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by

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On the Forecasting of Mortality Reduction Factors

A.E. Renshaw, S. Haberman

Abstract

Ways in which the so-called Lee-Carter time series approach to forecasting mortality patterns can be modified to forecast the possible future behaviour of mortality reduction factors are described. A comparison is drawn with an alternative regression type approach to the forecasting of mortality reduction factors, based on the same model structure. Case studies, illustrating different aspects of the methodology, making use of both the more recent mortality experience of UK male pensioner lives and the historical mortality experience of UK male annuitants, are presented.

Keywords: Mortality reduction factors; Time series; Forecasting

1. Introduction

Using what are essentially regression methods, formulated within the generalised linear modelling (GLM) framework, Renshaw and Haberman (2000) have described a method for modelling mortality reduction factors. While the method can be used for retrospectively monitoring the effectiveness of existing reduction factors, established regression patterns are also capable of extrapolation, as a means of predicting their possible future behaviour. As noted in Renshaw and Haberman (2000), it transpires that the model structures used in their paper are similar to, and, in certain circumstances, identical to those underpinning the Lee-Carter method for forecasting mortality (Lee 2000), although the Lee-Carter method is concerned with the forecasting of mortality trends without specific recourse to mortality reduction factors. However, the methods of fitting and extrapolating differ appreciably under the two approaches. In this paper, based on this common structure, we suggest ways in which the Lee-Carter methodology of fitting and forecasting mortality trends might be adapted for the construction of mortality reduction factors.

At the outset, we are faced with something of a presentational dilemma: whether to focus primarily on formulating the method in terms of reduction factors for the force of mortality or, alternatively, in terms of reduction factors for the probability of death. Given that the Lee-Carter singular value decomposition (SVD) method is formulated in terms of central mortality rates, and that the structure underpinning both the Lee-Carter methodology and the regression methods formulated in Renshaw and Haberman (2000) when targeting the force of mortality are identical, we opt for the former. However, given that current UK actuarial practice uses the forecasting of reduction factors for the probability of death, for practical reasons, we also address this issue, as a secondary consideration.

In Section 2, mortality reduction factors are characterised. An example, based on current UK usage, is reproduced. In Section 3, a brief summary of the core Lee-Carter methodology for forecasting established mortality patterns is given. In Section 4, we suggest ways in which the methodology might be modified in order to forecast mortality reduction factors. Given the close similarity between the model structures formulated in Section 4 and the GLM based model structures formulated in Renshaw and Haberman (2000), we have given a brief summary of the latter in Section 5. In Section 6, a detailed case study is presented, based on the recent male UK pensioner lives experience. In Section 7, the historical male UK annuitants experience is investigated by these methods. In Section 8, issues concerning the alignment and updating of mortality forecasts are discussed. Concluding comments are presented in Section 9.

2. Mortality Reduction Factors

Let

x denote age,

t denote time, with a well defined origin t = 0, and

μ_{xt} denote the force of mortality at age x and time t.

Then, in terms of μ_{xt}, a mortality reduction factor (RF) is characterised by the equation

$$\mu_{xt} = \mu_{x0} RF(x,t), \ \forall t \geq 0 \qquad (2.1)$$

subject to the constraint

$$RF(x,0) = 1 \ \forall x. \qquad (2.2)$$

The further constraints $1 \geq RF(x,t) > 0$, $\forall x$, $t > 0$, with non-increasing monotonicity for increasing t are implicit, otherwise, the reduction factor is an adjustment factor. It is envisaged that μ_{x0} is completely specified, typically in the form of a standard mortality table, while the problem is to forecast RF, so that future reductions in μ_{x0} might be predicted.

Alternatively, in terms of the conditional probability of death q_{xt}, between age x and x+1 at time t, and in keeping with UK actuarial practice, a RF may be characterised as

$$q_{xt} = q_{x0} RF(x,t), \ \forall t \geq 0$$

subject to the same constraints as above. Then typically, having determined μ_{x0} by graduating the appropriate crude mortality rates of a current base quadrennial period, using methods described in Forfar *et al* (1988) and Renshaw (1991), q_{x0} is computed by the numerical evaluation of the integral in the standard relationship:

$$q_{xo} = 1 - \exp(-\int_0^1 \mu_{x+s,0} ds) . \qquad (2.3)$$

By way of illustration, in this context, the mortality reduction factor

$$RF(x,t) = \alpha(x) + \{1 - \alpha(x)\}\{1 - f(x)\}^{t/20} \qquad (2.4)$$

for which

$$\alpha(x) = 0.13; \qquad x < 60$$

$$\alpha(x) = 1 + 0.87\frac{(x-110)}{50}; \qquad 60 \leq x \leq 110$$

and

$$f(x) = 0.55; \qquad x < 60$$

$$f(x) = \frac{0.55(110-x) + 0.29(x-60)}{50}; \qquad 60 \leq x \leq 110$$

has recently been published by the UK Continuous Mortality Investigation (CMI) Bureau (Section 6, pp 89-108, CMI Committee (1999)). This is designed to accompany newly published mortality tables, constructed by graduating the crude mortality rates of the base quadrennium 1991-1994. As explained, (CMI Committee (1999)), the reduction factor is constructed without recourse to any detailed modelling. It is designed rather with reference to a 20 year time span, with the origin t = 0 located one and a half years into the base quadrennium 1991-94. It is intended for application to all four of the pensioners' and annuitants', males-females, published mortality tables for the quadrennium 1991-94.

3. Lee-Carter Mortality Forecasting

The core structure of the so-called Lee-Carter mortality forecasting method (Lee and Carter (1992)), is the following model:

$$\log m_{xt} = \alpha_x + \beta_x \kappa_t \qquad (3.1)$$

where

m_{xt} denotes the central death mortality for age x at time t

α_x describes the shape of the age profile averaged over time

β_x describes the pattern of deviations from the age profile

κ_t describes the variation in the level of mortality with t,

subject to an additive error term on the RHS. The model is over-parameterised in the sense that the structure of the RHS is invariant under either of the following parameter transformations

$$\{\alpha_x, \beta_x, \kappa_t\} \mapsto \{\alpha_x, \beta_x / c, c\kappa_t\}$$

$$\{\alpha_x, \beta_x, \kappa_t\} \mapsto \{\alpha_x - c\beta_x, \beta_x, \kappa_t + c\}$$

for any constant c. So κ_t is determined up to a linear transformation, β_x is determined only up to a multiplicative constant, and α_x is determined only up to a linear adjustment. It is envisaged that a rectangular array of crude mortality rates is available for fitting the model structure. For this array, let

$$t = t_1, t_1 + 1, \ldots, t_1 + h - 1 = t_n$$

where $h = t_n - t_1 + 1$ specifies the range, then the Lee-Carter method normalises the parameter estimates by stipulating that

$$\sum_{t=t_1}^{t_n} \kappa_t = 0 \text{ and } \sum_{all\,x} \beta_x = 1 \text{ so that } \alpha_x = \log \prod_{t=t_1}^{t_n} m_{xt}^{1/h} \qquad (3.2)$$

is a least squares error estimator. The normalisation of β_x means that the resulting values indicate the relative rate of change of log mortality rates at different ages.

Let

(d_{xt}, e_{xt})

represent the data array, where

d_{xt} denote the number of deaths at age x and time t, and

e_{xt} denote the appropriate matching central exposures,

with crude central mortality rates

$r_{xt} = d_{xt} / e_{xt}$.

Then the Lee-Carter model fitting proceeds as follows:

- Estimate α_x as $\hat{\alpha}_x = \log \prod_{t=t_1}^{t_n} r_{xt}^{1/h}$, the logarithm of the geometric mean of the crude mortality rates, averaged over all t, for each age x.
- Compute the matrix of statistics $[z_{xt}] = [\log r_{xt} - \hat{\alpha}_x]$, then either
    1. estimate κ_t and β_x as the respective 1st right and 1st left singular vectors in the singular value decomposition (SVD) of the matrix $[z_{xt}]$, subject to the above constraints,

    or, as an approximation to SVD,

2. estimate κ_t as $\hat{\kappa}_t = \sum_{all,x} z_{xt}$, and then regress z_{xt} on $\hat{\kappa}_t \; \forall x$ (with no intercept term) to determine $\hat{\beta}_x$, as the estimated slope parameter.

- Finally the estimated κ_t are adjusted such that the actual total observed deaths $\sum_{all,x} d_{xt}$ equals the total expected deaths $\sum_{all,x} e_{xt} \, exp(\hat{\alpha}_x + \hat{\beta}_x \hat{k}_t)$, $\forall t$.

It is possible to implement SVD using a variety of standard statistical packages offering this facility, such as GENSTAT (Payne *et al.* (1993)).

Forecasts follow by modelling the values of κ_t as a time series using standard methods, with typically, the ARIMA(0,1,0) process, or random walk with drift, occupying a central role in this respect. For example, Tuljapurkar *et al* (2000) report that the factors κ_t "display highly linear long-term declines with superimposed short-term fluctuations". Denoting the resulting forecasts beyond the data time horizon as $\{\dot{\kappa}_{t_n+s} : s = 1,2,3,...\}$, forecast mortality rates are then computed as

$$\dot{m}_{x,t_n+s} = \exp(\hat{\alpha}_x + \hat{\beta}_x \dot{\kappa}_{t_n+s}); \; s = 1,2,3, \ldots$$

For an indication of the nature and complexity of the issues concerning the structure and magnitude of the errors in such forecasts, see appendix B of Lee and Carter (1992). In view of this discussion, we adopt the same stance as that taken by Lee and Carter, focusing solely on the error generated by the time series predictions, while ignoring the error generated by parameter estimation. For a fully comprehensive discussion of commendations, criticisms and extensions of the Lee-Carter methodology see Lee and Carter (1992), Bell (1997), Lee (2000), Lee and Miller (2000).

4. Modified Lee-Carter Methodology for Forecasting Mortality Reduction Factors.

Taking logarithms of both equation (2.1) and the constraint (2.2) implies that the reduction factor is characterised by

$$\log \mu_{xt} = \log \mu_{x0} + \log RF(x,t), \text{ s.t. } \log RF(x,0) = 0 \;\; \forall x; \; t \geq 0. \qquad (4.1)$$

Then, in the spirit of the Lee-Carter method, we note that it is possible to re-express (4.1) as

$$\log \mu_{xt} = \alpha_{x0} + \beta_x \kappa_t, \text{ s.t. } \kappa_0 = 0, \; t \geq 0,$$

where

$$\mu_{x0} = \exp \alpha_{x0}, \;\; RF(x,t) = \exp(\beta_x \kappa_t). \qquad (4.2)$$

Unlike the original Lee-Carter formulation, however, here we have a well defined origin $t = 0$, calibrated in calendar time, at which the age specific mortality profile α_{x0} is specified (and hence the introduction of the additional suffix to highlight this feature). Thus, parameter uniqueness is specified by a different set of conditions from (3.2), namely

$$\kappa_0 = 0, \; \sum_{all,x} \beta_x = 1, \; \alpha_{x0} = \log \mu_{x0}.$$

Although the foregoing specifications are motivated by the desire to make future adjustments to the values of a *standard mortality table* $\{\mu_{x0}\}$, it is possible to generalise these model specifications by allowing any appropriate age profile α_{x0}, valid at time $t = 0$, whether graduated or not, and to introduce the potential for smoothing within a more general framework, as required. Also, given the discrete

and potentially unordered nature of the β_x, there is the added need to incorporate a mechanism for smoothing the β_x s if a tractable and practicable formula for *RF* is to result. These considerations, together with the recognition that, as in the UK experience for instance, the age profile α_{x0} may well derive from data grouped by individual calendar year in the close vicinity of t = 0, we adapt the Lee-Carter methodology as described below.

We begin by focusing on the rectangular array of data, as defined in Section 3. For this array, let

$$t = t_1,\ t_1+1,\ \ldots,\ t_1+h-g=t_m,\ t_1+h-g+1,\ \ldots,\ t_1+h-1=t_n$$

where $h = t_n - t_1 + 1$ specifies the range of the data array in calendar years and $g = t_n - t_m + 1$ is a small positive integer, specifying the range of a subinterval. Typically g = 4 for the UK quadrennial graduated experiences. Site the origin in the appropriate calendar year t_0, located at or adjacent to the centre of the subinterval (t_m, t_n). Typically $t_0 = t_n - [\frac{g}{2}]$ where $[\frac{g}{2}]$ denotes the integer part of $\frac{g}{2}$. Obviously $t_m \le t_0 \le t_n$, with equality iff g =1. Model fitting can then proceed as follows:

Step1 Estimate α_{x0} as any of the following

(i) $\hat{\alpha}_{x0} = \log \mu_{x0}$, where the standard mortality table $\{\mu_{x0}\}$ is based on the grouped experience observed for the g years from t_m to t_n, inclusive.

(ii) $\hat{\alpha}_{x0} = \log \dfrac{\sum_{t=t_m}^{t_n} d_{xt}}{\sum_{t=t_m}^{t_n} e_{xt}}$, based on the grouped experience observed for the g years from t_m to t_n, inclusive.

(iii) $\hat{\alpha}_{x0} = \log \prod_{t=t_m}^{t_n} r_{xt}^{1/g}$, based on the geometric mean of the crude age specific mortality rates for years t_m to t_n, inclusive.

Step2 Allow for the further smoothing of $\hat{\alpha}_{x0}$, as required, under Step 1(ii) or Step 1(iii).

Step3 Estimate (β_x, κ_t) by forming the statistics $z_{xt} = \log r_{xt} - \hat{\alpha}_{x0}$ and applying either SVD, or its approximation, as described in Section 3.

Step4 Adjust the $\hat{\kappa}_t$ estimates such that the total observed deaths $\sum_{all,x} d_{xt}$ equal the total expected deaths $\sum_{all,x} \hat{d}_{xt}$, $\forall t$. (See Step 7 below).

Step5 Translate $\hat{\kappa}_t$ into $\hat{\kappa}_t - \hat{\kappa}_{t_0}$.

Step6 Allow for the smoothing of $\hat{\beta}_x$, if required.

Step7 Check for goodness of fit by computing and monitoring the residuals

$$\frac{d_{xt} - \hat{d}_{xt}}{\sqrt{\hat{d}_{xt}}}, \text{ where } \hat{d}_{xt} = e_{xt} \exp(\hat{\alpha}_{x0} + \hat{\beta}_x \hat{\kappa}_t).$$

Comments:

1) The case $g = 1$, implies that Step 1(ii) and Step 1(iii) are identical and that the forecasts match up with the most recent observed death rates, as discussed on page 88 of Lee (2000).
2) Step 1(iii) is suggested, as a viable alternative to Step 1(ii), in the spirit of the Lee-Carter method.
3) Step 1(ii), coupled with Step 2, is potentially equivalent to Step 1(i).
4) It is envisaged that the data cells are grouped and classified by individual year of age (as well as by individual calendar year), so that the introduction of the smoothing option at Step 6 is essential, if the resulting predictions are to be well ordered with respect to individual year of age.
5) As reported later, we have experimented by making Step 4 optional, only to find this unsatisfactory.
6) There is a total of 20 cases generated by Steps 1 to 3 and Step 6.
7) Although presented throughout in terms of the reduction factors for μ_{x0}, these methods also apply equally for the reduction factors for q_{x0}.

Forecasts follow next by modelling the values of κ_t, as determined in Step 5, as a time series, with the ARIMA (0,1,0) process playing a central role. For this particular process

$$\kappa_t = \kappa_{t-1} + \lambda + \varepsilon_t, \tag{4.3}$$

where ε_t denotes error, so that κ_t is forecast to change linearly by incremental amounts of λ. In general, denoting the forecasts as $\{\dot{\kappa}_{t_n+s} : s > 0\}$, the reduction factor is then forecast by

$$\log RF(x, t_n + s) = \hat{\beta}_x \dot{\kappa}_{t_n+s}.$$

Specifically, for the ARIMA (0,1,0) process

$$\dot{\kappa}_{t_n+s} = \hat{\kappa}_{t_n} + \hat{\lambda} s \tag{4.4}$$

so that

$$RF(x, t_n + s) = \exp\{\hat{\beta}_x(\hat{\kappa}_{t_n} + \hat{\lambda} s)\}. \tag{4.5}$$

Other ARIMA processes might be more appropriate for modelling κ_t and generating forecast values, such as the ARIMA (0,1,1) used in Section 7. Standard methodology is used for estimating the forecasts and the associated prediction errors (for example, see Hamilton (1994)).

5. Mortality Reduction Factors by Regression Methods

The regression methods for modelling mortality reduction factors described in Renshaw and Haberman (2000) are given a GLM setting in order to conform with current UK actuarial graduation practice as described in Forfar *et al* (1988) and Renshaw (1991). Expressed in terms of the Lee-Carter notation of Section 4, the structures underpinning the methods described in Renshaw and Haberman (2000), have the form

$$RF(x,t) = \frac{g^{-1}\{g(\exp\alpha_{x0}) + \beta_x \kappa_t\}}{\exp\alpha_{x0}}, \ \kappa_{t_0} = 0 \tag{5.1}$$

where g is a pre-specified link function. We have then proceeded to fit such structures by setting $\kappa_t = t - t_0$ and using quasi-likelihood methods to estimate $\{\beta_x\}$. This is done on the basis of either Poisson or binomial over-dispersed response models, as the case may be. A detailed account of these methods, together with a number of illustrative case studies, is given in Renshaw and Haberman (2000). In particular, when targeting reduction factors for μ_{xt}, the log link function is selected, so that the structure (5.1) reduces trivially to

$$RF(x,t) = \exp(\beta_x \kappa_t) \qquad (5.2)$$

which is identical to the Lee-Carter structure of equation (4.2).

6. An Analysis of the Recent UK Male Pensioners Experience

6.1 Introduction

By way of illustration, we consider the UK male pensioner lives experience. The data comprise the numbers of deaths with matching exposures, cross-classified by individual year of age, from age 60 to age 100, and by individual calendar year, from 1983 to 1994, both inclusive. The data are analysed relative to the base quadrennium 1991-94, for which the standard mortality table (PML92) applies (CMI Committee 1999), based on the formula

$$\mu_{x0} = GM(2,3) = a_0 + a_1 x' + \exp\{b_0 + b_1 x' + b_2(2x'^2 - 1)\},\ x' = \frac{x - 70}{50}. \qquad (6.1)$$

The formula has been determined using the graduation techniques described in Forfar *et al* (1988), applied to the quadrennial crude mortality rates, with central exposures, (as represented by Step1 (ii) above), but including a small amount of data outside the age range 60 to 100. For completeness, the parameter estimates are reproduced in Table 1. As already noted in Section 2, this is accompanied by the reduction formula (2.4), centred mid-way through 1992. This, however, operates on q_{x0} rather than μ_{x0}, a complication which we address later in this study. Instead, to begin with, we focus on the construction of a reduction factor for μ_{x0}. In the notation of Section 4, $t_1 = 1983,\ t_n = 1994,\ t_0 = 1992,\ t_m = 1991,\ h = 12,\ g = 4.$

6.2 Forecasting a RF for the Force of Mortality

To investigate the performance of a variety of the options discussed in Section 4, we present results for the following cases:

method of determining $\exp\alpha_{x0}$	*case*
graduated mortality rates (Table PML92)	1 (A or B)
smoothed grouped crude mortality rates	2 (A or B)
smoothed geometric mean of annual crude rates	3 (A or B)

where A or B indicates whether $\{\beta_x, k_t\}$ are determined in Step3 by SVD (coded A) or by its approximation (coded B). In all six cases, the sequence $\{\beta_x\}$ is smoothed by parametric methods, thereby resulting in a simple formula for RF. Also, in all six cases, the method of determining $\exp\alpha_{x0}$ relates to the quadrennium 1991-94.

The values of α_{x0}, plotted against x, for all cases are displayed in the separate frames of Fig. 1. In Case 1, the log crude mortality rates for the quadrennium 1991-

94, have been superimposed. In Cases 2 & 3, we have superimposed quadratic curves, fitted by weighted least squares, against the background of the computed values of α_{x0}. When fitting the quadratics, the outlying points at ages 97 to 100 (inclusive) have been zero weighted. Thus, in all cases, the values of α_{x0} are effectively generated as

$$\alpha_{x0} = \log \mu_{x0}$$

where μ_{x0} is given by equation (6.1), and where the non-linear refinement coefficients a_0 and a_1, are pre-set to zero in Cases 2 & 3. Details of the fitted curves are reproduced in Table 1. Obviously in Cases 2 & 3, the detailed refinements, normally associated with the graduation of a standard mortality table (Forfar *et al* (1988)), have been ignored.

The values of κ_t, determined immediately after Step 4, but prior to Step 5, are presented next in Table 2. It is of interest to note that in all six cases the 'natural' zero occurs closest to in the year 1993. In keeping with current UK practice (Section 6, pp 89-108, CMI Committee (1999)) however, we set the origin in the year 1992, before translating the values of κ_t accordingly.

As implied in the comments to Section 4, we have experimented by omitting Step 4, in order to assess its impact. For comparison purposes with Table 2, we reproduce the values of κ_t, determined in Step 3, prior to Step 4, in Table 3. Here, by comparison with Table 2, there is less consistency between the cases and the trends are much more erratic, thereby illustrating the material impact of Step 4.

In Fig. 2, the values of β_x, as determined in Step 3, are plotted against x, for each case separately. Then, the plotted points are smoothed, by fitting the straight line

$$\beta_x = c_0 + c_1 x + error \qquad (6.2)$$

using the method of weighted least squares. The outliers at ages 60, 61, 64 and 100 are omitted when fitting the straight lines in Case 1A, together with the additional points at ages 96 & 98 in Cases 1B and 1C; while the outliers at ages 60, 64, 96, 97, 99 and 100 are omitted in Cases 2A, 2B and 2C. Details of the fitted lines are reproduced in Table 4.

Finally at the model fitting stage, diagnostic checks on the quality of the fit are conducted. These take the form of a graphical analysis of the residuals defined in Step7. By way of example, separate residual plots against both calendar year and age, for the Case1A, are reproduced in the upper frames of Fig. 3. The satisfactory, well balanced (between positive and negative residuals), trend free nature of these plots is representative of the other five cases. In general, we have found that such plots are particularly sensitive to the choice of options at the various modelling steps, as illustrated by the other residual plots in this paper.

Forecasts are made next, using the time-series analysis outlined at the end of Section 4. In keeping with UK actuarial practice, we focus on 20 year forecasts, fitting an ARIMA (0,1,0) process in each case. By way of illustration, the estimated values of κ_t, together with the 20 year forecasts and prediction intervals, for the Case 1A, are reproduced in the lower left hand frame of Fig. 3. For this process, equation (4.5) coupled with equation (6.2) implies that the predicted reduction factor is given by

$$RF(x, t_n + s) = \exp\{\gamma_0 + \gamma_1 x + (\gamma_2 + \gamma_3 x)s\} \qquad (6.3)$$

where $\gamma_i = c_i \kappa_{t_n}$, i = 0, 1; $\gamma_i = c_{i-2}\lambda$, i = 2, 3. The estimates of λ, as defined in equation (4.3), and the estimates of $\{\gamma_i\}$, for all six cases, are reproduced in Table 5 and Table 6 respectively.

It is interesting to compare these parameter estimates for λ with the corresponding values for national populations. Tuljapurkar *et al* (2000) thus find that λ is in the range of 0.26 to 0.50 for the U.S., Canada and European G7 countries but 0.79 for Japan. The figure for the U.K. is 0.34. The estimates in Table 5 are higher than this level- this is not unexpected given the particular nature of the subpopulation investigated here and the underlying selection processes operating (males with pensions from insured pensions schemes).

The 20-year predicted log reduction factor for the Case 1A, together with its prediction limits are illustrated in the lower right hand frame of Fig. 3. The predicted values are calculated on substituting s = 18 in equation (6.3) and selecting the relevant values γ_i from Table 6. In addition, the equivalent regression based predictions, using the methods described in Renshaw and Haberman (2000), have been superimposed for comparison purposes. These latter predictions are obtained by setting $\kappa_t = t - t_0$ in equation (5.2), so that

$$\log RF(x, t_0 + s) = \beta_x s, \; t_0 = 1992, \; s = 20$$

and where β_x is parameterised as

$$\beta_x = \psi_0 + \psi_1 x. \qquad (6.4)$$

The estimates for ψ_0 and ψ_1, presented in Table 7, are determined by the quasi-likelihood methods associated with over-dispersed independent Poisson response variables. At an interim stage in this analysis, the graph of equation (6.4) is superimposed against the background of the individually estimated values of $\{\beta_x\}$, also obtained by these methods, in the upper frame in Fig. 4. Unlike Fig. 2, none of the data cells are zero weighted when fitting equation (6.4) under this approach. For completeness, the residual plots associated with the fitted version of equation (6.4) are reproduced in the lower frames of Fig. 4.

6.3 Forecasting a RF for the Probability of Death

Given the practical importance of forecasting reduction factors for the probability of death, we investigate this case next. It involves working with initial exposures, formed by adding half the number of deaths to the corresponding central exposures, and applying the methods of Section 4 to q_{xt}, rather than to μ_{xt}. However, there is an additional complication, which means that Case 1 (A or B), involving the estimation of $\alpha_{x0} = \log q_{x0}$ from standard mortality tables, is no longer viable. This is because the standard mortality table $\{q_{x0}\}$ (PML92Base: CMI Committee (1999)) is constructed using equation (2.3) and not via the direct graduation of the actual quadrennial crude mortality rates, based on initial exposures, as is effectively the case with Cases 2 & 3 (A or B). The manifestation of this effect is vividly demonstrated in the lower frames of Fig. 5, in which the residual plots for Case 1A, are reproduced. Given that these plots display a disproportionate number of negative residuals (336 negatives to 156 positives, representing 68.3% of all residuals), coupled with the above explanation as to why this happens, the study proceeds on the basis of Cases 2 & 3 (A or B). In addition, the separate curves for $\alpha_{x0} (= \log q_{x0})$ and β_x against x, used in the construction of these residuals, are given in the upper frames of Fig. 5.

Notably, in the upper left hand frame, with the exception of just four of the superimposed log crude mortality rates (at ages 60, 65, 95 and 96), all of the plotted log crude mortality rates lie below their smoothed standard values: a feature which largely explains the unsatisfactory nature of the associated residual plots.

Turning therefore to Cases 2 & 3 (A or B), the corresponding values of α_{x0}, together with fitted quadratics, in which the outliers at ages 97 to 100 are again excluded from the least squares fitting process, are presented in the separate upper frames of Fig. 6. Hence, the values of α_{x0}, used in the subsequent modelling process, are generated as

$$\alpha_{x0} = \log q_{x0} = b_0 + b_1 x' + b_2 (2x'^2 - 1),\ x' = \frac{x - 70}{50} \qquad (6.5)$$

where the parameter values are tabulated in Table 8.

The values of κ_t, generated immediately after Step 4, but prior to Step 5, are presented in Table 9. Although the 'natural' origin is again closest to 1993, we set the origin at 1992 in order to conform with current UK practice and translate the entries of Table 9 accordingly, before proceeding with the time series analysis of all four cases.

The values of β_x, determined in Step 3, are plotted against x together with the fitted straight line, equation (6.2), in the middle and lower frames of Fig. 6. The outliers at ages 60,61,64,96,98,100 are omitted when fitting the straight lines in Cases 2A and 3A, and the outliers at ages 60,64,96,97,99,100 are omitted in Cases 2B and 3B. Details of the fitted lines, equation (6.2), are presented in Table 10, for all four cases.

The residual plots for Case 2A are reproduced in the upper frames of Fig. 7. These plots may be considered as typical of the other three cases. While it is reasonable to suggest that there is potential for improvement in the distribution of residuals, when plotted against age (particularly at the highest ages: where the outlying nature of the plotted α_{x0} values in the upper right hand frame of Fig. 6 is probably responsible for this effect), we recall that the whole data set, comprising a total of 492 cross-classified cells, is modelled by just six parameters. In addition, there are 220 negative residuals, representing 44.7% of all residuals, which is an improvement on the situation discussed earlier for Fig. 5.

We again focus on 20 year forecasts in the time series analysis of $\{\kappa_t\}$, while illustrating the Case 2A in detail. Thus, in the lower left hand frame of Fig. 7, the values of κ_t together with their 20 year forecasts and prediction limits, generated by the ARIMA(0,1,0) process, are plotted against t. The values of the drift parameter λ, of equation (4.3), together with the parameters γ_i of equation (6.3), which define the predicted reduction factors, are tabulated in Table 11 and Table 12 respectively, for all four cases.

The predicted reduction factor for the Case 2A is illustrated in the lower right hand frame of Fig. 7. Here, together with the prediction limits, we have plotted the 20 year predicted values of the log reduction factor based on equation (6.3) with s = 18 and the values of γ_i selected from the first column of Table 12. In addition, both the 20 year log reduction factor prediction of the CMI Bureau based on equation (2.4) and the regression based prediction of Renshaw and Haberman (2000) have been superimposed for comparison purposes. The latter is again obtained by setting $\kappa_t = t - t_0$, this time in equation (5.1), so that

$$\log RF(x,t_0+s) = \log[g^{-1}\{g(\exp\alpha_{xo}) + \beta_x s\}] - \alpha_{x0};\ t_0 = 1992,\ s = 20$$

where β_x is given by equation (6.4). This time, modelling proceeds on the basis of over-dispersed independent binomial responses, in combination with the log-odds link function

$$g(q_{x0}) = \log\left\{\frac{q_{xo}}{1-q_{x0}}\right\},\ \text{where}\ q_{x0} = \exp\alpha_{x0}.$$

The estimates for ψ_0 and ψ_1 are presented in Table 13. The predicted reduction factors in the lower right hand frame of Fig. 7 are of a similar shape with respect to age for the Lee-Carter and GLM based methods. The non-linear form of the CMI Bureau reduction factor is highlighted. As in the previous analysis, the graph of equation (6.4) is superimposed against the background of the individually estimated values of $\{\beta_x\}$, in the upper frame in Fig. 8, and the residual plots associated with the fitted version of equation (6.4) are reproduced in the lower frames of Fig. 8. As in the previously reported GLM regression analysis, none of the points in the upper frame of Fig. 8 have been zero weighted.

7. An Analysis of the Historical UK Male Annuitants' Experience

7.1 Introduction

In this study, we focus on the UK male lives annuitants' experience for policy duration 1+ years. The data, comprising the numbers of deaths with matching central exposures, are again cross-classified by individual year of age, from age 60 to age 100, and by individual calendar year, this time ranging from 1946 to 1994, with the exception of 1968, 1971 and 1975, for which the data are missing. In addition, unlike the previous male pensioner study, there are a small number of empty data cells together with a few other cells with exposure but no reported deaths, occurring at the extremities of the age range. As a consequence, it is necessary to allocate zero weights to such cells, as part of the SVD process. Subject to this provision, modelling can proceed as before. Thus for this study $t_1 = 1946$, $t_n = 1994$, $t_0 = 1992$, $t_m = 1991$, $h = 49$ and $g = 4$.

By way of illustration, we focus on a mortality reduction factor for μ_{xt}, defined, in the first instance, relative to the standard mortality table μ_{x0} (IML92) based on the 1991-94 quadrennium (CMI Committee 1999) and which is again given by formula (6.1). The parameter estimates for the standard table IML92 are reproduced in Table 14 and the corresponding values of

$$\alpha_{x0} = \log\mu_{x0},$$

together with the matching log crude mortality rates for the quadrennium 1991-94 are illustrated in the top left hand frame of Fig. 9.

7.2 Modified Lee-Carter Method

The analysis proceeds on the basis of the approximation to SVD (therefore coded 1B) in order to facilitate readily the allocation of zero weights, where appropriate. The resulting values of β_x, plotted against x in the upper right hand frame of Fig. 9, are again smoothed by fitting the straight line (6.2) by the method of least squares, in which the relative outliers at ages 60, 61, 63, 64, 81 and 98 have been

zero weighted. The parameter estimates for the fitted line, together with their standard errors, are given in Table 15.

The values of κ_t are estimated in accordance with Step 3, Step 4 and Step 5 of Section 4, and plotted against t in the bottom left hand frame of Fig. 9.

The resulting residual plots against both calendar year and age are reproduced in the central frames of Fig. 9. Note in particular the gaps in the first of these plots corresponding to the missing data, while it is plausible to conjecture that the shearing effect at the lower ages, in the second plot, is caused by the obvious outliers in this age range, displayed in the upper two frames of Fig. 9.

The 20 year forecasts for the time series $\{\kappa_t\}$ are also displayed in the bottom left hand frame of Fig. 9, by way of illustration. Such forecasts are potentially of considerable interest in the pricing and reserving of annuity contracts. This time, however, analysis of the time series $\{\kappa_t\}$ suggests that a more appropriate model is the ARIMA (0,1,1) process, for which

$$\kappa_t = \kappa_{t-1} + \lambda + \varepsilon_t - \theta\varepsilon_{t-1} \qquad (7.1)$$

and this model has been used to generate the forecast values for κ_t. The parameter estimates and their standard errors are presented in Table 16. The main effect of introducing the highly significant moving average parameter θ in this particular analysis, is to narrow the prediction interval by an appreciable amount. As with the ARIMA (0,1,0) process used previously, κ_t is forecast to change linearly by incremental amounts of λ under this process also, but with the exception of the first year beyond the time horizon. Finally in this analysis, we display the resulting 20 year forecast for the log mortality reduction factor, in the bottom right hand frame of Fig. 9.

7.3 Original Lee-Carter Method

It is of interest to compare these forecasts, centred on the standard mortality table IML92, with those obtained using the 'original' Lee-Carter methodology, as outlined in Section 3. Firstly, the α_xs are estimated as the logarithms of the geometric means of the crude mortality rates, averaged over all t, for each age x. The resulting values are displayed in the upper left hand frame of Fig. 10. Not surprisingly in view of their mode of construction, we note that these display a more regular pattern at the extremities of the age range, than their counterpart in Fig. 9. Deviating slightly from the Lee-Carter methodology, it is further possible to smooth the plotted values of α_x, as shown. This is done using the method of weighted least squares to fit the cubic

$$\alpha_{x0} = b_0 + b_1 x' + b_2(2x'^2 - 1) + b_3(4x'^3 - 3x'),\ x' = \frac{x - 70}{50} \qquad (7.2)$$

to the plotted points, with the exception of the final point at age 100. The details are presented in Table 17. Recall that this describes the shape of the age profile, averaged over the whole time range, from 1946 to 1994, but unlike the age profile of the standard mortality table IML92, is not associated with an identifiable location in calendar time.

The individual values of β_x, corresponding to the un-smoothed α_x, are displayed in the upper right hand frame of Fig. 10. We remark that these display a greater degree of dispersion than their counterparts in Fig. 9. The straight line, equation (6.2), is again fitted to the plotted points by the method of weighted least

squares, this time on omitting the points in the age ranges ages 60 to 64 and 98 to 100, both inclusive. Details of the fitted line are given in Table 18.

The residual plots in the central frames of Fig. 10 relate to the fitted model under the smoothed versions of α_x and β_x (displayed in the upper frames of Fig. 10). These plots are also representative of the fitted model using the un-smoothed versions of α_x and β_x. Indeed, the residual plots are potentially 'better' in this latter case, since a greater number of parameters are involved in the fit.

The values of κ_t, which are near identical whether the α_xs and β_xs are smoothed or not, are displayed in the lower frame of Fig. 10. The time series $\{\kappa_t\}$ is modelled as an ARIMA (0,1,1) process, details of which, for both the 'smoothed' and 'un-smoothed' versions of κ_t, are presented in Table 19. The resulting (common) forecasts, together with the prediction limits, are added to the lower frame in Fig. 10.

Given that there is no basis for comparing mortality reduction factors under the modified and original Lee-Carter methodology, we compare instead the year 2012 forecasts for the force of mortality. This is done graphically, on the log scale, in Fig. 11. Here the predicted force of mortality is depicted, together with the prediction limits, for the modified Lee-Carter method centred on the 1992 CMI standard table IML92. In addition, the predictions based on the original Lee-Carter method, for both the smoothed and un-smoothed separate versions of α_x and β_x (of the upper frames of Fig. 10), have been superimposed in Fig. 11, for comparison purposes. It is apparent that the main source of variation in Fig. 11, between the predictions generated by the 'smoothed' and 'un-smoothed' original versions of the original Lee-Carter method, at the extreme ages, is largely attributable to the lack of smoothness in the estimated point β_xs, as depicted in the upper right hand frame of Fig. 10.

8. Updating Mortality Reduction Factors Forecasts

The updating of mortality reduction factor forecasts, on the basis of new data, is of potential interest. One method of proceeding is by refitting the model to the augmented data sets sequentially, as the latest data become available. Data to the year 1996 are available to illustrate this, for the male pensioner experience of Section 6.

We begin by reproducing the (Case 1A) values of κ_t, together with their ARIMA (0,1,0) forecasts, in the upper frame of Fig. 12. Recall that the same details are also depicted in the lower left hand frame of Fig. 4. Then, in the middle two frames of Fig. 12, the values of κ_t together with their ARIMA (0,1,0) forecasts, for the respective augmented data sets (from 1983) to 1995 and to 1996, are depicted. Note in particular the alignment of the forecasts with the most recent estimated value of κ_t, *viz* $\hat{\kappa}_{t_n}, \hat{\kappa}_{t_n+1}, \hat{\kappa}_{t_n+2}$, as the case may be: a noteworthy feature of the methodology, implicit in equation (4.4) and leading to equation (4.5). Finally, the resulting sequential generated 20 year mortality reduction factor forecasts, which are visibly sensitive to the alignment of the κ_t forecasts, are depicted in the lower frame of Fig. 12.

9. Concluding Comments

- Forecasts of mortality rates (particularly at the older ages) have important implications for the financing of public programmes of retirement pensions

and support for the elderly and for private financial security systems aiming to provide pensions, annuities and long term care insurance. It is important, therefore, to have available scientifically sound methods of modelling time dynamics and of forecasting.

- The Lee-Carter method for forecasting mortality is flexible. It can be modified, as described in this paper, to provide an alternative method to GLM regression methods for predicting mortality reduction factors. It has an immediate potential to offer greater flexibility for dealing with calendar year effects than the equivalent regression method. The results of Section 6, however, between the modified Lee-Carter method and the GLM approach, are similar.
- Both the Lee-Carter method and GLM approach provides a formal statistical model for mortality change so that prediction intervals for forecasts can be determined. The computation of prediction errors under the Lee-Carter method would appear to be more complex than under the equivalent GLM approach.
- Unlike the other published applications of the original Lee-Carter method, in which, to our knowledge, the raw data are grouped and classified by quinquennial year of age, in this paper the raw data are classified by individual year of age. Hence provision for smoothing the values of $\{\alpha_x\}$ and $\{\beta_x\}$ is necessary if the resulting mortality predictions are to be well ordered with respect to age. The method of smoothing is not however necessarily restricted to parametric methods, as here.
- The adjustment to κ_t, involving the equating of the total observed and total expected deaths in each calendar year (Step 4 of Section 4), is shown to be critical in establishing both a trend the time series $\{\kappa_t\}$ and consistency between the different methods of determining $\{\kappa_t\}$. This constraint is not imposed under the GLM approach.
- The computed $\{\kappa_t\}$ are naturally supportive of a time origin sited in the third, rather that the second year into the base quadrennium, for the case studies investigated.

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Table 1
Parameter estimates and (standard errors): equation (6.1)

	$100a_0$	$100a_1$	b_0	b_1	b_2
Case 1	-0.0081	-0.07	-4.67509	5.629188	-1.2
Case 2	0	0	-4.470	5.499	-0.9896
			(0.1347)	(0.1040)	(0.1395)
Case 3	0	0	-4.540	5.568	-1.050
			(0.1217)	(0.0939)	(0.1261)

Table 2
Step 4 κ_t estimates

Case	1A	1B	2A	2B	3A	3B
1983	6.987	7.375	6.578	7.509	6.731	7.551
1984	5.787	6.114	5.492	6.282	5.639	6.339
1985	6.142	6.495	6.815	6.663	5.963	6.713
1986	4.278	4.528	4.118	4.728	4.257	4.801
1987	4.956	5.249	4.730	5.437	4.871	5.501
1988	3.532	3.744	3.429	3.949	3.561	4.029
1989	2.080	2.207	2.102	2.427	2.221	2.519
1990	0.856	0.909	0.978	1.131	1.085	1.233
1991	1.428	1.517	1.485	1.720	1.597	1.817
1992	0.444	0.472	0.577	0.670	0.674	0.769
1993	-0.044	-0.047	0.125	0.145	0.222	0.253
1994	-2.253	-2.400	-1.892	-2.204	-1.829	-2.091

Table 3
Step 3 κ_t estimates

Case	1A	1B	2A	2B	3A	3B
1983	5.320	5.784	4.874	5.361	5.202	5.637
1984	2.412	5.899	1.889	5.475	2.187	5.752
1985	7.717	6.985	6.875	6.561	7.279	6.837
1986	3.369	4.863	2.845	4.439	3.143	4.716
1987	3.407	5.606	2.908	5.182	3.274	5.459
1988	1.560	3.357	1.428	2.933	1.697	3.210
1989	3.231	1.755	3.024	1.331	3.231	1.608
1990	-3.156	-1.167	-2.661	-1.591	-2.604	-1.314
1991	0.934	0.907	0.934	0.483	1.104	0.759
1992	1.657	-1.879	1.912	-2.303	2.025	-2.026
1993	1.572	-1.642	1.735	-2.066	1.825	-1.789
1994	-2.293	-2.914	-1.847	-3.338	-1.801	-3.061

Table 4
Parameter estimators and (standard errors): equation (6.2)

	c_0	c_1		c_0	c_1
Case 1A	0.07153	-.000608	Case 1B	0.07198	-.000631
	(0.0103)	(.000126)		(.00945)	(.000119)
Case2A	0.07790	-.000658	Case 2B	0.07731	-.000695
	(0.0087)	(.000108)		(.0096)	(.000121)
Case 3A	0.08328	-.000729	Case 3B	0.08287	-.000763
	(0.0084)	(.000105)		(0.0094)	(.000119)

Table 5
Parameter estimates and (standard error): ARIMA(0,1,0)

Case	1A	1B	2A	2B	3A	3B
λ	-0.840	-0.889	-0.770	-0.883	-0.778	-0.877
	(0.298)	(0.316)	(0.271)	(0.313)	(0.274)	(0.310)

Table 6
Parameter values: equation (6.3)

Case	1A	1B	2A	2B	3A	3B
γ_0	-0.1612	-0.1728	-0.1544	-0.1704	-0.1523	-0.1733
γ_1	.00137	.00151	.00130	.00153	.00133	.00160
γ_2	-0.0601	-0.0640	-0.0600	-0.0683	-0.0648	-0.0727
γ_3	.000511	.000567	.000507	.000614	.000567	.000669

Table 7
Parameter estimates and (standard errors): equation (6.4)

ψ_0	ψ_1
-0.0578	0.000495
(.00622)	(.0000801)

Table 8
Parameter estimates and (standard errors): equation (6.5)

	b_0	b_1	b_2
Case 2	-4.637	5.412	-1.143
	(0.1338)	(0.1032)	(0.1386)
Case 3	-4.708	5.481	-1.205
	(0.1207)	(0.0931)	(0.1250)

Table 9

Step 4 κ_t estimates

Case	2A	2B	3A	3B
1983	6.447	7.151	6.588	7.202
1984	5.393	5.991	5.529	6.053
1985	5.700	6.339	5.835	6.395
1986	4.046	4.506	4.175	4.582
1987	4.662	5.198	4.793	5.266
1988	3.399	3.794	3.523	3.875
1989	2.086	2.333	2.201	2.425
1990	0.990	1.109	1.095	1.209
1991	1.485	1.665	1.595	1.761
1992	0.600	0.674	0.697	0.771
1993	0.141	0.159	0.239	0.264
1994	-1.848	-2.081	-1.778	-1.972

Table 10

Parameter estimators and (standard errors): equation (6.2)

	c_0	c_1		c_0	c_1
Case2A	0.08583	-.000767	Case 2B	0.08406	-.000778
	(.00811)	(.000100)		(.00958)	(.000120)
Case 3A	0.09069	-.000830	Case 3B	0.08947	-.000844
	(.00785)	(.000097)		(.00937)	(.000118)

Table 11

Parameter estimates and (standard error): ARIMA(0,1,0)

Case	2A	2B	3A	3B
λ	-0.754	-0.839	-0.760	-0.834
	(0.266)	(0.297)	(0.268)	(0.295)

Table 12

Parameter values: equation (6.3)

Case	2A	2B	3A	3B
γ_0	-0.2101	-0.2315	-0.2244	-0.2035
γ_1	.00188	.00214	.00205	.00192
γ_2	-0.0647	-0.0705	-0.0689	-0.0746
γ_3	.000578	.000653	.000631	.000704

Table 13

Parameter estimates and (standard errors): equation (6.4)

ψ_0	ψ_1
-0.0590	0.000483
(.00483)	(.0000865)

Table 14
Parameter estimates and (standard errors): equation (6.1)

	$100a_0$	$100a_1$	b_0	b_1	b_2
Case 1	0.014429	-0.040629	-4.399861	5.568973	-0.654909

Table 15
Parameter estimators and (standard errors): equation (6.2)

c_0	c_1
0.07992	-0.000703
(.00323)	(.0000393)

Table 16
Parameter estimates and (standard error): equation (7.1)

λ	θ
-0.457	0.742
(0.124)	(0.117)

Table 17
Parameter estimates and (standard errors): equation (7.2)

c_0	c_1	c_2	c_3
-1.775	0	1.601	-1.419
(.0587)	-	(.0612)	(.0236)

Table 18
Parameter estimators and (standard errors): equation (6.2)

c_0	c_1
0.05010	-0.000291
(.0117)	(.000143)

Table 19
Parameter estimates and (standard error): equation (7.1)

smoothed		un-smoothed	
λ	θ	λ	θ
-0.382	0.762	-0.381	0.768
(.0993)	(.113)	(.0971)	(.112)

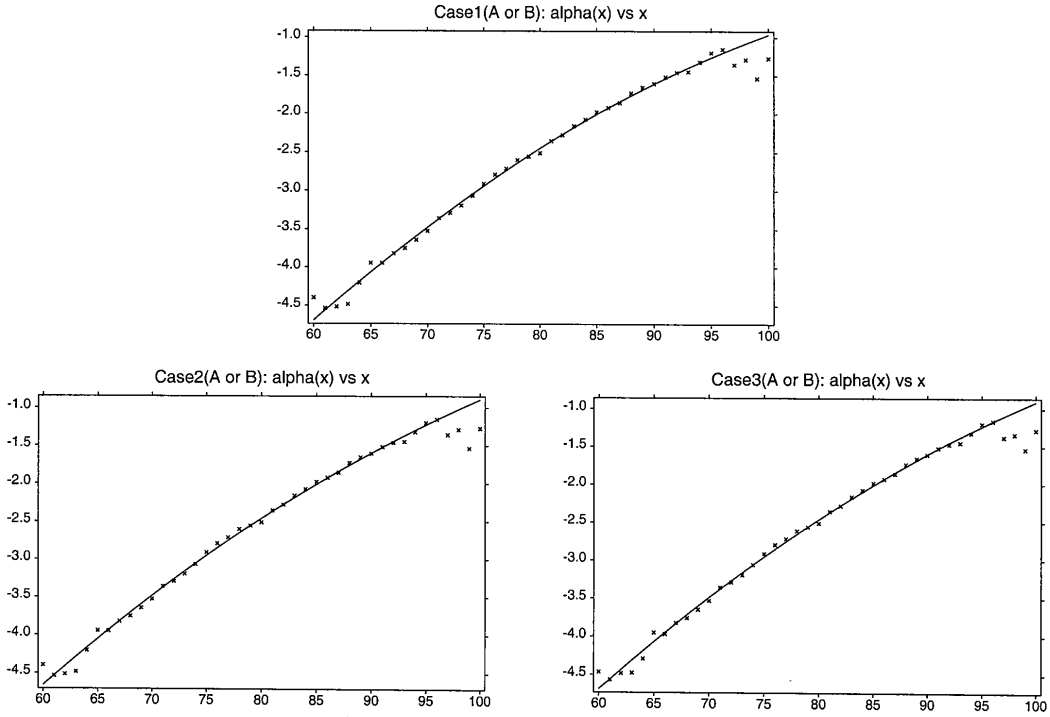

Fig. 1. RF for μ_x, various α_x

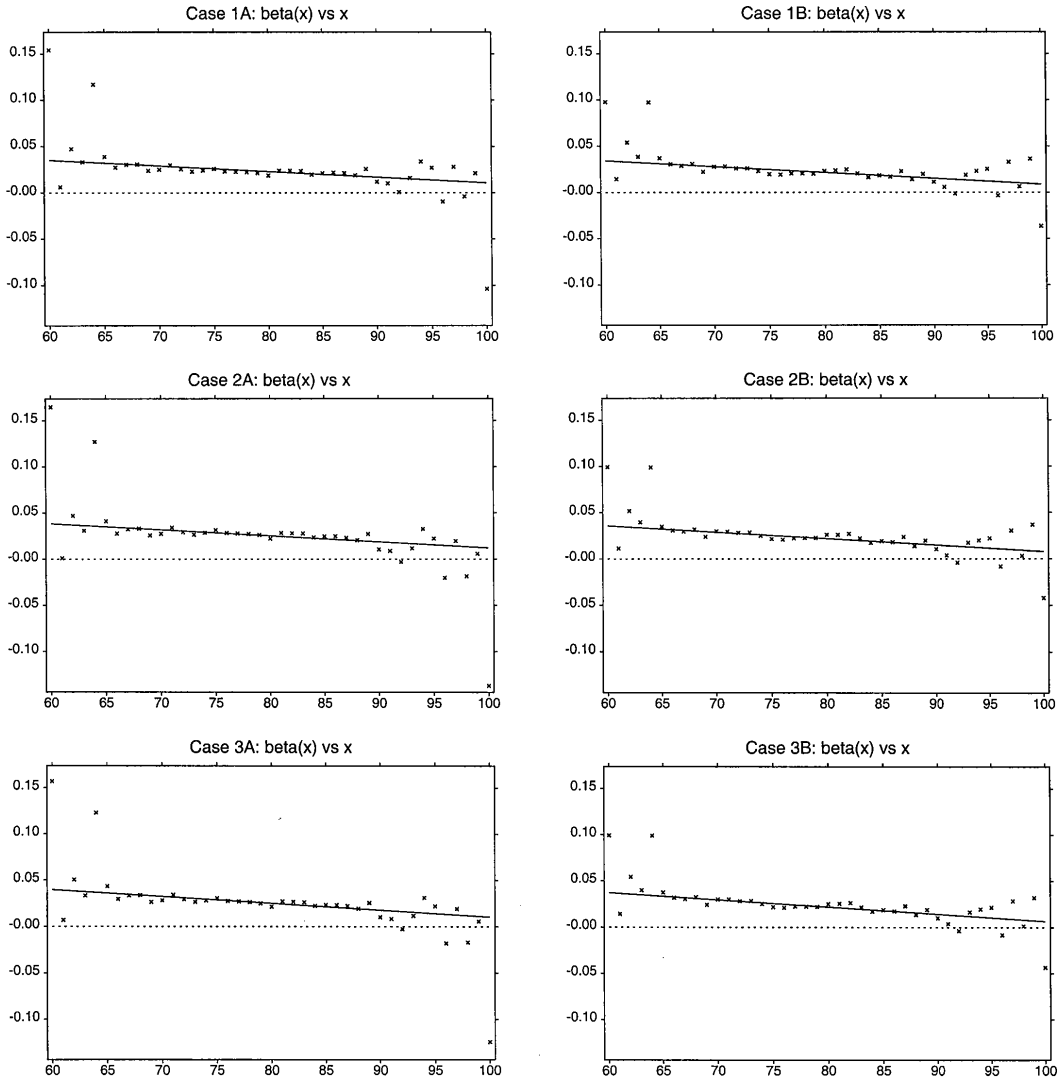

Fig. 2. RF for μ_x, various β_x

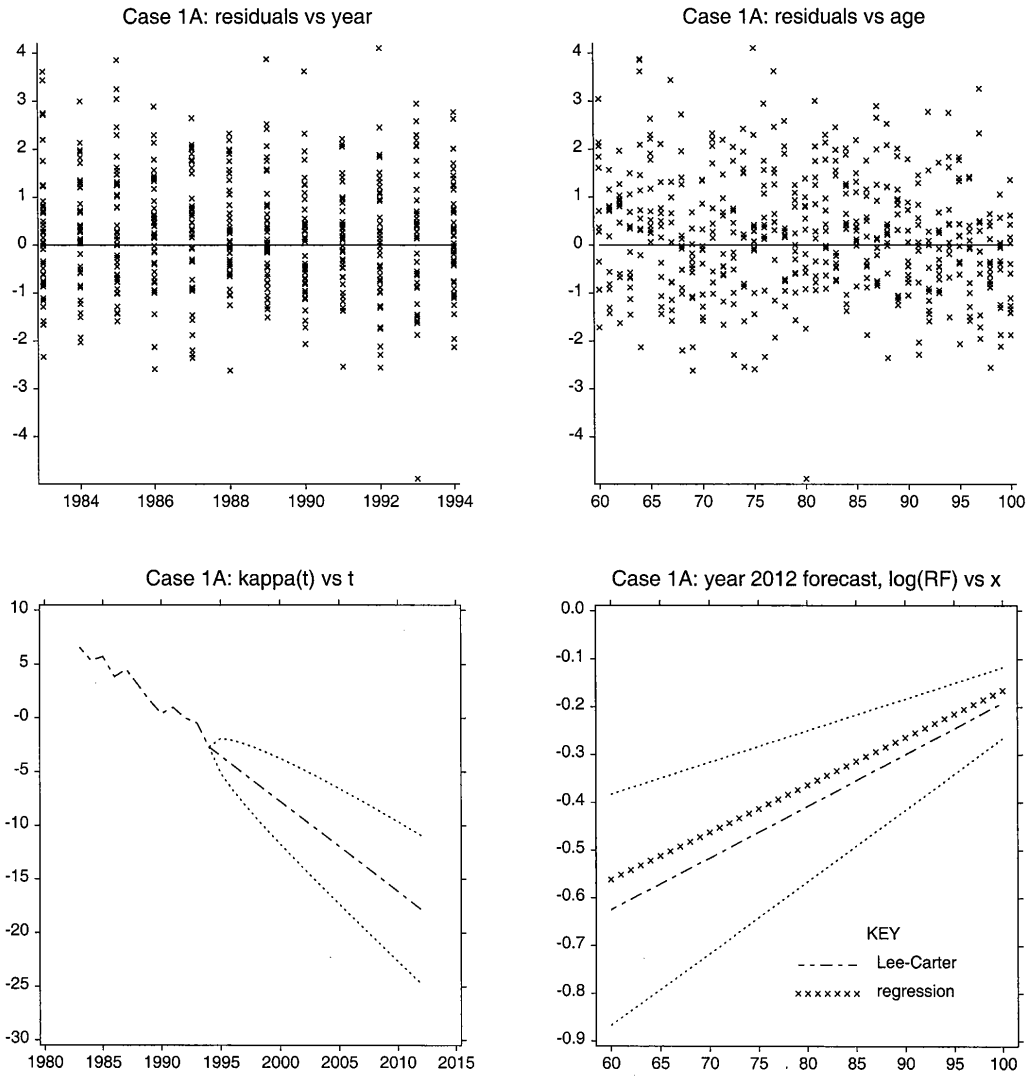

Fig. 3. RF for μ_x, specific residuals, κ_t, forecasts

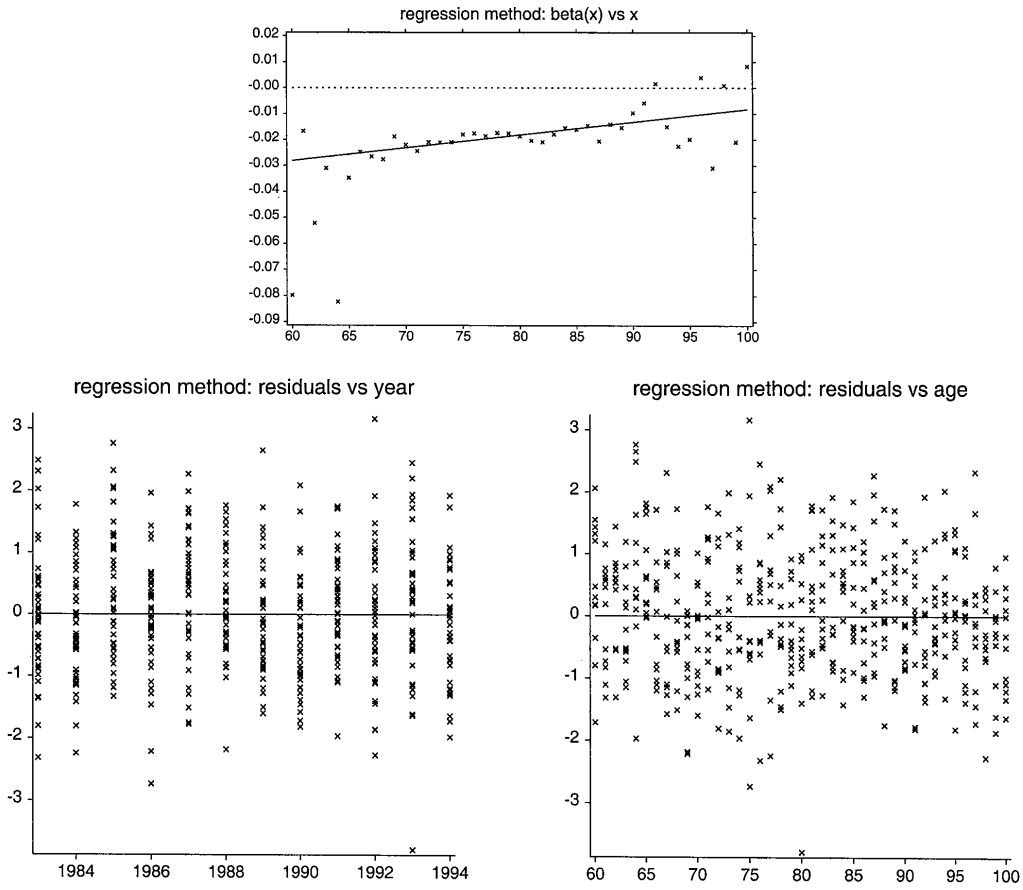

Fig. 4. RF for μ_x, regression approach, β_x, residuals

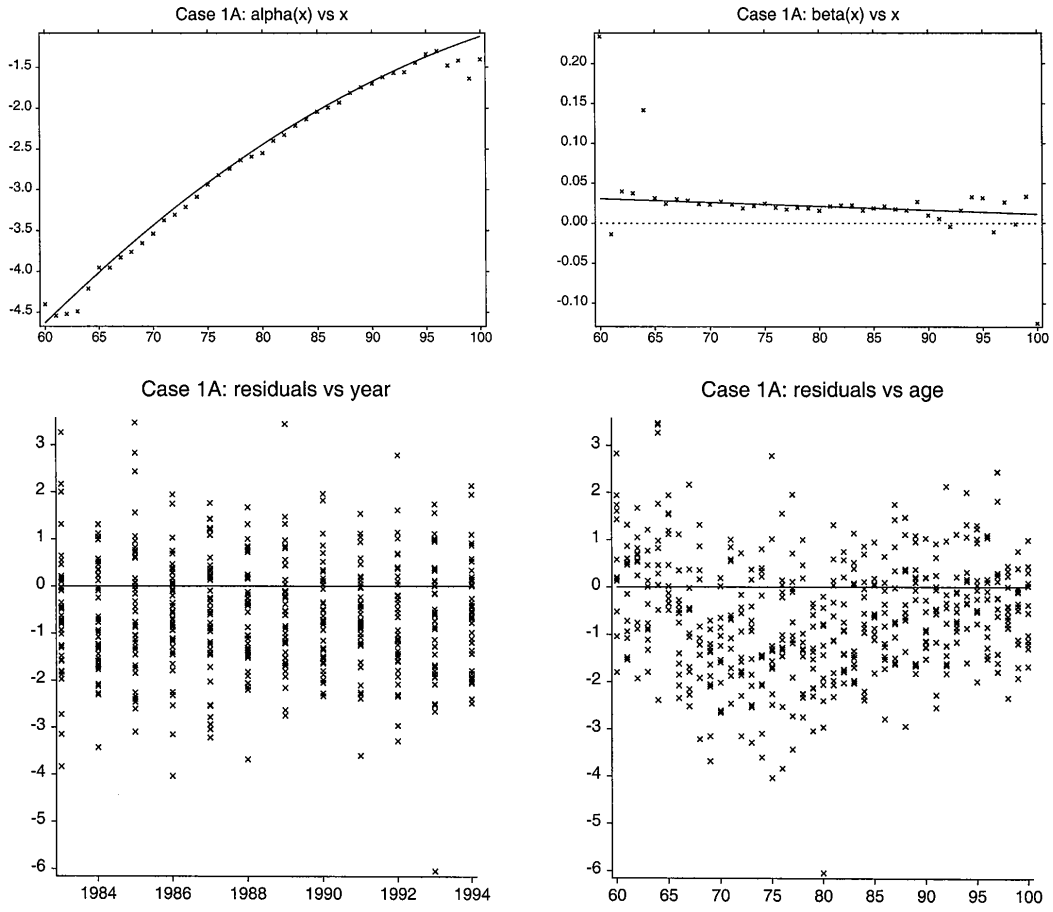

Fig. 5. RF for q_x, specific α_x, β_x, residuals

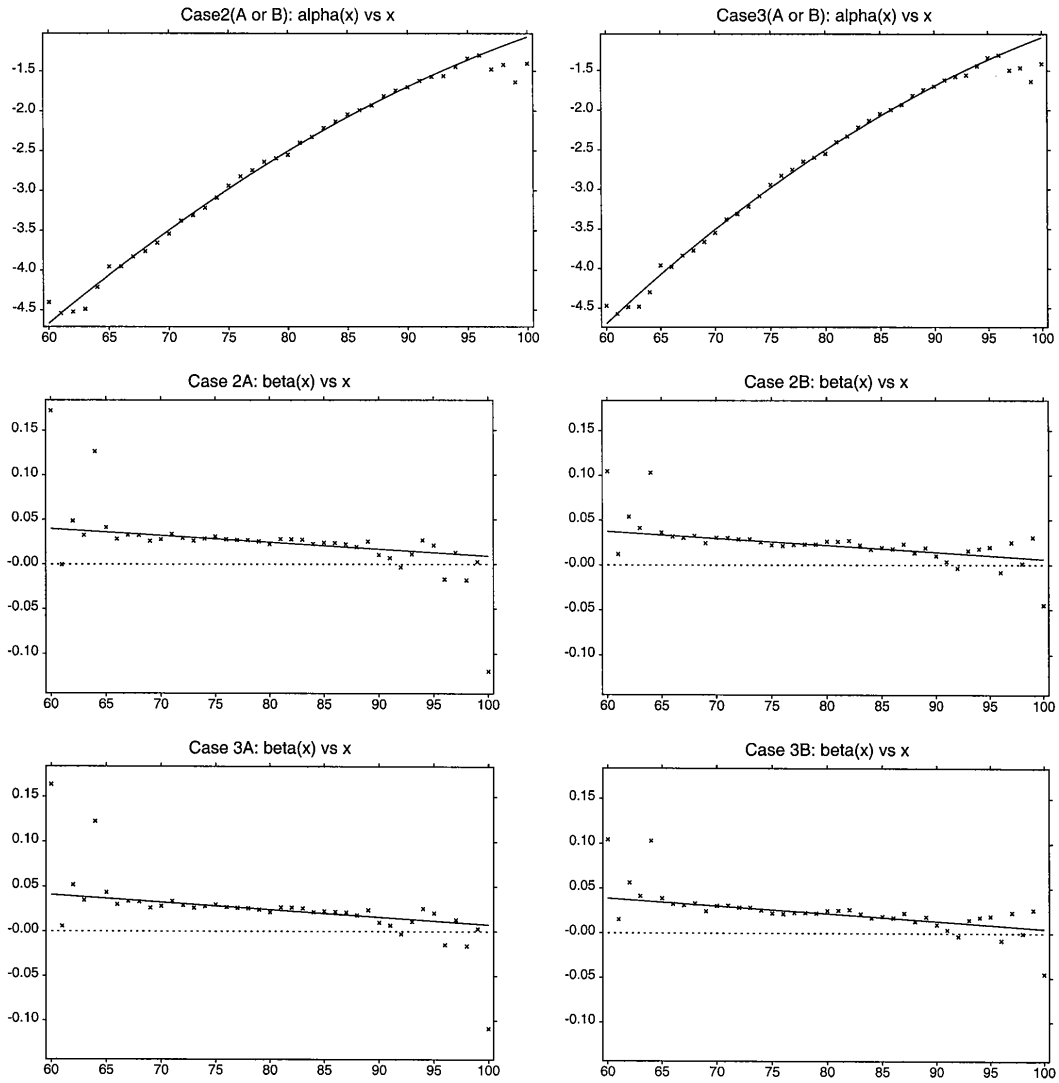

Fig. 6. RF for q_x, various α_x, β_x

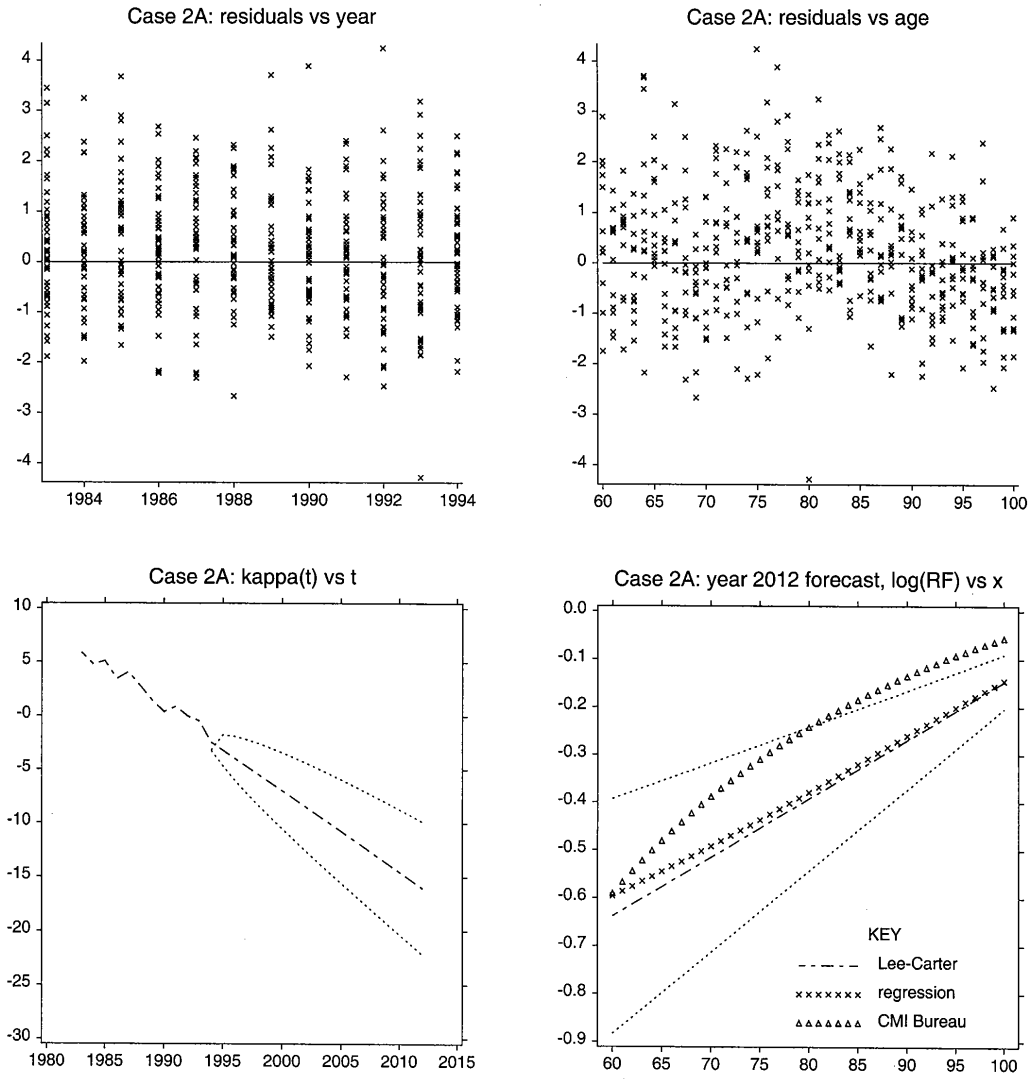

Fig. 7. RF for q_x, specific residuals, κ_t, forecasts

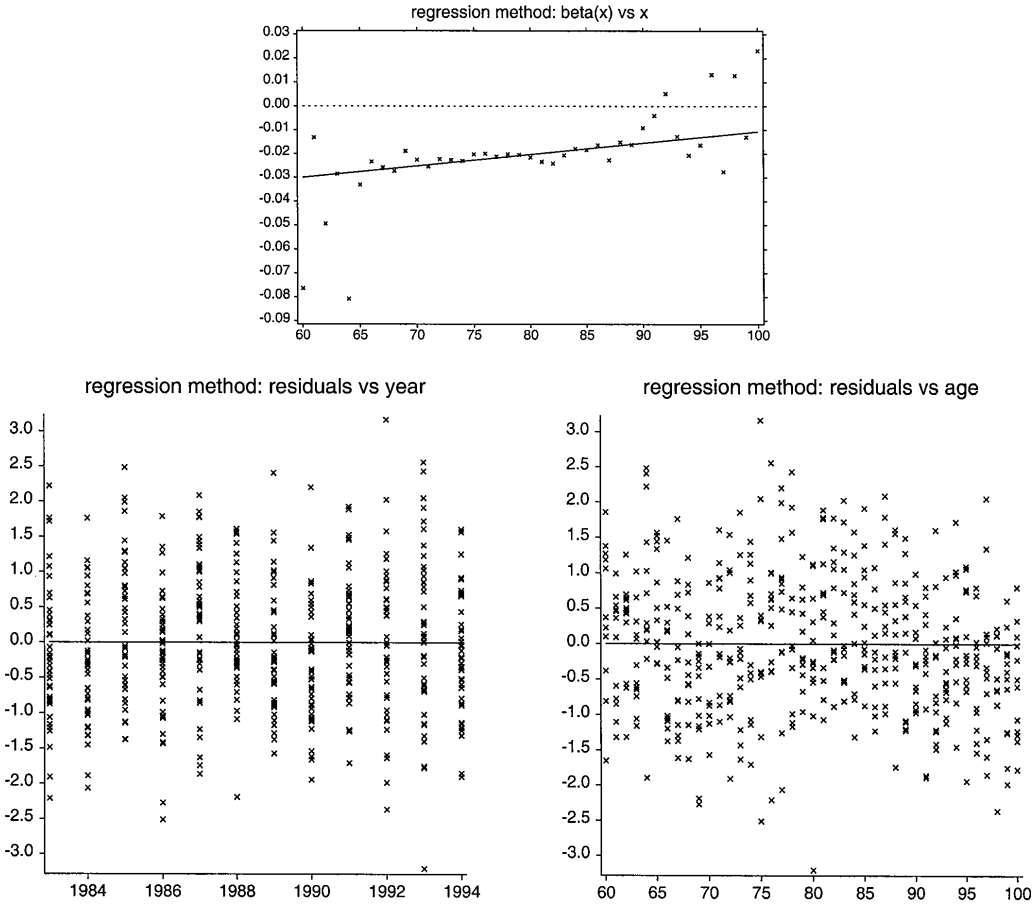

Fig. 8. RF for q_x, regression approach, β_x, residuals

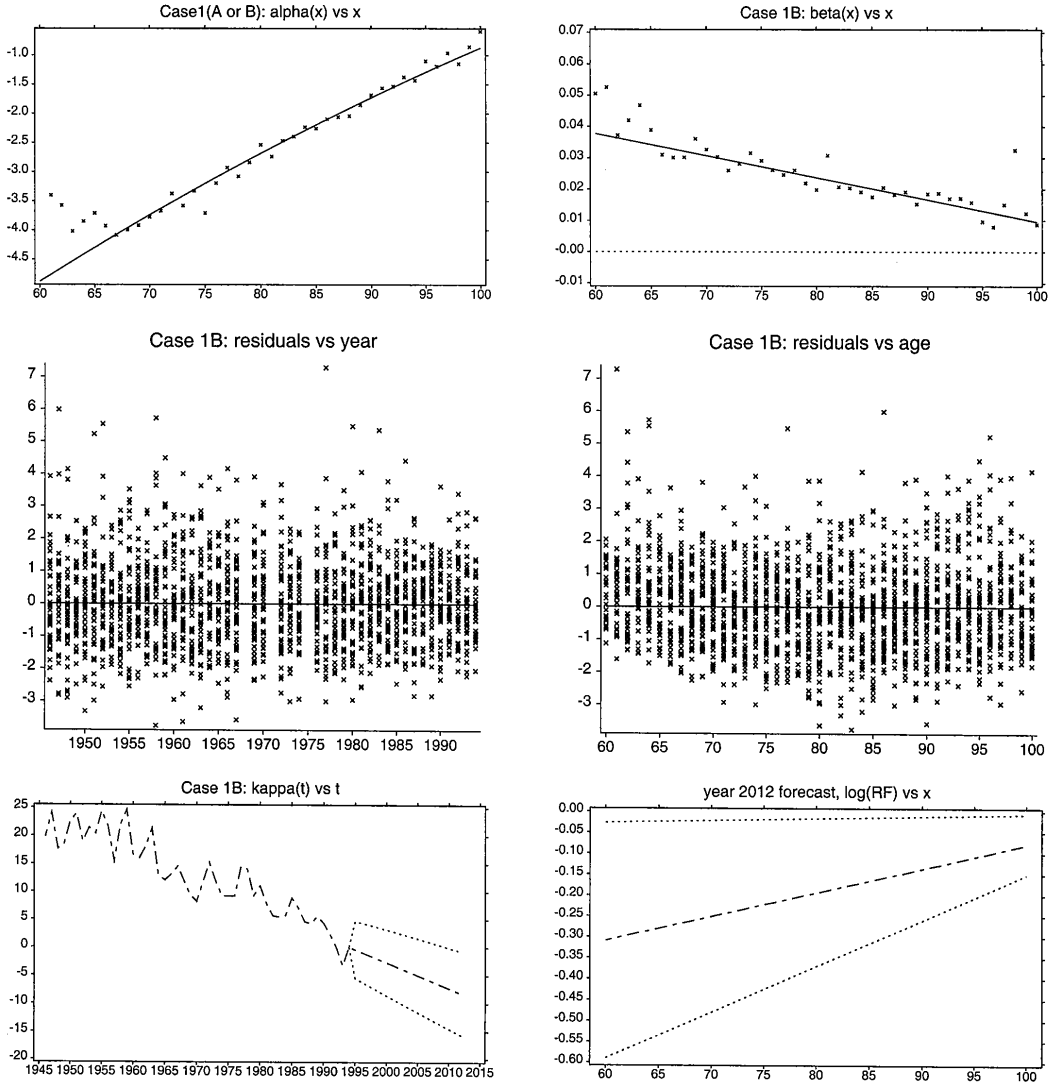

Fig. 9. RF for μ_x, specific α_x, β_x, residuals, κ_t, forecasts

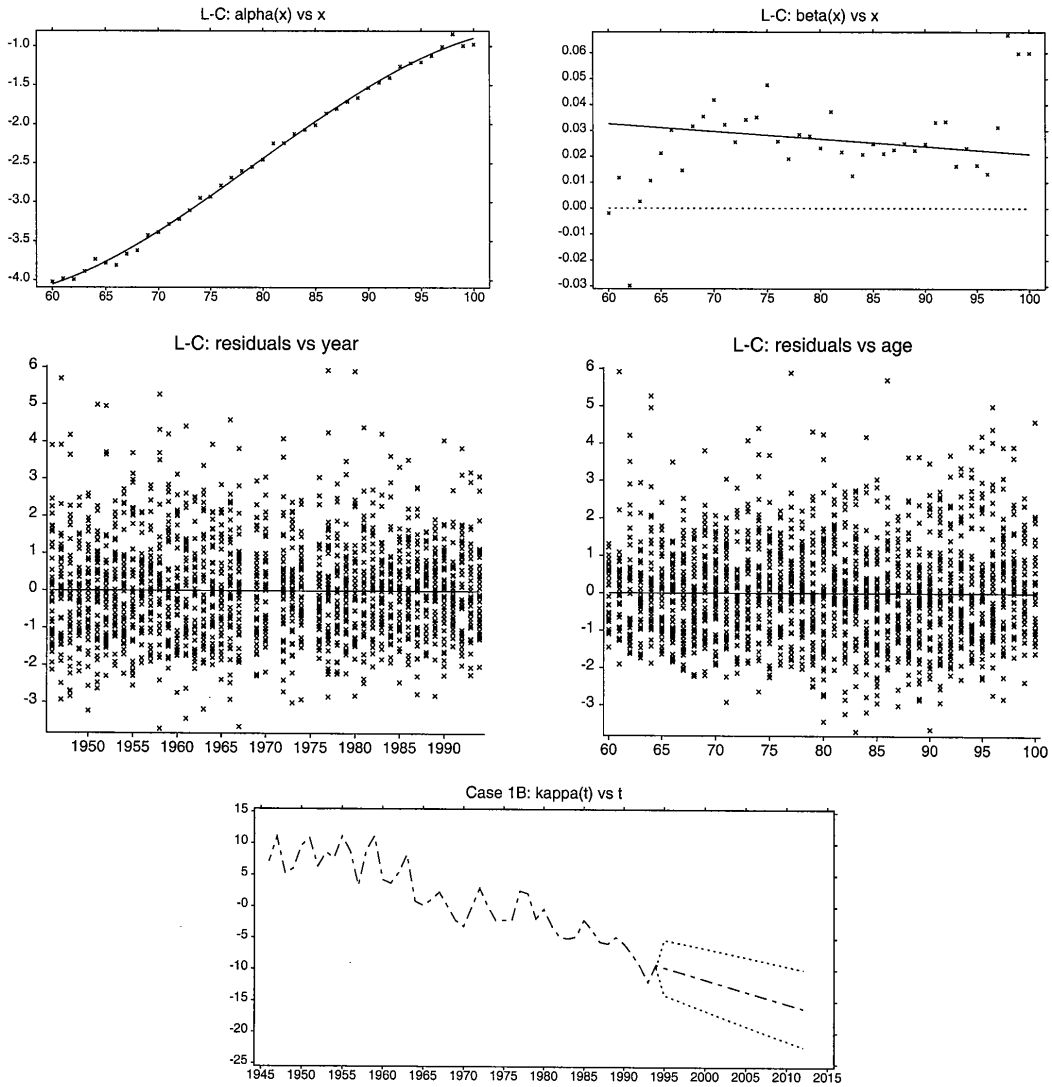

Fig. 10. Forecasting μ_x, L-C approach, α_x, β_x, residuals, κ_t

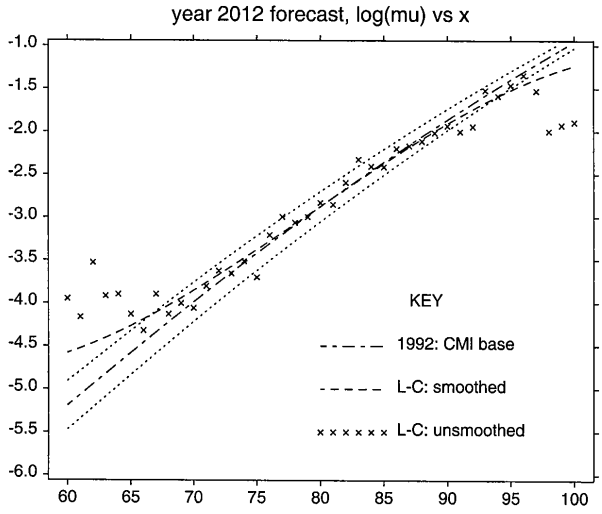

Fig. 11. Forecasting μ_x, comparative forecasts

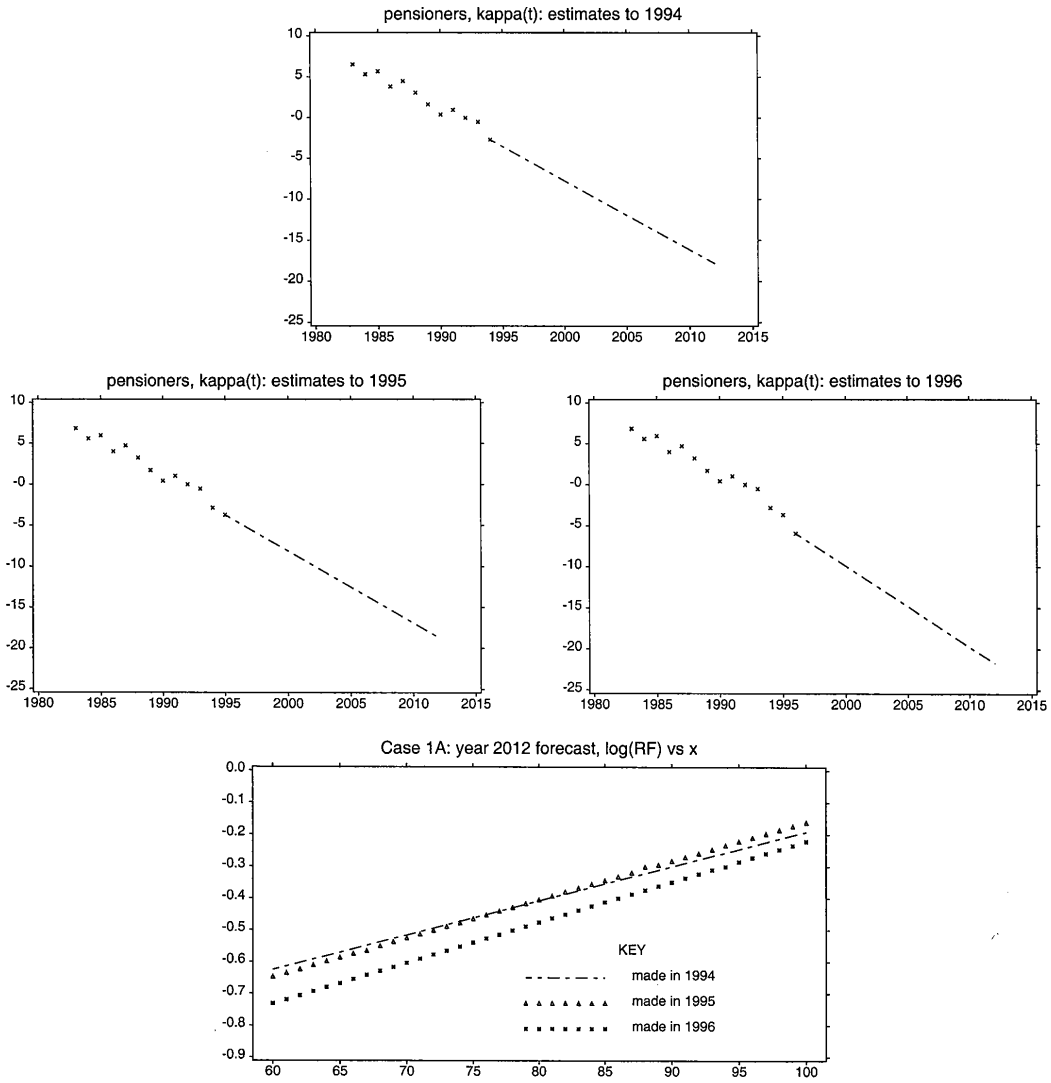

Fig. 12. Case 1A: estimated κ_t values with various forecasts

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