CONTRIBUTION AND SOLVENCY RISK IN A DEFINED BENEFIT PENSION SCHEME

by

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Abstract

This paper presents a stochastic investment model for a defined benefit pension scheme, in the presence of IID real rates of return. The spread method of adjustment to the normal cost is used. Two types of risk are identified, the "contribution rate risk" and the "solvency risk" which are concerned respectively with the stability and the security of the pension fund. A performance criterion is introduced to deal with the simultaneous minimisation of these two risks, using the spread period (M) as the control variable. A full numerical investigation of the optimal values of M is discussed and the optimal choices are presented with the help of tables and figures. The results lead to practical conclusions about the optimal funding strategy and, hence, about the optimal choice of the contribution rate subject to the constraints needed for the convergence of the performance criterion. A fuller discussion of the results can be found in Megaloudi (1998).

1. Introduction

1.1 Risk in a Defined Benefit Pension Scheme

As defined by Lee (1986), occupational pension schemes are arrangements by means of which employers or groups of employers provide pensions and other benefits to their employees. We are interested in defined benefit pension schemes where the benefits promised are the defined quantity and the contributions are the dependent variable. The determination of these contributions takes place through the valuation process, which is performed by the actuary at regular intervals.

The method by which the scheme is valued and the contribution rate determined is called the actuarial funding method. In this paper, we shall consider individual funding methods. In the light of the particular situation revealed by the valuation process, appropriate action will be taken by way of an adjustment to the contribution rate so as to remove the shortfall or to use the surplus. For individual funding methods, the most common ways of dealing with this adjustment are the spread method and the amortisation of losses method (Haberman (1994)). We will consider the spread method under which the unfunded liability is spread into the future over a certain period. The choice of this period, which is called the spread period, depends on the required balance between the different types of risk facing the pension scheme.

We will investigate two types of risk. The first one is the "contribution rate risk". According to Lee (1986), the sponsor of the scheme will look for a contribution plan which will not be disturbed by significant changes so that the contribution rate will remain reasonably stable in the future. The second type of risk is the "solvency risk". As Lee (1986) explains, the trustees and the employees will be concerned that the accumulated assets represent reasonable security for the growing pension rights of the members, independently of the employer, at any time or when the scheme is wound up. In this paper, we will use a mathematical model to represent the financial structure of a defined benefit pension scheme under various investments returns. We will consider methods for controlling the above types of risk by using the spread period as our control variable.

1.2 Formulation of the Problem

The approach described is based on Haberman (1997a,1997b). We use control theory in a stochastic environment to formulate the problem. The optimal contribution rate will be determined by minimising a quadratic performance criterion, that includes both the contribution rate risk and the solvency risk. The problem is described as follows using a discrete time formulation.

We wish to find the contribution rates C(s),C(s+1),...,C(T-1) over the finite time period (s, T) which minimise the quadratic performance criterion

$$J_T = E\{\sum_{t=s}^{T-1} v^t\left[(C(t) - CT(t))^2 + (1-\theta)(F(t) - FT(t))^2\right]\} \qquad (1)$$

The first term represents the contribution rate risk and the second term the solvency risk.

The expectation operator is necessary because we are interested in the stochastic case so as to recognise the random nature of investment returns. (In a continuous time formulation, the mathematical approach would be based on an integral version of (1)). We use the notation:

C(t)= contribution rate for the time period (t, t+1).
F(t)= fund level at time t, measured in terms of the market value of the assets.
CT(t)= contribution target for the period (t, t+1).
FT(t)= fund target for the period (t, t+1).
$v = (1+i)^{-1}$ where i is the valuation rate of interest
θ= weighting factor to reflect the relative importance of the solvency risk against the contribution rate risk.

We argue that the actuarial funding methods would normally specify appropriate values for CT(t) and FT(t) in order to control the pace of funding . Hence, we choose CT(t)= EC(t) and FT(t)= EF(t) as appropriate target values.

So, equation (1) becomes

$$J_T = \sum_{t=s}^{T-1} v^t[\theta VarC(t) + (1-\theta)VarF(t)] \qquad (2)$$

According to Owadally and Haberman (1995), in this presentation, the risk of the pension fund is defined as a "time-weighted" sum of the weighted average of the future variances of the fund level and contribution rate. θ is determined according to which of the variability of the fund or the contribution is more important for the employer. This balance will influence the choice of the funding strategy, since some methods (e.g. prospective benefit methods) aim more at stabilising the contribution rate, whereas some others (e.g. accrued benefit methods) have as their main purpose to fund the actuarial liability. $v = (1+i)^{-1}$ is used to discount the variances. A high i indicates that more emphasis is placed on the shorter-term position of the pension fund rather than the longer-term. Therefore, this is a mechanism for weighting in time. In this paper, we will use a discount factor $w \neq v$ in the definition of J_∞ to reach more general conclusions.

1.3 The Mathematical Problem

We consider the behaviour of C(t) by using a stochastic investment model of a defined benefit pension scheme. Its main features are a stationary population and independent and identically distributed rates of return. As noted earlier, we shall work in discrete time (t=0,1,2,...).

When an actuarial valuation takes place, the actuary estimates C(t) and F(t) based only on the active and retired members of the scheme at time t under these assumptions:

• The population is stationary (constant size and age distribution year after year)-see assumptions below.
• The valuation interest rate is fixed and is i.
• The contribution income and benefit outgo cash flows occur at the start of each scheme year.
• Valuations are carried out at annual intervals.
The following recurrence relations for the pension fund's assets and the actuarial liability hold:

$$F(t+1)=(1+i(t+1))(F(t)+C(t)-B(t)) \qquad (3)$$
$$AL(t+1)=(1+i)(AL(t)+NC(t)-B(t)) \qquad (4)$$

for $t=0,1,2,\ldots$

Further notation used is:
$i(t+1)$= rate of investment return earned during the period $(t, t+1)$, defined in a manner consistent with the definition of $F(t)$.

$AL(t+1)$= actuarial liability at the end of the period $(t, t+1)$ in respect of the active and retired members.
$B(t)$= overall benefit outgo for the period $(t, t+1)$.
$NC(t)$= normal cost for the period $(t, t+1)$.

We make the following further simplifying assumptions:
1. The experience is in accordance with all the features of the actuarial basis, except for investment returns.
2. The population is stationary from the start. We could alternatively assume that the population is growing at a fixed, deterministic rate.
3. There is no promotional salary scale. Salaries increase at a deterministic rate of inflation. This inflation component is used to reduce the assumed rate of investment return to give a real rate of investment return. We also assume that benefits in payment increase at the same rate as salaries and then consider all variables to be in real terms.
4. Following the previous assumption, $i(t+1)$ is the real rate of investment return earned during the period $(t, t+1)$ and $Ei(t)=i$ where i is the real valuation rate of interest. This means that contributions are assumed to be invested in future at the average rate of interest. We also define $\sigma^2=Var(i(t))$.
5. The earned rates of return $i(t)$ are independent, identically distributed random variables with $Prob(i(t)>-1)=1$.
6. The initial value of the fund at time zero is known i.e. $Prob[F(0)=F_0]=1$ for some F_0.

Assumptions 1-3 imply that the following parameters are constant with respect to time t (after dividing all monetary amounts relating to time t by $(1+I(t))$ where $I(t)$ is the rate of salary inflation during the period $(t, t+1)$):

$$NC(t)=NC$$
$$AL(t)=AL$$
$$B(t)=B$$

Then, combining (4) with these, we obtain:

$$AL=(1+i)(AL+NC-B) \qquad (5)$$

1.4 Individual Funding Methods

According to Haberman (1994), for an individual funding method, the unfunded liability denotes the difference between the plan's actuarial liability and its assets.

$$UL(t)= AL(t)-F(t) \qquad (6)$$

where UL(t) = unfunded liability at time t and
AL(t) is the total actuarial liability in respect of all members at time t.

These methods involve an actuarial liability and a normal cost which is then adjusted to deal with the unfunded liability. There are a number of choices for the ADJ(t) term. We will consider the spread method, under which:

$$C(t)= NC(t)+ADJ(t) \qquad (7)$$

$$ADJ(t)= kUL(t) \qquad (8)$$

where ADJ(t)= the adjustment to the contribution rate at time t
NC(t)= the total normal cost at time t
$k= 1/\ddot{a}_{\overline{M|}}$ calculated at the valuation rate of interest.
M= the "spread period".

So, the unfunded liability is spread over M years and k can be thought of as a penalty rate of interest that is being charged on it. The choice of M, as we will see later on, is of great importance and influences the funding strategy.

The above definition of ADJ(t) implies that the spread period is always the same whether there is a surplus or a deficit. According to Winklevoss(1993), this may not always be the case in practice with a shorter spread period being used to eliminate deficiencies than for surpluses (This is investigated by Haberman and Smith (1997) using simulation).

Finally, from (6), (7), (8) and the previous assumptions:

$$C(t)= NC+k(AL-F(t)) \qquad (9)$$

1.5 Moments of C(t) and F(t)

Dufresne (1988) has shown that, given our mathematical formulation,

$$EF(t)= q^{t} F_0 +AL(1- q^{t})$$
$$= q^{t} F_0 + r(1-q^{t})/(1-q) \quad ,t\geq 0 \qquad (10)$$

where $q= (1+i)(1-k)$, $r= (1+i)(k-d)AL$.

So from (9)

$$EC(t)= NC+k(AL-EF(t)) \qquad (11)$$

and $\lim_{t\to\infty} EF(t)= AL$ $\qquad \lim_{t\to\infty} EC(t)= NC$ provided $0<q<1$(i.e. $M>1$)

He also proves that

$$VarF(t)=b\sum_{j=1}^{t} a^{t-j} (EF(j))^2 \quad ,t\geq 1 \qquad (12)$$

where $b=\sigma^2 v^2$ and $a= q^2(1+b)=(1-k)^2(1+i)^2(1+\sigma^2 v^2)= (1-k)^2((1+i)^2+\sigma^2)$

$$\mathrm{VarC(t)}= k^2\,\mathrm{VarF(t)} \qquad (13)$$

and $\lim_{t\to\infty} \mathrm{VarF(t)}=\dfrac{bAL^2}{1-a}$ $\quad \lim_{t\to\infty} \mathrm{VarC(t)}= k^2 \dfrac{bAL^2}{1-a}$

provided $a<1 \Rightarrow (1-k)^2((1+i)^2+\sigma^2)<1$ which places restrictions either on the choice of σ^2 or on the choice of the spread period.

1.6 The General Form

If we substitute (2) for $T=\infty$, $s=0$ and replace v^t by w^t, then

$$J_\infty=\sum_{t=0}^{\infty} w^t[\theta\,\mathrm{VarC(t)}+(1-\theta)\mathrm{VarF(t)}] \qquad (14)$$

From equation (13)

$$J_\infty=\sum_{t=0}^{\infty} w^t\left[\theta k^2+(1-\theta)\right]\mathrm{VarF(t)} \qquad (15)$$

For the case $F_0 \neq AL$, equations (10), (12) and (15) lead to:

$$J_\infty=\frac{\left(\theta k^2+1-\theta\right)}{1-wa}\sigma^2 v^2 w\left[\frac{z^2q^2}{1-wq^2}+\frac{AL^2}{1-w}+\frac{2zALq}{1-wq}\right] \qquad (16)$$

where $z= F_0 -AL$ and $w=\dfrac{1}{1+j} \neq v$.

So, we wish to find the value(s) of k (or equivalently the spread period, as $k=1/\ddot{a}_{\overline{M}|}$) which minimises the above equation. Then, we can find the optimal C(t) via equation (9).

We note that $q=(1+i)(1-k)\Rightarrow k=1-qv$ and $a=q^2(1+\sigma^2 v^2)$
and since $q\to k$ is a 1:1 mapping with domain (0,1) and image set (d,1),
it is convenient to reparametrise J_∞ in terms of q. We write

$$g(q)=\frac{[\theta(1-qv)^2+1-\theta][z^2q^2(1-w)(1-wq)+AL^2(1-wq)(1-wq^2)+2zALq(1-w)(1-wq^2)]}{[1-wq^2(1+\sigma^2v^2)](1-wq^2)(1-wq)}$$

We need to solve $\dfrac{dJ_\infty}{dq}=0$ or $\dfrac{dg(q)}{dq}=0$ (17) in order to find the optimal values of q.

2. Risk as a Time-Weighted Mechanism

2.1 Introduction

We wish to solve equation (17) and find the values of q at which J_∞ is minimised. For all the calculations, we assume for convenience (and without loss of generality) that AL=1. The requirement that $a<1$ for the convergence of (12) and hence of (16) would mean:

$a<1 \;\Rightarrow\; q^2(1+b)<1 \;\Rightarrow\; q<(1+b)^{-1/2}$ since $q>0$

So the solutions of the above equation should be restricted to values such that:

$q<q_{max}<1$ where $q_{max}=(1+b)^{-1/2}$, $b=\sigma^2 v^2$.

We verify that the chosen values of q are the minimum points (not the maximum) by detailed calculations, as demonstrated by the relevant graphs of $J\infty$ plotted against q. If $q>q_{max}$, the solution is chosen to be q_{max}.

In any particular case, calculation of the minimising value(s) of q allows us to find the corresponding value of M from $q=(1+i)(1-k) \Rightarrow M=-\log(1-\dfrac{d}{1-qv})/\log(1+i)$. The tables in section 2.3 onwards provide the optimal values of M as a function of i, σ, j and θ (to the nearest integer) and the values of M which are marked with * correspond to q_{max}.

2.2 The Maximum Feasible Values of the Spread Period

As noted earlier, the requirement $a<1$ (for convergence) places a restriction on the choice of q. So, the optimal values of q must be restricted to values such that $q<q_{max}$ where $q_{max}=(1+b)^{-1/2}$, $b=\sigma^2 v^2$. Table 2.2.1 provides values of the maximum spread period M_{max} which correspond to q_{max} for different combinations of σ and i.

Table 2.2.1
Maximum Spread Period, M_{max}, such that a<1

	σ								
i	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
.01	535	318	223	112	66	42	30	22	17
.03	218	145	111	68	46	33	25	19	15
.05	144	99	78	51	37	28	21	17	14

Table 2.2.1 indicates the extent to which M_{max} decreases as σ and i each increase.

2.3 Initial Funding Level of 0%

Tables 2.3.1-2.3.7 provide the optimal values of the spread period M^0 for F_0=0 and different combinations of σ, i, j and θ.

Table 2.3.1
Values of M^0 when F_0=0 (z=-AL)

i=1%									
j=1%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	535*	318*	1	1	1	1	1	1	1
.25	535*	318*	1	1	1	1	1	1	1
.5	535*	318*	2	2	2	2	2	2	2
.75	535*	318*	223*	2	2	2	2	2	2
.85	535*	318*	223*	3	3	3	3	3	3
.95	535*	318*	223*	5	5	5	5	5	4
1	535*	318*	223*	112*	58	29	18	13	10

Table 2.3.2
Values of M^0 when F_0=0 (z=-AL)

i=3%									
j=1%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	218*	145*	1	1	1	1	1	1	1
.25	218*	145*	1	1	1	1	1	1	1
.5	218*	145*	2	2	2	2	2	2	2
.75	218*	145*	111*	2	2	2	2	2	2
.85	218*	145*	111*	3	3	3	3	3	3
.95	218*	145*	111*	5	5	5	4	4	4
1	218*	145*	111*	68*	30	18	13	10	8

Table 2.3.3
Values of M^0 when $F_0=0$ (z=-AL)

i=3%									
j=3%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	218*	145*	111*	1	1	1	1	1	1
.25	218*	145*	111*	68*	1	1	1	1	1
.5	218*	145*	111*	68*	2	2	2	2	2
.75	218*	145*	111*	68*	2	2	2	2	2
.85	218*	145*	111*	68*	3	3	3	3	3
.95	218*	145*	111*	68*	46*	6	5	5	5
1	218*	145*	111*	68*	46*	33*	23	15	11

Table 2.3.4
Values of M^0 when $F_0=0$ (z=-AL)

i=3%									
j=5%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	218*	145*	111*	68*	1	1	1	1	1
.25	218*	145*	111*	68*	46*	1	1	1	1
.5	218*	145*	111*	68*	46*	2	2	2	2
.75	218*	145*	111*	68*	46*	3	3	2	2
.85	218*	145*	111*	68*	46*	33*	3	3	3
.95	218*	145*	111*	68*	46*	33*	7	6	5
1	218*	145*	111*	68*	46*	33*	25*	19*	14

Table 2.3.5
Values of M^0 when $F_0=0$ (z=-AL)

i=5%									
j=3%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	144*	99*	78*	1	1	1	1	1	1
.25	144*	99*	78*	51*	1	1	1	1	1
.5	144*	99*	78*	51*	2	2	2	2	2
.75	144*	99*	78*	51*	2	2	2	2	2
.85	144*	99*	78*	51*	3	3	3	3	3
.95	144*	99*	78*	51*	37*	5	5	5	4
1	144*	99*	78*	51*	37*	28*	19	12	9

Table 2.3.6
Values of M^0 when F_0=0 (z=-AL)

i=5%									
j=5%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	144*	99*	78*	51*	1	1	1	1	1
.25	144*	99*	78*	51*	37*	1	1	1	1
.5	144*	99*	78*	51*	37*	2	2	2	2
.75	144*	99*	78*	51*	37*	3	3	2	2
.85	144*	99*	78*	51*	37*	28*	3	3	3
.95	144*	99*	78*	51*	37*	28*	7	6	5
1	144*	99*	78*	51*	37*	28*	21*	17*	12

Table 2.3.7
Values of M^0 when F_0=0 (z=-AL)

i=5%									
j=10%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	144*	99*	78*	51*	37*	28*	21*	1	1
.25	144*	99*	78*	51*	37*	28*	21*	1	1
.5	144*	99*	78*	51*	37*	28*	21*	2	2
.75	144*	99*	78*	51*	37*	28*	21*	17*	3
.85	144*	99*	78*	51*	37*	28*	21*	17*	5
.95	144*	99*	78*	51*	37*	28*	21*	17*	14*
1	144*	99*	78*	51*	37*	28*	21*	17*	14*

Tables 2.3.1-2.3.7 indicate that there is a rapid change in M when we increase σ. For example, we note that when i=1%, j=1% and θ =0, the optimal spread period M^0=318 when σ=.03, but M^0 =1 when $\sigma \geq .05$. As we noted in paragraph 2.2, when σ increases, the corresponding maximum feasible spread period M_{max} decreases. There is a value of σ, σ^* (with corresponding maximum feasible spread period M^*_{max}) for which J_∞ is minimised at both M^*_{max} and M_{min}, which is much shorter than M^*_{max}.
For $M_{min}<M<M^*_{max}$, J_∞ is higher than at the end points. So, when the choice of σ makes M_{max} lower than M^*_{max} ($\sigma>\sigma^*$), J_∞ is only minimised at M_{min}. When M can be longer than M^*_{max}, J_∞ decreases. Hence, when the choice of σ makes M_{max} higher than M^*_{max} ($\sigma<\sigma^*$), the optimal choice is M_{max}.

The critical values of σ^* for different values of j, θ and i are shown in Table 2.3.8. We notice the dependence of σ^* on these parameters.

Table 2.3.8
Critical Values of σ* when F_0=0

		F_0=0						
		θ						
i	**j**	**0**	**.25**	**.5**	**.75**	**.85**	**.95**	**1**
.01	**.01**	.04	.043	.046	.051	.055	.064	.14
.03	**.01**	.045	.046	.047	.053	.056	.066	.13
.03	**.03**	.097	.105	.11	.125	.135	.16	.245
.03	**.05**	.14	.155	.167	.186	.205	.23	.33
.05	**.03**	.098	.105	.115	.12	.135	.16	.24
.05	**.05**	.147	.157	.17	.19	.205	.24	.33
.05	**.10**	.255	.273	.295	.33	.345	.38	.5

Hence, for low values of σ, the optimal choice of M is to make M as large as possible.

Tables 2.3.1-2.3.7 also indicate that for $\sigma < \sigma^*$, this optimal choice of M does not depend on θ as neither $q_{max}= (1+\sigma^2 v^2)^{-1/2}$ nor $M_{max} = -\log(1-\frac{d}{1-q_{max}v})/\log(1+i)$ depends on θ. On the other hand, for $\sigma > \sigma^*$, increasing θ causes a rapid change in M^0. For example, when i=1%, j=1% and σ=.05, the optimal spread period M^0=1 for θ =.25, but M^0=223 for θ =.85.

We next consider equation (16) as a function of θ , 0 < θ < 1. We recall that θ controls the balance between the solvency risk and the contribution rate risk. The risk (as represented by J_∞) is a decreasing function of θ but this decrease in risk is significant only for large values of q (i.e. large values of M). So, when θ increases, the risk decreases but this downward shift in risk is not smooth. J_∞ decreases considerably for large values of M and remains approximately the same for small values of it, making the optimal spread period longer.

Tables 2.3.1-2.3.7 indicate that for $\sigma < \sigma^*$, when a higher discounting factor is used (a lower j) the optimal choice M^0 remains the same (=M_{max}). This is easily explained as neither $q_{max}=(1+\sigma^2 v^2)^{-1/2}$ nor $M_{max}=-\log(1-\frac{d}{1-q_{max}})/\log(1+i)$ depends on j. On the other hand, for $\sigma > \sigma^*$, we observe that the optimal choice M^0 becomes shorter when j is decreased. For example, when σ=.15, θ=.25 and i=3%, M^0=1 when j=3%, but M^0=46 when j=5%. Figure 2.3.1 shows the graph of J_∞ for these two cases.

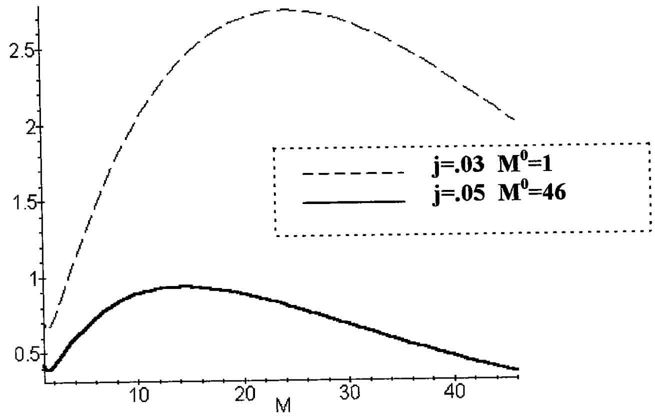

Figure 2.3.1: Graph of J_∞ when σ=.15, θ=.25 and i=3%

We observe that when j rises, the risk as represented by J_∞ decreases. This downward shift in risk is much more significant for large values of M, making the optimal spread period longer. Hence, when j=.03 the risk J_∞ is minimised for M=1. When j=.05 (more emphasis is placed on the shorter-term state of the pension fund), the risk remains approximately the same for M=1 but decreases considerably for M=46 and the optimal spread period becomes M^0=46.

We will try to explain these results in a different way. We consider equation (16) as a function of j and substitute z=-AL, i=3%, σ=.15 and θ=.25 for convenience. Figure 2.3.2 shows the graph of this function for different values of M, chosen carefully in the light of the earlier results.

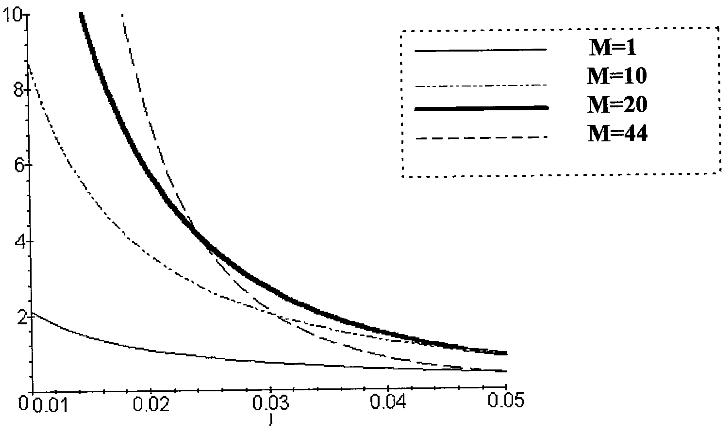

Figure 2.3.2: Graph of J_∞(j) when σ=.15, θ=.25 and i=3%

Figure 2.3.2 demonstrates what we have already claimed. The risk as represented by J_∞ is a decreasing function of j and this decrease in risk becomes more significant as the values of M become larger.

From Tables 2.3.1-2.3.7, it can also be seen that an increase in the assumed rate of return i causes a significant decrease in M^0 when $\sigma<\sigma^*$ and a slight decrease in M^0 when $\sigma>\sigma^*$. We recall that when $\sigma<\sigma^*$, the optimal choice is M_{max} which depends on I ($M_{max} = -\log(1-\frac{d}{1-q_{max}})/\log(1+i)$) and which changes in the way that the Table 2.2.1 shows. When $\sigma>\sigma^*$, the optimal choice is shorter and remains the same or decreases slightly when i increases. For example, when j=5%, θ=1 and σ=.35, M^0=14 for i=3% but M^0=12 for i=5%. Figure 2.3.3 demonstrates these two cases.

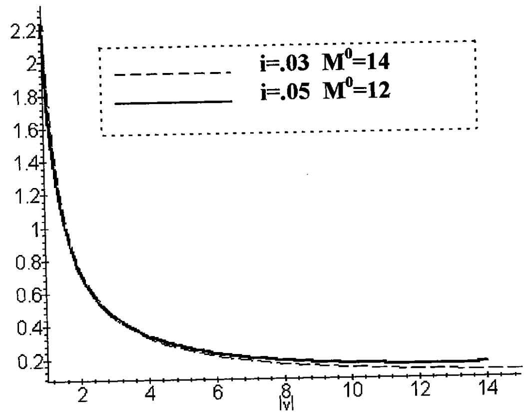

Figure 2.3.3: Graph of J_∞ when σ=.35, θ=1 and j=5%

It is demonstrated in Figure 2.3.3 that, in response to an increase in i, J_∞ remains approximately the same for low values of M and slightly increases for high values of M.

We consider equation (16) as a function of i and substitute z=-AL, σ=.35, θ=1 and j=5% for convenience. Figure 2.3.4 shows this function for different values of M.

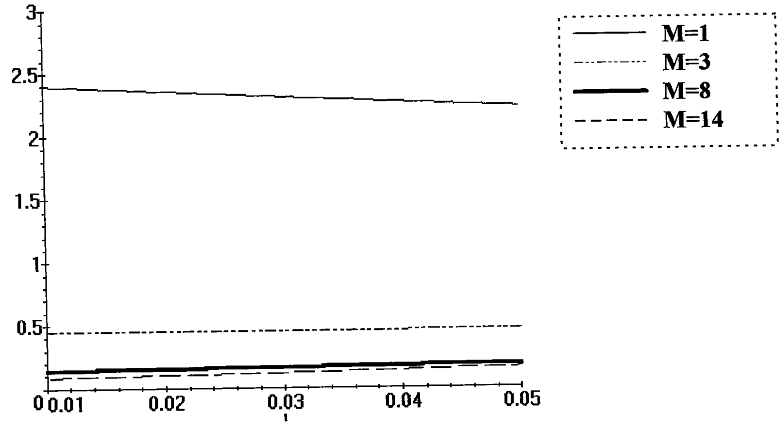

Figure 2.3.4: Graph of $J_\infty(i)$ when σ=.35, θ=1 and j=5%

Figure 2.3.4 demonstrates the sensitivity of J_∞ to changes in i. Therefore, it explains the fact that the optimal choice is influenced only slightly when the assumed rate of return changes. It also indicates that for the case when the rate of interest used in the discounting term j is equal to the valuation rate of interest i (see Tables 2.3.1,2.3.3,2.3.6), the changes in J_∞ are due to changes in the discounting rate of interest.

2.4 Initial Funding Level of 25%

Tables 2.4.1-2.4.7 provide the optimal values of the spread period M^0 for $F_0=\frac{1}{4}AL$ and different combinations of σ, θ, j and i.

Table 2.4.1

Values of M^0 when $F_0=\frac{1}{4}AL$ ($z=-\frac{3}{4}AL$)

	i=1%								
	j=1%								
	σ								
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	5	4	4	4
1	535*	318*	223*	112*	52	28	18	13	9

Table 2.4.2

Values of M^0 when $F_0=\frac{1}{4}AL$ ($z=-\frac{3}{4}AL$)

i=3%									
j=1%									
σ									
θ	.01	.03	.05	.1	.15	.2	.25	.3	.35
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	4	4	4	4
1	218*	145*	111*	68*	26	17	13	10	8

Table 2.4.3

Values of M^0 when $F_0=\frac{1}{4}AL$ ($z=-\frac{3}{4}AL$)

i=3%									
j=3%									
σ									
θ	.01	.03	.05	.1	.15	.2	.25	.3	.35
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	218*	2	2	2	2	2	2	2	2
.85	218*	145*	111*	3	3	3	3	3	3
.95	218*	145*	111*	6	6	5	5	5	4
1	218*	145*	111*	68*	46*	33*	21	14	10

Table 2.4.4

Values of M^0 when $F_0=\frac{1}{4}AL$ ($z=-\frac{3}{4}AL$)

i=3%									
j=5%									
σ									
θ	.01	.03	.05	.1	.15	.2	.25	.3	.35
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	218*	145*	111*	2	2	2	2	2	2
.75	218*	145*	111*	3	3	3	2	2	2
.85	218*	145*	111*	68*	3	3	3	3	3
.95	218*	145*	111*	68*	46*	7	6	5	5
1	218*	145*	111*	68*	46*	33*	25*	19*	13

Table 2.4.5

Values of M^0 when $F_0=\frac{1}{4}AL$ ($z=-\frac{3}{4}AL$)

i=5%									
j=3%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	144*	2	2	2	2	2	2	2	2
.85	144*	99*	78*	3	3	3	3	3	3
.95	144*	99*	78*	6	5	5	5	4	4
1	144*	99*	78*	51*	37*	27	16	11	9

Table 2.4.6

Values of M^0 when $F_0=\frac{1}{4}AL$ ($z=-\frac{3}{4}AL$)

i=5%									
j=5%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	144*	99*	78*	2	2	2	2	2	2
.75	144*	99*	78*	3	3	2	2	2	2
.85	144*	99*	78*	51*	3	3	3	3	3
.95	144*	99*	78*	51*	37*	6	6	5	5
1	144*	99*	78*	51*	37*	28*	21*	16	11

Table 2.4.7

Values of M^0 when $F_0=\frac{1}{4}AL$ ($z=-\frac{3}{4}AL$)

i=5%									
j=10%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	144*	99*	78*	51*	1	1	1	1	1
.25	144*	99*	78*	51*	37*	1	1	1	1
.5	144*	99*	78*	51*	37*	2	2	2	2
.75	144*	99*	78*	51*	37*	28*	3	3	3
.85	144*	99*	78*	51*	37*	28*	21*	4	4
.95	144*	99*	78*	51*	37*	28*	21*	17*	7
1	144*	99*	78*	51*	37*	28*	21*	17*	14*

If we compare Tables 2.3.1-2.3.7 with Tables 2.4.1-2.4.7, we observe that for a higher initial funding level, the optimal spread period presents an abrupt decrease for many combinations of σ, i, j and θ. For example, when i=1%, j=1% σ=.01 and θ=0 the optimal spread period is M^0=535 when F_0=0, and M^0=1 when $F_0=\frac{1}{4}AL$. The initial funding level of 25% leads to a shorter optimal choice of spread period when the other parameters are such that we do not have the case of M_{max}. For example, for the combination of j=10%, i=5%, σ=.01, the higher initial funding level of 25% is not sufficient to change the optimal choice which remains M^0=144. Hence, for this case, the effect of the high value of j is more significant than the one of the initial funding level.

If $M^0=M_{max}$ ($\sigma<\sigma^*$), we observe that the optimal choice does not change either for a higher or for a lower value of j as M_{max} does not depend on j. On the other hand, M_{max} depends on i. So, for the $\sigma<\sigma^*$ cases, the changes in the optimal spread period correspond exactly to Table 2.2.1.

When the optimal spread period is shorter than M_{max} ($\sigma>\sigma^*$), an increase in j leads to a higher optimal choice. For example, when i=5%, θ=.25 and σ=.01, M^0=1 for j=5% but M^0=144 for j=10%. Figure 2.4.1 demonstrates J_∞ plotted against M.

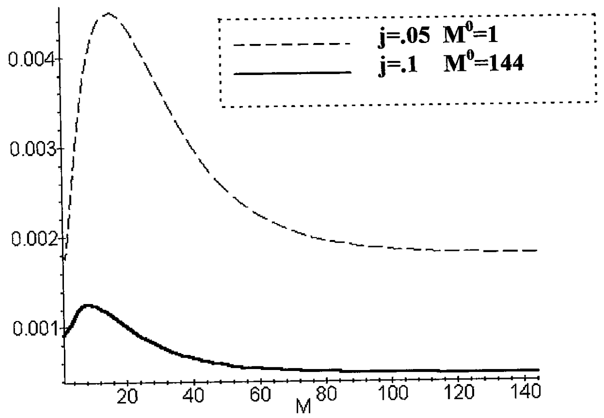

Figure 2.4.1: Graph of J_∞ when σ=.01, θ=.25 and i=5%

Figure 2.4.1 demonstrates what we have already claimed. The risk as represented by J_∞ is a decreasing function of j and is more sensitive in changes in j for the higher values of M (see also Figure 2.3.2). Therefore, when we are more interested in the shorter-term position of the pension fund (j=10%), the risk decreases to a greater extent for M=144 than for M=1 and the optimal spread period becomes M^0=144.

For an initial funding level of 25% and for the $\sigma>\sigma^*$ cases, an increase in the assumed rate of return causes the same results as for an initial funding level of 0%. The optimal choice does not change at all or decreases slightly. Therefore, when the valuation rate of interest i is equal to the rate of interest used in the discounting term j (see Tables 2.4.1, 2.4.3, 2.4.6), changes in M^0 are in response to changes in j.

Tables 2.4.1-2.4.7 also illustrate that the increase in θ causes an abrupt change in the optimal spread period. The reason for this change has already been discussed. The risk as represented by J_∞ is a decreasing function of θ and the level of this decrease is considerable only for the case of high M. Therefore, when θ increases, the risk decreases for high values of M and the optimal choice becomes longer. For example, when i=1% and σ=.01, M^0=1 for θ =0, but M^0=535 for θ =1.

Table 2.4.8 presents the critical values of σ^* for these combinations of parameters where they exist. The dependence of σ^* on these parameters is clear.

Table 2.4.8

Critical Values of σ * when $F_0=\frac{1}{4}AL$

$F_0=\frac{1}{4}AL$								
		θ						
i	**j**	**0**	**.25**	**.5**	**.75**	**.85**	**.95**	**1**
.01	**.01**	-	-	-	-	-		.125
.03	**.01**	-	-	-	-	-	-	.11
.03	**.03**	-	-	-	.02	.055	.095	.215
.03	**.05**	-	-	.055	.095	.12	.17	.32
.05	**.03**	-	-	-	.025	.06	.096	.195
.05	**.05**	-	-	.06	.99	.125	.175	.29
.05	**.10**	.12	.155	.185	.23	.26	.305	.45

2.5 Initial funding level of 50%

Tables 2.5.1-2.5.7 indicate the optimal spread period M^0 for $F_0=\frac{1}{2}$ AL and different combinations of θ, σ, j and i.

Table 2.5.1
Values of M^0 when $F_0=\frac{1}{2}AL$ ($z=-\frac{1}{2}AL$)

i=1%									
j=1%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	5	4	4	4
1	535*	318*	223*	112*	47	26	17	12	9

Table 2.5.2
Values of M^0 when $F_0=\frac{1}{2}AL$ ($z=-\frac{1}{2}AL$)

i=3%									
j=1%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	4	4	4	4
1	218*	145*	111*	36	23	17	12	10	8

Table 2.5.3
Values of M^0 when $F_0=\frac{1}{2}AL$ ($z=-\frac{1}{2}AL$)

i=3%									
j=3%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	6	6	6	5	5	5	5	4	4
1	218*	145*	111*	68*	46*	29	18	13	10

Table 2.5.4
Values of M^0 when $F_0=\frac{1}{2}AL$ ($z=-\frac{1}{2}AL$)

i=3%									
j=5%									
σ									
θ	.01	.03	.05	.1	.15	.2	.25	.3	.35
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	2	2	2	2	2
.95	7	7	6	6	6	6	5	5	5
1	218*	145*	111*	68*	46*	33*	25*	17	12

Table 2.5.5
Values of M^0 when $F_0=\frac{1}{2}AL$ ($z=-\frac{1}{2}AL$)

i=5%									
j=3%									
σ									
θ	.01	.03	.05	.1	.15	.2	.25	.3	.35
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	5	5	4	4
1	144*	99*	78*	51*	35	20	14	10	8

Table 2.5.6
Values of M^0 when $F_0=\frac{1}{2}AL$ ($z=-\frac{1}{2}AL$)

i=5%									
j=5%									
σ									
θ	.01	.03	.05	.1	.15	.2	.25	.3	.35
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	6	6	6	6	6	5	5	5	4
1	144*	99*	78*	51*	37*	28*	20	14	10

Table 2.5.7

Values of M^0 when $F_0=\frac{1}{2}AL$ ($z=-\frac{1}{2}AL$)

i=5%									
j=10%									
σ									
θ	.01	.03	.05	.1	.15	.2	.25	.3	.35
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	3	3	3	3	3	3	3	3	3
.85	4	4	4	4	4	4	4	3	3
.95	144*	99*	78*	51*	10	8	7	6	6
1	144*	99*	78*	51*	37*	28*	21*	17*	14*

The results presented by Tables 2.5.1-2.5.7 show less dramatic variation than would be expected (from the results in sections 2.3 and 2.4) because of the higher initial funding level. For low values of θ, we observe that the optimal choice is not affected by changes in σ, i or j. For example, when $\theta=.5$, $M^0=2$ for each value of i, j and σ investigated.

When θ increases (except for the cases of $\theta=.95$ and $\theta=1$), the optimal choice increases slightly. Given the initial funding level of 50%, changes in j and/or σ do not cause any rapid increase in M^0. The range of the optimal values is low.

For higher values of θ ($\theta=.95$ or $\theta=1$) and when $\sigma>\sigma^*$, an increase in j leads to a longer optimal choice. For example, when i=5%, $\theta=.95$ and $\sigma=.1$, $M^0=5$ for j=3%, $M^0=6$ for j=5% but $M^0=51$ for j=10%. When $\sigma<\sigma^*$, a change in value of j does not cause any changes in M^0 ($=M_{max}$ for these cases) contrary to the assumed rate of return which affects the optimal choice (M_{max} depends on i). So, when i increases, the maximum feasible spread period decreases, as can be seen from Table 2.2.1. For $\sigma>\sigma^*$, M^0 does not change markedly in response to changes in i. Hence, in the case of i=j (see Tables 2.5.1, 2.5.3, 2.5.6), the optimal choice is affected considerably only by j.

Table 2.5.8 shows the values of σ^* (where they exist) for combinations of i, j and θ.

Table 2.5.8

Critical Values of σ^* when $F_0=\frac{1}{2}AL$

$F_0=\frac{1}{2}AL$								
	i	.01	.03	.03	.03	.05	.05	.05
	j	.01	.01	.03	.05	.03	.05	.1
θ	.95	-	-	-	-	-	-	.105
	1	.12	.07	.17	.25	.145	.24	.43

2.6 Initial Funding Level of 75%

Tables 2.6.1-2.6.7 show the optimal spread period M^0 for $F_0=\frac{3}{4}$ AL and different combinations of θ, σ, j and i.

Table 2.6.1

Values of M^0 when $F_0=\frac{3}{4}AL$ ($z=-\frac{1}{4}AL$)

i=1%									
j=1%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	5	4	4	4
1	535*	318*	223*	89	42	25	17	12	9

Table 2.6.2

Values of M^0 when $F_0=\frac{3}{4}AL$ ($z=-\frac{1}{4}AL$)

i=3%									
j=1%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	4	4	4	4
1	44	41	38	28	21	16	12	9	8

Table 2.6.3

Values of M^0 when $F_0=\frac{3}{4}AL$ $(z=-\frac{1}{4}AL)$

i=3%									
j=3%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	5	4	4	4
1	218*	145*	111*	68*	40	24	16	12	9

Table 2.6.4

Values of M^0 when $F_0=\frac{3}{4}AL$ $(z=-\frac{1}{4}AL)$

i=3%									
j=5%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	6	6	6	5	5	5	5	5	4
1	218*	145*	111*	68*	46*	33*	23	15	11

Table 2.6.5

Values of M^0 when $F_0=\frac{3}{4}AL$ $(z=-\frac{1}{4}AL)$

i=5%									
j=3%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	4	4	4	4
1	64	58	49	32	22	16	12	10	8

Table 2.6.6

Values of M^0 when $F_0=\frac{3}{4}AL$ ($z=-\frac{1}{4}AL$)

i=5%									
j=5%									
σ									
θ	.01	.03	.05	.1	.15	.2	.25	.3	.35
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	5	5	4	4
1	144*	99*	78*	51*	37*	24	16	12	9

Table 2.6.7

Values of M^0 when $F_0=\frac{3}{4}AL$ ($z=-\frac{1}{4}AL$)

i=5%									
j=10%									
σ									
θ	.01	.03	.05	.1	.15	.2	.25	.3	.35
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	7	7	7	7	6	6	6	5	5
1	144*	99*	78*	51*	37*	28*	21*	17*	14*

With an initial funding level of 75%, the results are similar to these of section 2.5 (50% funding). For low values of M, an increase in j and/or in θ is not sufficient to cause any dramatic change in M^0 for all values of σ. The effect of the valuation rate of interest i is also minor (except for the case of θ=1).

When θ=1, Tables 2.6.1-2.6.7 indicate some rapid changes in M^0. For example, when $\sigma=.01$, $M^0=44$ for i=3% and j=1% and $M^0=535$ for i=1% and j=1%. Given the high initial funding level, it is observed that the effect of the valuation rate of interest is more significant than it was for lower initial funding levels. This is also illustrated in Figure 2.6.1 which presents the graph of J_∞ as a function of i.

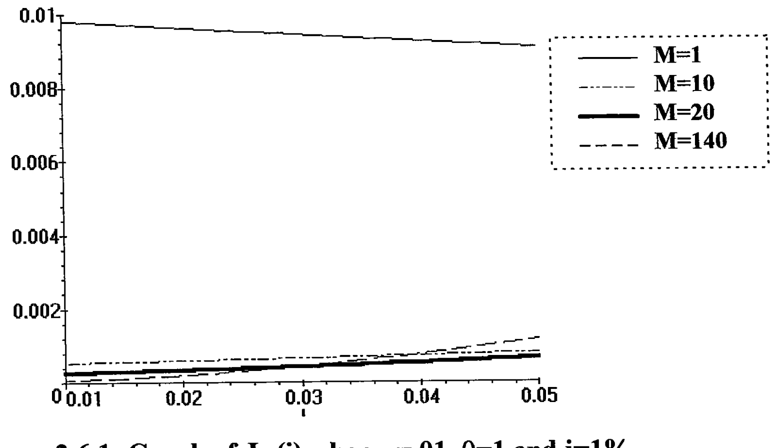

Figure 2.6.1: Graph of $J_\infty(i)$ when σ=.01, θ=1 and j=1%

Figure 2.6.1 demonstrates that the risk as represented by J_∞ is an increasing function of i for high values of M. Therefore, an increase in i leads to an upwards shift in the risk for the long spread periods, making the optimal choice shorter.

When θ increases, the optimal choice increases. When θ =1, for low values of σ, the risk is minimised when $M^0=M_{max}$. For higher values of σ, the optimal choice decreases. Table 2.6.8 indicates the critical values of σ^* when θ=1.

Table 2.6.8

Critical Values of σ* when $F_0=\frac{3}{4}AL$

$F_0=\frac{3}{4}$ AL						
	i	.01	.03	.03	.05	.05
	j	.01	.03	.05	.05	.1
θ	1	.095	.13	.24	.17	.37

It is clear that the influence of the high initial funding level is more significant than any of the other parameters, making the optimal choice shorter for most cases, when compared with the F_0=0 and $F_0=\frac{1}{4}$AL cases discussed earlier. We also observe that the higher is the initial funding level, the lower is the value of σ^*.

2.7 Initial Funding Level of 100%

Tables 2.7.1-2.7.7 present the values of M^0 when F_0=AL and for different combinations of θ, σ, j and i.

Table 2.7.1
Values of M^0 when F_0=AL (z=0)

i=1%									
j=1%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	4	4	4	4
1	466	252	163	71	38	24	16	12	9

Table 2.7.2
Values of M^0 when F_0=AL (z=0)

i=3%									
j=1%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	4	4	4	4	4
1	32	31	30	24	19	15	12	9	7

Table 2.7.3
Values of M^0 when F_0=AL (z=0)

i=3%									
j=3%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	4	4	4	4
1	195	122	89	49	30	20	15	11	9

Table 2.7.4
Values of M^0 when F_0=AL (z=0)

i=3%									
j=5%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	5	4	4	4
1	218*	145*	111*	68*	46*	33*	20	14	10

Table 2.7.5
Values of M^0 when F_0=AL (z=0)

i=5%									
j=3%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	4	4	4	4
1	26	26	25	21	17	14	11	9	7

Table 2.7.6
Values of M^0 when F_0=AL (z=0)

i=5%									
j=5%									
σ									
θ	**.01**	**.03**	**.05**	**.1**	**.15**	**.2**	**.25**	**.3**	**.35**
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	4	4	4	4
1	130	85	64	39	26	18	13	10	8

Table 2.7.7
Values of M^0 when F_0=AL (z=0)

					i=5%				
					j=10%				
					σ				
θ	.01	.03	.05	.1	.15	.2	.25	.3	.35
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	5	5	4	4
1	144*	99*	78*	51*	37*	28*	21*	17*	13

When the initial unfunded liability is zero, a different value of the interest rate used in the discounting process is not sufficient to alter the optimal choice-except for the case of θ=1. From Tables 4.6.1-4.6.4, it can be seen that for low values of θ, the results do not depend on σ, i or j. For example, when θ=.5, M^0=2 for each value of σ, i and j.

When θ increases, there is a slight increase in the optimal choice, as for the other cases of $-\frac{1}{2}AL \leq z < 0$. When we are only concerned with stabilising the contribution rate (θ=1), the optimal choice, as previously, is as long as possible. Therefore, for $\sigma<\sigma^*$, $M^0=M_{max}$ decreases when i rises but remains the same when j changes. For $\sigma>\sigma^*$, M^0 is shorter. Given the initial funding level of 100%, the optimal spread period is more sensitive to changes in the valuation rate of interest. For example, when σ=.01, j=3% and θ=1, M^0=26 for i=5%, but M^0=195 for i=3%. Therefore, when the valuation rate of interest i is equal to the rate of interest used in the discounting term j (see Tables 2.7.1, 2.7.3, 2.7.6), changes in the optimal choice are due to i.

For the specific cases tabulated, when θ=1, i=3% and j=5%, σ^*=.21 and when θ=1, i=5% and j=10%, σ^*=.33.

3. Interpretation of the Results

3.1 Minimising the Solvency Risk

If $\theta = 0$, we are minimising the solvency risk. The degree of security will depend on the speed with which the shortfall is removed by means of special contributions. In this case, the best course of action would be to pay the full amount of the shortfall as it arises without spreading any payments into the future (i.e. $M^0=1$). But this may not always be attractive, or even possible, from the employer's point of view.

However, Tables 2.3.1-2.3.7 show that when the assets in hand are much less than the initial liability ($F_0=0$), the optimal spread period is much longer, especially, for low values of σ. In particular, for $\sigma<\sigma^*$, the optimal choice is the maximum feasible spread period M_{max} which decreases as the mean return i increases. When $\sigma>\sigma^*$, the optimal spread period is equal to 1, independently of the set of parameters i, j and z.

3.2 Minimising the Contribution Rate Risk

If $\theta = 1$, we are minimising the contribution rate risk. We are concerned with stabilising the contribution rate by spreading the unfunded liability for as long as possible. As Owadally and Haberman (1995) argue, stable contributions enable the employer to plan cash flows and to predict tax relief in respect of the contributions, and lead to stable pension costs. Therefore, in order to make the call on the employer's resources stable, the actuary should choose the period for the extinguishing of the unfunded liability to be as long as possible, otherwise the range of variation of the contribution rates is increased.

The length of the spread period decreases as σ increases. For $\sigma<\sigma^*$, the optimal choice is M_{max}. For $\sigma>\sigma^*$, M^0 becomes shorter according to the particular combinations of i, j and z. When the initial funding level (represented by z) decreases, the contribution rate required rises. If our objective is one of minimising the contribution rate risk, then the optimal spread period increases.

For the $\sigma<\sigma^*$ cases, the optimal choice $M^0=M_{max}$ does not change whatever the value of the interest rate used in the discounting process (M_{max} does not depend on j). For $\sigma>\sigma^*$, a higher value of j leads to a longer optimal choice. An increase in j means that greater emphasis is being placed on the shorter-term state of the pension fund. For a funding deficit (low value of F_0), this means that a higher adjustment to the contribution rate is required. If we are concerned with minimising the contribution rate risk, a higher value of M^0 should be chosen so as to reduce the variation in the contribution rate. The higher is the initial funding deficit, the greater is the impact of j on the optimal choice.

The results are also sensitive to changes in the interest rate (i). The optimal choice M^0 decreases when i increases for each value of σ. For $\sigma<\sigma^*$, the changes in M^0 arising from changes in i correspond to Table 2.2.1. For the $\sigma>\sigma^*$ cases, the extent to which the results are affected by changes in the investment assumption depends on the initial level of assets. If the pension fund had no assets ($F_0=0$), the impact on M^0 would be less. If the initial funding level were high, an increase in i would lead to a greater interest obtained on the assets and, consequently to a lower contribution and a smaller optimal choice. Hence, increasing i has a larger impact on the optimal choice when the initial funding level as represented by z is high.

Dufresne (1988) considers the trade-off between the limiting variances of the contribution rate risk and of the fund level, recognising that improved security may imply regularly adjusted contribution rates and, conversely, stable contribution rates may be achieved by a greater variation in the fund level. He minimises the ultimate level of these variances and finds a region for M, (1,M*)

where $M^* = \begin{cases} -\log(1-d/k^*)/\log(1+i), & i \neq 0, \\ 1+1/\sigma^2, & i = 0. \end{cases}$, $k^* = \begin{cases} 0 & if \quad y<1, \\ 1-1/y & if \quad y>1. \end{cases}$

and $y=(1+i)^2+\sigma^2$.

He calls this region an optimal one, in the sense that for $M>M^*$, both limiting variances are increased and for $M\leq M^*$, the limiting variance of the contribution rate risk increases and the limiting variance of the fund level decreases.

Therefore, we may consider as a measure of the contribution rate risk the variance of C(t) in the limit, i.e. VarC(∞). We recall from (1.5) that VarC(∞)= $k^2\frac{bAL^2}{1-a}$

where $a=q^2(1+b)=(1-k)^2[(1+i)^2+\sigma^2]=(1-k)^2y$, $y=(1+i)^2+\sigma^2$, $b=\sigma^2v^2$.

Dufresne(1988) shows that k^* is the value for which VarC(∞) or $\alpha(k)=\frac{k^2AL^2}{1-(1-k)^2y}$ is minimised.

According to our formulation, for $\theta=1$, the contribution rate risk is defined as $J_\infty=\sum_{t=0}^{\infty} w^t VarC(t)$ where w is the discounting factor and $VarC(t)=k^2VarF(t)$.

So:

$$J_\infty=\frac{k^2bv_d}{1-wa}\left[\frac{z^2q^2}{1-wq^2}+\frac{AL^2}{1-w}+\frac{2zALq}{1-wq}\right]$$

Hence, we must find the values of k for which J_∞ or

$$\beta(k)=\frac{k^2[z^2q^2(1-w)(1-wq)+AL^2(1-w^2)(1-wq)+2zALq(1-wq^2)(1-w)]}{[1-w(1-k)^2y](1-wq^2)1-wq)}$$

is minimised.

We consider the case of z=-AL, for convenience. Figure 3.2.1 illustrates that the spread period M^0 which minimises J_∞ is longer than the spread period M^* which Dufresne defines as the optimal choice for minimising VarC(∞).

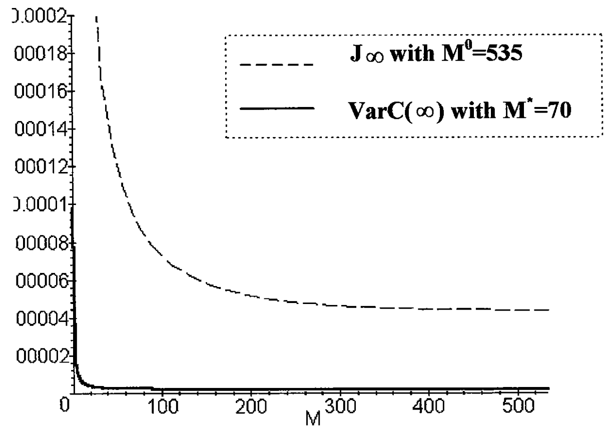

Figure 3.2.1: Graph of VarC(∞) and J_{∞} when i=1% and σ=.01

In our formulation of the problem, the criterion of minimising the contribution rate risk is defined as a time-weighted sum of the VarC(t). Hence, the discounting factor is the weight applied to the variance which means that for i>0 (w<1), more emphasis is placed on the shorter term variances. On the other hand, minimising VarC(∞) means that we consider only the ultimate situation (t→∞). If we want to approach Dufresne's case, we should put more weight on the future by choosing a discounting factor w, such that

w→1 (i→0). Then $\beta(k) \rightarrow \dfrac{k^2AL^2}{1-(1-k)^2y} = \alpha(k)$ when w→1.

Therefore, $\lim_{w\to 1} k^0 = k^* \Rightarrow \lim_{w\to 1} M^0 = M^*$.

3.3 Minimising the Risk: the General Case (0< θ <1)

Complete security or complete stability is not always an overriding principle (0< θ <1). In this case, it is necessary to consider how quickly a particular contribution arrangement relating to the pension scheme liability would meet this liability, in order to build up security of the benefits.

When σ^* exists, we observe that, for $\sigma < \sigma^*$, the optimal choice is M_{max} because we want to spread the unfunded liability for as long as possible. For $\sigma > \sigma^*$, M^0 is much shorter and depends on the particular combinations of i, j, z and θ.

From the Tables in Section 2 it can be seen that, when θ increases, the optimal spread period increases because we are more concerned with the criterion of stability. The sensitivity of the risk to θ depends on the particular values of the other parameters. Tables 2.3.1-2.3.7 indicate that, for low values of σ, the optimal choice (=M_{max}) is independent of θ (F_0=0). Table 2.4.3 shows that, for $\sigma<\sigma^*$, M^0 is very sensitive to changes in θ ($F_0=\frac{1}{4}AL$) and Table 2.5.3 shows that M^0 increases slightly when θ increases for each value of θ ($F_0=\frac{1}{2}AL$).

When the initial funding level (F_0) rises, the risk as represented by J_∞ is minimised for shorter spread periods. We consider equation (16) as a function of z, $-AL \le z \le 0$. J_∞ is an increasing function of z and is more sensitive to changes in z for high values of q (i.e. high values of M). For low values of q, the risk is approximately constant. So, when the initial funding level rises, the risk J_∞ remains approximately constant for low values of M, but it increases to a substantial extent for high values of M. With the objective of minimising the risk J_∞, the optimal spread period becomes shorter. The extent to which the optimal choice is decreased, when the initial funding level rises, depends on the particular combination of the other parameters. From the comparison of Tables 2.3.4 and 2.4.4, it can be observed that, for low values of σ, a lower initial funding level makes the optimal spread period jump from a low value to become equal to the maximum feasible spread period M_{max}. For higher values of σ, the effect of the initial funding level is minor.

The higher is the initial level of assets, the greater is the impact of the assumed rate of return (i) on the optimal choice. Because of the interest earned on the plan's high initial funding level, the criterion of security is satisfied when a small spread period is chosen, without leading to variations in contribution rate. Hence, the values of the optimal period range from 1 to 6 when z=0, as Tables 2.7.1-2.7.7 indicate.

With the objective of placing more emphasis on the shorter-term state of the pension fund (a higher value of j), the results become more dependent on j, the lower is the initial funding level and for $\sigma > \sigma^*$ (for $\sigma < \sigma^*$, $M^0=M_{max}$ is independent of j). When the short-term state of the pension fund is to be emphasised and a large initial unfunded liability exists, minimisation of the risk J_∞ can be achieved by small changes to the contribution rate. This means that longer spread periods should be chosen, as illustrated by Tables 2.3.5 and 2.3.7.

We consider the case where we wish to place more emphasis on the longer-term in more detail. According to our formulation

$$J_\infty = \sum_{t=0}^{\infty} w^t[\theta \mathrm{VarC(t)} + (1-\theta)\mathrm{VarF(t)}]$$

where w is the discounting factor.

So:

$$J_\infty = \frac{(\theta k^2 + 1 - \theta)}{1 - wa}\sigma^2 v^2 w \left[\frac{z^2 q^2}{1 - wq^2} + \frac{AL^2}{1-w} + \frac{2zALq}{1-wq}\right]$$

Hence, we must find the values of k for which J_∞ or

$\phi(k)=\frac{(\theta k^2 + 1 - \theta)}{k^2}\beta(k)$ is minimised.

If we are interested in the long-term position of the pension fund, we could remove the time weighting factor completely and choose a discounting factor w, such that $w \to 1$.

Then $\phi(k)=\frac{\theta k^2+1-\theta}{1-(1-k)^2 y}$ (as previously we assume for convenience that AL=1).

In order to find the optimal values of k, k^0, we should solve the equation:

$$\frac{d\varphi(k)}{dk}=\theta y k^2+\left[y(1-2\theta)+\theta\right]k-y(1-\theta)=0 \qquad (18)$$

where k should be restricted (for convergence) to values such that:

$k_{min}<k<1$ where $k_{min}=1-\frac{1}{\sqrt{y}}$.

Case I: If $\theta=0$, $k^0=1 \Rightarrow M^0=1$.

Case II: If $\theta=1$,

$k^0=1-\frac{1}{y}=k^*$ and the optimal spread period is the corresponding M^* as derived by Dufresne (1988).

This case has been discussed in detail in paragraph 3.2.

Case III: If $0<\theta<1$,

k^0 is the conventional positive root of equation (18) with $1-\frac{1}{\sqrt{y}}<k^0\leq 1$.

In any particular case, calculation of the values of k^0 allows us to find the corresponding values of M from $k=1/\ddot{a}_{\overline{M}|} \Rightarrow M=-\log(1-\frac{d}{k})/\log(1+i)$.

Tables 3.3.1-3.3.3 present the values of M^0 for combinations of i, σ and θ. It is worth noting that the results are independent of the initial funding level as represented by z (because they are determined by the limiting values, as $t \to \infty$).

Table 3.3.1
Values of M^0 when $j \to 0$

i=1%									
	σ								
θ	.01	.03	.05	.1	.15	.2	.25	.3	.35
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	4	4	4	4
1	70	66	60	42	28	19	14	11	8

Table 3.3.2
Values of M^0 when $j \to 0$

i=3%									
	σ								
θ	.01	.03	.05	.1	.15	.2	.25	.3	.35
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	4	4	4	4
1	24	23	23	20	16	13	10	8	7

Table 3.3.3
Values of M^0 when $j \to 0$

i=5%									
	σ								
θ	.01	.03	.05	.1	.15	.2	.25	.3	.35
0	1	1	1	1	1	1	1	1	1
.25	1	1	1	1	1	1	1	1	1
.5	2	2	2	2	2	2	2	2	2
.75	2	2	2	2	2	2	2	2	2
.85	3	3	3	3	3	3	3	3	3
.95	5	5	5	5	5	4	4	4	4
1	16	15	14	13	11	10	8	7	6

Tables 3.3.1-3.3.3 indicate that, for $0<\theta<1$, the values of the optimal spread period are low and do not depend on i or σ. Therefore, with equal time-weighting so that $j \to 0$ and $w \to 1$, minimisation of the risk J_∞ is achieved by a low value of the spread period, irrespective of the initial funding level.

4. Conclusions

From the results in Section 2 , it is clearly seen that, the lower is the initial funding level (F_0), the greater is the range of the optimal choices. Hence, when the funding strategy involves a low initial funding level, a high choice of M is necessary (under particular choices of the other parameters) in order to remove the initial funding deficit without great variation in the contribution rate.

The choice of the parameter θ is of great importance as it reflects which of the variability of the fund or of the contribution is required to be more stable from the employer's point of view. The results are presented for different values of θ and allow a comparison of the optimal choices for valuation methods with different principal objectives (e.g. more emphasis on stable contribution rates or on funding of the actuarial liability).

The use of a discounting factor $w \neq v$ clarifies the effect of the assumed rate of return (i) and of the rate of interest for discounting variances (j) on the range of the optimal spread periods. The results support the conclusion that, the greater is the departure from a 100% initial funding level, the less is the effect of the assumed rate of return. Given a high initial funding level, an alteration of the assumed rate of return does have an important effect on the range of the optimal spread periods. In particular, the effect of the interest earned on the pension plan's assets leads to a small range of optimal choices.

The rate of return used in the discounting process (j) indicates which of the short-term or the long-term state of the pension fund is to be more emphasised. The conclusion is that, the lower is the initial funding level, the greater is the impact of j on the range of the optimal choices. We also demonstrate that, in the long term, the risk as represented by J_∞ is independent of the initial funding level and the range of the optimal spread periods is much diminished.

Finally, it is seen that, the range of the optimal spread periods is large for particular combinations of the parameters. For these cases and for low values of σ, the optimal choice is to make M as large as possible. The critical values of σ, which make the optimal spread period jump from a low value to the maximum feasible spread period M_{max}, are shown in Section 2.

It is worth noting that, in the UK, the values of the spread period used from most pension schemes in practice may be different from what the tables in Section 2 indicate. As far as the solvency risk is concerned, the Minimum Funding Requirement (MFR) rules, as given in GN27, place a restriction on the choice of M. In particular, if the funding level (which is defined as the ratio of assets to liabilities) is less than 90%, the deficit has to be eliminated within 1 year. If the funding level is more than 90%, the unfunded liability has to be removed within 5 years. When F_0=0 and $\sigma<\sigma^*$, Tables 2.3.1-2.3.7 show that the spread period which minimises the risk J_∞ is longer than is allowed by the MFR rules. For all the other cases tabulated, the results presented inSection 2 are consistent with MFR rules.

Haberman (1994) explains that the values of M chosen in practice are unlikely to be greater than the average remaining membership period-with an average age of membership of 40-45 and a normal retirement age of 65 the choice of M would be in the range of 20-25. Therefore, the optimal spread periods presented in Section 2, for $\sigma<\sigma^*$, are too long to be used in practice. For $\sigma>\sigma^*$, if we are interested in minimising the risk J_∞, values of M which are much smaller than this indicative range 20-25 traditionally chosen by practitioners are optimal, as the results of this paper indicate.

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