. There are no pensioners at the outset. New entrants enter at age 20. The average cohort survival pattern, assumed to remain invariant, is according to the service and life tables summarized below. The progression of individual salaries, ignoring salary escalation, is according to the salary scale indicated below.

In order to illustrate the effect of changes in the population size and the parameters, the calculations have been made for the following six variants. The implied coefficients of variation of *exp(?(t))* (*CV1*) and of *exp(-d(t))* (*CV2*) are indicated. The value of *Ac(20,0)* is proportional to the initial population; for the population of 10,000, its value is 291.

Variant	Initial population	?	p	d	a	CV1	CV2
A	10,000	0.01	0.025	0.03	0.055	7.08%	7.08%
A1	1,000	0.01	0.025	0.03	0.055	7.08%	7.08%
A2	100	0.01	0.025	0.03	0.055	7.08%	7.08%
B	10,000	0.01	0.0205	0.03	0.0595	2.24%	2.24%
C	10,000	0.01	0.0205	0.03	0.0305	2.24%	17.30%
D	10,000	0.01	0.0495	0.03	0.0595	17.30%	2.24%

Tables 1 to 5 give detailed results for A, the main variant. Table 6 illustrates the sensitivity of the results to population size and Table 7 the effect of the parameters on the results. Table 8 shows the General Average Premiums for the different variants.

Initial population data, mortality assumptions and salary scale

Age Group	Initial Popn	Initial Sal Bill [$'000]	Age	Salary scale	Service Table	Age	Life Table
20-25	1415	1875	**20**	100	1000	**65**	1000
25-30	1339	2587	**25**	165	995	**70**	861
30-35	1265	3092	**30**	221	989	**75**	677
35-40	1193	3402	**35**	267	982	**80**	463
40-45	1121	3539	**40**	302	972	**85**	254
45-50	1047	3525	**45**	328	958	**90**	101
50-55	967	3365	**50**	344	936	**95**	25
55-60	878	3082	**55**	350	903	**100**	0
60-65	776	2721	**60**	350	851		
Totals	10000	27188	**65**	350	775		

Table 1. Demographic projections, Variant A

(t)	EA	VA	ER	VR	CovRA	CVA	CVR	corr(RA)	ER/EA	EDR	VDR	CVDR
0	10000	0	0	0	0	0.00%	0.00%	0	0.00%	0.00%	0.000000	
10	11051	879	1293	276	0	0.27%	1.28%	0	11.70%	11.70%	0.000002	1.31%
20	12214	5542	2137	795	0	0.61%	1.32%	0	17.50%	17.50%	0.000006	1.45%
30	13498	20009	2537	1099	0	1.05%	1.31%	0	18.80%	18.80%	0.000010	1.67%
40	14918	52903	2811	1257	0	1.54%	1.26%	0	18.84%	18.85%	0.000014	1.99%
50	16487	113940	3106	1384	799	2.05%	1.20%	0.0636	18.84%	18.85%	0.000019	2.30%
60	18221	204711	3433	2345	7716	2.48%	1.41%	0.3522	18.84%	18.85%	0.000020	2.38%
70	20137	334211	3794	4916	22276	2.87%	1.85%	0.5496	18.84%	18.85%	0.000021	2.41%
80	22255	516308	4193	9095	44983	3.23%	2.27%	0.6564	18.84%	18.85%	0.000021	2.44%
90	24596	769444	4634	15131	77860	3.57%	2.65%	0.7216	18.84%	18.85%	0.000022	2.47%
100	27182	1118077	5122	23685	124526	3.89%	3.00%	0.7652	18.84%	18.86%	0.000022	2.50%
110	30041	1594556	5660	35653	189881	4.20%	3.34%	0.7964	18.84%	18.85%	0.000023	2.54%
120	33201	2241578	6256	52229	280439	4.51%	3.65%	0.8196	18.84%	18.86%	0.000024	2.58%

Table 2. Financial projections (static salary basis), Variant A

(t)	ES	EB	VS	VB	CovBS	CVS	CVB	E(Bc)	VBc	CovBcS	CVBc
0	27188	0	0	0	0			0	0	0	
10	30047	245	4828	10	0	0.23%	1.32%	617	691	0	4.26%
20	33208	919	23158	131	0	0.46%	1.24%	1364	3833	0	4.54%
30	36700	1886	97953	523	0	0.85%	1.21%	2261	10523	0	4.54%
40	40560	3070	304327	1321	0	1.36%	1.18%	3332	22082	0	4.46%
50	44825	4386	726647	2500	3434	1.90%	1.14%	4143	23008	16168	3.66%
60	49540	5319	1364261	5596	33134	2.36%	1.41%	4578	29272	60755	3.74%
70	54750	5986	2278626	12244	95651	2.76%	1.85%	5060	37464	126862	3.83%
80	60508	6623	3569542	22685	193148	3.12%	2.27%	5592	48187	222559	3.93%
90	66872	7319	5369850	37739	334315	3.47%	2.65%	6180	62228	358647	4.04%
100	73905	8089	7855861	59074	534686	3.79%	3.00%	6830	80619	549523	4.16%
110	81677	8940	11260960	88926	815308	4.11%	3.34%	7549	104704	814319	4.29%
120	90267	9880	15893360	130270	1204140	4.42%	3.65%	8342	136238	1178395	4.42%

Note: Expected values in $1,000; variances and co-variances in $1,000x1,000

Table 3a. COV(S(t), S(u)), Variant A ($1,000x1,000)

(u)

(t)	10	20	30	40	50	60	70	80	90	100	110	120
10	4828	5695	8809	13092	17106	19020	21020	23231	25674	28374	31358	34656
20	5695	23158	40717	64446	88651	102796	113758	125722	138944	153557	169707	187555
30	8809	40717	97953	162890	233761	284327	320467	354367	391637	432825	478346	528654
40	13092	64446	162890	304327	454477	577698	671988	750722	829932	917217	1013682	1120292
50	17106	88651	233761	454477	726647	964816	1163587	1329207	1479408	1635331	1807319	1997396
60	19020	102796	284327	577698	964816	1364261	1716440	2022400	2290833	2545188	2813297	3109173
70	21020	113758	320467	671988	1163587	1716440	2278626	2787364	3242137	3654913	4056610	4483804
80	23231	125722	354367	750722	1329207	2022400	2787364	3569542	4291712	4951218	5564388	6171897
90	25674	138944	391637	829932	1479408	2290833	3242137	4291712	5369850	6381217	7320275	8209137
100	28374	153557	432825	917217	1635331	2545188	3654913	4951218	6381217	7855861	9257041	10575400
110	31358	169707	478346	1013682	1807319	2813297	4056610	5564388	7320275	9257041	11260960	13185230
120	34656	187555	528654	1120292	1997396	3109173	4483804	6171897	8209137	10575400	13185230	15893360

Table 3b. COV(B(t) ,B(u)), Variant A ($1,000x1,000)

(u)

(t)	10	20	30	40	50	60	70	80	90	100	110	120
10	10	6	2	0	0	0	0	0	0	0	0	0
20	6	131	69	17	1	0	0	0	0	0	0	0
30	2	69	523	242	53	2	0	0	0	0	0	0
40	0	17	242	1321	566	116	4	0	0	0	0	0
50	0	1	53	566	2500	1250	614	512	561	620	685	757
60	0	0	2	116	1250	5596	4867	4604	4901	5410	5979	6608
70	0	0	0	4	614	4867	12244	12459	12974	14138	15619	17261
80	0	0	0	0	512	4604	12459	22685	24254	26020	28544	31539
90	0	0	0	0	561	4901	12974	24254	37739	41306	44904	49402
100	0	0	0	0	620	5410	14138	26020	41306	59074	65483	71700
110	0	0	0	0	685	5979	15619	28544	44904	65483	88926	99318
120	0	0	0	0	757	6608	17261	31539	49402	71700	99318	130269

Table 3c. COV(B(t), S(u)), Variant A ($1,000x1,000)

(t) \ (u)	10	20	30	40	50	60	70	80	90	100	110	120
10	0	0	0	0	0	0	0	0	0	0	0	0
20	233	0	0	0	0	0	0	0	0	0	0	0
30	301	596	0	0	0	0	0	0	0	0	0	0
40	235	713	1039	0	0	0	0	0	0	0	0	0
50	437	1422	2894	4122	3434	3795	4194	4635	5123	5661	6257	6915
60	1493	6288	13573	22523	30604	33134	36619	40470	44727	49431	54629	60375
70	2191	11017	27139	48344	71160	87053	95651	105711	116829	129116	142695	157702
80	2539	13581	36843	71771	112748	148649	175089	193148	213461	235911	260722	288142
90	2810	15203	42634	88329	148513	208536	261198	302554	334315	369475	408333	451278
100	3106	16808	47368	100093	175754	261289	345668	421323	483475	534686	590920	653067
110	3432	18575	52358	110944	197430	303749	421733	539019	645998	736859	815308	901055
120	3793	20529	57864	122622	218614	339811	485386	646849	808383	957782	1087958	1204140

Table 4. Discounted financial projections, Variant A

(t)	EDS	EDB	ETDS	ETDB	VTDS	VTDB	CovTDBTDS	CovTDBDS	CovTDBDB	CovTDSDS	CovTDSDB
0	2718	0	2718	0	0	0	0	0	0	0	0
10	2225	18	27114	75	9769199	145	35890	5579	45	1346944	11001
20	1822	50	47088	433	61922520	10577	781196	55947	1549	3937569	108919
30	1492	76	63441	1091	166833700	102562	4001629	175764	9037	6486460	333219
40	1221	92	76830	1952	317439400	436999	11398900	344469	26086	8459370	640006
50	1000	97	87792	2919	500367900	1206683	23775510	525857	51458	9715577	949653
60	818	87	96767	3851	701564700	2460270	40228420	673150	72278	10302890	1105143
70	670	73	104115	4651	908839500	4027816	58694000	758407	82929	10346790	1130524
80	548	60	110131	5309	1112756000	5727774	77612530	786821	86124	9990323	1092889
90	449	49	115057	5848	1306658000	7442042	96050780	773741	84691	9365280	1024626
100	367	40	119089	6289	1486305000	9095385	113431800	733242	80257	8580768	938873
110	301	32	122391	6651	1649388000	10640350	129412100	676302	74024	7720736	844820
120	246	26	125094	6947	1795041000	12050340	143822500	611058	66883	6846101	749157

Note: Expected values in $10,000; variances and co-variances in $10,000x10,000

Table 5. Premiums, Variant A

	PAYG system		TFP system			AVP system		RRP system	
(t)	Classical	Stochastic	Classical	Stochastic	Period	Classical	Stochastic	Classical	Stochastic
10	0.82%	0.83%	2.05%	2.20%	**(0,10)**	0.28%	0.30%	0.61%	0.71%
20	2.77%	2.83%	4.11%	4.42%	**(0,20)**	0.92%	1.03%	1.46%	1.72%
30	5.14%	5.26%	6.16%	6.63%	**(0,30)**	1.72%	1.97%	2.32%	2.78%
40	7.57%	7.80%	8.22%	8.85%	**(0,40)**	2.54%	2.97%	3.14%	3.80%
50	9.78%	10.14%	9.24%	9.84%	**(0,50)**	3.33%	3.95%	3.88%	4.74%
60	10.74%	11.14%	9.24%	9.82%	**(0,60)**	3.98%	4.77%	4.44%	5.44%
70	10.93%	11.36%	9.24%	9.80%	**(0,70)**	4.47%	5.39%	4.82%	5.92%
80	10.95%	11.37%	9.24%	9.78%	**(0,80)**	4.82%	5.83%	5.09%	6.25%
90	10.95%	11.38%	9.24%	9.75%	**(0,90)**	5.08%	6.16%	5.30%	6.49%
100	10.95%	11.39%	9.24%	9.73%	**(0,100)**	5.28%	6.41%	5.45%	6.67%
110	10.95%	11.39%	9.24%	9.70%	**(0,110)**	5.43%	6.59%	5.57%	6.81%
120	10.95%	11.40%	9.24%	9.68%	**(0,120)**	5.55%	6.74%	5.66%	6.92%

Table 6. Comparison of Variants A A1 A2 (CVDR and stochastic premi ums)

	CVDR			PAYG			TFP				AVP			RRP	
(t)	A	A1	A2	A	A1	A2	A	A1	A2		A	A1	A2	A	A1	A2
10	1.3%	4.2%	13.1%	0.8%	0.9%	1.0%	2.2%	2.5%	3.5%	(0,10)	0.3%	0.3%	0.3%	0.7%	0.7%	0.8%
20	1.5%	4.6%	14.5%	2.8%	3.0%	3.4%	4.4%	5.1%	7.2%	(0,20)	1.0%	1.0%	1.1%	1.7%	1.7%	1.8%
30	1.7%	5.3%	16.6%	5.3%	5.5%	6.5%	6.6%	7.7%	11.0%	(0,30)	2.0%	2.0%	2.1%	2.8%	2.8%	2.9%
40	2.0%	6.3%	19.5%	7.8%	8.3%	10.3%	8.9%	10.3%	15.1%	(0,40)	3.0%	3.0%	3.1%	3.8%	3.8%	3.9%
50	2.3%	7.3%	22.2%	10.1%	11.0%	14.6%	9.8%	11.2%	16.3%	(0,50)	3.9%	4.0%	4.1%	4.7%	4.8%	4.9%
60	2.4%	7.5%	22.7%	11.1%	12.1%	16.7%	9.8%	11.1%	16.4%	(0,60)	4.8%	4.8%	5.0%	5.4%	5.5%	5.6%
70	2.4%	7.6%	22.9%	11.4%	12.4%	17.6%	9.8%	11.1%	16.4%	(0,70)	5.4%	5.4%	5.6%	5.9%	5.9%	6.1%
80	2.4%	7.7%	23.1%	11.4%	12.4%	18.1%	9.8%	11.0%	16.6%	(0,80)	5.8%	5.9%	6.0%	6.2%	6.3%	6.4%
90	2.5%	7.8%	23.3%	11.4%	12.5%	18.8%	9.8%	11.0%	16.7%	(0,90)	6.2%	6.2%	6.3%	6.5%	6.5%	6.7%
100	2.5%	7.9%	23.6%	11.4%	12.5%	19.6%	9.7%	10.9%	17.0%	(0,100)	6.4%	6.4%	6.6%	6.7%	6.7%	6.8%
110	2.5%	8.0%	23.9%	11.4%	12.5%	20.7%	9.7%	10.8%	17.3%	(0,110)	6.6%	6.6%	6.8%	6.8%	6.8%	7.0%
120	2.6%	8.1%	24.2%	11.4%	12.6%	22.1%	9.7%	10.7%	17.8%	(0,120)	6.7%	6.8%	6.9%	6.9%	6.9%	7.1%

Table 7. Comparison of Variants B C D (CVDR and stochastic premiums)

	CVDR			PAYG			TFP				AVP			RRP	
(t)	B	C	D	B	C	D	B	C	D		B	C	D	B	C	D
10	1.3%	1.3%	1.4%	0.8%	0.8%	0.8%	2.2%	2.2%	2.2%	(0,10)	0.3%	0.3%	0.3%	0.6%	0.8%	0.6%
20	1.3%	1.3%	2.0%	2.8%	2.8%	2.8%	4.4%	4.5%	4.4%	(0,20)	1.0%	1.2%	1.0%	1.5%	2.1%	1.5%
30	1.4%	1.4%	3.1%	5.2%	5.2%	5.4%	6.6%	6.7%	6.7%	(0,30)	1.8%	2.3%	1.8%	2.5%	3.5%	2.5%
40	1.4%	1.4%	4.5%	7.7%	7.7%	8.1%	8.8%	8.9%	9.0%	(0,40)	2.7%	3.6%	2.7%	3.4%	5.0%	3.4%
50	1.3%	1.3%	6.0%	10.0%	10.0%	10.7%	9.8%	9.9%	10.2%	(0,50)	3.5%	5.0%	3.6%	4.2%	6.4%	4.2%
60	1.3%	1.3%	7.0%	11.0%	11.0%	12.0%	9.8%	9.9%	10.2%	(0,60)	4.3%	6.1%	4.3%	4.8%	7.5%	4.8%
70	1.3%	1.3%	8.1%	11.2%	11.2%	12.5%	9.7%	9.9%	10.3%	(0,70)	4.8%	7.0%	4.8%	5.2%	8.2%	5.2%
80	1.2%	1.2%	9.3%	11.2%	11.2%	12.8%	9.7%	9.9%	10.4%	(0,80)	5.2%	7.7%	5.2%	5.5%	8.7%	5.5%
90	1.2%	1.2%	10.8%	11.2%	11.2%	13.1%	9.7%	9.9%	10.6%	(0,90)	5.5%	8.1%	5.5%	5.7%	9.0%	5.7%
100	1.1%	1.1%	12.4%	11.1%	11.1%	13.6%	9.7%	9.9%	10.8%	(0,100)	5.7%	8.4%	5.7%	5.9%	9.2%	5.9%
110	1.1%	1.1%	14.3%	11.1%	11.1%	14.1%	9.7%	9.9%	11.0%	(0,110)	5.9%	8.7%	5.9%	6.0%	9.4%	6.0%
120	1.1%	1.1%	16.4%	11.1%	11.1%	14.9%	9.7%	9.9%	11.4%	(0,120)	6.0%	8.9%	6.0%	6.1%	9.5%	6.1%

Note to Tables 5, 6 and 7: The RRP system premium relates to the reserve ratio of 5.

Table 8. Limiting values of discounted financial projections and GAP

Variant	ETDS	ETDB	VTDS	VTDB	CovTDBTDS	GAP (classical)	GAP (stochastic)
A	137415	8296	2645254877	20719273	229947655	6.04%	7.35%
A1	13742	830	26950756	209307	2330706	6.04%	7.36%
A2	1374	83	319328	2304	26430	6.04%	7.45%
B	137415	8296	234251067	1766318	19973878	6.04%	6.51%
C	137415	8296	51928158764	533555426	5210354509	6.04%	9.97%
D	137415	8296	313130969	2040346	24619168	6.04%	6.52%

Note: Expected values in $10,000; variances and co-variance in $10,000x10,000
Note to tables 5 to 8: Stochastic premiums are based on the Normal distribution and correspond to the 95th percentile (e = 0.05)

Cass Business School
City of London

FACULTY OF ACTUARIAL SCIENCE AND STATISTICS

Actuarial Research Papers since 2001

---

135. Renshaw A. E. and Haberman S. On the Forecasting of Mortality Reduction Factors. February 2001.
ISBN 1 901615 56 1

136. Haberman S., Butt Z. & Rickayzen B. D. Multiple State Models, Simulation and Insurer Insolvency. February 2001. 27 pages.
ISBN 1 901615 57 X

137. Khorasanee M.Z. A Cash-Flow Approach to Pension Funding. September 2001. 34 pages.
ISBN 1 901615 58 8

138. England P.D. Addendum to "Analytic and Bootstrap Estimates of Prediction Errors in Claims Reserving". November 2001. 17 pages.
ISBN 1 901615 59 6

139. Verrall R.J. A Bayesian Generalised Linear Model for the Bornhuetter-Ferguson Method of Claims Reserving. November 2001. 10 pages.
ISBN 1 901615 62 6

140. Renshaw A.E. and Haberman. S. Lee-Carter Mortality Forecasting, a Parallel GLM Approach, England and Wales Mortality Projections. January 2002. 38 pages.
ISBN 1 901615 63 4

141. Ballotta L. and Haberman S. Valuation of Guaranteed Annuity Conversion Options. January 2002. 25 pages.
ISBN 1 901615 64 2

142. Butt Z. and Haberman S. Application of Frailty-Based Mortality Models to Insurance Data. April 2002. 65 pages.
ISBN 1 901615 65 0

143. Gerrard R.J. and Glass C.A. Optimal Premium Pricing in Motor Insurance: A Discrete Approximation. (Will be available 2003).

144. Mayhew, L. The Neighbourhood Health Economy. A systematic approach to the examination of health and social risks at neighbourhood level. December 2002. 43 pages.
ISBN 1 901615 66 9

145. Ballotta L. and Haberman S. The Fair Valuation Problem of Guaranteed Annuity Options: The Stochastic Mortality Environment Case. January 2003. 25 pages.
ISBN 1 901615 67 7

146. Haberman S., Ballotta L. and Wang N. Modelling and Valuation of Guarantees in With-Profit and Unitised With-Profit Life Insurance Contracts. February 2003. 26 pages.

ISBN 1 901615 68 5

147. Ignatov Z.G., Kaishev V.K and Krachunov R.S. Optimal Retention Levels, Given the Joint Survival of Cedent and Reinsurer. March 2003. 36 pages.

ISBN 1 901615 69 3

148. Owadally M.I. Efficient Asset Valuation Methods for Pension Plans. March 2003. 20 pages.

ISBN 1 901615 70 7

149. Owadally M.I. Pension Funding and the Actuarial Assumption Concerning Investment Returns. March 2003. 32 pages.

ISBN 1 901615 71 5

150. Dimitrova D, Ignatov Z. and Kaishev V. Finite time Ruin Probabilities for Continuous Claims Severities. August 2003.

151. Iyer Subramaniam. Application of Stochastic Methods in the Valuation of Social Security Pension Schemes. 38 pages. August 2003.

ISBN 1 901615 72 3

Statistical Research Papers

1. Sebastiani P. Some Results on the Derivatives of Matrix Functions. December 1995. 17 Pages.

ISBN 1 874 770 83 2

2. Dawid A.P. and Sebastiani P. Coherent Criteria for Optimal Experimental Design. March 1996. 35 Pages.

ISBN 1 874 770 86 7

3. Sebastiani P. and Wynn H.P. Maximum Entropy Sampling and Optimal Bayesian Experimental Design. March 1996. 22 Pages.

ISBN 1 874 770 87 5

4. Sebastiani P. and Settimi R. A Note on D-optimal Designs for a Logistic Regression Model. May 1996. 12 Pages.

ISBN 1 874 770 92 1

5. Sebastiani P. and Settimi R. First-order Optimal Designs for Non Linear Models. August 1996. 28 Pages.

ISBN 1 874 770 95 6

6. Newby M. A Business Process Approach to Maintenance: Measurement, Decision and Control. September 1996. 12 Pages.

ISBN 1 874 770 96 4

7. Newby M. Moments and Generating Functions for the Absorption Distribution and its Negative Binomial Analogue. September 1996. 16 Pages.

ISBN 1 874 770 97 2

8. Cowell R.G. Mixture Reduction via Predictive Scores. November 1996. 17 Pages.

ISBN 1 874 770 98 0

9. Sebastiani P. and Ramoni M. Robust Parameter Learning in Bayesian Networks with Missing Data. March 1997. 9 Pages.
ISBN 1 901615 00 6

10. Newby M.J. and Coolen F.P.A. Guidelines for Corrective Replacement Based on Low Stochastic Structure Assumptions. March 1997. 9 Pages.
ISBN 1 901615 01 4.

11. Newby M.J. Approximations for the Absorption Distribution and its Negative Binomial Analogue. March 1997. 6 Pages.
ISBN 1 901615 02 2

12. Ramoni M. and Sebastiani P. The Use of Exogenous Knowledge to Learn Bayesian Networks from Incomplete Databases. June 1997. 11 Pages.
ISBN 1 901615 10 3

13. Ramoni M. and Sebastiani P. Learning Bayesian Networks from Incomplete Databases. June 1997. 14 Pages.
ISBN 1 901615 11 1

14. Sebastiani P. and Wynn H.P. Risk Based Optimal Designs. June 1997. 10 Pages.
ISBN 1 901615 13 8

15. Cowell R. Sampling without Replacement in Junction Trees. June 1997. 10 Pages.
ISBN 1 901615 14 6

16. Dagg R.A. and Newby M.J. Optimal Overhaul Intervals with Imperfect Inspection and Repair. July 1997. 11 Pages. ISBN 1 901615 15 4

17. Sebastiani P. and Wynn H.P. Bayesian Experimental Design and Shannon Information. October 1997. 11 Pages. ISBN 1 901615 17 0

18. Wolstenholme L.C. A Characterisation of Phase Type Distributions. November 1997. 11 Pages. ISBN 1 901615 18 9

19. Wolstenholme L.C. A Comparison of Models for Probability of Detection (POD) Curves. December 1997. 23 Pages. ISBN 1 901615 21 9

20. Cowell R.G. Parameter Learning from Incomplete Data Using Maximum Entropy I: Principles. February 1999. 19 Pages. ISBN 1 901615 37 5

21. Cowell R.G. Parameter Learning from Incomplete Data Using Maximum Entropy II: Application to Bayesian Networks. November 1999. 12 Pages ISBN 1 901615 40 5

22. Cowell R.G. FINEX : Forensic Identification by Network Expert Systems. March 2001. 10 pages. ISBN 1 901615 60X

23. Cowell R.G. When Learning Bayesian Networks from Data, using Conditional Independence Tests is Equivalant to a Scoring Metric. March 2001. 11 pages. ISBN 1 901615 61 8

Faculty of Actuarial Science and Statistics

Actuarial Research Club

The support of the corporate members

CGNU Assurance
Computer Sciences Corporation
English Matthews Brockman
Government Actuary's Department
KPMG
PricewaterhouseCoopers
Swiss Reinsurance
Watson Wyatt Partners

is gratefully acknowledged.

ISBN 1-901615-72-3