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Pension Funding and the Actuarial Assumption Concerning Investment Returns.

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Pension Funding and the Actuarial Assumption Concerning Investment Returns

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Abstract

An assumption concerning the long-term rate of return on assets is made by actuaries when they value defined-benefit pension plans. There is a distinction between this assumption and the discount rate used to value pension liabilities, as the value placed on liabilities does not depend on asset allocation in the pension fund. The more conservative the investment return assumption is, the larger planned initial contributions are, and the faster benefits are funded. A conservative investment return assumption, however, also leads to long-term surpluses in the plan, as is shown for two practical actuarial funding methods. Long-term deficits result from an optimistic assumption. Neither outcome is desirable as, in the long term, pension plan assets should be accumulated to meet the pension liabilities valued at a suitable discount rate. A third method is devised that avoids such persistent surpluses and deficits regardless of conservatism or optimism in the assumed investment return.

Keywords: Actuarial valuation, funding method, intervaluation gains and losses.

1 Introduction

Actuaries periodically value defined benefit pension plans to recommend suitable contribution rates. A number of valuation assumptions are made for this purpose concerning various uncertain factors affecting the value of pension obligations and the funding for these obligations. This set of valuation assumptions is usually called the valuation basis. Different bases may be required for different purposes. For example, in certain jurisdictions, technical solvency bases may be specified by regulation. There may also be a different set of projection assumptions, usually scenario-based or stochastic, to investigate pension benefit amendments, asset-liability management or other issues.

Actuarial valuations for funding purposes, that is, with the objective of recommending a contribution rate are considered in this paper. A deterministic valuation basis is typically employed. Factors of a demographic nature about which assumptions are made include the mortality of plan participants at various ages, as well as their disability and withdrawal rates from the plan. Assumptions about economic factors such as price and wage inflation are also required when pensions are a function of final or career-average salary and when they are indexed with price inflation. An assumption about investment returns on the pension plan assets is also made.

If the pension liability exceeds the plan assets, then an unfunded liability (or deficit) exists. The unfunded liability varies over time as actual experience generally does not unfold exactly according to actuarial valuation assumptions. Suitable methods of pension funding generate a schedule of contributions that satisfies two objectives. First, unfunded liabilities must be paid off and there must be enough funds to pay benefits as and when they are due. Second, the contributions that are required from the sponsor and members of the plan must be stable over time.

In this paper, we investigate the effect on pension funding of deviation of actual expe-

rience from the actuarial investment return assumption. The relevance of this assumption is discussed in section 2. A simple model is described in section 3. It is used to investigate pension funding under two common funding methods, in sections 4 and 5, and under a variation described in section 6 which has the useful property of yielding full funding independently of the investment return assumption. Finally, a numerical example is given in section 7.

A list of important symbols is given here for ease of reference:

AL	actuarial liability
B	benefit paid every year
C_t	pension contribution paid at start of year $(t, t+1)$
F_t	market value of pension plan assets at time t
i	actual rate of return on plan assets
i_A	actuarial assumption for rate of return on plan assets
i_L	actuarial assumption for rate to discount pension liabilities
K	parameter in spreading of gains and losses (equation (20))
K_1, K_2	parameters in modified spreading of gains and losses (equation (36))
L_t	actuarial intervaluation loss in year $(t-1, t)$
m	amortization period for gains and losses in section 4 (equation (13))
n	amortization period for initial unfunded liability (equation (10))
NC	normal cost or normal contribution rate
P_t	payment for initial unfunded liability at time t (equation (10))
S_t	supplementary contribution paid at the start of year $(t, t+1)$
u, u_A, u_L	$1+i$, $1+i_A$, $1+i_L$ respectively
U_t	unamortized part of initial unfunded liability at time t (equation (11))
UL_t	unfunded liability $= AL - F_t$
v, v_A, v_L	$(1+i)^{-1}$, $(1+i_A)^{-1}$, $(1+i_L)^{-1}$ respectively

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2 Investment Return Assumption

The actuarial investment return assumption, henceforth denoted by i_A, is an assumption concerning the long-term rate of return on pension plan assets. Funding for pension benefits involves the substitution of contribution income (from plan participants and sponsor) by investment income from accumulated assets. It is well-known that the choice of i_A (and indeed of other valuation assumptions) affects the incidence of contribution payments and pace of funding: see for example Berin (1989, p. 93) and Trowbridge and Farr (1976, p. 27). The more optimistic the investment return assumption is, the larger the investment return is assumed to be in any given year, and the smaller the contribution that is initially required. If insufficient assets are eventually accumulated compared to the pension liability (that is, if a deficit emerges), then higher contributions than otherwise necessary will eventually be required. Conversely, the more conservative i_A is, the larger the contribution that is initially required and, if surpluses emerge, smaller contributions than otherwise necessary, will eventually be required. Thus, the schedule of contribution payments is accelerated the more i_A is conservative, and it is slowed down the more i_A is optimistic. The actuarial choice of i_A is therefore a means of controlling the pace of funding in the pension plan (Daykin, 1976; Trowbridge and Farr, 1976, p. 27).

Another key actuarial valuation assumption is the interest rate assumption (i_L) used to discount pension liabilities. As pension liabilities are not generally traded, they must be priced by comparison with similar asset cash flows. In theory, pension liabilities should be valued using market discount rates, suitably risk-adjusted, or at the rates implied in asset portfolios that are dedicated or matched by cash flow to these liabilities. In practice, more approximate methods are used. Pension liabilities have a long duration and are usually discounted at a single term-independent discount rate which is typically based on corporate bond yields to reflect the risk of default from the sponsor.

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In classical actuarial valuation methodology (for example, Trowbridge and Farr, 1976), i_A and i_L are identical. More recent actuarial practice distinguishes between the two assumptions: see for example Actuarial Standard of Practice No. 27 of the Actuarial Standards Board (1996) in the United States. The U.S. pension accounting standard FAS87 also distinguishes between the liability discount rate and the assumption for the "expected long-term rate of return on plan assets". Thornton and Wilson (1992) refer to a "dual-interest" valuation method, used in the United Kingdom, whereby i_A is a "best-estimate assumption" of investment return on the actual asset portfolio and i_L is a "prudent estimate" of investment return based on a hypothetical asset portfolio that matches pension liabilities.

The distinction between the pension liability discount rate assumption and the investment return assumption is often blurred in practice because it is assumed that they are numerically equal. Actuarial Standard of Practice No. 27 of the U.S. Actuarial Standards Board (1996) states that "generally, the appropriate discount rate is the same as the investment return assumption". This presumes that the pension fund is invested in assets that closely match or hedge or immunize the pension liability so that approximately equal discount rates apply to both asset and liability cash flows. In practice, asset allocation may involve a mismatch between assets and liabilities. For example, asset managers may have a rate-of-return objective involving a benchmark portfolio or index set without reference to the liabilities (McGill *et al.*, 1996, p. 659). It is also generally difficult to hedge pension liabilities perfectly with normal market instruments, because of the risk of default from the plan sponsor and because final-salary pensions are related to economic wage inflation.

In this paper, the assumed rates on assets and liabilities (i_A and i_L respectively) are taken to be conceptually distinct (although they could be numerically equal). The aim of this paper is to investigate the effect on pension funding of actual investment returns being different from the assumed investment return on assets.

3 Model

A simplified model of a defined benefit pension plan is used here. For details of the model, refer to Dufresne (1988, 1989). A stationary pension plan population is assumed, with fixed mortality and withdrawal rates at different ages. The only benefit that is provided in the model plan is a final-salary pension paid at normal retirement age. There is no inflation on salaries and it is also postulated that actuarial valuation assumptions remain unchanged over time. This leads to a significant simplification in that the payroll, the pension benefit B paid out every year, as well as the combination of actuarial liability AL and normal cost NC generated by a given actuarial cost method, are constant. Trowbridge (1952) shows that an equation of equilibrium holds:

$$AL = (1 + i_L)(AL + NC - B), \tag{1}$$

where i_L is the interest rate used to discount pension liability cash flows. (Alternatively, one may assume that benefits in payment are indexed with wage inflation so that, when measured net of wage inflation, the payroll as well as B, AL and NC are all constant. All quantities must then be considered net of wage inflation.)

Assuming that contributions C_t and benefits B are paid at the start of year $(t, t + 1)$, the value of the pension fund F_t at time t follows a simple recurrence relation:

$$F_{t+1} = (1 + i)(F_t + C_t - B), \tag{2}$$

where i is the actual rate of return earned on the pension plan assets. The unfunded liability is defined as the excess of actuarial liability over assets:

$$UL_t = AL - F_t. \tag{3}$$

It is assumed that all actuarial valuation assumptions, other than i_A, are borne out by experience. In other words, demographic and economic experience unfold in accordance

with actuarial valuation assumptions, except that the actual investment rate of return i may differ from the assumed investment rate of return i_A.

An intervaluation loss L_t during year $(t,\ t+1)$ is the change in unfunded liability as a result of actual experience deviating from actuarial valuation assumptions (Dufresne, 1989). A gain is defined as a negative loss. More specifically, an *asset* loss is the unexpected increase in unfunded liability that is attributable to the actual investment return being less than the investment return assumption. The contribution that is paid at the start of year $(t,\ t+1)$ is equal to the normal cost NC plus a supplementary contribution S_t which is paid to amortize past intervaluation losses and any initial unfunded liability:

$$C_t \;=\; NC \,+\, S_t \tag{4}$$

Letting $v_L = (1 + i_L)^{-1}$, it follows from equations (1)–(4) that

$$UL_{t+1} \;=\; AL + (1+i)(UL_t - S_t - v_L AL). \tag{5}$$

Actual experience does not deviate from actuarial assumptions except possibly in investment returns. Therefore, only asset gains or losses occur. An expression for the asset loss is obtained by Dufresne (1989) as follows. Had a rate of return of i_A been earned on the plan assets (instead of the actual rate of return i), the unfunded liability at the end of year $(t,\ t+1)$ would have been $UL_{t+1}^A = AL + (1 + i_A)(UL_t - S_t - v_L AL)$, by comparison with equation (5). Therefore the intervaluation loss in year $(t,\ t+1)$ is

$$L_{t+1} \;=\; UL_{t+1} - UL_{t+1}^A \tag{6}$$

$$=\; UL_{t+1} - AL - (1 + i_A)(UL_t - S_t - v_L AL) \tag{7}$$

$$=\; (i - i_A)(UL_t - S_t - v_L AL). \tag{8}$$

Equation (8) shows that the asset intervaluation loss L_{t+1} in year $(t,\ t+1)$ arises because the actual return on assets in that year (i) is different from the assumed return (i_A).

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Equation (7) may be rewritten as

$$UL_{t+1} - u_A UL_t = L_{t+1} - u_A(S_t - (v_A - v_L)AL), \tag{9}$$

where $u_A = 1 + i_A$ and $v_A = (1 + i_A)^{-1}$.

The supplementary contribution S_t in equation (4) pays off over time past intervaluation losses as well as any initial unfunded liability at time 0. The initial unfunded liability may arise because of past service liabilities, or because of a change in the valuation basis or an amendment to benefit rules.

Assume henceforth that $L_t = 0$ for $t \leq 0$, $UL_t = 0$ for $t < 0$, and that the initial unfunded liability UL_0 is amortized over a finite period of n years at rate i_A by means of payments

$$P_t = \begin{cases} UL_0 / \ddot{a}_{\overline{n|}}, & 0 \leq t \leq n - 1, \\ 0, & t \geq n. \end{cases} \tag{10}$$

In equation (10), $\ddot{a}_{\overline{n|}} = (1 - v_A^n)/(1 - v_A)$ denotes the present value of an annuity-certain of term n payable in advance and calculated at rate i_A. The unamortized part of the initial unfunded liability at time t is

$$U_t = \begin{cases} UL_0 \ddot{a}_{\overline{n-t|}} / \ddot{a}_{\overline{n|}}, & 0 \leq t \leq n - 1, \\ 0, & t \geq n. \end{cases} \tag{11}$$

Observe that

$$u_A U_t - U_{t+1} = u_A P_t. \tag{12}$$

4   Amortizing Gains and Losses

Dufresne (1989) describes a funding method whereby the supplementary contribution S_t, in equation (4), is calculated to amortize past intervaluation gains and losses. His analysis may be extended by allowing for a distinction between the liability valuation rate (i_L)

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and the investment return assumption (i_A), as well as by explicitly amortizing the initial unfunded liability:

$$S_t = \sum_{j=0}^{m-1} \frac{L_{t-j}}{\ddot{a}_{\overline{m}|}} + (v_A - v_L)AL + P_t. \tag{13}$$

In equation (13), $\ddot{a}_{\overline{m}|} = (1 - v_A^m)/(1 - v_A)$ is the present value of an annuity-certain over m years payable in advance and calculated at assumed rate i_A. The supplementary contribution consists of level amortization payments for intervaluation losses over the past m years, an adjustment for the difference between assumed rates on assets and liabilities, as well as an amortization payment for the initial unfunded liability.

Replacing S_t from equation (13) into equation (9) and using equation (12) yields

$$(UL_{t+1} - U_{t+1}) - u_A(UL_t - U_t) = L_{t+1} - u_A \sum_{j=0}^{m-1} \frac{L_{t-j}}{\ddot{a}_{\overline{m}|}}. \tag{14}$$

The unfunded liability at the end of the year is therefore the accumulation of the unfunded liability at the start of the year plus the loss that emerges during the year less the accumulated value of payments made in respect of past losses.

It is easily verified that the solution of equation (14) is

$$UL_t - U_t = \sum_{j=0}^{m-1} \frac{\ddot{a}_{\overline{m-j}|}}{\ddot{a}_{\overline{m}|}} L_{t-j}. \tag{15}$$

For details of this solution, see Dufresne (1989). Note also equation (12) for the initial unfunded liability and recall that the annuities are valued at rate i_A.

When the funding method in equation (13) is used, a unit loss that emerged j years ago is completely paid off if $j \geq m$, but further payments of $1/\ddot{a}_{\overline{m}|}$ for the next $m - j$ years are outstanding if $0 \leq j \leq m - 1$. The present value of these payments is $\ddot{a}_{\overline{m-j}|}/\ddot{a}_{\overline{m}|}$. Equation (15) shows that the unfunded liability is the present value of the payments that remain to be made in respect of losses that are not yet paid off, together with the unamortized part of the initial unfunded liability.

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As in Dufresne (1989), replace S_t from equation (13) and UL_t from equation (15) into equation (8), and use equation (12), to obtain:

$$L_{t+1} = (i - i_A) \left[\sum_{j=0}^{m-1} L_{t-j} \left(\ddot{a}_{\overline{m-j|}} - 1 \right) / \ddot{a}_{\overline{m|}} - v_A (AL - U_{t+1}) \right]. \tag{16}$$

If the actual rate of return on plan assets in a given year is the same as the assumed rate of return (that is, if $i = i_A$), no intervaluation loss emerges in that year ($L_t = 0 \; \forall t$ from equation (16)) and the unfunded liability consists only of the unamortized part of the initial unfunded liability ($UL_t = U_t$ for $t \geq 0$ from equation (15)).

Dufresne (1989) obtains a sufficient condition for the convergence of $\{L_t\}$, $\{UL_t\}$ and $\{S_t\}$ as $t \to \infty$. The following result is due to Dufresne (1989).

RESULT 1 *Provided that* $|i - i_A| \sum_{j=0}^{m-1} (\ddot{a}_{\overline{m-j|}} - 1) / \ddot{a}_{\overline{m|}} < 1$,

$$\lim_{t \to \infty} L_t = \frac{-(i - i_A) v_A AL}{1 - (i - i_A) \sum_{j=0}^{m-1} (\ddot{a}_{\overline{m-j|}} - 1) / \ddot{a}_{\overline{m|}}}, \tag{17}$$

$$\lim_{t \to \infty} UL_t = \sum_{j=0}^{m-1} \frac{\ddot{a}_{\overline{m-j|}}}{\ddot{a}_{\overline{m|}}} \lim_{t \to \infty} L_t, \tag{18}$$

$$\lim_{t \to \infty} S_t = \frac{m}{\ddot{a}_{\overline{m|}}} \lim_{t \to \infty} L_t + (v_A - v_L) AL. \tag{19}$$

The only differences between equations (17)–(19) and the results of Dufresne (1989) are that the annuities are valued at rate i_A here and there is an explicit term for the difference between i_A and i_L in equation (19). Equations (17)–(19) follow from equations (16), (15) and (13). (Recall that $U_t = 0$ for $t \geq n$ from equation (11) since the initial unfunded liability is amortized over a finite period n.)

COROLLARY 1 *Assume that* $|i - i_A| \sum_{j=0}^{m-1} (\ddot{a}_{\overline{m-j|}} - 1) / \ddot{a}_{\overline{m|}} < 1$.
If $i_A = i$, *then* $\lim UL_t = 0$. *If* $i_A > i$, *then* $\lim UL_t > 0$. *If* $i_A < i$, *then* $\lim UL_t < 0$.

Corollary 1 confirms the observations made in section 2: if the actuarial investment return assumption is optimistic (that is, $i_A > i$), then a persistent deficit occurs ($\lim UL_t >$

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0); on the other hand, if the investment return assumption is conservative (that is, $i_A < i$), then a persistent surplus occurs ($\lim UL_t < 0$). Note also that, if $i_A \neq i$, $\lim UL_t$ depends on the period m over which gains and losses are amortized.

5   Spreading Gains and Losses

Dufresne (1988) discusses another funding method that is used to determine contributions. This method is widely used in the United Kingdom and is also implicit in actuarial cost methods such as the Aggregate and Frozen Initial Liability methods (Trowbridge and Farr, 1976, p. 85). The equations in Dufresne (1988) may also be extended to allow for the distinction between the rate at which liabilities are discounted and the investment return assumption, as well as for the separate treatment of the initial unfunded liability.

The supplementary contribution paid in year $(t, t + 1)$ is

$$S_t = \sum_{j=0}^{\infty}(1 - K)K^j u_A^j L_{t-j} + (v_A - v_L)AL + P_t, \tag{20}$$

where $0 \leq K < v_A$. In this alternative method, a unit loss is paid off by means of a sequence of exponentially declining payments, $\{(1 - K)K^j u_A^j, \ j = 0, 1, \ldots\}$, the unit loss being paid off in perpetuity since $\sum_{j=0}^{\infty}(1 - K)K^j u_A^j \cdot v_A^j = 1$. The larger the parameter K, the slower the loss is paid off. The loss is never completely defrayed, except in the limit as $t \to \infty$, but Trowbridge and Farr (1976) point out that this is not a weakness as intervaluation losses occur randomly in practice and are never completely removed. This funding method is commonly referred to as "spreading" gains and losses, by contrast with the method in section 4 which involves amortizing gains and losses (McGill *et al.*, 1996, p. 525; Berin, 1989, p. 18; Dufresne, 1988).

Replacing S_t from equation (20) into equation (9) and using equation (12) yields

$$(UL_{t+1} - U_{t+1}) - u_A(UL_t - U_t) = L_{t+1} - u_A \sum_{j=0}^{\infty}(1 - K)K^j u_A^j L_{t-j}. \tag{21}$$

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Recall that $L_t = 0$ for $t \leq 0$, $UL_t = 0$ for $t < 0$, and $UL_0 = U_0$. It is easily verified, from equation (21), that

$$UL_t - U_t = \sum_{j=0}^{\infty} K^j u_A^j L_{t-j}. \tag{22}$$

Compare equation (15) when losses are amortized to equation (22) when losses are spread.

Equation (22) is sensible since, for a unit loss that emerged j years ago, the following sequence of payments is outstanding: $\{(1 - K)K^l u_A^l, \; l = j, j + 1, \ldots\}$. The present value of these payments is $\sum_{l=j}^{\infty} (1 - K)K^l u_A^l \cdot v_A^{l-j} = K^j u_A^j$. Equation (22) thus shows that, at any time t, the unfunded liability is the present value of payments yet to be made in respect of all past and present losses, together with the unamortized part of the initial unfunded liability.

The supplementary contribution S_t in this method may be calculated directly as a proportion $1 - K$ of the unfunded liability, together with an adjustment for the difference between assumed rates on assets and liabilities and for the separate amortization of the initial unfunded liability. Comparing equations (20) and (22),

$$S_t = (1 - K)(UL_t - U_t) + (v_A - v_L)AL + P_t. \tag{23}$$

For simplicity, Dufresne (1988) disregards the separate treatment of initial unfunded liability and the distinction between i_A and i_L and considers only $S_t = (1 - K)UL_t$. Dufresne (1988) also states that the parameter K is usually calculated as $K = 1 - 1/\ddot{a}_{\overline{M}|}$. M is typically between 1 and 10 years in the United Kingdom. Thus, if $M = m$, the first payment made in respect of a unit loss is $1/\ddot{a}_{\overline{m}|}$ under both the amortization and spreading funding methods (equations (13) and (20) respectively).

Replace S_t from equation (20) and UL_t from equation (22) into equation (8), and use equation (12), to obtain:

$$L_t = (i - i_A) \left[\sum_{j=0}^{\infty} K^{j+1} u_A^j L_{t-j} - v_A(AL - U_{t+1}) \right]. \tag{24}$$

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Compare equation (16) when losses are amortized to equation (24) when losses are spread. If the actuarial assumption as to the rate of investment return on plan assets equals the actual rate of return (that is, if $i = i_A$), then no loss emerges ($L_t = 0 \; \forall t$ from equation (24)) and the unfunded liability consists only of the unamortized part of the initial unfunded liability ($UL_t = U_t$ for $t \geq 0$ from equation (22)).

From equation (24),

$$L_{t+1} - u_A K \, L_t \;=\; (i - i_A) \left[K \, L_t - v_A(AL - U_{t+1}) + K(AL - U_t) \right], \tag{25}$$

which is a first-order linear difference equation that simplifies to

$$L_{t+1} - u K \, L_t \;=\; -v_A(i - i_A) \left[(AL - U_{t+1}) - u_A K(AL - U_t) \right]. \tag{26}$$

Recall from equation (11) that $U_t = 0$ for $t \geq n$. Provided $|uK| < 1$, it follows from equation (26) that

$$\lim_{t \to \infty} L_t \;=\; -AL(i - i_A) v_A \frac{1 - u_A K}{1 - uK}. \tag{27}$$

In equation (20), K was defined to be such that $0 \leq K < v_A$. Provided $|uK| < 1$, the right hand side of equation (22) is also absolutely convergent and

$$\lim_{t \to \infty} UL_t \;=\; (1 - u_A K)^{-1} \lim_{t \to \infty} L_t. \tag{28}$$

$\lim S_t$ may be found from equations (23) and (28). This is summarised in the following result.

RESULT 2 *Provided $|uK| < 1$,*

$$\lim_{t \to \infty} L_t \;=\; -AL(i - i_A) v \frac{v_A - K}{v - K}, \tag{29}$$

$$\lim_{t \to \infty} UL_t \;=\; AL \frac{v - v_A}{v - K}, \tag{30}$$

$$\lim_{t \to \infty} S_t \;=\; AL(1 - K) \frac{v - v_A}{v - K} \;+\; AL(v_A - v_L). \tag{31}$$

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In contrast with Dufresne (1988), we have allowed for separate amortization of the initial unfunded liability and also for the possibility that the actuarial assumptions i_A and i_L are different, and we have also derived equations pertaining to the intervaluation loss L_t. Result 2 may alternatively be obtained, as in Dufresne (1988), by substituting S_t from equation (23) into equation (5) giving a first-order difference equation

$$(UL_{t+1} - U_{t+1}) - uK(UL_t - U_t) = (1 - uv_A)(AL - U_{t+1}), \tag{32}$$

which solves to

$$UL_t - U_t = (1 - uv_A) \sum_{j=0}^{t-1} (uK)^j (AL - U_{t-j}). \tag{33}$$

Corollary 2 hereunder follows directly from equation (30):

COROLLARY 2 *Assume that* $|uK| < 1$*. If* $i_A = i$*, then* $\lim UL_t = 0$*. If* $i_A > i$*, then* $\lim UL_t > 0$*. If* $i_A < i$*, then* $\lim UL_t < 0$*.*

Compare Corollary 1 with Corollary 2. Under both amortization and spreading, the choice of the actuarial investment return assumption i_A affects the long-term funding status of the pension plan. Note also from equation (30) that, when $i_A \neq i$, $\lim UL_t$ depends on the parameter K that is used to spread gains and losses.

6 Modified Spreading of Gains and Losses

If the actual investment return deviates from the actuarial investment return assumption, then persistent underfunding or overfunding will occur in the long term, as shown in Corollaries 1 and 2 in both of the preceding methods. Persistent deficits jeopardize the security of pension benefits for plan members since, in the event of sponsor insolvency, there will not be enough funds to meet benefit obligations. On the other hand, excessive surpluses are also undesirable as funds are being diverted from productive activity in the

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company. Plan participants may also demand that surpluses be distributed to them in the form of improved benefits (McGill *et al.*, 1996, p. 592-4).

In practice, the emergence of persistent surpluses or deficits causes actuaries to revise their actuarial valuation assumptions. Nevertheless, it is of interest to devise a funding method that avoids systematic surpluses and deficits.

Suppose that a constant stream of intervaluation losses of size $\ell > 0$ occurs in the pension plan. If losses are being amortized as in the method of section 4, then a positive unfunded liability (that is, a deficit) occurs since, from equation (15) and for $t \geq n$,

$$UL_t = \ell \sum_{j=0}^{m-1} \frac{\ddot{a}_{\overline{m-j}}}{\ddot{a}_{\overline{m}}} > 0. \tag{34}$$

Likewise, a deficit occurs if losses are being spread, as in section 5, since, from equation (22) and for $t \geq n$,

$$UL_t = \ell/(1 - u_A K) > 0. \tag{35}$$

This suggests a variation on the spreading of losses. Consider a new funding method, which is referred to henceforth as "modified spreading of gains and losses", where supplementary contributions are calculated to pay off intervaluation losses and the initial unfunded liability as follows:

$$S_t = \sum_{j=0}^{\infty} (\alpha_1 K_1^j - \alpha_2 K_2^j) u_A^j L_{t-j} + (v_A - v_L) AL + P_t \tag{36}$$

where

$$\alpha_1 = (1 - u_A K_1)(1 - K_1)/u_A(K_2 - K_1), \tag{37}$$

$$\alpha_2 = (1 - u_A K_2)(1 - K_2)/u_A(K_2 - K_1), \tag{38}$$

and where $0 \leq K_1 < v_A$ and $0 \leq K_2 < v_A$ and $K_1 \neq K_2$.

In this method, a unit loss is liquidated by means of an infinite sequence of payments

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$\{(\alpha_1 K_1^j - \alpha_2 K_2^j)u_A^j, \ j = 0, 1, \ldots\}$ and is paid off in perpetuity since

$$\sum_{j=0}^{\infty}(\alpha_1 K_1^j - \alpha_2 K_2^j)u_A^j \cdot v_A^j \ = \ \frac{\alpha_1}{1-K_1} - \frac{\alpha_2}{1-K_2} \ = \ 1. \tag{39}$$

Replacing S_t from equation (36) into equation (9) and using equation (12) yields

$$(UL_{t+1} - U_{t+1}) - u_A(UL_t - U_t) \ = \ L_{t+1} \ - \ u_A\sum_{j=0}^{\infty}(\alpha_1 K_1^j - \alpha_2 K_2^j)u_A^j L_{t-j}. \tag{40}$$

Now define

$$\beta_1 \ = \ (1 - u_A K_1)/u_A(K_2 - K_1), \tag{41}$$

$$\beta_2 \ = \ (1 - u_A K_2)/u_A(K_2 - K_1). \tag{42}$$

Noting that $\alpha_1 = \beta_1(1 - K_1)$ and $\alpha_2 = \beta_2(1 - K_2)$ and $\beta_1 - \beta_2 = 1$, the right hand side of equation (40) may be rewritten as

$$
\begin{aligned}
L_{t+1} \ &+ \ u_A\sum_{j=0}^{\infty}(\beta_1 K_1^{j+1} - \beta_2 K_2^{j+1})u_A^j L_{t-j} \ - \ u_A\sum_{j=0}^{\infty}(\beta_1 K_1^j - \beta_2 K_2^j)u_A^j L_{t-j} \\
&= \ u_A\sum_{j=-1}^{\infty}(\beta_1 K_1^{j+1} - \beta_2 K_2^{j+1})u_A^j L_{t-j} \ - \ u_A\sum_{j=0}^{\infty}(\beta_1 K_1^j - \beta_2 K_2^j)u_A^j L_{t-j} \\
&= \ \sum_{j=0}^{\infty}(\beta_1 K_1^j - \beta_2 K_2^j)u_A^j L_{t+1-j} \ - \ u_A\sum_{j=0}^{\infty}(\beta_1 K_1^j - \beta_2 K_2^j)u_A^j L_{t-j}, \tag{43}
\end{aligned}
$$

which, upon comparison with the left hand side of equation (40), yields

$$UL_t - U_t \ = \ \sum_{j=0}^{\infty}(\beta_1 K_1^j - \beta_2 K_2^j)u_A^j L_{t-j}. \tag{44}$$

Compare equations (15), (22) and (44).

Under the method of equation (36), for a unit loss that emerged j years ago, the following sequence of payments is yet to be made: $\{(\alpha_1 K_1^l - \alpha_2 K_2^l)u_A^l, \ l = j, j+1\ldots\}$. The present value of these outstanding payments is therefore

$$\sum_{l=j}^{\infty}(\alpha_1 K_1^l - \alpha_2 K_2^l)u_A^l \cdot v_A^{l-j} \ = \ \left[\frac{\alpha_1 K_1^j}{1-K_1} - \frac{\alpha_2 K_2^j}{1-K_2}\right]u_A^j \ = \ (\beta_1 K_1^j - \beta_2 K_2^j)u_A^j. \tag{45}$$

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Equation (44) thus shows that, at any time t, the unfunded liability is the present value of payments yet to be made in respect of all past and present losses, together with the unamortized part of the initial unfunded liability.

The following proposition is proven in the Appendix.

PROPOSITION 1 *Provided that*

$$\min(i, i_A) > -100\%, \tag{46}$$

$$i - i_A < 100\% + i_A, \tag{47}$$

$$0 \leq \min(K_1, K_2) < \max(K_1, K_2) < \min(v, v_A), \tag{48}$$

then

$$\lim_{t \to \infty} L_t = -AL(i - i_A)v, \tag{49}$$

$$\lim_{t \to \infty} UL_t = 0, \tag{50}$$

$$\lim_{t \to \infty} S_t = AL(v - v_L). \tag{51}$$

The sufficient conditions (46)–(48) in Proposition 1 are not very restrictive. (Necessary and sufficient conditions are discussed in the Appendix.) Condition (46) is easily satisfied under normal economic conditions. Condition (47) also holds in practice. Long-run economic growth means that the actuarial assumption i_A as to the long-term rate of return on plan assets is positive ($i_A > 0$). Condition (47) then requires that the actuarial investment return assumption i_A does not underestimate the actual return on assets i by 100% or more. Condition (48) is also easily met in practice. For example, if $\max(i, i_A) = 15\%$, then $0 \leq K_1 < 0.87$ and $0 \leq K_2 < 0.87$ with $K_1 \neq K_2$ means that condition (48) holds.

COROLLARY 3 *Assume that conditions (46)–(48) hold. Then,* $\lim UL_t = 0$, *irrespective of whether* $i_A = i$ *or* $i_A > i$ *or* $i_A < i$.

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Compare Corollaries 1, 2 and 3. Corollary 3 states that, under the modified spreading funding method described by equation (36), the pension plan is fully funded in the long term, irrespective of the deviation of the investment return assumption from the actual return on the pension plan assets (provided that the mild conditions (46)–(48) hold). Furthermore, $\lim UL_t$ is independent of the funding method parameters K_1 and K_2.

The choice of i_A affects the progression of funding in the short term, but i_A does not affect the funding position asymptotically. In fact, one could arbitrarily set $i_A = i_L$ as under the classical actuarial valuation methodology described in section 2 and effectively dispense with an investment return assumption i_A that is distinct from the rate i_L at which the pension liability is valued.

Corollary 3 may be explained as follows. Suppose that a constant stream of intervaluation losses of size $\ell > 0$ occurs in the pension plan. Recall that this results in a persisting deficit when losses are being either amortized or spread: see equations (34) and (35) respectively. By contrast, under the method of equation (36), a constant stream of losses of size $\ell \neq 0$ results in zero unfunded liability because, from equation (44) and for $t \geq n$,

$$UL_t = \ell \sum_{j=0}^{\infty} (\beta_1 K_1^j - \beta_2 K_2^j) u_A^j = \ell \left[\frac{\beta_1}{1 - u_A K_1} - \frac{\beta_2}{1 - u_A K_2} \right] = 0, \qquad (52)$$

where we use equations (41) and (42).

It was shown that the spreading method of equation (20) could be calculated more directly in terms of the unfunded liability, in equation (23). This may also be achieved here. The following proposition is proven in the Appendix.

PROPOSITION 2 *The funding method described in equation (36) is equivalently achieved by calculating supplementary contributions as follows:*

$$S_t = \lambda_1 (UL_t - U_t) + \lambda_2 \sum_{j=0}^{\infty} (UL_{t-j} - U_{t-j}) + (v_A - v_L)AL + P_t, \qquad (53)$$

where $\lambda_1 = 1 - u_A K_1 K_2$ and $\lambda_2 = v_A(1 - u_A K_1)(1 - u_A K_2)$.

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Trowbridge and Farr (1976, p. 62) state that "easy computations" are a desirable characteristic of a funding method. Equation (53) provides a straightforward way of computing contributions from year to year as only the historic sum of unfunded liabilities need be stored and updated.

Compare equations (23) and (53). The second term on the right hand side of equation (53) represents a historic sum (without interest) of past unfunded liabilities. Contributions are therefore paid until surpluses and deficits cancel each other out and the unfunded liability is zero. Modified spreading of gains and losses, in the representation of equation (53), is similar to a method described by Balzer (1982) in the context of a general insurance system (see also Taylor, 1987). Balzer (1982) refers to a summation term similar to the second term on the right hand side of equation (53) as supplying an "integral action" which adjusts for a "persisting stream of unpredicted claims".

7 Numerical Example

An illustration of the previous results is given here and is based on the following:

Demographic projections: Mortality: English Life Table No. 12 (males). Plan population: stationary with single entry age of 20 and single retirement age of 65.

Salary: Constant throughout working lifetime.

Benefit: A level pension at age 65 paying 2/3 of annual salary.

Economic projections: No inflation. Assets earn a constant rate of return of 4.5%.

Initial unfunded liability: Zero. (Alternatively, assume that UL_0 is being separately amortized as in equation (10) and that $UL_t - U_t$, rather than UL_t is evaluated below.)

Actuarial valuations: Frequency: yearly. Actuarial cost method: unit credit.

Actuarial assumptions: Fixed with valuation assumptions $i_L = 4\%$, $i_A = 1\%$, 4.5% and 6%. Other valuation assumptions are identical to projection assumptions.

Valuation data: Number of entrants and payroll are calculated such that the yearly benefit outgo B is normalized to 1. Actuarial liability $AL = 16.94$, normal cost $NC = 0.3486$, both expressed as a proportion of B.

Funding method parameters: Amortization: $m = 5$. Spreading: $K = 1 - 1/\ddot{a}_{\overline{5}|}$. Modified spreading: $K_1 = K$, $K_2 = 0.8$.

When $i = i_A = 4.5\%$, numerical work (not shown here) shows that neither gain nor loss arises and the funded ratio (that is, ratio of fund value to actuarial liability) remains at 100%, while the contribution paid is equal to the normal cost, for all three methods. This accords with Corollaries 1, 2 and 3 when $i_A = i$.

When $i = 4.5\%$ and $i_A = 6\%$, the investment return assumption is optimistic. Fund values (as a percentage of actuarial liability) and contributions (as a percentage of normal cost) over time are exhibited in Table 1. See also Figure 1. A contribution that is equal to 11.8% of normal cost is required initially under all three methods. Under amortization, the required contribution levels off at 86.6% of normal cost and an unfunded liability of 4.3% of actuarial liability remains. Under spreading, the contribution rises steadily to 93.3% of normal cost and an unfunded liability of 7.5% of actuarial liability is left eventually. Under modified spreading, the required contribution stabilizes at about 78% of normal cost with the plan being fully funded eventually. This therefore agrees with Corollaries 1, 2 and 3 when $i_A > i$: long-run deficits occur under amortization and spreading, but not under modified spreading. Furthermore, numerical experiments suggest that the long-run unfunded liabilities that occur under amortization and spreading are larger, the larger the deviation between actual and assumed returns.

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Note that the pension fund is ultimately in balance under all three methods. For example, under amortization, using units of yearly benefit outgo, a fund of $95.7\% \times 16.94 = 16.21$ yields investment income of $16.21 \times 4.5\% = 0.7295$ at the end of the year. At the start of the year, the present value of this income is $0.7295/1.045 = 0.698$. Contribution income is $86.6\% \times 0.3486 = 0.301$. Total income is $0.698 + 0.301 = 1$ which balances the benefit of 1 that is paid out. The balance occurs at different levels under the three methods. Under modified spreading, the fund is eventually in equilibrium in such a way that the pension plan is fully funded.

When $i = 4.5\%$ and $i_A = 1\%$, a conservative investment return assumption is being made. See Table 2 and Figure 2. A large contribution (more than double the normal cost) is required initially under all three funding methods. Intervaluation gains lead initially to falling contributions under all three methods (at about the same rate). Ultimately, the lowest contribution (at only 35% of normal cost) is generated when spreading is used, but this is at the expense of a large surplus in the pension fund of 20% of actuarial liability. On the other hand, the surplus is only 5% of actuarial liability within 10 years, and under 1% within 20 years, when modified spreading is used. This also agrees with Corollaries 1, 2 and 3 when $i_A < i$.

8 Conclusion

The investment return assumption made by actuaries when valuing defined benefit pension plans and its relevance to the pace of funding for pension benefits was discussed in section 2. It was argued that this assumption is theoretically distinct from the discount rate that is used to value pension liabilities, although they may be equal in practice. A simplified model pension plan was posited in Section 3, where actuarial liability, normal cost and benefit outgo were constant. The only intervaluation gains and losses allowed in the model resulted

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from actual investment return deviating from the actuarial investment return assumption.

Two practical funding methods were described in sections 4 and 5 and it was shown, in both cases, that a conservative investment return assumption leads to a long-term surplus whereas an optimistic investment return assumption leads to a long-term deficit. Both long-term surpluses and deficits were deemed to be undesirable. Surpluses may entail expensive demands for benefit enhancements from plan members during wage negotiations and also involves the diversion of capital away from projects within the sponsoring corporation. Deficits may endanger the security of pension benefits should the plan sponsor become insolvent. A funding method was devised and described in section 6 that avoids such persistent surpluses and deficits, under mild stability conditions, independently of the conservatism or optimism in the actuarial investment return assumption. A simple way of implementing this funding method was derived in terms of the historic sum of past unfunded liabilities. A numerical illustration of these results was provided in section 7.

The analysis in this paper yielded closed-form mathematical solutions but this required simplistic modelling assumptions. Future research should relax these restrictive assumptions. First, only asset gains and losses were considered. Mortality, withdrawal, inflation and other factors are also variable and should be incorporated in the model. Second, these factors are uncertain and intervaluation gains and losses are random. A stochastic approach following Dufresne (1988, 1989), who investigates the moments of various quantities in the pension fund when investment returns are random, should be illuminating. It will enable a more realistic comparison of the various funding methods to be made in terms of the variance of fund values and contributions. The efficient choice of parameters K_1 and K_2 under modified spreading of gains and losses can also then be investigated.

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Appendix

Proof of Proposition 1

It is easy to show, from equations (37), (38), (41) and (42) that

$$\alpha_1 - \alpha_2 \; = \; 1 + v_A - K_1 - K_2, \tag{54}$$

$$\alpha_1 K_2 - \alpha_2 K_1 \; = \; v_A - K_1 K_2, \tag{55}$$

$$\beta_1 - \beta_2 \; = \; 1, \tag{56}$$

$$\beta_1 K_2 - \beta_2 K_1 \; = \; v_A. \tag{57}$$

Replace S_t from equation (36) and UL_t from equation (44) into equation (8), and use equation (12), to obtain:

$$L_{t+1} \; = \; (i - i_A) \left[\sum_{j=0}^{\infty} \left[(\beta_1 - \alpha_1) K_1^j - (\beta_2 - \alpha_2) K_2^j \right] u_A^j L_{t-j} - v_A(AL - U_{t+1}) \right]. \tag{58}$$

This may be rewritten using the lag or backward shift operator B as follows:

$$B^{-1} L_t \; = \; (i - i_A) \left[\frac{\beta_1 - \alpha_1}{1 - u_A K_1 B} L_t - \frac{\beta_2 - \alpha_2}{1 - u_A K_2 B} L_t - B^{-1} v_A(AL - U_t) \right]. \tag{59}$$

Note from equations (54)–(57) that

$$(\beta_1 - \alpha_1) - (\beta_2 - \alpha_2) \; = \; K_1 + K_2 - v_A, \tag{60}$$

$$(\beta_1 - \alpha_1) K_2 - (\beta_2 - \alpha_2) K_1 \; = \; K_1 K_2. \tag{61}$$

Multiply both sides of equation (59) by $(1 - u_A K_1 B)(1 - u_A K_2 B)B$ and use the two equations above:

$$(1 - u_A K_1 B)(1 - u_A K_2 B) L_t = (i - i_A) \left[(K_1 + K_2 - v_A) B \, L_t - u_A K_1 K_2 B^2 L_t \right.$$
$$\left. - (1 - u_A K_1 B)(1 - u_A K_2 B) v_A(AL - U_t) \right]. \tag{62}$$

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Collect terms in L_t on the left hand side to obtain a second order linear difference equation for L_t:

$$\left\{1 - B\left[uK_1 + uK_2 - uv_A + 1\right] + B^2\left[uu_AK_1K_2\right]\right\}L_t$$
$$= -(1 - u_AK_1B)(1 - u_AK_2B)v_A(i - i_A)(AL - U_t). \quad (63)$$

Difference equation (63) has a quadratic characteristic equation,

$$P(z) = z^2 - z[uK_1 + uK_2 - uv_A + 1] + uu_AK_1K_2 = 0, \quad (64)$$

whose roots must be less than one in magnitude for $\{L_t\}$ to converge as $t \to \infty$. Necessary and sufficient conditions for this for a general quadratic equation are given by Marden (1966):

$$|P(0)| < 1 \quad \Rightarrow \quad |uu_AK_1K_2| < 1, \quad (65)$$

$$P(1) > 0 \quad \Rightarrow \quad uv_A(1 - u_AK_1)(1 - u_AK_2) > 0, \quad (66)$$

$$P(-1) > 0 \quad \Rightarrow \quad uv_A[2u_A(v + u_AK_1K_2) - (1 - u_AK_1)(1 - u_AK_2)] > 0. \quad (67)$$

It is now shown that inequalities (65)–(67) follow from the sufficient conditions in Proposition 1. Note first that condition (46) may be rewritten as $0 < \min(u, u_A) \leq \max(u, u_A)$. Conditions (46) and (48) thus imply that

$$0 \leq \min(u, u_A)\min(K_1, K_2) < \max(u, u_A)\max(K_1, K_2) < 1. \quad (68)$$

Hence, inequality (65) follows from sufficient conditions (46) and (48).

Next, note from the inequalities (68) that

$$0 < 1 - \max(u, u_A)\max(K_1, K_2) < 1 - \min(u, u_A)\min(K_1, K_2) \leq 1, \quad (69)$$

and therefore that

$$0 < (1 - u_AK_1) \leq 1 \quad \text{and} \quad 0 < (1 - u_AK_2) \leq 1. \quad (70)$$

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Hence, inequality (66) follows from sufficient conditions (46) and (48).

Finally, condition (47) may be written as $u < 2u_A$ or $2u_A v > 1$, by virtue of condition (46). It follows from inequalities (70) that

$$(1 - u_A K_1)(1 - u_A K_2) \leq 1 < 2u_A v \leq 2u_A v + 2u_A^2 K_1 K_2 \tag{71}$$

$$\Rightarrow 2u_A(v + u_A K_1 K_2) - (1 - u_A K_1)(1 - u_A K_2) > 0. \tag{72}$$

Hence, inequality (67) follows from sufficient conditions (46), (47) and (48).

Let the roots of the characteristic equation (64) be ν_1 and ν_2. If $\nu_1 \neq \nu_2$, L_t in equation (63) has a solution of the form $L_t = A\nu_1^t + B\nu_2^t + L$, where A, B, $L \in \mathbb{R}$. If sufficient conditions (46)–(48) hold, then $|\nu_1| < 1$ and $|\nu_2| < 1$, the sequence $\{L_t\}$ converges to L and, furthermore, the series $\sum_{j=0}^{\infty}(L_j - L)$ is absolutely convergent.

Assuming convergence, it is clear from equation (63) that

$$
\begin{aligned}
L = \lim_{t \to \infty} L_t &= \frac{-(1 - u_A K_1)(1 - u_A K_2)v_A(i - i_A)AL}{1 - [uK_1 + uK_2 - uv_A + 1] + [uu_A K_1 K_2]} \\
&= -AL(i - i_A)v\frac{v_A(1 - u_A K_1)(1 - u_A K_2)}{v_A(1 - u_A K_1)(1 - u_A K_2)} = -AL(i - i_A)v, \quad (73)
\end{aligned}
$$

which proves equation (49).

The limit in equation (51) is obtained by resorting to equation (36):

$$S_t = \sum_{j=0}^{\infty}(\alpha_1 K_1^j - \alpha_2 K_2^j)u_A^j(L_{t-j} - L) + \sum_{j=0}^{\infty}(\alpha_1 K_1^j - \alpha_2 K_2^j)u_A^j L$$

$$+ (v_A - v_L)AL + P_t. \tag{74}$$

As $t \to \infty$, the first sum on the right hand side of equation (74) vanishes since both $\sum_{j=0}^{\infty}(\alpha_1 K_1^j - \alpha_2 K_2^j)u_A^j$ and $\sum_{j=0}^{\infty}(L_j - L)$ are absolutely convergent and their Cauchy product is also absolutely convergent. As $t \to \infty$, the second sum on the right hand side of equation (74) converges to

$$\left(\frac{\alpha_1}{1 - u_A K_1} - \frac{\alpha_2}{1 - u_A K_2}\right) \times L = v_A \times -AL(i - i_A)v = -AL(v_A - v), \tag{75}$$

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where use is made of equations (37) and (38). P_t also vanishes as $t \to \infty$ from equation (10). Hence, $\lim_{t \to \infty} S_t = -AL(v_A - v) + AL(v_A - v_L) = AL(v - v_L)$.

Finally, the limit in equation (50) is obtained by taking limits on each term on the right hand side of equation (44) which may be rewritten as follows:

$$UL_t - U_t = \sum_{j=0}^{\infty}(\beta_1 K_1^j - \beta_2 K_2^j)u_A^j(L_{t-j} - L) + \sum_{j=0}^{\infty}(\beta_1 K_1^j - \beta_2 K_2^j)u_A^j L. \qquad (76)$$

As $t \to \infty$, the first sum on the right hand side of equation (76) vanishes since both $\sum_{j=0}^{\infty}(\beta_1 K_1^j - \beta_2 K_2^j)u_A^j$ and $\sum_{j=0}^{\infty}(L_j - L)$ are absolutely convergent and their Cauchy product is also absolutely convergent. As $t \to \infty$, the second sum on the right hand side of equation (76) converges to zero since

$$\sum_{j=0}^{\infty}(\beta_1 K_1^j - \beta_2 K_2^j)u_A^j = \frac{\beta_1}{1 - u_A K_1} - \frac{\beta_2}{1 - u_A K_2} = 0, \qquad (77)$$

where use is made of equations (41) and (42). Hence, $\lim_{t \to \infty} UL_t = 0$. $\square$

Proof of Proposition 2

Rewrite equation (44) in terms of the lag or backward shift operator B:

$$\begin{aligned} UL_t - U_t &= \left[\frac{\beta_1}{1 - u_A K_1 B} - \frac{\beta_2}{1 - u_A K_2 B} \right] L_t \\ &= \frac{(\beta_1 - \beta_2) - (\beta_1 K_2 - \beta_2 K_1)u_A B}{(1 - u_A K_1 B)(1 - u_A K_2 B)} L_t. \end{aligned} \qquad (78)$$

Using equations (56) and (57), the numerator on the right hand side of the above equation simplifies and

$$UL_t - U_t = \frac{1 - B}{(1 - u_A K_1 B)(1 - u_A K_2 B)} L_t. \qquad (79)$$

Likewise, rewrite equation (36) in terms of the backward shift operator B and use

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equations (54) and (55) to simplify:

$$S_t = \left[\frac{\alpha_1}{1 - u_A K_1 B} - \frac{\alpha_2}{1 - u_A K_2 B}\right] L_t + (v_A - v_L)AL + P_t$$

$$= \frac{(1 + v_A - K_1 - K_2) - (v_A - K_1 K_2)u_A B}{(1 - u_A K_1 B)(1 - u_A K_2 B)} L_t + (v_A - v_L)AL + P_t. \quad (80)$$

Cancel L_t from equations (79) and (80) and simplify:

$$S_t - (v_A - v_L)AL - P_t$$

$$= \frac{(1 + v_A - K_1 - K_2) - (v_A - K_1 K_2)u_A B}{1 - B}(UL_t - U_t)$$

$$= \left[1 - u_A K_1 K_2 + \frac{v_A(1 - u_A K_1)(1 - u_A K_2)}{1 - B}\right](UL_t - U_t)$$

$$= (1 - u_A K_1 K_2)(UL_t - U_t) + v_A(1 - u_A K_1)(1 - u_A K_2)\sum_{j=0}^{\infty}(UL_{t-j} - U_{t-j}), \quad (81)$$

which is equation (53). $\square$

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Time	Fund value (%)			Contribution (%)		
	A	S	MS	A	S	MS
0	100.0	100.0	100.0	11.8	11.8	11.8
2	97.4	97.4	97.6	42.5	39.7	55.6
4	96.0	95.8	96.7	72.5	58.1	78.2
6	95.7	94.6	96.6	87.0	70.1	88.8
8	95.7	93.9	96.8	86.6	78.1	92.9
10	95.7	93.4	97.2	86.6	83.3	93.4
12	95.7	93.1	97.7	86.6	86.7	92.3
14	95.7	92.9	98.1	86.6	89.0	90.5
16	95.7	92.8	98.4	86.6	90.5	88.4
18	95.7	92.7	98.8	86.6	91.4	86.5
20	95.7	92.6	99.0	86.6	92.1	84.8
25	95.7	92.6	99.5	86.6	92.9	81.7
30	95.7	92.5	99.7	86.6	93.2	79.8
35	95.7	92.5	99.8	86.6	93.3	78.8
40	95.7	92.5	99.9	86.6	93.3	78.3
45	95.7	92.5	100.0	86.6	93.3	78.0
50	95.7	92.5	100.0	86.6	93.3	77.8

Table 1: Fund value (per cent of actuarial liability) and contribution (per cent of normal cost) when $i_A = 6\%$ and $i = 4.5\%$ for amortization (A), spreading (S) and modified spreading (MS).

Time	Fund value (%)			Contribution (%)		
	A	S	MS	A	S	MS
0	100.0	100.0	100.0	238.8	238.8	238.8
2	106.3	106.3	105.7	169.1	175.9	124.1
4	110.2	110.7	107.5	96.5	132.3	66.3
6	111.2	113.8	107.3	57.1	102.2	41.7
8	111.2	115.9	106.3	54.6	81.3	35.2
10	111.3	117.3	105.1	54.1	66.9	37.7
12	111.3	118.3	103.8	54.1	56.9	44.0
14	111.3	119.0	102.8	54.1	50.0	51.2
16	111.3	119.5	102.0	54.1	45.3	57.8
18	111.3	119.9	101.3	54.1	42.0	63.4
20	111.3	120.1	100.9	54.1	39.7	67.7
25	111.3	120.4	100.3	54.1	36.6	74.2
30	111.3	120.5	100.1	54.1	35.3	76.7
35	111.3	120.6	100.0	54.1	34.9	77.5
40	111.3	120.6	100.0	54.1	34.7	77.7
45	111.3	120.6	100.0	54.1	34.6	77.7
50	111.3	120.6	100.0	54.1	34.5	77.7

Table 2: Fund value (per cent of actuarial liability) and contribution (per cent of normal cost) when $i_A = 1\%$ and $i = 4.5\%$ for amortization (A), spreading (S) and modified spreading (MS).
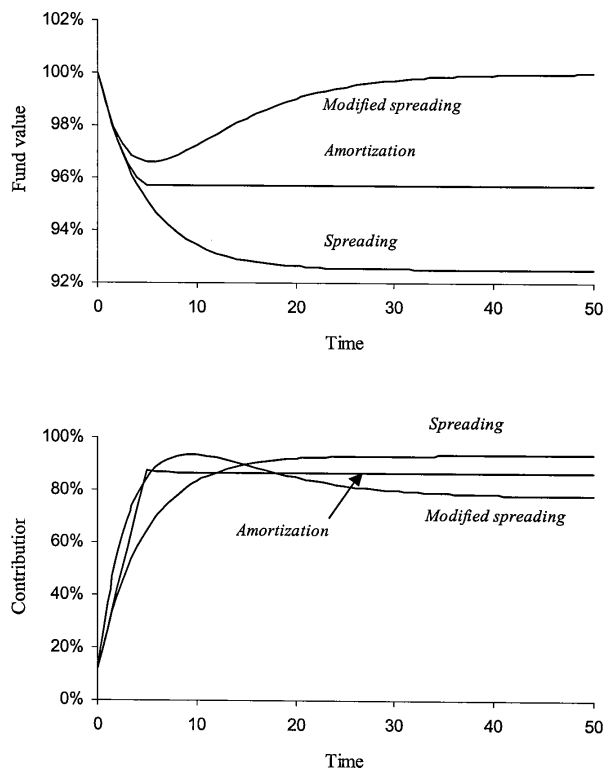
Figure 1: Fund value (per cent of actuarial liability) and contribution (per cent of normal cost) against time (years) when $i_A = 6\%$ and $i = 4.5\%$ for amortization, spreading and modified spreading.

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