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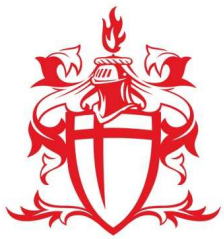
Non-Hermitian Hamiltonians in Field Theory

Paulo Eduardo Gonçalves de Assis

PhD Thesis



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A meus pais.

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Muito obrigado.

Declaration

The work presented in this thesis is based on investigations believed to be original and carried out at the Centre for Mathematical Science, City University London, in collaboration with Prof. Andreas Fring. It has not been presented elsewhere for a degree, diploma, or similar qualification at any university or similar institution. I have clearly stated my contributions, in jointly-authored works, as well as referenced the contributions of other people working in the area.

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Abstract

This thesis is centred around the role of non-Hermitian Hamiltonians in Physics both at the quantum and classical levels. In our investigations of two-level models we demonstrate [1] the phenomenon of fast transitions developed in the $\mathcal{PT}$-symmetric quantum brachistochrone problem may in fact be attributed to the non-Hermiticity of evolution operator used, rather than to its invariance under $\mathcal{PT}$ operation. Transition probabilities are calculated for Hamiltonians which explicitly violate $\mathcal{PT}$-symmetry. When it comes to Hilbert spaces of infinite dimension, starting with non-Hermitian Hamiltonians expressed as linear and quadratic combinations of the generators of the $su(1,1)$ Lie algebra, we construct [2] Hermitian partners in the same similarity class. Alongside, metrics with respect to which the original Hamiltonians are Hermitian are also constructed, allowing to assign meaning to a large class of non-Hermitian Hamiltonians possessing real spectra. The finding of exact results to establish the physical acceptability of other non-Hermitian models may be pursued by other means, especially if the system of interest cannot be expressed in terms of Lie algebraic elements. We also employ [3] a representation of the canonical commutation relations for position and momentum operators in terms of real-valued functions and a noncommutative product rule of differential form. Besides exact solutions, we also compute in a perturbative fashion metrics and isospectral partners for systems of physical interest. Classically, our efforts were concentrated on integrable models presenting $\mathcal{PT}$-symmetry. Because the latter can also establish the reality of energies in classical systems described by Hamiltonian functions, we search for new families of nonlinear differential equations for which the presence of hidden symmetries allows one to assemble exact solutions. We use [4] the Painlevé test to check whether deformations of integrable systems preserve integrability. Moreover we compare [5] integrable deformed models, which are thus likely to possess soliton solutions, to a broader class of systems presenting compacton solutions. Finally we study [6] the pole structure of certain real valued nonlinear integrable systems and establish that they behave as interacting particles whose motion can be extended to the complex plane in a $\mathcal{PT}$-symmetric way.

1 Introduction

Physical objects observed in nature can be characterized by certain properties which are measurable with the employment of appropriate devices. Some of these intrinsic attributes, like weight, are more tangible than others, such as time. From the very concrete idea of carrying a bag full of feathers or one full of stones we can move to deeper ideas, say for instance, why does time always increase? Despite all the conceptual differences between physical quantities, one property is common to all: wherever you look, whatever you measure, they are all specified by real numbers.

For this simple reason one tends to question the physical reality of descriptions based on complex numbers, of the form $z = x + \imath y$, with $\imath = \sqrt{-1}$ the imaginary number. Nonetheless, ruling out all complex analysis formalism from the scope of physics is too drastic an approach. Complex functions might be used perfectly well to formulate a mathematical representation of natural phenomena. For example, a set of pendulums disposed in a line side by side connected through elastic bands at their extremities is governed by a sinusoidal interaction. Such a Hamiltonian system is known as the sine-Gordon model which is one amongst many representatives of affine A_r ($SU(r+1)$) Toda field models with $r \in \mathbb{N}$, corresponding to $r = 1$. The Hamiltonian density function is real in such a case but for $r > 1$ this ceases to be true. For $r = 2$ for instance one has a system defined by

$$\mathcal{H}(x,t) = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2\right] - \frac{m^2}{\beta^2}\sum_{j=0}^{r}\left(e^{\imath\beta\alpha_j\cdot\phi} - 1\right), \tag{1.1}$$

where $\phi = (\phi_0(x,t), \phi_1(x,t), \phi_2(x,t))$ is a three-component field, $\alpha_0 = (-1,0,1)$, $\alpha_1 = (1,-1,0)$, $\alpha_2 = (0,1,-1)$ and $\beta, m \in \mathbb{R}$ are constants.

The whole family of complex A_r affine Toda field theories [1] (1.1) have been proven [7, 8] to have solitonic configurations with real mass, or equivalently, energy, regardless of the reality of the Hamiltonian density. Therefore, the simple example just presented indicates

[1] Real affine Toda field theories are obtained from (1.1) by taking $\beta \to -\imath\beta$.

attention is required when classical Hamiltonians are complex-valued functions as they might have physical meaning. This question however was not systematically investigated until the results found in [9]. Curiously, the development in this area was originated at a quantum level before the analogous problem was studied classically, differently from what normally happens since the underlying structures of quantum theories are frequently more involved and the traditional description of objects in terms of trajectories must be replaced.

An important feature of the subatomic particles is that they behave in a very counterintuitive way from a macroscopic perspective. A successful description of such systems relies on a probabilistic interpretation attached to a vector space so that interference patterns intrinsic to the particle-wave duality can be reproduced. The primordial assumption in quantum mechanics is to assign a measurable physical configuration, or a quantum state, to a vector living in a Hilbert space $\mathcal{H}$. These vector spaces can be finite-dimensional, like the one spanned by $\{(1,0),(0,1)\}$, or infinite-dimensional like the space spanned by $\{\sin(nx)\ ,\ n \in \mathbb{N}\}$.

By definition, the Hilbert space is a complete linear vector space $\mathcal{H}$ endowed with an inner product defined on the set of complex numbers,

$$\xi, \psi \in \mathcal{H} \quad \longrightarrow \quad \langle \xi | \psi \rangle \in \mathbb{C}, \tag{1.2}$$

where $|\psi\rangle$ belongs to a vector space and $\langle \xi|$ to its dual vector space.

Regarding the dynamics of quantum states, they evolve according to the Hamiltonian operator,

$$\hat{H}|\psi(t)\rangle = \imath \frac{\partial}{\partial t}|\psi(t)\rangle. \tag{1.3}$$

Despite being governed by a deterministic equation as the one just above, unless a measurement is made, quantum states can only be determined in a probabilistic sense so that states are often described to be in a linear combination of measurable states, the eigenstates. The vectors themselves have no physical significance but their scalar product is fundamental when interpreting the probability of finding the system in a certain quantum state.

These ideas make it natural to demand that elements in this Hilbert space must satisfy:

- **P1.** Positivity: $\langle \psi | \psi \rangle \geq 0 \quad$ and $\quad \langle \psi | \psi \rangle = 0 \iff \psi = 0.$
- **P2.** Hermiticity: $\langle \xi | \psi \rangle^* = \langle \psi | \xi \rangle$.

- **P3.** Linearity: $\langle\xi|c_1\psi_1 + c_2\psi_2\rangle = c_1\langle\xi|\psi_1\rangle + c_2\langle\xi|\psi_2\rangle, \qquad c_i \in \mathbb{C}.$

In this vector space it is customary to define the *norm* of a vector ψ as $||\psi|| \equiv \sqrt{\langle\psi|\psi\rangle}$, and given a transition amplitude between two states, $\langle\xi|\psi\rangle$, one interprets the quantity $||\langle\xi|\psi\rangle||^2 = \langle\xi|\psi\rangle\langle\psi|\xi\rangle$ as the probability that such a transition occurs. Similarly, the probability of having a measurement which indicates the system to be in the state $|\psi\rangle$ is given by $||\langle\psi|\psi\rangle||^2$. Note that a whole ray of vectors, i.e. vectors differing by a phase factor, can be used to describe the same physical state.

The Hermiticity property introduces the need of using Hermitian operators, e.g. [10, 11, 12]. Consider a linear operator $\mathcal{L}$ densely defined on a Hilbert space $\mathcal{H}$ which is also bounded,

$$||\mathcal{L}|\psi\rangle|| \leq \lambda||\psi|| \qquad \text{for some } \lambda > 0\ , \psi \in \mathcal{H}. \tag{1.4}$$

Then, $\mathcal{L}$ is said to be *Hermitian* if

$$\langle\xi|\mathcal{L}^\dagger|\xi\rangle = \langle\xi|\mathcal{L}|\psi\rangle, \quad \psi, \xi \in \mathcal{H} \tag{1.5}$$

where the Hermitian conjugation, denoted by the symbol †, corresponds to the action of the operator on the dual space,

$$\langle\xi|\mathcal{L}|\psi\rangle = \langle\xi|\mathcal{L}\psi\rangle \qquad \text{whereas} \qquad \langle\psi|\mathcal{L}^\dagger|\xi\rangle = \langle\mathcal{L}\psi|\xi\rangle. \tag{1.6}$$

As such operators were discussed by Dirac in a matrix form, e.g. [13], they are sometimes denoted *Dirac Hermitian* and the conjugation operation is just a transposition of the matrix followed by a complex conjugation. The concept of Hermiticity is closely related to that of *self-adjointness*. To observe the difference, let us first recall that $\mathcal{L}$ is *symmetric* if

$$\langle\xi|\mathcal{L}^\dagger|\xi\rangle = \langle\xi|\mathcal{L}|\psi\rangle, \quad \psi, \xi \in \mathcal{D}(\mathcal{L}), \tag{1.7}$$

where $\mathcal{D}(\mathcal{L})$ indicates the domain of $\mathcal{L}$. Bounded symmetric operators are then Hermitian. Sometimes no distinction is made between Hermitian and symmetric and instead call symmetric operators as Hermitian. The operator is classified as *self-adjoint* if it is symmetric and the domain of $\mathcal{L}$ equals the domain of $\mathcal{L}^\dagger$, i.e. a symmetric everywhere defined operator is self-adjoint. When $\mathcal{L}$ is unbounded the Hilbert space is necessarily infinite dimensional and the domain of $\mathcal{L}$ need not be all of the Hilbert space, whereas for bounded operators $\mathcal{D}(\mathcal{L}^\dagger) = \mathcal{D}(\mathcal{L})$ and the distinction disappears. The importance of a self-adjoint operator is that its spectrum is a subset of the real axis and all operators with such spectra are self-adjoint.

Self-adjoint operators in a Hilbert space can also be used to define the concept of *observables* in quantum mechanics. Although the notion of generalized observables as positive operator-valued measures in a Hilbert space can be introduced as discussed in [14], one usually defines an observable simply as a self-adjoint operator in a Hilbert space. The eigenvalues of an observable represent the possible values encountered during the measurement procedure. The energy characteristic values are obtained by diagonalizing the Hamiltonian operator $\hat{H}$ in (1.3) and similarly all physical quantities - position, momentum, spin, etc - are represented in quantum systems as observables, or self-adjoint operators. Following the tradition in most of the physics community, we shall not distinguish between self-adjointness and Hermiticity from now on.

By non-Hermitian operators we mean those operators which are not self-adjoint and consequently do not necessarily have real spectra. Non-Hermitian Hamiltonians appear frequently in the study of quantum systems and are usually interpreted as effective Hamiltonians associated with dissipative models when they posses complex spectra. However, also non-Hermitian Hamiltonians whose spectra were believe to be real have emerged sporadically in the literature. This illustrates that whereas Hermitian operators must have real eigenvalues, so that complex characteristic values can only appear in non-Hermitian systems, the latter can also present real spectra. In other words, by restricting their investigations to self-adjoint operators physicists miss out potentially significant setups. From this observation rises the interest in determining under what conditions will non-Hermitian Hamiltonians generate real eigenvalues.

This question had been neglected for decades until a deeper debate was set alight in the last ten years. It has been understood recently that the reality of the spectra can be explained in terms of a concept known as unbroken $\mathcal{PT}$-symmetry [9], whose name stems from the physical operation of simultaneous space and time reflection. In the occasion when these ideas were proposed a whole class of non-Hermitian Hamiltonians with real spectra, generalizing in a way the harmonic oscillator, were identified and analyzed. The simultaneous invariance of the Hamiltonian and its eigenfunctions under parity and time reversal transformations is a specific example of an anti-linear symmetry for which spectral properties had already been established in a generic manner [15, 16]. $\mathcal{PT}$-symmetry however was conjectured [17] to be a reasonable candidate to substitute the Hermiticity postulate of quantum mechanics and despite being a more appealing formulation of the quantum theory, it has become more evident that it is not fundamental. However, in

practical terms one is usually not in a position to know all eigenfunctions and eigenvalues for a Hamiltonian and therefore $\mathcal{PT}$-symmetry furnishes a convenient mechanism to single out possibly relevant models just by examining the form of the Hamiltonian.

A broader approach to the microscopic world must instead deal with the use of various metrics characterizing different Hilbert spaces. The inherent freedom of choosing the metric in the quantum formalism may be used to redefine isospectral observable partners which are Hermitian with respect to a nontrivial metric [18]. This opens up the possibility that a non-Hermitian set of observables might be regarded as Hermitian with respect to a new metric. Given a Hamiltonian which is not Hermitian, the construction of the associated metric will guarantee not only the reality of its spectrum but also the existence of a consistent quantum framework in which one has unitary time evolution. This last property, essential in order to maintain the probabilistic interpretation of quantum mechanics since it assures conservation of probability, cannot be established simply from the reality of the spectrum. Such notions are the cornerstone of the framework known to some as Quasi-Hermiticity and to others as Pseudo-Hermiticity. For reasons which will be clear in the following chapter we call this Quasi-Pseudo-Hermiticity (sometimes abbreviated as QP-Hermiticity in the text) in an attempt of being more precise. It becomes clear that the determination of QP-metrics is fundamental in the study of non-Hermitian theories but this is not an easy task, comparable in some situations to the complete knowledge of the eigenstates, necessary in the approach based on $\mathcal{PT}$-symmetry. The knowledge of the metric can be useful to decide whether some of the examples of non-Hermitian Hamiltonians which have appeared in the literature in the past do indeed constitute consistent quantum systems, e.g. [19] and references therein, or if the same applies to newly proposed models, opening a vast universe of possibilities.

Many interesting non-Hermitian systems have been proposed, most of them theoretically but also experimentally. It still remains somewhat unclear how setups of this kind can be best employed in a laboratory although initial attempts have been made, e.g. [20, 21, 22]. Perhaps the most controversial consists of the quantum brachistochrone problem for non-Hermitian systems, investigated initially in [23]. Not surprisingly it is also one of the most exciting problems in the area. It consists of determining under which conditions the evolution of a system between two pre-defined states occurs in the least amount of time. By introducing $\mathcal{PT}$-symmetric non-Hermitian Hamiltonians as the generators of the time evolution it was observed that in principle transitions faster than in

Hermitian quantum mechanics could take place. Shortly after it was shown that such phenomenon could also happen for non-$\mathcal{PT}$-symmetric systems, even for dissipative ones, [1]. The key feature of this peculiar behaviour is that one uses the eigenstates of an equivalent Hermitian Hamiltonian as the initial and/or final states regarding the non-Hermitian evolution. As all interesting problems, it generated not only answers but also more fundamental questions, such as the intriguing possibility of mixing Hermitian and non-Hermitian frameworks.

Involving only two terminal states, the quantum brachistochrone problem can be easily formulated in terms a 2×2 system. For this reason it provides a favourable scheme to explore the features of non-Hermitian systems in a very clear way. Nevertheless it is crucial to understand quantum models with infinite dimensional representation as well, e.g., the aforementioned harmonic oscillator and possible complex extensions of it. As a simple example consider a Hamiltonian described by the following combinations of the bosonic creation $a^\dagger$ and annihilation a operators: $a^\dagger\ a^\dagger$ and $a\ a$. Unless these terms appear with the same coefficients the Hamiltonian will not be Hermitian. Surprisingly there is a region in the parameter space defined by such coefficients where the model admits real eigenvalues and a consistent quantum description with unitary time evolution. The complete understanding of this problem requires the explicit construction of a pseudo-quasi-metric and a Hermitian Hamiltonian partner in the same similarity class. This task is considerably more intricate in infinite dimensional settings than it was in the 2×2 model and perturbation theory can always be employed. However, a more fundamental comprehension resides on the knowledge of exact solutions and a few tools have been employed in this thesis with this aim in mind, for instance Lie algebraic concepts and Bogoliubov transformations [24, 25, 2] or Moyal products and a differential representation of the problem [26, 27, 28, 3]. Each of these illustrative methods present their own qualities and advantages depending on the situation, so that they allow one to tackle a considerable part of the models encountered in the literature.

The importance of $\mathcal{PT}$-symmetry, albeit originated in connection with subatomic objects, exceeds the realms of quantum physics. It is well understood for many decades that most macroscopic effects observed are ultimately consequence of a limiting manifestation of the quantum laws. One way of carrying out the transition from quantum to classical mechanics may be obtained by replacing the commutation relations, denoted by $[\,\cdot\,,\,\cdot\,]$, of coordinate and momentum operators $\hat{x}$ and $\hat{p}$ by the classical Poisson brackets $\{\,\cdot\,,\,\cdot\,\}_{PB}$

of the generalized coordinates x and p in the limit $\hbar \to 0$. In this procedure $\mathcal{PT}$-symmetry survives and can be used to guarantee the reality of the energies without referring to eigenstates anymore, provided assumptions on the fields are made. Therefore, as long as the classical Hamiltonian function is invariant under parity and time reversal the energy of the system will be real, for both point particles or field excitations, if surface terms are appropriately chosen to vanish.

Complex extensions of classical theories can then be constructed in a controlled way so as to preserve a physical meaning. $\mathcal{PT}$-symmetric deformations of classical field theories like the Korteweg - de Vries equation [29, 30, 31] have proved to be very promising as they allow for the construction of conserved charges. These lead to the investigation of whether such extensions can preserve integrability and, if so, under which conditions. In some of the $\mathcal{PT}$-deformed models the existence of compacton solutions, i.e., solitary waves with compact support, has been observed and the determination of integrable deformations indicate the system which allow for the existence of solitonic solutions. For this purpose, the Painlevé test [32] shows to be a very systematic way to discriminate between models which are integrable and those which are not [4, 5]. Nevertheless, starting with classical models and extending them to the complex plane in a $\mathcal{PT}$-symmetric fashion is not the only way to generate systems with such a symmetry. This natural occurrence is well exemplified in nonlinear field equations which admit singular solutions [6].

1.1 Outline

The progress just described is only a sample of all the contribution made by many authors in this area and corresponds to the objects investigated in this thesis, in a journey through non-Hermitian systems appearing in physical studies, both at quantum and classical levels. In recent years much has been understood about the feasibility of working with observables which are not self-adjoint in a quantum theory. This however, does not diminish the necessity of further investigations in this area or of disseminating the knowledge accumulated in this period. This work intends to explore the formalism which is becoming standard in the community to tackle some interesting points analyzed by the author during a postgraduate programme.

The thesis is organized as follows:

In Chapter 2 we discuss the basic ideas behind non-Hermitian Hamiltonians, firstly in connection with dissipative systems and then we explore the occurrence of real eigen-

values for such operators and we give particular attention to the notion of $\mathcal{PT}$-symmetric quantum mechanics and its historical importance. In the sequence we present alternative, and in some occasions equivalent to $\mathcal{PT}$-symmetry, methods which can be used to establish the reality of spectrum of certain non-Hermitian Hamiltonians. The analysis culminates with the concept of pseudo- and quasi-Hermiticity together with a redefinition of the appropriate scalar product.

Chapter 3 provides a good opportunity to explore the features considered in the previous chapter in a system which can be formulated easily in a two-dimensional Hilbert space, namely the non-Hermitian quantum brachistochrone problem. Despite such a simplicity the mathematical environment is far from trivial and the model discussed touches deep conceptual questions and presents interesting consequences to physical applications. We show that the use of non-Hermitian but $\mathcal{PT}$-symmetric evolution operators can generate fast transition in an appropriate setting and we demonstrate this phenomenon is not exclusive to $\mathcal{PT}$-invariant models but that instead it could be attributed simply to the inclusion of a non-Hermitian evolution. Section 3.4 represents the core ideas contained in [1]. We also present controversial points discussed in recent literature as well as alternative formulations.

Having explored the necessary ideas to investigate non-Hermitian Hamiltonians in the previous chapter, Chapter 4 brings important concepts in our quest for exact results. We argue about the power of symmetries in general with respect to such a goal and present some basic ideas concerning Lie algebras. The latter are shown to be a fundamental ingredient when studying integrable models, which are also discoursed about, followed by the characterization of solitons and compactons. Finally we discuss a concept similar to integrability, yet not directly related, namely solvability.

Then in Chapter 5 we investigate infinite-dimensional non-Hermitian Hamiltonians with real spectra having in mind the construction of Hermitian counterparts. First we try to do so with quasi-exactly solvable models formulated in terms of quadratic combinations of the $sl(2,\mathbb{R})$ Lie algebra by employing a similarity transformation. Then we discuss an equivalent problem whose underlying algebraic structure is that of $sl(1,1)$ generators represented in a Fock space. The mapping of the latter models onto Hermitian partners is also addressed under the perspective of a generalized Bogoliubov transformation and the different approaches are then compared. We conclude the chapter by presenting possible realizations of the $sl(1,1)$ algebra and physical systems which fit into the category explored.

Chapter 6 deals with a similar problem, namely of finding isospectral partners and suitable corresponding metrics for non-Hermitian Hamiltonians, but not by solving operator equations. Instead, operators are replaced by commutative real-valued functions and the usual noncommutative operator product is substituted by the so-called Moyal product. In this way, the problem is transformed into a differential form and Hamiltonians of maximally cubic dependence on x and p can be solved. Examples of such models include the imaginary cubic interaction and the Reggeon model, which cannot be expressed in terms of the Lie algebras discussed in the previous chapter. Besides, they are also interesting to discuss because for them no exact isospectral mapping is known even though the reality of the spectrum can be assured by other methods, such as $\mathcal{PT}$-symmetry.

The appearance of $\mathcal{PT}$-symmetric Hamiltonian systems described by real-valued field equations, with the use of Moyal brackets to recover the quantum uncertainty principle, motivates us in the following studies exhibited in Chapter 7. There, we start with systems described by Hamiltonian functions together with a Poisson bracket structure, therefore classical in nature. The equations of motion used, nonlinear and normally used to described hydrodynamical flow, are then deformed in a $\mathcal{PT}$-symmetric fashion and we investigate integrability properties in those systems by making use of the Painlevé test and searching for solitonic solutions.

Classical nonlinear dynamical systems remain the focus of our study throughout Chapter 8 also in connection with structures which are invariant under a $\mathcal{PT}$ operation. We show that poles in real solution of classical nonlinear differential equations behave as interacting particles which can be made $\mathcal{PT}$-symmetric if the Hamiltonian evolution is imposed to be constrained. In order to obtain the exact solution of this restricted motion for two- and three-particle systems, we develop a systematic procedure which enables us to construct solitonic solutions for the nonlinear equation. Moreover, we compare the $\mathcal{PT}$-symmetric interacting model obtained in the way just described with other deformations found in the literature.

We finish with Chapter 9, which summarizes our main results and brings concluding discussions.

2 Aspects of non-Hermitian Hamiltonians

2.1 Dissipative systems

Non-Hermitian Hamiltonians appear most frequently in the physics literature to describe dissipative systems, regarded as a result of an effective description. This picture arises when one has a complicated physical configuration and judges that it is sensible to isolate a handful of degrees of freedom from the totality. The outlined approach represents an initial step to have a tractable configuration formed only by the elements of interest as opposed to the larger number, possibly infinite, composing the complete setup. However, both subsystems, the smaller depicted structure and its complement, are usually inherently coupled so that there will occur exchanges between them. Therefore from the perspective of one of the subsystems one will find either loss or gain of particle number, energy or any other physical quantity. Dissipation thus emanates in a natural way and is an indication that the model used cannot be considered fundamental as a complementary description becomes necessary for taking the remaining environment into account. This is a crucial difference when compared to the non-Hermitian Hamiltonians which comprise the main body of this thesis.

In classical mechanics it is long understood the universality of dissipation, as a consequence of microscopic phenomena of increasing disorder. When the system is not in thermodynamical equilibrium and energy and matter can be exchanged between the system, considered open, and the environment, dissipation leads to heating. When extending to quantum theory, ideas concerning dissipative systems appear promptly in the study of open and closed channels in atomic physics, for instance [33, 34, 35]. To appreciate this, let us start by considering a Hamiltonian system $\hat{H}$ whose physical state is described by

two coexisting sets of variables, say X and Y, through a function $\psi(X,Y)$. More precisely, according to the formalism of quantum mechanics, the Schrödinger equation reads

$$\hat{H}\psi(X,Y) = E\psi(X,Y). \tag{2.1}$$

Suppose there is one other observable $\hat{O}$ acting only on the subset of parameters Y. Then the eigenvalue problem associated to such a new operator defines a complete set of eigenfunctions, which in turn can be used as a basis to expand the complete wavefunction of the system described by $\hat{H}$,

$$\hat{O}\phi_n(Y) = \omega_n\phi_n(Y) \qquad \Longrightarrow \qquad \psi(X,Y) = \sum_n \varphi_n(X)\phi_n(Y). \tag{2.2}$$

It is customary to define a channel as being characterized by the eigenvalue ω_n. If $\hat{O}$ is a linear operator commuting with the Hamiltonian, $[\hat{O},\hat{H}] = 0$, the Schrödinger equation (2.1) can be solved separately for each channel ω_n in a way that different channels are independent. Conversely, if $\hat{O}$ and $\hat{H}$ do not commute, one observes the coupling of the channels. Actually, it follows from (2.1) and (2.2) that

$$\sum_{n\neq m}\int dY\ \phi_m^*(Y)\ H\ \phi_n(Y)\ \varphi_n(X) + \int dY\ \phi_m^*(Y)\ H\ \phi_m(Y)\ \varphi_m(X) = E\ \varphi_m(X), \tag{2.3}$$

so that only if $\phi_n(Y)$ also diagonalize $\hat{H}$ will the terms $n \neq m$ have no contribution. Without loss of generality one can decompose the Hamiltonian as

$$\hat{H} = \hat{H}_X + \hat{H}_Y + V(X,Y), \tag{2.4}$$

where the coupling between the different sets of variables is concentrated in $V(X,Y)$. The eigenvalues of H_Y will in this situation define channels,

$$\hat{H}_Y\phi_n(Y) = e_n\phi_n(Y), \tag{2.5}$$

so that the Schrödinger equation furnishes

$$\sum_{n\neq m} V_{m,n}\varphi_n(X) = \left(\frac{d^2}{dX^2} + E - e_m - U(X) - V_{m,m}\right)\varphi_m(X). \tag{2.6}$$

where $V_{m,n} = \int dY\ \phi_m^*(Y)\ V(X,Y)\ \phi_n(Y)$ and $\hat{H}_X$ has been explicitly expressed in terms of a kinetic operator and a potential $U(X)$. As the interaction vanishes asymptotically,

$$\left(\frac{d^2}{dX^2} + E - e_m\right)\varphi_m(X) = 0, \tag{2.7}$$

and two different kinds of channels appear depending on their relative values with respect to E: if $e_m < E$ then we have unbounded motion and the eigenstates $\varphi_m(X)$ have an oscillatory behaviour at large distances and they are called *open channels*; contrarily, if $e_m > E$ we refer to them as *closed channels*, with bound motion and asymptotically attenuated eigenstates.

Considering a system of only two coupled channels equations

$$\left(\frac{d^2}{dX^2} + \epsilon_1 - V_1(X)\right)\varphi_1(X) = V_{1,2}\varphi_2(X)\ , \tag{2.8}$$

$$\left(\frac{d^2}{dX^2} + \epsilon_2 - V_2(X)\right)\varphi_2(X) = V_{2,1}\varphi_1(X)\ , \tag{2.9}$$

with $E - e_i = \epsilon_i$ and $V_i(X) = U_i(X) - V_{i,i}$ for $i = 1, 2$. We assume also channel 1 to be open, $\epsilon_1 > 0$, whereas channel 2 is closed, $\epsilon_2 < 0$, and we take $\varphi_0(X)$ to be a bound state, proportional $\frac{1}{\lambda}\varphi_2(X)$, in the closed channel with no coupling,

$$\left(\frac{d^2}{dX^2} + \epsilon_0 - V_2(X)\right)\varphi_0(X) = 0. \tag{2.10}$$

Combining (2.8) with (2.10) one is able to determine the value of λ in terms of the potentials and energies as well as to construct the asymptotic behaviour of open channel wavefunction,

$$\varphi_1(X) = \frac{1}{\sqrt{\pi k_1}}\sin\left(k_1 X + \gamma + \delta\right) \qquad \text{where} \qquad \epsilon_i = k_i^2. \tag{2.11}$$

The phase γ is basically a constant determined by the potential $V_1(X)$ and depending only very weakly on the energy level ϵ_1. The phase δ on the other hand has a more dramatic dependence on the energy levels,

$$\delta = -\arctan\left(\frac{\pi}{2}\frac{\Gamma}{\epsilon_2 - \epsilon_0 - \Delta}\right), \tag{2.12}$$

where Γ, the decay width, is specified by the interaction potentials with, again, an insignificant dependence on ϵ_1 when compared to the pole structure above. The position of the poles defines the resonance, $\epsilon_R = \epsilon_0 + \Delta$, around which the phase undergoes abrupt change. Because Δ is a transition element involving the interaction potentials, it is in general complex, attenuating the wavefunction (2.11). This process of gain and loss can be described by an effective Hamiltonian with complex eigenvalues, consequently non-Hermitian.

Dissipation then is shown to be a common feature in realistic problems not only at the classical level but also at the quantum level. Metastable states, those in equilibrium but susceptible to fall into lower-energy states with only slight interaction, for instance, were

considered already in [36, 37]. Thus, dissipative phenomena constitute a very important aspect of non-Hermitian quantum systems and these concepts can hardly be dissociated. But the physical description of systems by Hamiltonians which are not self-adjoint go beyond the scope of open systems.

Non-Hermitian Hamiltonians can be used as well to formulate a consistent conservative theory, with real energy and unitary evolution. As we shall see, the equivalence established between Hermitian and non-Hermitian Hamiltonians provides a mechanism to allow for considerable simplification when dealing with a certain problem by changing the Hilbert space metric. Although quantum models with real energies had been discussed in the literature from time to time, e.g. [38, 39, 40, 41, 42, 43, 44, 45, 7, 8, 46], it was only with the introduction of $\mathcal{PT}$-symmetry [9] to explain this occurrence that more articulate investigations on this theme came to life.

2.2 The role of $\mathcal{PT}$-symmetry in Physics

We start by presenting an important concept permeated in the whole of this work and which serves as a background in our investigations: $\mathcal{PT}$-symmetry. The study of $\mathcal{PT}$-symmetry is closely related to the desire of describing non-Hermitian quantum theories in a consistent framework. This symmetry is associated to systems which are invariant under simultaneous parity $\mathcal{P}$ and time reversal $\mathcal{T}$ operations, whence the name. Operationally these operators can be characterized by their action on elements of the quantum description. Some of the simplest fundamental objects are the coordinate and momentum operators, $\hat{x}$ and $\hat{p}$, respectively. The result of $\mathcal{PT}$ on these operators is given by:

$$\begin{aligned} \mathcal{P}\,\hat{x}\,\mathcal{P}^{-1} &= -\hat{x}\,, \quad \mathcal{P}\,\hat{p}\,\mathcal{P}^{-1} = -\hat{p}, \\ \mathcal{T}\,\hat{x}\,\mathcal{T}^{-1} &= \hat{x}\,, \qquad \mathcal{T}\,\hat{p}\,\mathcal{T}^{-1} = -\hat{p}, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \mathcal{T}^2 &= \mathbb{1}\,, \quad \mathcal{T}^{-1} = \mathcal{T} \\ \mathcal{P}^2 &= \mathbb{1}\,, \quad \mathcal{P}^{-1} = \mathcal{P}. \end{aligned} \tag{2.14}$$

The combined action of $\mathcal{P}$ and $\mathcal{T}$ can then be composed with the help of their commutativity property,

$$[\mathcal{P}, \mathcal{T}] = 0. \tag{2.15}$$

But they are not yet fully determined and to complete the characterization one must remember that the uncertainty principle, a central concept in the quantum world, implies

the non-commutativity of the $\hat{x}$ and $\hat{p}$ operators, representing generators of the Heisenberg algebra,

$$[\hat{x}, \hat{p}\,] \equiv \hat{x}\hat{p} - \hat{p}\,\hat{x} = \imath\hbar\mathbb{1}, \tag{2.16}$$

with $\hbar$ being the angular version of *Planck's constant*[2], $\hbar \equiv \frac{h}{2\pi}$, and $\mathbb{1}$ the identity operator. The requirement that the expression above (2.16) is $\mathcal{PT}$-symmetric indicates key properties in $\mathcal{PT}$-symmetric quantum mechanics, namely the *linear* and *anti-linear* nature of $\mathcal{P}$ and $\mathcal{T}$. The invariance of (1.3) under time-reversal leads to the choice that $\mathcal{T}$ should be the anti-linear operator,

$$\mathcal{P}\,\imath\,\mathcal{P}^{-1} = \imath \qquad \text{and} \qquad \mathcal{T}\,\imath\,\mathcal{T}^{-1} = -\imath. \tag{2.17}$$

Therefore,

$$\mathcal{PT}: \quad \hat{p} \to \hat{p}\,, \quad \hat{x} \to -\hat{x}\,, \quad \imath \to -\imath\,. \tag{2.18}$$

The importance of $\mathcal{PT}$-symmetry follows from the work of Bender and Boettcher [9], where a whole new class of non-Hermitian but $\mathcal{PT}$-symmetric Hamiltonians with real eigenvalues were introduced, namely

$$H = \hat{p}^2 + m^2\hat{x}^2 - (\imath\hat{x})^N \qquad \text{with} \;\; N \in \mathbb{R}, \tag{2.19}$$

which was actually motivated by the conjecture by Bessis and Zinn-Justin that the spectrum of the cubic Hamiltonian $H = p^2 + \imath x^3$ might be real [47].

An important and very familiar property of Hermitian operators is that their eigenvalues are always real, which means that complex eigenvalues can only arise in non-Hermitian Hamiltonians. One sees an incompatibility between Hermitian operators and complex eigenvalues, but no such statement relating non-Hermitian operators and real eigenvalues can be made. That is exactly what became evident in [9]. There it was shown the existence of a wide range in the parameter space ($N \geq 2$) for which the spectra of the Hamiltonians in (2.19) are real and positive, despite being associated to non-Hermitian operators.

As argued in section 2.1, non-Hermitian Hamiltonians have been treated in the literature often in the description of dissipative systems, with decaying wavefunctions and complex eigenvalues coming from effective Hamiltonians. The presence of unstable particles with decaying probability of detection announces the violation of an important property of a complete quantum theory: conservation of probability density, or equivalently, unitarity of the evolution operator. For this reason, non-Hermitian models are frequently

[2]Henceforth, throughout this work we will consider $\hbar = 1$.

associated to incomplete portrayals of a larger Hermitian setup. However, non-Hermitian Hamiltonians might in many cases be regarded as part of a fundamental description of physical phenomena, with real eigenvalues and unitary time evolution.

The results in [9] had a big impact in the theoretical physics community. Although until that point many examples of non-Hermitian systems whose energies were believed to be real had appeared in the literature, in a vast range of fields, the interest in comprehending and formulating a common framework for the unrelated existing examples had not yet been ignited. After its publication in 1998, scientific effort has been put together in order to clarify the description of quantum physics by non-Hermitian operators and to abandon somewhat widespread myths and misperceptions regarding the (un)acceptability of non-Hermitian systems as a description of fundamental theories. In [48] for example one finds the following comments extracted from [49]: "A non-Hermitian Hamiltonian is unacceptable partly because it may lead to complex energy eigenvalues, but chiefly because it implies a non-unitary S matrix, which fails to conserve probability and makes a hash of the physical interpretation".

The observation of real spectra in non-Hermitian Hamiltonians can be found already in the study of Reggeon field theory, for instance, [40, 41, 42]. Even examples closely related to (2.19) had been analyzed earlier: anharmonic oscillators with imaginary cubic self-interaction were investigated in [43] whereas in [44], the $-x^4$ potential. The publication of [18] also represented an important attempt in recent theory to attribute meaning to non-Hermitian operators in physics. Such systems continued to appear in different areas, establishing for example a real mass spectrum for affine Toda field theories [7, 8] and a connection between field theories and quantum spin chains [46]. Also integrable quantum spin chains described by Hamiltonians which are not self-adjoint were first studied by von Gehlen [45].

However their non-Hermicity remained as a secondary remark. An interesting situation where the non-Hermiticity had been observed is the Lee model. There exists a regime for which ghost states appear, that is, states with negative norm, and this is related to the need of a change in the metric used to approach this problem [50]. This general question of indefinite metric can be traced back to works of Dirac [38] and Pauli [39] more than six decades ago. In relatively old mathematical literature one also finds discussions about changes of metric in order to establish the reality of the spectrum of operators, e.g. [51, 52, 53], so that the recent activity puts an end to this long standing hiatus in this

area.

The systematic study and revival of such type of Hamiltonians, initiated roughly ten years ago, culminated with a formidable amount of contributions towards a better understanding of them, e.g, [54, 17, 55, 48, 56], as well as a periodic international conferences programme [57, 58, 59, 60, 61, 62, 63, 64]. More recently, experimental efforts have also been put forward in the area of $\mathcal{PT}$-symmetric quantum mechanics in order to clarify the nature of such problem [20, 21].

As discussed above, the presence of $\mathcal{PT}$-symmetry in a Hamiltonian theory defined by

$$H|\psi_n\rangle = E_n|\psi_n\rangle \tag{2.20}$$

provides a good indication that the theory admits a real spectrum. This happens because if both of the following expressions are valid,

$$[H, \mathcal{PT}] = 0 \qquad \text{and} \qquad \mathcal{PT}|\psi_n\rangle = |\psi_n\rangle, \tag{2.21}$$

then one has, due to the anti-linearity of $\mathcal{PT}$,

$$E_n|\psi_n\rangle = H|\psi_n\rangle = H\mathcal{PT}|\psi_n\rangle = \mathcal{PT}H|\psi_n\rangle = E_n^*\mathcal{PT}|\psi_n\rangle = E_n^*|\psi_n\rangle. \tag{2.22}$$

Unfortunately, the same anti-linearity implies that, although H and $\mathcal{PT}$ commute, the second property above (2.21) does not necessarily hold. In the case where the eigenstates of H are also invariant under $\mathcal{PT}$, and (2.21) is valid, situation known as *unbroken $\mathcal{PT}$-symmetry*, the reality of the eigenvalues is guaranteed. Shortly after the introduction of $\mathcal{PT}$-symmetry, the behaviour of classical models governed by $(\imath\hat{x})^\epsilon$-potential were studied in [54]. In a similar way as $\mathcal{PT}$-symmetry can be imposed in a quantum system by the requirement (2.21) one can have classical Hamiltonian functions invariant under the linear parity action and the anti-linear time reversal operation. In this situation, we have [31]

$$\begin{aligned} E &= \int_{-a}^{a} dx\, \mathcal{H}\left[u(x,t)\right] = -\int_{a}^{-a} dx\, \mathcal{H}\left[u(-x,t)\right] = \\ &= \int_{-a}^{a} dx\, \mathcal{H}\left[u(-x,t)\right] = \int_{-a}^{a} dx\, \mathcal{H}^*\left[u(x,t)\right] = \\ &= E^*, \end{aligned} \tag{2.23}$$

so that the energies of classical $\mathcal{PT}$-symmetric systems are real if the field configuration satisfy $\mathcal{H}\left[u(-x,t)\right] = \mathcal{H}^*\left[u(x,t)\right]$. This can only be checked once the solution $u(x,t)$ is constructed and it suggests the existence of broken and unbroken $\mathcal{PT}$-symmetry for classical models. Actually, it has been found [66, 67] that in some complexified dynamical

systems the breaking of $\mathcal{PT}$-symmetry can be felt as the corresponding real energy classical trajectories, which are closed and periodic in the unbroken phase, become unconfined and eventually run off to infinity.

Quantum theories present a further difficulty, which is the need to have unitary time evolution. This feature is recovered in $\mathcal{PT}$-symmetric non-Hermitian quantum mechanics by the introduction of a new scalar product for the wavefunctions, denoted by the so-called $\mathcal{CPT}$ inner product and defined by

$$\langle \xi | \psi \rangle^{\mathcal{CPT}} \equiv \int dx \; \xi^{\mathcal{CPT}}(x) \; \psi(x), \tag{2.24}$$

where

$$\xi^{\mathcal{CPT}} \equiv \int dy \; \mathcal{C}(x,y) \; \xi^*(-y), \tag{2.25}$$

is defined in terms of the $\mathcal{C}$-operator. The latter can be represented as a summation over the eigenfunctions $\phi_n(x)$ of the Hamiltonian in question,

$$\mathcal{C}(x,y) = \sum_{n=0}^{\infty} \phi_n(x)\phi_n(y), \tag{2.26}$$

satisfying

$$[\mathcal{C}, H] = 0 \; , \qquad [\mathcal{C}, \mathcal{PT}] = 0 \; , \qquad \mathcal{C}^2 = \mathbb{1} \; . \tag{2.27}$$

The latter property, that the square of $\mathcal{C}$ is the identity operator, served as a motivation to denote this as $\mathcal{C}$-operator because the charge conjugation operator in quantum field theory also satisfies it. However, there is no direct interpretation of (2.26) as the conjugation of any charge. Besides, in quantum field theory $[\mathcal{CPT}, H] = 0$ has to hold but not $[\mathcal{C}, H] = 0$ separately.

A better understanding of the change of metric in (2.24) will be provided in the next section. Before doing so we analyze an important example of non-Hermitian models possessing real eigenvalues.

2.2.1 A family of non-Hermitian Hamiltonians with real spectra

The analytic continuation of real physical systems into the complex plane is a principle which has turned out to be very fruitful, since many new features can be revealed in this manner which might otherwise be undetected. A famous and already classical example, proposed more than half a century ago, is for instance Heisenberg's programme of the analytic S-matrix [65].

There are two fundamentally different possibilities to view complex systems: one may either regard the complexified version just as a broader framework, as in the spirit of the analytic S-matrix, and restrict to the real case in order to describe the underlying physics or alternatively one may try to give a direct physical meaning to the complex models. With the former in mind, consider the following Hamiltonian

$$H = \hat{p}^2 + V(\hat{x}) \tag{2.28}$$

with the usual representation of the momentum operator in the coordinate representation $\hat{p} = -\imath\frac{d}{dx}$. The Schrödinger equation, $H\ \psi(x) = E\ \psi(x)$ associated to this problem is usually defined on the real axis,

$$-\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)\ , \qquad x \in \mathbb{R}. \tag{2.29}$$

However, as we introduce a complex potential such as [54]

$$V(z) = (\imath z)^\epsilon \qquad \text{with } \epsilon \in \mathbb{R}, \tag{2.30}$$

then the generalization of the problem involves considering wavefunctions defined on complex contours in the Argand-Gauss plane, $x \mapsto z = x + \imath y = re^{\imath\theta} \in \mathbb{C}$. This detachment away from the real line comes along with the need to maintain physical boundary conditions in the problem. The spectra of differential eigenvalue equations depend naturally on boundary conditions and consequently on the choice of the domain. One can use the desired vanishing of wavefunctions at infinite distances to select appropriate domains for the contours. For instance, assuming $\psi(z)$ to decay exponentially for $|z| \to \infty$, we can use $\psi(z) = e^{\varphi(z)}$ in the Schrödinger eigenvalue equation, in a WKB approximation approach. In the asymptotic limits with $\varphi''(z) \ll \varphi'(z)^2$, the dominant contributions in (2.29) lead to the equation

$$\varphi'(z)^2 - (\imath z)^\epsilon = 0\ , \qquad \text{for } |z| \to \infty, \tag{2.31}$$

which implies $\varphi(z) \sim \frac{2}{\epsilon+1}(\imath z)^{1+\frac{\epsilon}{2}} = \frac{2}{\epsilon+1}(\imath r e^{\imath\theta})^{1+\frac{\epsilon}{2}}$ asymptotically. As we want $\psi(z) \to 0$ for $|z| \to \infty$ we must require that $\mathfrak{Re}[\varphi(z)] < 0$, or equivalently, that the contour lies asymptotically within the wedges

$$\mathcal{W}_L(\epsilon) = \left\{\theta \mid -\frac{\pi}{2}\frac{\epsilon+8}{\epsilon+2} < \theta < -\frac{\pi}{2}\frac{\epsilon+4}{\epsilon+2}\right\}, \tag{2.32}$$

$$\mathcal{W}_R(\epsilon) = \left\{\theta \mid -\frac{\pi}{2}\frac{\epsilon}{\epsilon+2} < \theta < -\frac{\pi}{2}\frac{\epsilon-4}{\epsilon+2}\right\}, \tag{2.33}$$

known as Stokes sectors, bounded by the so-called *Stokes lines*. As the parameter ϵ increases, the regions defined by the wedges become more restricted. In fact, for the first

three values, covering for instance the usual harmonic oscillator and the imaginary cubic potential, the real axis is included in the region where the contour is defined but after the limiting case $\epsilon = 4$ the contours must be in the complex plane.

Taking into consideration these complex contours, Bender and Boettcher showed in [9] that this whole family of Hamiltonians admit real eigenenergies, as depicted by the figure 2.1, extracted from the referred work. From the infinite possible contours, the ones used in [9] are the lines where the wavefunctions vanish most rapidly, which in this particular case are in the centres of the wedges, and are denoted as *anti-Stokes lines*.

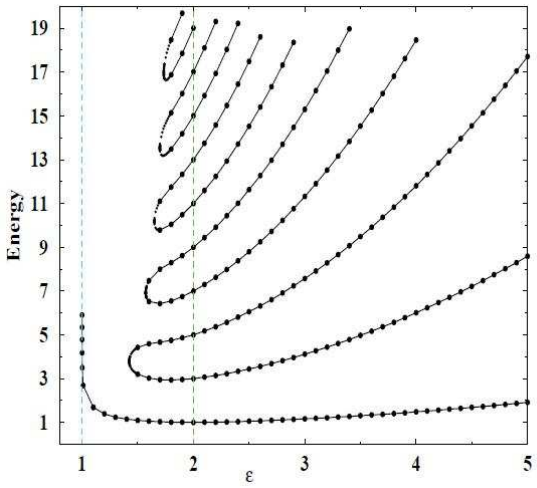

Figure 2.1: Qualitative behaviour of the energy spectra associated to the Hamiltonians $H = \hat{p}^2 - (i\hat{x})^\epsilon$ in terms of the parameter ϵ, extracted from [9].

The analysis carried out by the authors in [9] is separated in three regimes, depending on the value of ϵ:

i) For $\epsilon \geq 2$ all eigenvalues are real and positive, describing monotonically increasing functions with respect to the value of ϵ for each level and having the harmonic oscillator as a limit.

ii) Next comes a region where the first lower energy (energies) is (are) real but the higher excited levels become complex a neighbouring levels merge together in *exceptional* or *diabolical* points, the former being points where not only the eigenvalues but also the eigenvectors coalesce forming a Jordan block whereas for the latter the corresponding eigenvectors do not coincide and remain linearly independent. In this region the more excited states will have complex eigenvalues but there will be a cutoff energy below which

the energies will be real, so that at low energies the system behaves as nondissipative.

iii) Finally, for $\epsilon < 1$, the potential ceases to be confining and one does not look for bound states.

In terms of the $\mathcal{PT}$ symmetry introduced above, one can say that in the latter region this symmetry is completely broken, with no $\mathcal{PT}$-invariant eigenstate. For $\epsilon \geq 2$ the symmetry is unbroken, or exact, for all states, whereas in the intermediate region some eigenstates are $\mathcal{PT}$-symmetric and some are not. The cubic $\epsilon = 3$ non-Hermitian Hamiltonian [68, 69, 70] could so far be investigated perturbatively only. When it comes to exact results for this family the simplest known example consists of $\epsilon = 4$. An equivalent Hermitian Hamiltonian for the $-z^4$-potential,

$$H^{\epsilon=4} = p_z^2 - z^4, \tag{2.34}$$

was constructed in [71] by considering the following contour

$$z(x) = -2\imath\sqrt{1+\imath x}, \tag{2.35}$$

which behaves asymptotically as $z \sim e^{-\frac{\pi}{4}\imath} \in \mathcal{W}_R(4)$ and $z \sim e^{-\frac{3\pi}{4}\imath} \in \mathcal{W}_L(4)$, transforming the Hamiltonian (2.34) into

$$H^{\epsilon=4} = p_x^2 + \frac{1}{2}p_x + 16x^2 - 16 + \imath\left(xp^2 - 32x\right). \tag{2.36}$$

In [71] it was established that this Hamiltonian operator, with boundary conditions defined on (2.35) is equivalent to the following Hamiltonian defined on the real line

$$h^{\epsilon=4} = \frac{1}{64}p^4 + \frac{1}{2}p + 16x^2. \tag{2.37}$$

Due to the Hermiticity of this operator in the usual sense, the reality of the spectrum of (2.36) is guaranteed in a very elegant way. The original Hamiltonian, with potential $V(x) = -(\imath x)^\epsilon$ was already Hermitian for $\epsilon = 4$ but the implementation of the meaningful boundary conditions transformed it, so that the eigenvalues are bounded below. For other values of ϵ, the Hamiltonian will be non-Hermitian from the beginning but a mapping into a Hermitian partner as the one constructed above could still be possible. This operator just described, however, remains an exception since in many cases of interest it has not been possible to construct exact transformations between non-Hermitian Hamiltonians and isospectral Hermitian counterparts, with real eigenvalues.

Although perturbation theory can be applied, an exact but more elaborate proof of the reality of the spectrum of this family of quantum systems came a few years later in

[72, 73] with the use of an interesting equivalence between ordinary differential equations, such as the Schrödinger equation, and quantum integrable models [74, 75]. In some cases the spectral problem of spin chains are very well understood, from Bethe ansatz equations e.g., and the referred correspondence allows one to infer spectral information regarding differential equations.

2.3 Non-Hermitian Hamiltonians as fundamental theories

At this point it is clear that non-Hermitian Hamiltonians should play a role in the formulation of complete and fundamental quantum theories. We have already discussed the mechanism provided by $\mathcal{PT}$-symmetric quantum mechanics to rule out complex eigenvalues in non-Hermitian Hamiltonians and its key importance in the development of the subject. Regardless of the very physical interpretation of $\mathcal{PT}$ as opposed to the rather mathematical requirement of Hermiticity for observables, one must say that the parity and time reversal operators can have extended meanings. This leads to a deeper understanding of the central property of $\mathcal{PT}$-symmetry as attributed to its anti-linear nature. For this reason we try to establish in the following sections a natural introduction of non-Hermitian operators with real eigenvalues in the formalism of quantum mechanics.

2.3.1 Anti-unitary symmetries in Quantum Mechanics

As a consequence of properties **P1** and **P2** presented in the previous chapter, albeit being linear with respect to the vector space, the inner product is anti-linear with respect to the dual vector space: $\langle c_1\psi_1+c_2\psi_2|\xi\rangle = c_1^*\langle\psi_1|\xi\rangle+c_2^*\langle\psi_2|\xi\rangle$. Linearity and anti-linearity are closely related to unitary and anti-unitary transformations: whereas unitary operators are linear, anti-unitary operators are anti-linear. Because quantum mechanics is invariant under an operation O, acting on the states, which preserves the norm of scalar product, or equivalently, the probability density,

$$\langle\xi|O^\dagger O|\psi\rangle\langle\psi|O^\dagger O|\xi\rangle = \langle\xi|\psi\rangle\langle\psi|\xi\rangle. \tag{2.38}$$

it can be shown that this symmetry can either satisfy

$$\langle\xi|O^\dagger O|\psi\rangle = \langle\xi|\psi\rangle e^{\imath\phi}, \tag{2.39}$$

$$\langle c_1\,\xi|O^\dagger O|c_2\,\psi\rangle = \langle c_1\,\xi|c_2\,\psi\rangle e^{\imath\phi} = c_1^*c_2\langle\xi|\psi\rangle e^{\imath\phi} = c_1^*c_2\langle\xi|O^\dagger O|\psi\rangle, \tag{2.40}$$

$$\therefore O|c_i\,\psi\rangle = c_i\,O|\psi\rangle, \tag{2.41}$$

or

$$\begin{aligned} \langle \xi | O^\dagger O | \psi \rangle &= \langle \psi | \xi \rangle e^{\imath\varphi}, && (2.42) \\ \langle c_1\ \xi | O^\dagger O | c_2\ \psi \rangle &= \langle c_2\ \psi | c_1\ \xi \rangle e^{\imath\varphi} = c_2^* c_1 \langle \psi | \xi \rangle e^{\imath\varphi} = c_2^* c_1 \langle \xi | O^\dagger O | \psi \rangle, && (2.43) \\ \therefore O | c_i\ \psi \rangle &= c_i^*\ O | \psi \rangle. && (2.44) \end{aligned}$$

Here we have used arbitrary amplitude coefficients $c_i \in \mathbb{C}$ and phases $\phi, \varphi \in \mathbb{R}$. Whereas in the former situation O acts as a linear operator (2.41), in the latter it acts as an anti-linear operator (2.44).

If the phases vanish $\phi, \varphi = 0$ then the first set of transformations is **unitary** $O \equiv U$ leaving the scalar product invariant,

$$\langle \xi | U^\dagger U | \psi \rangle = \langle \xi | \psi \rangle, \tag{2.45}$$

and the second one $O \equiv A$, denoted **anti-unitary**, maps the scalar product into its complex conjugate

$$\langle \xi | A^\dagger A | \psi \rangle = \langle \psi | \xi \rangle = \langle \xi | \psi \rangle^*, \tag{2.46}$$

in such a way that A^2 behaves as a unitary operator. Anti-unitary transformations may then be formally decomposed in a unitary transformation U followed by a complex conjugation K:

$$A\left(c_i | \psi_i \rangle\right) = UK\left(c_i | \psi_i \rangle\right) = c_i^* UK | \psi_i \rangle = c_i^* A | \psi_i \rangle. \tag{2.47}$$

Wigner explored the features of anti-unitary operators in [15, 16] to establish the existence of invariant vectors under anti-unitary transformations and as noted in [76], it is the anti-linearity nature of the operators that explains the reality of spectra in $\mathcal{PT}$-symmetric quantum mechanics, as already indicated in (2.22).

Considering A^2 to possess a discrete spectrum[3] we may write

$$A^2 | \zeta(\omega, n) \rangle = \omega | \zeta(\omega, n) \rangle, \tag{2.48}$$

where $||\omega||^2 = 1$ because A^2 is a unitary operator and n denotes possible degeneracies. In fact, both unitary and anti-unitary operators have eigenvalues with unit modulus,

$$\begin{aligned} U|u_i\rangle = \upsilon_i |u_i\rangle \implies \langle u_i | u_i \rangle = \langle u_i | U^\dagger U | u_i \rangle = ||\upsilon_i||^2 \langle u_i | u_i \rangle \implies ||\upsilon_i||^2 = 1, \\ A|a_i\rangle = \alpha_i |a_i\rangle \implies \langle a_i | a_i \rangle = \langle a_i | A^\dagger A | a_i \rangle = ||\alpha_i||^2 \langle a_i | a_i \rangle \implies ||\alpha_i||^2 = 1. \end{aligned} \tag{2.49}$$

[3]The continuous spectrum analysis, attributed to Wigner, is alleged to present no major difference.

As a consequence of (2.48) we conclude that if $|\zeta(\omega,n)\rangle$ is an eigenstate of A^2 with eigenvalue ω, then $A|\zeta(\omega,n)\rangle$ is also an eigenstate of A^2 but with eigenvalue ω^*,

$$A^2\left(A|\zeta(\omega,n)\rangle\right)=\omega^*\left(A|\zeta(\omega,n)\rangle\right), \tag{2.50}$$

so that it is clear that $A|\zeta(\omega,n)\rangle$ is proportional to $|\zeta(\omega^*,n)\rangle$, or more precisely,

$$|\zeta(\omega^*,n)\rangle=\omega^{\frac{1}{2}}A|\zeta(\omega,n)\rangle \qquad \text{whereas} \qquad |\zeta(\omega,n)\rangle=(\omega^*)^{\frac{1}{2}}A|\zeta(\omega^*,n)\rangle, \tag{2.51}$$

and

$$A|\zeta(\omega,n)\rangle=(\omega^*)^{\frac{1}{2}}|\zeta(\omega^*,n)\rangle \qquad \text{whereas} \qquad A|\zeta(\omega^*,n)\rangle=\omega^{\frac{1}{2}}|\zeta(\omega,n)\rangle. \tag{2.52}$$

Therefore if $\omega\neq 1$ the states $|\zeta(\omega,n)\rangle$ and $A|\zeta(\omega,n)\rangle$ are orthogonal,

$$(\omega^*-1)\langle\zeta(\omega,n)|A|\zeta(\omega,n)\rangle=0, \tag{2.53}$$

and the dimension of the representation of A in this case is 2 in each of the n degenerate subspaces. From these expressions we see that for the two-dimensional representation obtained for $\omega\neq 1$ there is no linear combination of the so-called flipping states which would be invariant under the action of A. This would only be possible if $\omega=1$ because in this case $(\omega)^{\frac{1}{2}}$ would be real:

$A\left[\sigma_1\zeta(\omega,n)+\sigma_2\zeta(\omega^*,n)\right]=\sigma\left[\sigma_1\zeta(\omega,n)+\sigma_2\zeta(\omega^*,n)\right] \quad \Longleftrightarrow \quad \omega=1,\sigma_2=\sigma^*\sigma_1^*.$

Therefore, if $\omega=1$ we can have a one-dimensional representation of A, but if $\omega\neq 1$ then the irreducible representation is two-dimensional with two sets of vectors appearing: one formed by the invariant vectors under the action of the anti-unitary operator,

$$A|\zeta(1,n)\rangle=|\zeta(1,n)\rangle, \tag{2.54}$$

and other composed of those vectors which are not invariant,

$$A|\zeta(\omega,n)\rangle=(\omega^*)^{\frac{1}{2}}|\zeta(\omega^*,n)\rangle \qquad \text{with} \qquad \omega\neq 1. \tag{2.55}$$

These vectors can be used, together with their eigenvalues, $\omega_1,\omega_1^*,\omega_2,\omega_2^*,...$, to characterize the anti-unitary operator. Note that when $\omega=-1$ the states denoted by $|\zeta(-1,n)\rangle$ and $|\zeta(-1^*,n)\rangle$ are not equivalent but orthogonal instead.

Decomposing a state vector in the basis of the eigenstates of A^2

$$|\psi\rangle=\sum_n a(1,n)|\zeta(1,n)\rangle+\sum_{\omega,n}a(\omega,n)|\zeta(\omega,n)\rangle, \tag{2.56}$$

so that the action of the anti-linear operator results in

$$A|\psi\rangle = \sum_n a^*(1,n)|\zeta(1,n)\rangle + \sum_{\omega,n} a^*(\omega^*,n)\omega^{\frac{1}{2}}|\zeta(\omega,n)\rangle. \tag{2.57}$$

Imposing (2.56) to be invariant under the action of A, comparison with (2.57) provides us with the information that unless $\omega = 1$ we must have $a(\omega, n) = 0$. Thus, if the symmetry operator is anti-unitary, then the invariant states can be described by the invariant vectors $|\zeta(1,n)\rangle$.

When $\omega = 1$ the state $A|\zeta(1,n)\rangle$ might be a multiple of $|\zeta(1,n)\rangle$, in which case the representation is one-dimensional, or not, when we have a two-dimensional representation which is reducible. If we have states defining a one-dimensional representation

$$A|\psi\rangle = \lambda|\psi\rangle, \tag{2.58}$$

we introduce a new vector

$$|\tilde{\psi}\rangle = \lambda^{\frac{1}{2}}|\psi\rangle, \tag{2.59}$$

from which one can establish that

$$\begin{aligned} A|\tilde{\psi}\rangle &= A\left[(\lambda)^{\frac{1}{2}}|\psi\rangle\right] = (\lambda^{\frac{1}{2}})^* A|\psi\rangle = (\lambda^{\frac{1}{2}})^*\lambda|\psi\rangle = \lambda^{\frac{1}{2}}|\psi\rangle, \\ A|\tilde{\psi}\rangle &= |\tilde{\psi}\rangle, \end{aligned} \tag{2.60}$$

in a way that, due to the quantum equivalence of the physical states $|\psi\rangle$ and $|\tilde{\psi}\rangle = \lambda^{\frac{1}{2}}|\psi\rangle$ differing only by a phase factor, it is enough to consider vectors obeying (2.54). This means, for instance, there is no difference between anti-unitary symmetry ($\lambda = 1$) and anti-unitary anti-symmetry ($\lambda = -1$) in this sense. Therefore it is sufficient to consider:

Action of A^2	Action of A
$A^2\|\zeta(1,n)\rangle = \|\zeta(1,n)\rangle$	$A\|\zeta(1,n)\rangle = \|\zeta(1,n)\rangle$

Table 2.1: One-dimensional representation of the anti-unitary operator A.

On the other hand, the different two-dimensional representations can be separated in three different groups in the following table, as in [76]. But, as will be seen, these will not be as useful as (2.54) for the purpose of assuring the reality of the spectra in operators possessing an anti-unitary symmetry A.

Action of A^2	Action of A
$A^2\|\zeta(\omega,n)\rangle = \omega\|\zeta(\omega,n)\rangle$	$A\|\zeta(\omega,n)\rangle = (\omega^*)^{\frac{1}{2}}\|\zeta(\omega^*,n)\rangle$
$A^2\|\zeta(\omega^*,n)\rangle = \omega^*\|\zeta(\omega^*,n)\rangle$	$A\|\zeta(\omega^*,n)\rangle = (\omega)^{\frac{1}{2}}\|\zeta(\omega,n)\rangle$
$A^2\|\zeta(-1,n)\rangle = -\|\zeta(-1,n)\rangle$	$A\|\zeta(-1,n)\rangle = (-1^*)^{\frac{1}{2}}\|\zeta(-1^*,n)\rangle$
$A^2\|\zeta(-1^*,n)\rangle = -\|\zeta(-1^*,n)\rangle$	$A\|\zeta(-1^*,n)\rangle = (-1)^{\frac{1}{2}}\|\zeta(-1,n)\rangle$
$A^2\|\zeta(1,n)\rangle = \|\zeta(1,n)\rangle$	$A\|\zeta(1,n)\rangle = \|\zeta(1^*,n)\rangle$
$A^2\|\zeta(1^*,n)\rangle = \|\zeta(1^*,n)\rangle$	$A\|\zeta(1^*,n)\rangle = \|\zeta(1,n)\rangle$

Table 2.2: Two-dimensional representations of the anti-unitary operator A.

Reality of the spectra

Having established the possibility of invariant states under anti-unitary symmetries, we analyze the consequences of such a symmetry A when present in a Hamiltonian system H,

$$[H, A] = 0. \tag{2.61}$$

Assuming both operators to be diagonalizable we expect to find invariant vectors under A,

$$A|\zeta(1,n)\rangle = |\zeta(1,n)\rangle, \tag{2.62}$$

corresponding to the one-dimensional representation of A. If this happens, the system is said to possess an *exact* or *unbroken* anti-unitary symmetry. Then it is possible to diagonalize the Hamiltonian by these same eigenvectors:

$$H|\zeta(1,n)\rangle = E(n)|\zeta(1,n)\rangle \tag{2.63}$$

Combining (2.62) and (2.63), together with the commutativity of H and A, we may write, similarly as in (2.22)

$$\begin{aligned} E(n)|\zeta(1,n)\rangle &= H|\zeta(1,n)\rangle = HA|\zeta(1,n)\rangle = \\ &= AH|\zeta(1,n)\rangle = A\left[E(n)|\zeta(1,n)\rangle\right] = \qquad (2.64) \\ &= E(n)^* A|\zeta(1,n)\rangle = E(n)^*|\zeta(1,n)\rangle, \qquad (2.65) \end{aligned}$$

establishing the reality of the spectrum for any anti-unitary-symmetric operators in its unbroken phase, regardless of its Hermiticity properties. The discussion just presented

generalizes the idea of $\mathcal{PT}$-symmetric quantum mechanics to a broader class of any anti-linear symmetries, of which $\mathcal{PT}$ is just one specific example. But also because of the anti-linearity of A, even if the condition (2.61) holds, the eigenstates of the Hamiltonian will not necessarily be eigenvectors of A. If this is the case,

$$A|\zeta(\omega,n)\rangle \neq |\zeta(\omega,n)\rangle, \tag{2.66}$$

$$H|\zeta(\omega,n)\rangle = E(n)|\zeta(\omega,n)\rangle, \tag{2.67}$$

and we say that the Hamiltonian system is in a broken anti-linear-symmetric phase. Then, $A|\zeta(\omega,n)$ is also an eigenvector of H but associated to the eigenvalue $E(n)^*$,

$$\begin{aligned} HA|\zeta(\omega,n)\rangle &= AH|\zeta(\omega,n)\rangle = A\left[E(n)|\zeta(\omega,n)\rangle\right] = E(n)^* A|\zeta(\omega,n)\rangle, \\ H\left[A|\zeta(\omega,n)\rangle\right] &= E(n)^*\left[A|\zeta(\omega,n)\rangle\right], \end{aligned} \tag{2.68}$$

so that eigenvalues of anti-linear-symmetric operators come in complex conjugate pairs if the eigenvectors are not invariant under its action as well.

In [77] it was observed that if an operator H possesses an anti-linear symmetry (2.61), then

$$\det\left(H - E\mathbb{1}\right) = \det\left(AHA^{-1} - E\mathbb{1}\right) = \det\left(H^* - E\mathbb{1}\right), \tag{2.69}$$

so that H and H^* share the same set of eigenvalues, which means they are either real or come in complex conjugate pairs. It is also argued that such a symmetry implies that the secular equation is real and, furthermore, that if the secular equation is real then the operator is necessarily anti-linear-symmetric. Therefore (2.61) is not only a sufficient but also a necessary condition for the reality of the secular equation. For finite dimensional Hilbert spaces it is shown that it is not possible for a Hamiltonian which does not satisfy (2.61) to posses a real spectra. Such results can actually be extended to non-diagonalizable matrices, composed of Jordan-block submatrices, i.e. those which differ from diagonal block matrices by the presence of nonvanishing elements on the upper-diagonal. Although in these circumstances the number of eigenvectors is smaller than the dimension of the matrix, the off-diagonal entries do not affect the secular equation so that they are also real.

The implementation of anti-linear symmetries in non-Hermitian Hamiltonians can be very useful in many situations when one is interested in working with models presenting real energies. There are nonetheless other techniques to select or construct real spectra theory. In the sequence we present viable operational alternatives which do not rely on the presence of an anti-linear symmetry.

2.3.2 Darboux transformations and superpartner potentials

It is worth to mention the usefulness of *Darboux transformations* in super-symmetric quantum mechanics. They are key ingredients in a framework to construct almost isospectral Hamiltonians from known wavefunctions of given Hermitian Hamiltonians. Supersymmetry was introduced by Gel'fand and Likhtman in 1971 [78] and became a very important idea in fundamental physics, relating bosonic and fermionic states so that both sector can be treated in unified way through string theory, e.g. [79]. This unification of internal degrees of freedom combined with the combined approach towards space and time coordinates boosted generalizations of gravity known as supergravity [80]. But our concerns in this subsection will focus on supersymmetric properties for solving exactly spectral problems and discuss how it can be employed to deal with non-Hermitian Hamiltonians possessing real characteristic energies, especially $\mathcal{PT}$-symmetric ones. With this in mind we now present the main ideas of this approach [81].

Supersymmetry can be constructed by factorizing a Schrödinger operator according to [82]

$$H_1 = -\frac{d^2}{dx^2} + V_1(x) = \mathcal{A}^\dagger \mathcal{A}, \tag{2.70}$$

with generalized bosonic operators in terms of so-called superpotentials $W(x)$

$$\mathcal{A} = \frac{d}{dx} + W(x) \qquad \text{and} \qquad \mathcal{A}^\dagger = -\frac{d}{dx} + W(x) \tag{2.71}$$

satisfying $[\mathcal{A}, \mathcal{A}^\dagger] = 2W'(x)$. The ground state $\psi_0^{(1)}(x)$, with vanishing energy $E_0^{(1)} = 0$ satisfies

$$\mathcal{A}\psi_0^{(1)}(x) = 0 \qquad \text{so that} \qquad W(x) = -\frac{\psi_0'^{(1)}(x)}{\psi_0^{(1)}(x)}. \tag{2.72}$$

This means that once the ground state of a Hamiltonian operator is known the supersymmetric decomposition in terms of $\mathcal{A}$ and $\mathcal{A}^\dagger$ is formally determined. The use of this idea is more interesting when one considers a second Hamiltonian constructed from the previous one according to

$$H_2 = -\frac{d^2}{dx^2} + V_2(x) = \mathcal{A}\,\mathcal{A}^\dagger, \tag{2.73}$$

so that the superpartner potentials, with interactions $V_{1,2}(x) = W(x)^2 \mp W'(x)$, are related through a Darboux transformation. The intertwining properties of the bosonic operators,

$$H_2\,\mathcal{A} = \mathcal{A}\,H_1 \qquad \text{and} \qquad H_1\,\mathcal{A}^\dagger = \mathcal{A}^\dagger\,H_2 \tag{2.74}$$

can then be used to establish that H_1 and H_2 are almost isospectral in the sense that if $\psi_n^{(1)}(x)$ are eigenfunctions of the first Hamiltonian then $\mathcal{A}\psi_n^{(1)}(x)$ are eigenfunctions of the second one with the same energy, except of course the ground state, for which (2.72) holds. Therefore this method allows for the construction of a secondary system sharing eigenstates and eigenvalues,

$$\psi_n^{(1)}(x) = \frac{1}{\sqrt{E_{n-1}^{(1)}}}\mathcal{A}^\dagger\ \psi_{n-1}^{(2)}(x)\ , \qquad \psi_{n-1}^{(2)}(x) = \frac{1}{\sqrt{E_n^{(2)}}}\mathcal{A}\ \psi_n^{(1)}(x)\ , \tag{2.75}$$

and

$$E_{n-1}^{(2)} = E_n^{(1)}, \tag{2.76}$$

with the exception of the ground state unpairing. The vanishing of the ground state is an important ingredient in this construction and if this does not happen, one says supersymmetry is broken, resembling the broken and unbroken phases of $\mathcal{PT}$-symmetry. The existence of these two supersymmetric phases can be interpreted in terms of two nontrivial charge operators,

$$Q = \begin{pmatrix} 0 & 0 \\ \mathcal{A} & 0 \end{pmatrix} \qquad \text{and} \qquad Q^\dagger = \begin{pmatrix} 0 & \mathcal{A}^\dagger \\ 0 & 0 \end{pmatrix}, \tag{2.77}$$

whose action will annihilate the ground state vector

$$|0\rangle = \begin{pmatrix} \psi_0^{(1)}(x) \\ \psi_0^{(2)}(x) \end{pmatrix} \qquad \text{of the Hamiltonian} \qquad H = H_1 \oplus H_2 = \left\{Q, Q^\dagger\right\} \tag{2.78}$$

in both cases if and only if the vacuum state is unpaired and supersymmetry is unbroken. The supersymmetric partners constructed here were based on the knowledge of one ground state but one could also start from any m-th excited state so that there would be $m+1$ unpaired levels.

Supersymmetric quantum mechanics is a very rich area of study and its ideas have been successfully explored in connection with $\mathcal{PT}$-symmetric quantum mechanics [83, 84, 85, 86, 87, 88]. Although the formalism provided by supersymmetry has been combined with non-Hermitian Hamiltonians to construct real isospectral partners, it can be shown that formally diagonalizable operators with real eigenvalues can be mapped into each other via a similarity transformation [89], even for infinite dimensional Hilbert spaces, and that the latter procedure is equivalent to the presence of an anti-linear symmetry [90]. This consequently allows for a general treatment of non-Hermitian Hamiltonians with real eigenvalues by mapping them to Hermitian isospectral partners.

2.3.3 Dyson map and change of metric

The situations where Hamiltonians are not Hermitian but are not describing dissipative systems either can be formulated in a precise and consistent fashion so as to allow for unitary time evolution. The most effective and general way of doing so derives from a mapping of non-Hermitian Hamiltonians onto Hermitian counterparts by the action of a similarity transformation.

First, let us suppose we have a diagonalizable operator, for instance a Hamiltonian H. It is well known that the left- and right-eigenvectors, $\langle\phi_n|$ and $|\varphi_n\rangle$ respectively, are not related by a simple transposition followed by a complex conjugation unless the operator is Hermitian. Instead, these characteristic vectors are essentially different,

$$H|\varphi_n\rangle = E_n|\varphi_n\rangle \qquad \Longrightarrow \qquad \langle\varphi_n|H^\dagger = E_n^*\langle\varphi_n| \tag{2.79}$$

$$\langle\phi_m|H = E_m\langle\phi_m| \qquad \Longrightarrow \qquad H^\dagger|\phi_m\rangle = E_m^*|\phi_m\rangle, \tag{2.80}$$

but if $\langle\phi_n|$ is a left-egienvector of an H associated to a complex eigenvalue E_n then its Hermitian conjugate $|\phi_n\rangle$ can be considered as a right-eigenvector of $H^\dagger$ associated to the eigenvalue E_n^*. Left- and right-eigenvectors only coincide if the operator is self-adjoint, $H^\dagger = H$, and consequently the eigenvalues are real.

Projecting the state on the left side equation in (2.79) onto $\langle\phi_m|$ and acting with $|\varphi_n\rangle$ on the left side equation of (2.80) we obtain

$$\langle\phi_m|H|\varphi_n\rangle = E_n\langle\phi_m|\varphi_n\rangle \tag{2.81}$$

$$\langle\phi_m|H|\varphi_n\rangle = E_m\langle\phi_m|\varphi_n\rangle, \tag{2.82}$$

from the subtraction of which we arrive at the *bi-orthonormality* of the states and the completeness relation of this basis,

$$\langle\phi_m|\varphi_n\rangle = \delta_{mn} \qquad \text{and} \qquad \sum_n |\varphi_n\rangle\langle\phi_n| = \mathbb{1}. \tag{2.83}$$

Also, we may write

$$\langle\phi_m|H|\varphi_n\rangle = E_n\delta_{mn} \qquad \text{and} \qquad \langle\varphi_n|H^\dagger|\phi_m\rangle = E_n^*\delta_{mn}, \tag{2.84}$$

in such a way that if the eigenvalues are real, $E_n^* = E_n$, we have a necessary condition for the reality of the spectrum,

$$\langle\phi_m|H|\varphi_n\rangle = \langle\varphi_n|H^\dagger|\phi_m\rangle, \tag{2.85}$$

the non-Hermitian analogue of the Hermiticity condition. Only if left- and right-eigenstates are identical will the operator be considered Hermitian and $H^\dagger = H$.

Supposing we have a Hermitian Hamiltonian with the same eigenvalues, $E_n^* = E_n$,

$$h|\psi_n\rangle = E_n|\psi_n\rangle, \tag{2.86}$$

we will have an equivalence of the elements

$$\langle\phi_m|H|\varphi_n\rangle = \langle\psi_m|h|\psi_n\rangle = E_n\delta_{mn}. \tag{2.87}$$

Because the Hermitian, $h^\dagger = h$, and non-Hermitian, $H^\dagger \neq H$, Hamiltonians are isospectral we shall assume them to be related via a similarity transformation

$$h = \eta\, H\, \eta^{-1}, \tag{2.88}$$

where η is a linear operator known as a Dyson map [97]. Due to the Hermiticity of (2.88) we can write

$$H^\dagger = \rho\, H\, \rho^{-1}, \tag{2.89}$$

with

$$\rho = \eta^\dagger\eta, \tag{2.90}$$

so that ρ stays invariant up to a unitary transformation in the similarity transformation, $\eta \longrightarrow U\eta$, with $U^\dagger U = \mathbb{1}$. This transformation allows us to relate the eigenvectors of the non-Hermitian operator to those of the Hermitian one,

$$\langle\psi_m|h|\psi_n\rangle = \langle\phi_m|H|\varphi_n\rangle = \langle\varphi_m|\rho\, H|\varphi_n\rangle, \tag{2.91}$$

with

$$|\varphi_n\rangle = \eta^{-1}|\psi_n\rangle \qquad \text{and} \qquad |\phi_n\rangle = \rho|\varphi_n\rangle = \eta^\dagger|\psi_n\rangle. \tag{2.92}$$

The expression (2.91) above indicates the operator H represented in the basis of its eigenvectors can be regarded as Hermitian if one replaces the usual scalar product with the introduction of a new metric ρ :

$$\langle\,\cdot\,|\,\cdot\,\rangle \longrightarrow \langle\,\cdot\,|\,\cdot\,\rangle_\rho \equiv \langle\,\cdot\,|\,\rho\,|\,\cdot\,\rangle. \tag{2.93}$$

The Dyson transformation, not unitary, maps the usual Hilbert space $\mathcal{H}_1$, where the vectors $|\psi_n\rangle$ exist, into a new one $\mathcal{H}_\rho$, where the vectors $|\phi_n\rangle$ and $|\varphi_n\rangle$ are defined.

Using (2.83) we note that the metric can be reconstructed from the left-eigenvectors of the non-Hermitian operator and its inverse, from the right-eigenvectors,

$$\rho = \sum_n |\phi_n\rangle\langle\phi_n| \qquad \text{and} \qquad \rho^{-1} = \sum_n |\varphi_n\rangle\langle\varphi_n|, \tag{2.94}$$

and therefore we notice that the non-uniqueness in specifying the eigenvectors will reflect on the determination of the metric. In [91] there is a report on the ambiguity due to normalization of left- and right-eigenvectors leading to discrepancies in the literature [92, 93].

In order for a metric to be a well-defined metric it needs to be *linear*, *Hermitian*, *invertible*, *positive-definite* and *bounded*. Invertibility is established by the expression (2.94) above and positivity is proven by taking the eigenvalues r_i of the metric $\rho|R_i\rangle = r_i|R_i\rangle$ and showing that they are indeed all positive

$$r_i = \langle R_i|\rho|R_i\rangle = \sum_n \langle R_i|\phi_n\rangle\langle\phi_n|R_i\rangle = \sum_n ||\langle R_i|\phi_n\rangle||^2 > 0 \tag{2.95}$$

because ρ is invertible. Finally, boundedness is a consequence of the Hermiticity of ρ. In fact, the reality of the spectrum of an operator implies the Hermiticity of the associated observable with respect to some metric. For details of the proof of this theorem, see for example [56].

2.3.4 Quasi-Pseudo-Hermiticity

Although (2.85) was obtained as a necessary condition in order for a non-Hermitian Hamiltonian to have real eigenvalues, this is not necessarily sufficient. In fact, writing

$$\langle\phi_m|H|\varphi_n\rangle = \langle\varphi_m|\rho\, H|\varphi_n\rangle = \langle\varphi_n|H^\dagger\, \rho|\varphi_m\rangle = \langle\varphi_n|H^\dagger|\phi_m\rangle, \tag{2.96}$$

tells us only that

$$\rho\, H = H^\dagger\, \rho\ , \tag{2.97}$$

so that the usual Hermiticity condition is recovered as $\rho = \mathbb{1}$.

According to [51], this is not enough to guarantee the reality of the spectrum. Dieudonné starts with symmetrizable operator H, for which by definition exists a Hermitian operator $\rho = \rho^\dagger \neq 0$ such that (2.97) is valid. He then reduces the situation to the case where ρ is positive, i.e., $\langle R_i|\rho|R_i\rangle > 0$. This restricted class of operators contain Hermitian operators and was denoted *quasi-Hermitian*. It is shown that the spectra of such operators need not be real and that $H^\dagger$ need not be quasi-Hermitian. Only if ρ is furthermore assumed to

be invertible there exists an invertible operator $M = M^\dagger > 0$ satisfying $\rho = M^2$ so that (2.97) can be reexpressed as $MHM^{-1} = M^{-1}H^\dagger M$, allowing one to define a Hermitian operator h in terms of the non-Hermitian H,

$$h = MHM^{-1} = h^\dagger, \tag{2.98}$$

assuring the reality of the spectrum of the operators h and H. The difference between (2.88) and (2.98) is that in the latter it was not assumed that ρ was necessarily decomposed in Hermitian factors.

Theorems studied in [52, 53] state that if the operator H satisfy the *pseudo-Hermicity* condtion (2.89) and $0 \notin \mathrm{Cl}(W(\rho))$ [4], then the spectrum of H is real and H is similar to a self-adjoint operator. However these assumptions are not enough to establish the positivity of the metric as one wishes. Thus, to our purposes it is convenient to have the metric ρ as being a linear operator which is also invertible, Hermitian and positive definite.

The introduction of this new metric in order to make Hermitian the operators which are not self-adjoint modifies not only the Hamiltonian operator but all other observables as well. Consider a Hamiltonian system described by the generalized coordinate operators $\hat{x}$ and $\hat{p}$, $H(\hat{x}, \hat{p})$. Because the position and momentum operators are Hermitian with respect to the usual metric, a Hamiltonian $h(\hat{x}, \hat{p})$ can be transformed by an inverse Dyson map into a non-Hermitian operator:

$$H(\hat{x}, \hat{p}) = \eta(\hat{x}, \hat{p})^{-1}\, h(\hat{x}, \hat{p})\, \eta(\hat{x}, \hat{p}) = h(\eta^{-1}\, \hat{x}\, \eta\, , \eta^{-1}\, \hat{p}\, \eta\,). \tag{2.99}$$

Introducing the set of operators $\hat{X}$ and $\hat{P}$ so that

$$\hat{x} = \eta\, \hat{X}\, \eta^{-1} \qquad \text{and} \qquad \hat{p} = \eta\, \hat{P}\, \eta^{-1}\, , \tag{2.100}$$

we note that the non-Hermitian Hamiltonian expressed in terms of the usual observables is the same as the Hermitian Hamiltonian written with the new generalized coordinate operators Q. Naturally, the fundamental canonical commutation relation of quantum mechanics is preserved:

$$[\hat{x}, \hat{p}\,] = [\hat{X}, \hat{P}\,] = \imath, \tag{2.101}$$

Whereas $H, \hat{X}, \hat{P}$ are not Hermitian in $\mathcal{H}_1$, they are Hermitian in $\mathcal{H}_\rho$ and *mutatis mutandi* $h, \hat{x}, \hat{p}$ are only Hermitian in $\mathcal{H}_1$ but not in $\mathcal{H}_\rho$. This implies that the observables

[4] Here $W(\rho) = \{\langle R_i|\rho|R_i\rangle \ / \ ||R_i|| = 1\}$ and Cl denotes *closure* in the sense that if x is a point of closure of a subset of a topological space S if every neighbourhoood of x contains a point of S.

in the new Hilbert space are not $h, \hat{x}, \hat{p}$ but, instead, $H, \hat{X}, \hat{P}$. The equation

$$H(\hat{x}, \hat{p}) = h(\hat{X}, \hat{P}) \tag{2.102}$$

tells us that a non-Hermitian Hamiltonian H can be made Hermitian if we consider the new operators $\hat{X}, \hat{P}$ as the observables, so that our very physical problem will be described by these new operators. All problems can be formulated in $\mathcal{H}_1$ but using some other Hilbert space $\mathcal{H}_\rho$ might make the system easier to solve or even tractable at all as opposed to the formulation in $\mathcal{H}_1$.

Non-Hermitian position and momentum operators are commonly obtained when the commutation relations between them are deformed, leading to minimal uncertainty relations and minimal length at the Planck scale. In [94] it was demonstrated such type of deformed quantum mechanical systems may be treated in a similar framework as we have described.

There is a frequent confusion with regard to the terminology adopted in this framework of constructing a new metric with respect to which the system is Hermitian and the underlying similarity transformation. The following table helps us distinguish between the *pseudo-Hermiticity* condition, see [95, 96, 98], when ρ is linear, invertible and Hermitian but its positivity is not necessarily implied, and the *quasi-Hermiticity* condition [18, 51, 52, 53] when ρ is linear, Hermitian and positive but not necessarily invertible.

	$H^\dagger = \eta^\dagger \eta H (\eta^\dagger \eta)^{-1}$	$H^\dagger \rho = \rho H$	$H^\dagger = \rho H \rho^{-1}$
Hermiticity of ρ	✓	✓	✓
invertibility of ρ	✓	×	✓
positivity of ρ	✓	✓	×
definite metric	guaranteed	guaranteed	not conclusive
spectrum H	real [53]	real or complex [51]	real [53]
terminology for H	quasi-pseudo Hermitian	quasi-Hermiticity [51]	pseudo-Hermiticity [95]

Table 2.3: Differences between pseudo-Hermiticity, quasi-Hermiticity and quasi-pseudo-Hermiticity.

The conclusions that can be drawn from such assumptions are slightly different: whereas pseudo-Hermitian operators must have real eigenvalues, for quasi-Hermitian ones this is

not compulsory; on the other hand, pseudo-Hermiticity may lead to an indefinite metric, contrarily to quasi-Hermiticity, for which the existence of a positive definite metric is guaranteed. But for the construction of a consistent quantum framework, having a real spectrum is not enough; the metric defining the Hilbert space must also be appropriate. Thus, in order not to run any risks regarding the possibility that the eigenvalues might not be real or that the metric might not be appropriate, we should combine both pseudo- and quasi-Hermiticity concepts to obtain a conclusive formulation.

Introducing the $\mathcal{C}$-operator (2.26) in terms of the parity operator $\mathcal{P}$ and the metric ρ according to

$$\mathcal{C} = \mathcal{P}\rho \qquad \text{so that} \qquad \rho = \mathcal{P}\,\mathcal{C} \tag{2.103}$$

there is an equivalence between the following relations

$$H^\dagger = \rho\, H\, \rho^{-1} \qquad \Longleftrightarrow \qquad [H, \mathcal{C}] = 0 \tag{2.104}$$

Furthermore, if the Hamiltonian is said to be $\mathcal{PT}$-symmetric, meaning $[H, \mathcal{PT}] = 0$, we also have that both the $\mathcal{C}$-operator and the metric operator ρ are $\mathcal{PT}$-symmetric: $[\mathcal{C}, \mathcal{PT}] = [\rho, \mathcal{PT}] = 0$. Although the reality of the spectrum of a Hamiltonian would be guaranteed by the existence of an anti-linear symmetry of this operator, such as $\mathcal{PT}$ for instance, the introduction of the $\mathcal{C}$-operator brings unnecessary constraints, such as $\mathcal{C}^2 = \mathbb{1}$.

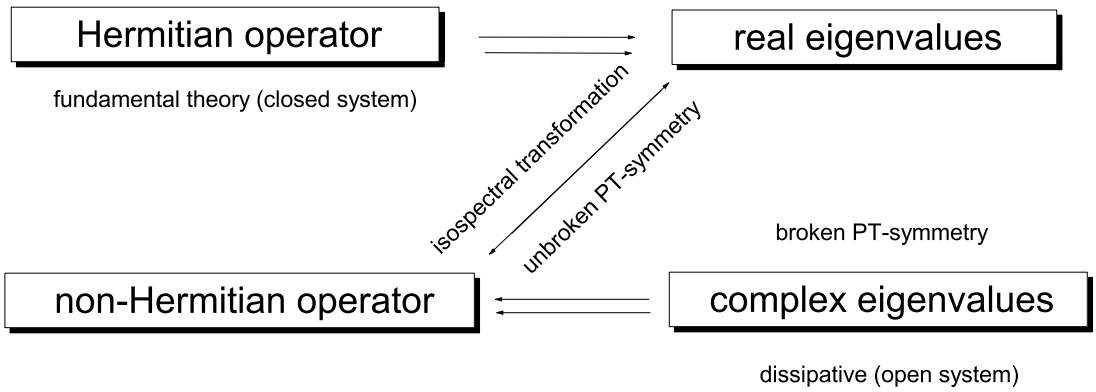

Figure 2.2: Summary of the possible eigenvalues of Hermitian and non-Hermitian operators.

In Figure 2.2 we see that whereas Hermitian operators generate necessarily real eigenvalues $E_n^* = E_n \in \mathbb{R}$, complex eigenvalues $E_n^* \neq E_n \in \mathbb{C}$ can only be originated from non-Hermitian operators. Therefore it is not possible to find Hermitian operators with complex eigenvalues. Nonetheless, non-Hermitian operators with real eigenvalues occur with certain frequency. Real eigenvalues in a physically accepted theory can be obtained, for instance if the system is invariant under $\mathcal{PT}$-symmetry, in its unbroken phase, or any other anti-linear symmetry.

In [77] it is argued that a Hamiltonian without such a symmetry cannot posses a completely real set of energy eigenvalues, so that this is both a sufficient and necessary condition for the reality of the spectrum. On the other hand, as deeply discussed in [56, 99], diagonalizable operators with a complete set of eigenvectors and real energies can be mapped into Hermitian operators by a similarity transformation, so that they can be regarded as quasi-pseudo-Hermitian operators. Moreover, it possible to construct for these systems an underlying anti-linear symmetry, the kind of which $\mathcal{PT}$ is a particular example. In [100] an example of non-diagonalizable $\mathcal{PT}$-symmetric Hamiltonian with no Hermitian counterpart is presented. The equivalence between quasi-pseudo-Hermiticity and the presence of an anti-linear symmetry was questioned more recently in [101], where the non-equivalence of such concepts was establish for bounded operators.

The analysis of Figure 2.3.4 does not relate dissipative systems and $\mathcal{PT}$-symmetric ones in its unbroken phase. In both regimes the systems are illustrations of non-Hermitian Hamiltonians possessing complex eigenvalues and therefore cannot be used as a fundamental theoretical description. Nonetheless, dissipative models are formulated from the beginning as an effective theory to describe settings for which there is a decrease in the probability of finding a certain state as time flows. Their eigenenergies are normally of the form

$$E_n = \epsilon_n^r - \imath\epsilon_n^i \,, \qquad \text{with} \qquad \epsilon_k^i > 0, \tag{2.105}$$

assuring that the evolution operator when action on the corresponding eigenstate will generate an exponential decay,

$$e^{-\imath E_i t} = e^{-\imath\epsilon_k^r t} e^{-\epsilon_k^i t}. \tag{2.106}$$

$\mathcal{PT}$-symmetric systems in the broken phase will differ from the latter due to the presence of eigenvalues with both positive and negative imaginary parts, thus leading not only to loss but also to gain. In principle a possible interpretation would be that these two

processes of increasing and decreasing probabilities would compensate each other overall. However, $\mathcal{PT}$-symmetric quantum mechanics is, differently from the dissipative approach, constructed to be a fundamental theory in the sense that the characteristic energies should be measurable. The occurrence of complex energies, however, contradicts our everyday experience of observing strictly real quantities. Complex eigenvalues appearing only in conjugate pairs lead to notable effects such as the reality of thermodynamic partition functions,

$$Z = \sum_n e^{-\beta E_n} = 2\sum_n e^{-\beta E_n^r} \cos\left(\beta E_n^i\right), \tag{2.107}$$

with possibly real and positive mean energies,

$$\langle E \rangle = -\frac{\partial \ln Z}{\partial \beta}, \tag{2.108}$$

where $\beta = \frac{1}{k_B T}$ is related to the inverse of the temperature by the Boltzmann constant k_B. But still the existence of complex expectation values pose serious questions to the physical meaningfulness of broken $\mathcal{PT}$-symmetric systems.

2.3.5 Perturbative approach

It is customary to express the Dyson operator in an exponential form,

$$\eta = e^{\mathcal{N}}, \tag{2.109}$$

so that, according to (2.90) one can express the metric as

$$\rho = e^{\mathcal{R}} \qquad \text{with} \qquad \mathcal{R} = \mathcal{N}^\dagger + \mathcal{N}. \tag{2.110}$$

The convenience of this formulation is that it allows for the use of the following relation

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots, \tag{2.111}$$

so that the equation satisfied by the metric operator can be written in the form

$$H^\dagger = H + [\mathcal{R},H] + \frac{1}{2!}[\mathcal{R},[\mathcal{R},H]] + \frac{1}{3!}[\mathcal{R},[\mathcal{R},[\mathcal{R},H]]] + \cdots. \tag{2.112}$$

Similarly, once $\mathcal{R}$ is determined it can be used to find $\mathcal{N}$ in (2.110) and then calculate the Hermitian counterpart,

$$h = H + [\mathcal{N},H] + \frac{1}{2!}[\mathcal{N},[\mathcal{N},H]] + \frac{1}{3!}[\mathcal{N},[\mathcal{N},[\mathcal{N},H]]] + \cdots. \tag{2.113}$$

Although (2.89) furnishes us with a precise expression to determine the metric operator, very few examples are known to present an exact solution to this problem. In most

situations one has no option but to rely on a perturbative approach, e.g. [68]. Assuming a non-Hermitian operator of interest H to be decomposed in a Hermitian part h_0 perturbed by a Hermitian term h_1 whose coupling is imaginary,

$$H = h_0 + \imath\, g\, h_1\ , \qquad \text{with} \qquad h_i^\dagger = h_i \qquad \text{and} \qquad g \in \mathbb{R}, \tag{2.114}$$

one can expand $\mathcal{R}$ in (2.110) in powers of g,

$$\mathcal{R} = \sum_{k=0}^{\infty} g^k \mathcal{R}_k. \tag{2.115}$$

The equation (2.112) can now be solved for each order of g. Considering that $\mathcal{R}_0 = 0$, so that in the Hermitian limit $g \to 0$ the metric becomes the identity operator $\rho \to \mathbb{1}$, a considerable simplification leads to

$$\begin{aligned}
[h_0, \mathcal{R}_1] &= 2\imath h_1\ , && (2.116)\\
[h_0, \mathcal{R}_2] &= 0\ , && (2.117)\\
[h_0, \mathcal{R}_3] &= -\frac{\imath}{6}[\mathcal{R}_1, [\mathcal{R}_1, h_1]]\ , && (2.118)\\
&\vdots\ , &&
\end{aligned}$$

necessary to construct the metric. Order by order an operator equation must be solved and one notices that at each order a term which commutes with h_0 may be added without any changes to (2.89). Therefore, there is an ambiguity when determining the metric.

2.3.6 Uniqueness of the metric

In quantum mechanics physical observables are represented mathematically by linear self-adjoint operators acting on a Hilbert space. For a particle system, two important operators are position $\hat{x}$ and momentum $\hat{p}$, satisfying the canonical commutation relation (2.16), which is impossible to be accomplished in finite dimensional spaces [102]. The Stone - von Neumann theorem, e.g. [103], establishes the uniqueness of the canonical commutation relation between position and momentum operators. From Stone's theorem the infinitesimal generator of a unitary group must be self-adjoint. There is a one-to-one correspondence between self adjoint operators and continuous one-parameter unitary groups which allows one to express (2.16) in the so-called Weyl form

$$e^{\imath s\hat{x}} e^{\imath r\hat{p}} = e^{\imath sr} e^{\imath r\hat{p}} e^{\imath s\hat{x}}, \tag{2.119}$$

and according to the Stone - von Neumann theorem, all such pairs of one parameter unitary groups are unitarily equivalent. Thus, in Hermitian quantum mechanics the observables are uniquely determined up to a unitary transformation.

Nevertheless, such a theorem is not valid if the system has an infinite number of degrees of freedom, so that non-equivalent representations may occur. When having a non-Hermitian Hamiltonian as the sole starting point there is not a unique Hermitian counterpart in the same similarity class associated to the adjoint action of one unique operator η. Consequently, also the metric operator ρ is not unique. The latter was pointed out for instance in [18, 26, 27] and exemplified in detail for the concrete example of the so-called Swanson Hamiltonian in [104, 2]. In fact, it is simple to see that any two non-equivalent metric operators, say ρ and $\bar{\rho}$, can be used as in [105] to construct a symmetry operator for the Hamiltonian, $S = \rho^{-1}\bar{\rho}$. Suppose we can find, for the same Hamiltonian H, more than one metric satisfying (2.89),

$$H^\dagger = \rho\, H\, \rho^{-1} \qquad \text{and} \qquad H^\dagger = \bar{\rho}\, H\, \bar{\rho}^{-1} \qquad \Longleftrightarrow \qquad [H, \rho^{-1}\bar{\rho}\,] = 0. \tag{2.120}$$

Then S is a symmetry of the system: $[S, H] = 0$. Likewise, this ambiguity can be related to the non-equivalent Hermitian counterparts

$$h = \eta\, H\, \eta^{-1} \qquad \text{and} \qquad \bar{h} = \bar{\eta}\, H\, \bar{\eta}^{-1} \qquad \Longleftrightarrow \qquad [s, h] = [\bar{s}, \bar{h}] = 0, \tag{2.121}$$

with symmetry operators $s = (\bar{\eta}\ \eta^{-1})^\dagger(\bar{\eta}\ \eta^{-1})$ and $\bar{s} = s^\dagger$. We may also use (2.120) to show that $\tilde{\rho} \equiv H^\dagger\, \rho\, H$ is also a pseudo-Hermitian metric operator, and so are

$$\tilde{\rho} = \left(H^\dagger\right)^n \rho\, H^n \qquad \text{for } \ n \in \mathbb{N}, \tag{2.122}$$

but because $\rho = \eta^\dagger\eta$ we can write $\tilde{\rho} = \tilde{\eta}^\dagger\tilde{\eta}$ with $\tilde{\eta} \equiv \eta\, H^n$ and

$$\tilde{h} = \tilde{\eta}\, H\, \tilde{\eta}^{-1} = \eta\, H\, \eta^{-1} = h. \tag{2.123}$$

Thus we encounter here an infinite amount of new solutions $\tilde{\rho}$ for the metric, all leading to the same Hermitian counterpart. The choice of the metric as a degree of freedom in quasi-pseudo quantum formulation introduces the non-uniqueness in the specification of observables, which must be Hermitian with respect to the chosen metric in order to ensure the reality of their spectra. Even for the cases where different metrics and Hermitian partners are admissible there are various ways to select a unique solution. After noticing that a same physical setup may be represented by different isospectral operators, one possibility of solving the uniqueness problem would be to specify more observables in the non-Hermitian system [18].

2.3.7 Non-Hermitian time evolution

The evolution of states under the action of a time-independent Hamiltonian can be easily obtained as one remembers that because $|\varphi_n\rangle$ and $|\phi_n\rangle$ are eigenvectors of H and $H^\dagger$ respectively, (2.79) and (2.79), the temporal behaviour of general combinations of these eigenstates will satisfy the following Schrödinger equations

$$\imath\frac{\partial}{\partial t}|\varphi(t)\rangle = H|\varphi(t)\rangle \qquad \Longrightarrow \qquad -\imath\frac{\partial}{\partial t}\langle\varphi(t)| = \langle\varphi(t)|H^\dagger, \tag{2.124}$$

$$\imath\frac{\partial}{\partial t}|\phi(t)\rangle = H^\dagger|\phi(t)\rangle \qquad \Longrightarrow \qquad -\imath\frac{\partial}{\partial t}\langle\phi(t)| = \langle\phi(t)|H. \tag{2.125}$$

The possibility of using this Dyson map for time-dependent problems, with $H = H(t)$, was considered in [106] by studying time-dependent Hamiltonians which could be made Hermitian with the choice of a time-independent metric. The problem of time-dependent non-Hermitian systems was then analyzed in a sequence of manuscripts, such as [107, 108]. First it was observed that conservation of probability for non-Hermitian systems, using that $H(t)|\varphi(t)\rangle = \imath\frac{\partial}{\partial t}|\varphi(t)\rangle$, would imply

$$\begin{aligned}\frac{d}{dt}\langle\phi(t)|\varphi(t)\rangle &= \frac{d}{dt}\langle\varphi(t)|\rho(t)|\varphi(t)\rangle = \\ &= \langle\varphi(t)|\left(H^\dagger(t)\rho(t) - \rho(t)H(t) - \imath\frac{d\rho(t)}{dt}\right)|\varphi\rangle = 0\end{aligned} \tag{2.126}$$

or simply

$$H^\dagger(t) = \rho(t)H(t)\rho(t)^{-1} - \imath\frac{d\rho(t)}{dt}\rho(t)^{-1} \tag{2.127}$$

But if we want $H(t)$ to be Hermitian with respect to the metric $\rho(t)$, then we should have $H^\dagger(t) = \rho(t)H(t)\rho(t)^{-1}$, indicating the necessity of having a time-independent metric in (2.127). Later it was showed that requiring a time-dependent Dyson map to hold for any time, $h(t) = \eta(t)H(t)\eta(t)^{-1}$, we can have a unitary time evolution as long as

$$\imath\frac{\partial}{\partial t}|\varphi(t)\rangle = \left(H(t) - \imath\frac{\partial\eta(t)}{\partial t}\eta(t)^{-1}\right)|\varphi(t)\rangle, \tag{2.128}$$

$$\imath\frac{\partial}{\partial t}|\phi(t)\rangle = \left(H^\dagger(t) + \imath\frac{\partial\eta(t)}{\partial t}\eta(t)^{-1}\right)|\phi(t)\rangle, \tag{2.129}$$

but, as was later clarified, the time evolution in this case is not determined by the Hamiltonian, but instead, by a modified operator which is not an observable if $\eta(t)$ is not an observable.

Questions concerning time evolution are very important, for example, in the problem discussed in the next chapter.

3 The quantum brachistochrone for non-Hermitian Hamiltonians

3.1 The classical brachistochrone problem

An interesting question where many aspects of the framework just described can be explored consists of the *quantum brachistochrone* problem. As the name indicates, it is a quantum version of the classical *brachistochrone* problem, which consists of determining the curve that minimizes the time it takes for an object to go between points in different heights along a vertical plane under the action of the gravitational field without friction. It has its origin probably in Galileo's *Two New Sciences*, from 1638. The terminology comes from the Greek words $\beta\rho\alpha\chi\acute{\upsilon}$ [brachi] = short (from which follows that $\beta\rho\alpha\chi\iota\sigma\tau o\varsigma$ [brachistos]= shortest) and $\chi\rho\acute{o}\nu o\varsigma$ [chronos] = time.

Johann Bernoulli announced in *Acta Eruditorum* in June 1696 to have a solution of this problem, challenging other scientists to publish their results within six months. By January 1697 only Gottfried Leibniz had communicated his solution but, after a prorogation, in May 1697 other three mathematicians presented their findings: Isaac Newton (anonymously), Jakob Bernoulli and Ehrenfried Walther von Tschirnhaus. Guillaume de l'Hôpital's contribution appeared in a later edition of the *Acta*. Johann Bernoulli noted the similarities between that problem and Christiaan Huygens' 1659 *tautochrone* or *isochrone* curve, the one for which the time taken by a similar object (sliding without friction in uniform gravity) to its lowest point is independent of its starting point, and concluded the form of the trajectory in question should also be a segment of a *cycloid* [109, 110]. The techniques employed in this occasion served as seeds for the development of what became the calculus of variations by Leonhard Euler and Joseph-Louis de Lagrange.

As observed more recently in [111] also in classical mechanics the analytic continuation

of certain objects may lead to a faster evolution in some sense. It is argued that a simple harmonic oscillator whose trajectories are allowed to be in the complex plane present a similar feature. Considering ω to be the natural frequency of the motion and restricting one's attention to the situations where the energy is constant, $H = E = \frac{1}{2}\omega^2$, the system is described by

$$H = \frac{1}{2}\left(p^2 + \omega^2 x^2\right) \qquad \Longrightarrow \qquad x(t) = A\cos(\omega t) + \sqrt{1 - A^2}\sin(\omega t)\ , \tag{3.1}$$

with $p = \dot{x} = \frac{dx}{dt}$ and $A \in \mathbb{R}$ so that the energy constraint imposes a maximum limit for the amplitude of the motion on the real line. In other words, if $|A| > 1$, the motion will be described by ellipses in the complex plane. Interestingly, the shortest time it takes to travel between the extreme points on the real line of the trajectories (from $x = -A$ to $x = A$) is always half a period, $\tau = \frac{\pi}{\omega}$, regardless of the separation $2A$ of these two points.

The intrinsic existence of a complex vector space describing quantum states might represent a more realistic possibility to achieve faster transitions. We analyse in the rest of this chapter the quantum version of the problem here described and discuss how non-Hermitian evolution could bring about faster evolution.

3.2 Hermitian evolution in Quantum Physics

Centuries after the study of the classical brachistochrone problem, a similar question in the context of quantum theory might have an impact as strong as its classical predecessor, potentially in fields such as quantum computing, quantum information and quantum cryptography. Quantum states are known to evolve when a suitable Hamiltonian is applied on them. Given an initial state $|\psi_I\rangle$ and a final state $|\psi_F\rangle$, it is an important question to ask under which circumstances will a Hamiltonian H develop one into the other in the shortest amount of time τ according to a quantum mechanical setting,

$$|\psi_F\rangle = e^{-\imath H\tau}|\psi_I\rangle. \tag{3.2}$$

This question was discussed in [112, 113, 114, 115] and generalized to non-Hermitian systems by Bender, Brody, Jones and Meister in [23], where it was found that when non-Hermitian Hamiltonians are $\mathcal{PT}$-symmetric this passage time could in principle be made arbitrarily small. Because in the brachistochrone problem the only relevant Hilbert subspace is the one generated by the initial and final states, it is enough to study 2×2

models. The analysis of these 2-level systems also provide a good environment to explore the ideas concerning non-Hermitian Hamiltonians with real eigenvalues discussed in previous chapters.

To understand the problem, let us first exploit the fact that any 2×2-matrix M can be decomposed in terms of Pauli matrices as

$$M = \mu_0 \mathbb{1} + \mu \cdot \sigma \ , \qquad \text{with} \qquad \mu_i \in \mathbb{C} \qquad \text{and} \qquad i = 0, 1, 2, 3, \tag{3.3}$$

to observe that the most general 2-dimensional Hermitian Hamiltonian system can be represented by the following matrix

$$H_0 = \lambda_0 \mathbb{1} + \Omega \cdot \sigma = \begin{pmatrix} \lambda_0 + R\cos\theta & R\, e^{-\imath\xi}\sin\theta \\ R\, e^{\imath\xi}\sin\theta & \lambda_0 - R\cos\theta \end{pmatrix}, \tag{3.4}$$

where $\Omega = R(\sin\theta\cos\xi, \sin\theta\sin\xi, \cos\theta)$ and σ is a vector composed of the Pauli matrices $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, with, as in the standard convention,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.5}$$

Then, the eigenvalues and eigenvectors of this operator is easy to calculate

$$H_0|\varphi^0_\pm\rangle = \lambda^0_\pm|\varphi^0_\pm\rangle, \qquad \langle\phi^0_\pm|H_0 = \lambda^0_\pm\langle\phi^0_\pm|, \tag{3.6}$$

with $\lambda^0_\pm = \lambda_0 \pm R$, implying that the transition frequency between these levels is given by $\omega_0 \equiv \lambda^0_+ - \lambda^0_- = 2R$ and the right- and left-eigenvectors are

$$\begin{array}{rclcrcl} |\varphi^0_+\rangle & = & \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{\imath\xi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} & , & |\varphi^0_-\rangle & = & \begin{pmatrix} -e^{-\imath\xi}\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \quad , \\ \langle\phi^0_+| & = & \left(\cos\left(\frac{\theta}{2}\right), e^{-\imath\xi}\sin\left(\frac{\theta}{2}\right)\right) & , & \langle\phi^0_-| & = & \left(-e^{\imath\xi}\sin\left(\frac{\theta}{2}\right), \cos\left(\frac{\theta}{2}\right)\right) \quad , \end{array} \tag{3.7}$$

satisfying bi-orthonormality conditions: $\langle\phi^0_\pm|\varphi^0_\pm\rangle = 1$ and $\langle\phi^0_\pm|\varphi^0_\mp\rangle = 0$. Obviously, due to the Hermiticity of H_0, right- and left-eigenvectors are simply related through a Hermitian conjugation: $\langle\phi^0_\pm| = |\varphi^0_\pm\rangle^\dagger$.

The evolution operator associated to this system can be written as

$$U_0 = e^{-\imath H_0 t} = e^{-\imath\lambda_0\mathbb{1}t}e^{-\imath\Omega\cdot\sigma t} = e^{-\imath\lambda_0 t}\left(\cos\left(\frac{\omega_0 t}{2}\right)\mathbb{1} - \frac{2\imath}{\omega_0}\sin\left(\frac{\omega_0 t}{2}\right)\Omega\cdot\sigma\right). \tag{3.8}$$

Although the states associated to the above eigenvectors, $|\varphi^0_\pm\rangle$ and $\langle\phi^0_\pm|$, are orthogonal they do not evolve in time under the action of U_0 as they are stationary. Thus if

one wants to transform states under the evolution generated by H_0 one should use linear combinations of its eigenvectors. Familiar orthogonal states in a 2-dimensional space are

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cos\left(\frac{\theta}{2}\right)|\varphi_+^0\rangle - e^{\imath\xi}\sin\left(\frac{\theta}{2}\right)|\varphi_-^0\rangle, \tag{3.9}$$

$$|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \cos\left(\frac{\theta}{2}\right)|\varphi_-^0\rangle + e^{-\imath\xi}\sin\left(\frac{\theta}{2}\right)|\varphi_-^0\rangle, \tag{3.10}$$

and $\langle\pm| = |\pm\rangle^\dagger$, in whose basis the matrix (3.4) was expressed. Simpler orthogonal states which are not stationary can be constructed from symmetric and anti-symmetric combinations of the eigenstates,

$$|\varphi_{S/A}^0\rangle = \frac{|\varphi_+^0\rangle \pm |\varphi_-^0\rangle}{\sqrt{2}} \qquad \text{and} \qquad \langle\phi_{S/A}^0| = \frac{\langle\phi_+^0| \pm \langle\phi_-^0|}{\sqrt{2}}. \tag{3.11}$$

Then we can formulate the quantum brachistochrone problem in a slightly less stringent form than in (3.2), by instead just considering the physically relevant matrix element

$$||\langle\phi_F|\varphi_I\rangle|| = ||\langle\phi_S^0|U_0|\varphi_A^0\rangle|| = \frac{||e^{-\imath\lambda_+^0 t}\langle\phi_+^0|\varphi_+^0\rangle - e^{-\lambda_-^0 t}\langle\phi_-^0|\varphi_-^0\rangle||}{2} = \frac{||e^{-\imath\omega_0 t} - 1||}{2}, \tag{3.12}$$

indicating the complete transition between these general orthogonal states cannot occur in a period of time smaller than $\tau = \frac{\pi}{\omega_0}$. This optimal transition time depends on the energy gap between the initial and final states. Considering this gap to be fixed, one state cannot be transformed into an orthogonal one in a time smaller than a fixed value τ when one works within a purely Hermitian framework. As we have seen, non-Hermitian systems can be treated as Hermitian models as long as a different metric defining the Hilbert space is used. This tells us that, if the energies are real, in a purely non-Hermitian framework the passage time also has a lower bound. However, as we shall explain later, according to the findings of [23], a hybrid framework could allow one to violate this speed limit in transition time. Bender, Brody, Jones and Meister extended the existing approach to the quantum brachistochrone problem and demonstrated the existence of a faster evolution if one does not insist in working only with Hermitian Hamiltonians but replaces the Hermiticity condition by the requirement that the system should be invariant under $\mathcal{PT}$-symmetry. Another interesting consideration which emerges is the possibility of this phenomenon to occur also when the energies are not real, with $\mathcal{PT}$-symmetry definitely broken. This question was answered by us in [1] and will be more explored in a future section.

3.3 Fast transitions with $\mathcal{PT}$-symmetric non-Hermitian Hamiltonians

To proceed note that the Hermiticity in (3.4), however, will only be present if $\lambda_0, R, \theta, \xi \in \mathbb{R}$, as assumed so far. It is enough to complexify θ in order to work with a non-Hermitian Hamiltonian. Before doing so, it should be mentioned that (3.4) can be re-parametrized with

$$\lambda_0 = \frac{s+u}{2}, \qquad R = r\sqrt{1+\left(\frac{s-u}{2r}\right)^2}, \qquad \theta = \arctan\left(\frac{2r}{s-u}\right), \tag{3.13}$$

giving the form of Hamiltonian studied in [23, 1]

$$H_0 = \begin{pmatrix} s & re^{-\imath\xi} \\ re^{\imath\xi} & u \end{pmatrix}. \tag{3.14}$$

A convenient transformation to make (3.4) non-Hermitian consists of

$$\lambda_0 = r\cos\gamma, \qquad R = \sqrt{\rho^2 - r^2\sin^2\gamma}, \qquad \theta = \frac{\pi}{2} - \imath\,\eta, \qquad \xi = 0, \tag{3.15}$$

with $\sinh\eta = \frac{r}{R}\sin\gamma$ and $\cosh\eta = \frac{\rho}{R}$. As a consequence the Hamiltonian takes the form

$$H_N = \begin{pmatrix} re^{\imath\gamma} & \rho \\ \rho & re^{-\imath\gamma} \end{pmatrix}, \tag{3.16}$$

clearly not Hermitian. This, nonetheless, represents a large class of $\mathcal{PT}$-symmetric systems in two dimensions, where $\mathcal{T}$ is as usual accomplished by a complex conjugation operation and the parity transformation can be represented, for example, by $\mathcal{P} = \sigma_1$ as it intertwines the states $|+\rangle$ and $|-\rangle$ in (3.9) into each other.

Introducing a new parameter δ by $r\sin\gamma = \rho\sin\delta$, characteristic values and vectors,

$$H_N|\varphi_\pm^N\rangle = \lambda_\pm^N|\varphi_\pm^N\rangle, \qquad \langle\phi_\pm^N|H_N = \lambda_\pm^N\langle\phi_\pm^N|, \tag{3.17}$$

take a simpler form, with real eigenvalues $\lambda_\pm^N = r\cos\gamma \pm \rho\cos\delta$, real transition frequency $\omega_N \equiv 2\rho\cos\delta$ and right- and left-eigenvectors

$$|\varphi_\pm^N\rangle = \frac{1}{\sqrt{2\cos\delta}}\begin{pmatrix} \pm e^{\pm\imath\frac{\delta}{2}} \\ e^{\mp\imath\frac{\delta}{2}} \end{pmatrix}, \qquad \langle\phi_\pm^N| = \frac{1}{\sqrt{2\cos\delta}}\left(\pm e^{\pm\imath\frac{\delta}{2}}, e^{\mp\imath\frac{\delta}{2}}\right), \tag{3.18}$$

satisfying

$$\mathcal{PT}|\varphi_+^N\rangle = |\varphi_+^N\rangle \qquad \text{and} \qquad \mathcal{PT}\left(\imath|\varphi_-^N\rangle\right) = \left(\imath|\varphi_-^N\rangle\right). \tag{3.19}$$

As expected from the reality of the eigenvalues $\lambda_\pm$, the eigenstates are in the unbroken phase of $\mathcal{PT}$. A two-dimensional representation of $\mathcal{PT}$ would involve the flipping pair $\frac{|\varphi_+^N\rangle+|\varphi_-^N\rangle}{\sqrt{2}}$ and $\frac{|\varphi_+^N\rangle-|\varphi_-^N\rangle}{\sqrt{2}}$ but these states are not eigenvectors of the Hamiltonian H_N. On the other hand, if $\rho < r\sin\gamma$ then we would have complex eigenvalues and $\mathcal{PT}$-symmetry would be broken as δ becomes complex.

It becomes evident that $\langle\phi_\pm^N| \neq |\varphi_\pm^N\rangle^\dagger$ and the metric with respect to which H_N can be regarded as Hermitian is calculated to be

$$\rho_N = |\phi_+^N\rangle\langle\phi_+^N| + |\phi_-^N\rangle\langle\phi_-^N| = \frac{1}{\cos\delta}\begin{pmatrix} 1 & -\imath\sin\delta \\ \imath\sin\delta & 1 \end{pmatrix} = \rho_N^\dagger, \tag{3.20}$$

with eigenvalues $\sec\delta \pm \tan\delta$, to establish orthonormality,

$$\langle\varphi_\pm^N|\rho_N|\varphi_\pm^N\rangle = \langle\phi_\pm^N|\rho_N|\phi_\pm^N\rangle = 1 \qquad \text{and} \qquad \langle\varphi_\pm^N|\rho_N|\varphi_\mp^N\rangle = \langle\phi_\pm^N|\rho_N|\phi_\mp^N\rangle = 0, \tag{3.21}$$

as well as the associated Dyson map

$$\eta_N = \frac{1}{\sqrt{\cos\delta}}\begin{pmatrix} -\sin\left(\frac{\delta}{2}\right) & \imath\cos\left(\frac{\delta}{2}\right) \\ -\imath\cos\left(\frac{\delta}{2}\right) & -\sin\left(\frac{\delta}{2}\right) \end{pmatrix}. \tag{3.22}$$

This is useful to determine the Hermitian counterpart using (2.88),

$$h_N = r\cos\gamma\mathbb{1} - \rho\cos\delta\sigma_1 = \begin{pmatrix} r\cos\gamma & -\rho\cos\delta \\ -\rho\cos\delta & r\cos\gamma \end{pmatrix}, \tag{3.23}$$

naturally with the same eigenvalues $\lambda_\pm^N$ and right- and left-eigenvectors related simply by a Hermitian conjugation,

$$|\psi_\pm^N\rangle = \frac{\imath}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp 1 \end{pmatrix} = \eta_N|\varphi_\pm^N\rangle. \tag{3.24}$$

which can be used to express the familiar orthogonal states $|\pm\rangle$ as

$$|\pm\rangle = \frac{|\psi_-^N\rangle \pm |\psi_+^N\rangle}{\imath\sqrt{2}}. \tag{3.25}$$

When not acting on eigenstates with the operator $e^{-\imath h_N t}$ or $e^{-\imath H_N t}$ one has to turn the infinite sum of operators into a matrix multiplication. For this one can express h_N or H_N in terms of Pauli matrices, the operation with $e^{-\imath h_N t}$ or $e^{-\imath H_N t}$ on a state reduces to a simple matrix multiplication by using the identity

$$e^{\varphi\mu\cdot\sigma} = \cos\varphi\mathbb{1} + \imath\sin\varphi\mu\cdot\sigma. \tag{3.26}$$

The non-Hermitian evolution operator and its Hermitian counterpart are given by

$$U_N = e^{-\imath H_N t} = \frac{e^{-\imath r\cos\gamma t}}{\cos\delta}\begin{pmatrix} \cos\left(\delta-\frac{\omega_N t}{2}\right) & -\imath\sin\left(\frac{\omega_N t}{2}\right) \\ -\imath\sin\left(\frac{\omega_N t}{2}\right) & \cos\left(\delta+\frac{\omega_N t}{2}\right)\end{pmatrix}, \tag{3.27}$$

$$u_N = e^{-\imath h_N t} = e^{-\imath r\cos\gamma t}\begin{pmatrix} \cos\left(\frac{\omega_N t}{2}\right) & \imath\sin\left(\frac{\omega_N t}{2}\right) \\ \imath\sin\left(\frac{\omega_N t}{2}\right) & \cos\left(\frac{\omega_N t}{2}\right)\end{pmatrix}. \tag{3.28}$$

Using as initial and final states $|\psi^N_{I/F}\rangle = |\pm\rangle$, corresponding to the bi-orthogonal states

$$\langle\phi^N_I| = \langle\psi^N_I|\eta^N = \frac{1}{\cos\delta}\begin{pmatrix} -\sin\left(\frac{\delta}{2}\right) \\ \imath\cos\left(\frac{\delta}{2}\right)\end{pmatrix}^T, \tag{3.29}$$

$$|\varphi^N_I\rangle = \eta_N^{-1}|\psi^N_I\rangle = \frac{1}{\cos\delta}\begin{pmatrix} \sin\left(\frac{\delta}{2}\right) \\ -\imath\cos\left(\frac{\delta}{2}\right)\end{pmatrix}, \tag{3.30}$$

$$\langle\phi^N_F| = \langle\psi^N_F|\eta^N = \frac{1}{\cos\delta}\begin{pmatrix} -\imath\cos\left(\frac{\delta}{2}\right) \\ -\sin\left(\frac{\delta}{2}\right)\end{pmatrix}^T, \tag{3.31}$$

$$|\varphi^N_F\rangle = \eta_N^{-1}|\psi^N_F\rangle = \frac{1}{\cos\delta}\begin{pmatrix} \imath\cos\left(\frac{\delta}{2}\right) \\ \sin\left(\frac{\delta}{2}\right)\end{pmatrix}, \tag{3.32}$$

the standard transition probability between initial and final states is

$$\frac{||\langle\psi^N_F|u_N|\psi^N_I\rangle||^2}{||\langle\psi^N_F|\psi^N_F\rangle||\,||\langle\psi^N_I|\psi^N_I\rangle||} = \frac{||\langle\phi^N_F|U_N|\varphi^N_I\rangle||^2}{||\langle\phi^N_F|\phi^N_F\rangle||\,||\langle\varphi^N_I|\varphi^N_I\rangle||} = \tag{3.33}$$

$$= \frac{||\langle\varphi^N_F|\rho_N U_N|\varphi^N_I\rangle||^2}{||\langle\varphi^N_F|\rho_N|\varphi^N_F\rangle||\,||\langle\varphi^N_I|\rho_N|\varphi^N_I\rangle||} = \tag{3.34}$$

$$= \frac{||\langle\phi^N_F|U_N\rho_N|\phi^N_I\rangle||^2}{||\langle\phi^N_F|\rho_N|\phi^N_F\rangle||\,||\langle\phi^N_I|\rho_N|\phi^N_I\rangle||} = \tag{3.35}$$

$$= \sin^2\left(\frac{\omega_N t}{2}\right), \tag{3.36}$$

providing one with the same shortest passage time, $\tau = \frac{\pi}{\omega_N}$, as long as the energy separation is kept the same $\omega_N = \omega_0$. As a consequence the energy uncertainty relation is preserved both in the Hermitian and non-Hermitian framework. The Hermitian evolution can be used to connect different sorts of states, such as in

$$\frac{||\langle\psi^N_F|u_N|\varphi^N_I\rangle||^2}{||\langle\psi^N_F|\psi^N_F\rangle||\,||\langle\varphi^N_I|\varphi^N_I\rangle||} = \cos^2\left(\frac{\delta-\omega_N t}{2}\right), \tag{3.37}$$

$$\frac{||\langle\phi^N_F|u_N|\psi^N_I\rangle||^2}{||\langle\phi^N_F|\phi^N_F\rangle||\,||\langle\psi^N_I|\psi^N_I\rangle||} = \cos^2\left(\frac{\delta+\omega_N t}{2}\right), \tag{3.38}$$

$$\frac{||\langle\phi^N_F|u_N|\varphi^N_I\rangle||^2}{||\langle\phi^N_F|\phi^N_F\rangle||\,||\langle\varphi^N_I|\varphi^N_I\rangle||} = \sin^2\left(\frac{\omega_N t}{2}\right), \tag{3.39}$$

so that the transition between bi-orthogonal states under a Hermitian evolution (3.39) provides again the same fixed passage time. The time it takes for the other transitions to occur has a different value and this may be attributed to the fact that those initial and final states are not orthogonal with respect to either of the metrics. More interesting, as shown below, is to calculate transition probabilities between mixed states under a non-Hermitian evolution, with the appropriate metric:

$$\frac{||\langle\psi_F^N|\rho_N U_N|\varphi_I^N\rangle||^2}{||\langle\psi_F^N|\rho_N|\psi_F^N\rangle||\;||\langle\varphi_I^N|\rho_N|\varphi_I^N\rangle||} = \cos^2\left(\frac{\delta+\omega_N t}{2}\right), \tag{3.40}$$

$$\frac{||\langle\psi_I^N|\rho_N U_N|\varphi_F^N\rangle||^2}{||\langle\psi_I^N|\rho_N|\psi_I^N\rangle||\;||\langle\varphi_F^N|\rho_N|\varphi_F^N\rangle||} = \cos^2\left(\frac{\delta-\omega_N t}{2}\right), \tag{3.41}$$

$$\frac{||\langle\psi_F^N|\rho_N U_N|\psi_I^N\rangle||^2}{||\langle\psi_F^N|\rho_N|\psi_F^N\rangle||\;||\langle\psi_I^N|\rho_N|\psi_I^N\rangle||} = \sin^2\left(\delta+\frac{\omega_N t}{2}\right). \tag{3.42}$$

This shows that, according to (3.42), using the non-orthogonality of $|+\rangle$ and $|-\rangle$ with respect to ρ_N,

$$\langle-|\rho_N|+\rangle = \imath\tan\delta, \tag{3.43}$$

an initial state $|\psi_I^N\rangle$ can be transformed into its orthogonal (in the Hermitian sense) state $|\psi_F^N\rangle$ in a variable passage time if the system is under the action of a non-Hermitian Hamiltonian. The tunable transition time,

$$\tau_N = \frac{\pi}{\omega_N} - \frac{2\delta}{\omega_N}, \tag{3.44}$$

can be made arbitrarily small by suitable choices of δ, differently from what happened for the Hermitian evolution. Note that in the limiting case $\delta \to \frac{\pi}{2}$ the travel time becomes negligible in principle. In this situation the initial and final states, (3.30) and (3.32), become almost coincident, but not parallel with respect to the appropriate metric (3.20). This interesting phenomenon of fast transition could well be, and it was in [23], conjectured to be a consequence of the $\mathcal{PT}$-symmetry present in the model, in a similar fashion as it can be used to prove the reality of some non-Hermitian Hamiltonians. Nonetheless, our results in [1] demonstrate similar effects take place also when the evolution is governed by non-Hermitian Hamiltonians with complex eigenvalues, meaning the phenomenon is in principle also observed when $\mathcal{PT}$-symmetry cannot be present.

3.4 Dissipative evolution

For that purpose we modify the Hamiltonian in a way to make it genuinely dissipative. In order to achieve this we must break the $\mathcal{PT}$-symmetry not only for the wavefunction

but also for the Hamiltonian. As discussed in section 2.1, such type of Hamiltonians result for instances as effective Hamiltonians say as a result of the coupling two non-degenerate states to some open channel [34, 33]

$$H_D = \begin{pmatrix} E+\epsilon & 0 \\ 0 & E-\epsilon \end{pmatrix} - \imath \begin{pmatrix} re^{\imath\gamma} & \rho \\ \rho & re^{-\imath\gamma} \end{pmatrix}, \tag{3.45}$$

with $E, \epsilon, r, \rho, \gamma \in \mathbb{R}$ but not providing here any concrete meaning for the parameters as we would like to keep our treatment as generic as possible. Note that this Hamiltonian does not simply correspond to going to the regime of broken $\mathcal{PT}$-symmetry for the Hamiltonian (3.16) of the previous section. Instead H_D reduces to H_N in the simultaneous limit $E, \epsilon \to 0$ and $r, \rho \to \imath$ and its eigenvalues are complex, $\lambda_{\pm}^D = E - \imath r\cos\gamma \pm \sqrt{(\epsilon + r\sin\gamma)^2 - \rho^2}$, but the energy gap, $\omega_D \equiv \lambda_+^D - \lambda_-^D = 2\sqrt{(\epsilon + r\sin\gamma)^2 - \rho^2}$ might be either real or imaginary depending on the sign of the radicand. If

$$(\epsilon + r\sin\gamma)^2 > \rho^2, \tag{3.46}$$

then

$$\omega_D = 2\sqrt{(\epsilon + r\sin\gamma)^2 - \rho^2} = \omega_R \in \mathbb{R}, \tag{3.47}$$

and we can introduce for convenience the parameter δ_R as $\sin\delta_R = \frac{\rho}{\epsilon + r\sin\gamma}$. Contrarily, if

$$(\epsilon + r\sin\gamma)^2 < \rho^2, \tag{3.48}$$

then

$$\omega_D = 2\sqrt{(\epsilon + r\sin\gamma)^2 - \rho^2} = 2\imath\sqrt{\rho^2 - (\epsilon + r\sin\gamma)^2} = \imath\,\omega_C \in \imath\,\mathbb{R} \tag{3.49}$$

and we can introduce the parameter δ_C as $\sin\delta_C = \frac{\epsilon + r\sin\gamma}{\rho}$. Note that this Hamiltonian does not simply correspond to going to the domain of broken $\mathcal{PT}$-symmetry for the Hamiltonian of the previous section. We separate our following analysis in these two complementary regimes.

The left- and right-eigenvectors of $H_D = H_{R/C}$ can be better expressed in the form

$$\langle \phi_+^R|^T = |\varphi_+^R\rangle = \frac{1}{\sqrt{\cos\delta_R}} \begin{pmatrix} \cos\left(\frac{\delta_R}{2}\right) \\ -\imath\sin\left(\frac{\delta_R}{2}\right) \end{pmatrix}, \tag{3.50}$$

$$\langle \phi_-^R|^T = |\varphi_-^R\rangle = \frac{1}{\sqrt{\cos\delta_R}} \begin{pmatrix} \sin\left(\frac{\delta_R}{2}\right) \\ -\imath\cos\left(\frac{\delta_R}{2}\right) \end{pmatrix}, \tag{3.51}$$

for real transition frequencies, whereas for imaginary ones we have

$$\langle\phi_{\pm}^{C}|^{T}=|\varphi_{\pm}^{C}\rangle=\frac{1}{\sqrt{2\cos\delta_{R}}}\begin{pmatrix} e^{\mp\imath\frac{\delta_{C}}{2}} \\ \mp e^{\pm\imath\frac{\delta_{C}}{2}} \end{pmatrix} \tag{3.52}$$

From the considerations in the previous section it is clear that the Dyson operator is vital for the computations of the matrix elements occurring in the quantum brachistochrone problem, especially when one wishes to evolve eigenstates of a Hermitian Hamiltonian with a time-evolution operator associated with a non-Hermitian system. However, since for the case at hand the Hamiltonian H_D is now genuinely complex there cannot exist any similarity transformation, which relates it to a Hermitian Hamiltonian. Nonetheless, we can use the other property of ρ, namely that it can be utilized to introduce a physically well-defined inner product. This means we can seek a transformation such that each set of eigenstates (3.50) and (3.52) become orthonormal with regard to this product. The metrics of our interest can be determined to be

$$\rho_{R/C}=|\phi_{+}^{R/C}\rangle\langle\phi_{+}^{R/C}|+|\phi_{-}^{R/C}\rangle\langle\phi_{-}^{R/C}|=\frac{1}{\cos\delta_{R/C}}\begin{pmatrix} 1 & -\imath\sin\delta_{R/C} \\ \sin\delta_{R/C} & 1 \end{pmatrix}, \tag{3.53}$$

with pseudo/quasi-orthonormality conditions becoming

$$\langle\varphi_{\pm}^{R/C}|\rho_{R/C}|\varphi_{\pm}^{R/C}\rangle=1 \qquad \text{and} \qquad \langle\varphi_{\pm}^{R/C}|\rho_{R/C}|\varphi_{\mp}^{R/C}\rangle=0. \tag{3.54}$$

The Dyson operator

$$\eta_{R/C}=\frac{1}{\sqrt{\cos\delta_{R/C}}}\begin{pmatrix} 1 & -\imath\sin\delta_{R/C} \\ \sin\delta_{R/C} & 1 \end{pmatrix}, \tag{3.55}$$

furnishes us with the two isospectral counterparts, associated to the different regimes (R/C) of (3.45),

$$h_{R}=\eta_{R}\,H_{D}\,\eta_{R}^{-1} \;=\; \begin{pmatrix} E-\imath\cos\gamma-\frac{\omega_{R}}{2} & 0 \\ 0 & E-\imath\cos\gamma+\frac{\omega_{R}}{2} \end{pmatrix}, \tag{3.56}$$

$$h_{C}=\eta_{C}\,H_{D}\,\eta_{C}^{-1} \;=\; \begin{pmatrix} E-\imath\cos\gamma & \frac{\omega_{C}}{2} \\ \frac{\omega_{C}}{2} & E-\imath\cos\gamma \end{pmatrix}. \tag{3.57}$$

Interestingly, although $\eta_{R/C}$ cannot be used to construct Hermitian partners because the similarity transformation must preserve the complex eigenvalues, η_{R} happens to diag-

onalize the original operator. The eigenvectors of h_R and h_C are trivial to determine:

$$\begin{aligned} |\psi_+^R\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |\psi_-^R\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \\ |\psi_+^C\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad , \quad |\psi_-^C\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad , \end{aligned} \tag{3.58}$$

from which we observe that, despite the fact that $h_{R/C}^\dagger \neq h_{R/C}$ these matrices are still symmetric in the sense that $h_{R/C}^T = h_{R/C}$, so that having real eigenvectors the latter form orthogonal bases: $\langle \psi_\pm^{R/C} | \psi_\pm^{R/C} \rangle = 1$ and $\langle \psi_\pm^{R/C} | \psi_\mp^{R/C} \rangle = 0$.

It can also be noticed that the up and down states $|\pm\rangle$ are stationary with respect to the evolution generated by h_R. Thus we consider as initial and final states the orthonormal combinations

$$|\psi_{I/F}^{R/C}\rangle = \frac{|\psi_-^{R/C}\rangle \pm |\psi_+^{R/C}\rangle}{\sqrt{2}} = |\psi_\pm^{C/R}\rangle, \tag{3.59}$$

and the corresponding transformed states can be calculated with the help of (2.92). In order to calculate the transition probabilities we are interested in, we write

$$u_R = e^{-\frac{\Gamma}{2}t} e^{-\imath E t} \begin{pmatrix} e^{\frac{\imath \omega_R t}{2}} & 0 \\ 0 & e^{-\frac{\imath \omega_R t}{2}} \end{pmatrix}, \tag{3.60}$$

$$u_C = e^{-\frac{\Gamma}{2}t} e^{-\imath E t} \begin{pmatrix} \cosh\left(\frac{\omega_C t}{2}\right) & \sinh\left(\frac{\omega_C t}{2}\right) \\ \sinh\left(\frac{\omega_C t}{2}\right) & \cosh\left(\frac{\omega_C t}{2}\right) \end{pmatrix}, \tag{3.61}$$

with $\Gamma = 2r\cos\gamma$ and $U_{R/C} = \eta_{R/C}^{-1}\, u_{R/C}\, \eta_{R/C}$. Some of the possible elements calculated for the evolution under the action of u_R are

$$\frac{||\langle \psi_F^R | u_R | \psi_I^R \rangle||^2}{||\langle \psi_F^R | \psi_F^R \rangle||\, ||\langle \psi_I^R | \psi_I^R \rangle||} = e^{-\Gamma t} \sin^2\left(\frac{\omega_R t}{2}\right) \tag{3.62}$$

$$\frac{||\langle \psi_F^R | u_R | \varphi_I^R \rangle||^2}{||\langle \psi_F^R | \psi_F^R \rangle||\, ||\langle \varphi_I^R | \varphi_I^R \rangle||} = e^{-\Gamma t} \cos^2\left(\frac{\delta_R + \omega_R t}{2}\right), \tag{3.63}$$

$$\frac{||\langle \phi_F^R | u_R | \psi_I^R \rangle||^2}{||\langle \phi_F^R | \phi_F^R \rangle||\, ||\langle \psi_I^R | \psi_I^R \rangle||} = e^{-\Gamma t} \cos^2\left(\frac{\delta_R - \omega_R t}{2}\right), \tag{3.64}$$

$$\frac{||\langle \phi_F^R | u_R | \varphi_I^R \rangle||^2}{||\langle \phi_F^R | \phi_F^R \rangle||\, ||\langle \varphi_I^R | \varphi_I^R \rangle||} = e^{-\Gamma t} \sin^2\left(\frac{\omega_R t}{2}\right), \tag{3.65}$$

showing us again that the transition between orthogonal (or bi-orthogonal) states always

takes the same amount of time whereas the evolution according to U_R,

$$\frac{||\langle\varphi_F^R|\rho_R U_R|\varphi_I^R\rangle||^2}{||\langle\varphi_F^R|\rho_R|\varphi_F^R\rangle||\ ||\langle\varphi_I^R|\rho_R|\varphi_I^R\rangle||} = e^{-\Gamma t}\sin^2\left(\frac{\omega_R t}{2}\right), \tag{3.66}$$

$$\frac{||\langle\psi_F^R|\rho_R U_R|\varphi_I^R\rangle||^2}{||\langle\psi_F^R|\rho_R|\psi_F^R\rangle||\ ||\langle\varphi_I^R|\rho_R|\varphi_I^R\rangle||} = e^{-\Gamma t}\cos^2\left(\frac{\delta_R-\omega_R t}{2}\right), \tag{3.67}$$

$$\frac{||\langle\psi_I^R|\rho_R U_R|\varphi_F^R\rangle||^2}{||\langle\psi_I^R|\rho_R|\psi_I^R\rangle||\ ||\langle\varphi_F^R|\rho_R|\varphi_F^R\rangle||} = e^{-\Gamma t}\cos^2\left(\frac{\delta_R+\omega_R t}{2}\right), \tag{3.68}$$

$$\frac{||\langle\psi_F^R|\rho_R U_R|\psi_I^R\rangle||^2}{||\langle\psi_F^R|\rho_R|\psi_F^R\rangle||\ ||\langle\psi_I^R|\rho_R|\psi_I^R\rangle||} = e^{-\Gamma t}\sin^2\left(\delta_R-\frac{\omega_R t}{2}\right) \tag{3.69}$$

indicates the time consumed to transform $|\psi_I^R\rangle$ into its orthogonal state $|\psi_F^R\rangle$ may be varied as δ_R changes, in an analogous fashion to (3.44),

$$\tau_R = \frac{\pi}{\omega_R} + \frac{2\delta_R}{\omega_R}, \tag{3.70}$$

For the complex transition frequency we computed

$$\frac{||\langle\psi_F^C|u_C|\psi_I^C\rangle||^2}{||\langle\psi_F^C|\psi_F^C\rangle||\ ||\langle\psi_I^C|\psi_I^C\rangle||} = e^{-\Gamma t}\sinh^2\left(\frac{\omega_C t}{2}\right) \tag{3.71}$$

$$\frac{||\langle\psi_F^C|u_C|\varphi_I^C\rangle||^2}{||\langle\psi_F^C|\psi_F^C\rangle||\ ||\langle\varphi_I^C|\varphi_I^R\rangle||} = e^{-\Gamma t}\frac{1}{2}\left(\cosh\left(\frac{\omega_C t}{2}\right)+\cos\delta_C\right), \tag{3.72}$$

$$\frac{||\langle\phi_F^C|u_C|\psi_I^C\rangle||^2}{||\langle\phi_F^C|\phi_F^C\rangle||\ ||\langle\psi_I^C|\psi_I^C\rangle||} = e^{-\Gamma t}\frac{1}{2}\left(\cosh\left(\frac{\omega_C t}{2}\right)+\cos\delta_C\right), \tag{3.73}$$

$$\frac{||\langle\phi_F^C|u_C|\varphi_I^C\rangle||^2}{||\langle\phi_F^C|\phi_F^C\rangle||\ ||\langle\varphi_I^C|\varphi_I^C\rangle||} = e^{-\Gamma t}\sinh^2\left(\frac{\omega_C t}{2}\right), \tag{3.74}$$

as well as

$$\frac{||\langle\varphi_F^C|\rho_C U_C|\varphi_I^C\rangle||^2}{||\langle\varphi_F^C|\rho_C|\varphi_F^C\rangle||\ ||\langle\varphi_I^C|\rho_C|\varphi_I^C\rangle||} = e^{-\Gamma t}\sinh^2\left(\frac{\omega_C t}{2}\right), \tag{3.75}$$

$$\frac{||\langle\psi_F^C|\rho_C U_C|\varphi_I^C\rangle||^2}{||\langle\psi_F^C|\rho_C|\psi_F^C\rangle||\ ||\langle\varphi_I^C|\rho_C|\varphi_I^C\rangle||} = e^{-\Gamma t}\frac{1}{2}\left(\cosh\left(\frac{\omega_C t}{2}\right)+\cos\delta_C\right), \tag{3.76}$$

$$\frac{||\langle\psi_I^C|\rho_C U_C|\varphi_F^C\rangle||^2}{||\langle\psi_I^C|\rho_C|\psi_I^C\rangle||\ ||\langle\varphi_F^C|\rho_C|\varphi_F^C\rangle||} = e^{-\Gamma t}\frac{1}{2}\left(\cosh\left(\frac{\omega_C t}{2}\right)+\cos\delta_C\right), \tag{3.77}$$

$$\frac{||\langle\psi_F^C|\rho_C U_C|\psi_I^C\rangle||^2}{||\langle\psi_F^C|\rho_C|\psi_F^C\rangle||\ ||\langle\psi_I^C|\rho_C|\psi_I^C\rangle||} = e^{-\Gamma t}\frac{1}{2}\left(\cosh\left(\frac{\omega_C t}{2}\right)-\cos(2\delta_C)\right). \tag{3.78}$$

Similar conclusions can be drawn, as depicted in the figures 3.1, where it has been depicted the behaviour of the two quantities contained in (3.78), namely

$$P_{IF} = e^{-\Gamma t}\frac{1}{2}\left(\cosh\left(\frac{\omega_C t}{2}\right)-\cos(2\delta_C)\right), \tag{3.79}$$

and

$$P0_{IF} = \frac{1}{2}\left(\cosh\left(\frac{\omega_C t}{2}\right)-\cos(2\delta_C)\right). \tag{3.80}$$

Whereas the expression (3.79) has a nontrivial time-dependence, so that the determination of the passage time requires the solution of a transcendental equation, one can obtain from (3.80) that

$$\tau_C = \frac{1}{\omega_C}\text{arccosh}\left(2P0_{IF} + \cos(2\delta_C)\right), \tag{3.81}$$

with the possibility of controlling the minimum time interval by varying δ_C. Instead of solving the transcendental equations associated to (3.79) we can verify graphically, in Figure 3.1, that the time it takes to go from one state to another is tunable even in the presence of dissipation. For a fixed amplitude of the transition frequency and certain values of r and γ it can be noticed that changing the separation ϵ of levels before the dissipative coupling in (3.45) one can see that this variation will crucially affect the time it takes for a desired transition to be complete.

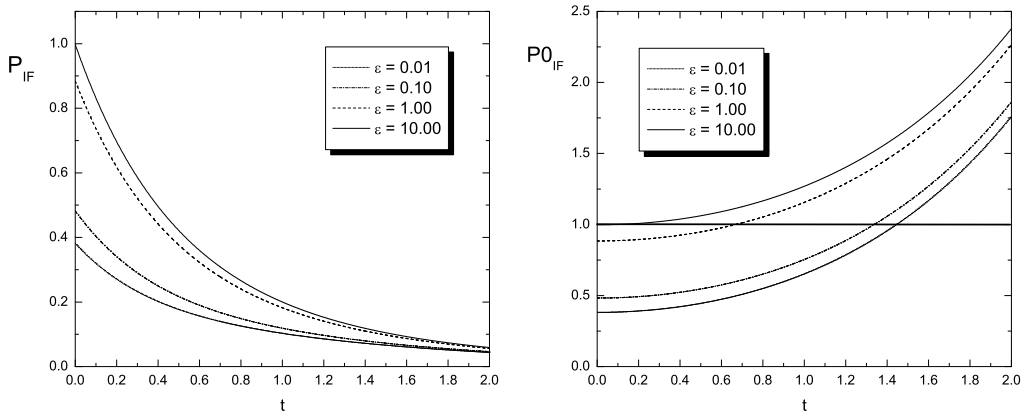

Figure 3.1: Behaviour, as a function of time, of two natural quantities appearing in $||\langle\psi_F^C|\rho_C U_C|\psi_I^C\rangle||^2$ for the specific values $\omega_C = 1, r = 1, \gamma = \frac{\pi}{8}$ and various values of ϵ. Disregarding the dissipative decay the transition $P0_{IF} = 1$ can be accomplished in a variable interval.

Therefore, by not insisting on the $\mathcal{PT}$-symmetry of the Hamiltonian by allowing it to be completely broken as we consider an effective Hamiltonian with complex characteristic energies and dissipative behaviour, we found that the same intriguing feature as observed in [23]. The observations that a quantum brachistochrone evolution may occur when one projects between orthonormal states, which are not eigenstates of the non-Hermitian Hamiltonian associated to the time evolution operator, irrespective of whether this Hamiltonian is $\mathcal{PT}$-symmetric or not. Clearly, it would be desired to have a more formal and

generic proof for these phenomena, rather than case-by-case studies but the latter are enough to make such claims.

In [116] the result of Bender et al has been challenged. However, the author has just missed this point and only argued on the two equivalent formulations of equation (3.33), which evidently always yields the same fixed passage time (3.44). Similar considerations are made in [117], where the author uses interesting geometrical arguments but does not address the crucial equation (3.42) and therefore reaches the wrong conclusion.

3.5 Geometry of the state space for the quantum brachistochrone

From Schrödinger equation (2.124) one can write

$$||\langle\phi(t)|\varphi(t+dt)\rangle||^2 = 1 - \Delta E^2\, dt^2\ , \tag{3.82}$$

where

$$\Delta E^2 \equiv \langle\phi(t)|H^2|\varphi(t)\rangle - \langle\phi(t)|H|\varphi(t)\rangle^2\ , \tag{3.83}$$

so that the energy uncertainty ΔE provides a measure of the speed of transition between states, separated by $ds = \Delta E\ dt$, reinforcing the need to keep the energy gap between states fixed. In the case of a Hermitian Hamiltonian $\langle\phi(t)| = |\varphi(t)\rangle^\dagger$ the expression above is known as the *Anandan-Aharonov* relation [118]. For pseudo- and quasi-Hermitian Hamiltonians $\langle\phi(t)| \neq |\varphi(t)\rangle^\dagger$, but the resultant value of ΔE will remain the same, as a consequence of the equivalence between purely Hermitian and purely pseudo/quasi-Hermitian systems. This is precisely the result obtained by Mostafazadeh in [117, 119]. There he uses the notion of projective Hilbert space, a space of all the rays representing quantum states and endowed with a geometric structure, to calculate the distance element ds^2 in terms of the projection operator,

$$\Lambda_\rho = \frac{|\varphi\rangle\langle\phi|}{\langle\phi|\varphi\rangle}, \tag{3.84}$$

as

$$ds^2_\rho = \frac{1}{2} tr\,(d\Lambda_\rho\ d\Lambda_\rho) = \frac{\langle\phi|\varphi\rangle\langle d\phi|d\varphi\rangle - ||\langle\phi|d\varphi\rangle||^2}{||\langle\phi|\varphi\rangle||^2}. \tag{3.85}$$

expression which is usually associated with Fubini-Study metric.

Using the equations of motion satisfied by the states $|\varphi(t)\rangle$ and $\langle\phi(t)|$ in (2.124) and (2.125), one may describe the vector $n(t) = (n_1(t), n_2(t), n_3(t))$, with $n_i(t) = \langle\phi(t)|\sigma_i|\varphi(t)\rangle$,

according to

$$\frac{d}{dt}n(t) = 2\Omega \times n(t), \tag{3.86}$$

which implies that the behaviour of the 2-level state can be made equivalent to the motion of a vector on a Bloch sphere described by Ω [5]. For the non-Hermitian Hamiltonian (3.16) we have $\Omega \cdot \Omega = \rho^2 \cos\delta$ after making use of the change of parameters introduced around (3.15), but Nesterov showed that for some particular form of non-Hermitian Hamiltonians with real eigenvalues the form of this quantity $\Omega \cdot \Omega$ specifies a one-sheeted two dimensional hyperboloid rather than a Bloch sphere. This difference could explain geometrically the possibility of achieving faster evolution than in Hermitian quantum mechanics [120].

3.6 Interpretation of the physical setup

As a *Gedankenexperiment* proposed in [23] the initial state could be prepared in a Stern-Gerlach filter and, after the non-Hermitian evolution, a second Stern-Gerlach apparatus could confirm the desired state, i.e., the direction of the spin with respect to σ_3 in the usual Hilbert space. However, in $\mathcal{H}_\rho$ this projection observable is described by

$$\Sigma_3^{N(R/C)} = \eta_{N(R/C)}^{-1}\, \sigma_3\, \eta_{N(R/C)} = \frac{1}{\cos\delta_{(R/C)}} \begin{pmatrix} -1 & \imath \sin\delta_{(R/C)} \\ \imath \sin\delta_{(R/C)} & 1 \end{pmatrix}, \tag{3.87}$$

with eigenvectors $\left(\sin\left(\frac{\delta_{(R/C)}}{2}\right), -\imath\cos\left(\frac{\delta_{(R/C)}}{2}\right)\right)$ and $\left(\imath\cos\left(\frac{\delta_{(R/C)}}{2}\right), \sin\left(\frac{\delta_{(R/C)}}{2}\right)\right)$ associated to the measurements ± 1. Therefore, the quasi-pseudo -up and -down states are only orthogonal in the regular sense when precisely the tuning parameter vanish. This demonstrates in a very clear way that although the faster evolution must be driven by an intermediate non-Hermitian Hamiltonian the measurement must be made in the usual Hermitian framework so that the direct comparison makes sense.

Because in this scenario the governing Hamiltonian changes from a Hermitian to a non-Hermitian one, the Hamiltonian becomes explicitly time dependent. This means we have to describe that setting by using $H(t) = h_0$, with $h_0^\dagger = h_0$, for $|t| > T$, and $H(t) = h_0 + g h_1$, with $h_1^\dagger \neq h_0$, for $t < |T|$, considering the non-Hermitian Hamiltonian acts between the instants $-T$ and T:

$$H(t) = h_0 + g\, h_1\, \theta(T - |t|), \tag{3.88}$$

[5] The Bloch sphere is a two-dimensional geometric representation of a state in a two-level quantum system.

with $\theta(x)$ denoting the Heaviside step-function.

Albeit a time-dependent problem, a time-independent metric is enough for our analysis. A standard example of $H(t)$ is for instance the Stark-LoSurdo Hamiltonian describing an atom in an external electric field. In [55, 106] the situation where also the unperturbed system was taken to be non-Hermitian. Concerning the fact that as soon as the non-Hermitian Hamiltonian comes into play the operator with respect to which the orthogonal states had been orthogonal loses its status as a physical observable, we note that no measure is to be made under the effect of the non-Hermitian evolution. Instead, measurements would be made when the non-Hermitian Hamiltonian is turned off.

Interpreting the specific brachistochrone setup as a subsystem of a larger Hermitian system, the authors in [121] confirmed the compatibility of a fast evolution with geometric analysis. However, since in the generalization of the measurement process proposed by them observables are defined as a whole conjugacy class of Hermitian operators, the state obtained after the measurement of an eigenvalue could not be determined because eigenstates would not coincide despite being associated to operators in the same similarity class [123]. More recently, Günther and Samsonov [124] used ideas from positive operator-valued measures (POVM). e.g. [122], to provide a new possibility to achieve this process in a laboratory. They use the fact that left- and right-eigenvectors of non-Hermitian operators ($\langle\phi_n|$ and $|\varphi_n\rangle$) are not simply related by a Hermitian conjugation operation to construct from all states $|\phi_n\rangle$ and $|\varphi_n\rangle$ an over-complete orthogonal basis whose dimension doubles. The coincidence problem mentioned in the end of section 3.3 is completely avoided by such an approach. Moreover, their study confirms the need of a Hermitian framework surrounding the non-Hermitian evolution.

It could be that the switching between Hilbert spaces could not be achieved instantaneously and the possibilities here presented would be violated, but problems which make us ask more fundamental questions, like the mixing of Hilbert spaces, are definitely interesting and deserve attention. In this sense, the quantum brachistochrone provides a very convenient framework for these investigations due to the simplicity of the Hilbert space. Non-Hermitian problems in infinite dimensional Hilbert spaces will be discussed in the sequence but before doing so we introduce a few important concepts which will permeate our analysis or at least serve as driving interest, namely solvability and integrability.

4 Symmetries, Integrability and Solvability

4.1 The importance of symmetries in Physics

We have seen in the previous chapters the useful consequences of anti-unitary transformations in fundamental description of nature, with $\mathcal{PT}$-symmetry as a notable example. A great deal of problems studied in undergraduate books present some sort of symmetry which allows for a complete solution. It is well known that a three-body problem in general has no exact solution but symmetries play a crucial role when the degrees of freedom in a problem increase in a sense that they might be the reason why some special cases are exactly solved. The Kepler problem of motion under central forces, the harmonic oscillator in any finite dimension or Lagrange and Euler tops are examples commonly examined in classical mechanics, e.g. [125].

Symmetries in general have the ability to greatly simplify physical problems and therefore deserve a special place in theoretical physics due to the fact that they are intimately related to conservation laws. This is essentially the content of Noether's theorem stating that whenever we find a continuous symmetry of the action I describing the problem in terms of the Lagrangean function L, then there must exist an associated conserved charge. This can be seen by applying Hamilton's least action variational principle and obtaining the corresponding equations of motion

$$I = \int dt\, L\left(\{q_i, \dot{q}_i\}; t\right) \qquad \Longrightarrow \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0\,, \quad i = 1, .., n, \tag{4.1}$$

so that whenever the system does not depend on q_j and therefore $\frac{\partial L}{\partial q_i} = 0$ a conserved quantity is found to be $\frac{\partial L}{\partial \dot{q}_j}$. Energy, momentum and angular momentum are crucial physical quantities whose conservation can be measured experimentally and which are closely

related to symmetries, namely the invariance under the flow of time, space translation and rotation, respectively.

Because Noether's theorem is formulated in terms of actions it requires the existence of a Lagrangian framework underlying the system so that it does not apply to dissipative systems. For such situations, the occurrence of differentiable symmetries is not a guarantee that conserved quantities can be associated to the problem. The conservation laws have a crucial property of restricting the phase space of a problem, facilitating its solution by reducing the number of degrees of freedom. However there are situations in field theory for which an infinite number of conserved quantities might be found even if no Lagrangian symmetry associated to it. In such cases, one talks about hidden symmetries and the latter can be described in terms of Lie algebraic structures.

A bit less studied than classical particle systems, nonlinear equations such as Korteweg - de Vries [29], Boussinesq [126], nonlinear Schrödinger [127], sine-Gordon and Toda theories, e.g. [128], have also been completely integrated due to the existence of symmetries. Likewise, in quantum mechanics a considerably small class of problems can have their complete spectral properties determined, with known wavefunctions and eigenvalues, such as the Hydrogen atom or the harmonic oscillator. There are even situations where only part of the spectrum can be calculated exactly, which is already some nontrivial achievement.

Symmetries are useful in physical sciences to describe crystal structures or to classify fundamental particles since with the determination of the symmetry properties of a system many of its features may be extracted. A very important theorem, due to Coleman and Mandula [129], states that the only conserved quantities in a physical theory must be invariant under Lorentz transformations, i.e. those space-time transformations relating the coordinates of objects in different reference frames according to Special Relativity. As a consequence, as long as some requirements are fulfilled, realistic theories can only involve symmetries which do not mix internal symmetry groups, like spin, with Poincaré group symmetries, which are composed of translations, rotations and Lorentz transformations. The Coleman-Mandula theorem therefore correspond to important restrictions in nature. A surprising combination of space-time with internal symmetries can be achieved by supersymmetry, e.g. [130], where symmetries are not only formulated in terms of commutation relations but also with anti-commutation relations.

One of their interesting properties is that if you apply successive symmetries to an object the composed action will still be a symmetry, allowing them to considered to form

a group, as will be discussed as follows. Actually, the branch of science responsible for providing a general fabric to study symmetries is known as group theory.

4.1.1 Lie groups and Lie algebras

Groups constitutes an important concept to understand mathematical structures in a system. A group G is formed by group elements g_i together with a composition law $\circ$ describing how these elements combine and satisfy a few defining properties [131]:

- Closure: $g_1, g_2 \in G \implies g_1 \circ g_2 \in G$.
- Associativity: $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$.
- Identity element: $\exists\, \mathbb{1} \quad | \, g_i \circ \mathbb{1} = \mathbb{1} \circ g_i = g_i, \quad \forall \quad g_i \in G$.
- Inverse element: $\exists\, g_i^{-1} \quad | \, g_i \circ g_i^{-1} = g_i^{-1} \circ g_i = \mathbb{1}, \quad \forall \quad g_i \in G$.

Groups may be identified as either finite or infinite. The former, as the name indicates, are composed of a finite number of elements, such as the cyclic group, whose elements Z satisfy $Z^n = \mathbb{1}$, and the symmetric group of permutations of a finite set of elements. On the other hand, the latter have infinitely many members and some of which might also be continuous, for instance rotations, parameterized by a set of continuous variables $\{x_i\}$. However, not all of them are continuous, like integers or rationals under addition. In the case of continuous infinite groups, if the functions describing combinations of the group elements are analytic one has a *Lie group*. Each point of such groups might be approximated locally by an Euclidean plane tangent to it, so that each point can be characterized by a combination of tangent fields,

$$A(x) = \sum_i A_i(x) \frac{\partial}{\partial x_i}. \tag{4.2}$$

A composition of two such elements gives

$$A(x) \circ B(x) = \sum_{i,j} \left(A_i(x) B_j(x) \frac{\partial^2}{\partial x_i \partial x_i} + B_j(x) \frac{\partial A_i(x)}{\partial x_j} \frac{\partial}{\partial x_i} \right), \tag{4.3}$$

in a way that

$$A(x) \circ B(x) - B(x) \circ A(x) = \sum_{i,j} \left(B_j(x) \frac{\partial A_i(x)}{\partial x_j} - A_j(x) \frac{\partial B_i(x)}{\partial x_j} \right) \frac{\partial}{\partial x_i}, \tag{4.4}$$

has the same form as (4.2), being closed under commutation operation,

$$C(x) \equiv [A(x), B(x)] = \sum_i C_i(x) \frac{\partial}{\partial x_i}. \tag{4.5}$$

For this reason it is said that Lie brackets, defined by $[X, Y] = X \circ Y - Y \circ X$, are the most important operations defined on vector fields (4.2). An algebra $\mathcal{G}$ with the anti-symmetric composition defined by commutation relations, called a *Lie algebra*, e.g. [132], satisfies

- Bilinearity: $[xX, yY + zZ] = xy[X, Y] + xz[X, Z]$,
- Anti-symmetry: $[X, Y] = -[Y, Z]$,
- Jacobi's identity: $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$,

for any vector fields $X, Y, Z \in \mathcal{G}$ and constants x, y, z.

We say two algebras are isomorphic, denoted by $\mathcal{H} \simeq \mathcal{G}$, if they satisfy the same commutation relations. Using the exponential map we can associate group elements to the generators of the algebra,

$$g = e^{\mathcal{G}} = e^{i\bar{\mathcal{G}}}, \tag{4.6}$$

so there will be a number equal to $\dim\mathcal{G}$, the dimension of the vector space associated to $\mathcal{G}$, of generators T_a for the algebra satisfying

$$[T_a, T_b] = f^c_{ab} T_c, \tag{4.7}$$

with $a, b, c = 1, ..., \dim\mathcal{G}$ for some values f^c_{ab} denoted structure constants which characterize completely the algebra and the local action of the associated group. If the corresponding vector space is of infinite dimension one has an infinite dimensional algebra, with an infinite number of generators, such as Kac-Moody algebras or Virasoro algebras, e.g. [133]

However, there are in principle different ways to reproduce the commutation relations (4.7) with the use of non-commuting operators, such as matrices or differential operators for example. Each of these sets of elements which reproduce the commutation relations of the algebra constitute a representation of it, so that the representation of the generators in general is not unique. Denoting $d(g)$ a matrix with entries $d^c_b(g)$ representing the element $g \in G$, the *adjoint representation* is given by $g\ T_a\ g^{-1} = d^b_a(g) T_b$ and has some interesting properties, such as $d(g_1 \circ g_2) = d(g_1)d(g_2)$ and $d^c_b(T_a) = f^c_{ab}$. It also allows for the construction of the $Killing form$ of an algebra,

$$\kappa_{ab} = \mathrm{tr}\left(d(T_a)d(T_b)\right). \tag{4.8}$$

An algebras is considered to be *simple* if it has no invariant abelian subalgebras. If the algebra is *semi-simple*, or in other words if it is a direct sum of commuting simple algebras,

then the Killing form will be positive definite if the algebra is compact [134, 135]. Compact algebras, differently from noncompact ones, admit finite dimensional representations.

4.1.2 Representation theory of semi-simple Lie algebras

Given a semi-simple Lie algebra $\mathcal{G}$, we call a *Cartan subalgebra* any maximal subalgebra composed of commuting generators H_i with $i = 1, ..., r \equiv \text{rank}\mathcal{G}$. Then, the complement of this subalgebra is formed by *step operators* E_α, diagonalizing the Cartan subalgebra generators,

$$[H_i, H_j] = 0 \ , \qquad [H_i, E_\alpha] = \alpha_i E_\alpha \ , \quad \text{with} \quad i = 1, ..., r. \tag{4.9}$$

in the Cartan-Weyl representation. The eigenvalues α_i are components of the so called *roots* α and the vector space where they live can be divided into hyperplanes perpendicular to them. The regions within hyperplanes are denoted *Weyl chambers* and the reflections of the roots on these Weyl planes, known as Weyl reflections, form a closed group of transformations. The Weyl reflection σ_b of a certain root α_a with respect to the hyperplane perpendicular to α_b can be expressed as [6]

$$\sigma_b(\alpha_a) = \alpha_a - 2\frac{\alpha_a \cdot \alpha_b}{\alpha_b \cdot \alpha_b}\alpha_b. \tag{4.10}$$

Note that the dimension of the vector space associated to the σ_a does not have to equal the rank of the algebra. Also the choice in representing the α_b will determine the representation of the algebra itself so that roots are key ingredients in the construction of algebraic representations. Apart from the dimension of the representation, all the information contained in the roots turns out to be encoded in a very simple way by *Cartan matrices* K with entries

$$K_{ab} = \frac{2\alpha_a \cdot \alpha_b}{\alpha_b^2}, \tag{4.11}$$

so that they can be used to characterize a particular algebra. The eigenstates of H_i are denoted by $|\mu\rangle$ and are associated to eigenvalues μ_i are components of vectors μ called weights, satisfying

$$H_i|\mu\rangle = \mu_i|\mu\rangle \ , \qquad \frac{2\alpha_a \cdot \mu_b}{\alpha_a^2} = \Delta_{ab} \ \in \mathbb{Z}. \tag{4.12}$$

If $\Delta_{ab} = \delta_{ab}$, the Kronecker delta, the weights, denoted by λ_i are called fundamental weights.

[6] Note that whereas indices i, j denote the components of a root, the indices a, b are used to distinguish different roots.

Algebraic structure of $sl(2,\mathbb{C}) = sl(2,\mathbb{R}) \oplus su(2) \simeq su(1,1) \oplus su(2)$

The finite dimensional non-trivial simple Lie algebras can be classified according to their Cartan matrices and therefore separated into four infinite families, $A_n (n \geq 1)$, $B_n (n \geq 2)$, $C_n (n \geq 3)$ and $D_n (n \geq 4)$, plus the possible algebras E_6, E_7, E_8, F_4, G_2. The former four correspond to groups with a classical matrix interpretation, namely the general linear, the unitary, the orthogonal and the symplectic groups:

General Linear Group, $GL(n)$ **:** group of $n \times n$ invertible matrices g, i.e. with $\det g \neq 0$;

Unitary Group $U(n)$ **:** group of $n \times n$ invertible matrices U, i.e. $U^\dagger \, U = \mathbb{1}$;

Orthogonal Group $O(n)$ **:** group of $n \times n$ orthogonal matrices O, i.e. $O^T \, O = \mathbb{1}$;

Symplectic Group $Sp\,(2n)$ **:** group of $2n \times 2n$ symplectic matrices S, i.e. $S^T \, \Omega \, S = \Omega$, with $\Omega^T = -\Omega$ an anti-symmetric matrix;

and the associated special groups $SL(n)$, $SU(n)$ and $SO(n)$ obtained by fixing the determinants of the matrices to be unity. In this new notation one has $A_n = SU(n+1), B_n = SO(2n+1), C_n = Sp(n), D_n = SO(2n)$.

A clear example of such objects is found in the study of the group formed by two-dimensional invertible matrices with real entries. Restricting to determinants equal to 1, we have

$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad | \quad a,b,c,d \in \mathbb{R} \quad | \quad ad - bc = 1 \right\} \tag{4.13}$$

Because $SL(2,\mathbb{R})$ is part of the special group, having unit determinant, the algebra generators of this group must be traceless. A common basis to represent a general element of the associated algebra $sl(2,\mathbb{R})$ is given by

$$H = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \tag{4.14}$$

satisfying

$$[H, E_\pm] = \pm E_\pm \, , \qquad [E_+, E_-] = 2H \, , \qquad \text{with} \qquad H^\dagger = H \, , E_\pm^\dagger = E_\mp \, . \tag{4.15}$$

If instead of real elements one works with matrices with complex entries, the group $SL(2,\mathbb{C})$ is somewhat similar and the same basis (4.14) can be used for the associated

algebra, $sl(2,\mathbb{C})$. Real combination R_i of (4.14) can be still be used as generators for $sl(2,\mathbb{R})$, for instance

$$R_1 = H = \frac{1}{2}\sigma_3 \ , \quad R_{2,0} = \frac{1}{2}\left(E_+ \pm E_-\right) = \frac{1}{2}e^{\imath\pi\frac{(1\mp 1)}{2}}\sigma_{1,2} \ , \tag{4.16}$$

and $K_i \equiv -\imath R_i$ so that different commutation relations satisfied are

$$\begin{array}{rclrclrcl} [R_1,R_2] &=& R_0, & [R_2,R_0] &=& -R_1, & [R_0,R_1] &=& -R_2, \\ [K_1,K_2] &=& -\imath K_0, & [K_2,K_0] &=& \imath K_1, & [K_0,K_1] &=& \imath K_2, \end{array} \tag{4.17}$$

with

$$\begin{array}{rclrclrcl} R_0^\dagger &=& -R_0, & R_1^\dagger &=& R_1, & R_2^\dagger &=& R_2, \\ K_0^\dagger &=& K_0, & K_1^\dagger &=& -K_1, & K_2^\dagger &=& -K_2. \end{array} \tag{4.18}$$

Because there are three generators, we have a three-dimensional adjoint representation of this algebra,

$$\mathrm{ad}\,(R_k)_{ij} = [R_k,R_i]_{R_j} \qquad \text{for} \ \ i,j,k = 0,1,2, \tag{4.19}$$

where $[X,Y]_Z$ means the Z-component of the commutator $[X,Y]$. The Killing form, in (4.8), becomes $\kappa = \mathrm{diag}(2,-2,2)$ so that $sl(2,\mathbb{R})$ corresponds to a non-compact algebra, as mentioned after the aforementioned equation, (4.8).

On the other hand, compact algebras appear when one takes complex combinations C_i of (4.14), such as

$$C_0 = \imath H \ , \quad C_{1,2} = \frac{\imath}{2}e^{-\imath\pi\frac{(1\mp 1)}{2}}\left(E_+ \pm E_-\right) \ , \qquad L_i \equiv -\imath C_i, \tag{4.20}$$

so that

$$\begin{array}{l} C_0 = \frac{\imath}{2}\sigma_3 \ , \quad C_1 = \frac{\imath}{2}\sigma_1 \ , \quad C_2 = \frac{\imath}{2}\sigma_2 \ , \\ L_0 = \frac{1}{2}\sigma_3 \ , \quad L_1 = \frac{1}{2}\sigma_1 \ , \quad L_2 = \frac{1}{2}\sigma_2 \ , \end{array} \tag{4.21}$$

satisfying

$$\begin{array}{rclrclrcl} [C_1,C_2] &=& -C_0, & [C_2,C_0] &=& -C_1, & [C_0,C_1] &=& -C_2, \\ [L_1,L_2] &=& \imath L_0, & [L_2,L_0] &=& \imath L_1, & [L_0,L_1] &=& \imath L_2, \end{array} \tag{4.22}$$

together with

$$\begin{array}{rclrclrcl} C_0^\dagger &=& -C_0, & C_1^\dagger &=& -C_1, & C_2^\dagger &=& -C_2, \\ L_0^\dagger &=& L_0, & L_1^\dagger &=& L_1, & L_2^\dagger &=& L_2. \end{array} \tag{4.23}$$

Because the Killing form in this case is $\kappa = \mathrm{diag}(2,2,2)$ we conclude the algebra $sl(2,\mathbb{C})$ admits compact representations.

A well known compact algebra emerging from the study of two-dimensional matrices is $su(2)$. The elements of the group $SU(2)$ are 2×2 unitary complex matrices with

$\det = 1$, so that according to (4.6) the generators of the associated algebra are traceless and anti-Hermitian,

$$su(2) = \left\{ \mathcal{G} \quad | \quad \mathcal{G} \in sl(2,\mathbb{C}) \quad | \quad \mathcal{G}^\dagger = -\mathcal{G} \right\}. \tag{4.24}$$

with $\bar{\mathcal{G}}^\dagger = \bar{\mathcal{G}}$. Consequently, remembering that the Pauli matrices are Hermitian we can obtain an appropriate basis by taking C_i (or J_i) in (4.20) and

$$su(2) = \text{span}\left\{ \imath\sigma_1, \imath\sigma_2, \imath\sigma_3, \right\} = \text{span}\left\{ \imath\sigma_2 \right\} \oplus \text{span}\left\{ \imath\sigma_1, \imath\sigma_3, \right\}. \tag{4.25}$$

On the other hand, the group $SU(1,1)$ is composed of complex unitary matrices with $\det = 1$ in a Minkoswki space in the sense that instead of the identity operator $\mathbb{1}$ one uses the $sigma_3$-matrix,

$$e^{\mathcal{G}^\dagger}\ \sigma_3\ e^{\mathcal{G}} = \sigma_3 \qquad \text{with } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.26}$$

so that the generators of the associated algebra satisfy

$$su(1,1) = \left\{ \mathcal{G} \quad | \quad \mathcal{G} \in sl(2,\mathbb{C}) \quad | \quad \mathcal{G}^\dagger = -\sigma_3\, \mathcal{G}\, \sigma_3^{-1} \right\}. \tag{4.27}$$

Thus we construct generators of $su(1,1)$ from combinations of (4.14)

$$\rho_0 = \imath H\ , \quad \rho_\pm = \frac{e^{\imath\frac{(1\mp 1)}{2}}}{\sqrt{2}} \left(E_+ \pm E_- \right)\ , \tag{4.28}$$

satisfying (4.27) and from which other complex combinations can be taken, for example R_i (or K_i) in (4.21). The latter, albeit not obeying (4.27), have its Killing form equivalent to $\kappa = \text{diag}(2,4,-4)$, indicating the algebra is still non-compact.

Considering certain algebra generators M_0, M_1, M_2 which satisfy the commutation relations

$$[M_1, M_2] = \imath\lambda M_0, \qquad [M_2, M_0] = \imath M_1, \qquad [M_0, M_1] = \imath M_2. \tag{4.29}$$

we see from (4.17) and (4.22) that the choice $\lambda = 1$ corresponds to the $su(2)$-Lie algebra, whose generators are denoted by $M_i \equiv L_i$, whereas $\lambda = -1$ represents $sl(2,\mathbb{R}) \simeq su(1,1)$, whose generators are taken to be $M_i \equiv J_i$ for $sl(2,\mathbb{R})$ or $M_i \equiv K_i$ for $su(1,1)$, depending on their conjugation relations. Then, the operators $M_0, M_\pm = M_1 \pm \imath M_2$ commute according to

$$[M_0, M_\pm] = \pm M_\pm, \qquad [M_+, M_-] = 2\lambda M_0. \tag{4.30}$$

The J_i generators are commonly taken to be

$$J_- = \partial_x, \qquad J_0 = x\partial_x - \frac{n}{2}, \qquad J_+ = x^2\partial_x - nx, \quad n \in \mathbb{Z}. \tag{4.31}$$

The corresponding Casimir operators, objects with the striking property of commuting with every element of the algebra, are given by

$$C = M_0^2 + \frac{\lambda}{2}\{M_+, M_-\}. \tag{4.32}$$

The algebras associated to the choices $\lambda = \pm 1$ appear recurrently in physics, for example in the study of angular momentum or bosonic interaction respectively. The dramatic difference between them lies in their representation theory: whereas the former, compact, admits finite dimensional representation the same is not true for the latter and all its unitary irreducible representations are infinite dimensional.

According to Schur's lemma [136] the Casimir operator in any irreducible representation is a multiple of the identity operator. Choosing the representation basis to be that of the simultaneous eigenstates of the two commuting operators M_0 (L_0 or K_0) and C,

$$L_0|\ell, \mu_\ell\rangle = \mu_\ell|\ell, \mu_\ell\rangle \qquad \text{and} \qquad C|\ell, \mu_\ell\rangle = \ell(\ell+1)|\ell, \mu_\ell\rangle, \tag{4.33}$$

$$K_0|k, \mu_k\rangle = \mu_k|k, \mu_k\rangle \qquad \text{and} \qquad C|k, \mu_k\rangle = k(k-1)|k, \mu_k\rangle, \tag{4.34}$$

with $\mu_\ell = 0, \pm\frac{1}{2}, ..., \pm\ell$, and $\mu_k = 0, 1, 2, ...$, so that the ladder operators act in this basis in different ways,

$$L_\pm|\ell, \mu_\ell\rangle = \sqrt{(\ell \mp \mu_\ell)(\ell \pm \mu_\ell + 1)}|\ell, \mu_\ell \pm 1\rangle, \tag{4.35}$$

and

$$K_\pm|k, \mu_k\rangle = \left(\mu_k \pm \left(\frac{1}{2} - \imath k\right)\right)|k, \mu_k \pm 1\rangle. \tag{4.36}$$

Starting from vacua states satisfying

$$L_-|\ell, -\ell\rangle = 0 \qquad \text{and} \qquad K_-|k, 0\rangle = 0, \tag{4.37}$$

all other states in the respective representations can be calculated by the action of the raising operators

$$|\ell, \mu_\ell\rangle = \sqrt{\frac{(\ell - \mu_\ell)!}{(\ell + \mu_\ell)!(2\ell)!}} L_+^{\ell+\mu_\ell}|\ell, -\ell\rangle, \tag{4.38}$$

$$|k, \mu_k\rangle = \sqrt{\frac{\Gamma(2k)}{\mu_k!\Gamma(2k + \mu_k)}} K_+^{\mu_k}|k, 0\rangle. \tag{4.39}$$

A simple but instructive example of the power of symmetries in physical theories can be found in the Hydrogen atom. Besides the usual approach to determine its solution, one can also employ representation theory of Lie algebras in a direct way as shown for

instance in [137]. Consider then the following realization of the $su(1,1)$ algebra generators in terms of differential operators,

$$K_0 = \left(\frac{d^2}{dy^2} + \frac{a}{y^2} - \frac{y^2}{16}\right) \quad \text{and} \quad K_\pm = \left(\frac{d^2}{dy^2} + \frac{a}{y^2} + \frac{y^2}{16}\right) \pm \frac{1}{2}\left(y\frac{d}{dy} + \frac{1}{2}\right), \tag{4.40}$$

which allow to express differential operators of the form

$$D_O = \left(\frac{d^2}{dy^2} + \frac{a}{y^2} + by^2 + c\right), \tag{4.41}$$

as an $su(1,1)$ element. After a transformation of variables followed by a gauge transformation the Hydrogen atom eigenvalues equation with angular momentum l and energy levels characterized by the quantum number n can be re-cast in the form of $K_0|n,m,l\rangle = \epsilon_n|n,m,l\rangle$. From the abstract analysis of the $su(1,1)$ algebra the problem is now essentially solved once the eigenstates of K_0 were already determined. This exemplifies at the same time the power of representation-independent Lie algebraic results as well as the importance of symmetries when searching for solutions in a problem. Because the Hydrogen atom could be re-expressed in terms of an $su(1,1)$ generator, the abstract knowledge of the eigenstates of K_0 can be immediately transferred to solve this problem. Similarly any other system, no matter how physically unrelated they might be, which can be shown to be just a different representation of K_0 will also allow for a simplification in the search for a solution.

4.2 Integrability

Integrable models are frequently regarded in the classical sense as systems of differential equations for which exact solutions can be found, with particular attention given to nonlinear equations, which are more difficult to solve than linear ones in general. In other words, one regards as integrable systems of differential equations which may be integrated despite the absence of linearity and the principle of superposition. Physically, these models are important because they describe a vast amount of natural phenomena in nonlinear theories. In fact nonlinearity is by no means an exception in the world and therefore the applicability of nonlinear systems is enormous in the real world. Also from a mathematical perspective integrability also offers a rich underlying structure, such as infinite dimensional Lie algebras, of the Kac-Moody type for instance, together with their representation theory and connections with the Riemann-Hilbert problem, e.g. [138], making it an exciting field of contact between physicists and mathematicians.

Thus, in principle a problem with enough symmetries can lead to exact solutions. Consider for instance an n-dimensional particle system described by a Hamiltonian function

$$H\left(\{q_i, p_i\}; t\right) = \sum_{i=1}^{n} \dot{q}_i p_i - L\left(\{q_i, \dot{q}_i\}; t\right), \tag{4.42}$$

related to the Lagrangian through a Legendre transformation, so that the dynamical equations of motion in terms of canonical coordinates are given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \{q_i, H\}_{PB} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} = \{p_i, H\}_{PB}, \tag{4.43}$$

where $i = 1, ..., n$ so that the system has $2n$ degrees of freedom, and the Poisson brackets are defined by

$$\{f, g\}_{PB} \equiv \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \right). \tag{4.44}$$

If there are n known functionally independent conserved quantities $F_i = F_i(q, p)$ in involution, that is

$$\dot{F}_i = \{H, F_i\}_{PB} = 0 \quad \text{and} \quad \{F_i, F_j\}_{PB} = 0, \tag{4.45}$$

then according to Liouville's theorem one can construct a canonical transformation $(q_i, p_i) \rightarrow (F_i, \omega_i)$ such that the integrals of motion are among the new coordinates, denoted by action and angle variables. The system can then be solved or integrated completely by quadratures, solving a finite number of equations. The ideas just presented are the core of Liouville's integrability and are suitable for finite dimensional configurations. Besides this concept of complete integrability, one can also distinguish cases between partial integrability and superintegrability, when the number of functionally independent quantities in involution is less, or more respectively, than the number of degrees of freedom.

For systems with infinite degrees of freedom this framework implies not only the need for an infinite set of charges but also an infinite number of steps before complete integration is achieved. Note that for continuous systems described by a Hamiltonian density $\mathcal{H}(x)$ the dynamical equations can be put in the form of non-canonical coordinates. Combining the noncovariant canonical coordinates q_i, p_i into a set of $2n$ generalized coordinates y^μ such that $y^i = q_i$ and $y^{n+1} = p_i$. Thus the fundamental Poisson brackets [128] take the form

$$\left\{y^i, y^{n+j}\right\} = \delta_{ij} \quad \text{or} \quad \{y^\mu, y^\nu\} = \epsilon^{\mu\nu} \ \text{ with } \epsilon^{\mu\nu} = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \tag{4.46}$$

and the dynamical equations of motion are written covariantly

$$\dot{y}^\mu = \{y^\mu, H\} = \epsilon^{\mu\nu}\partial_\nu H. \tag{4.47}$$

For a general set of noncanonical coordinates, such as in the case of constrained variables, a covariant formalism provides a similar description but $\epsilon^{\mu\nu}$ must be replaced by a coordinate dependent tensor $f^{\mu\nu}(y)$ obeying a few properties, see e.g. [128], so that in a continuum system one gets

$$\dot{u}(x) = \{u(x), H\} = \int dy\ f(x,y)\frac{\delta H}{\delta u(y)} \tag{4.48}$$

where δ denotes a functional derivative, defined when taking the continuous limit of the differential

$$dF = \sum_{m=1}^{M} \frac{\partial F(y_0)}{\partial y_n} dy_n, \tag{4.49}$$

when $M \to \infty$ and the discrete elements y_n become infinitely close to each other in a finite region, from a to b,

$$dF = \int_a^b dx \frac{\delta F[y(0)]}{\delta y(x)} \delta y(x). \tag{4.50}$$

The bracket structure of (4.48) is specified by

$$\{u(x), u(y)\} = f(x,y), \tag{4.51}$$

and

$$\{A[u], B[u]\} = \int\int dx\ dy\ \frac{\delta A}{\delta u(x)} f(x,y) \frac{\delta B}{\delta u(y)}. \tag{4.52}$$

However, as was shown by Poincaré most Hamiltonian systems are not integrable, indicating this might not be the best framework to explore such models. In fact, Liouville's approach is not the only definition of integrability and there are various nonequivalent alternative formulations, sometimes equivalent, for instance those related to hidden symmetries. The conservation of infinite charges during the dynamical evolution of integrable field theories gives rise to a very interesting kind of solutions, the *solitons*, discussed in more detail in the next section. Because the existence of such special objects depends deeply on the occurrence of an infinite number of conserved quantities, solitons are sometimes regarded as an equivalent concept to integrability. This operational approach allows labeling as integrable all models which admit solitons.

Solitary waves were already known for a long time, e.g. [139], when interest in classical integrable systems reappeared after the celebrated work of Fermi, Pasta and Ulam (FPU)

[140]. The numerical experiments conducted by them on vibrating strings containing non-linear terms indicated a surprising result: the emergence of non-ergodic behaviour, exhibiting complicated but almost exactly periodic motion. Such observation indicated the existence of nontrivial conservation laws restricting the physical phase space. The considerable difference between integrable and non-integrable models lies in the essentially distinct behaviour between regular and irregular motion, giving a deeper dimension to the simplistic characterization of integrable models as those which can be integrated exactly to a closed form. There is actually the imprecise but widespread characterization of completely integrable models as being those whose solution can be written in terms of known function although the logic usually goes in the reverse way, that is, functions are defined by equations they satisfy. Thus, there are deeper structures behind such systems.

The hidden symmetry in the FPU simulation can be explained using the continuum limit of the governing equations for the strings, namely the Korteweg - de Vries (KdV) equation, a relationship attributed to Kruskal and Zabusky [141]. The KdV equation,

$$u_t + \left(\alpha u_{xx} + \beta u^2\right)_x = 0 \qquad \text{or} \qquad u_t + \left[\alpha\partial_x^3 + \frac{2\beta}{3}(2u\partial_x + u_x)\right] u = 0 \tag{4.53}$$

with is a Hamiltonian system satisfying (4.48), with Hamiltonian density given by

$$\mathcal{H}(x) = \frac{\alpha}{2}u_x^2 + \frac{\beta}{3}u^3. \tag{4.54}$$

The left-hand side equation in (4.53) can be cast as

$$u_t = \frac{\partial}{\partial x}\frac{\delta H}{\delta u}, \tag{4.55}$$

from which comparison with (4.48) gives

$$f(x, y) = \partial_x \delta(x - y). \tag{4.56}$$

Because the second possibility in representing the KdV equation, as shown in the right-hand side of (4.53), there is another compatible Poisson structure, namely

$$f(x, u) = \left[\alpha\partial_x^3 + \frac{2\beta}{3}(2u\partial_x + u_x)\right] \delta(x - y), \tag{4.57}$$

describing the system. Therefore one says that it admits a bi-Hamiltonian formulation and Magri's theorem guarantees an infinite hierarchy of higher order commuting bi-Hamiltonian systems [142, 143]. What is more, the Poisson structure (4.57) being closely related to a classical version of a Virasoro algebra [144] allows for the construction of an infinite number of conservation laws.

The KdV, employed to describe the behaviour of waves of liquids in shallow depths, is historically also very important in the development of methods to solve exactly non-linear problems. The technique introduced by Gardner, Greene, Kruskal and Miura [145] represents the foundations of what is nowadays known as *inverse scattering method*, an analogue of Fourier analysis for nonlinear equations, through the solution of associated integral equations. It is based on the introduction of a linear operator determined in phase space and evolving the system isospectrally.

Later a representation of the KdV equation in terms of non-commuting operators was found by Lax [146], who used it to explain its integrability in terms of a linear isospectral evolution. The Lax operators introduced then,

$$L = u + \frac{3\alpha}{\beta}\frac{\partial^2}{\partial x^2} \quad \text{and} \quad M = (4\alpha)\frac{\partial^3}{\partial x^3} + 2\beta u\frac{\partial}{\partial x} + \beta u_x, \tag{4.58}$$

make the KdV equation equivalent to

$$\frac{\partial L}{\partial t} - [M, L] = 0. \tag{4.59}$$

This formulation is valid only for $(1+1)$-dimensional systems but has the advantage that (4.59) is equivalent to an isospectral evolution of L generated by M,

$$L(t) = U(t)L(0)U(t)^{-1} \qquad \Longleftrightarrow \qquad \frac{\partial U}{\partial t} = MU, \tag{4.60}$$

so that the eigenvalues λ_i of L are consntants of motion $\dot{\lambda}_i = 0$ in involution, $\{\lambda_i, \lambda_j\}_{PB} = 0$. Consequently, the traces of powers of L are conserved charges,

$$\frac{d}{dt}\mathrm{tr}L^n = \sum_i \dot{\lambda}_i^n = 0. \tag{4.61}$$

Soon these ideas were applied to other dynamical systems, making of Lax operators a general approach. More recently, Lax pairs were expressed in terms of a zero-curvature condition,

$$F_{\mu\nu} \equiv [\partial_\mu + A_\mu, \partial_\nu + A_\nu] = 0 \tag{4.62}$$

with $\mu, \nu = 0, 1$, representing just a compatibility condition expressed in terms of $\partial_0 = \partial_t, \partial_1 = \partial_x$, for two linear problems,

$$\begin{aligned} D_x\psi = 0\ , \quad &\text{with} \quad D_x \equiv \partial_x + A_x, \\ D_t\psi = 0\ , \quad &\text{with} \quad D_t \equiv \partial_t + A_t. \end{aligned} \tag{4.63}$$

This formulation is very fruitful in the study of Toda models and self-dual Yang-Mills equations and methods to solve nonlinear integrable systems have been extended

by Zakharov and Shabat [147] with the "dressing" method and variations allowing for the construction of solitons. The zero-curvature condition can be combined with Hirota method in a systematic approach to construct multi-soliton solutions in an effective way [148, 149, 150, 151]. Hirota method consists of expressing the problem in Hirota's bilinear form, e.g. [152, 153],

$$P(D)\tau \cdot \tau = 0 \tag{4.64}$$

which is a polynomial equation in Hirota's D-operator, defined by

$$e^{\delta D_x} f \cdot g = \sum_{k=0}^{\infty} \frac{\delta^k}{k!} D_x^n f(x) \cdot g(x) = f(x+\delta) g(x-\delta). \tag{4.65}$$

Hirota's operator therefore differ from the Leibniz differential operator by the presence of an alternating sign, responsible for the vanishing of

$$D_x^{2n+1} f \cdot f = 0 \,, \quad \text{if} \quad n \in \mathbb{N}. \tag{4.66}$$

For instance, considering a change of variables into τ-functions

$$u = \frac{6\alpha}{\beta} \frac{\partial^2}{\partial x^2} \ln \tau \tag{4.67}$$

the equation (4.53) can be reexpressed in terms of DH-operators as

$$D_x \left(D_t + \alpha D_x^3 \right) \tau \cdot \tau = 0. \tag{4.68}$$

The latter can be solved by expanding the τ-functions as

$$\tau = 1 + \epsilon \tau_1 + \epsilon^2 \tau_2 + \cdots \tag{4.69}$$

in a way that one can solve equations for each of the τ_i successively in exact form, i.e. non-perturbatively, order by order in ϵ, in a way that a truncation is achieved. The construction of a KdV soliton leads to

$$\tau = 1 + A \exp(ax - a^3 \alpha t), \tag{4.70}$$

with a and A free parameters.

Although many formulations have been presented above, from Liouville to inverse scattering, Lax pairs, zero-curvature and solitons, it remains unsolved how to determine a priori whether a given system is integrable or not. We discussed that Liouville's theorem might not be convenient for infinite dimensional configurations but also the question regarding the possibility of finding a Lax, or zero-curvature, representation for a certain

equation is difficult to achieve. It is certified that equations solved by the inverse scattering method, or any other linear and isospectral equivalent, all have an infinite hierarchy of local conservation laws or, in other words, continuity equations associated to it. Therefore the existence of nontrivial integrals of motion is indeed a good indicator of integrable behaviour but in principle one must find an infinite number of them before claiming integrability.

A more feasible approach is known as the Painlevé test, whose origins lie in the analysis of singularities in the solutions of differential equations and go back more than a century [154]. In analytic theory of ordinary differential equations one important class of system consists of those for which the location of any (algebraic, logarithmic or essential) singularity of their solutions is independent of initial conditions. Kowalesvki observed for the first time that there is a connection between the integrability of a system and the singularity structure of its solution as she was able to isolate an integrable case out of such class of equations. The system studied by her consisted of the motion of a rigid body about a fixed point and under the influence of gravitational force, i.e., a gravitational top.

Painlevé then classified second order nonlinear ordinary differential equations with no freedom in the singularities with regard to initial conditions, leading to the six celebrated Painlevé equations, denoted by the numbering from I to VI and the construction of new functions from the solutions of ordinary differential equations [155]. The very notion of a function implies immediately that the solutions one is seeking ought to be single-valued, which leads to a natural definition in terms of the movable singularities, that is to say those whose location depend on the initial conditions of the problem[7]: An ODE whose (general) solutions have no movable critical[8] singularities is said to possess the (generalized) Painlevé property [157, 158, 159, 160]. The classification of possible solutions to this problem can be organised into equivalence classes obtained from linear fractional (Möbius) transformations and has been completed only to some degree. The classification of algebraic ODEs with Painlevé property of order greater than two is still an open problem, albeit some partial results exist [161, 162, 163].

[7] Note that Fuch's theorem states that the solution of a linear second order ODE with variable coefficients can only be singular at the points where the coefficients are singular, e.g. [156]. Because these singularities are determined by the differential equations, they are said to be fixed and one has control over them. Differently from linear ODEs, for which only fixed singularities are admissible, nonlinear ODEs and PDEs may as well present singularities which are sensitive to the initial conditions.

[8] A critical singularity is multivalued in its neighbourhood.

Ablowitz and Segur observed that reductions of integrable equations lead to Painlevé equations, at least indirectly. Later it was conjectured that all ordinary differential equations related to integrable partial differential equations have the Painlevé property [164]. Weiss, Tabor and Carnevale developed a test to detect the existence of the Painlevé property without having to mention ordinary differential equations [32]. The Painlevé test will be discussed in a future chapter as we employ it to detect possible systems possessing integrability properties. The disadvantage of such a procedure is that it does not indicate direct ways to integrate the system. Therefore after the integrability of a system is established, possibly by carrying out the Painlevé test, one needs to combine the appropriate methods to construct multi-soliton solutions.

Before finishing this section it should be mentioned that in nature one encounters exactly solvable models which are more closely related to quantum integrable systems than classical ones. Among the most relevant methods to tackle these problems one finds the Bethe Ansatz approach based on Yang-Baxter equations and the quantum inverse scattering problem. The Bethe Ansatz is a method for finding exact solutions of one-dimensional quantum many-body models first introduced in the context of the one-dimensional antiferromagnetic Heisenberg model, e.g. [165]. By studying a quantum generalization of the zero curvature condition and formulating the so called quantum inverse scattering method one ends up with a new structure revealed, the R matrix. The latter readily leads to the celebrated Yang-Baxter equation, which is equivalent to factorization condition for the scattering-matrix. Thus, in the context of $(1+1)$-dimensional quantum field theories the notion of integrability is usually used synonymously to the factorization of the S-matrix, which can be achieved simply by making use of one non-trivial charge [166].

It can be seen that, unlike as in most scenarios when one compares quantum and classical theories, the latter appear to be more complicated in this particular regard. The richness of classical solitonic solutions are discussed next.

4.2.1 Solitons and Compactons

The importance of integrable systems lies not only in the mathematical possibility of constructing exact solutions, but also in the physical occurrence of localized and stable packets of energy travelling in space, often referred to as solitonic solutions as opposed to a disordered regime. Due to the severe restriction in their phase space, solitons are surprisingly stable under multi-soliton scattering processes. The terminology originates

from their solitary wave behaviour, in the sense that they are localized waves retaining their shapes as they propagate.

In a more precise sense, even though not consensually, solitons are defined as nonperturbative solutions of nonlinear equations which **i)** propagate with constant speed preserving their shape, **ii)** represent a localized distribution of energy without dissipating throughout the evolution and **iii)** whose only effect during a collision with another soliton is a mere relative displacement with respect to the place it would be had the scattering not occurred, or equivalently they are stable under collision up to a time delay. The nonlinearity is very important for the solution because the stability of the solution after a multi-soliton interaction may be seen as a compromise between nonlinearity and dispersive features of the equation of motion. Not to mention the absence of the superposition principle in a way that two solitons are not insensitive to each other but instead they interact.

Solitons appear in nature in many forms. Originally, the first description of a soliton was due to John Scott Russell, a Scottish naval engineer who observed in 1834 a solitary wave travalling a long distance without dissipating. The solitary wave observed by Scott Russell [167], when he was riding by the Grand Union Canal at Hermiston in Glasgow, were later shown to be described by the KdV equation with its soliton solution. Other forms of solitary waves in fluids are the impressive *morning glory* clouds formed typically in the Gulf of Carpentaria in northern Australia and which can extend for many kilometers. In smaller scales they also emerge in nonlinear optics, particularly the Kerr effect, when electric field of a light wave changes the index of refraction of the medium leading to solitary behaviour in to fiber optics. Due to very similar mathematical formulation (the Gross-Pitaevskii equation [168, 169]), optical solitons are present in Bose-Einstein condensates [171, 172] too. It is important to notice that simple solitary waves without the solitonic requirements may exist in nonintegrable systems.

The importance of solitons can be seen by the secondary concepts it originated: topological solitons, vector solitons, peakons, breathers and oscillons and compactons for instance. Topological solitons correspond to solitons whose stability is not due to integrability but because there are topological constraints restricting its behaviour, such as the existence of degenerate vacua. Examples of such a class of solitons include kinks in one dimension, lumps and vortices in two dimensions, monopoles and instantons in three dimensions and monopoles in four dimensions. Vector solitons are solitary waves with multiple polarization components, more frequent in optical setups. Peakons are solitons

with discontinuous first derivatives whereas breathers and oscillons are long living soliton-like solutions with localized and oscillatory behaviour. But oscillons are classical solution in nonintegrable field theories; unlike breathers, they radiate, interact inelastically and do not exhibit periodic oscillations in time.

Finally we have compactons [173], introduced recently as solitonic-like solutions with compact support, so that they do not have exponential tails stretching up to infinity and have a finite extension instead. The paradigmatic example of an equation with compact solutions is presented in [173],

$$u_t + (u^2 + (u^2)_{xx})_x = 0, \tag{4.71}$$

having solutions given by

$$u(x,t) = \begin{cases} \frac{4v}{3}\cos^2\left(\frac{x-vt}{4}\right) \ , & \text{if } |x-vt| \leq 2\pi, \\ 0, & \text{if } |x-vt| \geq 2\pi, \end{cases} \tag{4.72}$$

with v a free parameter. This function has a discontinuous second derivative at the end points $x = \pm 2\pi + vt$ but the equation of motion involves only $u_t, (u^2)_x, (u^2)_{xxx}$, which are all smooth even at the edges.

Studies on compactons [173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183] indicate they preserve the remarkable property that they scatter elastically, remaining entirely in shape after a collision [174, 175]. However, due to their finite wings, the interaction between two compactons has a short range when compared to solitons. Besides, compactons when scattered give also rise to a compacton anti-compacton pair with low relative amplitude, at least according to numerical simulations [173]. Besides the soliton analogues of compactons also breather type solutions have been found [180, 182] and more recently also higher dimensional compactons were constructed [183, 184]. These results, like the ones numerically observed by Fermi, Pasta and Ulam, indicate for surprising stability properties and demands for a better comprehension based on non-computational analysis.

In a recent investigation Bender, Cooper, Khare, Mihaila and Saxena [185] have found compacton solutions for $\mathcal{PT}$-symmetric extensions of generalized KdV equations. However, the relation between integrability and compactons does not seem to have been established, and $(1+1)$-dimensional integrable theories only solitons and breather play an important role. The possible dependence of compactons on integrability motivates our study on $\mathcal{PT}$-symmetrically deformed equations presenting compacton solutions. The classification of integrable systems presenting compacton solutions may lead to a better understanding of the structure behind the stability of compactons.

4.3 Solvability

In a previous section integrable models have been discussed and now we talk about solvable systems, the former implying that the amount of conserved quantities equals the degrees of freedom in the system roughly and the latter referring to a situation in which the spectra can be determined explicitly.

We have mentioned a few examples of problems whose spectral analysis can be fully carried out and argued that they belong to a comparatively small class, denoted by *exactly solvable* models. Being less demanding could in principle lead to a broader range of systems and this is accomplished by requiring only a finite part of the spectrum to be completely determined. These are called *quasi-exactly solvable* models.

A complete Hamiltonian can be represented in the form of an infinite dimensional matrix, which means that the knowledge on the spectrum depends on one's capability to diagonalize it. For a handful of problems there is a natural basis appearing in which the matrix assumes a diagonal form, as is the case of the harmonic oscillator, the Coulomb potential or even Morse and Pöschel-Teller potentials. Nonetheless, the infinite dimensionality of a matrix represents serious difficulties in its diagonalization as the problem becomes non-algebraic [186]. In this picture, quasi-exactly solvable models correspond to those having a block subspace with a finite dimension so that an exact solution can be obtained by diagonalizing for this finite subspace even though nothing being said about the infinite dimensional complementary space.

Whereas exactly solvable models, with all energy levels and corresponding wavefunctions determined explicitly, are of enormous importance they represent a huge restriction on the possible physical setups. With this in mind, quasi-exactly systems combine positive features of completely determined models in order to tackle an ampler variety of problems. For example, the quartic oscillator

$$H = p^2 + a_2 x^2 + a_4 x^4 \tag{4.73}$$

has no exact solution and cannot even be studied in a standard perturbative way because the anharmonic corrections are so big they must not be neglected and in situations like the knowledge of a fraction of the spectrum would already be helpful. The information coming from the finite parts of the solutions can then be used in the efforts to solve the complete spectrum exactly or at least perturbatively, allowing for a deeper understanding of underlying symmetries.

One can define a series of subspaces of an algebra $\mathcal{G}$ as below

$$\mathcal{G}_{(0)} = \mathcal{G} \ , \quad \mathcal{G}_{(1)} = [\mathcal{G}_{(0)}, \mathcal{G}_{(0)}] \ , \quad \cdots \ , \quad \mathcal{G}_{(n)} = [\mathcal{G}_{(n-1)}, \mathcal{G}_{(n-1)}] \ , \tag{4.74}$$

and denote *solvable algebras* those for which $\mathcal{G}_{(k)} = 0$ for some value k. This means the series does not extend indefinitely. Quasi-exactly solvable models can be established through a connection with Lie algebras admitting finite dimensional representations [187]. This notion was introduced by Turbiner [188] demanding that the action of quasi-exactly solvable operators on the space of polynomials leaves it invariant. More specifically when taking the operator to be a Hamiltonian operator H acting on the space of polynomials of order n

$$V_n = \text{span}\{1, x, x^2, x^3, x^4, ..., x^n\}, \tag{4.75}$$

as H: $V_n \mapsto V_n$, it preserves by definition the entire flag $V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n \subset \cdots$. Models respecting this property are referred to as *exactly solvable*.

Whenever exactly solvable Hamiltonians can be written in terms of combinations of first-order differential operators generating a finite dimensional Lie algebra, it is said they are of *Lie algebraic type* [189]. For quasi-exactly solvable problems at least for part of spectrum it is possible to analyse the hidden symmetries by means of partial algebraization, for instance using the representation theory of $sl(2, \mathbb{R})$-algebras. Quasi-exactly solvable models may be tackled with the use of a polynomial ansatz for the wavefunctions, point canonical transformations or Darboux transformations, for instance [190], and they have appeared in the context of $\mathcal{PT}$-symmetric quantum mechanics for the first time in [191]. Also many-body problems closely related to integrable Calogero models can be analysed in the framework of quasi-exactly solvability [192].

5 Mapping non-Hermitian into Hermitian Hamiltonians

5.1 Lie algebras and similarity transformations with operators

Many interesting and important physical problems may be formulated in terms of group theory and Lie algebras. For very general treatments one can take these descriptions as a starting point and generic frameworks, such that particular models simply result as specific choices of representations. The virtue of this kind of approach is that it allows for a high degree of universality and it has turned out to be especially fruitful throughout the branches of physics, particularly in the context of integrable and solvable models. In [193], e.g., the author study non-Hermitian Hamiltonians related to $SO(2,1)$ and solvable Scarf potentials and in [25] an $su(1,1)$ algebraic approach is used to describe the Swanson Hamiltonian [24]. Here we will extend such type of treatment to quasi-pseudo-Hermitian Hamiltonian systems.

Since Hermitian Hamiltonians are guaranteed to have real spectra, one obvious method to search for non-Hermitian Hamiltonians with real eigenvalues is to determine Hermitian counterparts belonging to the same similarity class. Since the metric operator ρ in (2.89) is of central importance in this approach many attempts have been made to construct it when given only a non-Hermitian Hamiltonian. It is very important to know the pseudo-Hermitian metric, since it determines the complete quantum formalism of the system, but in general it cannot be constructed exactly. So far one has only succeeded to compute exact expressions for the metric and isospectral partners in very few cases.

In Chapter 3 we examined a two-level system, for which the left- and right-eigenvectors can be determined easily both in the Hermitian and non-Hermitian frameworks. The

construction of metric operators indicated by (2.94) together with a complete description of the non-Hermitian system does not represent large technical efforts in that situation. However for infinite dimensional Hilbert spaces, diagonalization is seldom trivial. Although some examples may be treated within this formalism, such as the $\mathcal{PT}$-symmetric infinite square well [194, 195, 196] or a Delta-potental with complex coupling [197], in situations like these generally one must resort to other means of calculating the metric operator other than the spectral method.

Besides the knowledge of the complete set of eigenstates of the operator, one must perform infinite sums or integrals and establish convergence in order to determine ρ. Because usually one does not have all the eigenfunctions at one's disposal it is necessary to resort to more pragmatic techniques, such as perturbation theory [106, 68, 69, 198]. To overcome the difficulty in finding exact solutions, one may rely on perturbative calculations, in an approach similar to that described after equation (2.114) with an $\mathcal{R}$-expansion for the metric operator, so that one has to solve commutation relations order by order in the coupling constant.

Due to expression (2.94) it would be opportune to work with models for which wavefunctions could be constructed exactly. Thus, a logical step is to turn our attention to solvable models, i.e., those preserving the vector space V_n of polynomials of order n. More useful would be if we could formulate the problem in terms of Lie algebras so that a general treatment could be obtained with specific models coming as special choices of the representation. In order to be more concrete we have to identify V_n as the representation space of some specific Lie algebra. The simplest choice is to involve the only rank one Lie algebra $sl(2,\mathbb{C})$, which contains the compact real form $su(2)$ and the non-compact real form $sl(2,\mathbb{R})$, isomorphic to $su(1,1)$. We will focus here on these examples.

5.2 Hamiltonians of $sl(2,\mathbb{R})$-Lie algebraic type

The three generator J_0, J_+ and J_- of $sl(2,\mathbb{R})$ satisfy the same commutation relations as (4.30) with $\lambda=-1$,

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = -2J_0, \qquad \text{and} \qquad J_0^\dagger, J_\pm^\dagger \notin \{J_0, J_\pm\}. \tag{5.1}$$

As possible realisation for this algebra one may take for instance the differential operators

$$J_- = \partial_x, \qquad J_0 = x\partial_x - \frac{n}{2}, \qquad J_+ = x^2\partial_x - nx, \quad n \in \mathbb{Z}, \tag{5.2}$$

allegedly attributed to Sophus Lie, see e.g. [188]. Clearly the action of this algebra on the space of polynomials (4.75) leaves it invariant.

According to the above specified notions, a quasi-exactly solvable Hamiltonian of Lie algebraic type is therefore of the general form

$$H_J = \sum_{l=0,\pm} \kappa_l J_l + \sum_{n,m=0,\pm} \kappa_{nm} : J_n J_m :, \qquad \kappa_l, \kappa_{nm} \in \mathbb{R}, \tag{5.3}$$

where we introduced the ordering

$$: J_n J_m := \begin{cases} J_n J_m & \text{for } n \geq m \\ 0 & \text{for } n < m \end{cases} \tag{5.4}$$

to avoid unnecessary double counting[9]. This means the Hamiltonian H_J involves nine real constants κ, plus a possible overall shift in the energy. It is evident from the representation (5.2) that when $\kappa_+ = \kappa_{++} = \kappa_{+0} = 0$ the model becomes exactly solvable in the sense specified in section 4.3. For the given representation (5.2) the $\mathcal{PT}$-symmetry may be implemented trivially by rescaling $J_\pm \to \tilde{J}_\pm = \pm\imath J_\pm$ and $J_0 \to \tilde{J}_0 = J_0$, which leaves the algebra (5.1) unchanged. Taking the algebra in this representation will leave the real vector space of $\mathcal{PT}$-symmetric polynomials

$$V_n^{\mathcal{PT}} = \text{span}\left\{1, \imath x, x^2, \imath x^3, x^4, ..., e^{\imath \frac{n\pi}{2}} x^n\right\}, \tag{5.5}$$

invariant. Since by construction the Hamiltonian $H_{\tilde{J}}$ and the wavefunctions are $\mathcal{PT}$-symmetric, as they are polynomials in $V_n^{\mathcal{PT}}$, the eigenvalues for these systems must be real by construction [15, 199]. Nonetheless, to determine the explicit similarity transformation remains a challenge.

A simple explicit example for $H_{\tilde{J}}$ with $\kappa_{00} = -4$, $\kappa_+ = -2\zeta = \kappa_-$, $\zeta \in \mathbb{R}$, an overall energy shift by $M^2 + \zeta^2$ and all remaining coefficients equal to zero,

$$H_g^- = -4J_0^2 - 2\imath\zeta(J_+ - J_-) + M^2 + \zeta^2, \tag{5.6}$$

was recently studied by Bagchi et al [200, 201] and shown to arise as a gauged version from the $\mathcal{PT}$-symmetric potential

$$V(x) = -\left(\zeta \sinh 2x - \imath M\right)^2, \tag{5.7}$$

[9]By setting some of the arrangements to zero our normal ordering prescription differs slightly from the ordinary one, but this is simply convention here and has no bearing on our analysis.

after a change of variables $x \to \frac{1}{2}\log x$. The first energy levels together with their corresponding wavefunctions were constructed and the typical real energy spectrum for unbroken $\mathcal{PT}$-symmetry and complex conjugate pairs for broken $\mathcal{PT}$-symmetry was found. However, even for this simple version of (5.3) a general treatment leading to the complete eigenvalue spectrum and a well defined metric has not been carried out. This is what we intend to achieve next.

5.3 $sl(2,\mathbb{R})$ Metrics and Hermitian counterparts

At this point it is not even very clear what the Hamiltonian in (5.3) describes in concrete physical terms, as this interpretation can only be made once one knows the metric with respect to which it is Hermitian and consequently the meaningful observables. Thus, we start by considering first the Hamiltonian $H_{\tilde{J}}$ in the context of the pseudo/quasi-Hermitian programme discussed in a previous chapter and try to solve the equation

$$h_{\tilde{J}} = \eta H_{\tilde{J}} \eta^{-1} = h_{\tilde{J}}^{\dagger} \tag{5.8}$$

for η. As a general ansatz we start with the non-Hermitian operator

$$\eta = e^{2\varepsilon[\tilde{J}_0 + \lambda(\tilde{J}_+ + \tilde{J}_-)]} \neq \eta^{\dagger}, \qquad \text{with} \qquad \varepsilon, \lambda \in \mathbb{R}, \tag{5.9}$$

so that a positive definite metric is constructed as $\rho = \eta^{\dagger}\eta$. The adjoint action of the Dyson metric above (5.9) on the generators $\tilde{J}_i$ is calculated without difficulties when the commutation relations (5.1) are employed, so that the parameters in it can be determined.

One of the simplest cases to consider for expressions of $H_{\tilde{J}}$ is the purely linear one, i.e. when all κ_{nm} vanish. An example of how to transform the non-Hermitian Hamiltonian $H_{\tilde{J}}$ to a Hermitian Hamiltonian is given when the parameters in the model are related as

$$\kappa_0 = \pm 2\sqrt{\kappa_- \kappa_+} \qquad \text{and} \qquad \frac{\tanh\chi}{\chi/\varepsilon} = \pm \frac{\sqrt{\kappa_+}}{\sqrt{\kappa_+} + 2\lambda\sqrt{\kappa_-}}, \tag{5.10}$$

with $\chi = \varepsilon\sqrt{1-4\lambda^2}$. The Hermitian Hamiltonian counterpart is subsequently computed to

$$h_{\tilde{J}} = \left(\pm\frac{1}{2\lambda}\kappa_0 + \kappa_+ + \kappa_-\right)\tilde{J}_-. \tag{5.11}$$

Another interesting simple example is obtained by setting all terms involving the generator J_+ in (5.3) to zero, that is taking $\kappa_+ = \kappa_{++} = \kappa_{+0} = \kappa_{+-} = 0$, and keeping $J_- \to \tilde{J}_- = -\imath J_-$. In this case we are led to the relations

$$\kappa_0 = -(n+1)\kappa_{00}, \qquad \kappa_- = -\frac{n}{\lambda}\mu_{00}, \qquad \kappa_{--} = \frac{\mu_{00}}{\lambda^2}, \qquad \kappa_{0-} = \frac{2}{\lambda}\kappa_{00} \tag{5.12}$$

together with

$$\tanh\chi = \frac{\chi}{\varepsilon}. \tag{5.13}$$

The non-Hermitian Hamiltonian $H_{\tilde{J}}$ is then transformed to the Hermitian Hamiltonian

$$h = \kappa_{00}\tilde{J}_0^2 - \kappa_0\tilde{J}_0, \tag{5.14}$$

with $0 < |\lambda| < \frac{1}{2}$. These examples demonstrate that it is possible to carry out the above mentioned programme for some specific realisations of the $sl_2(\mathbb{R})$-Lie algebra, albeit not in complete generality and in a generic representation independent manner.

As we indicated, the representation (5.2) is ideally suited with regard to the question of solvability. However, the Hermiticity properties for the J's are not straightforward to determine within a Lie algebraic framework, since the Hermitian conjugates of the J's cannot be written in terms of the original generators. This feature makes the representation (5.2) rather inappropriate for the determination of the Hermiticity properties of the Hamiltonian $H_{\tilde{J}}$ in generality. As a consequence we may carry out our programme only for specific representations using directly some concrete operator expressions and not in a generic representation independent way. We will therefore consider a slightly different type of algebra.

5.4 Hamiltonians of $su(1,1)$-Lie algebraic type and generalized Swanson models

The above mentioned problems do not occur when we express our Hamiltonian in terms of the isomorphic $su(1,1)$-Lie algebra, whose generators are K_0, K_1 and K_2 or the combinations K_0, $K_\pm = K_1 \pm \imath K_2$.

The operators K_0, K_1 and K_2 also satisfy the same commutation relations as (4.17). Consequently K_0, $K_\pm = K_1 \pm \imath K_2$ satisfy an isomorphic algebra to (5.1)

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_+, K_-] = -2K_0 \qquad \text{and} \qquad K_0^\dagger = K_0, K_\pm^\dagger = K_\mp. \tag{5.15}$$

As a special case we consider the two boson representation

$$K_0 = \frac{1}{2}\left(a^\dagger a + \frac{1}{2}\right), \qquad K_+ = \frac{1}{2}a^\dagger a^\dagger, \qquad K_- = \frac{1}{2}aa, \tag{5.16}$$

with $K_0^\dagger = K_0, K_\pm^\dagger = K_\mp$, and the $a, a^\dagger$ are the standard bosonic annihilation and creation operators,

$$a = \frac{\hat{x} + \imath\hat{p}}{\sqrt{2}} \qquad \text{and} \qquad a^\dagger = \frac{\hat{x} - \imath\hat{p}}{\sqrt{2}}, \tag{5.17}$$

and $N = a^\dagger a$ being the occupation number operator. In this representation we may realize the $\mathcal{PT}$ operation as

$$\mathcal{PT}: \quad a \to -a, \quad a^\dagger \to -a^\dagger \quad \Leftrightarrow \quad x \to -x, \quad p \to p, \;\; \imath \to -\imath \tag{5.18}$$

From (5.17), a differential representation in x-space can be easily obtained by the usual identification $\hat{p} = -\imath\partial_x$. We may also represent the algebra (4.30) with $\epsilon = -1$ using generators made up from two distinguishable bosons labeled by 1 and 2

$$K_0 = \frac{1}{2}\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right), \qquad K_+ = a_1^\dagger a_2^\dagger, \qquad K_- = a_1 a_2, \tag{5.19}$$

where $K_0^\dagger = K_0, K_\pm^\dagger = K_\mp$ and $\mathcal{PT}$ can then be described through

$$\mathcal{PT}: \quad a_1 \to -a_1, \quad a_1^\dagger \to -a_1^\dagger, \quad a_2 \to -a_2, \quad a_2^\dagger \to -a_2^\dagger. \tag{5.20}$$

In analogy to (5.3) we may then consider a Hamiltonian of Lie algebraic type in terms of the $su(1,1)$-generators

$$H_K = \sum_{l=0,\pm} \mu_l K_l + \sum_{n,m=0,\pm} \mu_{nm} : K_n K_m :, \qquad \mu_l, \mu_{nm} \in \mathbb{R}, \tag{5.21}$$

where we have used the same conventions for the ordering as in equation (5.4),

$$: K_n K_m := \begin{cases} K_n K_m & \text{for } n \geq m, \\ 0 & \text{for } n < m. \end{cases} \tag{5.22}$$

Despite the fact that $K_0^\dagger, K_\pm^\dagger \in \{K_0, K_\pm\}$, as opposed to $J_0^\dagger, J_\pm^\dagger \notin \{J_0, J_\pm\}$, in general this Hamiltonian is not Hermitian; whenever $\mu_+ \neq \mu_-$, $\mu_{++} \neq \mu_{--}$ or $\mu_{+0} \neq \mu_{0-}$ we have $H_K^\dagger \neq H_K$. Our main aim in [2] was to identify a subset of Hamiltonians H_K, which despite being non-Hermitian possess a real eigenvalue spectrum.

The part of the Hamiltonian H_K linear in the generators K corresponds to the Hamiltonian recently studied by Quesne [25], who constructed an explicit metric operator for this Hamiltonian together with its Hermitian isospectral partner. For the particular representation (5.16) this reduces to the so-called Swanson Hamiltonian [24], for which various metric operators were constructed previously by Musumbu et al [104]. Here we shall extend the analysis to the case involving bilinear combinations, staying as generic as possible without appealing to any particular representation.

5.5 $su(1,1)$ Metrics and Hermitian counterparts

We shall now see that the Hamiltonians H_K in (5.21) allow for a more general treatment as the problems of the previous section may be circumvented. In analogy to (5.8) let us therefore solve the equation

$$h_K = \eta H_K \eta^{-1} = h_K^\dagger \tag{5.23}$$

for the Dyson operator η. To start with we take a similar operator ansatz for the similarity transformation as the one chosen in [104, 25]

$$\eta = \exp(2\varepsilon K_0 + 2\nu_+ K_+ + 2\nu_- K_-), \tag{5.24}$$

where the parameters $\varepsilon, \nu_+, \nu_-$ are left variable for the time being. Noting that the eigenvalue spectrum of η is given by

$$\exp\left[\left(n+\frac{1}{2}\right)\theta\right] \qquad \text{with} \qquad \theta \equiv \sqrt{\varepsilon^2 - 4\nu_+\nu_-}, \tag{5.25}$$

one can ensure the positivity of η when $\varepsilon^2 > 4\nu_+\nu_-$. Furthermore, this operator is clearly also linear and invertible, other essential properties for the metric operator $\rho = \eta^\dagger\eta$. Observe that this corresponds to a generalized squeeze transformation, e.g. [203],

$$S(\nu) = \exp\left(\nu a^{\dagger 2} - \nu^* a^2\right), \tag{5.26}$$

found when quantizing the electromagnetic field, when $\epsilon = 0, \nu_+ = \nu, \nu_- = -\nu^*$ so that it becomes unitary and the metric remains unaltered.

Using the ansatz (5.24), we have to compute its action on H_K in order to solve (5.23). In fact, the adjoint action of η on each of the $su(1,1)$-generators K_i, with $i = 0, \pm$, can be computed exactly. We find

$$\eta\, K_l\, \eta^{-1} = t_{l0}K_0 + t_{l-}K_- + t_{l+}K_+ \qquad \text{for } l = 0, \pm, \tag{5.27}$$

where the constant coefficients are

$$\begin{aligned}
t_{00} &= 1 - 8\nu_+\nu_- \frac{\sinh\theta^2}{\theta^2}, && (5.28)\\
t_{\pm\pm} &= \left(\cosh\theta \pm \varepsilon\frac{\sinh\theta}{\theta}\right)^2, && (5.29)\\
t_{\pm\mp} &= 4(\nu_\pm)^2\frac{\sinh\theta^2}{\theta^2}, && (5.30)\\
t_{0\pm} &= \mp 2\nu_\mp \frac{\sinh\theta}{\theta}\left(\cosh\theta \pm \varepsilon\frac{\sinh\theta}{\theta}\right), && (5.31)\\
t_{\pm 0} &= \pm 4\nu_\pm \frac{\sinh\theta}{\theta}\left(\cosh\theta \pm \varepsilon\frac{\sinh\theta}{\theta}\right).
\end{aligned}$$

We could have been also more generic by making a more general ansatz for the expressions for η, such as for instance allowing in addition bilinear combinations in the arguments of the exponential. In fact, as we will show in section 5.5.5, we are certain that more general types of metric operators must exist. Another very natural version of this ansatz would be to start with a Iwasawa [10] decomposed expression for η, such as

$$\bar{\eta} \equiv e^{2\kappa_+ K_+} e^{2\kappa_0 K_0} e^{2\kappa_- K_-}. \tag{5.32}$$

The action of this similarity transformation on a $su(1,1)$ Hamiltonian requires the knowledge of

$$\bar{\eta}\, K_l\, \bar{\eta}^{-1} = \bar{t}_{l0} K_0 + \bar{t}_{l-} K_- + \bar{t}_{l+} K_+ \qquad \text{for } l = 0, \pm, \tag{5.33}$$

and the coefficients are shown to be

$$\begin{aligned}
\bar{t}_{00} &= 1 - 8\kappa_+\kappa_- e^{-2\kappa_0}\ , \\
\bar{t}_{+-} &= 4\kappa_-^2 e^{-2\kappa_0}\ , \\
\bar{t}_{-+} &= 4\kappa_+^2 e^{-2\kappa_0}, \\
\bar{t}_{++} &= -8\kappa_+\kappa_- + 16\kappa_+^2\kappa_-^2 e^{-2\kappa_0} + e^{2\kappa_0}\ , \\
\bar{t}_{--} &= e^{-2\kappa_0}\ , \\
\bar{t}_{0+} &= -2\kappa_+(1 + 4\kappa_+\kappa_- e^{-2\kappa_0}), \\
\bar{t}_{0-} &= -2\kappa_+ e^{-2\kappa_0}\ , \\
\bar{t}_{+0} &= -4\kappa_-(1 - 4\kappa_+\kappa_- e^{-2\kappa_0})\ , \\
\bar{t}_{-0} &= -4\kappa_+ e^{-2\kappa_0},
\end{aligned} \tag{5.34}$$

from which one concludes that in order to have $\bar{\eta} = \eta$ we should impose that $\bar{t}_{ij} = t_{ij}$, leading to

$$\kappa_0 = -\log\left(\cosh\theta - \epsilon\frac{\sinh\theta}{\theta}\right), \tag{5.35}$$

$$\kappa_+ = -\frac{2\nu_+\frac{\sinh\theta}{\theta}}{\cosh\theta - \epsilon\frac{\sinh\theta}{\theta}}\ , \tag{5.36}$$

$$\kappa_- = -\frac{2\nu_-\frac{\sinh\theta}{\theta}}{\cosh\theta - \epsilon\frac{\sinh\theta}{\theta}}. \tag{5.37}$$

Albeit not compulsory, Hermiticity for the η operator may be guaranteed when we take from the very beginning $\nu_+ = \nu$, $\nu_- = \nu^*$ and $\varepsilon \in \mathbb{R}$ together with the Hermiticity conditions for the Lie algebraic generators as specified in the sequence of (5.15).

[10] Because $su(1,1)$ is a noncompact algebra; if the algebra was compact the equivalent decomposition would be named after Gauss.

With the help of these exact relations we evaluate the adjoint action of η on the Hamiltonian H_K

$$h_K \equiv \eta H_K \eta^{-1} = \sum_{l=0,\pm} \hat{\mu}_l K_l + \sum_{n,m=0,\pm} \hat{\mu}_{nm} : K_n K_m : . \tag{5.38}$$

It is evident from (5.27) that the general structure of the Hamiltonian will not change, albeit with a different set of constants $\hat{\mu}$, which are rather lengthy and we will therefore not report them here explicitly. However, they simplify when we impose the constraint that the resulting Hamiltonian ought to be Hermitian. The condition (5.23) leads to the six constraints

$$\hat{\mu}_0 = \hat{\mu}_0^*, \qquad \hat{\mu}_{00} = \hat{\mu}_{00}^*, \qquad \hat{\mu}_{+-} = \hat{\mu}_{+-}^*, \tag{5.39}$$

$$\hat{\mu}_+ = \hat{\mu}_-^*, \qquad \hat{\mu}_{++} = \hat{\mu}_{--}^*, \qquad \hat{\mu}_{+0} = \hat{\mu}_{0-}^*. \tag{5.40}$$

The first set of three equations (6.42) on the reality of $\hat{\mu}_0$, $\hat{\mu}_{00}$ and $\hat{\mu}_{+-}$ is simply satisfied by the condition $\nu \in \mathbb{R}$, implying the Hermiticity of η. Introducing the variables

$$\lambda = \frac{\nu}{\varepsilon} \qquad \text{and} \qquad Y = \varepsilon \frac{\tanh \theta}{\theta}, \tag{5.41}$$

the remaining three equations (6.48) may be converted into simpler, albeit still lengthy, equations

$$\begin{aligned} 0 \;=\; & \mu_+ - \mu_- + 2\,Y\,\left[\mu_+ + \mu_- + 2\,\lambda\,(\mu_{++} + \mu_{--} - \mu_0 - \mu_{00})\right] \\ & + 12\,Y^2\,\lambda\,\left[\mu_{++} - \mu_{--} + \lambda\,(\mu_{0-} - \mu_{+0})\right] \\ & - 2\,Y^3\,\left\{\mu_+ + \mu_- - 2\,\lambda\,\left[\mu_0 + \mu_{00} + 3\,(\mu_{--} + \mu_{++})\right]\right. \\ & \left. - \lambda^2\,\left[8\,\mu_{0-} - 4\,(\mu_- + \mu_+ - 2\,\mu_{+0})\right] + 8\,\lambda^3\,(\mu_{++} + \mu_{--} + \mu_0 - \mu_{00} - 2\,\mu_{+-})\right\} \\ & + Y^4\,\left(1 - 4\,\lambda^2\right)\,\left\{\mu_- - \mu_+ + 4\,\lambda\,\left[\mu_{++} - \mu_{--} + \lambda\,(\mu_{0-} - \mu_- + \mu_+ - \mu_{+0})\right]\right\}, \end{aligned} \tag{5.42}$$

$$\begin{aligned} 0 \;=\; & \mu_{++} - \mu_{--} - 2\,Y\,\left[\lambda\,(\mu_{0-} + \mu_{+0}) - 2\,(\mu_{--} + \mu_{++})\right] \\ & + 6\,Y^2\,\left[\mu_{++} - \mu_{--} + \lambda\,(\mu_{0-} - \mu_{+0})\right] \\ & - 2\,Y^3\,\left[3\,\lambda\,(\mu_{+0} + \mu_{0-}) + 4\,\lambda^3\,(\mu_{+0} + \mu_{0-}) - 8\,\lambda^2\,(\mu_{00} + \mu_{+-}) - 2\,(\mu_{++} + \mu_{--})\right] \\ & + Y^4\,\left(1 - 4\,\lambda^2\right)\,\left\{\mu_{++} - \mu_{--} - 2\,\lambda\,\left[\mu_{+0} - \mu_{0-} + 2\,\lambda\,(\mu_{--} - \mu_{++})\right]\right\}, \end{aligned} \tag{5.43}$$

$$
\begin{aligned}
0 \;=\;& \mu_{+0} - \mu_{0-} + 2\,Y\,\left[\mu_{0-} + \mu_{+0} + 4\,\lambda\,\left(\mu_{++} + \mu_{--} - \mu_{00} - \mu_{+-}\right)\right] \\
& + 24\,Y^2\,\left[\lambda(\mu_{++} - \mu_{--}) + \lambda^2\,\left(\mu_{0-} - \mu_{+0}\right)\right] \\
& - 2\,Y^3\,\left\{\mu_{+0} + \mu_{0-} - 4\,\lambda\,\left[\mu_{00} + \mu_{+-} + 3(\mu_{++} + \mu_{--})\right] - 12\lambda^2\,\left(\mu_{0-} + \mu_{+0}\right)\right. \\
& \left. +\;16\lambda^3\,\left(\mu_{++} + \mu_{--} - \mu_{00} - \mu_{+-}\right)\right\} \\
& + Y^4\,\left(1 - 4\,\lambda^2\right)\,\left\{\mu_{0-} - \mu_{+0} + 4\,\lambda\,\left[\lambda\,\left(\mu_{0-} - \mu_{+0}\right) + 2\,\left(\mu_{++} - \mu_{--}\right)\right]\right\}.
\end{aligned}
\tag{5.44}
$$

We will now systematically discuss the solutions for these three equations together with their implications on the metric operator and the corresponding isospectral pairs of Hamiltonians.

5.5.1 Non-Hermitian linear term and Hermitian bilinear combinations

The simplest modification with regard to the purely linear case, treated previously in [104, 25], is to perturb it with Hermitian bilinear combinations. This means we may assume the equalities $\mu_{++} = \mu_{--}$ and $\mu_{+0} = \mu_{0-}$ in order to determine the relations between the remaining constants from (5.42), (5.43) and (5.44). We find that (5.43) and (5.44) are solved solely by demanding

$$
\mu_{++} = \mu_{--} = \frac{\lambda^2(\mu_{00} + \mu_{+-})}{1 + 2\lambda^2} \qquad \text{and} \qquad \mu_{+0} = \mu_{0-} = \frac{2\lambda(\mu_{00} + \mu_{+-})}{1 + 2\lambda^2}, \tag{5.45}
$$

without any further constraint on Y. Solving subsequently equation (5.42) for Y yields the constraint

$$
\varepsilon\frac{\tanh 2\theta}{\theta} = \frac{\lambda(\mu_- - \mu_+)}{\lambda(\mu_- + \mu_+) + 2\lambda^2(\mu_{+-} - \mu_0) - 2\mu_{++}}. \tag{5.46}
$$

Considering (5.24) we note that the positivity of η^2 requires $|\lambda| < \frac{1}{2}$ as a further restriction on the domain of λ. Notice that when we send all coefficients μ_{nm} with $n, m \in \{0, \pm\}$ resulting from bilinear combinations to zero, we recover precisely the constraint found in [104], see equation (9) therein. These equations parametrize the metric and are enough to compute the Hermitian counterpart via equation (5.23). We will not report the expression here as they are rather lengthy and can be obtained as a reduction from the more general setting to be treated below.

5.5.2 Hermitian linear term and non-Hermitian bilinear combinations

Reversing the situation of the preceding subsection we may consider the Hamiltonian H_K with Hermitian linear part, i.e. $\mu_+ = \mu_-$, and non-Hermitian part involving bilinear

combinations. In this case we can solve the equations (5.42), (5.43) and (5.44) by

$$\begin{aligned} \mu_+ &= \mu_- = \lambda(\mu_0 + \mu_{00} - \mu_{++} - \mu_{--}), && (5.47)\\ \lambda(\mu_{+0} - \mu_+) &= \lambda^2(\mu_{+-} - \mu_0) + \mu_{++}, && (5.48)\\ \lambda(\mu_{0-} - \mu_-) &= \lambda^2(\mu_{+-} - \mu_0) + \mu_{--}, && (5.49) \end{aligned}$$

together with

$$\varepsilon \frac{\tanh 2\theta}{\theta} = \frac{\mu_{++} - \mu_{--}}{2\lambda\mu_+ + 2\lambda^2(\mu_{+-} - \mu_0) - (\mu_{++} + \mu_{--})}. \qquad (5.50)$$

This case does not reduce to any situation treated in the literature before.

Let us now embark on the general setting in which the linear as well as the terms in H_K involving bilinear combinations are taken to be non-Hermitian. We will find two different types of solutions, one being reducible to the foregoing two cases and the other being intrinsically non-Hermitian and not reducible to any of the previous cases. Reducible is meant in the sense that the limit of the relevant parameters going to zero is well defined.

5.5.3 Generic non-Hermitian reducible Hamiltonian

Taking now H_K to be genuinely non-Hermitian, we find that the equations (5.42), (5.43) and (5.44) are solved subject to the three constraints

$$\begin{aligned} \mu_{++} - \mu_{--} &= \lambda(\mu_{+0} - \mu_{0-}), && (5.51)\\ \mu_{--} - \lambda\mu_{0-} &= \lambda^2(\mu_{++} + \mu_{--} - \mu_{+-} - \mu_{00}),\\ 2\mu_+\mu_{--} - \mu_-(\mu_{++} + \mu_{--}) &= \lambda[(\mu_{++} - \mu_{--})(\mu_{++} + \mu_{--} - \mu_0 - \mu_{00}) + \mu_{0-}(\mu_+ - \mu_-)] , \end{aligned}$$

concomitantly with

$$\varepsilon \frac{\tanh 2\theta}{\theta} = \frac{\lambda(\mu_- - \mu_+) + \mu_{++} - \mu_{--}}{\lambda(\mu_- + \mu_+) + 2\lambda^2(\mu_{+-} - \mu_0) - (\mu_{++} + \mu_{--})}. \qquad (5.52)$$

We note that taking $\mu_{++} = \mu_{--}$ and $\mu_{+0} = \mu_{0-}$ or $\mu_+ = \mu_-$ these constraints reduce precisely to the ones previously treated in the sections 5.5.1 or 5.5.2, respectively. A further interesting specialization of this general case is the one involving purely bilinear combinations, which may be obtained for $\mu_- = \mu_+ = \mu_0$ in (5.51) and (5.52). For the situation in which the Hamiltonian does not contain any generators of the type K_-, i.e. $\mu_- = \mu_{--} = \mu_{0-} = 0$ we find

$$\lambda\mu_{+0} = \mu_{++}, \quad \mu_{00} = \mu_{++} - \mu_0, \quad \mu_{+-} = \mu_0, \quad \varepsilon = \frac{\operatorname{arctanh}\sqrt{1 - 4\lambda^2}}{2\sqrt{1 - 4\lambda^2}}, \qquad (5.53)$$

and when H_K does not contain any generators of the type K_+, i.e. $\mu_+ = \mu_{++} = \mu_{0+} = 0$ the equations simplify to

$$\lambda\mu_{0-} = \mu_{--}, \quad \mu_{00} = \mu_{--} - \mu_0, \quad \mu_{+-} = \mu_0, \quad \varepsilon = -\frac{\operatorname{arctanh}\sqrt{1-4\lambda^2}}{2\sqrt{1-4\lambda^2}}. \tag{5.54}$$

Another trivial consistency check is obtained when we add to the Swanson model a multiple of the Casimir operator $C = K_0^2 - \frac{1}{2}\{K_+, K_-\}$ and consider

$$H_C = H_l + \kappa C = (\mu_0 + \kappa)K_0 + \mu_+K_+ + \mu_-K_- + \kappa K_0^2 - \kappa K_+K_- \qquad \text{for } \kappa \in \mathbb{R}. \tag{5.55}$$

Since the Casimir operator is Hermitian and commutes with η no further constraint should result from this modification when compared with the non-Hermitian linear case. In fact, the linear case together with the constraining equations will produce the Casimir operator. When $\mu_{+-} \neq 0$ we can satisfy the constraints (5.51) by $\mu_{+-} = -\mu_{00}$ and setting all remaining μ's with double subscripts to zero, which is obviously satisfied by (5.55), together with

$$\varepsilon\frac{\tanh 2\theta}{\theta} = \frac{\mu_- - \mu_+}{\mu_- + \mu_+ - 2\lambda\mu_0}. \tag{5.56}$$

We conclude this section by making use of the constraining equation (5.52) and re-express the operator η in (5.24) purely as as a function of $\lambda \in [-\frac{1}{2}, \frac{1}{2}]\backslash\{0\}$

$$\eta(\lambda) = \exp\left[\frac{K_0 + \lambda(K_+ + K_-)}{\sqrt{1-4\lambda^2}}\operatorname{arctanh}F(\lambda)\right], \tag{5.57}$$

where

$$F(\lambda) := \sqrt{1-4\lambda^2}\frac{\lambda(\mu_- - \mu_+) + \mu_{++} - \mu_{--}}{\lambda(\mu_- + \mu_+) + 2\lambda^2(\mu_{+-} - \mu_0) - (\mu_{++} + \mu_{--})}, \tag{5.58}$$

subject to the constraints (5.51).

Hermitian counterpart

Using the explicit solution (5.57) we can compute the Hermitian counterpart h_K using the formula (5.23). As expected from similar calculations previously carried out in this context the explicit non-Hermitian Hamiltonian turns out to be rather complicated when compared to the fairly simple non-Hermitian Hamiltonian (5.21). Nonetheless, it may be

computed exactly and we find the coefficients in (5.38) to be given by

$$\hat{\mu}_0 = A_0 + 2(\mu_- - \mu_+)\lambda B_0, \tag{5.59}$$

$$\hat{\mu}_+ = \hat{\mu}_- = \lambda A_+ + \frac{1}{2}\left[\mu_- - \mu_+ + 2(\mu_{--} - \mu_{++})\lambda\right] B_0, \tag{5.60}$$

$$\hat{\mu}_{00} = \frac{1}{2\lambda} A_{00} + 2(\mu_{--} - \mu_{++})B_0, \tag{5.61}$$

$$\hat{\mu}_{+-} = A_{+-} + (\mu_{--} - \mu_{++})B_0, \tag{5.62}$$

$$\hat{\mu}_{++} = \hat{\mu}_{--} = \lambda A_{++} + \frac{1}{2}(\mu_{--} - \mu_{++})B_0, \tag{5.63}$$

$$\hat{\mu}_{+0} = \hat{\mu}_{0-} = 2A_{++} + \left(\frac{1+4\lambda^2}{2\lambda}\right)(\mu_{--} - \mu_{++})B_0, \tag{5.64}$$

where we further abbreviated

$$A_0 = \mu_0 - \frac{2\lambda}{1-4\lambda^2}\frac{(\mu_- - \mu_+)(\mu_{--} + 3\mu_{++} - 2\lambda\mu_{+0})}{(\mu_{--} - \mu_{++})}, \tag{5.65}$$

$$\begin{aligned} A_{00} = {} & \frac{1}{(1-4\lambda^2)(\mu_{--} - \mu_{++} - \lambda(\mu_- - \mu_+))}\left[2(\mu_{--}\mu_+ - \mu_{++}\mu_-)\right. \\ & -2\lambda(\mu_{--} - \mu_{++})(\mu_0 + \mu_{++} + \mu_{--}) + 2\lambda^2(\mu_- - \mu_+)(\mu_{--} + \mu_{++} + \mu_{+-}) + \\ & \left.-8\lambda^3(\mu_{--} - \mu_{++})(\mu_{--} + \mu_{++} - \mu_{+-}) + 8\lambda^4(\mu_- - \mu_+)(\mu_{--} + \mu_{++} - \mu_{+-})\right], \end{aligned} \tag{5.66}$$

$$\begin{aligned} A_{+-} = {} & \frac{1}{(1-4\lambda^2)(\mu_{--} - \mu_{++} - \lambda(\mu_- - \mu_+))}\left\{(\mu_{++} - \mu_{--})(\mu_{--} + \mu_{++} - \mu_{+-})\right. \\ & -\lambda\left[\mu_{+-}(\mu_- - \mu_+) - (\mu_- + \mu_+)(\mu_{--} - \mu_{++})\right] \\ & \left.-2\lambda^2(\mu_{--} - \mu_{++})(\mu_{+-} + \mu_0) + 4\lambda^3(\mu_- - \mu_+)\mu_{+-}\right\}, \end{aligned} \tag{5.67}$$

$$\begin{aligned} A_+ = {} & \frac{1}{(1-4\lambda^2)\lambda(\mu_{--} - \mu_{++})}\left\{-\lambda\left[\mu_{--}^2 - (\mu_- - \mu_+)\mu_{+0} + 2\mu_{--}\mu_{++} - 3\mu_{++}^2\right]\right. \\ & \left.+\mu_+(\mu_{--} + \mu_{++}) - 2\mu_-\mu_{++} - 2\lambda^2(\mu_{--} - \mu_{++})(\mu_- + \mu_- - \mu_{+0})\right\}, \end{aligned} \tag{5.68}$$

$$A_{++} = \frac{\lambda\mu_{+0} - \mu_{++} - 2\lambda^2(\mu_{--} + \mu_{++})}{(1-4\lambda^2)\lambda}, \tag{5.69}$$

$$B_0 = 2\frac{\sqrt{2\mu_{++}(\mu_{--} + \mu_{++}) - \lambda(\mu_{--} + 3\mu_{++})\mu_{+0} + \lambda^2\left[\mu_{+0}^2 + (\mu_{--} - \mu_{++})^2\right]}}{(1-4\lambda^2)(\mu_{--} - \mu_{++})}. \tag{5.70}$$

Clearly this Hamiltonian does not constitute an obvious starting point, whereas the non-Hermitian Hamiltonian H_J is fairly simple and natural to consider. We could also express the Hermitian version in a simple fashion by solving (5.59)-(5.70) for the μs, such that instead H_J would acquires a complicated form. However, the construction procedure

itself is only meaningful in the direction $H_J \to h_J$ and not $h_J \to H_J$ as we are only interested in finding Hermitian counterparts to non-Hermitian operators.

5.5.4 Generic non-Hermitian non-reducible Hamiltonian

Remarkably in contrast to the previously analysed purely linear case there exists a second non-equivalent type of solution. We find that (5.42), (5.43) and (5.44) are also solved by the four constraints

$$\begin{aligned}
\mu_+ - \mu_- &= 2\lambda(\mu_{++} - \mu_{--}), && (5.71)\\
\mu_{+0} - \mu_{0-} &= 2(\mu_+ - \mu_-), && (5.72)\\
\mu_{+0} &= 2\mu_+ + 2(\mu_{+-} - \mu_0)\lambda, && (5.73)\\
\mu_+ &= \lambda(\mu_0 + \mu_{00} + 2\mu_{++}) - 2\lambda^2(\mu_- - \mu_+ + \mu_{+0}) \quad , && (5.74)
\end{aligned}$$

conjointly with

$$\varepsilon \frac{\tanh 4\theta}{\theta} = \frac{\mu_{--} - \mu_{++}}{\mu_{--} + \mu_{++} + \lambda(\mu_+ - \mu_- - \mu_{+0})}. \qquad (5.75)$$

Notice that this solution can not be reduced to the cases of a non-Hermitian linear term plus Hermitian bilinear combination or a Hermitian linear term plus a non-Hermitian bilinear combination as discussed in sections 5.5.1 or 5.5.2, respectively. This is seen from (5.71) and (5.72) as $\mu_+ = \mu_-$ implies $\mu_{++} = \mu_{--}$, $\mu_{+0} = \mu_{0-}$ and vice versa, such that it is impossible to convert one part into a Hermitian one while keeping the other non-Hermitian.

As for the foregoing set of constraints there are some interesting subcases. For instance, we can consider again the situation where the Hamiltonian does not contain any generators of the type K_-, i.e. $\mu_- = \mu_{--} = \mu_{0-} = 0$. Then the constraints simplify to

$$\begin{aligned}
2\lambda\mu_{++} &= \mu_+, \qquad \mu_{+0} = 2\mu_+, \qquad \mu_{00} = 2\lambda\mu_+ - \mu_0, \qquad \mu_{+-} = \mu_0, \qquad (5.76)\\
\varepsilon &= -\frac{1}{4\sqrt{1-4\lambda^2}} \operatorname{arctanh}\left(\frac{\sqrt{1-4\lambda^2}}{1-2\lambda^2}\right).
\end{aligned}$$

Similarly, if the Hamiltonian does not contain any generators of the type K_+, i.e., $\mu_+ = \mu_{++} = \mu_{0+} = 0$, the constraints reduce to

$$\begin{aligned}
2\lambda\mu_{--} &= \mu_-, \qquad \mu_{0-} = 2\mu_-, \qquad \mu_{00} = 2\lambda\mu_- - \mu_0, \qquad \mu_{+-} = \mu_0 \qquad (5.77)\\
\varepsilon &= \frac{1}{4\sqrt{1-4\lambda^2}} \operatorname{arctanh}\left(\frac{\sqrt{1-4\lambda^2}}{1-2\lambda^2}\right).
\end{aligned}$$

Note that also for this reduced case the solutions (5.53) and (5.76) as well as (5.54) and (5.77) are different.

As before we can also in this case use of the constraining equation (5.52) and re-express the operator η in (5.24) purely as a function of $\lambda \in [-\frac{1}{2}, \frac{1}{2}]\backslash\{0\}$

$$\eta(\lambda) = \exp\left[\frac{K_0 + \lambda(K_+ + K_-)}{2\sqrt{1-4\lambda^2}}\text{arctanh}G(\lambda)\right], \tag{5.78}$$

where

$$G(\lambda) := \sqrt{1-4\lambda^2}\frac{(\mu_{--} - \mu_{++})}{\mu_{--} + \mu_{++} + \lambda(\mu_+ - \mu_- - \mu_{+0})}, \tag{5.79}$$

subject to the constraints (5.71) and (5.71).

Hermitian counterpart

Using again the explicit solution (5.57) we can compute the Hermitian counterpart h_K using the formula (5.78). Once more the expressions are quite cumbersome

$$\begin{aligned}
\hat{\mu}_0 &= C_0 + 4\lambda^2 D_0, && (5.80)\\
\hat{\mu}_+ &= \hat{\mu}_- = C_+ + 2\lambda D_0, && (5.81)\\
\hat{\mu}_{00} &= 2(C_{00} + 4\lambda^2 D_0), && (5.82)\\
\hat{\mu}_{+-} &= C_{+-} + 4\lambda^2 D_0, && (5.83)\\
\hat{\mu}_{++} &= \hat{\mu}_{--} = \lambda C_{++} + (1-2\lambda^2)D_0, && (5.84)\\
\hat{\mu}_{+0} &= \hat{\mu}_{0-} = 2(C_{++} + 2\lambda D_0), && (5.85)
\end{aligned}$$

where further abbreviated

$$\begin{aligned}
C_0 &= \frac{\mu_0 - \lambda(\mu_- - \mu_+) - 4\lambda^2(\mu_{++} + \mu_0) + 2\lambda^3(\mu_- + \mu_+) + 4\lambda^4(\mu_{+-} - \mu_0)}{1-4\lambda^2}, && (5.86)\\
C_{00} &= \frac{\mu_{00}}{2} - \frac{2\lambda^2\left[\mu_{--} + \mu_{++} - \lambda\mu_{+0} - 2\lambda^2(\mu_{--} - \mu_{++})\right]}{1-4\lambda^2}, && (5.87)\\
C_{+-} &= C_0 + \mu_{+-} - \mu_0, && (5.88)\\
C_+ &= \frac{\mu_+ - 2\lambda\mu_{++} - \lambda^2(\mu_- + \mu_+) + 2\lambda^3(\mu_{+-} - \mu_0)}{1-4\lambda^2}, && (5.89)\\
C_{++} &= \frac{\mu_{+0} - 4\lambda\mu_{++} - 2\lambda^2\mu_{+0} - 4\lambda^3(\mu_{--} - \mu_{++})}{2(1-4\lambda^2)}, && (5.90)\\
D_0 &= \frac{1}{2\ (4\,\lambda^2 - 1)}\left\{4\,\mu_{--}\,\mu_{++} + \lambda^2\ \left[\mu_{+0}^2 + 8\,\mu_{++}\ (\mu_{++} - \mu_{--})\right]\right. && (5.91)\\
&\quad \left. -2\,\lambda\,\mu_{+0}\ (\mu_{--} + \mu_{++}) + 4\,\lambda^3\,\mu_{+0}\ (\mu_{--} - \mu_{++}) + 4\,\lambda^4\,(\mu_{--} - \mu_{++})^2\right\}^{1/2}.
\end{aligned}$$

Again this demonstrates the general feature that some fairly simple non-Hermitian Hamiltonians possess quite complicated isospectral Hermitian counterparts.

A simpler metric, the case $\lambda = 0$

In the previous discussion we have excluded the case $\lambda = 0$, which equals $\nu = 0$ in our ansatz for the metric (5.24). This case may be dealt with separately and in fact is fairly easy, as η simplifies considerably because it only depends on the generator K_0. In this situation also the constraints turn out to be far simpler,

$$\mu_{--}\mu_{+}^2 = \mu_{++}\mu_{-}^2, \qquad \mu_{--}\mu_{+0}^2 = \mu_{++}\mu_{0-}^2 \qquad \text{with} \qquad \varepsilon = \frac{1}{8}\ln\frac{\mu_{--}}{\mu_{++}}, \tag{5.92}$$

and even the Hermitian counterpart Hamiltonian becomes fairly compact too

$$\begin{aligned} h_\varepsilon &= \mu_0 K_0 + \mu_+ e^{2\varepsilon}(K_+ + K_-) + \mu_{00}K_0^2 + \mu_{+-}K_+K_- + \mu_{++}e^{4\varepsilon}(K_+^2 + K_-^2) \\ &\quad + \mu_{+0}e^{2\varepsilon}(K_+K_0 + K_0K_-). \end{aligned} \tag{5.93}$$

This suggests that the simple metric $\eta = e^{2\varepsilon K_0}$ may be employed as an easy transformation also for other more complicated Hamiltonians.

Two further simple cases $\lambda = \pm\frac{1}{2}$

Finally let us also investigate the other boundary values for the parameter λ, that is $\lambda = \pm\frac{1}{2}$. In this case the constraints are

$$\begin{aligned} \mu_{++} &= \pm(\mu_+ - 2\mu_-) + \frac{(\mu_- - \mu_+)\mu_{+-}}{\mu_{+0} - \mu_{0-} + 2(\mu_- - \mu_+)} \pm \frac{(\mu_- - \mu_+)(\mu_{0-} - 2\mu_- \pm \mu_0)}{\mu_{0-} - \mu_{+0} - 2(\mu_- - \mu_+)} \\ &\quad + \frac{\mu_0 + \mu_{00}}{2} + \frac{(\mu_{0-} - \mu_{+0} - 2(\mu_- - \mu_+))(\mu_{+0} - 2(\mu_0 + \mu_{00} - 2\mu_-))}{4(\mu_{0-} - 2\mu_- \pm (\mu_0 - \mu_{+-}))}, \end{aligned} \tag{5.94}$$

$$\begin{aligned} \mu_{--} &= \mp\mu_+ + \frac{(\mu_- - \mu_+)\mu_{+-}}{\mu_{+0} - \mu_{0-} + 2(\mu_- - \mu_+)} + \frac{(\mu_- - \mu_+)(\mu_0 + \mu_{0-} - 2\mu_-)}{\mu_{0-} - \mu_{+0} - 2(\mu_- - \mu_+)} \\ &\quad + \frac{\mu_0 + \mu_{00}}{2} + \frac{(2(\mu_0 + \mu_{00}) \mp (\mu_{0-} + 4\mu_+))(\mu_{0-} - \mu_{0+} + 2(\mu_+ - \mu_-))}{4(\mu_{+0} - 2\mu_+ \pm (\mu_0 - \mu_{+-}))}, \end{aligned} \tag{5.95}$$

$$\varepsilon = \frac{\mu_{0-} - \mu_{+0} - 2(\mu_- - \mu_+)}{2(\mu_{0-} + \mu_{+0} - 2(\mu_+ + \mu_-) \pm 2(\mu_0 - \mu_{+-}))}. \tag{5.96}$$

The general Hermitian counterpart turns out to have a very complicated form, but there are some simple special cases, such as

$$H_{\frac{1}{2}} = K_+ - K_- - K_0 + K_0^2 + K_+K_- + K_+K_0 + K_0K_- + \frac{11}{2}K_+^2 + \frac{1}{2}K_-^2, \tag{5.97}$$

which is mapped into the Hermitian form

$$h_{\frac{1}{2}} = \mp\frac{13}{16}(K_+ + K_-) - \frac{23}{16}K_0 + \frac{5}{8}K_0^2 + \frac{13}{16}K_+K_- \mp \frac{11}{8}(K_+K_0 + K_0K_-) + \frac{61}{32}(K_+^2 + K_-^2), \tag{5.98}$$

with $\varepsilon = \mp\frac{1}{4}$ for $\lambda = \pm\frac{1}{2}$.

At this point we have investigated all possible solutions to the system (5.42)-(5.44), associated to the construction of a pseudo-Hermitian metric and Hermitian partner Hamiltonian (5.21) according to the ansatz (5.24). Despite a few cumbersome expressions, this method proves to be very convenient when an exact solution is needed and allowed us to recover familiar results of the literature as limits of our very general Hamiltonian. But chronologically linear Hamiltonians of the type (5.21) were first examined with the use of Bogoliubov transformations, a procedure analysed as follows.

5.5.5 Generalised Bogoliubov transformation in the construction of metrics

Bogoliubov transformations were first introduced with the purpose of understanding the pairing interaction in superconductivity [204] and have been generalized thereafter in many different ways, as for instance in [205]. In the present context they have been applied by Swanson [24] as an alternative method to establish the reality of the spectrum of a non-Hermitian Hamiltonian. Instead of constructing an explicit similarity transformation one can make a constraining assumption about its form. The simplest assumption to make is that the counterpart is of a harmonic oscillator type. We will now demonstrate how the Hamiltonian H_K can be transformed into such a form by means of a generalized Bogoliubov transformation and how this method relates to one we have been discussing so far.

In a matrix form, the similarity transformation (5.27) looks like

$$\begin{pmatrix} \hat{K}_0 \\ \hat{K}_+ \\ \hat{K}_- \end{pmatrix} = \begin{pmatrix} t_{00} & t_{0+} & t_{0-} \\ t_{+0} & t_{++} & t_{+-} \\ t_{-0} & t_{-+} & t_{--} \end{pmatrix} \begin{pmatrix} K_0 \\ K_+ \\ K_- \end{pmatrix} \tag{5.99}$$

where $\hat{K}_i \equiv \eta\, K_i\, \eta^{-1}$ for $i = 0, \pm$ and its determinant is simply unity. On the other hand, following [24], the generators of the the $su(1,1)$-algebra are linearly expressed in terms of new operators $\check{K}_i$ according to a generalized Bogoliubov transformation,

$$\begin{pmatrix} K_0 \\ K_+ \\ K_- \end{pmatrix} = \begin{pmatrix} \gamma\beta + \delta\alpha & \beta\delta & \alpha\gamma \\ 2\alpha\beta & \beta^2 & \alpha^2 \\ 2\gamma\delta & \delta^2 & \gamma^2 \end{pmatrix} \begin{pmatrix} \check{K}_0 \\ \check{K}_+ \\ \check{K}_- \end{pmatrix}. \tag{5.100}$$

The constraint

$$\beta\gamma - \alpha\delta = 1, \tag{5.101}$$

reduces the number of parameters in the transformation from 4 to 3 and gives a determinant equal to unity as well, so that key properties on the behaviour of $\check{K}_i$ can be imposed. Inverting the relation (5.100), upon using (5.101) we may also express the $\check{K}_0, \check{K}_\pm$ in terms of $K_0, K_\pm$:

$$\begin{pmatrix} \check{K}_0 \\ \check{K}_+ \\ \check{K}_- \end{pmatrix} = \begin{pmatrix} \gamma\beta + \delta\alpha & -\gamma\delta & -\alpha\beta \\ -2\gamma\alpha & \gamma^2 & \alpha^2 \\ -2\delta\beta & \delta^2 & \beta^2 \end{pmatrix} \begin{pmatrix} K_0 \\ K_+ \\ K_- \end{pmatrix} . \tag{5.102}$$

The comparison between (5.99) and (5.102) above makes evident that one might associate a similarity transformation to a generalized Bogoliubov transformation, with the following conditions being respected

$$\begin{aligned} \alpha &= -2\nu_- \frac{\sinh\theta}{\theta} , \qquad \beta = \cosh\theta - \epsilon\frac{\sinh\theta}{\theta}, \\ \delta &= +2\nu_+ \frac{\sinh\theta}{\theta} , \qquad \gamma = \cosh\theta + \epsilon\frac{\sinh\theta}{\theta}, \end{aligned} \tag{5.103}$$

and consequently $(\beta\gamma - \alpha\delta)^3 = 1$. Without any additional restriction we are able to decompose the $\check{K}_i$ generators into new creation and annihilation operators c and d,

$$\check{K}_0 = \frac{1}{2}\left(cd + \frac{1}{2}\right) , \quad \check{K}_+ = \frac{1}{2}c^2 , \quad \check{K}_- = \frac{1}{2}d^2 , \quad [c,d] = 1 \tag{5.104}$$

being related to the usual bosonic operator through a linear combination,

$$\begin{pmatrix} a \\ a^\dagger \end{pmatrix} = \begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} d \\ c \end{pmatrix} \qquad \Longleftrightarrow \qquad \begin{pmatrix} d \\ c \end{pmatrix} = \begin{pmatrix} \beta & -\delta \\ -\alpha & \gamma \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}, \tag{5.105}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. This is the interesting feature that will allow to diagonalize the system, despite the fact that neither c, d nor $\check{K}_i$ are Hermitian. We do not even require the same Hermiticity conditions as the conventional operators $a = (a^\dagger)^\dagger$, and in general $c \neq d^\dagger$. For our purposes we do nonetheless impose a definite behaviour under the $\mathcal{PT}$-transformation. Noting that $\mathcal{PT}$: $a, a^\dagger \to -a, -a^\dagger$ implies that $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ or $i\mathbb{R}$, such that

$$\mathcal{PT} \ : \ c, d \to -c, -d \quad \text{or} \quad \mathcal{PT} \ : \ c, d \to \ \ c, d \, , \tag{5.106}$$

respectively.

Replacing in H_K the generators $K_0, K_\pm$ by the newly defined generators $\check{K}_0, \check{K}_\pm$, we can re-express the Hamiltonian into the form

$$H_K = \sum_{l=0,\pm} \check{\mu}_l \check{K}_l + \sum_{n,m=0,\pm} \check{\mu}_{nm} : \check{K}_n \check{K}_m : . \tag{5.107}$$

Notice that due to the identity $8\check{K}_+\check{K}_- = 8\check{K}_0\check{K}_0 - 8\check{K}_0 + 1$ not all coefficients $\check{\mu}_l$, $\check{\mu}_{nm}$ are uniquely defined although this ambiguity will not play any role in our analysis as the relevant equations will be insensitive to these redefinitions.

However, if one can write the non-Hermitian Hamiltonian, for example as a result of constraints in the original coefficients in the operator, in the form of

$$\begin{aligned} H &= \check{\mu}_0\check{K}_0 + \check{\mu}_{00}\check{K}_0^2 + \check{\mu}_{+-}\check{K}_+\check{K}_- \\ &= \frac{1}{2}\check{\mu}_0\left(\check{N}+\frac{1}{2}\right) + \frac{1}{4}\check{\mu}_{00}\left(\check{N}^2+\check{N}+\frac{1}{4}\right) + \frac{1}{4}\check{\mu}_{+-}(\check{N}^2-\check{N}) \end{aligned} \tag{5.108}$$

then, diagonalization is possible by analogy between $N = a^\dagger\, a$ and $\check{N} = cd$:

$$\check{N}|\check{n}\rangle = n|\check{n}\rangle \tag{5.109}$$

In analogy to the harmonic oscillator case we may now easily construct the eigensystem for this Hamiltonian. Defining the states

$$|\check{n}\rangle = \frac{1}{\sqrt{n!}}c^n|\check{0}\rangle \qquad \text{with} \qquad d|\check{0}\rangle = 0, \tag{5.110}$$

we have $\check{N}|\check{n}\rangle = n|\check{n}\rangle$. Note that imposing unbroken $\mathcal{PT}$-symmetry for the states $|\check{n}\rangle$ requires that $\mathcal{PT}$: $c \to c$, which in turn implies the coefficients $\alpha, \beta, \gamma, \delta$ must be purely imaginary. In this situation the parameters $\varepsilon, \nu_\pm$, cannot be real as assumed so far for the Hermiticity of the Dyson operator. With this in mind, the pseudo-Hermitian approach in principle allows for real eigenvalues in the $\mathcal{PT}$-symmetric broken phase, where this cannot be guaranteed.

Demanding further that the Hamiltonian in terms of the new generators $\check{K}_0$, $\check{K}_\pm$ acquires the form of a harmonic oscillator plus a Casimir operator means we have to set the constants $\check{\mu}_+$, $\check{\mu}_-$, $\check{\mu}_{++}$, $\check{\mu}_{--}$, $\check{\mu}_{+0}$, $\check{\mu}_{0-}$ to zero. Expressing these constraints through the original constants in (5.21) yields the equations

$$\mu_{++}y^4 + \mu_{+0}y^3 + (\mu_{+-}+\mu_{00})y^2 + \mu_{0-}y + \mu_{--} = 0, \tag{5.111}$$

$$\mu_{--}z^4 + \mu_{0-}z^3 + (\mu_{+-}+\mu_{00})z^2 + \mu_{+0}z + \mu_{++} = 0, \tag{5.112}$$

$$\begin{aligned} &\mu_{+0}y^3z + 4\mu_{++}y^3 + 2(\mu_{+-}+\mu_{00})y^2z + 3\mu_{+0}y^2 + \\ &\quad + 3\mu_{0-}yz + 2(\mu_{+-}+\mu_{00})y + 4\mu_{--}z + \mu_{0-} = 0, \end{aligned} \tag{5.113}$$

$$\begin{aligned} &\mu_{0-}yz^3 + 4\mu_{--}z^3 + 2(\mu_{+-}+\mu_{00})yz^2 + 3\mu_{0-}z^2 + \\ &\quad + 3\mu_{+0}yz + 2(\mu_{+-}+\mu_{00})z + 4\mu_{++}y + \mu_{+0} = 0, \end{aligned} \tag{5.114}$$

$$
\begin{aligned}
&(\mu_{0-}-\mu_-)yz^3+(2\mu_{+-}+\mu_{00}-\mu_0)yz^2+2\mu_{--}z^3+\\
+&(2\mu_{+0}-\mu_+)yz+(\mu_-+\mu_{0-})z^2+(\mu_0+\mu_{00})z+2\mu_{++}y+\mu_+ &= 0 \qquad (5.115)\\
&(\mu_{+0}-\mu_+)y^3z+(2\mu_{+-}+\mu_{00}-\mu_0)y^2z+2\mu_{++}y^3+\\
+&(2\mu_{0-}-\mu_-)yz+(\mu_++\mu_{+0})y^2+(\mu_0+\mu_{00})y+2\mu_{--}z+\mu_- &= 0 \qquad (5.116)
\end{aligned}
$$

where we abbreviated

$$
y=\frac{\alpha}{\gamma} \qquad \text{and} \qquad z=\frac{\delta}{\beta}, \tag{5.117}
$$

so that the number of parameters diminishes but care is then recommended when examining certain limiting cases. We will now systematically solve the six equations following (5.111). When $\alpha,\delta\neq 0$ the equations reduce to the simpler form

$$
\begin{aligned}
z^2(\mu_{00}+\mu_{+-}) &= \mu_{++}(1+4yz+y^2z^2), && (5.118)\\
z^2(\mu_{+-}-\mu_0) &= \mu_{++}(1+yz)^2+\mu_+z^3+\mu_-z, && (5.119)\\
\mu_{--}z^2 &= \mu_{++}y^2, && (5.120)\\
\mu_{+0}z &= -2\mu_{++}(1+yz), && (5.121)\\
\mu_-z &= \mu_+y, && (5.122)\\
\mu_{0-}z &= \mu_{+0}y. && (5.123)
\end{aligned}
$$

Similarly as in the previous section the solutions fall into different classes distinguished by vanishing linear or bilinear combinations.

Genuinely non-Hermitian non-reducible Hamiltonian

We start to solve the six constraints (5.118)-(5.123) for the generic case by demanding $\mu_+,\mu_-\neq 0$ and $\mu_{++},\mu_{--},\mu_{+0},\mu_{0-}\neq 0$. We find the unique solution

$$
\begin{aligned}
\mu_- &= \frac{y}{z}\mu_+, \quad \mu_{--}=\frac{\mu_-^2}{\mu_+^2}\mu_{++}, \quad \mu_{0-}=\frac{\mu_-}{\mu_+}\mu_{+0}, \quad y=\frac{\pm\vartheta-\mu_{+0}/4}{\mu_{++}}, && (5.124)\\
\mu_{+-} &= \mu_0-\frac{\mu_+\mu_{+0}}{2\mu_{++}}+\frac{\mu_{+0}^2}{4\mu_{++}}, \qquad \mu_{00}=-\mu_0+\frac{\mu_+\mu_{+0}}{2\mu_{++}}+\frac{2\mu_-\mu_{++}}{\mu_+}, && (5.125)
\end{aligned}
$$

with the abbreviation $\vartheta:=\sqrt{\mu_{+0}^2/16-\mu_{++}^2\mu_+/\mu_-}$. The Hamiltonian H_K in (5.21) or equivalently the Hamiltonians $H_{\check{K}}$ in (5.107) can now be expressed entirely in terms of the number operator $\check{N}=cd$ and acquires the simple form

$$
H_K = \frac{\vartheta^2}{\mu_{++}}(\check{N}^2+\check{N})\pm\frac{\vartheta(\mu_{+0}-2\mu_+)}{2\mu_{++}}\left(\check{N}+\frac{1}{2}\right)+\frac{3\mu_0}{16}-\frac{3\mu_+\mu_{+0}}{32\mu_{++}}+\frac{\mu_{+0}^2}{16\mu_{++}}-\frac{5\mu_-\mu_{++}}{8\mu_+}. \tag{5.126}
$$

The requirement that the spectrum is real and bounded from below imposes the further constraints

$$\mu_{++} > 0 \qquad \text{and} \qquad \mu_- \mu_{+0}^2 > 16 \mu_{++}^2 \mu_+ . \tag{5.127}$$

It is now interesting to compare this result with our previous construction for the isospectral counterpart in section 5.5.4. Using the constraints (5.124) and (5.125) we may solve the conditions (5.71)-(5.71) for the similarity transformation needed to be able to construct a well defined metric operator by

$$\mu_{+0} = 2\mu_+ \qquad \text{and} \quad \lambda = \frac{\mu_+^2}{2(\mu_+ + \mu_-)\mu_{++}}, \tag{5.128}$$

such that (5.75) acquires the form

$$\varepsilon \frac{\tanh 4\theta}{\theta} = \frac{2(\mu_-^2 - \mu_+^2)\mu_{++}^2}{2(\mu_-^2 + \mu_+^2)\mu_{++}^2 - \mu_+^4}. \tag{5.129}$$

Thus upon these constraints, the two constructions coincide, if besides (5.127) we also demand that $\mu_+^2 \leq \mu_{++} |\mu_+ + \mu_-|$ since $|\lambda| \leq \frac{1}{2}$. This means that in this situation we do not only have an explicit similarity transformation, a well defined metric and a Hermitian counterpart, but in addition we know the exact eigenspectrum and eigenfunctions. Relaxing these conditions it also implies that there must be a larger class of similarity transformations not covered by the ansatz (5.24) for the operator η. As already mentioned we might be losing out on some possibilities by demanding η to be Hermitian.

A further natural generalisation would be to include also bilinear combinations into the argument of the exponential in the expression for η.

Hermitian linear term and non-Hermitian bilinear combinations

It seems natural that we mimic the same cases as for the construction of the metric in previous sections. However, when tuning the linear term to be Hermitian by demanding $\mu_+ = \mu_-$ the constraints (5.120), (5.122) and (5.123) imply that $\mu_{++} = \mu_{--}$ and $\mu_{+0} = \mu_{0-}$, such that also the terms involving bilinear combinations becomes Hermitian. The case $\mu_+ = \mu_- = 0$ is special since the last equation in (5.125) yields $\mu_+/\mu_- = (\mu_0 + \mu_{00})/(2\mu_{++})$. Using this and still demanding that $\mu_{++}, \mu_{--}, \mu_{+0}, \mu_{0-} \neq 0$, the solutions to (5.118)-(5.123) become

$$\mu_{--} = \frac{(\mu_0 + \mu_{00})^2}{4\mu_{++}}, \qquad \mu_{0-} = \frac{\mu_{+0}}{2\mu_{++}} (\mu_0 + \mu_{00}), \qquad \mu_{+-} = \mu_0 + \frac{\mu_{+0}^2}{4\mu_{++}}, \tag{5.130}$$

$$y = \frac{\pm\bar{\vartheta} - \mu_{+0}/4}{\mu_{++}}, \qquad z = y \frac{2\mu_{++}}{\mu_0 + \mu_{00}}, \tag{5.131}$$

with the abbreviation $\bar{\vartheta} := \sqrt{\mu_{+0}^2/16 - \mu_{++}\left(\mu_0 + \mu_{00}\right)/2}$. The Hamiltonian H_K in (5.21), (5.107) can be expressed again entirely in terms of the number operator and acquires the simple form

$$H_K \;= \frac{\bar{\vartheta}^2}{\mu_{++}}(\check{N}^2 + \check{N}) \pm \frac{\bar{\vartheta}\mu_{+0}}{2\mu_{++}}\left(\check{N} + \frac{1}{2}\right) + \frac{\mu_{+0}^2}{16\mu_{++}} - \frac{5}{16}\mu_{00} - \frac{\mu_0}{8}. \tag{5.132}$$

The requirement that the spectrum is real and bounded from below yields in this case the additional constraints

$$\mu_{++} > 0 \qquad \text{and} \qquad \mu_{+0}^2 > 8\mu_{++}\left(\mu_0 + \mu_{00}\right). \tag{5.133}$$

Interestingly when demanding (5.130) and (5.131), we cannot solve the constraints in section 5.4 and therefore can not construct a metric with the ansatz (5.24) in this case. This is a clear indication that the metric proposed is not the most general.

Non-Hermitian linear case and Hermitian bilinear combinations

Reversing the setting of the previous section we may now demand the bilinear combinations to be Hermitian, $\mu_{++} = \mu_{--}$ and $\mu_{+0} = \mu_{0-}$. This is equally pathological as now the linear term becomes also Hermitian by (5.120), (5.122) and (5.123). Nonetheless, a non-trivial limit is obtained with $\mu_{++} = \mu_{--} = \mu_{+0} = \mu_{0-} = 0$ and requiring $\mu_+, \mu_- \neq 0$. We may then solve (5.118)-(5.123) by

$$\mu_- = \frac{y}{z}\mu_+, \qquad \mu_{+-} = -\mu_{00}, \qquad y = \frac{\pm\tilde{\vartheta} - (\mu_0 + \mu_{00})/2}{\mu_+}, \tag{5.134}$$

with the abbreviation $\tilde{\vartheta} := \sqrt{(\mu_0 + \mu_{00})^2/4 - \mu_+\mu_-}$. Once again the Hamiltonian H_K in (5.21) and (5.107) can be expressed entirely in terms of the modified number operator simplifying it to

$$H_K \;= \pm\tilde{\vartheta}\left(\check{N} + \frac{1}{2}\right) - \frac{3\mu_{00}}{16}. \tag{5.135}$$

The eigenspectrum is real and bounded from below when we discard the minus sign in (8.79) and impose the condition

$$(\mu_0 + \mu_{00})^2 > 4\mu_+\mu_-. \tag{5.136}$$

When setting $\mu_{00} = \mu_{+-} = 0$ these expressions reduce precisely to those found in [24] for the purely linear case. Comparing now with the construction in section 5.5.4, we find that (5.45) is solved by the conditions (5.134), if we further demand that

$$\mu_{00} + \mu_{+-} = 0, \tag{5.137}$$

such that (5.46) becomes

$$\varepsilon \frac{\tanh 2\theta}{\theta} = \frac{\mu_- - \mu_+}{\mu_- + \mu_+ - 2\lambda(\mu_{00} + \mu_0)}. \tag{5.138}$$

We may also put further restrictions on the generalized Bogoliubov transformation (5.105) itself by setting some of the constants to zero.

Asymmetric generalized Bogoliubov transformation with $\delta = 0$

Let us now set the α in (5.105) to zero. Then the equations (5.111) are solved by

$$\mu_+ = \mu_{++} = \mu_{+0} = 0, \quad \mu_{0-} = -\frac{2\mu_{--}}{y}, \quad \mu_{00} = -\mu_0 - \frac{\mu_-}{y}, \quad \mu_{+-} = \frac{\mu_{--}}{y^2} - \mu_{00}. \tag{5.139}$$

In this situation the transformed Hamiltonian H_K (5.107) can be expressed as

$$H_K = \frac{\mu_{0-}^2}{16\mu_{--}}(\check{N}^2 - \check{N}) + \frac{\mu_{0-}\mu_-}{4\mu_{--}}\left(\check{N} + \frac{1}{8}\right) + \frac{3\mu_0}{16}. \tag{5.140}$$

Once again we may compare with the construction in section 5.5.4. The operator η can be constructed when we demand

$$\mu_{+-} = \mu_0 \qquad \text{and} \qquad \lambda = -\frac{1}{2y} \tag{5.141}$$

together with

$$\varepsilon = \frac{1}{4\sqrt{1 - \frac{1}{y^2}}} \mathrm{ArcTanh}\left(\frac{2y^2\sqrt{1 - \frac{1}{y^2}}}{1 - 2y^2}\right). \tag{5.142}$$

The meaningful interval $\lambda \in [-\frac{1}{2}, \frac{1}{2}]/\{0\}$ is now translated into the condition $y \in [-1, 1]/\{0\}$.

Asymmetric generalized Bogoliubov transformation with $\alpha = 0$

We may also put further constraints on the transformation (5.105) itself. Then the equations (5.111) are solved by

$$\mu_- = \mu_{--} = \mu_{0-} = 0, \quad \mu_{+0} = -\frac{2\mu_{++}}{z}, \quad \mu_{00} = -\mu_0 - \frac{\mu_+}{z}, \quad \mu_{+-} = \frac{\mu_{++}}{z^2} - \mu_{00}. \tag{5.143}$$

Now the transformed Hamiltonian H_K (5.107) can be expressed as

$$H_K = \frac{\mu_{+0}^2}{16\mu_{++}}(\check{N}^2 - \check{N}) + \frac{\mu_{+0}\mu_+}{4\mu_{++}}\left(\check{N} + \frac{1}{8}\right) + \frac{3\mu_0}{16}. \tag{5.144}$$

The comparison with the construction in section 5.5.4 yields now that the operator η can be constructed when we demand

$$\mu_{+-} = \mu_0 \qquad \text{and} \qquad \lambda = -\frac{1}{2z} \tag{5.145}$$

as well as

$$\varepsilon = \frac{1}{4\sqrt{1-\frac{1}{z}}}\text{ArcTanh}\left(\frac{2z^2\sqrt{1-\frac{1}{z^2}}}{1-2z^2}\right) \tag{5.146}$$

Now $\lambda \in [-\frac{1}{2},\frac{1}{2}]/\{0\}$ is translated to the condition $z \in [-1,1]/\{0\}$.

As a trivial consistency we observe that for $\alpha = \delta = 0$, i.e. when $y = z = 0$, the transformation (5.105) becomes the identity and we have the vanishing of all coefficients except for μ_0, μ_{00} and μ_{+-}. Thus the initial Hamiltonian is already Hermitian and just corresponds to the harmonic oscillator displaced by a Casimir operator. The configuration when the constants μ_+, $\mu_-, \mu_{++}, \mu_{--}, \mu_{+0}$ and μ_{0-} vanish is obviously of little interest.

For completeness we also comment on the case $yz = -1$ for which we may also find an explicit solution. However, in this situation the coefficients in front of $\check{K}_0^2$ and $\check{K}_0$ are not positive and consequently this scenario is of little relevance.

5.5.6 Some concrete realisations of the generalised Swanson Hamiltonian

Let us finish our generic discussion with a few comments related to some concrete realisations of the algebras discussed. Using the usual representation of the $su(1,1)$ is probably the aforementioned two-boson representation (5.16) with the familiar identifications for the creation and annihilation operators in terms of differential operators in x-space it is then straightforward to express H_K in terms maximally quartic in the position and momentum operators, albeit not in its most general form,

$$H_{xp} = \gamma_0 + \gamma_1\hat{x}^2 + \gamma_2\hat{p}^2 + \gamma_3\hat{x}^4 + \gamma_4\hat{p}^4 + \imath\gamma_5\hat{x}\hat{p} + \gamma_6\hat{x}^2\hat{p}^2 + \imath\gamma_7\hat{x}\hat{p}^3 + \imath\gamma_8\hat{x}^3\hat{p}. \tag{5.147}$$

The coefficients γ_i in (5.147) and the μ_l, $\mu_{n,m}$ in (5.21) are related as

$$\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\\ \gamma_3\\ \gamma_4\\ \gamma_5\\ \gamma_6\\ \gamma_7\\ \gamma_8\end{pmatrix} = \frac{1}{16}\begin{pmatrix} 0 & -4 & 4 & -2 & 3 & 3 & 1 & 2 & -2\\ 4 & 4 & 4 & 0 & -6 & 6 & -4 & -5 & 1\\ 4 & -4 & -4 & 0 & 6 & -6 & -4 & -1 & 5\\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1\\ 0 & -8 & 8 & -4 & 12 & 12 & -4 & 4 & -4\\ 0 & 0 & 0 & 2 & -6 & -6 & 2 & 0 & 0\\ 0 & 0 & 0 & 0 & 4 & -4 & 0 & -2 & 2\\ 0 & 0 & 0 & 0 & -4 & 4 & 0 & -2 & 2\end{pmatrix}\begin{pmatrix}\mu_0\\ \mu_+\\ \mu_-\\ \mu_{00}\\ \mu_{++}\\ \mu_{--}\\ \mu_{+-}\\ \mu_{+0}\\ \mu_{0-}\end{pmatrix}. \tag{5.148}$$

Since the determinant of the matrix in (5.148) is non-vanishing we may also express the μ_l, $\mu_{n,m}$ in terms of the γ_i, which then translates the constraining equations and the coefficient occurring in the Hermitian counterparts too. It is interesting to note that the argument $2\varepsilon[K_0 + \lambda(K_+ + K_-)]$ in the exponential of the operator η becomes $\varepsilon\left[\frac{1}{2}(\hat{x}^2 + \hat{p}^2) + \lambda(\hat{x}^2 - \hat{p}^2) + 1\right]$, such that at the boundaries of the interval in which λ takes its values $\lambda = \frac{1}{2}$ and $\lambda = -\frac{1}{2}$ the operator and therefore the metric becomes a function only of $\hat{x}$ and $\hat{p}$, respectively.

There are various types of representations in terms of differential operators for this algebra as for instance the multi-boson representation

$$K_0 = k_0(N), \qquad K_+ = k_+(N)(a^\dagger)^n, \qquad K_- = k_-(N)(a)^n, \tag{5.149}$$

where the $a, a^\dagger$ are the usual bosonic annihilation and creation operators with $N = a^\dagger a$ being the number operator. The $k_0(N)$, $k_\pm(N)$ are functions of the latter and may be determined recursively for any number of bosons n involved [206]. The simplest case $n = 1$ yields the Holstein-Primakoff representation [207] with $K_0 = N + \frac{1}{2}$, $K_+ = \sqrt{N}a^\dagger$ and $K_- = a\sqrt{N}$. For $n = 2$ one obtains the very well known two boson representation (5.16). However, also the realisation for $n = 1$ in (5.149) plays an important role in physics for instance in the study of the Jaynes-Cummings model [208].

Many other representations exist, some of which mentioned in [25]. Instead of using (5.17) one can work with a generalised infinite dimensional representation of the bosonic operators such as

$$a = A(x)\frac{d}{dx} + B(x), \qquad a^\dagger = -A(x)\frac{d}{dx} - A'(x), \qquad \text{with} \qquad 2AB' - AA'' = 0, \tag{5.150}$$

as is done for generalized Swanson models in [209].

An interesting one is for instance one

$$K_0 = \frac{1}{4\xi}\left(-\frac{d^2}{dr^2} + \frac{g}{r^2} + \xi^2 r^2\right), \tag{5.151}$$

$$K_\pm = \frac{1}{4\xi}\left(\frac{d^2}{dr^2} - \frac{g}{r^2} + \xi^2 r^2 \mp \xi\left(2r\frac{d}{dr} + 1\right)\right), \tag{5.152}$$

Using this representation H_K may be expressed as a differential operator in radial coordinates

$$\begin{aligned} H_R = {} & \rho_0 + \rho_1\frac{d^4}{dr^4} + \rho_2 r\frac{d^3}{dr^3} + \rho_3 r^2\frac{d^2}{dr^2} + \rho_4\frac{d^2}{dr^2} + \rho_5\frac{1}{r^2}\frac{d^2}{dr^2} + \rho_6 r^3\frac{d}{dr} + \\ & + \rho_7 r\frac{d}{dr} + \rho_8\frac{1}{r}\frac{d}{dr} + \rho_9\frac{1}{r^3}\frac{d}{dr} + \rho_{10}r^4\frac{d}{dr} + \rho_{11}r^2 + \rho_{12}\frac{1}{r^2} + \rho_{13}\frac{1}{r^4}, \end{aligned} \tag{5.153}$$

and its corresponding Hermitian counterpart, using the constraints (5.51), (5.52), (5.52), (5.52), is given by

$$\begin{aligned} h_R &= \tilde{\rho}_0 + \tilde{\rho}_1 \frac{d^4}{dr^4} + \tilde{\rho}_3 r^2 \frac{d^2}{dr^2} + \tilde{\rho}_4 \frac{d^2}{dr^2} + \tilde{\rho}_5 \frac{1}{r^2}\frac{d^2}{dr^2} + \\ &+ \tilde{\rho}_7 r \frac{d}{dr} + \tilde{\rho}_9 \frac{1}{r^3}\frac{d}{dr} + \tilde{\rho}_{10} r^4 \frac{d}{dr} + \tilde{\rho}_{11} r^2 + \tilde{\rho}_{12}\frac{1}{r^2} + \tilde{\rho}_{13}\frac{1}{r^4}. \end{aligned} \tag{5.154}$$

The ρs and $\tilde{\rho}$s may be computed explicitly, but this is not relevant for our purposes here. Keeping only linear terms in K in the Hamiltonian, we obtain

$$H = \frac{(\mu_0 - \mu_+ - \mu_-)}{4\xi}\left(-\frac{d^2}{dr^2} + \frac{g}{r^2}\right) + \frac{\mu_- - \mu_+}{4}\left(1 + 2r\frac{d}{dr}\right) + \frac{(\mu_0 + \mu_- + \mu_-)}{4}\xi r^2, \tag{5.155}$$

which under the action of

$$\eta = \exp\left[\frac{1}{\sqrt{1-4\lambda^2}}\text{ArcTanh}\left(\frac{(\mu_- - \mu_+)\sqrt{1-4\lambda^2}}{\mu_- + \mu_+ - 2\lambda\mu_0}\right)[K_0 + \lambda(K_+ + K_-)]\right] \tag{5.156}$$

transforms into

$$h = \frac{a+b}{\xi}\left(\frac{d^2}{dr^2} - \frac{g}{r^2}\right) + \left(\frac{1+2\lambda}{1-2\lambda}a + 3b\right)\xi r^2, \tag{5.157}$$

with parameters given by

$$a = \frac{1}{2(1+2\lambda)}\sqrt{\mu_+\mu_- - \lambda\mu_0(\mu_+ + \mu_-) + \lambda^2(\mu_0^2 + (\mu_- - \mu_+)^2)}, \tag{5.158}$$

$$b = \frac{\mu_0 - 2\lambda(\mu_+ + \mu_-)}{4(1-4\lambda^2)}. \tag{5.159}$$

Some mild variations of the representation (5.151), (5.152) can be used to obtain multi-particle systems, such as Calogero models but an easier multi-particle model covered by this formalism is of charged particles in a magnetic field [210], which results when taking as representation (5.19) with the $a_i^\dagger, a_i$ as the creation and annihilation of the i-th bosonic particle. It is straightforward to apply the above programme also to this type of system.

As a variation of the above idea we may also study multi-particle $\mathcal{PT}$-symmetric Hamiltonians, for which we do not mix different particle types implicitly within $su(1,1)$-generators, i.e. taking direct sums of Fock spaces, but consider instead systems of the type $su(1,1) \oplus su(1,1)$, such as

$$\begin{aligned} H_m &= \mu_0^{(1)}K_0^{(1)} + \mu_+^{(1)}K_+^{(1)} + \mu_-^{(1)}K_-^{(1)} + \mu_0^{(2)}K_0^{(2)} + \mu_+^{(2)}K_+^{(2)} + \mu_-^{(2)}K_-^{(2)} + \mu_{00}K_0^{(1)}K_0^{(2)} \\ &+ \mu_{+-}K_+^{(1)}K_-^{(2)} + \mu_{-+}K_-^{(1)}K_+^{(2)} + \mu_{++}K_+^{(1)}K_+^{(2)} + \mu_{--}K_-^{(1)}K_-^{(2)} + +\mu_{+0}K_+^{(1)}K_0^{(2)} \\ &+ \mu_{0-}K_0^{(1)}K_-^{(2)} + \mu_{0+}K_0^{(1)}K_+^{(2)} + \mu_{-0}K_-^{(1)}K_0^{(2)}, \end{aligned} \tag{5.160}$$

with the superscripts in the $K^{(i)}$ indicate the particle type. We may start with an ansatz of a similar type

$$\eta = \eta_1\, \eta_2 \qquad \text{with} \qquad \eta_i = \exp\left[2\varepsilon_i \left(K_0^{(i)} + \lambda_i (K_+^{(i)} + K_-^{(1)})\right)\right] \tag{5.161}$$

and it is then straightforward to show that the constraints

$$\mu_{00} = \frac{\mu_{++}}{\lambda_1 \lambda_2}, \qquad \mu_{-+} = \mu_{+-} = \mu_{--} = \mu_{++}, \tag{5.162}$$

$$\mu_{0+} = \mu_{0-} = \frac{\mu_{++}}{\lambda_1}, \qquad \mu_{+0} = \mu_{-0} = \frac{\mu_{++}}{\lambda_2}, \tag{5.163}$$

with

$$\varepsilon_i \frac{\tanh 2\theta_i}{\theta_i} = \frac{\mu_-^{(i)} - \mu_+^{(i)}}{\mu_-^{(i)} + \mu_+^{(i)} - 2\lambda_i \mu_0^{(i)}} \qquad \text{for } i = 1, 2, \tag{5.164}$$

convert the Hamiltonian H_m into a Hermitian one. Note that despite the fact that in H_m we have an interaction between different particle types the constraints are identical to the ones in the linear case for individual particles. In fact, H_m is indeed linear in $K^{(1)}$ and $K^{(2)}$ and the terms involving the products of $K^{(1)}$ and $K^{(2)}$ operators are Hermitian. Adding some genuinely bilinear combinations in the Hamiltonian is expected to generate a more intricate structure sheding light on interacting spins etc, but we leave this for future investigations as it goes beyond the simple comment we intended to make in this section regarding explicit realisations.

5.6 Hamiltonians of $su(2)$-Lie algebraic type and bosonic spin chains

As we have seen, the three $su(2)$ generators satisfy the following commutation relations

$$[L_0, L_\pm] = \pm L_\pm, \qquad [L_+, L_-] = 2L_0 \qquad \text{and} \qquad L_0^\dagger = L_0, L_\pm^\dagger = L_\mp, \tag{5.165}$$

for which one has the well known angular momentum representation in terms of differential operators,

$$L_0 = \frac{\partial}{\partial \phi} \;, \qquad L_\pm = \imath e^{\pm \imath \phi} \left(\pm \frac{\partial}{\partial \theta} + \imath \cot\theta \frac{\partial}{\partial \phi}\right) \tag{5.166}$$

with compact angles ϕ, θ, constitutes an infinite dimensional representation of this compact algebra. Another possible realization for the $su(2)$ algebra is given by

$$L_0 = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right), \qquad L_+ = a_1^\dagger a_2, \qquad L_- = a_2^\dagger a_1, \tag{5.167}$$

with $L_0^\dagger = L_0, L_\pm^\dagger = L_\mp$, used for number-conserving Bose-Hubbard models for which a fixed particle number $N = a_1^\dagger a_1 + a_2^\dagger a_2$ corresponds to a specified modulus of the angular momentum: $L^2 = \frac{N}{2}\left(\frac{N}{2}+1\right)$. In this representation we may realize $\mathcal{PT}$ operation as in [212],

$$\mathcal{PT}: \quad a_1 \to -a_1, \quad a_1^\dagger \to -a_1^\dagger, \quad a_2 \to -a_2, \quad a_2^\dagger \to -a_2^\dagger. \tag{5.168}$$

It is interesting to note that the combinations of these generators

$$\tilde{K}_0 = \frac{1}{2}\left(a_2^\dagger a_2 - a_1^\dagger a_1\right) \equiv -L_0, \qquad \tilde{K}_+ = a_2^\dagger a_1 \equiv L_-, \qquad \tilde{K}_- = -a_1^\dagger a_2 \equiv -L_+. \tag{5.169}$$

can be constructed so that they satisfy the commutation relation (4.30) with $\epsilon = -1$. However the algebras $su(2)$ and $su(1,1)$ are not equivalent, since the former corresponds to a compact group and allows for a finite dimensional representation while the latter arises from a noncompact group and has only infinite dimensional representations. The difference between such algebras in such representation becomes evident when we analyze the conjugation properties of these operators and see that they differ by a minus sign

$$\tilde{K}_0^\dagger = \tilde{K}_0, \quad \tilde{K}_\pm^\dagger = -\tilde{K}_\mp. \tag{5.170}$$

and the $\tilde{K}_i$ are not generators of $su(1,1)$ but of $su(2)$ instead, for which we have applied $L_0 \to -L_0$, $L_\pm \to \mp L_\mp$, which of course is not an automorphism. The L-operators can then be used to define analogous models to (5.3) and (5.21), namely

$$H_L = \sum_{l=0,\pm} \mu_l L_l + \sum_{n,m=0,\pm} \mu_{nm} :L_n L_m:, \qquad \mu_l, \mu_{nm} \in \mathbb{R}, \tag{5.171}$$

with the usual normal ordering prescribed in (5.4) and (5.22).

The generators of the compact $su(2)$-algebra, the Pauli matrices, are generally used to formulate richer interacting spin quantum chain models, very common in condensed matter physics to describe magnetization in materials. Non-Hermitian versions of such many-body models with $\mathcal{PT}$-symmetry defined on a lattice have been analysed recently, for instance in [213, 214], with the calculation of exact Bethe wavefunctions and pseudo-Hermitian metrics. The Yang-Lee problem, which is just an N-spin Heisenberg (anti-)ferromagnetic chain in the presence of a magnetic field with components in two perpendicular direction, was investigated in a $\mathcal{PT}$-symmetric framework in [92, 93] with pseudo-Hermitian metrics and Hermitian counterparts being calculated in [19] by considering one of the couplings in the Hamiltonian to be imaginary.

The algebraic structure, proving both mathematical tools to solve the problem and universal representation-independent results, is confirmed to be a successful approach

when searching for a Dyson map to define a fundamental theory based on a non-Hermitian Hamiltonian. Not always, however, can algebraic symmetries be identified in a system and therefore other techniques to such a purpose must be considered. In the next chapter we investigate the use of a different approach which does not rely on identifying an algebraic structure so that it can be implemented in principle to a greater variety of systems.

6 Moyal products and isospectral transformations

6.1 Moyal Products to construct metric operators

In the examples investigated in the previous chapter, the entire problem of finding a suitable metric with respect to which a non-Hermitian system is considered Hermitian was analyzed under the perspective of finding exact solutions by solving equations for operators $\hat{x}, \hat{p}$ or $a^\dagger, a$. Instead of doing so, this task can sometimes be considerably simplified if the operator equations are converted into differential equations using Moyal products [216], denoted here by $*$ products, and also sometimes called Weyl-Groenewold product [217], *star* products.

In this formalism operators expressed in terms of the aforementioned operators are replaced by real-valued functions and a new product composition rule is defined in order to restore the noncommutativity between position and momentum, which is an essential feature for the uncertainty principle of quantum mechanics. Moyal and Weyl-Groenewold brackets, designated respectively by [11]

$$[f,g]_* = f*g - g*f \qquad \text{and} \qquad [f,g]_\star = f\star g - g\star f, \tag{6.1}$$

can be seen as a deformation of the phase-space Poisson bracket (4.44) with the introduction of the factor $\imath\hbar$, as in (2.16). In fact this concept of Moyal-like brackets is simply an alternative representation for the canonical commutation relations satisfied by the operators $\hat{x}$ and $\hat{p}$. The Moyal product can be introduced by considering the possible irreducible representation of the Heisenberg algebra (2.16)

$$e^{\imath r\hat{p}}e^{\imath s\hat{x}} = e^{\imath rs}e^{\imath s\hat{x}}e^{\imath r\hat{p}}, \tag{6.2}$$

[11] Note that $*$ denote a Moyal product but * denotes the usual operation of complex conjugation.

with the introduction of the elements

$$\bar{v}(r,s) = e^{\imath r\hat{p}} e^{\imath s\hat{x}} \qquad \Longleftrightarrow \qquad \langle \bar{v}(r',s')|\bar{v}(r,s)\rangle = 2\pi\delta(r-r')\delta(s-s'), \tag{6.3}$$

whose completeness make it suitable to expand any operator:

$$F(\hat{x},\hat{p}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dr\ ds\ e^{\imath r\hat{p}} e^{\imath s\hat{x}}\ f(s,r). \tag{6.4}$$

Its kernel,

$$f(s,r) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\hat{x}\ d\hat{p}\ e^{-\imath s\hat{x}} e^{-\imath r\hat{p}}\ F(\hat{x},\hat{p}), \tag{6.5}$$

can be useful to construct a corresponding real-valued function,

$$F(x,p) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dr\ ds\ e^{\imath rp} e^{\imath sx}\ f(s,r). \tag{6.6}$$

For instance, whereas $\hat{x}^m\hat{p}^n$ is mapped into $x^m p^n + \imath^m n p^{n-1}$, $\hat{p}^n\hat{x}^m$ goes to $x^m p^n$ simply. Taking the product between of such operator functions we obtain

$$F(\hat{x},\hat{p})G(\hat{x},\hat{p}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dr\ ds\ dr'\ ds'\ f(r,s)g(r',s')e^{\imath(r+r')\hat{p}} e^{\imath(s+s')\hat{x}} e^{-\imath r's}, \tag{6.7}$$

an expression which indicates the possibility of replacing the operators by real-valued functions if the extra factor which appears, $e^{-\imath r's}$, is taken care with the introduction of a new product rule. The Moyal product,

$$* = e^{\imath \overleftarrow{\partial}_x \overrightarrow{\partial}_p}, \tag{6.8}$$

serves precisely to this purpose:

$$F(x,p) * G(x,p) = F(x,p) e^{\imath \overleftarrow{\partial}_x \overrightarrow{\partial}_p} G(x,p), \tag{6.9}$$

a distributive and associative map allowing one to reproduce the commutation relation included in (6.2), namely, $e^{\imath rp} * e^{\imath sx} = e^{\imath rs}\ e^{\imath sx} * e^{\imath rp}$. Moreover,

$$F(x,p)^\dagger = e^{\imath\partial_x\partial_p} F(x,p)^*, \tag{6.10}$$

so that an operator will be Hermitian if and only if $F(x,p)^* = e^{-\imath\partial_x\partial_p} F(x,p)$.

Nevertheless, the Heisenberg algebra can be represented in a more symmetry way:

$$e^{\imath r\hat{p}} e^{\imath s\hat{x}} e^{-\frac{\imath rs}{2}} = e^{\frac{\imath rs}{2}} e^{\imath s\hat{x}} e^{\imath r\hat{p}}. \tag{6.11}$$

Using a more symmetric definition of the Moyal product due to Groenewold,

$$\star = e^{\frac{\imath}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)}, \tag{6.12}$$

so that

$$F(x,p)\star G(x,p)=\sum_{k=0}^{\infty}\frac{(-\imath/2)^k}{k!}\sum_{j=0}^{k}(-1)^j\frac{k!}{j!(k-j)!}\partial_x^j\partial_p^{k-j}F(x,p)\;\partial_x^{k-j}\partial_p^j G(x,p), \quad (6.13)$$

we can successfully reproduce $e^{\imath rp}\star e^{\imath sx}\;e^{-\frac{\imath rs}{2}}=e^{\frac{\imath rs}{2}}\;e^{\imath sx}\star e^{\imath rp}$. One can use the basis

$$v(r,s)=e^{\imath(s\hat{x}+r\hat{p})}=e^{\frac{\imath}{2}s\hat{x}}e^{\imath r\hat{p}}e^{\frac{\imath}{2}s\hat{x}}=e^{\frac{\imath}{2}r\hat{p}}e^{\imath s\hat{x}}e^{\frac{\imath}{2}r\hat{p}} \quad (6.14)$$

and its completeness $\langle v(r',s')|v(r,s)\rangle=2\pi\delta(r-r')\delta(s-s')$ to expand operators:

$$F(\hat{x},\hat{p})=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dr\;ds\;e^{\imath(s\hat{x}+r\hat{p})}\;f(s,r) \quad (6.15)$$

with

$$f(s,r)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\hat{x}\;d\hat{p}\;e^{-\imath(s\hat{x}+r\hat{p})}\;F(\hat{x},\hat{p}), \quad (6.16)$$

and

$$F(x,p)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dr\;ds\;e^{\imath(sx+rp)}\;f(s,r). \quad (6.17)$$

From these expressions, one can determine that $f(s,r)^\dagger=f(-s,-r)^*$, and $F(\hat{x},\hat{p})$ will be $\mathcal{PT}$-symmetric if $f(s,r)^\dagger=f(s,-r)$. The product

$$F(\hat{x},\hat{p})G(\hat{x},\hat{p})\simeq\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dr\;ds\;dr'\;ds'\;f(r,s)g(r',s')e^{\imath[(r+r')\hat{p}+(s+s')\hat{x}]}e^{\frac{\imath}{2}(rs'-sr')}, \quad (6.18)$$

so that the operator product expression can be made isomorphic to real-valued functions multiplied by the $\star$-product:

$$F(\hat{x},\hat{p})\;G(\hat{x},\hat{p})\simeq F(x,p)\star G(x,p)=F(x,p)e^{\frac{\imath}{2}(\overleftarrow{\partial}_x\overrightarrow{\partial}_p-\overleftarrow{\partial}_p\overrightarrow{\partial}_x)}G(x,p). \quad (6.19)$$

This new definition has a few advantages over the previous, more asymmetrical, one. It was shown in [28], for instance, that the use of $*$ leads to more complicated differential equations than $\star$, and we have seen above that the Hermiticity condition has a less transparent equation when $*$ is used (6.10). Furthermore, an operator will be Hermitian in this formulation if the corresponding function is real:

$$F(\hat{x},\hat{p})^\dagger=F(\hat{x},\hat{p})\qquad\simeq\qquad F(x,p)^*=F(x,p). \quad (6.20)$$

Also, whereas $(F*G)^*\neq G^**F^*$ the new definition gives a more intuitive result,

$$(F(\hat{x},\hat{p})\;G(\hat{x},\hat{p}))^\dagger=G(\hat{x},\hat{p})^\dagger F(\hat{x},\hat{p})^\dagger\qquad\simeq\qquad(F(x,p)\star G(x,p))^*=G(x,p)^*\star F(x,p)^*. \quad (6.21)$$

As an interesting example we consider $F(x,p) = x^m p^n$ for which we compute the corresponding kernel as $f(s,r) = \imath^{m+n}\delta^{(m)}(s)\delta^{(n)}(r)$. The kernel $f(r,s) = -\delta'(r)\delta'(s)$ of $\frac{1}{2}\left(\hat{x}\,\hat{p} + \hat{p}\,\hat{x}\right)$ is mapped into the monomial $x\,p = p\,x$. Indeed, for instance,

$$x\,p = \frac{1}{2}\left(x \star p + p \star x\right) \quad \simeq \quad \frac{1}{2}\left(\hat{x}\,\hat{p} + \hat{p}\,\hat{x}\right) \tag{6.22}$$

$$x^2\,p = \frac{1}{3}\left(x^2 \star p + x \star p \star x + p \star x^2\right) \quad \simeq \quad \frac{1}{3}\left(\hat{x}^2\,\hat{p} + \hat{x}\,\hat{p}\,\hat{x} + \hat{p}\,\hat{x}^2\right) \tag{6.23}$$

$$x\,p^2 = \frac{1}{3}\left(p^2 \star x + p \star x \star p + x \star p^2\right) \quad \simeq \quad \frac{1}{3}\left(\hat{p}^2\,\hat{x} + \hat{p}\,\hat{x}\,\hat{p} + \hat{x}\,\hat{p}^2\right) \tag{6.24}$$

are based on the isomorphism $x^m \star p^n \simeq \hat{x}^m\,\hat{p}^n$, useful in our future applications. This method allows one to go from operator calculations to calculations involving differential equations, simplifying the problem in some cases. After performing the computations with the Moyal products, we have to translate our results back into the operator language. To do that we substitute each real-valued monomial by a totally symmetric combination of operators:

$$x^m\,p^n = p^n\,x^m \qquad \longmapsto \qquad S_{mn} = \frac{m!n!}{(m+n)!}\sum_{\pi}\hat{x}^m\,\hat{p}^n, \tag{6.25}$$

where π denote all possible permutations of the operators $\hat{x}$ and $\hat{p}$.

The isomorphism just explored can be used in the construction of metric operators and Hermitian Hamiltonian partners. First of all, we need to solve the right hand side of the isomorphic relation

$$\hat{H}(\hat{x},\hat{p})^\dagger\hat{\rho}(\hat{x},\hat{p}) = \hat{\rho}(\hat{x},\hat{p})\hat{H}(\hat{x},\hat{p}) \qquad \simeq \qquad H(x,p)^\dagger \star \rho(x,p) = \rho(x,p) \star H(x,p), \tag{6.26}$$

which is a differential equation for the "scalar metric function" $\rho(x,p)$. Taking as a starting point the non-Hermitian Hamiltonian $\hat{H}(\hat{x},\hat{p})$, we have to transform this quantity into a scalar expression $H(x,p)$ by replacing all occurring operator products with Moyal products. The order of the resulting differential equation will depend on the highest powers of x and p in $H(x,p)$. Having $\rho(x,p)$ at our disposal we have to:

i) solve the differential equation $\rho(x,p) = \eta(x,p)^\dagger \star \eta(x,p)$ for the "Dyson mapping function" $\eta(x,p)$;

ii) compute directly the scalar function associated to the Hermitian counterpart by evaluating

$$h(x,p) = \eta(x,p) \star H(x,p) \star \eta(x,p)^{-1}; \tag{6.27}$$

iii) finally convert the functions $\rho(x,p)$, $\eta(x,p)$ and $h(x,p)$ into operator-valued functions by reversing the isomorphism to construct the metric operators $\hat{\rho}(\hat{x},\hat{p})$, $\hat{\eta}(\hat{x},\hat{p})$, $\hat{h}(\hat{x},\hat{p})$ by replacing monomials in x and p by appropriate operator combinations.

Remember that the intrinsic arbitrariness in the metric could be eliminated if more observables were specified. For example, we can impose that either $\hat{x}$ or $\hat{p}$ would remain Hermitian with respect to the new metric. Interestingly, as expected, the differential equations obtained in such a case imply that the metric cannot depend on the other parameter:

$$x \star \rho(x,p) = \rho(x,p) \star x \qquad \longrightarrow \qquad \frac{\partial}{\partial p}\rho(x,p) = 0, \tag{6.28}$$

$$p \star \rho(x,p) = \rho(x,p) \star p \qquad \longrightarrow \qquad \frac{\partial}{\partial x}\rho(x,p) = 0. \tag{6.29}$$

In this chapter we will focus on employing these ideas when trying to interpret non-Hermitian Hamiltonians. As an application of the method just described we can establish the metric operators and isospectral Hermitian Hamiltonians for a master Hamiltonian of cubic order.

6.2 Generic cubic $\mathcal{PT}$-symmetric non-Hermitian Hamiltonians

The lattice version of a Reggeon field theory [40, 42], one early occurrence of non-Hermitian models, when restricted to a single site system, leads to a potential very similar to the complex cubic potential $\imath x^3$. Somewhat later it was found [43] that the latter model possesses a real spectrum on the real line, and more recently there was the surprising discovery of an entire family of non-Hermitian Hamiltonians, involving potentials $z^2(\imath z)^n$ for $n \geq 0$, admitting real spectra when the domains are appropriately continued to the complex plane. Nevertheless, an exact analysis of the cubic Hamiltonian still lacks in the literature. Our attempt here is to consider a differential representation of the canonical commutation relations in terms of real-valued functions to help solving the problem, as carried out for instance in [26, 27, 28, 3]. Yet we will try to be more generic and study a broader class of models, namely those involving cubic combinations of $\hat{x}$ and $\hat{x}$ generators.

Having in mind the Hamiltonians studied in chapter 5, we use the relations (5.17) to start with a Hamiltonian in terms of the operators a and $a^\dagger$. The most general $\mathcal{PT}$-symmetric Hamiltonian which is maximally cubic in creation and annihilation operators $a^\dagger, a$ is of the following form

$$\begin{aligned} H_c \;&=\; \lambda_1\, a^\dagger a \;+\; \lambda_2\, a^\dagger a^\dagger \;+\; \lambda_3\, a\, a \;+\; \lambda_4 \;+ \\ &+\; \imath\left(\lambda_5\, a^\dagger \;+\; \lambda_6\, a \;+\; \lambda_7\, a^\dagger a^\dagger a^\dagger \;+\; \lambda_8\, a^\dagger a^\dagger a \;+\; \lambda_9\, a^\dagger a\, a \;+\; \lambda_{10}\, a\, a\, a\right). \end{aligned} \tag{6.30}$$

with $\lambda_i \in \mathbb{R}$. Note that this model is not contained in the system described by (5.21), which has only even combinations of $a^\dagger, a$ in the two-boson representation. This means the Lie-algebraic analysis based on $su(1,1)$ algebra carried out previously cannot be directly employed here, and the present situation provides a good framework where the Moyal product technique can be used.

One can easily re-express the above Hamiltonian in terms of coordinate and momentum operators with the help of (5.17). For convenience we introduce a coupling constant $g \in \mathbb{R}$ in order to be able to treat the imaginary part as a perturbation of the Hermitian operator, $H_c(\hat{x},\hat{p}) = h_0(\hat{x},\hat{p}) + \imath g h_1(\hat{x},\hat{p})$, with $h_0(\hat{x},\hat{p})^\dagger = h_0(\hat{x},\hat{p})$ and $h_1(\hat{x},\hat{p})^\dagger = h_1(\hat{x},\hat{p})$. For the terms containing $\hat{x}$ and $\hat{p}$ we also use their totally symmetrized version, namely

$$\frac{1}{2}\left(\hat{x}\,\hat{p} + \hat{p}\,\hat{x}\right) = \frac{\{\hat{x}\,,\hat{p}\}}{2}, \tag{6.31}$$

$$\frac{1}{3}\left(\hat{x}^2\,\hat{p} + \hat{x}\,\hat{p}\,\hat{x} + \hat{p}\,\hat{x}^2\right) = \frac{\{\hat{x}^2\,,\hat{p}\}}{2}, \tag{6.32}$$

$$\frac{1}{3}\left(\hat{p}^2\,\hat{x} + \hat{p}\,\hat{x}\,\hat{p} + \hat{x}\,\hat{p}^2\right) = \frac{\{\hat{x}\,,\hat{p}^2\}}{2}, \tag{6.33}$$

in terms of anticommutators, i.e., $\{A,B\} = AB + BA$. The equivalent Hamiltonian is

$$H_c(\hat{x},\hat{p}) = \alpha_1\hat{p}^3 + \alpha_2\hat{p}^2 + \alpha_3\frac{\{\hat{x}^2\,,\hat{p}\}}{2} + \alpha_4\hat{p} + \alpha_5\hat{x}^2 + \alpha_6 + \tag{6.34}$$

$$+ \imath g\left(\alpha_7\frac{\{\hat{x}\,,\hat{p}^2\}}{2} + \alpha_8\frac{\{\hat{x}\,,\hat{p}\}}{2} + \alpha_9\hat{x}^3 + \alpha_{10}\hat{x}\right). \tag{6.35}$$

with $\alpha_i \in \mathbb{R}$. Expressions (6.30) and (6.34) are the same, with coefficients α_i being determined by the λ_i via a 10×10-matrix transformation, $\alpha = M\lambda$, where

$$M = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{3}{2\sqrt{2}} \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2\sqrt{2}g} & \frac{1}{2\sqrt{2}g} & \frac{1}{2\sqrt{2}g} & -\frac{3}{2\sqrt{2}g} \\
0 & -\frac{1}{g} & \frac{1}{g} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2\sqrt{2}g} & \frac{1}{2\sqrt{2}g} & \frac{1}{2\sqrt{2}g} & \frac{1}{2\sqrt{2}g} \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}g} & \frac{1}{\sqrt{2}g} & 0 & -\frac{1}{\sqrt{2}g} & -\frac{1}{\sqrt{2}g} & 0
\end{pmatrix} \tag{6.36}$$

The Hamiltonian H_c encompasses many models and for specific choices of the α_i it reduces to various well studied examples, such as the simple massive or massless $\imath x$-potential

[218], the Swanson Hamiltonian [24], the complex cubic potential [68], the transformed version of the $-x^4$-potential [71]. As we will show below, in addition it includes several interesting models which have not been studied in the context of non-Hermitian quantum physics, such as the single site lattice version of Reggeon field theory [41], a thirty years old model, and the transformed version of the $\pm x^6$-potential, which serves as a toy model to identify theories with vanishing cosmological constant [219]. The latter models have not been solved so far with regard to their metric operators and isospectral partners. Besides these models, H_c also includes many new models not considered so far, some of which are even solvable. To enable easy reference we summarize the various choices in the following table.

model $\diagdown$ constants	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
massive $\imath x$-potential	0	1	0	0	m^2	0	0	0	0	1
massive $\imath x^3$-potential	0	1	0	0	m^2	0	0	0	1	0
Swanson model	0	$\frac{\Delta}{2}$	0	0	$\frac{\Delta}{2}$	-$\frac{\Delta}{2}$	0	1	0	0
mapped $-z^4$-potential	0	$\frac{1}{2}$	0	$\frac{1}{4}-\frac{1}{2g}$	$\frac{g}{2}$	-$\frac{g}{2}$	$\frac{1}{2g}$	0	0	-1
lattice Reggeon	0	$\frac{\Delta}{2}$	0	0	$\frac{\Delta}{2}$	-$\frac{\Delta}{2}$	$\bar{g}$	0	$\bar{g}$	$-2\bar{g}$
H_{SSSR} (6.88)	$\bar{\lambda}$	$\frac{\Delta}{2}$	$\bar{\lambda}$	$-2\bar{\lambda}$	$\frac{\Delta}{2}$	-$\frac{\Delta}{2}$	0	0	0	0
H_{S1} (6.89)	$\bar{\lambda}$	$\frac{\Delta}{2}$	$\bar{\lambda}$	$-2\bar{\lambda}$	$\frac{\Delta}{2}$	-$\frac{\Delta}{2}$	$\bar{g}$	0	$\bar{g}$	$-2\bar{g}$
H_{S2} (6.92)	0	$\frac{\Delta}{2}$	0	0	$\frac{\Delta}{2}$	-$\frac{\Delta}{2}$	$\bar{g}$	0	0	$-2\bar{g}$
H_J (6.96)	0	$\frac{1}{2}$	0	$\frac{1}{4}-\frac{1}{2g}$	$192\lambda_1$	κ_1	$\frac{1}{2g}$	0	$\frac{64\lambda_1}{g}$	$\frac{\kappa_2}{g}$

Table 6.1: Special reductions of the Hamiltonian H_c, specified by coupling constants $m, \Delta, g, \bar{g}, \lambda, \bar{\lambda}, \lambda_1, \kappa_1, \kappa_2 \in \mathbb{R}$.

Now in order to determine the metric operator by using the differential equations we must first transform the operator (6.34) into a real valued function. As explained before we simply replace operators by functions and use the $\star$-product instead of operators product:

$$H_c(x,p) = \alpha_1 p^3 + \alpha_2 p^2 + \alpha_3 x^2 p + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + \tag{6.37}$$

$$+ \imath g\left(\alpha_7 x p^2 + \alpha_8 x p + \alpha_9 x^3 + \alpha_{10} x\right). \tag{6.38}$$

Substituting (6.37) into the right-hand side of the isomorphism into (6.26) yields the

third order differential equation for the "metric scalar function" $\rho(x,p)$

$$\left(\alpha_3 xp\,\partial_p+\alpha_5 x\,\partial_p+\frac{\alpha_3}{8}\partial_x\partial_p^2+\frac{\alpha_1}{8}\partial_x^3-\alpha_2 p\partial_x-\frac{3}{2}\alpha_1 p^2-\frac{\alpha_3}{2}x^2\partial_x-\frac{\alpha_4}{2}\partial_x\right)\rho(x,p) \quad =$$
$$=g\left(\alpha_9 x^3+\alpha_{10}x+\alpha_8 p\,x+\alpha_7 x\,p^2+\frac{\alpha_7}{2}p\,\partial_x\partial_p+\frac{\alpha_8}{4}\partial_x\partial_p+\frac{\alpha_7}{2}x\partial_x^2-\frac{3}{4}\alpha_9 x\partial_p^2\right)\rho(x,p) \quad (6.39)$$

After solving for the functions $\rho(x,p)$ we must replace the monomials in x and p for the associated symmetric combinations of operators. Then, the problem can be formulated in terms of $a^\dagger, a$ by just inverting the matrix M: $\lambda_i = M^{-1}\alpha_i$. Presently there are various ways to proceed, and simplifications can be made at this stage, such as assuming that either $\hat{x}$ or $\hat{p}$ is an observable in the non-Hermitian framework so that $\partial_p\rho(x,p)=0$ or $\partial_x\rho(x,p)=0$, respectively. Here, instead, we will assume $\rho(x,p)$ admits a perturbative expansion in g.

Making a generic ansatz for $\rho(x,p)$ as being an exponential whose argument is real and $\mathcal{PT}$-symmetric,

$$\rho(x,p)=e^{g\left(\beta_1 p^3+\beta_2 x^2 p+\beta_3 p^2+\beta_4 x^2+\beta_5 p\right)}, \quad (6.40)$$

we construct systematically all exact solutions of this form. Substituting the ansatz into (6.39) and reading off the coefficients in front of each monomial in x and p yields at each order in g ten different equations. By solving these equations we find five qualitatively different types of exact solutions characterized by some sets of constraints. We will now present these solutions.

6.2.1 Non-vanishing $\hat{p}\,\hat{x}^2$-term

Constraints 1

We consider the full Hamiltonian $H_c(x,p)$ in (6.37) and impose as the only constraint that the px^2-term does not vanish, i.e. $\alpha_3\neq 0$. For this situation we can solve the differential equation (6.39) exactly to all orders in perturbation theory for

$$H_c(x,p)=h_0(x,p)+\imath g\left(\frac{\alpha_1\alpha_9}{\alpha_3}p^2x+\frac{\alpha_2\alpha_9-\alpha_5\alpha_7}{\alpha_3}px+\alpha_9x^3+\frac{\alpha_4\alpha_9-\alpha_5\alpha_8}{\alpha_3}x\right) \quad (6.41)$$

where we imposed the additional constraints

$$\alpha_1\alpha_9=\alpha_3\alpha_7, \qquad \alpha_2\alpha_9=\alpha_5\alpha_7+\alpha_3\alpha_8 \qquad \text{and} \qquad \alpha_4\alpha_9=\alpha_5\alpha_8+\alpha_3\alpha_{10}. \quad (6.42)$$

In (6.41) we have replaced the constants α_7,α_8 and α_{10} using (6.42). The solution of the differential equation is the metric scalar function

$$\rho(x,p)=e^{-g\left(\frac{\alpha_7}{\alpha_3}p^2+\frac{\alpha_8}{\alpha_3}p+\frac{\alpha_9}{\alpha_3}x^2\right)}. \quad (6.43)$$

Since $\eta^2(x,p)$ is real it follows from (6.20) that the corresponding metric operator is Hermitian. Next we solve $\eta(x,p)\star\eta(x,p)=\rho(x,p)$ for $\eta(x,p)$. Up to order g^2 we find

$$\begin{aligned}\eta(x,p) &= 1-g\frac{\alpha_7p^2+\alpha_8p+x^2\alpha_9}{2\alpha_3}+ \\ &\quad +g^2\left(\frac{(p\alpha_7+\alpha_8)^2p^2+\alpha_9^2x^4}{8\alpha_3^2}+\frac{\alpha_9\left(\alpha_7+2\alpha_7p^2x^2+2\alpha_8px^2\right)}{8\alpha_3^2}\right)+\mathcal{O}(g^3).\end{aligned} \tag{6.44}$$

The corresponding Hermitian counterpart corresponding to this solution is computed by means of (6.27) to

$$\begin{aligned}h_c(x,p) &= \alpha_3px^2+\alpha_5x^2+\alpha_6+\frac{\alpha_3\alpha_7}{\alpha_9}p^3+\frac{(\alpha_5\alpha_7+\alpha_3\alpha_8)}{\alpha_9}p^2+ \\ &+ \frac{(\alpha_5\alpha_8+\alpha_3\alpha_{10})}{\alpha_9}p-g^2\frac{(2\,\alpha_7p\,+\alpha_8)\,\left(p\,(\alpha_7p\,+\alpha_8)+\alpha_9x^2\,+\alpha_{10}\right)}{4\,\alpha_3}+\mathcal{O}(g^4).\end{aligned} \tag{6.45}$$

Notice that since we demanded α_3 to be non-vanishing these solutions can not be reduced to any of the well studied models presented in table 6.1, but represent new types of solutions. We may simplify the above Hamiltonians by setting various αs to zero.

Demanding for instance that $\hat{x}$ is an observable in the non-Hermitian system we are forced by (6.43) to set $\alpha_7=\alpha_8=0$ and by (6.42) also $\alpha_1=\alpha_2=0$. The Hamiltonian in (6.41) then simplifies to

$$H_c(x,p)=\alpha_3px^2+\alpha_4p+\alpha_5x^2+\alpha_6+\imath g\left(\alpha_9x^3+\frac{\alpha_4\alpha_9}{\alpha_3}x\right). \tag{6.46}$$

Since $\rho(x,p)$ only depends on x in this case, we can compute exactly $\eta(x,p)=e^{-g\frac{\alpha_9}{2\alpha_3}x^2}$. The Hermitian counterpart results to

$$h_c(x,p)=h_0(x,p)=\alpha_3px^2+\alpha_4p+\alpha_5x^2+\alpha_6. \tag{6.47}$$

If we require on the other hand that $\hat{p}$ is an observable, we have to choose $\alpha_9\to 0$. However, in that case the constraints (6.42) imply that the non-Hermitian part of the Hamiltonian (6.41) vanishes, i.e. we obtain the trivial case $H_c(x,p)=h_0(x,p)$.

Constraints 2

In the construction of the previous solution some coefficients had to satisfy a quadratic equations in the parameters to guarantee the vanishing of the perturbative expansion. The other solution for this equation leads to the constraints $\alpha_1=\alpha_7=0$, such that the non-Hermitian Hamiltonian simplifies. If we now impose the additional constraints

$$\alpha_3\alpha_{10}=\alpha_4\alpha_9 \qquad \text{and} \qquad \alpha_3\alpha_8=2\alpha_2\alpha_9, \tag{6.48}$$

we can solve the differential equation (6.39) exactly. For

$$H_c(x,p) = \alpha_2 p^2 + \alpha_3 p x^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + \imath g \left(\frac{2\alpha_2\alpha_9}{\alpha_3} p x + \alpha_9 x^3 + \frac{\alpha_4\alpha_9}{\alpha_3} x \right), \quad (6.49)$$

we compute the exact scalar metric function to

$$\rho(x,p) = e^{-g\frac{\alpha_9}{\alpha_3}x^2}. \quad (6.50)$$

Clearly $\rho(\hat{x},\hat{p})$ is a Hermitian and positive definite operator, which follows from the facts that $\rho(x,p)$ and $\log\rho(x,p)$ are real, respectively. Notice the fact that the Hamiltonian (6.49) does not follow as a specialization of (6.41), since the constraints (6.48) do not result as a particular case of (6.42). The Hermitian Hamiltonian counterpart corresponding to (6.49) is computed with $\rho(x,p) = e^{-g\frac{\alpha_9}{2\alpha_3}x^2}$ by means of (6.27) to

$$h_c(x,p) = \alpha_2 p^2 + \alpha_3 p x^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + g^2 \frac{\alpha_2\alpha_9^2}{\alpha_3^2} x^2. \quad (6.51)$$

Once again we may simplify the above Hamiltonians by setting various αs to zero or other special values, except for the case $\alpha_9 \to 0$ for which the constraints (6.48) reduce the non-Hermitian part of the Hamiltonian (6.49) to zero.

Thus this case requires a separate consideration:

6.2.2 Non-vanishing $\hat{p}\,\hat{x}^2$-term and vanishing $\hat{x}^3$-term

Let us therefore embark on the treatment of the complementary case to the previous subsection, namely $\alpha_3 \neq 0$ and $\alpha_9 = 0$. For these constraints we can solve the differential equation (6.39) exactly for the Hamiltonian

$$H_c(x,p) = h_0(x,p) + \imath g \left(\alpha_7 p^2 x + \alpha_8 p x + \frac{\alpha_5(\alpha_3\alpha_8 - \alpha_5\alpha_7)}{\alpha_3^2} x \right), \quad (6.52)$$

when we impose one additional constraint

$$\alpha_{10}\alpha_3^2 = \alpha_5(\alpha_3\alpha_8 - \alpha_5\alpha_7). \quad (6.53)$$

The "metric scalar function"results to

$$\rho(x,p) = \rho(p) = e^{g\left(\frac{\alpha_7}{2\alpha_3}p^2 + \frac{\alpha_3\alpha_8 - \alpha_5\alpha_7}{{\alpha_3}^2}p\right)}. \quad (6.54)$$

Once again $\rho(\hat{x},\hat{p})$ is a Hermitian and positive definite operator, which follows again from the facts that $\rho(x,p)$ and $\log\rho(x,p)$ are real. Since $\rho(x,p)$ only depends on p, we

can simply take the square root to compute $\eta(p)$. Then the corresponding Hermitian counterpart is computed by means of (6.27) to

$$h_c = h_0 + g^2 \left(\frac{{\alpha_7}^2}{4\alpha_3} p^3 + \frac{2\alpha_3\alpha_7\alpha_8 - \alpha_5\alpha_7^2}{4{\alpha_3}^2} p^2 + \frac{{\alpha_3}^2{\alpha_8}^2 - {\alpha_5}^2{\alpha_7}^2}{4{\alpha_3}^3} p + \frac{\alpha_5 \left(\alpha_5\alpha_7 - \alpha_3\alpha_8\right)^2}{4{\alpha_3}^4} \right). \tag{6.55}$$

In fact we can implement the constraint (6.53) directly in the solution. The function

$$\rho(p) = (p\,\alpha_3 + \alpha_5)^{\frac{g\left({\alpha_5}^2\,\alpha_7 - \alpha_3\,\alpha_5\,\alpha_8 + {\alpha_3}^2\,\alpha_{10}\right)}{{\alpha_3}^3}} e^{g\left(\frac{\alpha_7}{2\alpha_3}p^2 + \frac{\alpha_3\alpha_8 - \alpha_5\alpha_7}{{\alpha_3}^2}p\right)} \tag{6.56}$$

solves (6.39) for the generic Hamiltonian (6.37) with the only constraint that $\alpha_3 \neq 0$ and $\alpha_9 = 0$. In this case the corresponding Hermitian counterpart is computed to

$$h_{\mathrm{c}}(x,p) = h_0 + g^2 \frac{\left(p^2\,\alpha_7 + p\,\alpha_8 + \alpha_{10}\right)^2}{4\,\left(p\,\alpha_3 + \alpha_5\right)}. \tag{6.57}$$

Implementing the constraint (6.53), the Hamiltonian (6.57) reduces to the one in (6.55). Similarly as the model of the previous subsection, these solutions can not be reduced to any of the well studied models presented in table 6.1, since α_3 is assumed to be non-vanishing.

6.2.3 Vanishing $\hat{p}\,\hat{x}^2$-term and non-vanishing $\hat{x}^2$-term

Next we consider the complementary case to the previous two section, that is we take $\alpha_3 = 0$ in (6.34). For this set up we can only find an exact solution when we demand in addition that $\alpha_5 \neq 0$ and $\alpha_9 = 0$. For the non-Hermitian Hamiltonian

$$H_c(x,p) = \alpha_1 p^3 + \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + \imath g \left(\alpha_7 p^2 x + \alpha_8 p x + \alpha_{10} x\right) \tag{6.58}$$

we can solve the differential equation (6.39) exactly by

$$\rho(x,p) = e^{g\left(\frac{\alpha_7}{3\,\alpha_5}p^3 + \frac{\alpha_8}{2\alpha_5}p^2 + \frac{\alpha_{10}}{\alpha_5}p\right)}. \tag{6.59}$$

Once again $\rho(x,p)$ only depends on p and we can simply take the square root to compute $\rho(p)$. Using (6.27) the corresponding Hermitian Hamiltonian is subsequently computed to

$$h_{\mathrm{c}}(x,p) = \alpha_1 p^3 + \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + g^2 \frac{\left(p^2\alpha_7 + p\alpha_8 + \alpha_{10}\right)^2}{4\alpha_5}. \tag{6.60}$$

Obviously these solutions can be reduced to various cases presented in table 6.1, notably the transformed $-z^4$-potential and the Swanson Hamiltonian.

6.2.4 Vanishing $\hat{p}$ $\hat{x}^2$-term and non-vanishing $\hat{p}$ -term or non-vanishing $\hat{p}^2$-term

Finally we consider the complementary case of the previous section by taking $\alpha_3 = 0$ and allowing α_5 to acquire any value. To be able to find an exact solution we need to impose the additional constraints

$$\alpha_1 = \alpha_7 = \alpha_9 = 0, \qquad \text{and} \qquad \alpha_4\alpha_8 = 2\alpha_2\alpha_{10}, \tag{6.61}$$

i.e. we consider the non-Hermitian Hamiltonian

$$H_c(x,p) = \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + ig\left(\alpha_8 px + \alpha_{10} x\right), \tag{6.62}$$

for which we can solve equation (6.39) by

$$\rho(x,p) = e^{-g\frac{\alpha_{10}}{\alpha_4}x^2} \qquad \text{for } \alpha_4 \neq 0, \tag{6.63}$$

$$\rho(x,p) = e^{-g\frac{\alpha_8}{2\alpha_2}x^2} \qquad \text{for } \alpha_2 \neq 0. \tag{6.64}$$

As $\rho(x,p)$ only depends on x, we can take the square root to compute $\eta(x)$ and subsequently evaluate the corresponding Hermitian counterpart using (6.27)

$$h_c(x,p) = \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + g^2\frac{\alpha_2\alpha_{10}^2}{\alpha_4^2}x^2 \qquad \text{for } \alpha_4 \neq 0, \tag{6.65}$$

$$h_c(x,p) = \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + g^2\frac{\alpha_8^2}{4\alpha_2}x^2 \qquad \text{for } \alpha_2 \neq 0. \tag{6.66}$$

The Hamiltonian in (6.62) can be reduced to the Swanson Hamiltonian. Notice that when we impose $\alpha_1 = \alpha_4 = \alpha_7 = \alpha_{10} = 0$ for the Hamiltonian in (6.58) and $\alpha_4 = \alpha_{10} = 0$ for the Hamiltonian in (6.62), they become both identical to the Swanson Hamiltonian. The corresponding solutions for the metric operators reduce to $\hat{\rho}(x,p) = e^{g\frac{\alpha_8}{2\alpha_5}p^2}$ and $\rho(x,p) = e^{-g\,\frac{\alpha_8}{2\alpha_2}x^2}$, respectively, which are the well known non-equivalent solutions for the Swanson Hamiltonian, see e.g. [104]. This means according to (2.120) we can identify a symmetry operator for the Swanson Hamiltonian as

$$S(\hat{x},\hat{p}) = e^{-g\frac{\alpha_8}{2\alpha_5}\hat{p}^2}e^{-g\frac{\alpha_8}{2\alpha_2}\hat{x}^2}. \tag{6.67}$$

Notice that $S(x,p)\star H(x,p) = H(x,p)\star S(x,p)$ is a more difficult equation to solve in this example than (6.26), since S is not of a simple exponential form as η^2.

Obviously we can convert our solutions into expressions using creation and annihilation operators by simply using the relation $\lambda = M^{-1}\alpha$. We include the following table

to summarize the kinds of Hamiltonians which admit exact metric of the generic type proposed by us.

model $\diagdown$ const	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
H_c (6.41)	α_1	α_2	$\neq 0$	α_4	α_5	α_6	$\frac{\alpha_1\alpha_9}{\alpha_3}$	$\frac{\alpha_2\alpha_9-\alpha_5\alpha_7}{\alpha_3}$	α_9	$\frac{\alpha_4\alpha_9-\alpha_5\alpha_8}{\alpha_3^2}$
H_c (6.49)	0	α_2	$\neq 0$	α_4	α_5	α_6	0	$\frac{2\alpha_2\alpha_9}{\alpha_3}$	α_9	$\frac{\alpha_4\alpha_9}{\alpha_3}$
H_c (6.52)	α_1	α_2	$\neq 0$	α_4	α_5	α_6	α_7	α_8	0	$\frac{\alpha_5(\alpha_3\alpha_8-\alpha_5\alpha_7)}{\alpha_3^2}$
H_c (6.58)	α_1	α_2	0	α_4	$\neq 0$	α_6	α_7	α_8	0	α_{10}
H_c (6.62)	0	α_2	0	$\neq 0$	α_5	α_6	0	$\frac{2\alpha_2\alpha_{10}}{\alpha_4}$	0	α_{10}
H_c (6.62)	0	$\neq 0$	0	α_4	α_5	α_6	0	α_8	0	$\frac{\alpha_4\alpha_8}{2\alpha_2}$

Table 6.2: Hamiltonians H_c admitting a Hermitian counterpart with respect to a new metric of the form (6.40).

6.3 The single site lattice Reggeon model

One interesting non-Hermitian but $\mathcal{PT}$-symmetric Hamiltonian which has been studied for over thirty years appears when the Reggeon field theory is considered on a lattice. The construction of a positive pseudo-Hermitian metric ρ with respect to which a consistent quantum formulation of this problem can be formulated was one of the reasons motivating our previous analysis with Moyal products.

The Reggeon field Hamiltonian,

$$H_{RFT} = \int d\vec{x} \left[\Delta\bar{\psi}\psi + \imath g'\bar{\psi}(\psi+\bar{\psi})\psi + \alpha'\nabla\bar{\psi}\cdot\nabla\psi \right], \tag{6.68}$$

with $\Delta, g', \alpha' \in \mathbb{R}$ and an equal rapidity commutation relation for the field operators $[\psi(\vec{x},\theta), \bar{\psi}(\vec{x}',\theta)] = \delta(\vec{x}-\vec{x}')$, can be put on a lattice with spacing l such that $\vec{x} = l\vec{n}$. Under some approximations, this system can be transformed into a quantum spin model, defined in terms of bosons on a lattice [40, 42],

$$H_{LR} = \sum_{\vec{n}} \left[\Delta a^\dagger_{\vec{n}} a_{\vec{n}} + \imath g a^\dagger_{\vec{n}}(a_{\vec{n}} + a^\dagger_{\vec{n}}) a_{\vec{n}} + \alpha \sum_{<\vec{m}>} (a^\dagger_{\vec{n}} - a^\dagger_{\vec{m}})(a_{\vec{n}} - a_{\vec{m}}) \right], \tag{6.69}$$

with $\Delta, g, \alpha \in \mathbb{R}$ and the summation $< \vec{m} >$ made on neighbouring sites. Here our interest will focus on the single site Reggeon lattice model, obtained in the low interaction limit ($\alpha_0 \to 0$),

$$H_{SSLR} = \Delta a^\dagger a + \imath g a^\dagger (a + a^\dagger) a. \tag{6.70}$$

The Hamiltonian above is clearly invariant under $\mathcal{PT} : a \to -a, a^\dagger \to -a^\dagger$ and its spectrum is real. Due to its non-Hermiticity we have interest in determining the meaningful metric which should be used in the consistent description of this problem. We have seen in the previous chapter that creation and annihilation operators can be used as generators of $su(1,1)$ operators in some representations, but the triple combinations appearing in the non-Hermitian part of the potential above vexes the use of a Lie-algebraic approach to this problem. One may find already in the old literature, e.g. [220], that the Hermitian conjugation of the Hamiltonian (6.70) can be achieved by an adjoint action with the the parity operator,

$$\mathcal{P} = e^{\imath \pi a^\dagger a} \qquad \text{so that} \qquad \mathcal{P} a \mathcal{P} = -a \quad \text{and} \quad \mathcal{P} a^\dagger \mathcal{P} = -a^\dagger, \tag{6.71}$$

but because this operator is not Hermitian it cannot be used as a metric operator to construct Hermitian counterparts as specified in (2.88) and (2.89). A Hermitian operator which has the same effect as $\mathcal{P}$ is

$$\bar{\mathcal{P}} = e^{\imath \frac{\pi}{2}(aa - a^\dagger a^\dagger)}, \tag{6.72}$$

but because its eigenvalues are not bounded from below we cannot guarantee its positivity so it might not be a legitimate metric. In fact, using $\sqrt{\bar{\mathcal{P}}}$ to construct a Hermitian counterpart to H_{SSLR}, we obtain

$$h_{SSLR} = e^{\imath \frac{\pi}{4}(aa - a^\dagger a^\dagger)} H_{SSLR} e^{-\imath \frac{\pi}{4}(aa - a^\dagger a^\dagger)} = -\Delta_0 a^\dagger a + g a (a + a^\dagger) a^\dagger, \tag{6.73}$$

which suffers exactly from non-positivity and unboundedness from below.

In order to try to find a more meaningful metric operator we resort to Moyal products. To do so, we must first re-write the single site lattice Regge Hamiltonian in terms of $\hat{x}$ and $\hat{p}$ instead of $a^\dagger, a$,

$$H_{SSLR}(\hat{x}, \hat{p}) = \frac{\Delta}{2}\left(\hat{p}^2 + \hat{x}^2 - 1\right) + \imath \bar{g}\left(\hat{x}^3 + \hat{p}^2 \hat{x} - 2\hat{x} + \imath \hat{p}\right), \tag{6.74}$$

where $\bar{g} = \frac{g}{\sqrt{2}}$. This is the massive complex cubic model of Bender and Boettcher [9], extended by the inclusion of the last three terms. The procedure from now on consists of

replacing the expressions in terms of operators by real-valued functions multiplied by the Moyal $\star$-product: $\hat{x} \to x,\ \hat{p} \to p,\ \hat{p}^2\hat{x} \to p^2 \star x = px^2 - \imath p$,

$$H_{SSLR}(x,p) = \frac{1}{2}\left(p^2 + x^2 - 1\right) + \imath\bar{g}\left(x^3 + p^2x - 2x\right), \tag{6.75}$$

where we have set $\Delta = 1$. This model does not fall in any of the classes we have discussed in the previous section, so we would have to solve this problem from the beginning. For these values presented here, the differential equation for the metric (6.39) reduces to

$$\left(2x\partial_p - 2p\partial_x\right)\rho(x,p) = g\left(4x^3 - 8x + 4p^2x + 2p\partial_x\partial_p - 3x\partial_p^2 - x\partial_x^2\right)\rho(x,p). \tag{6.76}$$

Due to its close resemblance to the cubic potential we do not expect this to be exactly solvable and therefore must recur to perturbation theory.

6.3.1 Perturbative solution

Because the initial Reggeon Hamiltonian we were interested in could not be solved exactly by the generic ansatz proposed to the metric, our task must be accomplished by assuming a perturbative expansion of $\rho(x,p)$ in terms of the coupling constant g specifying the perturbing terms,

$$\rho(x,p\,;g) = 1 + \sum_{n=1}^{\infty} g^n c_n(x,p), \tag{6.77}$$

with $\rho(x,p\,;g) = \eta(x,p\,;g)\star\eta(x,p\,;g)$. Combining this expansion in the differential equation for the pseudo-Hermitian metric function of the single lattice Regge model (6.76) we are left with equations for each order in the parameter g. Solving each of them successively allows us to determine the series expansion. It is evident that on the left hand side of the aforementioned differential equation we can always add to $c_n(x,p)$ any function of the Hermitian part of $H_{SSLR}(x,p)$. The ambiguities associated to the boundary conditions of the differential equation may be eliminated by assuming the dependence on the coupling constant to be

$$\eta(x,p\,;-g) = \eta(x,p\,;g)^{-1}, \tag{6.78}$$

and

$$h(x,p\,;-g) = h(x,p\,;g)\ \ ,\ \ H(x,p\,;-g) = H(x,p\,;g)^{\dagger}\ , \tag{6.79}$$

so that pseudo-Hermiticity is valid. We further impose that

$$\rho(x,p\,;g)\star\rho(x,p\,;-g) = 1. \tag{6.80}$$

This enables us to fix our solutions. We obtain recursively the first few coefficients

$$
\begin{aligned}
c_1(x,p) &= 2p^3 - 4p + 2px^2 \,, \qquad (6.81)\\
c_2(x,p) &= 2p^6 - 8p^4 + p^2 + 2x^2 - 8p^2x^2 + 4p^4x^2 + 2p^2x^4 \,,\\
c_3(x,p) &= \frac{4}{3}p^9 - 8p^7 - 10p^5 + 48p^3 - 8p + 16px^2 - 12p^3x^2 - 16p^5x^2 + 4p^7x^2 +\\
&- 2px^4 + \frac{4}{3}p^3x^6 - 8p^3x^4 + 2p^5x^4 \,,\\
c_4(x,p) &= \frac{2}{3}p^12 - \frac{16}{3}p^10 - 24p^8 + 152p^6 - 10p^4 - 144p^2 + 48x^2 - 36p^2x^2 +\\
&+ 208p^4x^2 - 56p^6x^2 - 16p^8x^2 - 26x^4 + 56p^2x^4 - 40p^4x^4 - 16p^6x^4 + 4p^8x^4 +\\
&- 8p^2x^6 + \frac{8}{3}p^{10}x^2 - \frac{16}{3}p^4x^6 + \frac{8}{3}p^6x^6 + \frac{2}{3}p^4x^8.
\end{aligned}
$$

Next we solve the differential equation for the Dyson map function $\eta(x,p;g)$, $\rho(x,p;g) = \eta(x,p;g) \star \eta(x,p\,;g)$ by making the ansatz

$$\eta(x,p\,;g) = 1 + \sum_{n=1}^{\infty} g^n q_n(x,p), \qquad (6.82)$$

and the coefficients in this expansion are determined to be

$$
\begin{aligned}
q_1(x,p) &= \frac{1}{2}c_1(x,p) \,, \qquad (6.83)\\
q_2(x,p) &= \frac{1}{4}c_2(x,p) \,,\\
q_3(x,p) &= \frac{1}{6}p^9 - p^7 - \frac{17}{4}p^5 + 16p^3 - 3p - \frac{15}{2}p^3x^2 + \frac{1}{2}p^7x^2 + \frac{1}{2}p^5x^4 +\\
&+ \frac{1}{6}p^3x^6 - \frac{13}{4}px^4 - p^3x^4 + 12px^2 - 2p^5x^2 \,,\\
q_4(x,p) &= -35p^2 + \frac{11}{8}p^4 + \frac{51}{2}p^6 - \frac{9}{2}p^8 - \frac{1}{3}p^{10} + \frac{1}{24}p^{12} - \frac{25}{4}p^2x^2 + \frac{39}{2}p^2x^4 +\\
&+ \frac{1}{24}p^4x^8 + \frac{1}{6}p^{10}x^2 - \frac{61}{8}x^4 - \frac{23}{2}p^4x^4 - \frac{25}{2}p^6x^2 + \frac{1}{4}p^8x^4 - \frac{7}{2}p^2x^6 +\\
&- \frac{1}{3}p^4x^6 + \frac{1}{6}p^6x^6 + 45p^4x^2 - p^8x^2 - p^8x^2 - p^6x^4 + 13x^2.
\end{aligned}
$$

We are now in the position to compute the Hermitian counterpart to $H_{SSLR}(x,p)$ by means of (2.88)

$$
\begin{aligned}
h_{SSLR}(x,p) &= \tfrac{1}{2}\left(x^2+p^2-1\right) + g^2\left(\tfrac{3}{2}(p^4+x^4) - 4(p^2+x^2) + 3p^2x^2 + 1\right) +\\
&- g^4\left(\tfrac{17}{2}p^6 - 34p^4 + 4p^2 + 8 + 4x^2 - 48p^2x^2 + \tfrac{41}{2}p^4x^2 - 14x^4 + \tfrac{31}{2}p^2x^4 + \tfrac{7}{2}x^6\right) + \mathcal{O}(g^6).
\end{aligned} \qquad (6.84)
$$

All the solutions we have constructed consist of real-valued functions so that we still have to recast them in terms of operators, so we replace the monomials in x and p by the respective totally symmetric combination of operator products. Finally, using the creation

and annihilation operators, we can express the Dyson operator, up to order g^2, as

$$\eta = 1 + \imath\sqrt{2}ga^\dagger(a^\dagger - a)a + g^2a^\dagger\left[a^\dagger(2a^\dagger a - a^\dagger a^\dagger - aa + 5)a - 2a^\dagger a^\dagger - 2aa + 2\right]a \quad (6.85)$$

and the Hermitian counterpart to the non-Hermitian Hamiltonian H_{SSLR} acquires the form

$$\begin{aligned} h_{SSLR} &= a^\dagger a + g^2a^\dagger(6a^\dagger a + 4)a + \\ &+g^4\left[a^\dagger a^\dagger(10a^\dagger a^\dagger + 10aa - 48a^\dagger a)aa + a^\dagger(20a^\dagger a^\dagger + 20aa - 120a^\dagger a)a - 32a^\dagger a\right] + \mathcal{O}(g^6). \end{aligned} \quad (6.86)$$

As for all previously constructed perturbative solutions, it would be highly desirable to investigate in more detail the convergence properties of them.

6.4 Solvable examples

After examining the single site lattice version of the Reggeon field theory and concluding that a generic metric with polynomial exponent (6.40) does not give rise to exact results so that instead one needs to recur to perturbation theory, we employ the Moyal product method to some solvable models closely related to $H_{SSLR}(x,p)$. For example,

$$H_{S1} = \Delta a^\dagger a + \imath g a^\dagger(a + a^\dagger)a + \imath\lambda a^\dagger(a - a^\dagger)a \quad (6.87)$$

is obtained from H_{SSLR} by adding a complementary potential which differs from the original one by a relative sign,

$$H_{SSSR} = \Delta a^\dagger a + \imath\lambda a^\dagger(a - a^\dagger)a. \quad (6.88)$$

This new Hamiltonian is equivalent to

$$H_{S1}(x,p) = \frac{\Delta}{2}\left(p^2 + x^2 - 1\right) + \bar{\lambda}\left(p^3 + px^2 - 2p\right) + \imath\bar{g}\left(x^3 + p^2x - 2x\right), \quad (6.89)$$

where $\bar{g} = \frac{g}{\sqrt{2}}$ and $\bar{\lambda} = \frac{\lambda}{\sqrt{2}}$, falling in one of the categories studied before, with parameters satisfying constraints (6.42). An exact solution for the scalar metric function can then be identified as

$$\rho(x,p) = e^{\frac{\bar{g}}{\bar{\lambda}}(x^2+p^2)}. \quad (6.90)$$

Considering the imaginary part in (6.87) as a perturbation of a real-valued Hamiltonian, this system exhibits an interesting strong-weak symmetry, in the sense that it is invariant under the following set of transformations:

$$\{a \to -a, a^\dagger \to a^\dagger, \Delta \to -\Delta, \bar{g} \to -\bar{\lambda}, \bar{\lambda} \to -\bar{g}\}. \quad (6.91)$$

A further example of a solvable model of the type (6.37) related to a single site Reggeon model is $H_{SSLR}(\hat{x},\hat{p}) - \imath\bar{g}\hat{x}^3$,

$$H_{S2}(x,p) = \frac{\Delta}{2}\left(p^2 + x^2 - 1\right) + \imath\bar{g}\left(p^2 x - 2x\right). \tag{6.92}$$

This is identical to H_c in (6.58) for certain values of the parameters, in a way that an exact solution is given by

$$\rho(x,p) = e^{-\frac{\bar{g}}{3\Delta}(p^3-6p)}. \tag{6.93}$$

In fact, this model can be matched with the transformed version of the $-z^4$-potential, for which the exact metric was constructed [71],

$$H_4(z,p_z) = \frac{1}{2}p_z^2 - \frac{g}{32}z^4 \qquad \longrightarrow \qquad H_4(x,p_x) = \frac{1}{2}p_x^2 + \frac{1}{4}p_x + \frac{g}{2}x^2 - \frac{g}{2} - \imath g x, \tag{6.94}$$

with z specified in (2.35). The transformed $-z^4$-Hamiltonian itself is exactly solved for being of the form (6.58).

A similar situation is found when constructing Hamiltonians with a space-time symmetry transformation between de-Sitter and anti-de-Sitter space of the type $x^\mu \longrightarrow \imath x^\mu$:

$$\text{dS} \longrightarrow \text{adS} \ : \qquad x \longrightarrow \imath x \quad , \quad p \longrightarrow -\imath p\,. \tag{6.95}$$

This transformation, introduced in [219], is interesting for relating vacuum solutions with positive cosmological constant to those with negative cosmological constant, so that it can only be a symmetry of the vacuum state if the cosmological constant is vanishing. The transformation from real coordinates into imaginary counterparts is likely to break the Hermiticity properties of the operators describing such a theory and the boundary conditions of the physical wavefunctions must be re-analyzed. For example, taking the $\mathcal{PT}$-symmetric Hamiltonian proposed in by Jackiw and investigated in [219]

$$H_J(z) = \lambda_0 p_z^2 + \lambda_1 z^6 + \lambda_2 z^2, \tag{6.96}$$

one notices the dS $\rightarrow$ adS map (6.95) takes $H_J \rightarrow -H_J$. As was noted in [219, 221], for $\lambda_0 = \frac{1}{2}, \lambda_1 = 2, \lambda_2 = -3$, the ground state wavefunction is simply $\psi_0 = \exp\left(-\frac{z^4}{2}\right)$ and H_J factorizes in a way that allows it to be interpreted as a bosonic part of a supersymmetric pair of Hamiltonians.

As discussed in [9, 221] the complexification of the coordinates can lead to the continuation of the Schrödinger equation away from the real axis. Assuming an exponential fall

off of the wavefunctions at infinity, one may choose any parametrization which remains asymptotically inside the wedges

$$\mathcal{W}_L = \left\{\theta| -\frac{7}{8}\pi < \theta < -\frac{5}{8}\pi\right\} \qquad \text{and} \qquad \mathcal{W}_R = \left\{\theta| -\frac{3}{8}\pi < \theta < -\frac{1}{8}\pi\right\}. \tag{6.97}$$

In fact, we can employ the same transformation (2.35) as the one which was used successfully for the $-z^4$-potential in [71], which has the desired asymptotic behaviour as $z \sim e^{-\imath\frac{\pi}{4}} \in \mathcal{W}_R$ and $z \sim e^{-\imath\frac{3\pi}{4}} \in \mathcal{W}_L$. This transformation maps $H_J(z)$ into

$$H_J(x) = \frac{\lambda_0}{2}p^2 - \frac{\lambda_0}{4}p + 192\lambda_1 x^2 - (6\lambda_1 + 4\lambda_2) + \imath\left(\frac{\lambda_0}{2}xp^2 - (4\lambda_2 + 192\lambda_1)x + 64x^3\right), \tag{6.98}$$

which allows us to interpret it as a perturbation of the exactly solvable model $H_4(x, p_x)$ in (6.94)

$$H_J(x)|_{\lambda_0=1,\lambda_1=\frac{g}{384},\lambda_2=\frac{g}{8}} = H_4(x, p_x) + \imath\frac{g}{6}x^3 - \frac{g}{6}. \tag{6.99}$$

We can also relate it to an exactly solvable model (6.58),

$$H_J(x)|_{\lambda_0=1,\lambda_1=2,\lambda_2=-3} = H_{(6.58)}(x, p_x) + 128\imath x^3. \tag{6.100}$$

In principle the purely linear Hamiltonian considered in [2] and discussed in section 5.3 of the previous chapter fits into the class of general $\mathcal{PT}$-symmetric Hamiltonians, when the constants therein are identified as $\alpha_3 = -\kappa_+$, $\alpha_4 = -\kappa_-$, $\alpha_6 = -(n+1)\kappa_0/2$, $\alpha_8 = \kappa_0$, $\alpha_{10} = -(n+1)\kappa_+$ and all remaining constants are taken to be zero.

However, none of the exactly solvable models obtained in there matches with $H_{\tilde{J}}$. Furthermore, relaxing the condition $\eta = \eta^\dagger$ as in the ansatz (5.9) allows to construct an exact Hermitian isospectral counterpart as we can see now.

6.5 Limitations of the method

Because the order of differential equation for $\rho(x, p)$ in (6.26) depend on the degree of the polynomial $H(x, p)$, the method just presented is not best suited for potentials written as expansions. Even simple sinusoidal potentials will generate, in principle differential equations of infinite order. Also if the Hamiltonian has any negative power term the derivatives will never truncate and again we will have a differential equation of infinite order for the metric function. This problem is overcome, however, if we use an ansatz for the form of the metric, just as we have done in the systems studied in this chapter.

For example, considering a $\mathcal{PT}$-symmetric Hamiltonian of the form

$$H = p^2 + \lambda \cos x + \imath g \sin x, \tag{6.101}$$

one can easily see that whereas a metric of the form $\rho(x,p) = \exp\left(\beta_{pp}p^2 + \beta_{xx}x^2 + \beta_p p + \beta_x x\right)$ will not provide a solution to the problem, a possible metric for this system has the form $\rho(x,p) = \exp\left(\frac{\cos x}{p}\right)$, whose positivity is not in general guaranteed. If the Hamiltonian is

$$H = p^2 + \lambda\frac{1}{x^2} + \imath g\frac{1}{x} + \imath k\frac{1}{x^3}, \tag{6.102}$$

then a good ansatz would have $\log x$, $\frac{1}{x}$, $\frac{1}{x^2}$, ... in the exponent. In fact, a simple calculation shows that a solution of (6.26) in this situation is $\rho(x,p) = \exp\left(-\frac{\log x}{p} + \frac{k}{2gpx^2}\right)$. Then, it becomes clear that although in principle the method is more appropriate to polynomial potentials of finite degree and non-negative powers, certain choices for the metric can allow one to construct solutions for these more complicated Hamiltonians.

Besides the different ways of determining the metric we have discussed, namely the spectral method, the Lie-algebraic approach and the use of Moyal products, there are other methods. For instance, in [222] Mostafazadeh considers a Hamiltonian $H = \frac{\hat{p}^2}{2m} + V(\hat{x},\hat{p}\,)$ in the pseudo-Hermiticity relation $H^\dagger\rho = \rho\ H$ together with the representation of the metric $\rho(x,y) \equiv \langle x|\rho|y\rangle$ to obtain a Klein-Gordon equation with variable mass for the metric:

$$\left[-\partial_x^2 + \partial_y^2 + 2m\left(V\left(x, \frac{d}{dx}\right)^* - V\left(y, \frac{d}{dy}\right)\right)\right]\rho(x,y) = 0. \tag{6.103}$$

This equation is very general but may also be very complicated to solve. Moreover, boundedness and invertibility are not a priori guaranteed by this method, so the solution calculated may not represent a meaningful metric, unfortunately.

7 Integrable $\mathcal{PT}$-symmetric deformations of classical models

7.1 $\mathcal{PT}$-symmetry in classical theories

So far in our journey through non-Hermitian Hamiltonian in Physics we have seen that $\mathcal{PT}$-symmetry, being an anti-linear operator, serves as a good guiding principle in order to select potential candidates to describe physical theories. We have in the previous chapters explored intrinsically quantum systems but noted that field equations appear naturally, which makes one wonder: is $\mathcal{PT}$-symmetry a concept interesting only at the quantum level or can it be useful when examining classical theories? In fact, it is nowadays very well understood that classical mechanics emerges simply as a macroscopic limit of the more fundamental quantum theory. Thus, it becomes paramount that the role of $\mathcal{PT}$-symmetry in classical systems is investigated as well. A quick look at the literature attests the importance of it in classical physics, e.g. [223, 224, 225, 30, 31, 226].

The virtue of $\mathcal{PT}$-symmetry, i.e. invariance under a simultaneous parity transformation $\mathcal{P} : x \to -x$ and time reversal $\mathcal{T} : t \to -t, \imath \to -\imath$, as mentioned, for a classical Hamiltonian is that it guarantees the reality of the energy due to its anti-linear nature [31]. When quantizing $\mathcal{H}$ one also needs to ensure $\mathcal{PT}$-symmetry of the corresponding wavefunctions in order to obtain real spectra as discussed in the first chapters.

There are various different ways to deform classical systems in a $\mathcal{PT}$-symmetric way. Given a $\mathcal{PT}$-symmetric PDE as a starting point, we adopt the deformation principle of [30, 31] to define new $\mathcal{PT}$-symmetric extensions of this model by replacing ordinary derivatives of $\mathcal{PT}$-symmetric functions by their deformed counterparts

$$\partial_x f(x) \to -\imath(\imath f_x)^\varepsilon =: f_{x;\varepsilon} \qquad \text{with } \varepsilon \in \mathbb{R}. \tag{7.1}$$

Clearly the original $\mathcal{PT}$-symmetry is preserved. In general the deformations will continue real derivatives into the complex plane, unless $\varepsilon = 2n-1$ with $n \in \mathbb{Z}$. We do not make use here of the possibility to deform also the higher derivatives via the deformation (**??**), i.e. replacing for instance $\partial_x^2 f(x)$ by the composition $f_{x;\varepsilon} \circ f_{x;\varepsilon}$, but simply define them as successive action of ordinary derivatives on one deformation only

$$\partial_x^n f(x) \to \imath^{\varepsilon-1} \partial_x^{n-1} (f_x)^\varepsilon = \partial_x^{n-1} f_{x;\varepsilon} =: f_{nx;\varepsilon}. \tag{7.2}$$

This deformation preserves the order of the PDE and we can now employ this prescription to introduce new $\mathcal{PT}$-symmetric models. One could also include deformations of the term involving the time derivative. The same substitution as in (7.1) can be implemented for the supersymmetric KdV equation, originally defined in [227],

$$\Phi_t + \alpha D^6 \Phi - \lambda D^2(\Phi D\Phi) + 2(\beta+\lambda) D\Phi D^2\Phi = 0, \tag{7.3}$$

with α, β, λ real constants, $\Phi(x,\psi)$ a fermionic superfield,

$$\Phi(x,\theta) = \xi(x) + \psi u(x), \tag{7.4}$$

and D denoting a superderivative defined by

$$D = \psi \partial_x + \partial_\psi. \tag{7.5}$$

Note that $\xi(x)$ are fermionic anti-commuting fields, ψ is an anti-commuting superspace variable and $u(x)$ is the usual bosonic commuting KdV field. Therefore, the equation (7.3) is equivalent to

$$u_t + \alpha u_{xxx} + 2\beta u u_x - \lambda \xi \xi_{xx} = 0, \tag{7.6}$$

$$\xi_t + \alpha \xi_{xxx} + (2\beta - \lambda) u \xi_x - \lambda \xi u_x = 0, \tag{7.7}$$

so that when $\lambda = 0$ or $\xi = 0$ equations (7.6) recovers the original KdV equation. Amongst the new families of $\mathcal{PT}$-symmetric extensions [228], some preserve supersymmetric invariance admit a Hamiltonian formulation.

The freedom of choice when implementing these extensions becomes evident if one observes two examples of deformations of the KdV equation found in [30] and [31]. The latter has proved to be more interesting for being a Hamiltonian system with the first conserved charges constructed in a more straightforward way. However, the existence of a

few quantities being preserved during the evolution of the system is not enough evidence of its integrability.

Here we will focus on the question of whether deformations preserving integrability are accomplishable. In other words, do integrable $\mathcal{PT}$-symmetric models allow for deformations which do not destroy the integrability? A positive answer to this question will naturally lead to new integrable models. We focus on two prototype models of integrable systems, the Burgers and the Korteweg-deVries (KdV) equation.

The formal construction of an infinite set of conserved charges can be guaranteed if one can associate the original problem to a linearized version in terms of Lax operators, but this task is far from trivial. Alternatively, there is an easily implemented procedure which can be used to rule out a large number of equations that cannot be integrable. This method is known as the Painelevé test and it leaves one with the few candidates which are likely to present integrability.

7.2 Integrability and the Painlevé test

There is clearly no doubt that integrability is an extremely desirable property to have in a physical system, as it usually leads to exact solvability rather than to mere perturbative results.

Classically the definitions of integrability are much more varied and non-uniform. In the general theory of differential systems there is Frobenius theorem for integrability regarding over-determined systems. In Hamiltonian dynamical systems, a common notion is the so-called Liouville integrability, which assumes for a system with n degrees of freedom (and $2n$ canonical coordinates) the existence of n analytic single valued global integrals of motion in involution, usually referred to as action-angle variables. The equations of motion are then separable and exact solutions can be obtained, at least in principle. A system of differential equations is said to be integrable when, given a sufficient amount of initial data, they are solvable via an associated linear problem. Other formulations of integrability have been presented previously, including those relying on hidden symmetric structures.

The problem with these definitions is that one does not know a priori whether a system is integrable or not without having computed all integrals of motion, mapped the problem to a linear one or actually solved the equations of motion. Integrability of a dynamical

system is a rare and usually delicate feature which is all but trivial to identify. A general method to identify integrable models before this, often very difficult, task is completed does not exist. The closest one may get to such a method is to check whether the system possesses the Painlevé property.

The first step to check the presence of the Painlevé property in the system consists of assuming that near a movable singularity $z = z_0$ the dominant behaviour of the solution is of the form $u(z) = \lambda_0(z - z_0)^\theta$, for some negative integer θ. The possible values for such a negative power can be determined by the substitution of $u(z)$ into the differential equation. Besides, the parameter λ_0 is consequently specified. If a dominant term proposed in this way is allowed then one may proceed to impose a whole Laurent expansion starting at the power θ just obtained:

$$u(z) = \sum_{k=0}^{\infty} \lambda_k (z - z_0)^{k+\theta}. \tag{7.8}$$

As we replace (7.8) in the equation of motion equations for each power in $(z - z_0)$ are constructed allowing us to determine all coefficients in the expansion order by order. Remembering that the position of the pole is in principle free, a small displacement, $z_0 \to z_0 + \epsilon$, will generate a leading contribution at order $\theta - 1$ because $\lambda_0(z - z_0 - \epsilon)^\theta \simeq \lambda_0(z - z_0)^\theta + \theta\lambda_0(z - z_0)^{\theta-1}\epsilon$, so that $r = -1$ is always a resonance, denoted as *universal* and it is associated to the freedom in the choice of initial conditions. Coming from an n-th order differential equation, we expect the system of algebraic equations for λ_k to enclose $n - 1$ free parameters. Whenever an arbitrary λ_r occur, a *resonance* $r \in \mathbb{N}$ is said to have been found. All other negative resonances should be ignored because they violate the hypothesis that θ is the leading power. Non-integer resonances must not be present either because they indicate the presence of branch points in the solution expansion, so that it becomes multi-valued.[12]

If for each leading behaviour, that is to say for all possible α, one finds less than $n - 1$ resonances then the solutions are not generic in the sense that not all initial conditions are allowed, indicating the ansatz misses an essential part of the solution. This is most probably due to a leading singularity undetected by this method. On the other hand, if all these conditions analyzed are observed with all necessary degrees of freedom are present we may have a suitable convergent Laurent expansion. This process is called *Painlevé*

[12]Rational resonances might be allowed if the weak Pailevé property is checked, but this shall not be considered here. These cases could in principle be dealt with by rescaling the field so that only integer powers occur and the Painlevé test can be carried out in the usual sense.

test. When the radius of convergence of the series is uniformly bounded below, so that the poles of any solution cannot coalesce to form a more complicated singularity elsewhere, the equation actually possesses the Painlevé property and integrability is conjectured. Nonetheless, this scheme simply provides a necessary condition for the absence of movable algebraic or logarithmic branch points, but movable essential singularities are not directly detected by the method.

Long after Kowalesvki's observations it was conjectured by Ablowitz, Ramani and Segur (ARS) [164] that a nonlinear ODE has the Painlevé property if it can be exactly reduced from a nonlinear PDE which is solvable by the inverse scattering method. To this day this conjecture has not been proven rigorously, but is supported by a huge amount of evidence. On one hand one has verified this property for almost all known integrable PDEs [32, 229, 230, 231] and in turn, which is more impressive, one has also used it to identify new integrable ODEs [232, 233].

For nonlinear PDEs, the situation is somewhat less structured but extrapolating the previous notions one defines: A PDE whose solutions have no movable critical singularities near any noncharacteric[13] manifold is said to possess the Painlevé property. In general this is difficult to establish, however, there exists a more applicable necessary, albeit not sufficient, condition for a PDE to possess the Painlevé property, which was developed by Weiss, Tabor and Carnevale [32]. They used the ARS algorithm for the occurrence of the Pailevé property to introduce a systematic procedure to detect possible integrable equations without referring to ODEs. It consists of formally expanding all solutions of a PDE in a power series around an arbitrary singularity manifold given by $\phi(x,t)=0$. Without loss of generality [159] this manifold can be taken to be $\phi(x,t)=x-\zeta(t)$. Thus, one starts with an expansion

$$u(x,t)=\sum_{k=0}^{\infty}\lambda_k(x,t)\phi(x,t)^{k+\theta}, \tag{7.9}$$

and checks the existence of enough resonances to accommodate the initial conditions in an analogous fashion as was done for ODEs. But because this algorithm presented is insensitive to the existence of essential singularities, the Painlevé test is a necessary but not a sufficient condition for integrability. If an equation satisfies the Painlevé property, then it is a prime candidate for being completely integrable. Once it is established that a

[13]On a characteristic manifold we can not apply Cauchy's existence theorem and therefore we do not have a unique solution for a given initial condition.

PDE passes the Painlevé test one needs to be cautious about the conclusions one can draw as it is only a necessary but not sufficient condition for the Painlevé property. In case one can also guarantee the convergence of the series the PDE possess the Painlevé property, which is taken as very strong evidence for the equation to be integrable. This step has only been carried out rigorously in very rare cases, e.g. in [234, 235]. We will adopt here the logic that a PDE which passes the Painlevé test and whose Painlevé expansion converges also possess the Painlevé property. We take this as a very good indication that the system is integrable.

7.3 $\mathcal{PT}$-symmetrically deformed Burgers' equation

Burgers' equation is extensively studied in fluid dynamics and integrable systems, as it constitutes the simplest PDE involving a nonlinear as well as a dispersion term. Its general form is

$$u_t + \left[\alpha u_x + \beta u^2\right]_x = 0, \tag{7.10}$$

but introducing $\sigma \equiv -\frac{\alpha}{2\beta}$ and a time scaling $t \to 2\beta t$, it gives rise to

$$u_t + uu_x = \sigma u_{xx}. \tag{7.11}$$

Burgers' equation is an integrable model and as such it possesses infinitely many symmetries, found with the Cole-Hopf transformation or with a Lax representation. Nonetheless it has only one local conserved density, clearly observed when written it in the form of a continuity equation (7.10).

Obviously equation (7.11) remains invariant under the transformation $t \to -t, x \to -x$, $u \to u$ and $\sigma \to -\sigma$. Taking the constant σ to be purely imaginary, i.e. $\sigma \in i\mathbb{R}$, this invariance can be interpreted as a $\mathcal{PT}$-symmetry, which was also noted recently by Yan [236]. A similar complex, albeit not $\mathcal{PT}$-symmetric, version of Burgers' equations plays an important role in the study of two-dimensional Yang-Mills theory with an SU(N) gauge group [237, 238]. The models considered there become $\mathcal{PT}$-symmetric after a Wick rotation, i.e $t \to \imath t$.

Let us now consider the $\mathcal{PT}$-symmetrically deformed Burgers' equation

$$u_t + uu_{x;\varepsilon} = i\kappa u_{xx;\mu} \qquad \text{with} \quad \kappa, \varepsilon, \mu \in \mathbb{R}, \tag{7.12}$$

or more explicitly,

$$u_t - \imath u(\imath u_x)^\varepsilon = \imath\kappa\mu(\imath u_x)^{\mu-1}u_{xx}, \tag{7.13}$$

where for the time being we allow two different deformation parameters ε and μ.

Our first objective is to test whether this set of equations passes the Painlevé test and this was accomplished in [4]. Following the method proposed in [32], we therefore assume that the solution of (7.12) acquires the general form of the Painlevé expansion

$$u(x,t) = \sum_{k=0}^{\infty} \lambda_k(x,t)\phi(x,t)^{k+\theta}. \tag{7.14}$$

Here $\theta \in \mathbb{Z}_-$ is the leading order singularity in the limit $\phi(x,t) = (\varphi(x,t) - \varphi_0) \to 0$, with $\varphi(x,t)$ being an arbitrary analytic function characterizing the singular manifold, φ_0 being an arbitrary complex constant which can be utilized to move the singularity mimicking the initial condition and the $\lambda_k(x,t)$ are analytic functions, which have to be computed recursively.

Leading order terms

As a starting point we need to determine all possible values for α by substituting the first term of the expansion (7.14), that is

$$u(x,t) \to \lambda_0(x,t)\phi(x,t)^{\theta} \qquad \text{as } \phi(x,t) \to 0, \tag{7.15}$$

into (7.12) and reading off the leading orders. For the three terms in (7.12) they are

$$u_t \sim \phi^{\theta-1} \quad , \quad uu_{x;\varepsilon} \sim \phi^{\theta+\theta\varepsilon-\varepsilon} \quad \text{and} \quad u_{xx;\mu} \sim \phi^{\theta\mu-\mu-1}. \tag{7.16}$$

In order for a non-trivial solution to exist the last two terms have to match each other in powers of ϕ, which immediately yields

$$\theta = \frac{\varepsilon - \mu - 1}{\varepsilon - \mu + 1} \in \mathbb{Z}_-. \tag{7.17}$$

Thus $\theta = -1$ and $\varepsilon = \mu$ is the only possible solution provided ε and μ are integers. Based on the leading order analysis the possibility of rational values for ε and μ can not be excluded as they might also produce negative integer values for θ, e.g. $\varepsilon = 1/3$ and $\mu = 2/3$ will produce $\theta = -2$. However, the deformation principle (**??**) for the function u in the form (7.14) or its derivatives will always lead to expressions which involve taking the root of an infinite sum. Consequently the Painlevé test in the spirit of [32] can not be performed. For integer values of ε and μ, this means we observe from the very onset of the procedure that only the models in which all x-derivatives are deformed with the same

deformation parameter have a chance to pass the Painlevé test. For that assumption we can therefore conclude already at this stage that one of the deformations of (7.11) studied in [236], i.e. $\varepsilon = 1$ and μ generic, can not pass the Painlevé test in the usual sense. Hence they do not possess the Painlevé property and are therefore not integrable for μ being integer. The case of non-integer values of μ remains inconclusive.

Recurrence relations

Substituting next the Painlevé expansion (7.14) for $u(x,t)$ with $\theta = -1$ into (7.12) with $\varepsilon = \mu$ gives rise to the recursion relations for the λ_k by identifying powers in $\phi(x,t)$. We find

$$\begin{aligned} &\text{at order } -(2\varepsilon+1): && \lambda_0 + 2\imath\varepsilon\kappa\phi_x = 0, \\ &\text{at order } -2\varepsilon: && \phi_t\delta_{\varepsilon,1} + \lambda_1\phi_x - \imath\kappa\varepsilon\phi_{xx} = 0, \\ &\text{at order } -(2\varepsilon-1): && \partial_x(\phi_t\delta_{\varepsilon,1} + \lambda_1\phi_x - \imath\kappa\varepsilon\phi_{xx}) = 0, \end{aligned} \tag{7.18}$$

such that

$$\lambda_0 = -2\imath\varepsilon\kappa\phi_x, \qquad \lambda_1 = (\imath\varepsilon\kappa\phi_{xx} - \phi_t\delta_{\varepsilon,1})/\phi_x \qquad \text{and} \qquad \lambda_2 \text{ is arbitrary.} \tag{7.19}$$

This means the number of free parameters, i.e. φ_0 and λ_2, at our disposal equals the order of the PDE, such that (7.12) passes the Painlevé test provided the series (7.14) makes sense and we can determine all λ_j with $j > 2$. To compute the remaining λ_j we need to isolate them on one side of the equation and those involving λ_k with $k < j$ on the other side. We expect to find some recursion relations of the form

$$g(j, \phi_t, \phi_x, \phi_{xx}, \ldots)\lambda_j = f(\lambda_{j-1}, \lambda_{j-2}, \ldots, \lambda_1, \lambda_0, \phi_t, \phi_x, \phi_{xx}, \ldots), \tag{7.20}$$

with g and f being some functions characteristic for the system under consideration. We will not present here these recursion relations for generic values of ε as they are rather cumbersome and we shall only present the first non-trivial deformation, that is the case $\varepsilon = 2$.

Resonances

For some particular values of j, say $j = r_1, \ldots, r_\ell$, we might encounter that the function g in (7.20) vanishes. Clearly this leads to an inconsistency and a failure of the Painlevé test unless f also vanishes. When this scenario occurs, it implies that the recursion

relation (7.20) does not fix λ_j and the compatibility conditions $g = f = 0$ lead to ℓ so-called resonances λ_{r_i} for $i = 1, \ldots \ell$. When $\ell + 1$ is equal to the order of the differential equation we can in principle produce a general solution which allows for all possible initial values, with the extra necessary degree of freedom coming from the specification of the pole. It might turn out that some missing free parameters are located before the start of the expansion (7.14), i.e. at $j < 0$, so-called negative resonances which can be treated following arguments developed in [239]. When not enough additional free parameters exist to match the order of the differential equation, the series is still of Painlevé type and is called *defective*.

It is straightforward to determine all possible resonances by following a standard argument. The coefficient of the leading order term has already been determined and is followed by an infinite number of terms in the series solution proposed. But if one is only interested in the resonances it is enough to search for the coefficients λ_r which are arbitrary. As we substitute

$$u_r(x,t) = \lambda_0(x,t)\phi(x,t)^\theta + \lambda_r(x,t)\phi(x,t)^{r+\theta} \tag{7.21}$$

into (7.12), using the expression for λ_0 from (7.19) and $\theta = -1$, we observe that the highest order terms in the equation behave as $\phi^{-2\varepsilon-1+r}$. The need that the coefficient of this term should vanish provides us with the condition

$$2^{\varepsilon-1}\varepsilon^\varepsilon\kappa^\varepsilon\phi_x^{2\varepsilon}\ \lambda_r(r+1)(r-2) = 0, \tag{7.22}$$

from which is clear to see that if $(r+1)(r-2) = 0$ then λ_r is allowed to assume any value. This necessary condition for a resonance to exist yields precisely two resonances, one at $r = 2$, corresponding to the third equation in (7.18), and the so-called universal resonance at $r = -1$. This means that are two coefficients whose equations might be neglected and we cease to have an over-determined system, so that at higher order we can not encounter any inconsistencies or possible breakdowns of the Painlevé test for any value of the deformation parameter ε. Note that the introduction of k terms with $0 < k < r$ in the expansion is unnecessary because the leading order equations would not depend on the λ_r we are looking for and the following order equations would give the recursive relations between λ_r and λ_k themselves, i.e., the complete solution in principle. Thus, in order to determine the resonances the ansatz (7.21) is all one needs.

The $\varepsilon = 2$ deformation

As already mentioned, the details of the recursion relation for generic values of ε are rather lengthy and we shall therefore only present the case $\varepsilon = 2$ explicitly. In that case the deformed Burgers' equation (7.12) becomes

$$u_t + \imath u u_x^2 + 2\kappa u_x u_{xx} = 0. \tag{7.23}$$

The substitution of the Painlevé expansion (7.14) into (7.23) and the subsequent matching of equal powers in ϕ then yields the recursion relation

$$\begin{aligned}
& \imath\lambda_0\phi_x^2\left\{\lambda_j\left[(2j-3)\lambda_0 - 2\imath((j-5)j+4)\kappa\phi_x\right] + 2\lambda_0\left(\lambda_0 + 2\imath\kappa\phi_x\right)\delta_{0,j}\right\} = \\
+ & \sum_{n,m=1}^{j}\left\{\lambda_{j-m-n-2}\lambda_{m,x}\lambda_{n;x} + (m-1)\lambda_m\phi_x\left[(n-1)\lambda_{j-m-n}\lambda_n\phi_x + 2\lambda_{j-m-n-1}\lambda_{n;x}\right]\right\} \\
+ & \sum_{n=1}^{j-1}\left\{2\lambda_{0,x}\left[(n-1)\lambda_{j-n-1}\lambda_n\phi_x + \lambda_{j-n-2}\lambda_{n;x}\right] - 2\lambda_0\phi_x\left[(n-1)\lambda_{j-n}\lambda_n\phi_x + \lambda_{j-n-1}\lambda_{n;x}\right]\right. \\
& -2\imath\kappa\left\{\lambda_{j-n,x}\left[\lambda_{n-3;xx} + (n-3)\left((n-2)\lambda_{n-1}\phi_x^2 + 2\lambda_{n-2,x}\phi_x + \lambda_{n-2}\phi_{xx}\right)\right]\right. \\
& \left.\left. + (j-n-1)\lambda_{j-n}\phi_x\left[\lambda_{n-2;xx} + (n-2)\left((n-1)\lambda_n\phi_x^2 + 2\lambda_{n-1,x}\phi_x + \lambda_{n-1}\phi_{\mathrm{xx}}\right)\right]\right\}\right\} \\
+ & 2\lambda_{0,x}\left[(j-5)j+6\right]\kappa\lambda_{j-1}\phi_x^2 + \lambda_{j-2}\left[2(j-3)\kappa\phi_{xx} + i\lambda_{0,x}\right] \\
- & 2\lambda_0\phi_x\left\{\lambda_{j-1}\left[(j-2)\kappa\phi_{xx} + i\lambda_{0;x}\right] + \kappa\left[\lambda_{j-2;xx} + 2(j-2)\phi_x\lambda_{j-1;x}\right]\right\} \\
+ & (j-4)\lambda_{j-3}\phi_t + \lambda_{j-4;t} + 2\kappa\lambda_{0,x}\left[\lambda_{j-3;xx} + 2(j-3)\phi_x\lambda_{j-2;x}\right],
\end{aligned} \tag{7.24}$$

which is indeed of the general form (7.20). Having brought all λ_j with $j > k$ to the left hand side of (7.24), we may now successively determine the λ_j to any desired order. Starting with the lowest value $j = 0$ the equation (7.24) reduces to

$$\lambda_0^2\phi_x^2\left(\lambda_0 + 4\imath\kappa\phi_x\right) = 0. \tag{7.25}$$

This leads to $\lambda_0 = -4\imath\kappa\phi_x$ and thus simply reproduces the expression in (7.19) for $\varepsilon = 2$. For $j = 1$ the equation (7.24) simplifies to

$$-\lambda_0^2\lambda_1\phi_x^2 = 2\lambda_0\phi_x\left[\imath\kappa\lambda_0\phi_{xx} + \left(\lambda_0 + 4\imath\kappa\phi_x\right)\lambda_{0;x}\right], \tag{7.26}$$

such that $\lambda_1 = 2\imath\kappa\phi_{xx}/\phi_x$, which coincides with (7.19) for $\varepsilon = 2$. When $j = 2$ the equation acquires the form

$$\begin{aligned}
\lambda_0\lambda_2\phi_x^2\left(\lambda_0 + 4\sigma\phi_x\right) &= 2\phi_x\lambda_{1;x}\lambda_0^2 - \lambda_{0;x}^2\lambda_0 + 2\lambda_1\phi_x\lambda_{0;x} - 2\imath\kappa\phi_{xx}\lambda_{0;x} - 4\imath\kappa\phi_x\lambda_{0;x}^2 \\
& \quad -2\imath\kappa\phi_x\left(\lambda_{0,xx} - 2\phi_x\lambda_{1;x}\right)\lambda_0.
\end{aligned} \tag{7.27}$$

It is evident that the left hand side vanishes identically and upon substitution of the values for λ_0 and λ_1. We can verify that this also holds for the right hand side of (7.27), thus leading to the first resonance at level 2 and therefore to an arbitrary parameter λ_2. One may now continue in this fashion to compute the expansion to any finite order, but before we embark on this task we make a few further simplification.

As the singularity has to be a noncharacteristic analytic movable singularity manifold, we employ the implicit function theorem and make a further assumption about the specific form of $\lambda_k(x,t) = \lambda_k(t)$ and $\phi(x,t) = x - \xi(t)$, with $\xi(t)$ being an arbitrary function. Then the equation (7.24) simplifies to a much more transparent form

$$8\kappa^2\left(8\kappa\delta_{0,j} + \imath(j-2)(j+1)\lambda_j(t)\right) = \sum_{n,m=1}^{j} \imath(1-m)(n-1)\lambda_m(t)\lambda_{j-m-n}(t)\lambda_n(t) \quad (7.28)$$
$$+ \sum_{n=1}^{j-1}\left[2\kappa(n-1)\left(n^2-n-j(n-2)+2\right)\lambda_{j-n}(t)\lambda_n(t)\right] + (j-4)\lambda_{j-3}(t)\xi'(t) - \lambda'_{j-4}(t).$$

Solving this equation recursively leads to the Painlevé expansion

$$u(x,t) = -\frac{4\imath\kappa}{\phi} + \lambda_2\phi + \frac{\xi'}{8\kappa}\phi^2 - \frac{\imath\lambda_2^2}{20\kappa}\phi^3 - \frac{\imath\lambda_2\xi'}{96\kappa^2}\phi^4 + \mathcal{O}(\phi^5). \quad (7.29)$$

Clearly we can use (7.28) to extend this expansion to any desired order. For the ordinary Burgers equation, i.e. $\varepsilon = 1$, there exist a simple choice for the free parameters, which terminates the expansion, such that one may generate Bäcklund and Cole-Hopf transformations in a very natural way. Unfortunately (7.29) does not allow an obvious choice of this form. Taking for instance $\lambda_2 = 0$ yields the expansion

$$\begin{aligned} u(x,t) &= -\frac{4i\kappa}{\phi} + \frac{\xi'\phi^2}{2^3\kappa} - \frac{i\xi'^2\phi^5}{7\times 2^8\kappa^3} + \frac{i\xi''\phi^6}{5\times 2^9\kappa^3} - \frac{\xi'^3\phi^8}{35\times 2^{13}\kappa^5} - \frac{23\xi'\xi''\phi^9}{385\times 2^{13}\kappa^5} - \frac{\xi^{(3)}\phi^{10}}{135\times 2^{14}\kappa^5} \\ &\quad + \frac{19i\xi'^4\phi^{11}}{3185\times 2^{18}\kappa^7} - \frac{51i\xi'^2\xi''\phi^{12}}{385\times 2^{19}\kappa^7} - \frac{i\left(43641\xi''^2 + 16460\xi'\xi^{(3)}\right)\phi^{13}}{779625\times 2^{20}\kappa^7} + \mathcal{O}(\phi^{14}). \quad (7.30) \end{aligned}$$

Being even more specific and assuming a solitary wave solution, the general form of the movable singularity is $\xi(t) = \omega t$, which gives

$$\begin{aligned} u(x,t) &= -\frac{4i\kappa}{\phi} + \frac{\omega\phi^2}{2^3\kappa} - \frac{i\omega^2\phi^5}{7\times 2^8\kappa^3} - \frac{\omega^3\phi^8}{35\times 2^{13}\kappa^5} + \frac{19i\omega^4\phi^{11}}{3185\times 2^{18}\kappa^7} + \frac{\omega^5\phi^{14}}{3185\times 2^{21}\kappa^9} \\ &\quad - \frac{561i\omega^6\phi^{17}}{2118025\times 2^{28}\kappa^{11}} - \frac{93\omega^7\phi^{20}}{3328325\times 2^{32}\kappa^{13}} + \frac{625011i\omega^8\phi^{23}}{53003575625\times 2^{38}\kappa^{15}} \\ &\quad + \frac{32971\omega^9\phi^{26}}{53003575625\times 2^{41}\kappa^{17}} - \frac{1509727i\omega^{10}\phi^{29}}{11501775910625\times 2^{46}\kappa^{19}} + \mathcal{O}(\phi^{30}). \quad (7.31) \end{aligned}$$

Clearly we can carry on with this procedure to any desired order in ϕ.

Convergence of the Painlevé expansion Having established that the deformed Burgers equations pass the Painlevé test for any value of the deformation parameter ε, let us now see whether the obtained series converges so that we can determine if these equations also posses the Painlevé property. It suffices to demonstrate this for some specific cases. Taking for this purpose $\lambda_2 = 0$, we can express the expansion (7.30) in the general form

$$u(x,t) = -\frac{4i\kappa}{\phi} + \phi \sum_{n=1}^{\infty} \alpha_n \phi^n \tag{7.32}$$

and employ Cauchy's root test, i.e. $\sum_{n=1}^{\infty} \gamma_n$ converges if and only if $\lim_{n\to\infty} |\gamma_n|^{1/n} \leq 1$, to establish the convergence of the series. We can easily find an upper bound for the real and imaginary parts of α_n

$$|\mathfrak{Re}\alpha_{3n-\nu}| \leq \frac{|\mathfrak{Re}p_{3n-\nu}(\xi', \xi'', \xi''', \ldots)|}{2^{3n+4-\nu}\Gamma(\frac{3n-\nu}{2})\,|\kappa|^{2n-1}} \qquad \text{for } \nu = 0, 1, 2, \tag{7.33}$$

where the $p_n(\xi', \xi'', \xi''', \ldots)$ are polynomials of finite order in t, that is $\sum_{n=0}^{\ell} \omega^n t^n$ with $\ell < \infty$ and $\omega \in \mathbb{C}$. The same expression holds when we the replace real part by the imaginary part on both sides of the inequality. We should also comment that this point of the proof is not entirely rigorous in the strict mathematical sense as we have only verified the estimate (7.33) up to order thirty and did not provide generic arguments for the validity of (7.33). Approximating now the gamma function in (7.33) by Stirling's formula as $n \to \infty$

$$\Gamma\left(\frac{n}{2}\right) \sim \sqrt{2\pi} e^{-n/2} \left(\frac{n}{2}\right)^{\frac{n-1}{2}} \tag{7.34}$$

we obtain

$$\lim_{n\to\infty} |\mathfrak{Re}\alpha_{3n-\nu}|^{\frac{1}{2}} \sim \frac{|\mathfrak{Re}p_{3n-\nu}|^{1/n}}{2^{3+\frac{4-\nu}{n}}(2\pi)^{\frac{1}{2n}} e^{-\frac{1}{2}} (\frac{3n-\nu}{2})^{\frac{1}{2}-\frac{1}{2n}}\,|\kappa|^{2-\frac{1}{n}}} = 0. \tag{7.35}$$

The same argument holds for the imaginary part, such that the series (7.32) converges for any value of κ and choices for $\xi(t)$ leading to finite polynomials $p_n(\xi', \xi'', \xi''', \ldots)$. It is straightforward to repeat the same argument for $\lambda_2 \neq 0$.

Alternatively we can identify the leading order term in (7.23) and integrate the deformed Burgers equation twice. In this way we change the ODE into an integral equation

$$u(x,t) = 2\kappa \left\{ g(t) + \int_{x_1}^{x} d\hat{x} \left[\frac{\imath}{2} + \frac{1}{u^2(\hat{x},t)} \left(f(t) + \int_{x_0}^{\hat{x}} d\tilde{x} \frac{u_t(\tilde{x},t)}{u_{\tilde{x}}(\tilde{x},t)} \right) \right] \right\}^{-1}, \tag{7.36}$$

where $g(t), f(t)$ are some functions of integration. When discretizing this equation, i.e. taking the left hand side to be $u_{n+1}(x,t)$ and replacing all the $u(x,t)$ on the right hand side

of this equation by $u_n(x,t)$, we may iterate (7.36) with $-4\imath\kappa/[x-\xi(t)]$ as initial condition and recover precisely the expansion (7.29). Exploiting the Banach fixed point theorem one may also use (7.36) as a starting point to establish the convergence of the iterative procedure and therefore the Painlevé expansion, similarly as was carried out for instance in [234, 235].

Reduction from PDE to ODE Making further assumptions on the dependence of $u(x,t)$ on x and t we can reduce the PDE to an ODE, and attempt to solve the resulting equation by integration. A common assumption is to require the solution to be of the form of a solitary wave $u(x,t)=\zeta(z)=\zeta(x-vt)$ with v being constant. When v is taken to be real, even solutions will be invariant under the original $\mathcal{PT}$-symmetry. With this ansatz the deformed Burgers' equation for $\varepsilon=2$ (7.23) acquires the form

$$-v\zeta_z+i\zeta\zeta_z^2+2\kappa\zeta_z\zeta_{zz}=0. \tag{7.37}$$

When $\zeta_z\neq 0$ we can re-write this equation as

$$\frac{d}{dz}\left(c-vz+\frac{i}{2}\zeta^2+2\kappa\zeta_z\right)=0, \tag{7.38}$$

which can be integrated to

$$\zeta(z)=e^{i\pi 5/3}(2v\kappa)^{1/3}\frac{\tilde{c}\;Ai'(\chi)+Bi'(\chi)}{\tilde{c}\;Ai(\chi)+Bi(\chi)} \tag{7.39}$$

with c, $\tilde{c}$ being constants, $\chi=e^{i\pi/6}(vz-c)(2v\kappa)^{-2/3}$ and $Ai(\chi)$, $Bi(\chi)$ denoting Airy functions.

We should point out that most of our arguments will still hold when we start in (7.12) with the usual Burgers equation, which has broken $\mathcal{PT}$-symmetry, i.e. with $\sigma=\imath\kappa\in\mathbb{R}$. However, when embarking on the computation of charges and in particular energies we expect to find a severe difference as then the $\mathcal{PT}$-symmetry has a bearing on the reality of the eigenvalues of the charges.

7.4 $\mathcal{PT}$-symmetrically deformed KdV equation

After presenting the Painlevé analysis for a $\mathcal{PT}$-symmetrically deformed Burgers' equation and establishing that the deformations introduced do not necessarily destroy the integrability of the system, we turn our attention to the KdV equation,

$$u_t+\left[\alpha u_{xx}+\beta u^2\right]_x=0, \tag{7.40}$$

already $\mathcal{PT}$-symmetric, being the first equation for which ϵ-deformations have been studied [30, 31]. Here we investigate the $\mathcal{PT}$-symmetrically deformed version of the KdV-equation with two different deformation parameters ε and μ

$$u_t - 6uu_{x;\varepsilon} + u_{xxx;\mu} = 0 \qquad \text{with} \quad \varepsilon, \mu \in \mathbb{R}, \tag{7.41}$$

where it has been specified that $\frac{2\beta}{\alpha} = 6$ and the time re-scaling $t \to \alpha t$ has been used, so that

$$u_t + \imath\mu(\mu-1)(\imath u_x)^{\mu-2}u_{xx}^2 + \mu(\imath u_x)^{\mu-1}u_{xxx} + 6\imath u(\imath u_x)^{\varepsilon} = 0. \tag{7.42}$$

The case $\mu = 1$ and ε generic was considered in [30] and the case $\varepsilon = 1$ and μ generic was studied in [31].

Leading order terms

As in the previous section we substitute $u(x,t) \to \lambda_0(x,t)\phi(x,t)^\theta$ into (7.41) in order to determine the leading order term. From

$$u_t \sim \phi^{\theta-1} \quad , \quad uu_{x;\varepsilon} \sim \phi^{\theta+\theta\varepsilon-\varepsilon} \quad \text{and} \quad u_{xxx;\mu} \sim \phi^{\theta\mu-\mu-2} \tag{7.43}$$

we deduce

$$\theta = \frac{\varepsilon - \mu - 2}{\varepsilon - \mu + 1} \quad \in \mathbb{Z}_-, \tag{7.44}$$

so that the only solution is $\theta = -2$ with $\varepsilon = \mu$ provided ε and μ are integers. This means neither the case $\mu = 1$ and ε generic nor the case $\varepsilon = 1$ and μ generic can pass the Painlevé test, but the hitherto uninvestigated deformation with $\varepsilon = \mu$ has at this point still a chance to pass it. The possibility of ε and μ being non-integer values remains inconclusive for the same reasons mentioned in the previous section.

Recurrence relations

Substituting the Painlevé expansion (7.14) for $u(x,t)$ with $\theta = -2$ into (7.41) with $\varepsilon = \mu$ gives rise to the recursion relations for the λ_k by identifying powers in $\phi(x,t)$. We

find at

$$\begin{aligned}
&\text{order } -(3\varepsilon+2): && \lambda_0 = \tfrac{1}{2}\varepsilon(3\varepsilon+1)\phi_x^2,\\
&\text{order } -(3\varepsilon+1): && \lambda_1 = -\tfrac{1}{2}\varepsilon(3\varepsilon+1)\phi_{xx},\\
&\text{order } -3\varepsilon: && \lambda_2 = \tfrac{\varepsilon(3\varepsilon+1)}{24}\left(\tfrac{4\phi_x\phi_{xxx}-3\phi_{xx}^2}{\phi_x^2}\right) + \delta_{\varepsilon,1}\tfrac{\phi_t}{6\phi_x},\\
&\text{order } -(3\varepsilon-1): && \lambda_3 = \tfrac{\varepsilon(3\varepsilon+1)}{24}\left(\tfrac{4\phi_x\phi_{xx}\phi_{xxx}-3\phi_{xx}^3-\phi_x^2\phi_{4x}}{\phi_x^4}\right) + \delta_{\varepsilon,1}\tfrac{\phi_t\phi_{xx}-\phi_x\phi_{xt}}{6\phi_x^3},\\
&\text{order } -(3\varepsilon-2): && \lambda_4 = \tfrac{\varepsilon(3\varepsilon+1)}{24}\left(\tfrac{6\phi_x\phi_{xx}^2\phi_{xxx}-\frac{15}{4}\phi_{xx}^4-\frac{3}{2}\phi_x^2\phi_{xx}\phi_{4x}}{\phi_x^6} + \tfrac{\phi_x\phi_{5x}-5\phi_{xxx}^2}{5\phi_x^4}\right).
\end{aligned} \tag{7.45}$$

Therefore the relation at order $-(3\varepsilon-2)$ becomes an identity only for $\varepsilon = 1$, which makes us suspect that also at higher order we will not encounter compatibility conditions and therefore will not have enough parameters equaling the order of the differential equation. To test whether new compatibility conditions arise at higher levels we can use the same general argument as in subsection 7.3.

Resonances

We try once again to match the first term in the expansion (7.14) with some term of unknown power. Using the expression for λ_0 in (7.45) and making the ansatz

$$u_r(x,t) = \lambda_0(x,t)\phi(x,t)^\theta + \lambda_r(x,t)\phi(x,t)^{r+\theta}, \tag{7.46}$$

with $\theta = -2$, we compute all possible values of r for which λ_r becomes a free parameter. Substituting $\tilde{u}(x,t)$ into (7.41) and reading off the terms of the highest order, i.e. $\phi^{-3\varepsilon-2+r}$, we find the necessary condition

$$\varepsilon^\varepsilon(-i)^{\varepsilon-1}(3\varepsilon+1)^{\varepsilon-1}\phi_x^{3\varepsilon}\,(r+1)\left[6(1+3\varepsilon)-2(2+3\varepsilon)r+r^2\right]\lambda_r = 0, \tag{7.47}$$

for a resonance to exist. Besides the presence of the expected universal resonance at $r=-1$, the bracket containing the quadratic term in r can be factorized as $(r-r_-)(r-r_+)$ with $r_\pm = -(2+3\varepsilon)\pm\sqrt{9\varepsilon^2-6\varepsilon-2}$, such that $r_\pm \in \mathbb{Z}$ for $9\varepsilon^2-6\varepsilon-2 = n^2$ with $n\in\mathbb{N}$. For the solution of this equation $\varepsilon_\pm = (1\pm\sqrt{n^2+3})/3$ to be an integer we need to solve a diophantine equation $3+n^2=m^2$ with $n,m\in\mathbb{N}$, which only admits $n=1$ and $m=2$ as solution. Thus the bracket only factorizes in the case $\varepsilon=1$ into $(r-6)(r-4)$. Hence, only in that case the system can fully pass the Painlevé test. Nonetheless, we may still be able to obtain a defective series if all remaining coefficients λ_j may be computed recursively. This is indeed the case as we demonstrate in detail for one particular choice of the deformation parameter.

$\varepsilon = 2$ deformation

For $\varepsilon = \mu = 2$ the deformed KdV equation (7.41) acquires the form

$$u_t - 6iuu_x^2 + 2iu_{xx}^2 + 2iu_x u_{xxx} = 0, \tag{7.48}$$

Since the expression become rather lengthy for generic values in the expansion we will present here only the case $\lambda_k(x,t) = \lambda_k(t)$ and $\phi(x,t) = x - \xi(t)$, with $\xi(t)$ being an arbitrary function. We find a recursion relation of the form (7.20)

$$\begin{aligned}
&-28\imath(1+j)(j^2-16j+42)\lambda_j(t) = -6\imath\sum_{n=1}^{j}\sum_{m=1}^{j-n-1}\{(m-2)(n-2)\lambda_m(t)\lambda_n(t)\lambda_{j-m-n}(t)\}\\
&+2\imath\sum_{n=1}^{j-1}\left\{[(7-k)n^3+(k-4)kn^2+(18-5k)kn+6k(5+k)-28(6+n)]\lambda_{j-n}(t)\lambda_n(t)\right\}\\
&+\lambda'_{j-6}(t)+(j-7)\lambda'_{j-5}(t).
\end{aligned} \tag{7.49}$$

The recursive solution of this equation leads to the expansion

$$\begin{aligned}
u(x,t) &= \frac{7}{\phi^2}+\frac{i\xi'\phi^3}{156}+\frac{(\xi')^2\phi^8}{192192}-\frac{\xi''\phi^9}{681408}+\frac{i(\xi')^3\phi^{13}}{73081008}-\frac{725i\xi'\xi''\phi^{14}}{216449705472}+\frac{i\xi'''\phi^{15}}{20262348288}\\
&\quad-\frac{340915(\xi')^4\phi^{18}}{23898859332927488}+\frac{1867(\xi')^2\xi''\phi^{19}}{758331543121152}+\mathcal{O}(\phi^{20}).
\end{aligned} \tag{7.50}$$

Thus we have obtained a solution of Painlevé type for the deformed KdV equation, albeit without enough free parameters, i.e. without the possibility to accommodate all possible initial values. This means we have a so-called defective series. As in the case of the deformed Burgers equation it is instructive to consider the series for solitary wave solutions, i.e. taking $\xi(t) = \omega t$, which yields

$$\begin{aligned}
u(x,t) &= \frac{7}{\phi^2}+\frac{i\omega\phi^3}{156}+\frac{\omega^2\phi^8}{192192}+\frac{i\omega^3\phi^{13}}{73081008}-\frac{340915\omega^4\phi^{18}}{23898859332927488}\\
&+\frac{391907i\omega^5\phi^{23}}{56760007181706436608}-\frac{38892808841\omega^6\phi^{28}}{5072600974623933411022602 24}+\mathcal{O}(\phi^{33}).
\end{aligned} \tag{7.51}$$

We find a similar behaviour for other values of ε.

For the KdV equation our findings suggest that its $\mathcal{PT}$-symmetric deformations are not integrable, albeit they allow for the construction of a defective series, as opposed to the deformation of the Burgers equation - when deforming both terms involving space derivatives, we found that the deformations of the Burgers equation pass the test. In specific cases we have also established the convergence of the series, such that the ϵ-Burgers equations have in addition the Painlevé property. Based on the conjecture by

Ablowitz, Ramani and Segur we take this as very strong evidence that these equations are integrable. Regarding these models as new integrable systems leads immediately to a sequence of interesting new problems related to features of integrability.

It is likely that these systems admit soliton solutions and it should be possible to compute the higher charges by means of Lax pairs, Dunkl operators or other methods. We should point out that most of our arguments will still hold when we start in (7.12) with the usual Burgers equation, which has broken $\mathcal{PT}$-symmetry, i.e. with $\sigma = \imath\kappa \in \mathbb{R}$. However, when embarking on the computation of charges and in particular energies we expect to find a severe difference as then the $\mathcal{PT}$-symmetry has a bearing on the reality of the eigenvalues of the charges.

7.5 $\mathcal{PT}$-symmetrically generalized KdV Hamiltonian: $\mathcal{H}_{l,m,p}$

It would clearly be very interesting to investigate other $\mathcal{PT}$-symmetrically integrable systems in the manner described above in order to establish their integrability. Particularly interesting are models presenting compactons (see e.g. [173, 174, 175, 176]) since it is nor clear how the existence of these solutions depend on integrability properties. For this reason we choose to analyze the model presented in the present section. The Hamiltonian density

$$\mathcal{H}_{l,m,p} = -\frac{2\beta u^l}{l(l-1)} - \frac{\imath^m \alpha}{2(m-1)} u^p (iu_x)^m \tag{7.52}$$

generalises the KdV equation (7.40), which is recovered when $m = 2, p = 0, l = 3$. It was introduced in [185] where it was considered $\alpha = 2\, g\, \imath^m$ and $\beta = \frac{1}{2}$.

The density $\mathcal{H}_{l,2,p}$ reduces to a modification of a Hamiltonian description [174, 175] of generalized KdV-equations [173] which are known to admit compacton solutions. For $l = 3$, $p = 0$ and $m = \varepsilon + 1$ one obtains a re-scaled version of the $\mathcal{PT}$-symmetrically deformed KdV-equation ($\varepsilon = 1$) introduced in [31],

$$u_t + uu_x + \imath\varepsilon(\varepsilon-1)(\imath u_x)^{\varepsilon-2} u_{xx}^2 + \varepsilon(\imath u_x)^{\varepsilon-1} u_{xxx} = 0. \tag{7.53}$$

The first $\mathcal{PT}$-symmetric extensions of the KdV-equation proposed in [30] cannot be obtained from (7.52) as they correspond to non-Hamiltonian systems. The equation of motion resulting from the variational principle

$$u_t = \left(\frac{\delta \int \mathcal{H} dx}{\delta u}\right)_x = \sum_{n=0}^{\infty} (-1)^n \left(\frac{d^n}{dx^n} \frac{\partial \mathcal{H}}{\partial u_{nx}}\right)_x, \tag{7.54}$$

for the Hamiltonian density $\mathcal{H}_{l,m,p}$ in (7.52) is

$$u_t + u^{l-2}u_x = -g i^m u^{p-2} u_x^{m-3} \times \tag{7.55}$$
$$\times \left[(m-2)mu^2u_{xx}^2 + 2mpuu_x^2u_xx + mu^2u_xu_{xxx} + (p-1)pu_x^4\right].$$

The most natural way to implement $\mathcal{PT}$-symmetry in (7.52) is to keep the interpretation from the standard KdV-equation and view the field u as a velocity, such that it transforms as $u \to u$. Then $\mathcal{H}_{l,m,p}$ is $\mathcal{PT}$-symmetric for real coupling constant g and all possible real values of l, m, p. Alternatively, we could also allow a purely complex coupling constant, i.e. $g \in i\mathbb{R}$, by transforming the field as $u \to -u$, such that $\mathcal{H}_{l,m,p}$ is $\mathcal{PT}$-symmetric when l is even and $p+m$ odd.

Because these $\mathcal{PT}$-symmetric models have been shown to posses compacton solutions one might wonder about the coexistence with solitons, which would be possible if integrability is present. Our goal here is to investigate whether this equation admits soliton solutions for some specific choices of the parameters l, m, p, and consequently determine whether it is possible to find solitons and compactons in the same model (7.55). We will repeat the steps carried out in the previous subsection and perform the Painlevé test as presented in [5]. A positive result of the latter, as discussed before, indicates integrability, which allows for the construction of solitons. The approach adopted here has the advantage of being considerably easier to implement than constructing explicitly the soliton solutions or the conserved charges, usually a formidable task.

For the matter of implementing the Painlevé test we assume the solutions can be expanded in a Laurent series of the form (7.14). We know one further demands that in the limit $\phi(x,t) \to 0$, the function $u(x,t)$ is meromorphic, such that the leading order singularity θ is a negative integer and the $\lambda_k(x,t)$ are analytic functions. The general procedure of the Painlevé test consists in substituting the expansion (7.14) into the equation of motion, (7.55) for the case at hand, and determining the functions $\lambda_k(x,t)$ recursively, with enough free parameters to match the order of the differential equation.

We compute λ_0 by substituting the first term in the expansion (7.14), i.e. $u(x,t) \to \lambda_0(x,t)\phi(x,t)^\theta$, into (7.55) and evaluating the values for all possible leading order singularities θ. The individual terms have the following leading order behaviour

$$u_t \sim \phi^{\alpha\theta-1} \quad , \quad u^{l-2}u_x \sim \phi^{\theta(l-1)-1} \quad \text{and all remaining} \sim \phi^{\theta(m+p-1)-m-1}. \tag{7.56}$$

Therefore the leading order terms may only be canceled if any of the following three

conditions hold:

i) $\theta - 1 = \theta(l-1) - 1 \leq \theta(m+p-1) - m - 1,$

which results from assuming that u_t and $u^{l-2}u_x$ constitute the leading order terms. In this case we obtain $l = 2$ and the inequality $\theta(2-m-p) \leq -m$. Thus θ remains undetermined.

ii) $\theta - 1 = \theta(m+p-1) - m - 1 \leq \theta(l-1) - 1,$

which corresponds to the assumption that $u^{l-2}u_x$ is the least singular term and matching the leading orders of all the remaining ones. Then we conclude that $l \leq 2$ and θ is fixed to $\theta = \frac{m}{(m+p-2)}$.

iii) $\theta(l-1) - 1 = \theta(m+p-1) - m - 1 \leq \theta - 1.$

which is the consequence of u_t being least singular term and the matching of the remaining ones. This means the leading order singularity of $u(x,t)$ is of the order

$$\theta = \frac{m}{p+m-l} \quad \in \mathbb{Z}^- \quad \text{and} \quad l \geq 2. \tag{7.57}$$

Canceling the leading order terms then yields

$$\lambda_0^{(n)} = e^{\frac{2\pi\imath n\theta}{m}} [gl(l-1)]^{-\frac{\theta}{m}} (\imath\theta\phi_x)^{-\theta}, \tag{7.58}$$

where $1 \leq n \leq p+m-l$ indicates the different roots of the determining equation.

In principle we could also envisage a scenario in which u_t and $u^{l-2}u_x$ are the least dominant terms and the leading order singularity is canceled by all the remaining terms. However, all these terms only differ by an overall numerical factor, such that λ_0 turns out to be zero in this case and we can therefore discard this case.

A key feature of the Painlevé test is the occurrence of so-called resonances, which arise whenever the coefficient in front of a specific λ_r in the recurrence relations becomes zero. This implies that λ_r can not be determined recursively. When in this case the remaining part of the recurrence relation becomes an identity, the λ_r becomes a free parameter, otherwise the Painlevé test fails. The possible values for r can be found by substituting

$$u_r(x,t) = \lambda_0(x,t)\phi(x,t)^\theta + \lambda_r(x,t)\phi(x,t)^{r+\theta}, \tag{7.59}$$

into (7.55) and computing all possible values of r for which λ_r becomes a free parameter. Considering the case *iii*) for integer values l, m, p the coefficients of the leading order $\phi^{r+\theta(l-1)-1}$ is proportional to

$$g^{\theta\frac{(2-l)}{m}}\phi_x^{\theta(2-l)+1}\,\lambda_r(r+1)(r+\theta l)[r+\theta(l-1)]. \tag{7.60}$$

This means that besides the so-called fundamental resonance at $r = 1$, we also find two more resonances at $r = -\theta l, \theta(1-l)$. Since the differential equation (7.55) is of order three all these models fully pass the Painlevé test provided $\lambda_{-\theta l}$ and $\lambda_{\theta(1-l)}$ can indeed be chosen freely.

The standard procedure to verify this would be now to derive the recursive equation resulting from combining (7.14) and (7.55). Clearly for generic values of l, m, p this will be extremely lengthy, but even for specific choices it is fairly complicated. It suffices, however, to compute the λ_k up to $k > -\theta l$. We will present these values for various examples for several choices of the parameters l, m, p corresponding to scenarios leading to solutions with qualitatively different kinds of behaviour.

7.5.1 Generalized KdV-equation: m=2

Cooper, Khare and Saxena [240] found that in the generalized KdV equation, i.e. $m = 2$, a necessary condition for compactons to be stable is to consider models with $2 < l < p+6$. This means none of the conditions *i*) or *ii*) for the leading order singularity to cancel can be satisfied. The special choice $l = p + 2, 0 < p \leq 2$ guarantees that the compacton solutions have in addition a width which is independent of their amplitude [174]. For that particular case also the condition *iii*) admits no solution, such that the Painlevé test fails.

However, for models which admit stable compacton solutions having a width depending on the amplitude we can find solutions to the condition *iii*) and proceed with the Painlevé test. For instance, $m = 2, p = 1, l = 5$ is such a choice. In this case we find from (7.57) that $\alpha = -1$ and the leading order singularity of the corresponding differential equation

is ϕ^{-5}. Computing now order by order the functions λ_k we find the two solutions

$$\begin{aligned}
\lambda_0^{\pm} &= \pm 2i\sqrt{5g}\phi_x, \quad \lambda_1^{\pm} = \mp i\sqrt{5g}\frac{\phi_{xx}}{\phi_x}, \\
\lambda_2^{\pm} &= \mp i\frac{\sqrt{5g}}{6}\frac{\left(3\phi_{xx}^2 - 2\phi_x\phi_{xxx}\right)}{\phi_x^3}, \\
\lambda_3^{\pm} &= \frac{3\phi_t\phi_x^2 \mp 4i\sqrt{5g^3}\left(6\phi_{xx}^3 - 6\phi_x\phi_{3x}\phi_{xx} + \phi_x^2\phi_{4x}\right)}{48g\phi_x^5}.
\end{aligned} \tag{7.61}$$

Crucially we observe next that $\lambda_4^{\pm}$ and $\lambda_5^{\pm}$ can be chosen arbitrarily. The remaining $\lambda_k^{\pm}$ for $k > 5$ can all be computed, but the expressions are all extremely cumbersome and we will therefore not report them here. Making, however, the further assumption on ϕ to be a travelling wave, i.e. $\phi(x,t) = x - \omega t$, simplifies the expressions considerably. Choosing $\lambda_4^{\pm} = \lambda_5^{\pm} = 0$ the two solutions for that scenario reduce to

$$\begin{aligned}
\lambda_{3\kappa+1}^{\pm} &= \lambda_{3\kappa+2}^{\pm} = 0 \qquad \text{for } \kappa = 0, 1, 2, \ldots \\
\lambda_0^{\pm} &= \pm 2i\sqrt{5g},\ \lambda_3^{\pm} = -\frac{\omega}{16g}, \\
\lambda_6^{\pm} &= \mp\frac{3i\omega^2}{3584\sqrt{5}g^{5/2}},\ \lambda_9^{\pm} = \frac{\omega^3}{573440g^4}, \\
\lambda_{12}^{\pm} &= \pm\frac{33i\omega^4}{1669857280\sqrt{5}g^{11/2}}, \\
\lambda_{15}^{\pm} &= -\frac{3\omega^5}{66794291200g^7}, \ldots
\end{aligned} \tag{7.62}$$

We conclude that the Painlevé test is passed for this choices of parameters, which means that besides stable compacton solutions, whose width depends on their amplitude, we also find genuine solitons in these models and, provided the series (7.14) converges, they are therefore integrable.

In the unstable compacton regime, i.e. $l \leq 2$ or $l \geq p+6$, the condition *iii*) can not be satisfied. Consequently we do not expect to find genuine soliton solutions. We have also verified this type of behaviour for other representative examples which we do not present here.

7.5.2 $\mathcal{PT}$-symmetric generalized KdV-equation

For the $\mathcal{PT}$-symmetric extensions of the generalized KdV-equation (7.55) the necessary condition for compactons to be stable was extended by Bender et al [185] to $2 < l < p+3m$. Thus also for generic values of m none of the conditions *i*) or *ii*) for the leading order singularity to cancel can be satisfied. Furthermore, the requirement for stable compacton

solutions to possess also a width which is independent of their amplitude was generalized in [185] to $l = p + m$. As for the special case $m = 2$ this value coincides with the leading order singularity resulting from the condition *iii*) tending to infinity and therefore the Painlevé test fails.

As in the previous case, for models which have stable compacton solutions whose width is a function of their amplitude the Painlevé test has a chance to pass, as one can find a value for the leading order singularity and potentially has the correct amount of resonances. We verify this for the example **m=3,p=1,l=7**, for which we obtain $\theta = -1$ and ϕ^{-7} as the leading order singularity in (7.55). Since $-\theta/m = 1/3$ in this case, we find now three non-equivalent solutions related to the different roots for the $\lambda_k^{(n)}$ with $n = 1, 2, 3$, of which the first terms are

$$\begin{aligned}
\lambda_0^{(n)} &= -ie^{2\pi in/3}(42g)^{1/3}\phi_x, \\
\lambda_1^{(n)} &= \frac{ie^{2\pi in/3}(21g)^{1/3}\phi_{xx}}{2^{2/3}\phi_x}, \\
\lambda_2^{(n)} &= \frac{ie^{2\pi in/3}(7g)^{1/3}\left(3\phi_{xx}^2 - 2\phi_x\phi_{xxx}\right)}{2(6)^{2/3}\phi_x^3}, \\
\lambda_3^{(n)} &= \frac{ie^{2\pi in/3}(7g)^{1/3}\left(6\phi_{xx}^3 - 6\phi_x\phi_{xxx}\phi_{xx} + \phi_x^2\phi_{xxxx}\right)}{4(6)^{2/3}\phi_x^5}.
\end{aligned} \tag{7.63}$$

From (7.60) we know that we should encounter resonances at the level 6 and 7, which is indeed the case as we find that $\lambda_6^{(n)}$and $\lambda_7^{(n)}$ can be chosen freely. The remaining $\lambda_k^{(n)}$ for $k > 7$ can all be computed iteratively and the Painlevé test is passed for this example.

For a travelling wave ansatz $\phi(x,t) = x - \omega t$ with the choice $\lambda_6^{(n)} = \lambda_7^{(n)} = 0$ the expressions simplify to

$$\begin{aligned}
\lambda_{5\kappa+1}^{(n)} &= \lambda_{5\kappa+2}^{(n)} = \lambda_{5\kappa+3}^{(n)} = \lambda_{5\kappa+4}^{(n)} = 0, \text{ for } \kappa = 0, 1, \ldots \\
\lambda_0^{(n)} &= -ie^{2\pi in/3}(42g)^{1/3},\ \lambda_5^{(n)} = \frac{e^{4\pi in/3}\omega}{36(42)^{1/3}g^{4/3}}, \\
\lambda_{10}^{(n)} &= \frac{17i\omega^2}{598752g^3}, \\
\lambda_{15}^{(n)} &= -\frac{53e^{\frac{2in\pi}{3}}\omega^3}{21555072(42)^{2/3}g^{14/3}}, \ldots
\end{aligned} \tag{7.64}$$

Thus we observe no qualitative difference in the $\mathcal{PT}$-symmetric extensions in comparison to the case $m = 2$ and find that also in this one may have stable compacton solutions, whose width depends on their amplitude and genuine solitons at the same time.

In the unstable compacton regime, that is $l \leq 2$ or $l \geq p + 3m$, the condition *iii*) can

not be satisfied and the Painlevé test fails. Once again we do not represent here other representative examples for which we obtained the same type of behaviour.

7.5.3 Deformations of Burgers equation: $m = 1, p = 1, l = 3$

Considering $m = 1, p = 1, l = 3$ in the equation of motion (7.55) is a very simple example leading to a Painlevé expansion for $u(x,t)$, which can even be truncated after the second term. As this type of behaviour is reminiscent of Bäcklund transformation generating solutions found in other models [32], we present this case briefly. For this choice (7.55) reduces to

$$u_t + uu_x - 2igu_{xx} + \frac{igu}{u_x^2}\left(u_{xx}^2 - u_x u_{xxx}\right) = 0, \tag{7.65}$$

which can be viewed as a deformation of Burgers equation corresponding to the first three terms. Proceeding as in the previous sections, we find the solution

$$u(x,t) = \frac{-6ig\phi_x}{\phi} + \frac{6ig\phi_{xx} - 3\phi_t}{2\phi_x}, \tag{7.66}$$

provided that ϕ satisfies the equation

$$\phi_x^2\phi_{tt} + \phi_t^2\phi_{xx} = 2\phi_{tx}\phi_t\phi_x. \tag{7.67}$$

A travelling wave $\phi(x,t) = x - \omega t$ is for instance a solution of (7.67), such that we obtain the simple expression

$$u(x,t) = \frac{6ig}{\omega t - x} + \frac{3}{2}\omega \tag{7.68}$$

for the solution of (8.49). Incidentally, the travelling wave solution for Burger's equation [32] conicides with (7.68).

7.5.4 Coexistence of solitons and compactons

In previous investigations [174, 175, 240] various criteria have been found, which separate the models $\mathcal{H}_{l,m,p}$ into three distinct classes exhibiting qualitatively diffferent types of compacton solutions, unstable compactons and stable compactons, which have either dependent or freely selectable width A and amplitude β. We have carried out the Painlevé test for various examples for each of these classes and found that all models which allow stable compactons for which the width cannot be chosen independently from their amplitude pass the Painlevé test. Assuming that the Painlevé expansion (7.14) converges

these models possess the Painlevé property [241] and allow therefore for genuine soliton solutions and are thus integrable. We found that the generalized KdV equation resulting from $\mathcal{H}_{l,2,p}$ and their $\mathcal{PT}$-symmetric extensions $\mathcal{H}_{l,m,p}$ have the same qualitative behaviour in the three different regimes. Remarkably the original compacton model [173], which is non-Hamiltonian and whose equation of motion differs from $\mathcal{H}_{l,2,p}$ only by some constants, was found not to possess the Painlevé property [242]. For convenience we summarize the different qualitative behaviours in the following table:

$\mathcal{H}_{l,m,p}$	compactons	solitons
$l = p + m$	stable, independent A, β	no
$2 < l < p + 3m$	stable, dependent A, β	yes
$l \leq 2$ or $l \geq p + 3m$	unstable	no

Table 7.1: The models $\mathcal{H}_{l,m,p}$ and their solutions.

Clearly our investigations do not constitute a complete mathematical proof as we based our findings on various representative examples for the different classes and it would be very interesting to settle this issue more rigorously with a generic argumentation not relying on case-by-case studies. At the same time such a treatment would probably provide a deeper understanding about the separation of the different models. Nonetheless, our findings provide enough evidence to make it worthwhile to investigate the models which pass the test with other techniques developed in the field of integrable models, whereas models which do not pass the test may be excluded from such investigations.

8 $\mathcal{PT}$-symmetric fields and particles

8.1 $\mathcal{PT}$-symmetric constraints on real fields

The fact that unbroken $\mathcal{PT}$-symmetry will allow for a consistent quantum mechanical description evidences that symmetric systems constitute well defined models which have been overlooked up to now. This idea has been adapted to classical systems as well and has been used to formulate various complex extensions of nonlinear wave equations. In the previous chapter we have explored $\mathcal{PT}$-symmetric extensions of some dynamical systems, such as Burgers and KdV equations, and we showed that some deformations preserve integrability, allowing for the existence of solitons and compacton solutions. Nonetheless, so far any such proposal lacks a direct physical meaning and the complexifications are generally introduced in a rather *ad hoc* manner.

Here we shall explore $\mathcal{PT}$-symmetry in a context where the complex extensions or deformations do not need to be imposed artificially, but instead we investigate whether this symmetry is already naturally present in the system, albeit hidden. We introduce another possibility to accomplish a $\mathcal{PT}$-symmetric extension consisting of restricting the motion to $\mathcal{PT}$-symmetric invariant submanifolds rather than deforming the equations themselves. This novel procedure could in principle be used in different situations but we shall work specifically with the emergence of complex multi-particle Calogero particles emerging from real nonlinear field equations, thus providing a well defined physical origin for these systems. After exploring these connections we are naturally led to complex $\mathcal{PT}$-symmetric Calogero systems.

8.2 Calogero models

The Calogero problem is possibly the first to be understood at the quantum level before the corresponding advances were obtained classically. It was originally proposed [243] as a one-dimensional problem of three particles interacting in pairs according an inverse squared distance potential together with a possible quadratic confining well,

$$\left[-\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}+\sum_{i\neq j}^{3}\frac{g}{(x_i-x_j)^2}+\sum_{i\neq j}^{3}\omega^2(x_i-x_j)^2\right]\psi=E\psi \qquad \text{with} \quad g,\omega\in\mathbb{R}. \tag{8.1}$$

It was noticed a reformulation of this problem in terms of a different set of coordinates allows for an exact solution based on separation of variables. The configuration is better described in terms of the centre of mass and polar Jacobi coordinates,

$$R = \frac{1}{3}\left(x_1+x_2+x_3\right), \tag{8.2}$$

$$r = \frac{1}{\sqrt{3}}\sqrt{(x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_1)^2}, \tag{8.3}$$

$$\phi = \arctan\left[\frac{\sqrt{3}(x_1-x_2)}{(x_1-x_3)+(x_2-x_3)}\right], \tag{8.4}$$

and consequently,

$$x_{1,2}=R+\frac{r\cos\phi}{\sqrt{6}}\pm\frac{r\sin\phi}{\sqrt{2}} \qquad \text{and} \qquad x_3=R-\sqrt{\frac{2}{3}}r\cos\phi. \tag{8.5}$$

The associated Schrödinger equation becomes simply

$$\left[-\frac{1}{3}\frac{d^2}{dR^2}-\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\left(\frac{d^2}{d\phi^2}-\frac{9g}{2\sin^2\phi}\right)+3\omega^2r^2\right]\psi(R,r,\phi)=E\psi(R,r,\phi), \tag{8.6}$$

from which we see there are three quantum numbers emerging and the centre of mass energy can be absorbed by a simple re-scaling of the energy levels. If one is interested in scattering properties rather than bound states, there is a slight simplification: as $\omega\to 0$, the Laguerre polynomials characterizing the radial part of the solution reduce to Bessel functions. Practically at the same time, Calogero examined the N-particle generalization of this problem [244] and managed to obtain general results for its ground state.

Only after the completion of this work Marchioro [245, 246] successfully constructed the classical trajectories of a system of three particles interacting in pairs by inverse-cube forces in the presence of harmonic forces, using conservation of both energy and the analogous of the angular quantity B,

$$\frac{1}{2}m\dot{r}^2+\frac{B^2}{r^2}=E \qquad \text{and} \qquad \frac{1}{2}mr^4\dot{\phi}^2+\frac{9g}{2\sin^2 3\phi}=B^2. \tag{8.7}$$

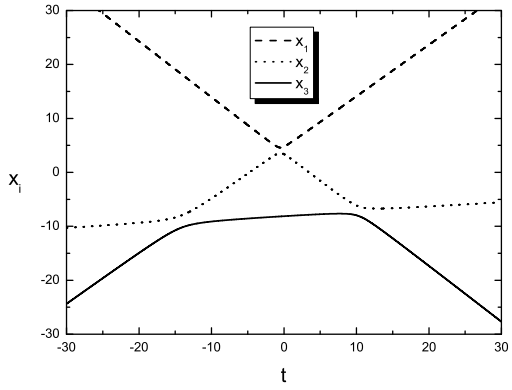

Figure 8.1: Classical scattering of three particles under pairwise inverse-square interaction.

The integrability of the classical N-particle Calogero scattering problem,

$$H_C = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{N} \frac{g}{(x_i - x_j)^2}, \tag{8.8}$$

was established later by Moser [247], using a Lax pair consisting of matrices L, M, with entries

$$L_{ij} = p_i\delta_{ij} + \frac{\imath\sqrt{g}}{x_i - x_j}(1 - \delta_{ij}), \tag{8.9}$$

$$M_{ij} = \sum_{k\neq i}^{N} \frac{\imath\sqrt{g}}{(x_i - x_k)^2}\delta_{ij} - \frac{\imath\sqrt{g}}{(x_i - x_j)^2}(1 - \delta_{ij}), \tag{8.10}$$

constructed in such a manner that the Lax equation

$$\frac{dL}{dt} + [M, L] = 0, \tag{8.11}$$

becomes equivalent to the Calogero equations of motion,

$$x_i = \frac{\partial H_C}{\partial p_i} \quad \text{and} \quad p_i = -\frac{\partial H_C}{\partial x_i} \qquad \Longrightarrow \qquad \ddot{x}_i = \sum_{j\neq i}^{N} \frac{2g}{(x_i - x_j)^3}. \tag{8.12}$$

Integrability follows in the standard fashion by noting that all quantities of the form $I_n = \mathrm{tr}(L^n)/n$ are integrals of motion and conserved in time by construction.

The classical solutions of a two-particle Calogero problem are given by

$$x_{1,2}(t) = 2R(t) \pm \sqrt{\frac{g}{E} + 4E(t - t_0)^2}, \tag{8.13}$$

with E, t_0 being initial conditions and $\dot{R}(t) = 0$ the centre of mass velocity. Relaxing this condition by allowing boosts will only shift the energy scale since the total momentum

is conserved. Depending therefore on the initial conditions we may have either real or complex solutions.

The three particle model, i.e. taking $N = 3$ in (8.8), is slightly more complicated. Marchioro [245] found the general solution by expressing the dynamical variables in terms of Jacobi relative coordinates R, X, Y in polar form via the transformations $R(t) = (x_1(t) + x_2(t) + x_3(t))/3$, $X(t) = r(t)\sin\phi(t) = (x_1(t) - x_2(t))/\sqrt{2}$ and $Y(t) = r(t)\cos\phi(t) = (x_1(t) + x_2(t) - 2x_3(t))/\sqrt{6}$. The variables may then be separated and the resulting equations are solved by

$$x_{1,2}(t) = R(t) + \frac{1}{\sqrt{6}} r(t)\cos\phi(t) \pm \frac{1}{\sqrt{2}} r(t)\sin\phi(t), \tag{8.14}$$

$$x_3(t) = R(t) - \frac{2}{\sqrt{6}} r(t)\cos\phi(t), \tag{8.15}$$

where

$$R(t) = R_0 + V_0\, t, \tag{8.16}$$

$$r(t) = \sqrt{\frac{B^2}{E} + 2E(t - t_0)^2}, \tag{8.17}$$

$$\phi(t) = \frac{1}{3}\cos^{-1}\left\{\varphi_0 \sin\left[\sin^{-1}(\varphi_0\cos 3\phi_0) - 3\tan^{-1}\left(\frac{\sqrt{2}E}{B}(t - t_0)\right)\right]\right\}. \tag{8.18}$$

The solutions involve 7 free parameters: The total energy E, the angular momentum type constant of motion B, the integration constants t_0, ϕ_0, R_0, V_0 and the coupling constant g, with the abbreviation $\varphi_0 = \sqrt{1 - 9g/2B^2}$. We note that, depending on the choice of these parameters, both real and complex solutions are admissible, a feature which might not hold for the Calogero system restricted to an invariant submanifold.

Calogero systems have become very important in theoretical physics, having been explored in various contexts ranging from condensed matter physics to cosmology, e.g. [248, 249, 250, 251]. An intriguing property of this sort of interacting models is that they appear very naturally in the study of nonlinear systems as the ones we have examined in the previous chapter, such as KdV or Boussinesq equations. This indicates that $\mathcal{PT}$-symmetric deformations carried out directly on equations of motion might not be the only interesting procedure to generate families of $\mathcal{PT}$-invariant waves. The main focus of our interest here are the complex extensions which have been studied recently in connection with $\mathcal{PT}$-symmetric models [252, 253, 254, 255, 256, 257].

8.3 Poles of nonlinear waves as interacting particles

The assumption of rational real valued functions as multi-soliton solutions of nonlinear wave equations was studied more than three decades ago by various authors, see e.g. [258]. We take some of these findings as a setting for the problem at hand. In order to illustrate the key idea we present what is probably the simplest scenario in which corpuscular objects emerge as poles of nonlinear waves, namely in the Burgers equation

$$u_t + \alpha u_{xx} + \beta (u^2)_x = 0. \tag{8.19}$$

Assuming that this equation admits rational solutions of the form

$$u(x,t) = \frac{2\alpha}{\beta} \sum_{i=1}^{N} \frac{1}{x - x_i(t)}, \tag{8.20}$$

it is straightforward to see that surprisingly the N poles interact with each other through a Coulombic inverse square force

$$\ddot{x}_i(t) = -2\alpha \sum_{j \neq i}^{N} \frac{1}{[x_i(t) - x_j(t)]^2}. \tag{8.21}$$

This pole structure survives even after making modifications in the ansatz for the wave equation, although the nature of the interaction may change. By acting on the second derivative in Burgers equation with a Hilbert transform

$$\hat{H}u(x) = \frac{1}{\pi} PV \int_{-\infty}^{+\infty} dz \frac{u(z)}{z - x}, \tag{8.22}$$

with PV standing for the principle value of the integral, we obtain the Benjamin-Ono equation [259, 260]

$$u_t + \alpha \hat{H} u_{xx} + \beta (u^2)_x = 0. \tag{8.23}$$

As shown in [261], the ansatz proposed for the equation above which will allow for similar conclusions has a slightly different form,

$$u(x,t) = \frac{\alpha}{\beta} \sum_{k=1}^{N} \left(\frac{\imath}{x - z_k(t)} - \frac{\imath}{x - z_k^*(t)} \right), \tag{8.24}$$

being, however, still a real valued solution with the only restriction that the complex poles satisfy complex Calogero equations of motion

$$\ddot{z}_k(t) = 8\alpha^2 \sum_{k \neq j}^{N} \frac{1}{(z_k(t) - z_j(t))^3}. \tag{8.25}$$

Note that there is a difference in the power laws appearing in (8.21) and (8.25), but more importantly that equation (8.20) has real poles, whereas (8.24) has complex ones. We stress once more that the field $u(x,t)$ is real in both cases. Hence, this viewpoint provides a nontrivial mechanism which leads to particle systems defined in the complex plane.

Interesting observations of this kind can be made for other nonlinear equations as well, but not always will the ansatz work directly, that is without any further requirements as in the previous cases. In some situations additional conditions might be necessary. Examples of nonlinear integrable wave equations for which such type of constraints occur are the KdV and the Boussinesq equations,

$$u_t + \left(\alpha u_{xx} + \beta u^2\right)_x = 0 \qquad \text{and} \qquad u_{tt} + \left(\alpha u_{xx} + \beta u^2 - \gamma u\right)_{xx} = 0, \tag{8.26}$$

respectively. For both of these equations one can have "N-soliton" solutions[14] of the form

$$u(x,t) = -6\frac{\alpha}{\beta}\sum_{k=1}^{N}\frac{1}{(x - x_k(t))^2}, \tag{8.27}$$

as long as in each case two sets of constraints are satisfied

$$\dot{x}_k(t) = -12\alpha\sum_{j\neq k}^{N}(x_k(t) - x_j(t))^{-2} \ , \qquad 0 = \sum_{j\neq k}^{N}(x_k(t) - x_j(t))^{-3}, \tag{8.28}$$

and

$$\ddot{x}_k(t) = -24\alpha\sum_{j\neq k}^{N}(x_k(t) - x_j(t))^{-3} \ , \qquad \dot{x}_k(t)^2 = 12\alpha\sum_{j\neq k}^{N}(x_k(t) - x_j(t))^{-2} + \gamma, \tag{8.29}$$

respectively. Naturally these constraints might be incompatible or admit no solution at all, in which case (8.27) would of course not constitute a solution for the wave equations (8.26). Notice that if the $x_k(t)$ are real or come in complex conjugate pairs the solution (8.27) for the corresponding wave equations is still real.

Airault, McKean and Moser provided a general criterion, which allows us to view these equations from an entirely different perspective, namely to regard them as constrained multi-particle systems [258]:

Given a multi-particle Hamiltonian $H(x_1,...,x_N,\dot{x}_1,...,\dot{x}_N)$ *with flow* $x_i = \frac{\partial H}{\partial \dot{x}_i}$ *and* $\dot{x}_i = -\frac{\partial H}{\partial x_i}$ *together with conserved charges* I_n *in involution with* H*, i.e. vanishing Poisson*

[14]Soliton is to be understood here in a very loose sense in analogy to the Painlevé type ideology of indestructible poles. In the strict sense not all solution possess the N-soliton solution characteristic, that is moving with a preserved shape and regaining it after scattering though each other.

brackets $\{H, I_n\} = 0$ *, then the locus of* $grad(I_n) = 0$ *is invariant with respect to time evolution. Thus it is permitted to restrict the flow to that locus provided it is not empty.*

Taking the Hamiltonian to be the Calogero Hamiltonian H_C it is well known that one may construct the corresponding conserved quantities from the Calogero Lax operator (8.9) as mentioned from $I_n = \mathrm{tr}(L^n)/n$. The first of these charges is just the total momentum, the next is the Hamiltonian followed by non trivial ones

$$I_1 = \sum_{i=1}^{N} p_i \ , \tag{8.30}$$

$$I_2 = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + g\sum_{i\neq j}^{N} \frac{1}{(x_i - x_j)^2} \ , \tag{8.31}$$

$$I_3 = \frac{1}{3}\sum_{i=1}^{N} p_i^3 + g\sum_{i\neq j}^{N} \frac{p_i + p_j}{(x_i - x_j)^2} \ , \tag{8.32}$$

$$\vdots$$

According to the above mentioned criterium we may therefore consider an I_3-flow restricted to the locus defined by $\mathrm{grad}(I_2) = 0$ or an I_2-flow subject to the constraint $\mathrm{grad}(I_3 - \gamma I_1) = 0$. Remarkably it turns out that the former viewpoint corresponds exactly to the set of equations (8.28), whereas the latter to (8.29) when we identify the coupling constant as $g = -12\alpha$. Thus the solutions of the Boussinesq equation are related to the constrained Calogero Hamiltonian flow, whereas the KdV soliton solutions arise from an I_3-flow subject to constraining equations derived from the Calogero Hamiltonian.

As our main focus is on the Calogero Hamiltonian flow and its possible complexifications we shall concentrate on possible solutions of the systems (8.29) and investigate whether these type of equations allow for nontrivial solutions or whether they are empty. It will be instructive to commence by looking first at the unconstrained system. Let us now elaborate further on the connection between the field equations and the particle system and restrict the general solution (8.14)-(8.16) by switching on the additional constraints in (8.29) and subsequently study the effect on the soliton solutions of the nonlinear wave equation. Notice that the second constraint in (8.29) can be viewed as setting the difference between the kinetic and potential energy of each particle to a constant. Adding all of these equations we obtain $H_C = N\gamma/2$, which provides a direct interpretation of the constant γ in the Boussinesq equation as being proportional to the total energy of the Calogero model.

8.4 The motion of Boussinesq singularities

The two particle system, i.e. $N = 2$, is evidently the simplest I_2-Calogero flow constrained with $\mathrm{grad}(I_3 - \gamma I_1) = 0$ as specified in (8.29). The solution for this system was already provided in [258],

$$x_{1,2}(t) = \kappa \pm \sqrt{\gamma(t-\tilde{\kappa})^2 - 3\alpha/\gamma}, \tag{8.33}$$

with $\kappa, \tilde{\kappa}$ taken to be real constants. In fact this solution is not very different from the unconstrained motion shown in the previous section (8.13). The restricted one may be obtained via an identification between the coupling constant and the parameter in the Boussinesq equation as $\kappa = 2R(t)$, $E = \gamma/4$, $\tilde{\kappa} = t_0$ and $g = -3\alpha/4$. The two soliton solution for the Boussinesq equation (8.27) then acquires the form

$$u(x,t) = -12\frac{\alpha}{\beta}\gamma\frac{\gamma(x-\kappa)^2 + \gamma^2(t-\tilde{\kappa})^2 - 3\alpha}{[\gamma(x-\kappa)^2 - \gamma^2(t-\tilde{\kappa})^2 + 3\alpha]^2}, \tag{8.34}$$

which, in principle, is still real-valued when keeping the constants to be real. When inspecting (8.34) it is easy to see that the two singularities repel each other on the x-axis as time evolves, thus mimicking a repulsive scattering process. However, we may change the overall behaviour substantially when we allow the integration constants to be complex, such that the singularities become regularized. In that case we observe a typical solitonic scattering behaviour, i.e. two wave packets keeping their overall shape while evolving in time and when passing though each other regaining their shape when the scattering process is finished, albeit with complex amplitude. A special type of complexification occurs when we take the integration constants $\kappa, \tilde{\kappa}$ to be purely imaginary, in which case (8.34) becomes a solution for the $\mathcal{PT}$-symmetrically constrained Boussinesq equation, with $\mathcal{PT} : x \to -x, t \to -t, u \to u$. We depict the described behaviour in figure 1 for some special choices of the parameters.

For larger numbers of particles the solutions have not been investigated and it is not even clear whether the locus of interest is empty or not. Let us therefore embark on solving this problem systematically. Unfortunately we can not simply imitate Marchioro's method of separating variables as the additional constraints will destroy this possibility. However, we notice that (8.29) can be represented in a different way more suited for our purposes. Differentiating the second set of equations in (8.29) and making use of the first

one, we arrive at the set of expressions

$$\sum_{k\neq j}^{N} \frac{(\dot{x}_k(t) + \dot{x}_j(t))}{(x_k(t) - x_j(t))^3} = 0, \tag{8.35}$$

which are therefore consistency equations of the other two.

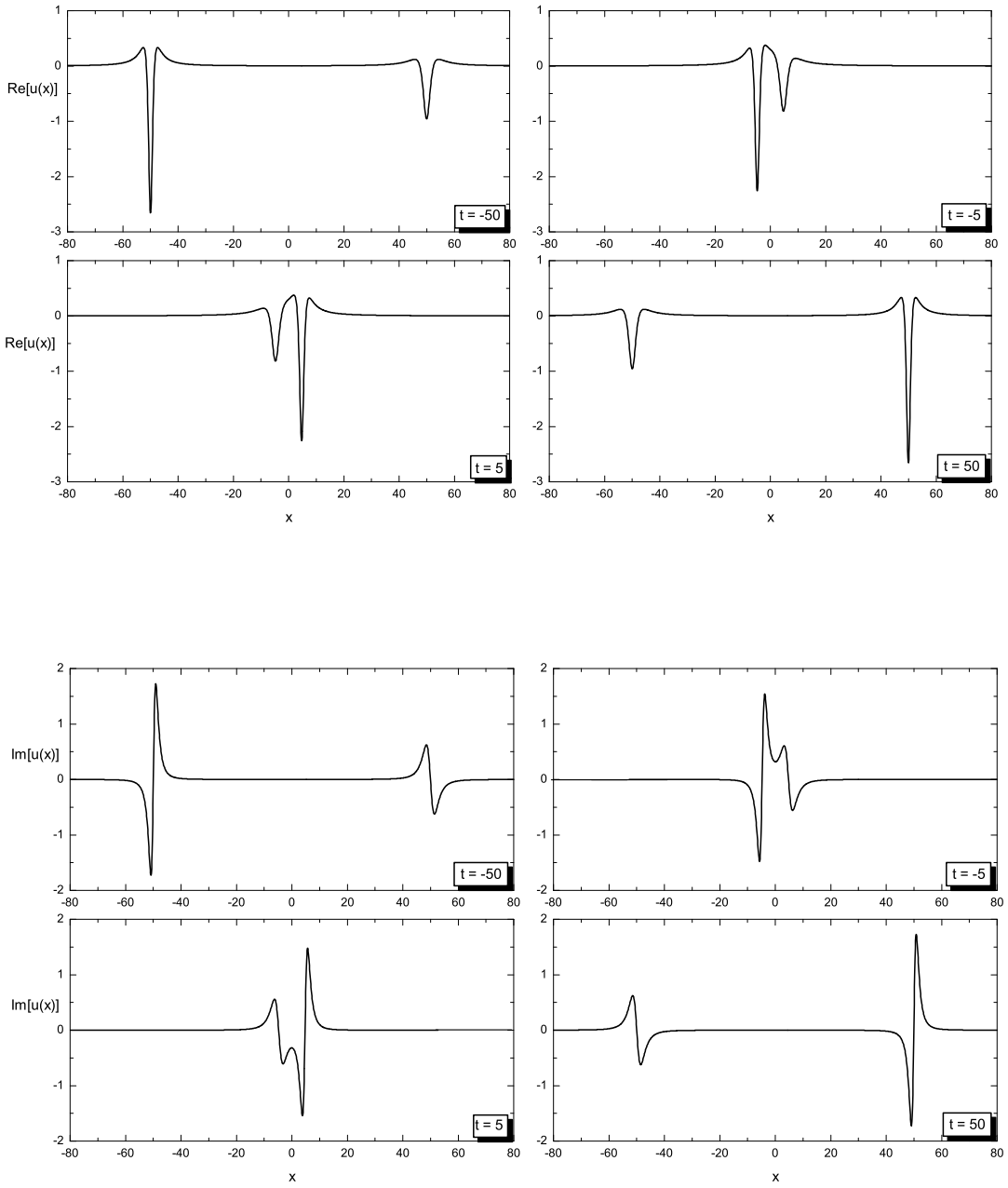

Figure 8.2: Time evolution of the real and imaginary parts of the constraint Boussinesq two soliton solution (8.34) with $\kappa = -\imath, \bar{\kappa} = 2\imath, \alpha = -\beta = \gamma = 1$.

We now focus on the case $N = 3$. Inspired by the general solution of the unconstrained three particle solution (8.14) and (8.15), we adopt an ansatz of the general form

$$x_{1,2}(t) = A_0(t) + A_1(t) \pm A_2(t), \tag{8.36}$$

$$x_3(t) = A_0(t) + \lambda A_1(t), \tag{8.37}$$

with $A_i(t)$, $i = 0, 1, 2$ being some unknown functions and λ a free constant parameter. We note that $\lambda \neq 1$, since otherwise the three coordinates could be expressed in terms of only two linearly independent functions, $A_0(t) + A_1(t)$ and $A_2(t)$, and we would not able to express the normal mode like functions $A_i(t)$ in terms of the original coordinates $x_i(t)$. Calogero's choice, $\lambda = -2$, in equation (8.15), allows an elegant map of Cartesian coordinates into Jacobi's relative coordinates, but other possibilities might be more convenient in the present situation. Here we keep λ to be free for the time being.

Substituting this ansatz for the $x_i(t)$ into the second set of equations in (8.29) and using the compatibility equation (8.35), we are led to six coupled first order differential equations for the unknown functions $A_0(t), A_1(t), A_2(t)$

$$\frac{(\dot{A}_0(t) + \lambda\dot{A}_1(t))^2 - \gamma}{2g} + \frac{1}{2A_+(t)^2} + \frac{1}{2A_-(t)^2} = 0, \tag{8.38}$$

$$\frac{(\dot{A}_0(t) + \dot{A}_1(t) \pm \dot{A}_2(t))^2 - \gamma}{2g} + \frac{1}{8A_2(t)^2} + \frac{1}{2A_\mp(t)^2} = 0, \tag{8.39}$$

$$\frac{2\dot{A}_0(t) + (\lambda+1)\dot{A}_1(t) + \dot{A}_2(t)}{A_-(t)^3} - \frac{2\dot{A}_0(t) + (\lambda+1)\dot{A}_1(t) - \dot{A}_2(t)}{A_+(t)^3} = 0, \tag{8.40}$$

$$\frac{\dot{A}_0(t) + \dot{A}_1(t)}{4A_2(t)^3} + \frac{2\dot{A}_0(t) + (\lambda+1)\dot{A}_1(t) \pm \dot{A}_2(t)}{A_\mp(t)^3} = 0. \tag{8.41}$$

For convenience we made the identifications $A_\pm(t) = A_2(t) \pm (\lambda - 1)A_1(t)$.

From the latter set of equations above, (8.40) and (8.41), we can now eliminate two of the first derivatives together with the use of the conservation of momentum. Depending on the choice, the remaining $\dot{A}_i(t)$ are eliminated with the help of the first three equations (8.38) and (8.39). The two equations left then become multiples of each other depending only on $A_1(t)$ and $A_2(t)$. Subsequently we can express $A_2(t)$, and consequently $\dot{A}_0(t), \dot{A}_1(t)$, in terms of $A_1(t)$ as the only unknown quantity. In this manner we arrive at

$$A_2(t) = \frac{\sqrt{-g - 4\gamma(\lambda-1)^2 A_1(t)^2}}{2\sqrt{3\gamma}}, \tag{8.42}$$

$$\dot{A}_0(t) = \sqrt{\gamma} + \frac{3g\sqrt{\gamma}(2+\lambda)}{(\lambda-1)[g + 16\gamma(\lambda-1)^2 A_1(t)^2]}, \tag{8.43}$$

$$\dot{A}_1(t) = \frac{9g\sqrt{\gamma}}{(1-\lambda)[g + 16\gamma(\lambda-1)^2 A_1(t)^2]}, \tag{8.44}$$

with $g = -12\alpha$. This means that once we have solved the differential equation (8.44) for $A_1(t)$ the complete solution is determined up to the integration of $\dot{A}_0(t)$ in (8.43) and a simple substitution in (8.42). In other words we have reduced the problem to solve the set of coupled nonlinear equations (8.29) to solving one first order nonlinear equation.

Let us now make a comment on the number of free parameters, that is integration constants, occurring in this solution. In the original formulation of the problem we have started with 3 second order differential equations, so that we expect to have 6 integration constants for the determination of x_1, x_2 and x_3. However, together with the additional 3 constraining equations this number is reduced to 3 free parameters. Finally we can invoke the conservation of total momentum from (8.30), which yields $3\ddot{A}_0(t) + (\lambda + 2)\ddot{A}_1(t) = 0$ and we are left with only 2 free parameters. We choose them here to be the two arbitrary constants attributed to the integration of $\dot{A}_0(t)$ in (8.43) and $\dot{A}_1(t)$ in (8.44), respectively.

In turn this also means that, without loss of generality, we may freely choose the constant λ introduced in (8.37). Indeed, keeping it generic we observe that the solutions for the $A_i(t)$ do not depend on it despite its explicit presence in the equations (8.42), (8.43) and (8.44). The most convenient choice is to take $\lambda = -2$ as in that case the equations simplify considerably.

Let us now solve (8.42), (8.43) and (8.44) and substitute the result into the original expressions (8.36) and (8.37) in order to see how the particles behave. We find

$$x_{1,2}(t) = c_0 + \sqrt{\gamma}t + \frac{1}{12}\left(\frac{g}{\xi(t)} - \frac{\xi(t)}{\gamma}\right) \pm \frac{\imath}{4\sqrt{3}}\left(\frac{g}{\xi(t)} + \frac{\xi(t)}{\gamma}\right), \tag{8.45}$$

$$x_3(t) = c_0 + \sqrt{\gamma}t - \frac{1}{6}\left(\frac{g}{\xi(t)} - \frac{\xi(t)}{\gamma}\right), \tag{8.46}$$

where for convenience we introduced the abbreviation

$$\xi(t) = \left[-54\gamma^2(\sqrt{\gamma}gt + c_1) + \sqrt{g^3\gamma^3 + [54\gamma^2(\sqrt{\gamma}gt + c_1)]^2}\right]^{\frac{1}{3}}. \tag{8.47}$$

The above mentioned two freely chosen constants of integration are denoted by c_0 and c_1. As in the two particle case, we may once again compare this solution with the unconstrained one in (8.14), (8.15) when considering the Jacobi relative coordinates

$$R(t) = c_0 + t\sqrt{\gamma}, \quad r^2(t) = -\frac{g}{6\gamma} \quad \text{and} \quad \tan\phi(t) = i\frac{g\gamma + \xi^2(t)}{g\gamma - \xi^2(t)}. \tag{8.48}$$

We observe that the solution is now constrained to a circle in the XY-plane with real radius when $g\gamma \in \mathbb{R}^-$. The values for $\phi(t)$ lead to the most dramatic consequence, namely

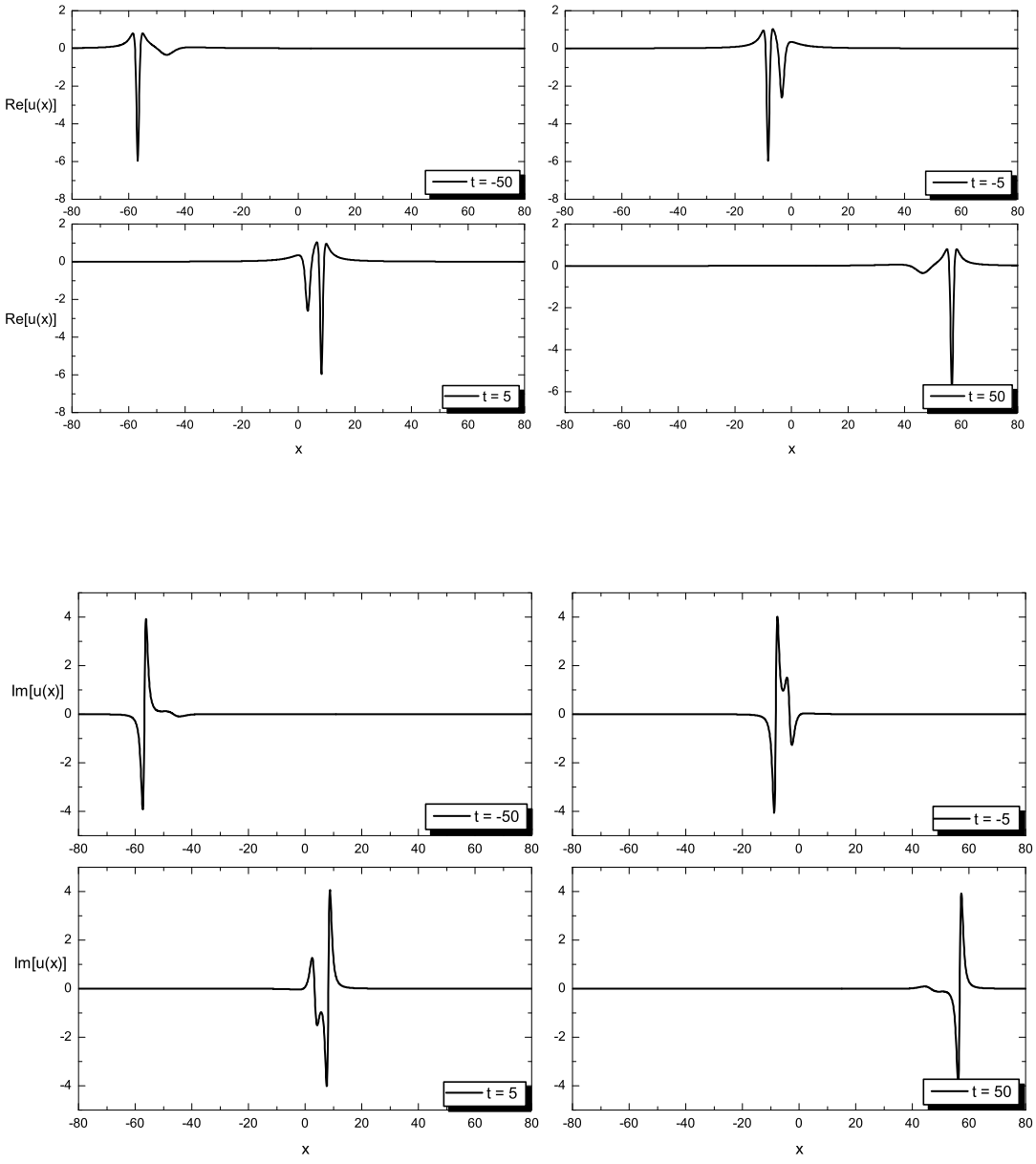

Figure 8.3: Time evolution of the real and imaginary parts of the constraint Boussinesq three soliton solution (8.49) with $c_0 = c_1 = \imath, \alpha = -\beta = \gamma = 1$.

that the particles are now forced to move in the complex plane, unlike as in unconstrained Calogero system or the $N = 2$ case where all options are open.

Interestingly, despite the poles being complex, we may still have real wave solutions for the Boussinesq equation. Provided that $\xi(t), \gamma, g, c_0, c_1 \in \mathbb{R}$ the pole $x_3(t)$ is obviously real whereas $x_1(t)$ and $x_2(t)$ are complex conjugate to each other, such that the ansatz

(8.27) yields a real solution

$$
\begin{aligned}
u(x,t) &= -\frac{6\alpha}{\beta}\frac{1}{\left(\varphi-\frac{1}{6}\left(\frac{g}{\xi(t)}-\frac{\xi(t)}{\gamma}\right)\right)^2}+ \\
&+ \frac{216\alpha}{\beta}\gamma^2\xi(t)^2\left[\frac{g^2\gamma^2-12g\gamma^2\varphi\xi(t)-4\gamma(18\gamma\varphi^2-g)\xi(t)^2+12\gamma\varphi\xi(t)^3+\xi(t)^4}{(g^2\gamma^2+6g\gamma^2\varphi\xi(t)+\gamma(36\gamma\varphi^2+g)\xi(t)^2-6\gamma\varphi\xi(t)^3+\xi(t)^4)^2}\right]
\end{aligned}
\tag{8.49}
$$

with $\varphi \equiv c_0 + \sqrt{\gamma}t - x$.

Due to the non-meromorphic form of $\xi(t)$ it is not straightforward to determine how the solutions transforms under a $\mathcal{PT}$-transformation. Nonetheless, the symmetry of the relevant combinations appearing in (8.45) and (8.46) can be analyzed well for $c_0, c_1 \in i\mathbb{R}$ and $\gamma > 0$. In that case the time reversal acts as $\mathcal{T} : \left(\frac{g}{\xi(t)} \pm \frac{\xi(t)}{\gamma}\right) \rightarrow \pm\left(\frac{g}{\xi(t)} \pm \frac{\xi(t)}{\gamma}\right)$, which implies $\mathcal{PT} : x_i(t) \rightarrow -x_i(t)$ for $i = 1, 2, 3$. Thus, the solutions to the constrained problem are not only complex, but in addition they can also be $\mathcal{PT}$-symmetric for certain choices of the constants involved.

8.5 Different constraints in nonlinear wave equations

It is clear from the above that the class of complex ($\mathcal{PT}$-symmetric) multi-particle systems which might arise from nonlinear wave equations could be much larger. We shall demonstrate this by investigating one further simple example which was previously studied in [262] and also refer to the literature [263] for additional examples. One very easy nonlinear wave equation which, because of its simplicity, serves as a very instructive toy model is

$$u_t + u_x + u^2 = 0. \tag{8.50}$$

We may now proceed as above and seek for a suitable ansatz to solve this equation, possibly leading to some constraining equations in form a multi-particle systems. Making therefore a similar ansatz for $u(x,t)$ as in (8.24) or (8.27) we take

$$u(x,t) = \sum_{i=1}^{N} \frac{1-\dot{z}_i(t)}{x-z_i(t)}. \tag{8.51}$$

It is then easy to verify that this solves the nonlinear equation (8.50) provided the $z_i(t)$ obey the constraints

$$\ddot{z}_i(t) = 2\sum_{j\neq i}^{N} \frac{(1-\dot{z}_i(t))(1-\dot{z}_j(t))}{z_i(t)-z_j(t)}. \tag{8.52}$$

We could now proceed as in the previous section and try to solve this differential equation, but in this case we may appeal to the general solution already provided in [262], where it was found that

$$u(x,t) = \frac{f(x-t)}{1+tf(x-t)}, \tag{8.53}$$

solves (8.50) for any arbitrary function $f(x)$ with initial condition $u(x,0) = f(x)$. Comparing (8.53) and (8.51) it is clear that the $z_i(t)$ can be interpreted as the poles in (8.53), which becomes singular when $x \to z_i(t) = t + f_i^{-1}(-1/t)$, with $i \in \{1, N\}$ labeling the different branches which could result when assuming that f is invertible but not necessarily injectively. Making now the concrete choice for f to be rational of the form

$$f(x) = \sum_{i=1}^{N} \frac{a_i}{\alpha_i - x}, \qquad \text{with } \alpha_i, a_i \in \mathbb{C}, \tag{8.54}$$

we can determine the poles concretely by inverting this function. First of all we obtain from the initial condition that

$$z_i(0) = \alpha_i, \qquad \dot{z}_i(0) = 1 + a_i \qquad \text{and} \qquad \ddot{z}_i(0) = \sum_{j\neq i}^{N} \frac{2a_i a_j}{\alpha_i - \alpha_j}. \tag{8.55}$$

The first two conditions simply follow from the comparison of (8.53) and (8.51), but also follow, as so does the latter, from taking the appropriate limit in (8.54). We note that the total momentum is conserved for this system $\sum_{i=1}^{N} \dot{z}_i(t) = N + \sum_{i=1}^{N} a_i$.

Let us now see how to obtain explicit expressions for the poles. Inverting (8.54) for $N = 2$ it is easy to find that for generic values of t the poles take on the form

$$z_{1,2}(t) = t + \frac{\bar{\alpha}_{12}}{2} + \frac{\bar{a}_{12}}{2}t \pm \frac{1}{2}\sqrt{\alpha_{12}^2 + 2\alpha_{12}a_{12}t + \bar{a}_{12}^2 t^2}, \tag{8.56}$$

where we introduced the notation $\alpha_{ij} = \alpha_i - \alpha_j$, $\bar{\alpha}_{ij} = \alpha_i + \alpha_j$ and analogously for $\alpha \to a$. We note that in the case $N = 2$ the constraint (8.52) can be changed into two-particle Calogero systems constraint with the identification $g = a_1 a_2 \alpha_{12}^2$.

Next we consider the case $N = 3$ for which we obtain the solution

$$z_1(t) = t - \frac{a(t)}{3} + s_+(t) + s_-(t), \tag{8.57}$$

$$z_{2,3}(t) = t - \frac{a(t)}{3} - \frac{1}{2}\left[s_+(t) + s_-(t)\right] \pm \imath\frac{\sqrt{3}}{2}\left[s_+(t) - s_-(t)\right], \tag{8.58}$$

where we abbreviated

$$
\begin{aligned}
s_\pm(t) &= \Big[r(t) \pm \sqrt{r^2(t)+q^3(t)}\Big]^{1/3}, && (8.59)\\
r(t) &= \frac{9a(t)b(t)-27c(t)-2a^3(t)}{54}, \quad q(t)=\frac{3b(t)-a^2(t)}{9}, && (8.60)\\
a(t) &= -a_1-\alpha_2-\alpha_3-t(a_1+a_2+a_3), && (8.61)\\
b(t) &= \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3+t[a_1\bar{\alpha}_{23}+a_2\bar{\alpha}_{31}+a_3\bar{\alpha}_{21}], && (8.62)\\
c(t) &= -t(a_1\alpha_2\alpha_3+a_2\alpha_3\alpha_1+a_3\alpha_1\alpha_2)-\alpha_1\alpha_2\alpha_3. && (8.63)
\end{aligned}
$$

In terms of Jacobi's relative coordinates this becomes

$$
R(t)=t-\frac{1}{3}a(t), \quad r^2(t)=6s_+(t)s_-(t) \quad \text{and} \quad \tan\phi(t)=i\frac{s_-(t)-s_+(t)}{s_-(t)+s_+(t)}, \qquad (8.64)
$$

which makes a direct comparison with the constrained Calogero system (8.48) straightforward. As the system (8.57), (8.58) involves more free parameters than the constrained Calogero system (8.48), we expect to observe some relations between the parameters α_i, a_i to produce the right number of free parameters. Indeed, we find that for

$$
a_i=-\frac{g}{2}\prod_{j\neq i}(\alpha_i-\alpha_j)^{-2} \qquad (8.65)
$$

and the additional constraints

$$
c_0=\frac{1}{3}\sum_{i=1}^{3}\alpha_i, \quad c_1=\frac{2}{27}\prod_{\substack{1\leq j<k\leq 3\\ j,k\neq l}}(\alpha_j+\alpha_k-2\alpha_l), \quad g=4\sum_{\substack{i=1\\ i<j}}^{3}\alpha_i\alpha_j-\alpha_i^2, \quad \gamma=1. \qquad (8.66)
$$

the two systems become identical. Thus we have obtained an identical singularity structure for two quite different nonlinear wave equations.

8.6 Complex Calogero systems and $\mathcal{PT}$-symmetric deformations

As we have seen in section 8.4 the solution of the constrained Calogero models studied has intrinsically complex nature, so that the emergence of particles in $\mathcal{PT}$-symmetric wave equations in connection with these systems enforces the need to analyse other complex extensions of Calogero models which have appeared in the literature. In order to do so we use the fact that the Calogero Hamiltonian presented in (8.8) can be expressed in terms

of a standard $(n+1)$-dimensional representation of the roots α in the A_n Lie algebra root system Δ,

$$H_C = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i\neq j}^{N} V(q_i - q_j) = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{\alpha\in\Delta} V(\alpha\cdot q). \tag{8.67}$$

For the A_2 algebra the standard three-dimensional representation of the roots is given by

$$\alpha_1 = (1,-1,0)\ , \qquad \alpha_2 = (0,1,-1)\ , \qquad \alpha_3 = \alpha_1 + \alpha_2\ , \tag{8.68}$$

which implies the coordinate vector is also three-dimensional, $q = (q_1(t), q_2(t), q_3(t))$ with each of the components representing a different particle, so that the three-body problem analyzed is reproduced exactly.

The simplest $\mathcal{PT}$-deformation of any model is obtained just by adding a $\mathcal{PT}$-invariant term to the original Hamiltonian. For a many-body situation, this was proposed for the first time in the framework of A_n Calogero models by introducing the Hamiltonian [252]

$$H(q,p) = H_C(q,p) + \sum_{i\neq j}^{N} \frac{\imath\tilde{g}p_i}{(x_i - x_j)^2}. \tag{8.69}$$

where $\tilde{g}, \in \mathbb{R}$ and $q, p \in \mathbb{R}^\ell$. The representation-independent formulation of (8.69) was achieved in [255] and takes the form:

$$\begin{aligned} H_\mu &= \frac{1}{2}p^2 + \sum_{\alpha\in\Delta} g_\alpha V(\alpha\cdot q) + \imath\mu\cdot p \\ &= \frac{1}{2}(p+\imath\mu)^2 + \sum_{\alpha\in\Delta} \hat{g}_\alpha V(\alpha\cdot q) + \imath\mu\cdot p, \end{aligned} \tag{8.70}$$

with the definition of the new vector $\mu = \frac{1}{2}\sum_{\alpha\in\Delta}\tilde{g}_\alpha f(\alpha\cdot q)\alpha$, where $f(x) = \frac{1}{x}$, $V(x) = f(x)^2$ and new effective coupling constants re-expressed in terms of short and long roots Δ_s and Δ_l:

$$\hat{g}_\alpha = \begin{cases} g_s + \alpha_s\tilde{g}_s\ , & \alpha\in\Delta_s, \\ g_l + \alpha_l\tilde{g}_l\ , & \alpha\in\Delta_l. \end{cases} \tag{8.71}$$

The deformation just described has the advantage of (i) allowing, without difficulties, the construction of a Hermitian counterpart of the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian (8.70) through a similarity transformation $H_\mu = \eta^{-1}H_C\ \eta$ with $\eta = e^{-q\cdot\mu}$; and (ii) establishing the integrability of the system by the introduction of a new Lax pair $L_\mu(p) = L_C(p+\imath\mu)$ and $M_\mu = M_C$. Thus it was shown that this simply corresponds to shifting the momenta in the standard Calogero Hamiltonian together with a re-definition of the coupling constant.

The second type of deformation [256] consists of replacing directly the set of ℓ-dynamical variables $q = \{q_1, \ldots, q_\ell\}$ and their conjugate momenta $p = \{p_1, \ldots, p_\ell\}$ by means of a deformation map $(q, p) \to (\tilde{q}, \tilde{p})$, whereby the map is constructed in such a way that the original invariance under the Weyl group $\mathcal{W}$ is replaced by an invariance under a $\mathcal{PT}$-symmetrically deformed version of the Weyl group $\mathcal{W}^{\mathcal{PT}}$.

In terms of roots the map is defined by replacing each root α by a deformed counterpart $\tilde{\alpha}$, $\alpha \to \tilde{\alpha}$, whereby the precise form of the deformation ensures the invariance under $\mathcal{W}^{\mathcal{PT}}$ as specified in [256]. Expanding the momenta in terms of the roots as $p = \sum_i \kappa_i \alpha_i$, with $\kappa_i \in \mathbb{R}$, the Hamiltonian becomes

$$H_C(q,p) \to H_{\mathcal{PT}}(\tilde{q},\tilde{p}) = \frac{1}{2}\sum_{i,j} \kappa_i \kappa_j \tilde{\alpha}_i \tilde{\alpha}_j + \frac{1}{2}\sum_{\tilde{\alpha}\in\tilde{\Delta}} \frac{g}{(\tilde{\alpha}\cdot q)^2}. \tag{8.72}$$

Before continuing we need to specify the root deformations and although there are various possibilities to complexify Calogero systems we shall focus on those which preserve not only $\mathcal{PT}$-symmetry but also the algebraic symmetry underlying these Hamiltonians. In [256] the invariance of Calogero models under Weyl reflections of the root system was used to introduce $\mathcal{PT}$-symmetric reflections $\mathcal{W}^{\mathcal{PT}}$ which leave the deformed Calogero Hamiltonians invariant, imposing that the new roots will be mapped onto its negative by the $\mathcal{PT}$-extended Weyl reflections, $\bar{\sigma}_i \equiv \sigma_i \mathcal{T}$, with σ_i given by (4.10). The root extension is obtained by introducing an imaginary part to the roots in terms of a combination of the fundamental weights λ_i associated to the root system in such a way that it is orthogonal to the real part,

$$\alpha_i \to \tilde{\alpha}_i = R(\varepsilon)\alpha_i + \imath I(\varepsilon)\sum_{j\neq i} \varsigma_j \lambda_j, \tag{8.73}$$

and we recover the original undeformed roots α_i in the limit where the deformation parameter vanishes $\epsilon \to 0$: $\lim_{\epsilon\to 0} R(\epsilon) = 1, \lim_{\epsilon\to 0} I(\epsilon) = 0$. It is easy to see that $\tilde{\sigma}_i(\tilde{\alpha}_i) = -\tilde{\alpha}_i$, as desired. In order to find the concrete forms for $\tilde{q}$ and $\tilde{p}$ we need to be more specific about the algebras involved. Let us therefore examine the models based on the rank 2 algebras A_2, B_2 and G_2. Depending on the dimensionality of the representation for the simple roots, we obtain either a two or a three particle systems and may therefore compare with the solutions found in the previous sections. In all cases the deformations of the simple roots α_1 and α_2 take on the general form

$$\tilde{\alpha}_1(\varepsilon) = R(\varepsilon)\alpha_1 + \imath I(\varepsilon) K_{12}\lambda_2, \qquad \text{and} \qquad \tilde{\alpha}_2(\varepsilon) = R(\varepsilon)\alpha_2 - \imath I(\varepsilon) K_{21}\lambda_1, \tag{8.74}$$

with λ_1, λ_2 being fundamental weights obeying $\frac{2\lambda_i\cdot\alpha_j}{\alpha_j^2}=\delta_{ij}$, the functions $R(\varepsilon)$, $I(\varepsilon)$ satisfying $\lim_{\varepsilon\to 0} R(\epsilon)=1$, $\lim_{\varepsilon\to 0} I(\epsilon)=0$ and $K_{ij}=\frac{2\alpha_i\cdot\alpha_j}{\alpha_j^2}$ are the entries of the associated Cartan matrix. Let us now take the following two dimensional representations for the simple roots and fundamental weights

$$\begin{aligned}
A_2:&\quad \alpha_1=(1,-\sqrt{3}),\qquad \alpha_2=(1,\sqrt{3}),\qquad \lambda_1=\tfrac{2}{3}\alpha_1+\tfrac{1}{3}\alpha_2,\qquad \lambda_2=\tfrac{1}{3}\alpha_1+\tfrac{2}{3}\alpha_2,\\
B_2:&\quad \alpha_1=(1,-1),\qquad \alpha_2=(0,1),\qquad \lambda_1=\alpha_1+\alpha_2,\qquad \lambda_2=\tfrac{1}{2}\alpha_1+\alpha_2,\\
G_2:&\quad \alpha_1=-(\tfrac{3}{2},\tfrac{\sqrt{3}}{2}),\qquad \alpha_2=(1,0),\qquad \lambda_1=2\alpha_1+3\alpha_2,\qquad \lambda_2=\alpha_1+2\alpha_2.
\end{aligned}\tag{8.75}$$

We easily verify that this reproduces the correct entries for the Cartan matrices

$$K_{A_2}=\begin{pmatrix}2&-1\\-1&2\end{pmatrix}\ ,\quad K_{B_2}=\begin{pmatrix}2&-2\\-1&2\end{pmatrix}\ ,\quad K_{G_2}=\begin{pmatrix}2&-3\\-1&2\end{pmatrix}\ .\tag{8.76}$$

Having constructed the deformed roots we compute next the deformed conjugate momenta and coordinates. In the representations (8.75) the kinetic energy term changes just by an overall factor as

$$\tilde{p}^2=\left[R(\varepsilon)^2-\nu_{\mathbf{g}}^2 I(\varepsilon)^2\right]p^2\qquad\text{with}\quad \nu_{A_2}=\frac{1}{\sqrt{3}}\ ,\ \nu_{B_2}=1\ ,\ \nu_{G_2}=-\sqrt{3}.\tag{8.77}$$

The specific choice $R(\varepsilon)=\cosh\varepsilon$ and $I(\varepsilon)=\nu_{\mathbf{g}}^{-1}\sinh\varepsilon$, used in [256], keeps the kinetic energy term completely invariant, in the sense that the original and deformed momenta are identical. This deformation is equivalent to a complexification directly at the coordinates level, so that the dual canonical coordinates $\tilde{q}$ are computed from

$$\tilde{\alpha}\cdot q=\tilde{q}\cdot\alpha,\qquad \alpha,q\in\mathbb{R},\ \tilde{\alpha},\tilde{q}\in\mathbb{R}\oplus\imath\mathbb{R}.\tag{8.78}$$

We find

$$\tilde{q}_1=R(\varepsilon)q_1-\imath|\nu_{\mathbf{g}}|I(\varepsilon)q_2,\qquad\text{and}\qquad \tilde{q}_2=R(\varepsilon)q_2+\imath|\nu_{\mathbf{g}}|I(\varepsilon)q_1.\tag{8.79}$$

We will now argue that (8.79) is always different from the constrained two particle solution of the Calogero model (8.33). In order to see this we recall first of all that for the solution to be $\mathcal{PT}$-symmetric we require $\kappa,\tilde{\kappa}\in i\mathbb{R}$. Equating now the sums $x_1+x_2=\tilde{q}_1+\tilde{q}_2$ we conclude that

$$q_1(t)=-q_2(t)=-\frac{||\kappa||\nu_{\mathbf{g}}}{I(\varepsilon)}=\text{const.}\tag{8.80}$$

Next we compute $(x_1-x_2)^2$, which yields

$$\gamma(t-\tilde{\kappa})^2-3\frac{\alpha}{\gamma}=\left(2R(\varepsilon)q_1(t)\right)^2.\tag{8.81}$$

This equation is inconsistent as the right had side is real and time independent, a condition which can not be achieved for the left hand side. This proves our statement that the deformation method suggested here is genuinely different from the proposal in [256] in the two particle case.

But the Calogero A_n pairwise interaction is only mimicked if the dimension of the roots is $n+1$. Keeping the deformed roots to be of the form (8.74), the three dimensional representations for the simple roots

$$A_2 : \alpha_1 = (1,-1,0), \quad \alpha_2 = (0,1,-1), \qquad G_2 : \alpha_1 = (-2,1,1), \quad \alpha_2 = (1,-1,0), \tag{8.82}$$

yield the same result for the kinetic energy term (8.77), but obviously have to produce different dual canonical coordinates $\tilde{q}$. In this case we obtain

$$\tilde{q}_1 = R(\varepsilon)q_1 - \imath\zeta_{\mathbf{g}} I(\varepsilon)(q_2 - q_3), \tag{8.83}$$

$$\tilde{q}_2 = R(\varepsilon)q_2 - \imath\zeta_{\mathbf{g}} I(\varepsilon)(q_3 - q_1), \tag{8.84}$$

$$\tilde{q}_3 = R(\varepsilon)q_3 - \imath\zeta_{\mathbf{g}} I(\varepsilon)(q_1 - q_2), \tag{8.85}$$

where $\zeta_{A_2} = 1/3$ and $\zeta_{G_2} = 1$. Equating these solutions with (8.45), (8.46) and solving the resulting equations for the q_i with $i = 1,2,3$, it is easy to argue that the q_i can not be made real, which establishes the claim that the solutions are also intrinsically different for the three particle model. In fact, there are no three-dimensional pairs of roots new β_1, β_2 such that $\beta_i \cdot q = \alpha_i \cdot x$ with $x = (x_1(t), x_2(t), x_3(t))$. This occurs because we have a problem of mapping three complex functions, $x_1(t), x_2(t), x_3(t)$, into the three real coordinates, $q_1(t), q_2(t), q_3(t)$, appearing in the original Calogero problem.

Before the publication of [6] there had been two different types of procedures to complexify Calogero models. As explained in section 8.5 one may either add $\mathcal{PT}$-symmetric terms to the original Hamiltonian [252], which have turned out to be simple shifts in the momenta [255] or one may directly deform the root system on which the formulation of the model is based [256]. In all these approaches the deformation is introduced in a rather ad hoc fashion. In this paper we have provided a novel mechanism, which has real solutions of physically motivated nonlinear wave equations as the starting point. The constrained motion of some solitonic solutions of these models then led to complex Calogero models, some of them being $\mathcal{PT}$-symmetric.

There are some obvious open problems left. For instance it would naturally be very interesting to study systems involving larger numbers of particles, which would correspond

to higher soliton solutions for the nonlinear wave equations. Clearly the study of different types of wave equations, such as the KdV etc and their $\mathcal{PT}$-symmetrically deformed versions would complete the understanding.

Our analysis is schematically summarized in the figure below.

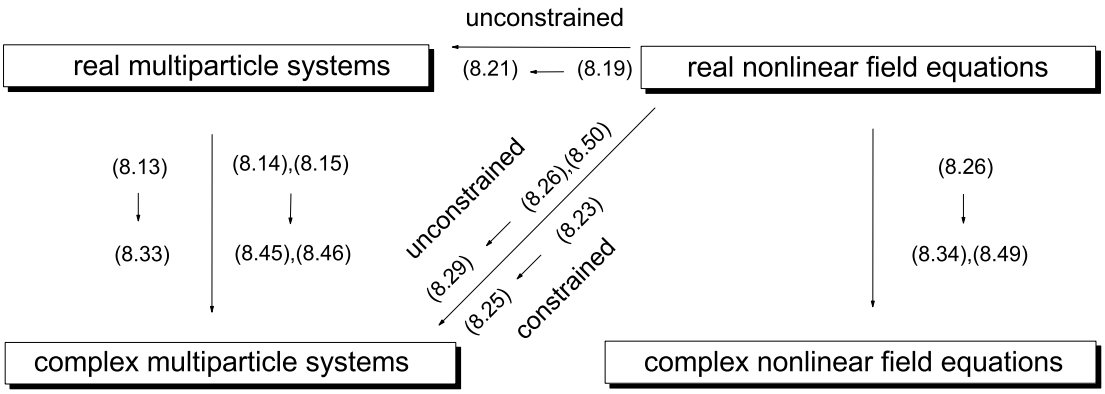

Figure 8.4: Schematic representation of the results obtained in [6].

9 Concluding remarks

In this thesis we have searched for exact results for theories described by non-Hermitian Hamiltonian operators in Quantum Physics and complex Hamiltonian functions in Classical Physics. Although it is widely believed that such systems can only be used to describe effective models, we have argued that they have achieved a more fundamental status in recent years. Motivated by the novel developments in $\mathcal{PT}$-symmetric Quantum Mechanics, due to the seminal work of Bender et al, we explore various aspects of non-Hermitian systems, from simpler formulations in finite dimensional Hilbert spaces to more elaborate systems of infinite dimension. The applicability of non-Hermitian Hamiltonians in the description of fundamental theories is achieved as isospectral transformations relating them to Hermitian counterparts are obtained. Such a process is usually followed by the construction of a new scalar product for the Hilbert space. We then use this freedom of choosing the metric defining the underlying quantum formalism in order to investigate the equivalence of certain Hamiltonian systems.

We have presented different possible approaches, for instance of unbroken $\mathcal{PT}$-symmetry, which is the invariance of both the Hamiltonian and its wavefunctions under simultaneous parity and time reversal transformation, and of quasi-pseudo-Hermiticity, when a positive definite metric can be used to construct an isospectral Hermitian counterpart. However, for either of the procedures, we have seen through examples in the thesis that establishing a non-Hermitian system as an acceptable physical theory involves considerable effort and exact results remain a small fraction of the cases studied so far.

Examples of problems for which the construction of a physical metric operator can be accomplished exactly are given by models which can be described in terms of Lie algebra generators. The underlying symmetry provided by the algebraic structure allows one to compute calculations involving noncommutative operators. We have considered Hamiltonians of $sl_2(\mathbb{R})$-Lie algebraic type and for some specific cases we constructed a

similarity transformation together with isospectral Hermitian counterparts. We indicated the difficulty these types of Hamiltonians pose with regard to the outlined programme, mainly due to the feature that the Hermitian conjugation does not close within the set of $sl_2(\mathbb{R})$-generators. Nonetheless, for specific realisations of the algebra the outlined programme may be carried out explicitly.

Instead, considering Hamiltonians of $su(1,1)$-Lie algebraic type, which generalize the Swanson model to some extent, circumvents these issues. We were able to construct systematically exact solutions for metric operators, which are of exponential form with arguments linear in the $su(1,1)$-generators. Our solutions fall into various subcases and are characterized by the constraints on the coupling constants in the model. In several cases we used the square root of the metric operator to construct the corresponding similarity transformation and its Hermitian counterparts. Alternatively we constructed the energy spectrum together with their corresponding eigenfunctions by means of generalized Bogoliubov transformations, which map the original Hamiltonians onto harmonic oscillator type Hamiltonians. The comparison between these two approaches exhibits agreement in some cases, but the overlap is not complete and we can obtain models which cannot be mapped to a harmonic oscillator type Hamiltonian by means of generalized Bogoliubov transformations and vice versa. On one hand this is probably due to our restrictive ansatz for the metric operator η by demanding it to be Hermitian and in addition assuming it to be of exponential form with arguments linear in the $su(1,1)$-generators. On the other hand we could of course also make a more general ansatz for the "target Hamiltonian" in the generalized Bogoliubov transformation approach.

In order to describe other models which cannot be formulated in terms of only $sl_2(\mathbb{R})$ or $su(1,1)$ algebra generators we have used other techniques to avoid solving more complicated operator equations. We have demonstrated that exploiting the isomorphism between operator and Moyal-like products allows one to convert the operator identities into manageable differential equations. This method was employed for the most generic cubic $\mathcal{PT}$-symmetric non-Hermitian Hamiltonians in the coordinate and momentum operators $\hat{x}, \hat{p}$ or equivalently the creation and annihilation operators $a, a^\dagger$. We systematically constructed various exact solutions for the metric operator of exponential form with $\mathcal{PT}$-symmetric, real and cubic argument. With these operators we were able to determine the corresponding similarity transformation and the corresponding Hermitian counterparts. Our solutions are characterised by various constraints on the ten parameters characterizing

the problem. Several of the quasi-solvable models may be reduced to previously studied systems, but some correspond to entirely new examples for quasi-solvable models.

Even when no obvious exact solution exists, perturbation theory can be carried out on the level of the differential equation to almost any desired order, but it usually remains unclear whether the perturbative series converges. There are obvious limitations for the Moyal bracket method as it works well only for potentials of polynomial form, as otherwise the differential equations will be of infinite order. In addition even for the differential equations of finite order the solutions have not been properly classified and one has to make various assumptions. A particularly important premise concerns the form of the ansatz for the metric. Some classes of solution are expected to be missed for an ansatz of a purely exponential form, as we have seen. The special cases considered, namely the single site lattice Reggeon model as well as the Jackiw model are both not quasi-exactly solvable within our framework. However, both of them may be understood as quasi-exactly solvable models perturbed by some complex cubic potential. Generalizations of the presented analysis to Hamiltonians of higher powers present no further complications and specifically interesting would be the study of a generic quartic Hamiltonian so that a precise comparison with the Lie-algebraic method could be carried out.

The mappings determined so far have presented considerable efforts. Even for situations where the mathematical structure can be considerably simpler, as is the case of two-level systems, so that calculations become more manageable, the effects of introducing non-Hermitian Hamiltonians as a fundamental description of nature are far from trivial. The quantum brachistochrone problem exemplifies well such a statement. The passage time needed for the evolution between two specified states has a lower bound τ, determined by the transition frequency between the corresponding energy levels, when the problem is formulated in terms of a Hermitian evolution generator. Nonetheless, when a non-Hermitian Hamiltonian is suitably coupled to the system the passage time can be made tunable. This means fast quantum transitions could be possible in an effect analogous to a wormhole phenomenon in the Hilbert space, with remarkable consequences from a practical point of view. Because this occurrence was first observed in $\mathcal{PT}$-symmetric models, $\mathcal{PT}$-symmetry was attributed to be the responsible feature for it.

However, we have demonstrated that similar effects can be reproduced also for systems in which the aforementioned symmetry is manifestly broken, i.e. dissipative systems. The fast evolutions obtained are clearly an effect of the coupling of a Hermitian with a non-

Hermitian system, which raises the question of how to combine various Hilbert spaces. Further developments regarding this problem have been intensively reported. Besides geometric analysis establishing, as expected, the equivalence between a completely Hermitian formulation and a completely non-Hermitian formulation, more recent results indicate that the quantum brachistochrone can be realized as subsystem of a larger Hermitian system living in a higher-dimensional Hilbert space. This is an example where theoretical advances come before those accomplished by experiments so that more direct evidences to clarify this problem still need to be obtained.

Because the elaboration of $\mathcal{PT}$-symmetric concepts originated in quantum mechanics, progress in this area has been achieved in a reverse fashion with respect to the usual advances of science, normally starting at a classical level. The use of $\mathcal{PT}$-symmetry ideas in classical theories have proven to be very fruitful, though. With special interest in integrable classical systems we have explored how such models can be deformed in a way so that $\mathcal{PT}$-symmetry is present. Furthermore, the search for the subclasses which still preserve integrability was pursued. As a consequence we have determined models of potential physical interest. Although integrability is usually established after an infinite number of conserved charges is computed or when a Lax pair is constructed, we opted for a more systematic approach capable of separating feasible candidates which are most likely integrable. With this purpose, we have carried out the Painlevé test for $\mathcal{PT}$-symmetric deformations of the Burgers equation and the KdV equation. When deforming both terms involving space derivatives, we found that the deformations of the Burgers equation pass the test. In specific cases we have also established the convergence of the series, such that these equations have in addition the Painlevé property. Based on the conjecture by Ablowitz, Ramani and Segur we take this as very strong evidence that these equations are integrable. Regarding these models as new integrable systems leads immediately to a sequence of interesting new problems related to features of integrability. It is very likely that these systems admit soliton solutions and it should be possible to compute the higher charges by means of other methods.

For the KdV equation our findings suggest that their $\mathcal{PT}$-symmetric deformations are not integrable, albeit they allow for the construction of a defective series. It would clearly be very interesting to investigate other $\mathcal{PT}$-symmetrically integrable systems in the described manner in order to establish their integrability. Nonetheless, we have shown that a different $\mathcal{PT}$-symmetric generalization of the KdV equation includes a family of

models possessing the Painlevé property. Remarkably the strong evidence of integrability in these systems are closely related to the existence of compacton solutions found in previous investigations. There, various criteria had been found, which separate the models into three distinct classes exhibiting qualitatively different types of compacton solutions: unstable compactons and stable compactons, which have either independent or dependent width and amplitude. We have demonstrated that the coexistence of solitons, present in integrable theories, and compactons can only happen in the latter situation.

Finally, we provided a novel method capable of generating complex $\mathcal{PT}$-symmetric many-body interacting systems. Rather than explicitly extending models to the complex plane, we have explored the possibilities of complex systems arising naturally from integrable nonlinear field equations. Because in some situations the poles in the wave solutions behave as interacting particles whose motion is constrained to a certain invariant submanifold with complex trajectories, $\mathcal{PT}$-symmetry might not have to be imposed but may emanate more naturally from real fields. This was accomplished as we investigated the pole structure of the Boussinesq equation and established that they behave as complex particles governed by constrained Calogero equations of motion. This observation was analyzed more closely for two and three particle systems, although more bodies could be investigated potentially through the systematic method developed. Solitonic solutions of the Boussinesq equation were obtained and the complex Calogero motion was shown to be inequivalent to previous complex deformations of Calogero systems found in the literature.

In conclusion this work confirmed the potential importance of studying non-Hermitian Hamiltonians, motivated by the recent interest in $\mathcal{PT}$-symmetric theories. The scope of such models is immense, covering possibly all fields of theoretical physics, with some illustrations presented here. We have no doubts investigations carried out by us, schematically presented in Figure 9.1, are of relevance to the community and hope they can inspire further developments.

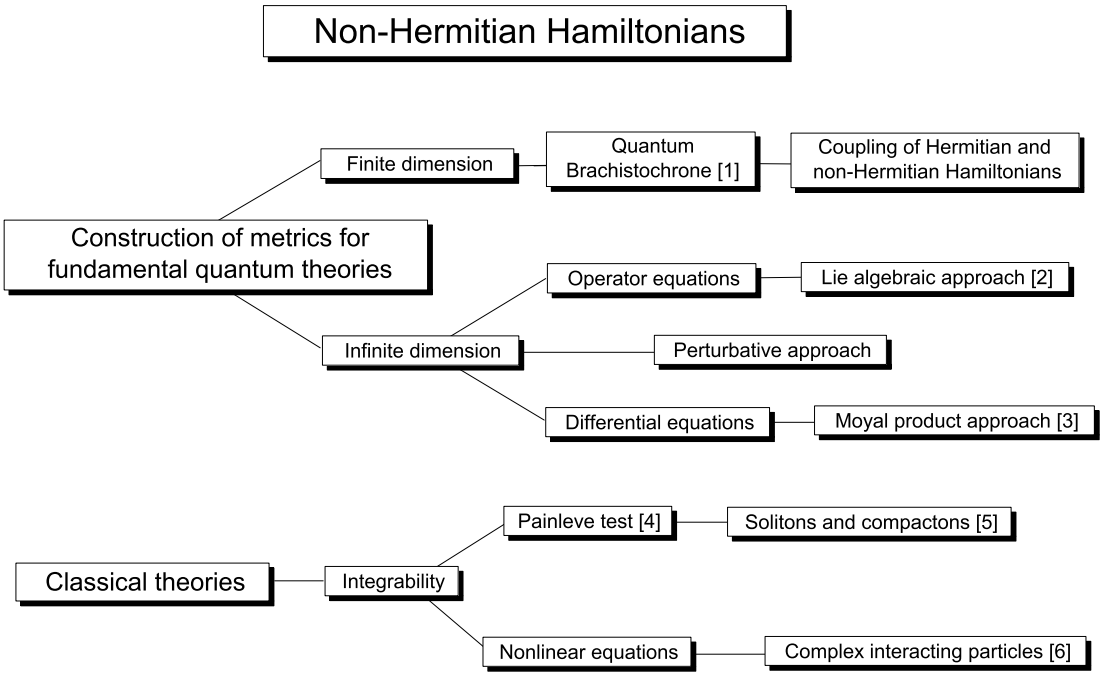

Figure 9.1: Graphic representation of our contributions to the field, both at quantum and classical levels.

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