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Moving Average Models for Interest Rates and Applications to Life Insurance Mathematics

by

D Velmachos and S Haberman

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Department of Actuarial Science and Statistics
City University
London

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Abstract

The paper examines the effect on standard functions from life insurance mathematics when not only mortality but also interest rates are treated as having a random component. The analysis is concerned with a particular type of stochastic model, viz the force of interest is modelled by an unconditional moving average process of order q. These explicit results may be seen as extensions to those of Frees (1990) who considered the case q=1.

1. INTRODUCTORY COMMENTS

Historically, the theory of life contingencies has developed from deterministic beginnings.

Random fluctuations in mortality, morbidity, interest and expenses, were ignored, although actuaries have implicitly attempted to allow for random fluctuations by using conservative assumptions for each of the factors entering a formulae. For example, in the calculation of the present values of the liabilities for a policy, we can assume that mortality follows an a priori known mortality table or that the variability due to mortality (i.e. variability in future lifetimes) can be ignored because of the presence of a very large number of identical liabilities in respect of different lives. Similarly, the interest rate may be assumed to be constant or an implicit allowance may be made by adopting a conservative estimate of future interest rates.

A first step forward in the development of the subject was to consider the time until decrement (death, disability, and so on) as a random variable in the calculation of actuarial functions, while the interest rate was assumed to be constant. This is a "semi-stochastic approach".

It is only since about 1970 that there has been interest in actuarial models which consider both the time until decrement and the investment rate of return as random variables.

Pollard (1971) and Boyle (1976) have considered interest rate fluctuations by treating the force of interest as a random variable. Boyle (1976) examined the case in which the force of interest in any year is a normal variable but uncorrelated with the force of interest in any other year. This is a natural and simple assumption to make and is explored further in section 2.

Pollard (1971) on the other hand modelled the force of interest by using a particular stationary autoregressive process of order two. Panjer and Bellhouse (1980) and Bellhouse and Panjer (1981) have developed a general theory for autoregressive models of the force of interest, including both continuous and discrete models. This theory has been developed for unconditional and conditional autoregressive processes of order one and two.

Giaccotto (1986) has developed an algorithm for evaluating present value functions when interest rates are assumed to follow an ARIMA process. Also Wilkie (1976), Waters (1978), Westcott (1981), de Jong (1984), Dhaene (1989) and Frees (1990) have considered stochastic interest models in the calculation of the standard actuarial functions of life insurance mathematics.

The application of such time series based stochastic interest rate models has become the response that actuaries provide to the criticism of other financial analysts that the "traditional" actuarial approach does not take into account the uncertainty of future values of interest rates.

A question that now arises is whether the stochastic nature which is used for the calculation of interest rates is correct. Many actuaries remain sceptical as to the use of a stochastic interest rate model believing that the results obtained owe more to the specific model used than to any underlying reality. This question of the significance of model sensitivity is the subject of current research and interested readers are referred to Wright (1997) for an investigation in the area of pension funding models.

However, in this paper, we will concentrate on models with a certain stochastic nature and the choice of an incorrect interest model will not be considered here.

As in Frees (1990), we use the sequence $\{\Delta_k\}$ to model a stochastic environment. Here, $\{\Delta_k\}$ represents the random force of interest in the k-th period. $\{\Delta_k\}$ can be interpreted as a one-period spot rate.

It is convenient to model the force of interest as a random quantity, in lieu of the effective interest or discount rate, due to the linear nature of correlation and autoregressive models and the multiplicative nature of compound interest.

In this paper, we consider only discrete time models, and for convenience we refer to time intervals as years.

2. INDEPENDENT INTEREST RATE MODEL

2.1 Introduction

In this section we will consider the case in which the $\{\Delta_k\}$ are independent, identically distributed (i.i.d) random variables.

With the interpretation of the $\{\Delta_k\}$ as one-period spot rates and the i.i.d assumption, then at time 0 the present value of one unit payable at time k is:

$$v_k = \prod_{s=1}^{k} exp\,(-\Delta_s) = exp(-\sum_{s=1}^{k} \Delta_s\,)$$

so that the logarithm of v_k is a random walk. This is a feature desirable from the viewpoint of the theory of financial economics because the random walk is a special case of a martingale, the structure of which does not permit riskless arbitrage (see Frees (1990)): this is a requirement of this theory. The i.i.d assumption also provides a benchmark assumption against which more complex models, such as those of section 3, can be compared.

We will also assume that $\{\Delta_k\}$ are distributed normally with mean μ and variance σ^2 so that $\Delta_k \sim N\,(\,\mu\,,\,\sigma^2)$. In this case $exp(-\Delta_k)$ is said to be lognormally distributed with parameters $-\,\mu\,,\,\sigma^2$. Then, $exp(-\Delta_k) \sim logN\,(-\mu\,,\,\sigma^2\,)$.

2.2 Some Basic Results

The moment generating function of Δ_k is

$E(exp(t\ \Delta_k)) = M_\Delta(t) = exp(t\mu + t^2\sigma^2/2).$

Thus with $t = -1$:

$$E(exp(-\Delta_k)) = M_\Delta(-1) = exp(-\mu + \sigma^2/2). \tag{2.1}$$

The present value at time 0 of 1 unit payable at time k is :

$v_k = e^{-\sum_{i=1}^{k} \Delta_i}$ for $k \geq 1$ and $v_k = 1$ for $k = 0$.

So the expected value of the present value is:

$$E(v_k) = E(exp(-\sum_{i=1}^{k} \Delta_i)) = E(exp(-\Delta_1)\ exp(-\Delta_2) \ldots exp(-\Delta_k)) \overset{\Delta\ i.i.d}{=}$$

$$E(exp(-\Delta_1))\ E(exp(-\Delta_2)) \ldots E(exp(-\Delta_k)) = (exp(\mu - \sigma^2/2))^k = exp(-k\ (\mu - \sigma^2/2)).$$

We set $d_1 = \mu - \sigma^2/2$ then $E(v_k) = exp(-k\ d_1).$ (2.2).

$$v_k^2 = exp(-2\sum_{i=1}^{k} \Delta_i) \Rightarrow E(v_k^2) = E(exp(-2\sum_{i=1}^{k} \Delta_i)) = (M_\Delta(-2))^k = exp(-k\ 2\ (\mu - \sigma^2)).$$

We set $d_2 = 2\ (\mu - \sigma^2)$ and then

$$E(v_k^2) = exp(-k\ d_2). \tag{2.3}$$

In general : $E(v_k^n) = E(exp(-n\sum_{i=1}^{k} \Delta_i)) = (M_\Delta(-n))^k = exp(-k(n\ \mu - n^2\sigma^2/2))$

and so $E(v_k^n) = exp(-k\ d_n)$ where $d_n = n\ \mu - n^2\sigma^2/2.$ (2.4)

Similarly, we find that: $E(v_s\, v_r) = exp(-r\ d_1)\ exp(-s\ (d_2 - d_1))$ $(s<r).$ (2.5)

2.3 Life Insurance Actuarial Functions

This section is based on the approach of Frees (1990) . We will assume that there is only one decrement, mortality. We define the integer-valued random variable K to be the curtate time of decrement, so that if T_x is the future time until decrement of a person aged x then $K < T_x < K + 1$.

We will use the standard notation: $P(K = k) = {}_{k/}q_x$ and $P(K>k) = {}_{k+1}p_x$ $k = 0,1,2.....$ for the discrete probability function and survival function respectively.

Also we will assume that K is independent of $\{\Delta_k\}$.

Two types of general contracts are considered: insurance and annuity contracts.

For the general insurance contract, a benefit b_{k+1} is taken to be payable at time $k+1$ at the end of the year of death, given that death occurs during the year $(k, k+1)$.

The random present value of this insurance benefit is $Z_{K+1} = v_{K+1}\, b_{K+1}$ where $b_{K+1} \in (-\infty, +\infty)$.

For the general annuity contract, payments c_s are payable at time s, at the beginning of each year up to and including the year of death.

The random present value of the annuity benefits is then:
$a(K) = \sum_{s=0}^{K} v_s c_s$ *where* $c_s \in (-\infty, +\infty)$.

By the law of conditional expectations, we have

$E[g(X)] = E[E[g(X)/Y]]$ so for the case of the life insurance contract:

$$E(Z_{K+1}) = E[E(v_{K+1}\, b_{K+1} \;/K=k\;)] = E[\exp(-(K+1)\, d_1)\, b_{K+1}] =$$

$$= \sum_{k=0}^{\infty} e^{-d_1(k+1)} b_{k+1\; k|}q_x \,. \tag{2.6}$$

$$E(Z_{K+1}^2) = E[E(v_K^2\; b_{K+1}^2 \;/K=k\;)] = E[\exp(-(K+1)\, d_2)\; b_{K+1}^2] =$$
$$= \sum_{k=0}^{\infty} e^{-d_2(k+1)} b_{k+1\; k|}^2 q_x \quad . \tag{2.7}$$

In a similar manner, we can use (2.4) to demonstrate that

$$E(Z_{K+1}^n) = \sum_{k=0}^{\infty} e^{-d_n(k+1)} b_{k+1\; k}^n|q_x \tag{2.8}$$

For the case of the annuity contract:

$$E[a(K)] = E[E(\sum_{s=0}^{K} v_s c_s \;/K=k\;)] = E[\sum_{s=0}^{K} e^{-d_1 s} c_s\,] = \sum_{s=0}^{\infty} c_s e^{-d_1 s}\; {}_s p_x \,. \tag{2.9}$$

$$E[a(K)^2] = E(\sum_{s=0}^{K} e^{-d_2 s} c_s^2\,) + 2\,E(\,\sum_{r=1}^{K} \sum_{s=0}^{r-1} e^{-d_1 s} e^{-(d_2-d_1)r} c_r c_s\,)\,. \tag{2.10}$$

$$E[Z_{K+1}\, a(K)] = E(b_{K+1} \sum_{s=0}^{K} e^{-s(d_2-d_1)} e^{-(K+1)d_1} c_s). \tag{2.11}$$

As an alternative approach, Waters (1978) provides simple expressions for the moments of compound interest functions which are useful for calculating the high order moments of certain actuarial functions which will be used later in some numerical examples.

Let $\tilde{a}_{\overline{n|}}$ be the stochastic equivalent for the annuity-certain $a_{\overline{n|}}$ then $\tilde{a}_{\overline{n|}} = \sum_{k=1}^{n} v_k$.

Let $\theta = exp(-\mu)$ $\varphi = exp(\sigma^2/2)$, then

$$E(\tilde{a}_{\overline{n|}}) = \sum_{k=1}^{n} \theta^k \varphi^k \tag{2.12}$$

and the following results are obtained by Waters (1978):

$$E(\tilde{a}_{\overline{n|}}^2) = \sum_{k=1}^{n} \theta^{2k} \varphi^{4k} + 2 \sum_{k=2}^{n} \sum_{j=1}^{k-1} \theta^{(k+j)} \varphi^{(k+3j)} \tag{2.13}$$

$$E(\tilde{a}_{\overline{n|}}^3) = \sum_{k=1}^{n} \theta^{3k} \varphi^{9k} + 3 \sum_{k=2}^{n} \sum_{j=1}^{k-1} (\theta^{(2k+j)} \varphi^{(4k+5j)} + \theta^{(k+2j)} \varphi^{(k+8j)}) +$$
$$6 \sum_{k=3}^{n} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \theta^{(i+j+k)} \varphi^{(5i+3j+k)} \tag{2.14}$$

$$E(\tilde{a}_{\overline{n|}}^4) = \sum_{k=1}^{n} \theta^{4k} \varphi^{16k} + 4 \sum_{k=2}^{n} \sum_{j=1}^{k-1} (\theta^{(k+3j)} \varphi^{(k+15j)} + \theta^{(3k+j)} \varphi^{(9k+7j)}) +$$
$$6 \sum_{k=2}^{n} \sum_{j=1}^{k-1} \theta^{(2k+2j)} \varphi^{(4k+12j)} + 12 \sum_{k=3}^{n} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} (\theta^{(i+j+2k)} \varphi^{(7i+5j+4k)} + \theta^{(i+2j+k)}$$
$$\varphi^{(7i+8j+k)} \theta^{(2i+j+k)} \varphi^{(12i+3j+k)}) + 24 \sum_{k=4}^{n} \sum_{j=3}^{k-1} \sum_{i=2}^{j-1} \sum_{h=1}^{i-1} \theta^{(h+i+j+k)} \varphi^{(7h+5i+3j+k)}$$
$$\tag{2.15}$$

$$E(\tilde{a}_{\overline{n|}} v_m) = \sum_{k=1}^{n} \theta^{(m+k)} \varphi^{(m+3k)} \qquad \text{if } m>n$$
$$= \sum_{k=1}^{m} \theta^{(m+k)} \varphi^{(m+3k)} + \sum_{k=1}^{n} \theta^{(m+k)} \varphi^{(k+3m)} \qquad \text{if } m<n.$$

We note also that

$\tilde{A}_x$ (Whole Life Assurance) , $\tilde{A}^1_{x:\overline{n|}}$ (Temporary Assurance)
$\tilde{A}_{x:\overline{n|}}$ (Endowment Assurance) , $\tilde{a}_x$ (Immediate Whole Life Annuity)
$\tilde{a}_x$ (Whole Life Annuity due) , $\tilde{a}_{x:\overline{n|}}$ (Immediate Temporary Annuity)

are the stochastic equivalents (i.e random variables) corresponding to the standard actuarial functions. In order to derive expressions for the moments of these random variables we use the rules of conditional expectation, as applied earlier.

We let $g(K)$ be the present value of the benefit for a standard life insurance policy issued to a life aged x. Then, for the case of a whole life policy,

$$g(K) = \{v_{K+1} \quad K \geq 0\} \quad \text{and} \quad E\,[g(K)^m] = \sum_{k=0}^{\infty} {}_{k/}q_x\, E[\, v_{k+1}{}^m] \,. \tag{2.16}$$

For the case of a temporary insurance policy:

$$g(K) = \left\{ \begin{matrix} v_{K+1} & K < n \\ 0 & K \geq n \end{matrix} \right\} \text{and } E\,[g(K)^m] = \sum_{k=0}^{n-1} {}_{k/}q_x\, E[\, v_{k+1}{}^m] \,. \tag{2.17}$$

For the case of an endowment insurance policy: $g(K) = \left\{ \begin{matrix} v_{K+1} & K < n \\ v_n & K \geq n \end{matrix} \right\}$ and

$$E\,[g(K)^m] = \sum_{k=0}^{n-1} {}_{k/}q_x E[\, v_{k+1}{}^m] + {}_np_x\, E[\, v_n{}^m]. \tag{2.18}$$

We let $h(K)$ be the present value of a standard annuity policy issued to a life aged x. Then, for the case of an immediate whole life annuity:

$$E\,[h(K)^m] = \sum_{k=1}^{\infty} {}_{k/}q_x\, E[\tilde{a}_{\overline{k|}}{}^m] \,. \tag{2.19}$$

For the case of an immediate temporary annuity then:

$$E\,[h(K)^m] = \sum_{k=1}^{n-1} {}_{k/}q_x\, E[\tilde{a}_{\overline{k|}}{}^m] + {}_np_x\, E[\tilde{a}_{\overline{n|}}{}^m].$$

Also we know that the present value of the whole life annuity-due is equal to the present value of the immediate whole life annuity plus 1 and the present value of the temporary annuity-due (age x, term n) is equal to the present value of the immediate temporary annuity (age x, term $n-1$) plus 1.

2.4 Examples

The above formulae (2.6-2.20) enable us to compute the moments of the present values and related functions for these simple insurances and annuities.

We will assume in the following calculations in this section and in section 3.6 that all lives are subject to the mortality experience of A 1967-70 (Ultimate).

Tables 2.1 - 2.5, on the following pages, present calculated values of the expectation, standard deviation, skewness and kurtosis for the case of μ=0.07 and for several values of x, n, (where appropriate) and σ for the respective cases of an endowment, temporary and whole life insurance, and a temporary and whole life annuity.

Some relevant graphs are also presented (Figures 2.1 - 2.3).

When $\sigma = 0$ the stochastic equivalents of the actuarial functions are equal to the normal actuarial functions calculated on a deterministic basis..

In each example, the standard deviation of the actuarial function increases as σ increases (as we can see in Fig. 2.1 for the Temporary Insurance), but the amount of increase is not significant. This feature occurs because the greater part of the variation in these functions is due to fluctuations in the age at death as opposed to fluctuations in the interest rates. Of course, this situation changes if we consider a large number of independent lives and then the fluctuations in interest rates would become more important.

The standard measures of skewness and kurtosis are calculated as $\frac{\mu_3}{\mu_2^{3/2}}$ and $\frac{\mu_4}{\mu_2^2}$ respectively

where μ_n is the n th central moment of the distribution. So, for a generic random variable X, we have that:

$$\text{Skewness of } X = \ldots = \frac{E(X^3) - 3E(X^2)E(X\ ) + 2E(X\ )^3}{VAR(X)^{3/2}}$$

$$\text{Kurtosis of } X = \ldots = \frac{E(X^4) - 4E(X^3)E(X\ ) + 6E(X^2)E(X\ )^2 - 3E(X\ )^4}{VAR(X)^2}.$$

We can see from the Tables and Figures that the values of skewness and kurtosis are both very large. This reflects the very skew and very sharply peaked shape of the density functions.

Figures 2.2 and 2.3 also show that an increase in σ is associated with decreases in the absolute values of the skewness and kurtosis. This is because, when σ increases, the density function will tend to be spread more evenly over its range.

Endowment Assurance

μ= 0.07

Age	σ= 0			σ= 0.03			
	20	30	40	20	30	40	← Term
25	0.250732	0.130961	0.075913	0.252958	0.132643	0.077098	
30	0.252214	0.135393	0.08456	0.254441	0.137091	0.085801	
35	0.255969	0.144352	0.099852	0.258195	0.146076	0.101177	**Expectation**
40	0.263217	0.159945	0.124037	0.26544	0.161706	0.125479	
45	0.275708	0.184779	0.159124	0.277924	0.186594	0.160713	
25	0.040116	0.054966	0.065193	0.052351	0.059008	0.066894	
30	0.043825	0.062549	0.075287	0.055305	0.066334	0.077113	
35	0.054743	0.07873	0.093565	0.064397	0.082066	0.095499	**Standard Deviation**
40	0.0721	0.101686	0.117287	0.079803	0.104664	0.119368	
45	0.094606	0.128931	0.143257	0.100758	0.131727	0.145534	
25	12.16509	9.478377	7.736863	5.611329	7.815982	7.334613	
30	10.2824	7.407445	5.818319	5.255787	6.36953	5.580754	
35	7.768355	5.463463	4.315624	4.887615	4.958812	4.19439	**Skewness**
40	5.698423	4.026704	3.264449	4.286018	3.795818	3.199596	
45	4.171295	2.977109	2.499181	3.511257	2.867781	2.4601	
25	165.9879	106.9121	76.5245	59.53698	82.9287	71.18357	
30	122.6754	68.83172	46.31082	50.67303	56.48902	43.71835	
35	71.7015	38.79347	26.66794	39.35196	34.27791	25.64169	**Kurtosis**
40	39.41627	21.9352	16.07935	27.59368	20.41533	15.65257	
45	21.8678	12.83286	10.20655	17.85284	12.30768	10.00679	

	σ= 0.05			σ= 0.07			
25	0.256965	0.13569	0.079256	0.263096	0.140397	0.082616	
30	0.258449	0.140165	0.088056	0.264581	0.144912	0.091564	
35	0.262201	0.149194	0.103582	0.26833	0.154005	0.107315	**Expectation**
40	0.26944	0.164891	0.128094	0.275559	0.1698	0.13214	
45	0.281912	0.189871	0.163589	0.288009	0.194915	0.168025	
25	0.070038	0.066259	0.070151	0.092462	0.077413	0.075626	
30	0.072355	0.073203	0.08058	0.094324	0.083908	0.086342	
35	0.079653	0.088218	0.099146	0.100186	0.098	0.105149	**Standard Deviation**
40	0.092743	0.110202	0.123267	0.111106	0.119116	0.129624	
45	0.111535	0.136932	0.149776	0.127524	0.145327	0.156635	
25	2.698064	5.816389	6.668243	1.744333	4.141832	5.800229	
30	2.68798	5.0323	5.187335	1.765969	3.815507	4.674605	
35	2.864348	4.238725	3.992629	1.921236	3.488757	3.727957	**Skewness**
40	2.935804	3.441217	3.092483	2.08443	3.031094	2.954378	
45	2.728892	2.694033	2.396992	2.097574	2.483059	2.319456	
25	21.15711	55.42012	62.40491	10.07848	33.57412	51.07552	
30	19.86285	41.04694	39.44889	9.942408	27.41702	33.91595	
35	19.09772	27.8804	23.93996	10.40519	21.2425	21.7187	**Kurtosis**
40	16.92069	18.06497	14.9508	10.49592	15.3121	14.05671	
45	13.16966	11.46342	9.686215	9.432219	10.42355	9.302338	

(Table 2.1)

Temporary Insurance

μ= 0.07

Age ▼	20	30	40	20	30	40	◀── Term
	σ= 0			σ= 0.03			
25	0.008802	0.015809	0.025669	0.00884	0.015927	0.025941	**Expectation**
30	0.013122	0.025255	0.040547	0.013187	0.025457	0.040988	
35	0.022484	0.042475	0.064463	0.022599	0.042814	0.065144	
40	0.039462	0.070528	0.099116	0.039662	0.071074	0.100106	
45	0.067541	0.112467	0.144796	0.067875	0.113299	0.146125	
25	0.06884	0.076295	0.079364	0.069311	0.077065	0.08039	**Standard Deviation**
30	0.080006	0.090104	0.092185	0.080649	0.091164	0.093552	
35	0.102151	0.112944	0.111433	0.103023	0.114334	0.113138	
40	0.133218	0.141417	0.133424	0.134365	0.14316	0.13544	
45	0.170499	0.171063	0.154938	0.171965	0.173165	0.157226	
25	8.953197	6.755749	5.724027	8.979497	6.745778	5.678192	**Skewness**
30	6.969304	4.922426	4.16433	6.998017	4.922136	4.134205	
35	5.097343	3.55962	3.145786	5.12375	3.563889	3.124647	
40	3.683315	2.603871	2.498685	3.704725	2.607934	2.479716	
45	2.630119	1.902682	2.058647	2.647343	1.905402	2.039504	
25	91.00436	58.89308	47.46284	91.45446	58.5813	46.68274	**Kurtosis**
30	56.76734	33.43919	27.64519	57.13931	33.28769	27.179	
35	31.20699	18.67758	17.00241	31.46779	18.62261	16.73788	
40	17.07158	11.08223	11.54311	17.22666	11.05037	11.36838	
45	9.522994	7.047866	8.39353	9.611581	7.020165	8.268706	

	σ= 0.05			σ= 0.07			
25	0.00891	0.016139	0.026435	0.009016	0.016464	0.027198	**Expectation**
30	0.013303	0.025819	0.041786	0.013481	0.026374	0.043018	
35	0.022806	0.043424	0.066378	0.023119	0.044358	0.068282	
40	0.040021	0.072057	0.101896	0.040567	0.073562	0.104653	
45	0.068476	0.114796	0.148527	0.069388	0.117088	0.152222	
25	0.070165	0.078477	0.082292	0.092462	0.077413	0.075626	**Standard Deviation**
30	0.081814	0.093106	0.096083	0.094324	0.083908	0.086342	
35	0.104604	0.116881	0.116295	0.100186	0.098	0.105149	
40	0.136446	0.146354	0.139173	0.111106	0.119116	0.129624	
45	0.174624	0.177017	0.161462	0.127524	0.145327	0.156635	
25	9.031695	6.734988	5.604097	9.123981	6.737299	5.513491	**Skewness**
30	7.054487	4.928545	4.088527	7.152976	4.956173	4.040998	
35	5.175067	3.577167	3.093909	5.2631	3.611769	3.065846	
40	3.746162	2.619827	2.45199	3.816844	2.649694	2.426167	
45	2.680645	1.91433	2.011029	2.737365	1.938216	1.982863	
25	92.4058	58.14589	45.39441	94.24417	57.83147	43.74943	**Kurtosis**
30	57.92817	33.11777	26.43958	59.45547	33.13837	25.57846	
35	32.01151	18.58814	16.32832	33.04223	18.71085	15.8816	
40	17.54876	11.03288	11.09886	18.15655	11.11486	10.80867	
45	9.796604	6.996641	8.076328	10.14826	7.032748	7.870315	

(Table 2.2)

Whole Life Assurance

μ= 0.07

Age ▼	σ= 0	σ= 0.03	
25	0.046933	0.047692	
30	0.062972	0.063904	
35	0.085592	0.086719	**Expectation**
40	0.116086	0.117421	
45	0.155641	0.157185	
25	0.0739	0.075033	
30	0.083739	0.08518	
35	0.100591	0.102317	**Standard Deviation**
40	0.122126	0.124116	
45	0.145857	0.148103	
25	6.241921	6.131449	
30	4.808983	4.725328	
35	3.769096	3.710624	**Skewness**
40	3.008606	2.966109	
45	2.398801	2.366687	
25	55.43961	53.88551	
30	34.97892	33.98406	
35	22.08254	21.53847	**Kurtosis**
40	14.50455	14.20148	
45	9.765005	9.591821	

	σ= 0.05	σ= 0.07	
25	0.04908	0.051254	
30	0.065604	0.068256	
35	0.088768	0.091954	**Expectation**
40	0.119843	0.123594	
45	0.159978	0.164288	
25	0.077178	0.080743	
30	0.087896	0.092365	
35	0.105554	0.110841	**Standard Deviation**
40	0.127834	0.133868	
45	0.152282	0.159025	
25	5.938401	5.662286	
30	4.583711	4.392437	
35	3.613735	3.48828	**Skewness**
40	2.896926	2.810612	
45	2.315474	2.25439	
25	51.18419	47.35429	
30	32.30988	30.07196	
35	20.6411	19.49152	**Kurtosis**
40	13.70986	13.10493	
45	9.316718	8.996194	

(Table 2.3)

Temporary Annuity

μ= 0.07

Age	20	30	40	20	30	40	Term
	σ= 0			σ= 0.03			
25	10.3242	11.96872	12.71783	10.36248	12.02552	12.78646	Expectation
30	10.29898	11.89758	12.58322	10.33708	11.95367	12.65011	
35	10.23709	11.75601	12.34796	10.2748	11.81078	12.41202	
40	10.11901	11.51195	11.97955	10.15599	11.56452	12.03943	
45	9.917046	11.12658	11.45024	9.952814	11.17582	11.5045	
25	0.618771	0.83109	0.975919	0.941087	1.232491	1.409108	Standard Deviation
30	0.679557	0.948195	1.127272	0.981233	1.311098	1.510452	
35	0.850512	1.192883	1.398156	1.105092	1.492901	1.711886	
40	1.118759	1.536578	1.747301	1.320614	1.773098	1.994143	
45	1.463675	1.940734	2.127609	1.62034	2.124356	2.318852	
25	-11.8378	-9.26744	-7.60488	-3.20914	-2.66065	-2.32707	Skewness
30	-9.91057	-7.20182	-5.70944	-3.13693	-2.54443	-2.17538	
35	-7.45813	-5.30802	-4.24368	-3.26187	-2.54737	-2.13481	
40	-5.46917	-3.91813	-3.22227	-3.21176	-2.41616	-2.01927	
45	-4.0046	-2.90335	-2.47788	-2.86105	-2.10398	-1.79248	
25	157.9609	102.9104	74.46677	31.19068	22.93274	18.77073	Kurtosis
30	114.7629	65.62365	44.96435	27.91833	19.39841	15.35806	
35	66.62529	36.97874	26.01169	24.54627	16.16924	12.64791	
40	36.66236	21.01373	15.80142	19.68412	12.61449	10.08173	
45	20.41433	12.385	10.1075	14.11653	9.149648	7.707116	

	σ= 0.05			σ= 0.07			
25	10.43104	12.12753	12.90996	10.53511	12.28306	13.09889	Expectation
30	10.40533	12.05439	12.77046	10.50892	12.20794	12.9545	
35	10.34233	11.90911	12.52724	10.44484	12.05901	12.70336	
40	10.22223	11.65891	12.14709	10.32276	11.80275	12.31153	
45	10.01687	11.2642	11.60199	10.11408	11.39883	11.75076	
25	1.346338	1.750425	1.98179	1.812798	2.359162	2.666662	Standard Deviation
30	1.3735	1.802147	2.04441	1.831577	2.392051	2.699517	
35	1.462496	1.931385	2.181837	1.896588	2.481522	2.78386	
40	1.628275	2.145003	2.388515	2.022877	2.636896	2.9188	
45	1.874877	2.429016	2.638916	2.219585	2.851979	3.088018	
25	-0.80788	-0.59959	-0.48214	0.064039	0.210994	0.295443	Skewness
30	-0.86139	-0.65989	-0.53392	0.026046	0.164144	0.249355	
35	-1.15124	-0.88719	-0.71997	-0.16288	-0.00165	0.102133	
40	-1.49783	-1.11829	-0.90806	-0.45788	-0.23156	-0.09767	
45	-1.6726	-1.20324	-0.99373	-0.73506	-0.42551	-0.2761	
25	9.591583	7.830894	7.015083	5.320939	4.942853	4.820204	Kurtosis
30	9.30434	7.462155	6.603544	5.272849	4.862703	4.721178	
35	9.72376	7.461756	6.484426	5.545584	4.937625	4.725592	
40	9.825548	7.180295	6.212775	5.881287	4.9792	4.704384	
45	8.800274	6.29753	5.582212	5.841431	4.767332	4.520536	

(Table 2.4)

Whole of Life Annuity

μ= 0.07

Age	σ= 0	σ= 0.03	
25	13.09734	13.17409	**Expectation**
30	12.8601	12.93279	
35	12.52551	12.59322	
40	12.07446	12.13625	
45	11.48938	11.54441	
25	1.093088	1.535606	**Standard Deviation**
30	1.238624	1.625412	
35	1.487895	1.803931	
40	1.806433	2.055049	
45	2.157448	2.349702	
25	-6.24192	-2.03259	**Skewness**
30	-4.80899	-1.91441	
35	-3.7691	-1.9269	
40	-3.00861	-1.88862	
45	-2.39881	-1.73235	
25	55.43965	15.87259	**Kurtosis**
30	34.97896	13.20502	
35	22.08257	11.30506	
40	14.50457	9.441296	
45	9.765023	7.493314	

Age	σ= 0.05	σ= 0.07	
25	13.31247	13.52481	**Expectation**
30	13.06377	13.26451	
35	12.71511	12.90171	
40	12.2474	12.41732	
45	11.64332	11.79431	
25	2.134861	2.863422	**Standard Deviation**
30	2.17524	2.85975	
35	2.282771	2.90236	
40	2.454176	2.993636	
45	2.671955	3.125019	
25	-0.38029	0.36631	**Skewness**
30	-0.43745	0.313379	
35	-0.6262	0.162477	
40	-0.83292	-0.0467	
45	-0.951	-0.24375	
25	6.496247	4.793086	**Kurtosis**
30	6.166195	4.696396	
35	6.124692	4.689028	
40	5.981172	4.669524	
45	5.485222	4.505019	

(Table 2.5)

Charts

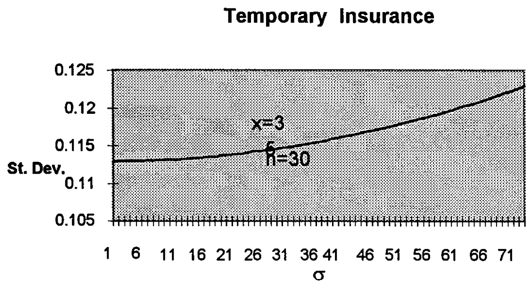

Fig. 2.1

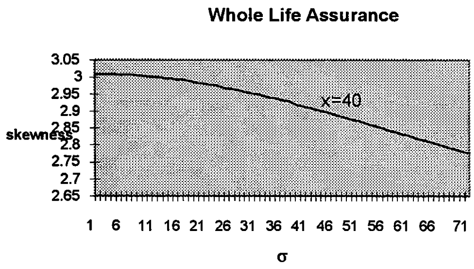

Fig. 2.2

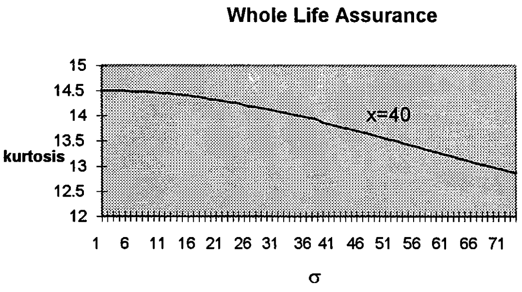

Fig. 2.3

2.6 Cumulative Distribution Function

The approach can be used to derive the cumulative distribution function (and hence the probability density function) for the random variables defined in section 2.3 as in Frees (1990). As an illustration of the method, we derive an expression for and calculate the cumulative distribution function for the present value of a whole life assurance, $\tilde{A}_x$ .Then, following Frees (1990):

$$F(y) = P(\tilde{A}_x \leq y) = E\,[\,P(v_{K+1} \leq y \,/K=k\,)\,] = \sum_{k=1}^{\infty} P(v_{k+1} \leq y)\ {}_{k/}q_x =$$

$$= \sum_{k=1}^{\infty} \Phi\Big(\frac{\log y + (k+1)\mu}{\sigma\sqrt{k+1}}\Big)\ {}_{k/}q_x$$

The following graph gives the Cum. Distr. Fun.of $\tilde{A}_x$ for $x=30$, $\mu = 0.05$, $\sigma = 0.07$

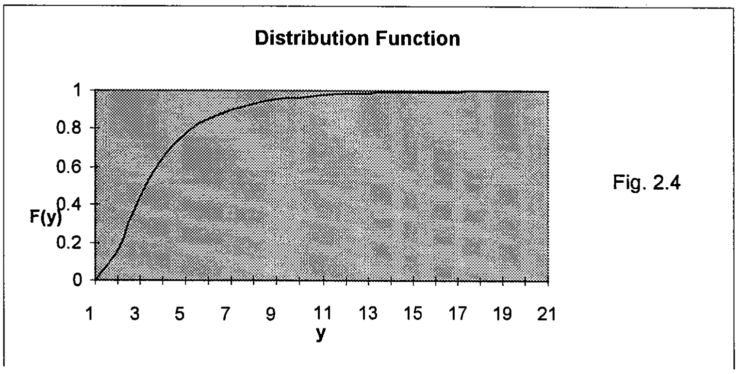

Fig. 2.4

The median is 0.119019, but the mean is 0.138238.

The fact that the median is less than the mean is an indication that the distribution is skewed to the right (as noted earlier).

2.6 Comments

In this section, formulae for the mean, variance, skewness, kurtosis and high order moments have been developed, for several types of life insurance and annuity contract, in the presence of independent and identically distributed investment returns.

However, we have to note that the i.i.d normality assumption will rarely be satisfied in practice. It does though serve as a useful benchmark. In the next section, we introduce a moving average representation of the force of interest.

3. DEPENDENT INTEREST RATE MODEL: The MOVING AVERAGE PROCESS MA(q)

3.1 Introduction

As Frees (1990) mentions, "… the assumption that the interest environment represented by the sequence $\{\Delta_k\}$ is i.i.d is a useful modification of the traditional assumption that $\{\Delta_k\}$ is deterministic . This modification permits volatility of interest rates in the model ".

In this section, we continue with this approach and assume that the interest rate model is stochastic but also dependent. It is assumed that the force of interest follows a moving average model of order q (q is a variable) . In this section we will generalise some of the ideas of Frees (1990) and discuss the general case of the $MA(q)$ model rather than the specific case of $q=1$.

This model $\{ MA(q) \}$ accounts for certain autocorrelation aspects of the sequence $\{\Delta_k\}$ and has the advantage of being tractable (in the mathematical sense) in terms of the calculation of insurance functions. The model has also been applied to pension funding problems by Haberman and Wong (1997) and Bédard and Dufresne (1998).

3.2 Basic Results

We now consider the unconditional $MA(q)$ model,

$$\Delta_k = \mu + \varepsilon_k + a_1\varepsilon_{k-1} + a_2\varepsilon_{k-2} + \ldots + a_q\varepsilon_{k-q} \ . \tag{3.1}$$

where $\{\varepsilon_k\}_{k-\infty}^{+\infty}$ is an i.i.d sequence with mean zero and variance σ^2. The coefficients $\{a_i\}(i=1,2,\ldots q)$ are usually constrained so that the roots of the characteristic equation $1-\sum_{i=1}^{q} a_i x^i = 0$ lie outside the unit circle. If $q=1$, this constraint becomes $-1<a_1<1$, while if $q=2$, the conditions become:

$$a_1 + a_2 < 1, \quad a_2 - a_1 < 1, \quad -1 < a_2 < 1.$$

These constraints are required so that the model is invertible, that is it can be alternatively expressed as an autoregressive model of infinite order, or as the limit of a sequence of autoregressive models of finite order - the invertibility requirement ensures that this sequence is convergent (Box and Jenkins, 1976). Invertibility is similar to the stationarity condition applied to autoregressive models. We define $M(t) = E[e^{\varepsilon t}]$ to be the moment generating function of ε, and assume that $M(t)$ exists.

The following proposition is an important building block for this section.

3.2.1 Proposition 1

$E[v_k] = C_1 e^{-kg_1}$ for $k=1,2,..$

$$g_1 = \mu - \ln(M(-(1+\sum_{i=1}^{q} a_i))) \tag{3.2}$$

$$C_1 = \frac{\left(\prod_{j=1}^{q-1} M(-(1+\sum_{i=1}^{j} a_i))\right) M(-1) \left(\prod_{j=1}^{q} M(-\sum_{i=j}^{q} a_i)\right)}{\left(M(-(1+\sum_{i=1}^{q} a_i))\right)^{q}}. \tag{3.3}$$

Proof

$\Delta_1 = \mu + \varepsilon_1 + a_1\varepsilon_0 + \ldots + a_q\varepsilon_{1-q}$
$\Delta_2 = \mu + \varepsilon_2 + a_1\varepsilon_1 + \ldots + a_q\varepsilon_{2-q}$
.
.
$\Delta_k = \mu + \varepsilon_k + a_1\varepsilon_{k-1} + \ldots + a_q\varepsilon_{k-q}$

By adding the above (and collecting terms along diagonals) we have:

$$\sum_{i=1}^{k} \Delta_i = k\mu + \sum_{i=1}^{k-q} \varepsilon_i\,(1+a_1+\ldots+a_q) + (1+a_1+\ldots+a_{q-1})\,\varepsilon_{k-q+1} +$$

$$+ (1+a_1+\ldots+a_{q-2})\,\varepsilon_{k-q+2} + \ldots + (1+a_1)\,\varepsilon_{k-1} + \varepsilon_k + (a_1+a_2+\ldots+a_q)\,\varepsilon_0 +$$

$$+ (a_2+\ldots+a_q)\,\varepsilon_{-1} + \ldots + (a_{q-1} + a_q)\,\varepsilon_{2-q} + a_q\varepsilon_{1-q}.$$

$$v_k = e^{-\sum_{i=1}^{k} \Delta_i} = e^{-k\mu} e^{-(1+a_1+\ldots+a_q)\sum_{i=1}^{k-q} \varepsilon_i} e^{-(1+a_1+\ldots+a_{q-1})\varepsilon_{k-q+1}} \cdots e^{-(1+a_1)\varepsilon_{k-1}} e^{-\varepsilon_k} \cdots$$
$$\cdots e^{-(a_1+\ldots+a_q)\varepsilon_0} \cdots e^{-(a_{q-1}+a_q)\varepsilon_{2-q}} e^{-a_q\varepsilon_{1-q}}. \tag{3.4}$$

Each component of v_k is independent so that we can write:

$$E[v_k] = e^{-k\mu}E[e^{-(1+a_1+...+a_q)\sum_{i=1}^{k-q}\varepsilon_i}]E[e^{-(1+a_1+...+a_{q-1})\varepsilon_{k-q+1}}]\cdots E[e^{-(1+a_1)\varepsilon_{k-1}}]E[e^{-\varepsilon_k}]\cdot\cdot$$

$$\cdots E[e^{-(a_1+...+a_q)\varepsilon_0}]\cdots E[e^{-(a_{q-1}+a_q)\varepsilon_{2-q}}]E[e^{-a_q\varepsilon_{1-q}}] =$$

$$= e^{-k\mu}(M(-(1+...+a_q)))^{k-q}M(-(1+...+a_{q-1}))\cdots$$
$$\cdots M(-1)M(-(a_1+...a_q))\cdots M(-a_q) =$$

$$= e^{-k\mu}e^{k\ln(M(-(1+...a_q)))}\frac{M(-(1+...+a_{q-1}))..M(-1)M(-(a_1+..a_q))..M(-a_q)}{(M(-(1+...+a_q)))^q} \tag{3.5}$$

$= e^{-k(\mu-\ln(M(-(1+...+a_q)))}C_1 = C_1e^{-kg_1}$ on introducing the above definitions of g_1 and C_1.

Note that in the special case of $a_1= a_2=... a_q= 0$, then $g_1=d_1$ leading to the i.i.d result given earlier in section 2.

3.2.2 Proposition 2

i) $E[v_k^n] = C_ne^{-kg_n}\quad n\geq 1$ (3.6)

ii) $E[v_sv_k] = Be^{-sg_2}e^{-(k-s)g_1}\quad s<k$ (3.7)

where $g_n = n\mu - \ln(M(-n(1+\sum_{i=1}^{q}a_i)))$

$$C_n = \frac{\left(\prod_{j=1}^{q-1}M(-nS_j)\right)M(-n)\left(\prod_{j=1}^{q}M(-n(S_q-S_{j-1}))\right)}{\left(M(-nS_q)\right)^q} \tag{3.8}$$

$$B = \frac{\left(\prod_{l=0}^{q-1}M(-2(S_l+\tfrac{1}{2}(S_q-S_l)))\right)M(-1)\left(\prod_{j=1}^{q}M(-2(S_q-S_{j-1}))\right)\left(\prod_{j=1}^{q-1}M(-S_j)\right)}{(M(-2S_q))^q\quad(M(-S_q))^q}. \tag{3.9}$$

and we use the additional shorthand notation $S_0=1$ and $S_j=1+\sum_{i=1}^{j}a_i$ for $j\geq 1$.

Proof

i) From equation (3.4), we obtain directly:

$$E[v_k^n] = e^{-k[n\mu - ln(M(-n(1+a_1+...+a_q)))]}\frac{M(-n(1+a_1+...+a_{q-1}))\cdots M(-n)M(-n(a_1+...+a_q))\cdots M(-na_q)}{M(-n(1+a_1+...+a_q))^q}$$

which reduces to (3.6) an substitution for g_n and S_j.

ii) If $s<k:\ E[v_s v_k] = E[e^{-\sum_{i=1}^{s}\Delta_i}e^{-\sum_{i=1}^{k}\Delta_i}] = E[e^{-\left(\sum_{i=1}^{s}\Delta_i+\sum_{i=1}^{k}\Delta_i\right)}]$

Then, proceeding in the same manner as for the proof of Proposition 1, we write:

$$\sum_{i=1}^{s}\Delta_i + \sum_{i=1}^{k}\Delta_i = 2\sum_{i=1}^{s}\Delta_i + \sum_{i=s+1}^{k}\Delta_i = (k\text{-}s)\mu + 2s\mu + 2(1+a_1+...+a_q)\sum_{i=1}^{s-q}\varepsilon_i ++$$

$$+ (2(1+...+a_{q\text{-}1})+a_q)\varepsilon_{s\text{-}q+1} + (2(1+...+a_{q\text{-}2})+a_{q\text{-}1}+a_q)\varepsilon_{s\text{-}q+2}+...+$$
$$+(2+a_1+...+a_q)\varepsilon_s + (1+...+a_q)\sum_{i=s+1}^{k-q}\varepsilon_i + (1+...+a_{q\text{-}1})\varepsilon_{k\text{-}q+1} +...+ \varepsilon_k +$$

$$+ 2(a_1+...a_q)\varepsilon_0 + 2(a_2+...+a_q)\varepsilon_{\text{-}1} +... + 2a_q\varepsilon_{1\text{-}q}.$$

Thus we can write

$$E[v_s v_k] = Be^{-s(2\mu - ln(M(-2(1+a_1+...+a_q))))}e^{-(k-s)(\mu - ln(M(-(1+...+a_q))))}$$
$$= Be^{-sg_2}e^{-(k-s)g_1}$$

on substituting for g_1 and g_2 and with B given by

$$B = M(-2(1+...+a_{q-1})-a_q)M(-2(1+...+a_{q-2})-a_{q-1}-a_q)\cdots M(-(2+a_1+...+a_q))\times$$
$$\frac{M(-(1+...+a_{q-1}))\cdots M(-1)M(-2(a_1+...+a_q))M(-2(a_2+...+a_q))\cdots M(-2a_q)}{M(-2(1+a_1+...+a_q))^q M(-(1+a_1+...+a_q))^q}.$$

As before if $a_1 = a_2 = ...\ a_q = 0$, then $g_n = d_n$ and $C_n = 1$ as in the i.i.d result of section 2.

3.3 Life Insurance Actuarial Functions

If we consider, as in section 2, the general insurance and annuity contracts with random present values, $Z_{K+1} = v_{K+1}\ b_{K+1}$ and $a(K) = \sum_{s=0}^{K} v_s c_s$ respectively, we can calculate again the mean and variance of Z_{K+1} and $a(K)$.
Thus,

Thus,

$$E[Z_{K+1}] = C_1 E[e^{-g_1(K+1)} b_{K+1}] \tag{3.10}$$

$$E[a(K)] = E[\sum_{s=0}^{K} v_s c_s] = E[c_0 + \sum_{s=1}^{K} v_s c_s] = c_0 + E[\sum_{s=1}^{K} v_s c_s] =$$

$$= c_0 + C_1 E[\sum_{s=1}^{K} e^{-g_1 s} c_s] \tag{3.11}$$

and

$$E[Z_{K+1}^2] = C_2 E[e^{-g_2(K+1)} b_{K+1}^2] \tag{3.12}$$

$$E[a(K)^2] = E[(c_0 + \sum_{s=1}^{K} v_s c_s)^2] = E[c_0^2 + 2c_0 \sum_{s=1}^{K} v_s c_s + \left(\sum_{s=1}^{K} v_s c_s\right)^2] =$$

$$c_0^2 + 2c_0 E[\sum_{s=1}^{K} v_s c_s] + E[\left(\sum_{s=1}^{K} v_s c_s\right) \tag{3.13}$$

where $E[\sum_{s=1}^{K} v_s c_s] = C_1 E[\sum_{s=1}^{K} e^{-g_1 s} c_s]$ and

$$E[(\sum_{s=1}^{K} v_s c_s)^2] = E[C_2 \sum_{s=1}^{K} e^{-g_2 s} c_s^2 + 2B \sum_{r=2}^{K} \sum_{s=1}^{r-1} e^{-g_1 s} e^{-(g_2 - g_1)r} c_s c_r].$$

3.4 Distribution of the Force of Interest

Under the simplifying assumption that $\varepsilon \sim N(0, \sigma^2)$, it is then straightforward to show that Δ_k follows a normal distribution with mean μ and variance $\sigma^2(1 + a_1^2 + ... + a_q^2)$.
We then note that increasing q leads to an increase in the variance of Δ_k.

3.5 Proposition 3

(a) If $\varepsilon \sim N(0, \sigma^2)$ and $0 < a_1 + a_2 + ... + a_q < 1$, then
   i) $g_1 < d_1$
   ii) $g_2 < d_2$

(b) If $\varepsilon \sim N(0, \sigma^2)$ and $\sum_{j=0}^{q-1} S_j (S_q - S_j) > 0$ then $C_n < 1$ for $n \geq 1$ and $B < 1$ (using the notation of Proposition 2).

Proof

(a) i) $d_1 = \mu - \sigma^2/2$. Then,

$$g_1 = \mu - ln(M(-(1 + a_1 + ... + a_q))) = \mu - ln(e^{\frac{1}{2}\sigma^2(1+...+a_q)^2}) =$$

$$= \mu - \frac{1}{2}\sigma^2(1 + ... + a_q)^2.$$

$$g_1 - d_1 = -\frac{1}{2}\sigma^2[(1 + ... + a_q)^2 - 1] .$$

Then, we note that $0 < a_1+a_2+...+a_q \Rightarrow g_1 < d_1$.

Remark:

If we take, for example, the cases of an MA(1) or MA(2) model we can also say that a positive autocorrelated environment leads to $g_1 < d_1$. To demonstrate this point we define ρ_i for i=1,2,... to be the autocorrelation function of the process. Then for:

- MA(1)
$\rho_1 = a_1/(1+a_1^2)$ and $\rho_1>0 \Rightarrow a_1>0$

- MA(2)

$\rho_1 = (a_1+a_1a_2)/(1+a_1^2+a_2^2)$ and $\rho_1>0 \Rightarrow a_1(1+a_2)>0$ (i)
$\rho_2 = a_2/(1+a_1^2+a_2^2)$ and $\rho_2>0 \Rightarrow a_2>0$ (ii)

Then $(i), (ii) \Rightarrow a_1>0 , a_2>0 \Rightarrow a_1 + a_2>0.$

This indicates that the actuary should use a higher interest assumption than for the corresponding i.i.d environment.

ii) $d_2=2\mu-2\sigma^2$. Then,

$g_2=2\mu - ln(M(-2\lambda))$ where $\lambda = 1 + \sum_{i=1}^{q} a_i > 1$ (because $\sum_{i=1}^{q} a_i > 0$)

$$g_2 = 2\mu - ln(e^{\frac{1}{2}\sigma^2 4\lambda^2}) = 2\mu - 2\sigma^2\lambda^2$$
$$g_2 - d_2 = 2\sigma^2 - 2\sigma^2\lambda^2 = 2\sigma^2(1 - \lambda^2) < 0 \Rightarrow g_2 < d_2 .$$

(b) With $S_0=1$ and $S_j = 1+\sum_{i=1}^{j} a_i$ for $j \geq 1$, it is straightforward to show that, if $\varepsilon \sim N(0,\sigma^2)$ so $M(t) = e^{\frac{1}{2}t^2\sigma^2}$, then

$$log\, C_1 = -\sigma^2\sum_{j=0}^{q-1} S_j(S_q - S_j)$$
$$log\, C_n = n^2\, log\, C_1 \text{ for n>1}$$

and $log\, B = 3\, log\, C_1$.

Then, if by assumption $\sum_{j=0}^{q-1} S_j(S_q - S_j) > 0,\quad C_n < 1,$ for n≥1, and $B<1$.

As remarked by Frees (1990, page 104) (but using the notation of this paper) " ...the interpretation is that in the case of reserves we have off-setting factors. For example in a positively autocorrelated interest environment $a_1 > 0$ thus $C_1<1$ and $g_1<d_1$. Now in general a lower interest factor means that the reserve is higher. However this is slightly offset in the calculation of reserves because we multiply by C_1 , a factor less than one ."

It is important to note that, because $\Delta \sim N(\mu, \sigma^2(1+a_1^2+...+a_q^2))$,if $\varepsilon\sim N(0,\sigma^2)$ then we can use the results that we have obtained in section 2 and utilise the formulae derived there to calculate the mean, standard deviation, skewness and kurtosis for the random present values of standard insurance and annuity contracts.

3.6 Examples

Tables 3.1 - 3.3, on the following pages, present the calculated values for the moments in the cases of a temporary insurance, endowment and whole life assurances in the MA(1) case (with $a_1=0.5$) with $\mu=0.07$ for different choices of x,n (where appropriate) and σ.

It is clear from the examples that, if we compare an $MA(q)$ model with $\varepsilon\sim N(0,\sigma^2)$ and parameters $\{a_k\}$ as in equation (3.1) and an i.i.d model with $\Delta_k \sim N(\mu,\sigma^2)$, the standard deviations of the actuarial functions in the MA(q) model are higher than for the $\Delta_k \sim N(\mu,\sigma^2)$ model. This result follows because the MA(q) model is also an $N(\mu,\sigma_q^2)$ with variance σ_q^2, given by $\sigma^2((1+a_1^2+...+a_q^2)$ which is greater than σ^2. So the variance in $MA(q)$ is higher than for the equivalent $N(\mu,\sigma^2)$. Also, as we mentioned in the previous section, an increase in σ_q will lead to a decrease in the skewness and kurtosis as demonstrated in Tables 3.1 - 3.3.

MA(1) Model

Temporary Insurance

μ= 0.07 **a_1= 0.5**

Age	20	30	40	20	30	40	← Term
	σ= **0**			σ= **0.03**			
25	0.008802	0.015809	0.025669	0.00885	0.015956	0.02601	**Expectation**
30	0.013122	0.025255	0.040547	0.013203	0.025507	0.041099	
35	0.022484	0.042475	0.064463	0.022628	0.042899	0.065316	
40	0.039462	0.070528	0.099116	0.039713	0.071211	0.100355	
45	0.067541	0.112467	0.144796	0.067959	0.113508	0.14646	
25	0.06884	0.076295	0.079364	0.06943	0.077261	0.080651	**Standard Deviation**
30	0.080006	0.090104	0.092185	0.080811	0.091432	0.0939	
35	0.102151	0.112944	0.111433	0.103243	0.114686	0.113573	
40	0.133218	0.141417	0.133424	0.134655	0.143602	0.135953	
45	0.170499	0.171063	0.154938	0.172335	0.173698	0.157808	
25	8.953197	6.755749	5.724027	8.986409	6.743711	5.667181	**Skewness**
30	6.969304	4.922426	4.16433	7.005533	4.922492	4.127158	
35	5.097343	3.55962	3.145786	5.130624	3.565309	3.119783	
40	3.683315	2.603871	2.498685	3.710288	2.609239	2.475344	
45	2.630119	1.902682	2.058647	2.651816	1.906336	2.035061	
25	91.00436	58.89308	47.46284	91.57622	58.51052	46.49361	**Kurtosis**
30	56.76734	33.43919	27.64519	57.24011	33.25586	27.06788	
35	31.20699	18.67758	17.00241	31.5379	18.61273	16.67545	
40	17.07158	11.08223	11.54311	17.26828	11.04479	11.3272	
45	9.522994	7.047866	8.39353	9.635416	7.014803	8.239292	

	σ= **0.05**			σ= **0.07**			
25	0.008937	0.016223	0.026631	0.00907	0.016633	0.027598	**Expectation**
30	0.013349	0.025962	0.042103	0.013572	0.026664	0.043664	
35	0.022887	0.043665	0.066868	0.023281	0.044844	0.069278	
40	0.040163	0.072445	0.102605	0.040849	0.074346	0.106096	
45	0.068712	0.115388	0.149478	0.06986	0.11828	0.154152	
25	0.070504	0.079043	0.083062	0.07218	0.081889	0.086996	**Standard Deviation**
30	0.082278	0.093885	0.097108	0.084568	0.097797	0.102332	
35	0.105233	0.117903	0.117574	0.108339	0.123029	0.124084	
40	0.137273	0.147636	0.140685	0.14136	0.154066	0.148385	
45	0.175681	0.178563	0.163177	0.180905	0.186324	0.171914	
25	9.054062	6.733352	5.577975	9.178035	6.748013	5.478238	**Skewness**
30	7.078501	4.933583	4.073612	7.21002	4.979339	4.027539	
35	5.196684	3.584425	3.084428	5.313443	3.636765	3.060715	
40	3.76356	2.626172	2.44337	3.857086	2.670927	2.420991	
45	2.694616	1.919302	2.001949	2.769617	1.955609	1.97583	
25	92.83429	58.02181	44.92987	95.39999	57.859	43.07325	**Kurtosis**
30	58.28404	33.08919	26.18473	60.41524	33.29791	25.27376	
35	32.25378	18.59865	16.19135	33.68066	18.86745	15.74431	
40	17.69189	11.04091	11.00921	18.53188	11.21512	10.72259	
45	9.879167	6.997242	8.012467	10.36652	7.089859	7.810302	

(Table 3.1)

MA(1) Model

Endowment Assurance

μ= 0.07 a_1 = 0.5

Age ▼ ◀— Term

σ= 0 σ= 0.03

Age	20	30	40	20	30	40	
25	0.250732	0.130961	0.075913	0.253517	0.133068	0.077398	Expectation
30	0.252214	0.135393	0.08456	0.255001	0.137519	0.086114	
35	0.255969	0.144352	0.099852	0.258755	0.14651	0.101511	
40	0.263217	0.159945	0.124037	0.265999	0.16215	0.125843	
45	0.275708	0.184779	0.159124	0.278481	0.18705	0.161114	
25	0.040116	0.054966	0.065193	0.055072	0.060021	0.067334	Standard Deviation
30	0.043825	0.062549	0.075287	0.057902	0.067289	0.077583	
35	0.054743	0.07873	0.093565	0.066663	0.082914	0.095996	
40	0.0721	0.101686	0.117287	0.081673	0.105423	0.119901	
45	0.094606	0.128931	0.143257	0.102284	0.13244	0.146115	
25	12.16509	9.478377	7.736863	4.879921	7.471243	7.237005	Skewness
30	10.2824	7.407445	5.818319	4.636545	6.146473	5.523126	
35	7.768355	5.463463	4.315624	4.449238	4.844685	4.164914	
40	5.698423	4.026704	3.264449	4.027286	3.741725	3.183879	
45	4.171295	2.977109	2.499181	3.375243	2.84175	2.450719	
25	165.9879	106.9121	76.5245	49.25991	78.07852	69.89253	Kurtosis
30	122.6754	68.83172	46.31082	42.80372	53.8772	43.09124	
35	71.7015	38.79347	26.66794	34.77302	33.26081	25.39263	
40	39.41627	21.9352	16.07935	25.49944	20.0582	15.54933	
45	21.8678	12.83286	10.20655	17.03316	12.18197	9.958938	

σ= 0.05 σ= 0.07

Age	20	30	40	20	30	40	
25	0.258547	0.1369	0.080116	0.266282	0.142865	0.08439	Expectation
30	0.260031	0.141386	0.088955	0.267767	0.1474	0.093414	
35	0.263783	0.150431	0.10454	0.271514	0.156524	0.109279	
40	0.27102	0.166154	0.129133	0.278737	0.172367	0.134264	
45	0.283487	0.19117	0.16473	0.291175	0.19755	0.170347	
25	0.076202	0.069126	0.071506	0.07218	0.081889	0.086996	Standard Deviation
30	0.078367	0.075941	0.082013	0.084568	0.097797	0.102332	
35	0.0852	0.0907	0.100646	0.108339	0.123029	0.124084	
40	0.097617	0.112452	0.124862	0.14136	0.154066	0.148385	
45	0.11571	0.139049	0.151503	0.180905	0.186324	0.171914	
25	2.282594	5.262726	6.426361	1.619904	3.642431	5.422903	Skewness
30	2.294437	4.641331	5.044494	1.636954	3.429743	4.451795	
35	2.488333	4.009545	3.919068	1.753803	3.223358	3.612668	
40	2.624981	3.320969	3.053758	1.889946	2.872383	2.89575	
45	2.516327	2.633357	2.374725	1.921062	2.398227	2.288712	
25	16.2816	48.07694	59.23725	8.510926	27.2527	46.17849	Kurtosis
30	15.58845	36.62766	37.90461	8.459048	23.15619	31.51925	
35	15.58903	25.85158	23.32122	8.84362	18.88824	20.75796	
40	14.55244	17.26216	14.69841	9.040327	14.23614	13.68631	
45	11.90856	11.16576	9.574239	8.391063	10.00475	9.159518	

(Table 3.2)

MA(1) Model

Whole Life Assurance

$\mu = 0.07$ $a_1 = 0.5$

Age	$\sigma = 0$	$\sigma = 0.03$	
25	0.046933	0.047885	**Expectation**
30	0.062972	0.06414	
35	0.085592	0.087004	
40	0.116086	0.117758	
45	0.155641	0.157574	
25	0.0739	0.075324	**Standard Deviation**
30	0.083739	0.08555	
35	0.100591	0.102759	
40	0.122126	0.124625	
45	0.145857	0.148676	
25	6.241921	6.104006	**Skewness**
30	4.808983	4.704835	
35	3.769096	3.696432	
40	3.008606	2.955872	
45	2.398801	2.359019	
25	55.43961	53.50037	**Kurtosis**
30	34.97892	33.74101	
35	22.08254	21.40668	
40	14.50455	14.12856	
45	9.765005	9.55051	

Age	$\sigma = 0.05$	$\sigma = 0.07$	
25	0.049635	0.052408	**Expectation**
30	0.066282	0.069659	
35	0.089585	0.093634	
40	0.120806	0.125564	
45	0.161087	0.166545	
25	0.078065	0.082739	**Standard Deviation**
30	0.089012	0.094846	
35	0.106879	0.113756	
40	0.12935	0.137175	
45	0.153982	0.1627	
25	5.86464	5.530547	**Skewness**
30	4.531234	4.30673	
35	3.57861	3.434947	
40	2.872314	2.775742	
45	2.297662	2.231308	
25	50.15696	45.54467	**Kurtosis**
30	31.69289	29.08251	
35	20.3174	19.01231	
40	13.53592	12.86876	
45	9.221833	8.88314	

(Table 3.3)

4. CONCLUDING COMMENTS

As noted by Frees (1990) and Dufresne (1992), moving average processes often lead to tractable results and are simpler to manipulate than the full ARMA processes while still incorporating dependence over time. This arises because of the relatively simple form of the covariance structure. In this paper, we demonstrate the tractability and the convenience in the case of standard present value calculations in a life insurance context. There is a duality between the standard AR and MA models which means that, in practice, it is often difficult to distinguish between them when fitting models to observational data (Frees (1990)). Indeed, any lack of fit with actual data from using MA(q) models may be offset by the simplifications arising from their use.

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