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Surfaces of general type via $\mathbb{Q}$-Gorenstein smoothing

by

Sohail Iqbal

Thesis

Submitted to the University of Warwick

for the degree of

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Declarations

Chapter 1 is introduction to my work. Chapter 2 and first section of Chapter 3 are of an expository nature. Aside from this I declare that, to the best of my knowledge, the material contained in this thesis is original work of the author except where otherwise indicated.

Sohail Iqbal
18 April 2011

Chapter 1

Introduction

Interest in studying surfaces of general type goes back to the works of Enriques and Castelnuovo in the early 20th century. Apart from finding new examples it is also vital to study their deformation types for classifying surfaces of general type. Our objective is to study construction and deformation equivalence of general type surfaces by studying their canonical rings. Apart from using other existing techniques, we mainly use $\mathbb{Q}$-Gorenstein smoothings. In particular we achieve the following:

- Construction of simply connected surfaces of general type with $p_g = 3$ and $2 \leq K^2 \leq 8$,
- $\mathbb{Q}$-Gorenstein smoothings of Godeaux surface X with $\operatorname{Tors} X = \mathbb{Z}_4$ leads to constructions of Campedelli surface with torsion $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, and possibly $\mathbb{Z}_4$,
- Further application of the same kind of techniques to construct additional surfaces of interest. In some cases, we can justify these constructions in part from computer algebra although for which complete rigorous

mathematical proof is still lacking,

- Examples of Godeaux and Campedelli surfaces containing certain non-intersecting curves. These constructions might lead to further new constructions.

We also considered all other possibilities of linking the existing constructions of canonical rings of Godeaux and Campedelli surfaces using $\mathbb{Q}$-Gorenstein smoothing.

There are many methods for construction of general type surfaces including Campedelli's method, and Godeaux's method which was later extended by Reid to make it more efficient. In 2007 Lee and Park introduced another method of construction called $\mathbb{Q}$-Gorenstein smoothing theory. We use the later two methods in our constructions.

In 1931 Lucien Godeaux gave a way of constructing surfaces with $p_g = 0$ and $K^2 = 1$. The construction was given as quotient of a quintic surface in $\mathbb{P}^3$ by a $\mathbb{Z}_5$ group action. Surfaces with these invariants are now called (numerical) Godeaux surfaces. Later on in 1978 M. Reid expanded Godeaux's method by giving a systematic way of studying canonical models of surfaces of general type. For a surface S of general type Reid's method consist of describing the canonical ring of the etale covering $T \to S$ corresponding to $\operatorname{Tors} S \subset \operatorname{Pic}(S)$. The canonical ring

$$R(T, K_T) = \bigoplus_{n \geq 0} H^0(T, nK_T)$$

can be constructed by studying the $(\mathbb{Z} \oplus \operatorname{Tors} S)$-graded ring

$$R(S, K_S, \operatorname{Tors} S) = \bigoplus_{\substack{n \geq 0, \\ \sigma \in \operatorname{Tors} S}} H^0(S, nK_S + \sigma).$$

The canonical ring $R(Y, K_Y)$ is invariant under $\operatorname{Tors} S$-action and under this action

$$R(S, K_S) = R(T, K_T)^{\operatorname{Tors} S}.$$

Using this method Reid constructed canonical models of Godeuax surfaces as $\mathbb{Z}_5$, $\mathbb{Z}_4$ and $\mathbb{Z}_3$ quotients. Using a slight variation of this method Rebecca Barlow in 1980 constructed Godeaux surface with $\operatorname{Tors} S = \mathbb{Z}_2$ and 0. Later on in 2009 M. Mendes Lopes, R. Pardini, M. Reid proved that the constructions of general type surfaces with $p_g = 0$, $K^2 = 2$ and fundamental group of order 8 given by Godeaux and Reid give a complete classification of these surfaces. Many other constructions used this method but the problem of constructing all surfaces of general type remains unsolved.

In 2007 Y. Lee and J. Park introduced a new method of construction namely $\mathbb{Q}$-Gorenstein smoothings. The method consists of the following steps: first we force chains of curves, representing resolution graph of special quotient singularities, on a projective surface. In second step we contract these chains of curves to get a singular surface containing special quotient singularities, called T-singularities. In the final step we study the $\mathbb{Q}$-Gorenstein smoothing of the singular surface. Lee and Park used this technique to construct many families of surfaces which include: simply connected surfaces of general type with $p_g = 0$, $K^2 = 2$ [LP07], and surfaces of general type with $p_g = 0$, $K^2 = 2$, $H_1 = \mathbb{Z}_2$, $\mathbb{Z}_3$ [LP09]. Later on H. Park, J. Park, D. Shin used this method to construct many families of general type surfaces.

The reverse process of $\mathbb{Q}$-Gorenstein smoothing also gives us a useful method. Clearly in the reverse process: we first deform a projective surface in a one parameter family, called $\mathbb{Q}$-Gorenstein deformation, to get a surface with special quotient singularities. Then we resolve these singularities to get a new surface, the surface contains resolution graph of the special quotient singularities.

In general for varieties of any dimension there are many methods of construction but giving explicit construction of graded rings is advantageous. Firstly because there are many techniques available for construction and manipulation videlicet projection-unprojection, Hilbert series, key varieties etc. Secondly many classes of algebraic varieties can be studied via this method including surfaces of general type, del Pezzo surfaces, canonical 3-folds, and Fano 3-folds etc. The graded ring R associated to a variety X with an ample divisor D is given by

$$R(X, D) = \bigoplus_{m \in \mathbb{Z}} H^0(X, mD)$$

such that $X = \operatorname{Proj} R(X, D)$. To the study canonical models we replace D by K_X. Alongside the fact that great deal of information about a surface can be obtained by studying its canonical ring, It is also in our advantage that there are many techniques available to study them.

So far there have been no attempts to study canonical rings of surfaces using $\mathbb{Q}$-Goresnstein smoothing. In this thesis, apart from using other important techniques, we apply $\mathbb{Q}$-Gorenstein smoothing and $\mathbb{Q}$-Gorenstein deformation to study canonical rings of general type surfaces. As a result we get the construction of canonical rings for some general type surfaces. There follows an overview of this thesis, chapter by chapter.

Chapter 2 gives a brief introduction to graded rings, surfaces of general type, and the related machinery to be used in coming chapters.

Chapter 3 introduces Steiner n-folds. For our constructions of surfaces in Chapter 4 we needed Steiner 3-folds but we discovered that the construction can be generalized to give a relatively easy construction of Steiner n-folds. An interesting phenomenon in the construction of Steiner 3-folds is the appearance of a Kummer surface, the discovery and method is independent of the work of Richmond [HW80].

Chapter 4 presents a way to construct simply connected surfaces of general type with $p_g = 3$, and $2 \leq K^2 \leq 8$. The method is based on $\mathbb{Q}$-Gorenstein deformation expressed in terms of higher dimensional key varieties. These key varieties are constructed using Steiner 3-folds from Chapter 3.

Chapter 5 links Godeaux surfaces T with $\operatorname{Tors} T = \mathbb{Z}_4$ and Campedelli surfaces with torsion $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, and $\mathbb{Z}_4$. We mainly use $\mathbb{Q}$-Gorenstein smoothing and give explicit constructions of these Campedelli surfaces by writing out canonical rings by generators and relations. The rings are described as unprojections. The explicit constructions of surfaces are expressed as sections of higher dimensional key varieties. The basic idea of the "key variety technique" is to construct a large "simple" variety containing lots of interesting and complicated varieties, and our required variety is usually given as a linear or quadric section inside it. As key varieties we constructed Fano n-folds ranging from codimension 2 to 6. In future, the same path can be used to study varieties in codimension 5 and 6 especially Fano 3-folds, Clabi–Yau 3-folds and 3-folds of general type.

Chapter 6 proposes some constructions based on $\mathbb{Q}$-Gorenstein smoothing. We find exceptional T-divisors, namely (-4)-curves, on some surfaces. In the first section we discuss the case of a Godeaux surface with $\pi_1 = \mathbb{Z}_5$, and find $5 \times (-4)$-curves on a quintic in $\mathbb{P}^3$. Finding such T-divisor leads us to a construction of a Campedelli surface with $\pi_1 = \mathbb{Z}_5$. Although we have a code in Magma which provides us a construction of Campedelli with $\pi_1 = \mathbb{Z}_5$ attempts at studying their canonical rings have been frustrating. In the second section we find a (-4)-curve in a Godeaux–Reid surface. This should lead us to surfaces of general type with $p_g = 0$, $K^2 = 3$ and various different fundamental groups. We intend to return to questions like these in the future.

Chapter 2

Preliminaries

Here we present some basic properties of general type surfaces, graded rings, Hilbert series, and related machinery.

2.1 Surfaces of general type

General type surfaces are important because they form the largest class of surfaces, for example, most of the Hilbert modular surfaces, almost every surface obtained as a complete intersection, etc. For our discussion we need the following terminology.

For divisors D_1, D_2, $D_1 \cdot D_2$ denotes the intersection number. Two Cartier divisors D_1 and D_2 are numerically equivalent, denoted $D_1 \equiv D_2$, if

$$D_1 \cdot C = D_2 \cdot C$$

for every irreducible curve C. For a nonsingular projective surface, S, the canonical divisor K_S and sheaf ω_S are defined by

$$\omega_S = \wedge^2 \Omega_{k(S)/k} = \mathcal{O}_S(K_S).$$

The geometric genus p_g of S is defined

$$p_g = \dim H^0(S, \mathcal{O}_S(K_S)),$$

and the irregularity q of S is defined

$$q(S) = \dim H^1(S, \mathcal{O}_S(K_S)) = \dim H^0(S, \Omega_{k(S)/k}).$$

By Serre Duality we have

$$\chi(\mathcal{O}_S) = p_g(S) - q(S) + 1.$$

A sheaf $\mathcal{F}$ on variety X is said to be ample if there exist an integer $n \geq 1$ and an closed embedding $i\colon X \hookrightarrow \mathbb{P}^n$ such that $\mathcal{F}^{\otimes n} \cong i^*(\mathcal{O}_{\mathbb{P}^N}(1))$. In case $n = 1$ $\mathcal{F}$ is called very ample.

Riemann-Roch formula

$$\chi(\mathcal{O}_S(D)) = \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - KD),$$

for any divisor D on S.

Adjunction formula

For a nonsingular curve C of genus g on S

$$C^2 + C \cdot K_S = 2g - 2.$$

which is consequence of Riemann–Roch formula and the following short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

Some surface singularities

Let S be a normal surface and $P \in S$ an isolated singularity.

Rational Singularity: The point $P \in S$ is rational if for a resolution $f\colon T \to S$, we have $R^i f_* \mathcal{O}_T = 0$ for $i > 0$. The following equivalent characterizations are useful.

For every divisor D supported on $f^{-1}P$

1. $\chi(\mathcal{O}_D) \geq 1$;
2. $H^1(\mathcal{O}_D) = 0$;
3. $p_a D \leq 0$;
4. $\deg\colon H^1(\mathcal{O}_D^*) = \operatorname{Pic} D \xrightarrow{\sim} \mathbb{Z}^k$, where k is the number of components of D.

We know that for any coherent sheaf $\mathcal{F}$ on S we have

$$H^p(f^{-1}U, \mathcal{F}) = 0 \quad \text{for } p \geq 2\,,$$

so we have $P \in S$ is rational if and only if $R^1 f_* \mathcal{O}_T = 0$.

Du Val singularity: An important class of surface rational singularities are Du Val singularities or canonical surface singularities. There are many ways of describing Du Val singularities (cf: [Dur79]), two of the characterizations are

(1) Absolutely isolated double point: $P \in S$ is an rational surface singularity with multiplicity 2, and has a resolution $S_n \to S_{n-1} \to \cdots \to S_1 \to S$ such that each step $S_i \to S_{i-1}$ is blow up of an isolated double point.

(2) Canonical class: There exist a resolution of singularities $f\colon T \to S$ such that $K_T = f^* K_S$. The resolution f is called crepant resolution.

Cyclic quotient singularities: It is another important class of rational singularities. The point $p \in S$ is a cyclic quotient singulary of type $\frac{1}{r}(a_1, a_2)$ if $p \in S$ is locally analytically isomorphic to $\mathbb{A}^2/\mathbb{Z}_r$, where $\mathbb{Z}_r$ acts by $(x, y) \to (\varepsilon^{a_1} x, \varepsilon^{a_2} y)$, for ε a primitive rth root of unity. For more details see [Reic].

Canonical model

If X is a minimal surface of general type, then the only curves C with $K_X \cdot C = 0$ are (-2)-curves, and there are only finitely many of these. There is a map $f\colon X \to S$ contracting all (-2)-curves to Du Val points. Here S is called the *canonical model* of X, and X is the minimal resolution of S.

2.1.1 Invariants for surfaces

For the birational geometry of curves we need only the geometric genus to fully classify them. In the case of minimal surfaces apart from the geometric genus $p_g = H^0(X, \mathcal{O}(K_X))$ we also need the degree K^2 for the process of classification. These are called numerical invariants. An auxiliary invariant to distinguish between surfaces with the same numerical invariants is the algebraic fundamental group $\pi_1^{\text{alg}}(X)$.

Fundamental Group

The algebraic fundamental group $\pi_1^{\text{alg}}(X)$ of a variety X is the inverse limit of the Galois groups of the finite etale covers of X. If X, X' are nonsingular complex varieties and $f\colon X \to X'$ is a birational map between them then $\pi_1(X) \cong \pi_1(X')$. The same is not true for singular varieties in general. But if x be a rational singularity of the surface X and $f\colon S \to X$ is a minimal resolution, then $\pi_1(S) \cong \pi_1(X)$.

For surface X, consider the following short exact sequence

$$o \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0\,,$$

the long exact sequence related to above sequence, after simplification, is given as

$$0 \longrightarrow H^1(X;\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*) \longrightarrow H^2(X,\mathbb{Z})$$

$$\longrightarrow H^2(X,\mathcal{O}_X) \longrightarrow H^2(X,\mathcal{O}_X^*).$$

The above sequence is useful for calculating invariants for algebraic surfaces. For surfaces with finite fundamental group $\pi_1(X)$, the first Betti number is

$$\operatorname{Rank}_{\mathbb{R}} H^1(X;\mathbb{Z}) \otimes \mathbb{R} = 0$$

which gives us $q = H^0(X, \mathcal{O}_X) = 0$.

2.2 Quotients by group action

A finite group G act on variety a X by algebraic automorphisms. For $x \in X$, $g \in G$ we write $g(x)$ for the image of x under the action of g on X.

The *fixed locus* of $g \in G$ is the set $X^G = \{x \in X : g(x) = x\}$. The *elliptic* elements $g \in G$ are those with nonempty fixed loci. The *elliptic subgroup* of G, denoted by E is the subgroup generated by elliptic elements.

Let X be a normal variety with $\pi_1^{alg}(X) = 1$, (respectively, $\pi_1(X) = 1$). Let G be a finite group acting on X with elliptic subgroup E. Let $Y = X/G$. Then $\pi_1^{alg}(Y) = G/E$ (respectively, $\pi_1(Y) = G/E$).

2.3 Graded rings

Let X be an algebraic variety and D be an ample divisor on X. We are interested in finding the generators and relations of the graded ring

$$R(X, D) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD)).$$

Here D is called a polarization of X. The graded summands of $R(X, D)$ are defined as

$$H^0(X, nD) = \{f \in k(X) \big| \operatorname{div} f + nD \geq 0\},$$

with the multiplication map

$$H^0(X, nD) \times H^0(X, mD) \to H^0(X, (m + n)D)$$

which induces the grading on $R(X, D)$. In terms of graded ring the canonical model of a surface X is

$$S = \operatorname{Proj} R(X, K_X),$$

where $R(X, K_X)$ is the canonical graded ring of the surface X defined as

$$R(X, K_X) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}(nK_X)).$$

2.3.1 Hilbert series

The Hilbert series is a generating function which records the dimension of each summand of $R(X, D)$, and is defined as

$$P_{X,D}(t) = \sum_{n=0}^{\infty} h^0(X, nD)t^n.$$

The Hilbert series is a coarser invariant than the free resolution.

2.4 Chow ring

The Chow ring of a smooth algebraic variety X of dimension n is

$$A(X) = \bigoplus_{i=0}^{n} A^i(X),$$

where the grading is given by codimension, such that $A^i(X)$ is the codimension i algebraic cycles modulo algebraic equivalence. Consider

$$\begin{aligned} N^1(X) &= A^1(X)/\equiv\,, \\ N_1(X) &= A^{n-1}(X)/\equiv\,, \end{aligned}$$

where $\equiv$ is numerical equivalence. We denote the numerical equivalence class of σ by $[\sigma]$. The group $N^1(X)$ is also called Néron–Severi group. It is a torsion free group and is given as a quotient of a subgroup of $H^2(X;\mathbb{Z})$.

Remark 2.4.1 In the logical structure of Mori theory a surface, S, is of general type if K_S nef (numerically effective) and $K_S^2 > 0$. For reasons coming from birational classification of surfaces we are only interested in surfaces with K_X nef [Rei97].

Chapter 3

Steiner varieties

In this chapter we introduce Steiner varieties or Steiner n-folds, and a way of constructing them. The name (Steiner variety) is picked due to the similarity in construction with the Steiner surfaces, that are obtained as projections of the Veronese surface V_4 in $\mathbb{P}^5$. Our Steiner varieties are defined over an algebraically closed field k, for simplicity we take $k = \mathbb{C}$.

We use these varieties in our specific geometric situation in a subsequent chapter.

3.1 Introduction

Steiner n-folds are rational varieties, not always projectively Gorenstein. For the construction of Steiner n-folds we start by taking the second Veronese embedding of $\mathbb{P}^n$, that is $V_n = v_2(\mathbb{P}^n)$ in $\mathbb{P}^N$ where $N = \binom{n+2}{2} - 1$. In the second step we choose a set of points on this Veronese variety. After successive projection from these points we get Steiner n-folds.

Definition 3.1.1 A Steiner n-fold W_d^n of degree d $(< 2^n)$ is a variety obtained by successively projecting the Veronese variety $v_2(\mathbb{P}^n)$ from a set of points $\{p_1, \ldots, p_{2^n-d}\}$ in general position.

We omit the dimension superscript in W_d^n whenever it is obvious. We start with an exposition on Veronese varieties.

3.1.2 Veronese Variety

Although we are interested in $v_2(\mathbb{P}^3)$, we describe the general Veronese varieties. A Veronese n-fold $v_r(\mathbb{P}^n)$ is the image of $\mathbb{P}^n$ under the Veronese map of degree r, which is a closed immersion of $\mathbb{P}^n$ using the line bundle $\mathcal{O}_{\mathbb{P}^n}(r)$, for some positive integer r. The defining equations of $v_r(\mathbb{P}^n)$ are always quadrics. To see these quadrics consider $\mathbb{P}^n_{x_0,x_1,\ldots,x_n}$ and set $H := \mathcal{O}_{\mathbb{P}^n}(r)$. The linear system $|H|$ embeds $\mathbb{P}^n$ as an n-fold $v_r(\mathbb{P}^n) := \Phi_{|H|}(\mathbb{P}^n) \subset \mathbb{P}^{N-1}$, where $N = \binom{n+r}{r}$. Give the homogeneous coordinates of $\mathbb{P}^{N-1}$ the lexicographic ordering, that is, the coordinates are $u_{i_0 i_1 \ldots i_n}$ so that Φ is defined by

$$u_{i_0 i_1 \ldots i_n} = x_0^{i_0} x_1^{i_1} \ldots x_n^{i_n}, \text{ whenever } \quad i_0 + i_1 + \cdots + i_n = r,$$

in this setting the defining equations of $v_r(\mathbb{P}^n)$ are given by

$$u_{i_0 i_1 \ldots i_n} u_{j_0 j_1 \ldots j_n} = u_{k_0 k_1 \ldots k_n} u_{l_0 l_1 \ldots l_n}, \tag{3.1.3}$$

such that

$$i_0 + j_0 = k_0 + l_0, i_1 + j_1 = k_1 + l_1, \ldots, i_n + j_n = k_n + l_n.$$

The case of $r = 2$ has a very nice determinantal representation of quadrics. This means that $v_2(\mathbb{P}^n)$ is defined by the 2×2 minors of the following symmetric matrix

$$M^n = \begin{pmatrix} u_{00} & u_{01} & u_{02} & \cdots & u_{0n} \\ & u_{11} & u_{12} & \cdots & u_{1n} \\ & & u_{22} & \cdots & u_{2n} \\ \text{Sym} & & & \ddots & \vdots \\ & & & & u_{nn} \end{pmatrix}, \tag{3.1.4}$$

where

$$u_{kl} = x_k x_l \qquad \text{for} \quad 0 \le k \le l \le n. \tag{3.1.4.1}$$

Invariants of $v_r(\mathbb{P}^n)$

The Hilbert polynomial of $v_r(\mathbb{P}^n)$ is given by

$$P_{v_r(\mathbb{P}^n)}(t) = \binom{n + tr}{tr},$$

and the leading term by

$$\frac{t^n r^n}{n!},$$

hence $\deg v_r(\mathbb{P}^n) = r^n$. The remaining invariants can be understood by the fact that the Veronese variety $v_r(\mathbb{P}^n)$ is isomorphic to $\mathbb{P}^n$.

3.1.5 Projection

For a projective variety $V \subset \mathbb{P}^n$ and a point p on V, the linear projection from p gives the image $\pi_p(V)$ of V on a hyperplane $H \subset \mathbb{P}^n$ not containing p, where

π_p is defined as

$$q \in V \xrightarrow{\pi_p} (L_1(q), \ldots, L_n(q)),$$

with $L_1, L_2, \ldots, L_n$ linear forms defining the point p. In terms of the homogeneous coordinate ring $k[V]$, projection from p is a simple elimination of a carefully chosen variable from $k[V]$. Let $S := \pi_p(V)$ be the image of the projection of V from p. Then

$$\deg(V) = m + d \deg(S),$$

where m is the multiplicity of V at p, and d is the degree of the map π_p. We are interested in the cases when $m = 1$, that is p is a nonsingular point of V. Roughly speaking, after projection, the point p is replaced by the projectivization of its tangent space. The map π_p extends uniquely to a morphism ϕ. If $\sigma_p \colon \widetilde{V} \to V$ is the blowup of V at the point p then we have

$$\begin{tikzcd} \widetilde{V} \arrow[d, "\sigma_p"'] \arrow[dr, "\phi"] & \\ V \arrow[r, dashed, "\pi_p"] & S \subset \mathbb{P}^{n-1}\,. \end{tikzcd} \tag{3.1.6}$$

The image E under π_p of the projectivised tangent space $\mathcal{T}_p$ at $p \in V$ is the same as the image under ϕ of the exceptional divisor $\sigma_p^{-1}(p) \subset \widetilde{V}$.

3.1.7 Points In General Position

Let $\{q_1, q_2, \ldots, q_m\}$ be a set of points on the Veronese n-fold $v_2(\mathbb{P}^n)$. If

$$\pi \colon v_2(\mathbb{P}^n) \dashrightarrow X$$

is the successive projection from these points then X has singularities coming from the projection but we do not want any other singularities on X. In other words we want a set of points $\{q_1, q_2, \ldots, q_m\}$ such that X does not get unexpected singularities. Such a set is called an *unnodal point set* in Dolgachev and Ortland [DO88]. Since the Veronese variety $v_2(\mathbb{P}^n)$ is isomorphic to $\mathbb{P}^n$, it is enough to find conditions on points in $\mathbb{P}^n$. We find the conditions by considering the blowup X_m of $\mathbb{P}^n$ at points $\{q_1, q_2, \ldots, q_m\}$. For practical purposes we follow the methods of Coble [Cob82] to find conditions. The conditions are then given in terms of the effective cone of X.

Discriminant Conditions

Let us denote $X_m = \mathcal{B}l_{q_1,q_2,\ldots,q_m}\mathbb{P}^n$, which then gives the following

$$X_m \xrightarrow{\sigma_m} X_{m-1} \xrightarrow{\sigma_{m-1}} \cdots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 = \mathbb{P}^n,$$

where σ_i is the blowup at q_i. Now consider the bilattice

$$N(X_m) = (N^1(X_m), N_1(X_m)),$$

called the Néron – Severi bilatice of X_m. Here $N^1(X_m)$ and $N_1(X_m)$ are abelian groups of finite rank equipped with a pairing

$$N^1(X_m) \times N_1(X_m) \to \mathbb{Z}\,,$$

defined by the intersection of cycles, that is, given by

$$(D, C) \mapsto D \cdot C.$$

In fact

$$N^1(X_m) = \mathbb{Z}H_0 + \mathbb{Z}H_1 + \cdots + \mathbb{Z}H_m\,,$$

and

$$N_1(X_m) = \mathbb{Z}L_0 + \mathbb{Z}L_1 + \cdots + \mathbb{Z}L_m\,,$$

where

$$\begin{aligned} H_0 &= (\sigma_m \circ \ldots \sigma_1)^{-1}(h_0), \quad \text{where } h_0 \text{ is a hyperplane in } \mathbb{P}^n \\ H_i &= (\sigma_i \circ \ldots \sigma_1)^{-1}(q_i), \quad \text{for } i = 1, \ldots, m\,, \end{aligned}$$

and

$$\begin{aligned} L_0 &= (\sigma_m \circ \ldots \sigma_1)^{-1}(l_0) \quad \text{where } l_0 \text{ is a line in } \mathbb{P}^n \\ L_i &= (\sigma_i \circ \ldots \sigma_1)^{-1}(l_i) \quad l_i \text{ is a line in } \sigma_i^{-1}(q_i)\, i = 1, \ldots, m\,. \end{aligned}$$

Moreover

$$\begin{aligned} H_0 \cdot L_0 &= 1, \\ H_i \cdot L_i &= -1 \quad \text{for } i \neq 0, \\ H_i \cdot H_j &= 0 \quad \text{for } i \neq j\,. \end{aligned}$$

Consider the hyperbolic lattice Ξ given by

$$\Xi = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_m,$$

where

$$\begin{aligned} e_0 \cdot e_0 &= 1, \\ e_i \cdot e_i &= -1, \quad \text{for } i \neq 0, \\ e_i \cdot e_j &= 0, \qquad \text{for } i \neq j. \end{aligned}$$

The Neron-Severi bilattice is isomorphic to (Ξ, Ξ) by the following map

$$\Phi = (\Phi^1, \Phi_1)\colon (\Xi, \Xi) \to (N^1(X_m), N_1(X_m)), \tag{3.1.8}$$

where

$$\begin{aligned} \Phi^1(e_i) &= H_i \quad \text{for } i = 0, \ldots, m, \\ \Phi_1(e_i) &= L_i \quad \text{for } i = 0, \ldots, m. \end{aligned}$$

A *root system* in a bilattice $L = (M_1, M_2)$ is a pair $(B, \check{B})$, where B and $\check{B}$ are subsets of M_1 and M_2 respectively, such that there is a bijection $B \to \check{B}$,

$\alpha \mapsto \check{\alpha}$, satisfying

$$\begin{aligned} \alpha \cdot \check{\alpha} &= -2, \\ \alpha \cdot \check{\beta} &\geq 0, \quad \text{for } \alpha, \beta \in B, \quad \alpha \neq \beta. \end{aligned}$$

A root system is *symmetric* if

$$\alpha \cdot \check{\beta} = \beta \cdot \check{\alpha}, \quad \text{for } \alpha, \beta \in B.$$

For $\alpha \in B$ the maps

$$\begin{aligned} s_\alpha \colon x_1 &\to x_1 + (x \cdot \check{\alpha})\alpha, \quad \text{for any } x_1 \in M_1, \\ \check{s}_\alpha \colon x_2 &\to x_2 + (x_2 \cdot \alpha)\check{\alpha}, \quad \text{for any } x_2 \in M_2, \end{aligned}$$

define linear involutions on M_1 and M_2 respectively. These involutions generate a subgroup W_B and $W_{\check{B}}$ of $\mathrm{GL}(M_1)$ and $\mathrm{GL}(M_2)$ respectively. In fact W_B and $W_{\check{B}}$ are isomorphic under $s_\alpha \mapsto s_{\check{\alpha}}$. We denote the W_B-orbit of B in M_1 (respectively of $\check{B}$ in M_2) by R_B (respectively $R_{\check{B}}$). The set R_B is partitioned as

$$R_B = R_B^+ \bigsqcup R_B^-,$$

where R_B^+ (respectively R_B^-) is the set of those elements R_B which can be written as linear combination of elements of B with non-negative (respectively non-positive) integral coefficients. Moreover we have

$$R_B^- = \{-\alpha \colon \alpha \in R_B^+\}.$$

For $m \geq n+1 \geq 3$ we have the *canonical root basis of type* $n > 1$ in the hyperbolic lattice Ξ given by

$$\begin{aligned} B_n &= \{\alpha_0, \ldots, \alpha_{m-1}\}, \\ \check{B}_n &= \{\check{\alpha}_0, \ldots, \check{\alpha}_{m-1}\}, \end{aligned}$$

where

$$\begin{aligned} \alpha_0 = &\quad e_0 - e_1 - \cdots - e_{n+1}, & \alpha_i = e_i - e_{i+1} \quad \text{for } i = 1, \ldots, m-1, \\ \check{\alpha}_0 = &\quad (n-1)e_0 - e_1 - \cdots - e_{n+1}, & \check{\alpha}_i = \alpha_i = e_i - e_{i+1} \quad \text{for } i = 1, \ldots, m-1. \end{aligned}$$

For the isomorphism of bilattices $\Phi\colon (\Xi, \Xi) \to (N^1(X_m), N_1(X_m))$, given in 3.1.8, we set

$$\begin{aligned} R_B(\Phi)^+ &= \{\alpha \in R_B : \Phi^1(\alpha) \text{ is effective }\}, \\ R_{\check{B}}(\Phi)^+ &= \{\alpha \in R_{\check{B}} : \Phi_1(\alpha) \text{ is effective }\}. \end{aligned}$$

The elements of $R_B(\Phi)^+$ and $R_{\check{B}}(\Phi)^+$ are called effective (or nodal) B-roots and $\check{B}$-roots respectively. The following is helpful in our further discussion

$$R_B(\Phi)^+ \subset R_B^+. \tag{3.1.9}$$

Conditions

We are now in a position to state the discriminant conditions. A point set $\{q_1, q_2, \ldots, q_m\}$ in $\mathbb{P}^n$ is unnodal if $R_B(\Phi)^+ = \emptyset$. As $R_B(\Phi)^+ \subset R_B^+$ we try to find description of R_B^+. A partial description of R_B^+ is given by:

Proposition 3.1.10 *[DO88] If $\alpha = a_0e_0 - a_1e_1 - \cdots - a_me_m$ is an element of R_B^+ then $\alpha_0 \geq 0$. If $a_0 = 0$, then $\alpha = e_i - e_j$ for some $1 \leq i < j \leq m$. If $a_0 > 0$ then the elements α satisfy the following conditions*

1. $a_i \geq 0$, *for all* i,
2. $(n-1)a_0^2 - a_1^2 - \cdots - a_m^2 = -2$,
3. $(n+1)a_0 - a_1 - \cdots - a_m = 0$,
4. $(n-1)a_0 < a_{i_1} + \cdots + a_{i_{n+1}}$ *if* $a_{i_1} \geq \cdots \geq a_{i_m}$, $i_j \in \{1, \ldots m\}$,
5. $(n-1)a_0 \geq a_{i_1} + \cdots + a_{i_n}$ *if* $a_0 > 1$, $a_{i_1} \geq \cdots \geq a_{i_m}$.

Example 3.1.11 In case of $\mathbb{P}^2$ a point set $\{p_1, p_2, ..., p_8\}$ is unnodal in $\mathbb{P}^2$ if no two points coincide, no three points are collinear, no six on a conic, no eight of them on a cubic having a node at one of them. In fact in the case of $n = 2$ and $m = 8$ the set R_B^+ is described by

1. $\alpha = e_i - e_j$, $1 \leq i < j \leq m$,
2. $\alpha = e_0 - e_i - e_j - e_k$, $i \neq j \neq k$, $i, j, k \neq 0$,
3. $2e_0 - e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5} - e_{i_6}$, $1 \leq i_k \leq i_s \leq m$ if $k < s$,
4. $3e_0 - e_1 - \cdots - e_8 - e_i$, $1 \leq i \leq 8$,

and to get $R_B(\Phi)^+ = \emptyset$ we impose a sufficient condition, that is, $R_B^+ = \emptyset$. This means we want none of the above conditions to be satisfied by the e_i.

Corollary 3.1.12 A point set in $\mathbb{P}^3$, with at most 7 points, is unnodal if

1. No two points are infinitely close.

2. No three points collinear.

3. No four points lie on a plane.

4. In case of 7 points, the points does not lie on a quadric with a singularity at one of these points.

Proof Using the above proposition the condition on R_B^+-roots in case of $\mathbb{P}^3$, that is, in the case of $n = 3$ and $m = 7$ are

$$\alpha = e_0 - e_i - e_j - e_k - e_l, \quad \text{where } 1 \leq i < j < k \leq 7,$$

$$\alpha = 2e_0 - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_i \quad \text{where } 1 \leq i \leq 7.$$

The conditions follow from above. $\square$

Remark 3.1.13 In fact discriminant conditions are required so that $-\frac{1}{2}K_{X_m}$ is ample. In the case of X_m we have

$$-K_{X_m} = (n+1)H - (n-1)H_1 - \cdots - (n-1)H_m.$$

Consider, for example, the case of $X_4 = \mathcal{B}l_{q_1,q_2,q_3,q_4}\mathbb{P}^3$ and assume the four points $q_1, \ldots, q_4$ lie on a plane, that is there is a divisor

$$D := H - H_1 - H_2 - H_3 - H_4 \in N^1(X_4),$$

and since

$$-\frac{1}{2}K_{X_4} = 2H + H_1 + \cdots + H_4,$$

so we have

$$(-\frac{1}{2}K_{X_4})^2 = 4H^2 + H_1^2 + \cdots + H_4^2 \in NE(X_4).$$

implying

$$(-\frac{1}{2}K_{X_4})^2 \cdot D \not> 0\,.$$

Similarly we can see in other cases. So we have a way of imposing conditions on points $q_1, \ldots, q_m \in \mathbb{P}^3$ such that $-K_{X_m}$ is nef. Also it can be seen that divisor $-\frac{1}{2}K_{X_m}$ is big if $m < 2^n$. It will be interesting to find these conditions without the use of root system.

3.2 Steiner n-folds

To construct a Steiner n-fold W^n_d of degree d ($< 2^n$) we start with the Veronese n-fold $v_2(\mathbb{P}^n)$. For simplicity we write $v_2(\mathbb{P}^n) = W_{2^n}$. In fact

$$W_{2^n} = \operatorname{Proj} R(\mathbb{P}^n, lK_{\mathbb{P}^n}),$$

where

$$R(\mathbb{P}^n, lK_{\mathbb{P}^n}) = k[u_{ij}]/r, \quad \text{for } 0 \le i \le j \le n, \tag{3.2.1}$$

here $l = \frac{-2}{n+1}$, the variables are u_{ij} are same as in (3.1.4.1), and r is the ideal generated by the 2×2 minors of the matrix 3.1.4.

From W_{2^n} we get W_{2^n-1}, $W_{2^n-2}\,, \ldots$ by projecting W_{2^n} successively from an unnodal point set with cardinality $1, 2, \ldots$ respectively.

It is evident that the coordinate points of $\mathbb{P}^n$ form an unnodal point set. In order to understand projections from coordinate points it is easy to use the following:

Proposition 3.2.2 *The projection of $W_{2^n} \subset \mathbb{P}^{N-1}$ from a coordinate point with $u_{pp} = 1$, is given by all but except those 2×2 minors of the matrix M^n*

(3.1.4) involving u_{pp}.

A Steiner n-fold is Gorenstein for n odd [DNH97, pages 634-635]. In dimension 1 the Steiner varieties can be completely understood by the fact that they are isomorphic to $\mathbb{P}^1$. In the following we discuss the two and three dimensional cases one by one.

3.2.3 Steiner surfaces

Recall that W_{2^2} is the Veronese surface defined by 2×2 minors of the matrix

$$M^2 = \begin{pmatrix} u_{00} & u_{01} & u_{02} \\ & u_{11} & u_{12} \\ \text{Sym} & & u_{22} \end{pmatrix}. \tag{3.2.4}$$

We can take three points to be the images of the coordinate points of $\mathbb{P}^2$ under the Veronese map. Projecting once and twice gives us $W_3 \subset \mathbb{P}^4$, $W_2 \subset \mathbb{P}^3$ respectively, where W_2 is given by

$$u_{01}u_{22} - u_{02}u_{12},$$

which is a nonsingular quadric which is isomorphic to Del Pezzo surface of degree 8. It is interesting to note that if the points are not in general position then this leads to singular Steiner surfaces and a complete classification is given in [CSS96].

3.3 Steiner 3-folds

The defining equations of W_8 are the 2×2 minors of the following symmetric matrix:

$$M^3 = \begin{pmatrix} u_{00} & u_{01} & u_{02} & u_{03} \\ & u_{11} & u_{12} & u_{13} \\ \text{Sym} & & u_{22} & u_{23} \\ & & & u_{33} \end{pmatrix}. \tag{3.3.1}$$

To construct the spaces W_{8-i}, for $1 \leq i \leq 7$, we project W_8 successively from an unnodal point set of cardinality i. We choose and order the points to be $p_i := v_2(q_i)$, where the q_i are given by

$$\begin{aligned} &q_1 = (1:0:0:0), \quad q_2 = (0:1:0:0), \quad q_3 = (0:0:1:0), \\ &q_4 = (0:0:0:1), \quad q_5 = (1:1:1:1), \end{aligned} \tag{3.3.2}$$

and

$$q_6 = (\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3), \quad q_7 = (\beta_0 : \beta_1 : \beta_2 : \beta_3), \tag{3.3.3}$$

such that none of the 2×2 minors of the following matrix vanish

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{pmatrix}, \tag{3.3.4}$$

and the q_i satisfy the conditions of corollary (3.1.12). Let

$$\pi_i \colon W_{9-i} \dashrightarrow W_{8-i}, \quad \text{for } 1 \leq i \leq 7, \tag{3.3.5}$$

be the projection map from the first i points. If

$$\mu_i \colon \widetilde{W}_{8-i} \to W_8, \quad \text{for} \quad 1 \leq i \leq 7, \tag{3.3.6}$$

is the blowup at the first i points then the π_i extend to morphisms

$$\widetilde{\pi}_i \colon \widetilde{W}_{8-i} \to W_{8-i}, \quad \text{for} \quad 1 \leq i \leq 7, \tag{3.3.7}$$

so that the following diagram commutes

$$\begin{tikzcd} \widetilde{W}_{8-i} \arrow[d, "\mu_i"] \arrow[dr, "\widetilde{\pi}_i"] & \\ W_8 \arrow[r, dashed, "\pi_i \circ \pi_{i-1} \ldots \circ \pi_1"] & W_{8-i}\,, \end{tikzcd} \tag{3.3.8}$$

for $1 \leq i \leq 7$. In this case the map $\widetilde{\pi}_i$ is given by the linear system of global sections of $\mathcal{O}_{\widetilde{W}_{8-i}}(-\frac{1}{2}K)$. Let

$$\sigma_i \colon X_i \to \mathbb{P}^3, \quad \text{for} \quad 1 \leq i \leq 7, \tag{3.3.9}$$

be the blowup of $\mathbb{P}^3$ at first i points from the set $\{q_1, q_2, \ldots, q_7\}$. As $W_8 \cong \mathbb{P}^3$ we have $X_i \cong \widetilde{W}_{8-i}$ for $1 \leq i \leq 7$. With these notation the W_{8-i} are given by:

$$W_{8-i} = \operatorname{Proj} R(X_i, -\frac{1}{2}K_{X_i}). \tag{3.3.10}$$

for $1 \leq i \leq 6$. If

$$H_i := (\sigma_i \circ \sigma_{i-1} \circ \ldots \sigma_1)^{-1}(q_i)$$

then

$$K_{X_i} = -(4)H_0 + 2H_1 + \cdots + 2H_i\,,$$

where H_0 is a hyperplane in $\mathbb{P}^3$. Now we discuss all Steiner 3-folds one by one. In other words we explain the seven projections of W_8 step by step, and we see the appearance of exceptional planes during this process.

Stage I: W_7

After projecting once we get W_7, which is defined by all the 2×2 minors of M^3 except those involving u_{00} by Proposition (3.2.2). Also W_7 can be considered as a $\mathbb{P}^1$-bundle $\mathbb{F}$ over $\mathbb{P}^2$ given by

$$\mathbb{F} \cong \mathbf{P}(\mathcal{E}), \tag{3.3.11}$$

where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$. Indeed $\mathcal{E}[\mathbb{P}^2] = k\,\langle 1, t_1, t_2, t_3\rangle$, and so

$$\mathrm{Sym}(\mathcal{E}) = k[z_1, z_2, z_3, t_1, t_2, t_3]/(z_i t_j - z_j t_i), \quad \text{for } i, j \in \{1, 2, 3\}. \tag{3.3.12}$$

Together with the fact that $\mathbf{P}(\mathcal{E}) \cong \mathbf{P}(\mathcal{E}\otimes\mathcal{O}(1))$ the map $\widetilde{\pi}_1$ is an isomorphism. Consider the following to get more insight

$$\begin{tikzcd} & \widetilde{W_7} \cong \mathcal{B}l_{p_0}\mathbb{P}^3 \arrow[dl, "\sigma_1"'] \arrow[d, "\mu_1"] \arrow[dr, "\widetilde{\pi}_1"] & \\ \mathbb{P}^3 \arrow[r, "\cong"] & W_8 \arrow[r, dashed, "\pi_1"] & W_7\,. \end{tikzcd} \tag{3.3.13}$$

In this case we get one exceptional plane $E_1 \subset W_7$.

Stage II: W_6

The case where we project from two points is different from the above because in this case $\widetilde{\pi}_2$ (given in 3.3.8) is not an isomorphism. In fact for $i > 1$ the map $\widetilde{\pi}_i$ is not an isomorphism.

Inside W_6 we have two planes namely E_1 and E_2, the exceptional planes. These planes intersect at a single point

$$E_1 \cap E_2 = \{p_{12}\},$$

where p_{12} is a coordinate point with $u_{01} = 1$. In fact p_{12} is a double point. To see this consider $L \subset \mathbb{P}^3$, the line joining two points q_1 and q_2. Let $L = A_1 \cap A_2$, where A_1 and A_2 are hyperplanes in $\mathbb{P}^3$. For

$$L \subset A_1 \subset \mathbb{P}^3$$

consider

$$0 \to \mathcal{N}_{L/_{A_1}} \to \mathcal{N}_{L/_{\mathbb{P}^3}} \to \mathcal{N}_{A_1/_{\mathbb{P}^3}}\big|_L \to 0$$

[GD, EGA IV 16.2.7 & 16.9.13], then the normal bundle of L is given by

$$\mathcal{N}_{L/_{\mathbb{P}^3}} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

Consider the blowup X_2 of $\mathbb{P}^3$ at q_1 and q_2. Let (using the notation of 3.3.9) $\widetilde{L}$ and $\widetilde{A}_1$ be the strict transforms, under the blowup map σ_2, of L and A_1 respectively. Using the same argument as above we get

$$\mathcal{N}_{\widetilde{L}/X_2} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

In the case of W_6 we have

$$W_6 = \operatorname{Proj} R(X_2, -\frac{1}{2}K_{X_2}).$$

Hence the proper transform $\widetilde{L}$ of L gets contracted to a double point in W_6. There is also another way of proving that the point p_{12} is a double point. For this consider the line L; under the Veronese embedding the line is

$$L \cong v_2(L) \subset \mathbb{P}^2_{u_{00},u_{01},u_{11}} \subset \mathbb{P}^9.$$

After projecting twice, that is, eliminating u_{00} and u_{11}, we are left with u_{01} only, and the local coordinates on the tangent spaces $\mathcal{T}_{p_1}$, $\mathcal{T}_{p_2}$ are $\{u_{01}, u_{02}, u_{03}\}$ and $\{u_{01}, u_{12}, u_{13}\}$ respectively. The dimension of the tangent space at this point is 4 because in the affine cover $u_{01} \neq 0$ the equations of W_6 reduce to only one quadratic equation making it a double point.

Hence the Steiner 3-fold W_6 contains two exceptional planes and a double point.

Stage III: W_5

For W_5, apart from the exceptional planes E_1, E_2 and E_3 we get one more plane E_{123} contained in W_5. This plane is really obtained from the plane Π_{123} defined by q_1, q_2, q_3 inside $\mathbb{P}^3$. And $v_2(\Pi_{123})$ is defined by the 2×2 minors of the following submatrix of M^3

$$\begin{pmatrix} u_{00} & u_{01} & u_{02} \\ u_{01} & u_{11} & u_{12} \\ u_{02} & u_{12} & u_{22} \end{pmatrix}. \qquad (3.3.14)$$

After projecting from the first three points, eliminating u_{00}, u_{11}, u_{22} from the above matrix we are left with only three coordinates u_{01}, u_{02}, u_{12} and no equation, forming a plane E_{123} contained in W_5. Hence the Steiner 3-fold W_5 contains four planes

$$E_1, \quad E_2, \quad E_3, \quad E_{123}\,,$$

and the three double points

$$p_{12}, \quad p_{13}, \quad p_{23},$$

where, as we have seen before, p_{ij} comes from the line $L_{ij} \subset \mathbb{P}^3$ passing through q_i and q_j.

In fact the plane E_{123} obtained by projecting $v_2(\Pi_{123})$ is given by the standard Cremona transformation, that blows up 3 points of Π_{123} then contracts 3 lines.

Stage IV: W_4

Projecting a fourth time gives us $W_4 \subset \mathbb{P}^5_{u_{01},u_{02},u_{03},u_{12},u_{13},u_{23}}$, defined by the following equations

$$u_{01}u_{23} = u_{02}u_{13} = u_{03}u_{12} \tag{3.3.15}$$

that is, complete intersection of two quadrics in $\mathbb{P}^5$. In this case we again get more planes than our naive expectation.

Proposition 3.3.16 *There are 8 planes contained in W_4.*

Proof W_4 can also be considered as image of the following quadratic transformation

$$[z_0 : z_1 : z_2 : z_3] \xrightarrow{\Phi} [z_0 z_1 : z_0 z_2 : z_0 z_3 : z_1 z_2 : z_1 z_3 : z_2 z_3] \subset \mathbb{P}^5 \tag{3.3.17}$$

To understand the structure of these planes consider a tetrahedron in $\mathbb{P}^3$ with vertices q_1, q_2, q_3, q_4. Each face of the tetrahedron maps to another plane in $\mathbb{P}^5$. The map Φ is not defined at the vertices so we blow up the vertices. The configuration of 8 planes is then given by the 4 exceptional planes and the four planes arising from the faces of the tetrahedron. □

The exceptional planes in W_4 intersect at the following ordinary double points

$$E_k \cap E_l = \{p_{kl}\} \quad \text{for} \quad k \neq l, \tag{3.3.18}$$

where

$$\begin{aligned} p_{12} &= (1:0:0:0:0:0), \quad p_{13} = (0:1:0:0:0:0), \\ p_{14} &= (0:0:1:0:0:0), \quad p_{23} = (0:0:0:1:0:0), \\ p_{24} &= (0:0:0:0:1:0), \quad p_{34} = (0:0:0:0:0:1). \end{aligned} \tag{3.3.19}$$

Stage V: W_3

Projection from a 5th point can be simplified under the following projective transformation T_1

$$\begin{pmatrix} y_{01} \\ y_{02} \\ y_{03} \\ y_{12} \\ y_{13} \\ y_{23} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_{01} \\ u_{02} \\ u_{03} \\ u_{12} \\ u_{13} \\ u_{23} \end{pmatrix}, \tag{3.3.20}$$

where y_{kl} are new coordinates. After the above transformation the fifth point becomes the coordinate point with $y_{01} = 1$. So to do the projection we just need to eliminate y_{01}, yielding W_3

$$\pi_5 : W_4 \dashrightarrow W_3 \subset \mathbb{P}^4, \tag{3.3.21}$$

where W_3 is defined by the following equation

$$y_{02}y_{13}(y_{23} - y_{03} - y_{12}) - y_{03}y_{12}(y_{23} - y_{02} - y_{13}). \tag{3.3.22}$$

The new exceptional plane E_5 intersects the others as follows

$$\begin{aligned} E_5 \cap E_1 &= \{(0:0:1:1:1)\}, \\ E_5 \cap E_2 &= \{(1:1:0:0:1)\}, \\ E_5 \cap E_3 &= \{(1:0:1:0:1)\}, \\ E_5 \cap E_4 &= \{(0:1:0:1:1)\}. \end{aligned}$$

The points given above are double points of W_3, as these are the images of lines passing through q_5 and q_i, for $i = 1, 2, 3, 4$, in $\mathbb{P}^3$. Since the map π_5 is an isomorphism outside the point of projection, so in addition to the above double points, there are 6 double points coming from the images of points in (3.3.19).

In addition to double points we get 15 planes. Five of these planes are $E_1, \ldots, E_5$, the exceptional planes, and the remaining 10 planes are

$$E_{ijk}, \quad \text{with } i,\ j,\ k \text{ distinct.}$$

Using the same argument as in above cases, these planes are images of Π_{ijk}, the planes passing through q_i, q_j and q_k inside $\mathbb{P}^3$.

Hence the Steiner 3-fold of degree three contains 10 double points and 15 planes.

Stage VI: W_2

To project again, from sixth point, we consider W_3 under the following projective transformation T_2

$$\begin{pmatrix} x_{02} \\ x_{03} \\ x_{12} \\ x_{13} \\ x_{23} \end{pmatrix} = \begin{pmatrix} 1/a & 0 & 0 & 0 & 0 \\ (\alpha_0(\alpha_1 - \alpha_3))/a & 1 & 0 & 0 & 0 \\ (\alpha_0(\alpha_0 - \alpha_2))/a & 0 & 1 & 0 & 0 \\ (\alpha_1(\alpha_0 - \alpha_3))/a & 0 & 0 & 1 & 0 \\ (\alpha_0\alpha_1 - \alpha_2\alpha_3)/a & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{02} \\ y_{03} \\ y_{12} \\ y_{13} \\ y_{23} \end{pmatrix}, \qquad (3.3.23)$$

and the fifth point becomes a coordinate point with $x_{02} = 1$ making the projection easy. In this case W_2 is double cover of $\mathbb{P}^3_{x_{03},x_{12},x_{13},x_{23}}$ branched at a quartic C_4. In other words

$$W_2\colon D_4 \subset \mathbb{P}(x_0, x_1, x_2, x_3, z),$$

where $\operatorname{wt} x_i = 1$ and $\operatorname{wt} z = 2$, and $D_4 = C_4 + z^2$. The following discussion explores the relation between W_2 and Kummer surfaces.

Remark 3.3.24 In fact one sees that W_2 contains 32 planes that map $2:1$ to 16 tangent planes to Kummer quartic C_4.

Kummer Surfaces

Let T be a two-dimensional torus, and let i be the involution automorphism. The quotient manifold $S = T/i$ is called the singular Kummer surface of the torus T. The fixed points of the involution i give rise to 16 ordinary double points on S. Let $\pi\colon \tilde{S} \longrightarrow S$ be the resolution of these singularities. The surface $\tilde{S}$ is a Kähler manifold with $\omega_{\tilde{S}} \simeq \mathcal{O}_{\tilde{S}}$, and $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 0$ implying that $\tilde{S}$ is a $K3$ surface. The converse situation is discussed in the following result, which is useful for our further discussion.

Theorem 3.3.25 ([Nik75]) *Let X be a Kähler $K3$ surface containing 16 nonsingular rational curves $E_1, \dots, E_{16}$ which do not intersect with each other. Then, up to isomorphism, there exist a complex torus T such that X is obtained from T by the above Kummer process. In particular X is a Kummer surface.*

Proposition 3.3.26 *The surface $C_4 \subset \mathbb{P}^3$ is a Kummer surface.*

Proof The quartic C_4 has 15 double points coming from the lines joining two of the points $\{q_1, \dots, q_6\}$. As we know that through $d+3$ points in general position in $\mathbb{P}^d$ there is a unique rational normal curve through them [Ver81].

So in this case there is a unique rational normal curve passing through these 6 points. After projection from these 6 points we get a 16th double point on C_4 coming from this rational normal curve. $\square$

Let E_6 be the exceptional plane coming from this projection, then

$$E_i \cap E_j = L_{ij} \quad i \neq j, \quad i, j \in \{0, 1, \dots, 5\}, \tag{3.3.27}$$

where L_{ij} are lines in $\mathbb{P}^3_{u_{03}, u_{12}, u_{13}, u_{23}}$. We do not have any further need for the equations of these lines, and so we omit them.

Stage VII: W_1

We do not discuss W_1, as in the coming chapter we are only concerned with Steiner 3-folds of degree ≥ 2.

It is interesting, though irrelevant to our purpose, to see that the exceptional planes in W_3, W_2, W_1 correspond to the Dynkin diagram of type A_5, D_6 and E_7, as a consequence of a result from [DO88, page 73].

Steiner 3-folds and Del Pezzo surfaces

In this section we discuss the link between Steiner 3-folds and Del Pezzo surfaces.

A general hyperplane section of $v_2(\mathbb{P}^3)$ corresponds to taking a quadric hypersurface in $\mathbb{P}^3$. So in the case of degree 8 Steiner 3-fold we have

$$W_8 \cap H \cong \mathbb{P}^1 \times \mathbb{P}^1,$$

that is a general hyperplane section of W_8 is a non-singular quadric which is isomorphic to a Del Pezzo surfaces of degree 8.

For $W_7, \ldots, W_3$ a general hyperplane section is a rational surface of degree n in $\mathbb{P}^n$, for $n = 7, 6, 5, 4, 3$ respectively. Hence a general hyperplane section of a Steiner 3-fold of degree $7, \ldots, 3$ gives a Del Pezzo surface of degree $7, 6, 5, 4, 3$.

Chapter 4

Surfaces of general type with $p_g = 3$, $2 \leq K^2 \leq 8$

In this chapter we present a way to construct simply connected surfaces of general type with the above mentioned invariants.

In 2007 Lee and Park introduced $\mathbb{Q}$-Gorenstein smoothing as a technique for construction of surface. Using this technique Lee, Park, and others gave interesting results about surfaces, including constructions of simply connected surfaces of general type with $p_g = 0$, $K^2 = 2, 3, 4$ [LP07; PPS11; PPS09].

In the consequent discussion we use $\mathbb{Q}$-Gorenstein smoothings to give explicit constructions of canonical rings, by giving their generators and relations, expressed in terms of higher dimensional key varieties.

4.1 Overview

Following a similar construction of Godeaux – Reid surfaces ([God31] and [Reid], [MLPR09]), we first choose a canonical model S. By studying its $\mathbb{Q}$-Gorenstein deformations we obtain surfaces with the required type and number of singularities. The important part of the process is to check that the central fibre of such a deformation does not contain more or worse singularities than those required. Resolving the singularities of the central fibre produces the required surfaces.

All of the process of construction of our surfaces is simplified by the use of key varieties. We use these key varieties to degenerate the surface S to a special surface with the required singularities. To get the resolution of this singular surface we again use key varieties. In fact, the surface S is contained in sections of a 6-dimensional key variety V_8, and is obtained by cutting V_8 by 4 elements of a very ample linear system. Moving these sections around in an appropriate way gives us the required singular fibre. The resolution of singularities is then obtained by using sections of another key variety.

We briefly review the technique of $\mathbb{Q}$-Gorenstein smoothing in two subsequent chapters, stating the relevant properties for the results in the corresponding chapter. This chapter contains the 1st part of this brief review (of the $\mathbb{Q}$-Gorenstein smoothing). Then we discuss surfaces of general type with $p_g = 3$ and $K^2 = 8$. Later in this chapter we construct key varieties $V_8, \ldots, V_2$ using the Steiner 3-folds introduced earlier. These key varieties are the necessary ingredient of our construction.

4.2 $\mathbb{Q}$-Gorenstein smoothing

For a normal projective surface X with quotient singularities, a $\mathbb{Q}$-Gorenstein smoothing is a one-parameter flat family of projective surfaces $\mathcal{X} \to \Delta$ over a small disk Δ, which satisfies the following three conditions:

(i) the general fibre X_t is a smooth projective surface,

(ii) the central fibre X_0 is X,

(iii) the canonical divisor $K_{\mathcal{X}/\Delta}$ is $\mathbb{Q}$-Cartier.

We say that X' is a $\mathbb{Q}$-Gorenstein smoothing of X if there exists such an $\mathcal{X}$ and $X' = \Psi^{-1}(t)$ for some $t \in \Delta$. In general, if X has quotient singularities, such a smoothing does not exist and it depends on the type of singularities of X. For a germ $(X_0, 0)$ of a quotient singularity, X_0 has a $\mathbb{Q}$-Gorenstein smoothing iff the singularity is either a rational double point or a T-singularity. A T-singularity is a cyclic quotient singularity of type $\frac{1}{dn^2}(1, dna-1)$, where a, d, n are integers and $\gcd(a, n) = 1$ [KSB88, page 314].

Example 4.2.1 A quotient singularity X_0 of type $\frac{1}{dn^2}(1, dna-1)$ is $X_0 = Y_0/\mathbb{Z}_n$, where $Y_0\colon (xy - z^{dn} = 0) \subset \mathbb{C}^3_{x,y,z}$ and the action of the group $\mathbb{Z}_n$ is given by

$$(x, y, z) \longmapsto (\varepsilon x, \varepsilon^{-1} y, \varepsilon^a z), \tag{4.2.2}$$

where ε is a primitive nth root of unity. A $\mathbb{Q}$-Gorenstein smoothing of X_0 is

$$X_t = Y/\mathbb{Z}_n, \quad Y := xy - z^{dn} + t = 0,$$

and the action of $\mathbb{Z}_n$ is given by

$$(x, y, z, t) \longmapsto (\varepsilon x, \varepsilon^{-1} y, \varepsilon^a z, t).$$

The resolution graph of a T-singularity which is not a rational double point is either one of the following:

$$\underset{-4}{\circ} \quad \text{and} \quad \underset{-3}{\circ} - \underset{-2}{\circ} - \cdots - \underset{-2}{\circ} - \underset{-3}{\circ}$$

or it can be obtained from the above using the following rule: if a singularity

$$\underset{-\alpha_1}{\circ} - \cdots - \underset{-\alpha_l}{\circ}$$

is of class T then so are

$$\underset{-2}{\circ} - \underset{-\alpha_1}{\circ} - \cdots - \underset{-\alpha_l - 1}{\circ}$$

and

$$\underset{-\alpha_1 - 1}{\circ} - \cdots - \underset{-\alpha_l}{\circ} - \underset{-2}{\circ}\ .$$

Let X be a surface with T-singularities. If $\Psi\colon \mathcal{X} \to \Delta$ is a global $\mathbb{Q}$-Gorenstein smoothing of X then the family $\mathcal{X}$ is flat and hence $K^2_{X_t} = K^2_X$, for $X_t = \Psi^{-1}(t)$.

Example 4.2.3 (Motivation) Let X be a surface of general type with unique singularity a T-singularity of type $\frac{1}{4}(1,1)$. In this case $2K_X$ is a Cartier divisor. If $f\colon \widetilde{X} \to X$ is a resolution of X then $K^2_{\widetilde{X}} = \frac{1}{4}(2K_X)^2 - 1$. It should be noted that this is not the case with other $\mathbb{Q}$-Gorenstein smoothings, for example, if X has a T-singularity of type $\frac{1}{9}(1,2)$ then $K^2_{\widetilde{X}} = K^2_X - 2$, where $3K_X$ is a Cartier divisor.

4.3 Surfaces with $p_g = 3, K^2 = 8$ and $\pi_1 = \mathbb{Z}_2$

Let S be a surface of general type with $p_g = 3, K_S^2 = 8$ and $\pi_1(S) = \mathbb{Z}_2$. Here $\pi_1 = H_1(S;\mathbb{Z}) = \operatorname{Tors} S \subset \operatorname{Pic} S$. Since we know that

$$b_1 = \operatorname{Rank} H^1(S;\mathbb{Z}) = 2h^{0,1},$$

so S is regular, that is, $H^1(\mathcal{O}_S) = 0$; then by [Bom73] there exists an unramified universal covering $\psi\colon \widetilde{S} \to S$ of degree 2 such that

$$K_{\widetilde{S}}^2 = 2K_S^2, \quad \chi(\mathcal{O}_{\widetilde{S}}) = 2\chi(\mathcal{O}_S).$$

Here $\widetilde{S}$ is simply connected. The sheaf of regular functions $\mathcal{O}_{\widetilde{S}}$ decomposes as

$$\psi_*\mathcal{O}_{\widetilde{S}} = \mathcal{O}_S \oplus \mathcal{O}_S(\sigma),$$

where $\sigma \in \operatorname{Tors} S$ is the nonidentity element. Since $H^1(\mathcal{O}_S) = 0$, $\widetilde{S}$ is a surface of general type with $p_g = 7$ and $K_{\widetilde{S}}^2 = 16$. Motivated by [Reid] and [MLPR09] we assume the canonical image T of $\widetilde{S}$ to be a complete intersection of four quadrics:

$$T = \operatorname{Proj} R(\widetilde{S}, K_{\widetilde{S}}) = \bigcap_{i=1}^{4} q_i \subset \mathbb{P}^6. \tag{4.3.1}$$

Lemma 4.3.2 *The surface T is a simply connected surface of general type with $p_g = 7$, $K_T^2 = 16$.*

Proof By using the adjunction formula, we deduce that $p_g = h^0(\mathcal{O}_T(K)) = 7$, and $K_T^2 = 16$. Also the canonical divisor is very ample and hence nef.

Together with the condition $K_T^2 > 0$, we see that T is a surface of general type. Finally T is a complete intersection so by the Lefschetz hyperplane theorem it is simply connected. $\square$

In the next discussion we prove that there exist a surface T with a fixed point free action of $\mathbb{Z}_2$ on T such that the quotient is S with $\pi_1 = \pi_1^{\mathrm{alg}} = \mathbb{Z}_2$. Let us assume that $\mathrm{Gal}(T/S) = \mathbb{Z}_2$; then the action of $\mathbb{Z}_2$ induces one on the spaces $H^i(T, \mathcal{F})$, for any equivariant coherent sheaf $\mathcal{F}$. From [Reid] we have

$$\chi(T, \psi^*\mathcal{F}) = \chi(S, \mathcal{F}) \otimes k[\mathbb{Z}_2],$$

where $k[\mathbb{Z}_2]$ is the group algebra of $\mathbb{Z}_2$. For $\mathcal{F} = \mathcal{O}_S(K_S)$ we obtain the following isomorphism

$$f\colon H^0(\mathcal{O}_T(K_T))) \oplus k \xrightarrow{\sim} \big(H^0(\mathcal{O}_S(K_S)) \oplus k\big) \otimes k[\mathbb{Z}_2],$$

since S and T are regular and $\mathcal{O}_T(K_T) = \psi^*(\mathcal{O}_S(K_S))$, as ψ is etale. The k on the left and right hand side correspond to $H^2(\mathcal{O}_T(K_T))$ and $H^2(\mathcal{O}_S(K_S))$ respectively. Moreover the group $\mathbb{Z}_2$ acts trivially on the k on left hand side. Hence $H^0(\mathcal{O}_T(K_T))$ splits as $I \oplus A$, where I is the invariant and A the anti-invariant part. Since $p_g(S) = 3$ we can take $I = H^0(\mathcal{O}_S(K_S)) = \langle y_1, y_2, y_3\rangle_k$, where by abuse of notation we take $y_i := f^*(y_i)$, for $i = 1, 2, 3$. If we set $A = \langle z_0, z_1, z_2, z_3\rangle_k$ then the projective space $\mathbb{P}^6$ in (4.3.1) has homogeneous coordinates $y_1, y_2, y_3, z_0, z_1, z_2, z_3$. For $\mathbb{Z}_2 = \langle i\rangle$, the action of i on $\mathbb{P}^6$ is simply the following

$$y_k \mapsto y_k,\; z_l \mapsto iz_k\,. \tag{4.3.3}$$

And

$$\operatorname{Fix} i = \mathbb{P}^2_+ \sqcup \mathbb{P}^3_- \,,$$

where $\mathbb{P}^2_+$ and $\mathbb{P}^3_-$ have homogeneous coordinates y_1, y_2, y_3 and z_0, z_1, z_2, z_3 respectively.

The four quadrics q_k are chosen to be invariant under the above action. Therefore

$$q_k \in \operatorname{Sym}^2 \langle y_1, y_2, y_3 \rangle \oplus \operatorname{Sym}^2 \langle z_0, \dots, z_3 \rangle \,, \tag{4.3.4}$$

for $k = 1, \dots, 4$. For general coefficients the surface T is nonsingular by Bertini's theorem, and

$$T \cap \operatorname{Fix}(i) = \emptyset. \tag{4.3.5}$$

Lemma 4.3.6 *There is a family of surfaces of general type with $p_g = 3$, $K^2 = 8$ with $\pi_1 = \mathbb{Z}_2$.*

Proof The canonical ring of S is given by

$$R(S, K_S) = \left[R(T, K_T)\right]^{\mathbb{Z}_2},$$

where the action of $\mathbb{Z}_2$ is defined is 4.3.3. $\square$

4.4 Key varieties

In this section we construct 6-dimensional key varieties. These key varieties are used to degenerate the surface S constructed in Section 4.3 to get a singular surface S_0^d with d singularities of type $\frac{1}{4}(1,1)$. Then the resolution of S_0^d is our required surface, and is obtained by the use of key varieties again.

We start from Steiner 3-folds W_d as constructed in the Chapter 3. Let U_d be a cone over W_d with vertex $\Lambda \cong \mathbb{P}^2_{y_1,y_2,y_3}$, with the homogeneous coordinate ring

$$\begin{aligned} k[U_d] &= k[W_d][y_1,y_2,y_3]/R_{W_d} \\ &= k[u_{ij},y_1,y_2,y_3]/R_{W_d} \ , \end{aligned}$$

where u_{ij} $(0 \le i \le j \le 3)$, and R_{W_8} are generators and relations of W_8 respectively. Consider the affine cone $\mathcal{C}U_d$ with a $\mathbb{C}^*$ action defined in the following way

$$\begin{aligned} y_i &\mapsto \lambda y_i, \quad \text{for } i = 1,2,3, \\ u_{ij} &\mapsto \lambda^2 u_{ij}, \quad \text{for } 0 \le i \le j \le 3. \end{aligned}$$

The key varieties V_d are then the quotient $\mathcal{C}U_d/\mathbb{C}^*$.

For the coordinates $y_1, y_2, y_3, z_0, \ldots, z_3$ of $\mathbb{P}^6$ consider the action of $\mathbb{Z}_2 = \langle i \rangle$ given by

$$\begin{aligned} y_i \mapsto y_i, &\qquad \text{for } 1 \le i \le 3 \\ z_j \mapsto i z_j, &\qquad \text{for } 0 \le i \le 3 \end{aligned}$$

(see 4.3.3). The homogeneous coordinate ring $k[V_8]$ is

$$\begin{aligned} k[V_8] &= k[\Lambda * W_8]\,, \\ &= k[y_1, y_2, y_3, u_{jk}]/(R_{W_8}), \\ &= \Big[k[y_1, y_2, y_3, z_0, \ldots, z_3]\Big]^{\mathbb{Z}_2}, \end{aligned}$$

where $u_{jk} = z_j z_k$ for $0 \le j \le k \le 3$, and R_{W_8} are 2×2 minors of the following matrix

$$M^3 = \begin{pmatrix} u_{00} & u_{01} & u_{02} & u_{03} \\ & u_{11} & u_{12} & u_{13} \\ \text{Sym} & & u_{22} & u_{23} \\ & & & u_{33} \end{pmatrix}. \tag{4.4.1}$$

In the case of $d = 8$ the key variety V_8 is the quotient $\mathbb{P}^6/\mathbb{Z}_2$.

The homogeneous coordinate ring of V_d is given by

$$k[V_d] = k[\Lambda * W_d] = k[y_i, u_{jk}]/R_{W_d}, \tag{4.4.2}$$

where u_{jk} and R_{W_d} are the generators and relations of $k[W_d]$ respectively (here we are using the same notations as in Section 3.2). The key varieties V_d are 6-dimensional varieties.

The family of surfaces S with $p_g = 3$ and $K^2 = 8$ obtained in Section 4.3 can also be obtained as sections of the key variety V_8. The coordinate ring of S is

$$k[S] = \Big[k[y_1, y_2, y_3, z_0, \ldots, z_3]/(q_1, q_2, q_3, q_4)\Big]^{\mathbb{Z}_2}, \tag{4.4.3}$$

where the action of $\mathbb{Z}_2$ is given in 4.3.3. The above implies

$$k[S] = k[V_8]/(H_1, \ldots, H_4), \tag{4.4.4}$$

where $H_1, \ldots, H_4 \in H^0(\mathcal{O}_{V_8}(2))$ are general elements. In other words

$$S = \bigcap_{i=1}^{4} H_i \cap V_8.$$

The space $H^0(\mathcal{O}_{V_8}(2))$ has dimension 16 and the general elements H_i are of the form

$$a^i + b^i, \tag{4.4.5}$$

where

$$a^i \in \operatorname{Sym}^2 \langle y_1, y_2, y_3 \rangle, \tag{4.4.6}$$

and

$$b^i = \sum_{0 \le k \le l \le 3} b^i_{kl} u_{kl} \,. \tag{4.4.7}$$

Let $A^i = [a^i_{lm}]_{1 \le l,m \le 3}$ be the symmetric matrix corresponding to a^i and $B^i = [b^i_{kl}]_{0 \le k \le l \le 3}$ the row matrix of coefficient of b^i. We also define

$$B = \begin{pmatrix} B^1 \\ \vdots \\ B^4 \end{pmatrix}. \tag{4.4.8}$$

For general H_i the surface S is nonsingular. If we move these sections in some family of parameters then we get degenerations of S. These degenerations of S may contain singularities. We want to do this so as to get T-singularities of type $\frac{1}{4}(1,1)$ on S. The following example shows that moving these sections may give us some other singularities.

Example 4.4.9 Suppose that we choose the H_i such that $S \cap \Lambda \neq \emptyset$, for example, by taking $a^i_{00} = 0$ for $i = 1, \ldots, 4$. Then $S \cap \Lambda = \{(1 : 0 : \ldots : 0)\}$ is a singular point. Note that this is not a hypersurface singularity and it is not the kind of singularity we are looking for.

The following is one possible way to find $\frac{1}{4}(1,1)$ singularities on X.

Proposition 4.4.10 *A point $p \in S \cap W_8$ is a singular point of S of type $\frac{1}{4}(1,1)$ provided the matrix B has rank ≥ 3 and for general a^i a general linear combination of the a^i is a nondegenerate quadratic form in y_1, y_2, y_3.*

Proof Without loss of generality we can take $p = (1 : 0 : \cdots : 0) \in W_8$. Three sections H_1, H_2, H_3 can be used to write

$$u_{0i} = f_i(u_{jk}, a^1, a^2, a^3), \tag{4.4.11}$$

where $i = 1, 2, 3$ and $1 \leq j \leq k \leq 3$. Using the above the fourth section H_4 can be written as

$$H_4 = g(a^1, \ldots, a^4, u_{jk}), \tag{4.4.12}$$

where $1 \leq j \leq k \leq 3$. Now consider the affine piece $u_{00} \neq 0$. The defining equations of V_8 are 2×2 minors of the following matrix:

$$\begin{pmatrix} u_{00} & u_{01} & u_{02} & u_{03} \\ & u_{11} & u_{12} & u_{13} \\ \text{Sym} & & u_{22} & u_{23} \\ & & & u_{33} \end{pmatrix}. \tag{4.4.13}$$

At $u_{00} \neq 0$,

$$u_{jk} = f_{jk}, \tag{4.4.14}$$

where

$$f_{jk} \in \mathrm{Sym}^2 \langle u_{01}, u_{02}, u_{03} \rangle \quad \text{for } 1 \le j \le k \le 3\,.$$

Then from (4.4.12) and (4.4.14) we get

$$H_4 \colon g(a^i, f_{jk}) = 0,$$

which, under conditions on a^i is a nondegenerate quadric. Hence $(1 : 0 : \cdots : 0)$ is at least a double point. Moreover, there is a $\mathbb{Z}_2$ group action on $\mathbb{A}_{u_{00} \neq 0}$ coming from the weight of the variable u_{00}. Under this action the point $p \in X$ becomes a singularity of type $\frac{1}{4}(1,1)$. $\square$

So we have a way of degenerating S using key varieties so that the central fibre has singularities of type $\frac{1}{4}(1,1)$. To find out the restrictions on the number of isolated singularities of type $\frac{1}{4}(1,1)$ we can introduce in this way, we observe the following:

Lemma 4.4.15 *We can degenerate S to S_0^d having d points of type $\frac{1}{4}(1,1)$, where d is $1, \ldots, 6$ or 8. Also 7 such points is not possible.*

Proof As we have seen earlier that the surface S can be given as

$$S = \bigcap_{i=1}^{4} H_i \cap V_8,$$

for general elements $H_1, \ldots, H_4 \in H^0(\mathcal{O}_{V_8}(2))$. The H_i have the form $a^i + b^i$ where a^i and b^i are given in (4.4.6) and (4.4.7) respectively. Since b^1, b^2, b^3 are general so form a regular sequence, the intersection $\bigcap_{i=1}^{3} H_i \cap W_8$ is a zero dimensional variety which consists of 8 points, say, $p_1, \ldots, p_8$. We can move the section H_4 to get the family of surfaces $\Psi^d \colon \mathcal{S} \to \Delta$, for some unit dist Δ, such that the general fibre $S_t = \Psi^{-1}(t)$ is

$$\bigcap_{i=1}^{4} H_i \cap V_8,$$

and the central fibre $S_0^d = \Psi^{-1}(0)$ is

$$\bigcap_{i=1}^{3} H_i \cap H_4^d \cap V_8, \tag{4.4.16}$$

where H_4^d is a specialization of H_4 such that

$$H_4^d(p_i) = 0 \quad \text{for } 0 \leq i \leq d.$$

In fact $d \neq 7$ since for $d = 7$ the matrix (4.4.8) has rank 3 and so the intersection

$$\bigcap_{i=1}^{4} H_3 \cap H_4^7 \cap W_8 \tag{4.4.17}$$

is 8 points rather than 7. Moreover we assume that $H_1, \ldots, H_4$ satisfy the conditions of Proposition (4.4.10) so that at each of the point $p_1, \ldots, p_d$ the intersection (4.4.16) is transversal, and so these points are isolated singularities of type $\frac{1}{4}(1,1)$ on S_0^d. $\square$

The points $p_1, \ldots, p_d$, for $d = 1, \ldots, 6, 8$, defined in the proof of above Lemma 4.4.15 are in general position by the following result:

Lemma 4.4.18 *Let Q_1, Q_2, Q_3 be three general quadrics in $\mathbb{P}^3$. Then the 8 points of intersection $Q_1 \cap Q_2 \cap Q_3$ are in general position, that is, satisfy conditions of Corollary 3.1.12.*

Proof Let $p_1, \ldots, p_8$ be the points of intersection of the quadrics Q_1, Q_2, Q_3 in $\mathbb{P}^3$. We want to show the following

1. all points are distinct,
2. No three points are collinear,
3. No four lie on a plane,
4. No 7 of these points lie on a quadric with singularity at one of the points.

Part 2 follows from the following fact: if three collinear points lie on a quadric in $\mathbb{P}^3$ then the line joining them is also contained in the quadric. For part 3; assume on contrary that the points $p_1, \ldots, p_4$ lie on the plane $\Pi \subset \mathbb{P}^3$. The restriction of $Q_1 \cap Q_2 \cap Q_3$ to Π is four points. So, up to projective transformation, we can say that Π is contained in one of the quadrics, say, Q_1. Hence Q_1 is a union of planes, a contraction. For part 4; assume the contrary, that is, there is quadric Q passing through the seven points $p_1, \ldots, p_7$ with multiplicity 2 at one of the points, say, p_1. Then the quadric Q can be written as linear combination of Q_1, Q_2, and Q_3. Hence the intersection $Q_1 \cap Q_2 \cap Q_3$ has multiplicity at least 2 at the point p_1, a contradiction. $\square$

Theorem 4.4.19 *For the family S of surface of general type with $p_g = 3$, $K^2 = 8$ and $\pi_1 = \mathbb{Z}_2$ there exists a degeneration $\Psi^d \colon \mathcal{S} \to \Delta$, where Δ is the unit disk, such that the general fibre is nonsingular and central fibre S_0^d has d isolated singularities of type $\frac{1}{4}(1,1)$. The possible values for d are $1, \ldots, 6, 8$.*

Proof By Lemma 4.4.15 there exist a degeneration $\Psi^d \colon \mathcal{S} \to \Delta$, where Δ is the unit disk, such that the central fibre $S_0^d = \Psi^{-1}(0)$ has d singularities

$p_1, \ldots, p_d$ of type $\frac{1}{4}(1,1)$, and moreover the only possible values of d are $1, \ldots, 6$ and 8. Since the points $p_1, \ldots, p_8$ are in general position, after an appropriate projective transformation, the points $p_1, \ldots, p_6$ are

$$p_i = v_2(q_i) \quad \text{for } 1 \leq i \leq 6,$$

where the points $q_1, \ldots, q_6$ are given in (3.3.2) and (3.3.3).

We need to check if these are the only singularities on S_0^d. Let $p \in S_0^d \backslash W_8$, and suppose

$$p = (\beta_1 : \beta_2 : \beta_3 : \alpha_{00} : \ldots : \alpha_{33}),$$

here $p \notin \Lambda$ since the a^i are general and so the restriction of S_0^d to Λ is empty. So one of the α_{ij} and one of the β_k must be nonzero. We take $\alpha_{00}, \beta_1 \neq 0$. Up to a projective transformation we can take p to be a coordinate point with $\alpha_{00} = \beta_1 = 1$. To get rid of the weight difference of the variables consider $v_2(\Lambda)$. Here $v_2(\Lambda)$ is defined by the 2×2 minors of

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ & t_{22} & t_{23} \\ \text{Sym} & & t_{33} \end{pmatrix}, \tag{4.4.20}$$

where $t_{ij} = y_i y_j$ for $1 \leq i \leq j \leq 3$. We do another projective transformation f given by

$$f(t_{ij}) = \begin{cases} t_{ij} - u'_{00} & \text{for } i = j = 1 \\ t_{ij} & \text{otherwise} \end{cases} \tag{4.4.21}$$

and $f(u_{ij}) = u_{ij}$. Under this transformation $f(p)$ is a coordinate point with

$u_{00} = 1$, and the matrix (4.4.20) becomes

$$\begin{pmatrix} t_{11} - u_{00} & t_{12} & t_{13} \\ & t_{22} & t_{23} \\ \text{Sym} & & t_{33} \end{pmatrix}. \tag{4.4.22}$$

In the affine cover $\mathbb{A}_{u_{00} \neq 0}$ the tangent space is defined by the following 13 linear equations,

$$\begin{array}{lll} u_{11} = 0, & u_{12} = 0, & u_{13} = 0, \\ u_{22} = 0, & u_{23} = 0, & u_{33} = 0, \\ H_1 = 0, & H_2 = 0, & H_3 = 0, \\ H_4^d = 0 & t_{22} = 0, & t_{23} = 0, \\ t_{33} = 0, & & \end{array} \tag{4.4.23}$$

where the last three equations come from matrix (4.4.22). So $\dim \mathcal{T}_{f(p)} = \dim \mathcal{T}_p = 2$, so $f(p)$ and hence p a regular point. $\square$

By the above proof it also becomes clear that the family $\Psi \colon \mathcal{S} \to \Delta$ is flat. Since for $S_t = \Psi^{-1}(t)$ the divisor $2K_{S_t}$ is Cartier, we have:

Corollary 4.4.24 The degeneration $\Psi \colon \mathcal{S} \to \Delta$ of S is a $\mathbb{Q}$-Gorenstein smoothing of S_0^d.

4.5 Surfaces of general type with $p_g = 3$ and $2 \leq K^2 \leq 7$

Let us denote by Z^d the surfaces of general type with $p_g = 3$ and $K^2 = 8 - d$, for $1 \leq d \leq 6$. To construct Z^d we consider the degeneration $\Psi^d \colon \mathcal{S} \to \Delta$, such that the general fibre S_t of the degeneration is a nonsingular surface with $p_g = 3$, $K^2 = 8$, $\pi_1 = \mathbb{Z}_2$, and the central fibre S_0^d has $d \times \frac{1}{4}(1,1)$ singularities.

Such a degeneration exists by Theorem (4.4.19). The central fibre S_0^d of Ψ^d is

$$S_0^d = \bigcap_{i=1}^{3} H_i \cap H_4^d \cap V_8\,, \tag{4.5.1}$$

where the H_4^d satisfy

$$H_4^d(p_k) = 0, \quad \text{for } 1 \leq k \leq d, \tag{4.5.2}$$

where the points $p_1, \ldots, p_d \in H_1 \cap H_2 \cap H_3 \cap W_8$ are isolated singular points of S_0^d. The points $p_1, \ldots, p_6$ are in general position so we can take p_k to be

$$v_2(q_k) = p_k, \quad \text{for } k = 1, \ldots, 6, \tag{4.5.3}$$

where the $q_i \in \mathbb{P}^3$ are given by

$$\begin{aligned} &q_1 = (1:0:0:0), \quad q_2 = (0:1:0:0), \quad q_3 = (0:0:1:0), \\ &q_4 = (0:0:0:1), \quad q_5 = (1:1:1:1), \quad q_6 = (\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3). \end{aligned} \tag{4.5.4}$$

Remark 4.5.5 We stop at $d = 6$ as by Lemma (4.4.15) $d = 7$ is impossible, and $d = 8$ leads to a surface with $p_g = 3$, $K^2 = 0$ which is not of general type, hence outside the scope of this work.

Let us denote the minimal resolution of S_0^d by Z^d, $1 \leq d \leq 6$ then:

Lemma 4.5.6 *The resolution* $\pi\colon Z^d \dashrightarrow S_0^d$ *is* $Z^d = \bigcap_{i=1}^3 H_i \cap H_4^d \cap V_{8-d}$.

Proof To resolve we use projection, that is, elimination of a certain variable. To get the resolution of S_0^d we use the key variety V_{8-d}. In fact we have seen in the construction of Steiner 3-folds that W_{8-d} is obtained after projection from d points of W_8. We can take the d points in general position to be $p_1, \ldots, p_d$. Now V_{8-d} is the cone over W_{8-d} with vertex $\Lambda \cong \mathbb{P}^2$ hence V_{8-d} is projection of V_8 from the points $p_1, \ldots, p_d$. The surface S_0^d from (4.5.1) is

$$S_0^d \;=\; \bigcap_{i=1}^{3} H_i \cap H_4^d \cap V_8\,.$$

The projection leaves H_1, H_2, H_3, H_4^d unchanged as these does not involve the variables to be eliminated. And so the blowup $Z^d \to S_0^d$ is obtained as

$$Z^d \;=\; \bigcap_{i=1}^{3} H_i \cap H_4^d \cap V_{8-d}. \qquad \square$$

For the fundamental group of Z^d we have:

Lemma 4.5.7 *The surfaces Z^d are simply connected for $1 \leq d \leq 6$.*

Proof We can write

$$\begin{aligned} k[S_0^d] &= k[V_8]/(H_1, H_2, H_3, H_4^d), \\ &= k[\Lambda * W_8]/(H_1, H_2, H_3, H_4^d), \quad \text{from 4.4.2} \\ &= k[y_1, y_2, y_3, u_{jk}]/(R, H_1, H_2, H_3, H_4^d), \end{aligned}$$

where $0 \leq j \leq k \leq 3$, and R represents the 2×2 minors of the matrix (4.4.1). So we have

$$k[S_0^d] \cong \Big[k[y_1, y_2, y_3, z_0, z_1, z_2, z_3]/(q_i)\Big]^{\mathbb{Z}_2},$$

where

$$\begin{aligned} q_i &= a^i + b^i, \qquad \text{for } 1 \le i \le 3, \\ q_4 &= a^4 + \beta^4 \end{aligned} \tag{4.5.8}$$

such that a^i and b^i are the same as in (4.4.5) and

$$\beta^4 = \sum_{0 \le k \le l \le 3} b^4_{kl} z_k z_l\,,$$

satisfying

$$\beta^4(q_k) = 0 \quad \text{for } 1 \le k \le d,$$

and the points $q_1, \ldots, q_d$ come from (4.5.3). Here the action of $\mathbb{Z}_2 = \langle i \rangle$ is given by $y_k \mapsto y_k$ and $z_k \mapsto i z_k$. Under this action, $\operatorname{Fix} i = \mathbb{P}^2_{y_1,y_2,y_3} \bigsqcup \mathbb{P}^3_{z_0,z_1,z_2,z_3}$. By the Lefschetz hyperplane theorem $\bigcap_{i=1}^4 q^i$ is simply connected. Also we require $\bigcap_{i=1}^4 q^i \cap \mathbb{P}^3_{z_0,\ldots,z_3} \neq \emptyset$. So $\pi_1^{\mathrm{alg}}(S_0^d) \cong \mathbb{Z}_2/E$, where E is the elliptic subgroup which in this case is $\mathbb{Z}_2$. So the S_0^d are simply connected for $1 \le d \le 6$.

Since the singularities of S_0^d are rational, by the Van Kampen theorem [Bar82] and the fact that the resolution of a $\frac{1}{4}(1,1)$ singularity is a (-4)-curve we get that the Z^d are simply connected. $\square$

Theorem 4.5.9 *The surfaces Z^d, for $1 \le d \le 6$, are simply connected surfaces of general type with $p_g = 3$, $K^2 = 8 - d$.*

Proof The Hilbert series of V_8 is given by

$$P_r(V_8) = \frac{(r^4 + 6r^2 + 1)}{(1 - r^2)^4\,(1 - r)^3}\,.$$

Since the family $\Psi^d\colon \mathcal{S} \to \Delta$ is flat each S_t^d has the same Hilbert series given by

$$P_r(S^d) = \frac{(r^4 + 6r^2 + 1)}{(1-r)^3}\,.$$

Thus the adjunction number is 24 and so $\mathcal{O}(K_{S^d}) \cong \mathcal{O}_{\mathbb{P}(1^3,2^{10})}(1)\big|_{S_d}$. We know that the genus is a birational invariant, so $p_g(Z^d) = 3$, for $1 \leq d \leq 6$. The embedding is canonical hence these are surfaces of general type. $\square$

Chapter 5

$\mathbb{Q}$-Gorenstein smoothing of Godeaux with $\pi_1 = \mathbb{Z}_4$

5.1 Introduction

In this chapter we find a relation between Godeaux surface with $\pi_1 = \mathbb{Z}_4$ and Campedelli surfaces with different fundamental groups. In other words we construct Campedelli surfaces with $\pi_1 = \mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ by starting from Godeaux surface with $\pi_1 = \mathbb{Z}_4$ using $\mathbb{Q}$-Gorenstein smoothing, and unprojection. We also give a construction of Campedelli surface with π_1 containing a copy of $\mathbb{Z}_4$, and we discuss the difficulties in calculating the exact fundamental group in this case.

Our first object is construct Godeaux surface with $\pi_1 = \mathbb{Z}_4$ containing a (-4)-curve. In second stage we contract the curve which gives us a singular surface. Smoothing the surface gives us a Campedelli surface. The different fundamental groups of Campedelli surface stems from different constructions of Godeaux surface with $\pi_1 = \mathbb{Z}_4$ containing a (-4)-curve.

The constructions are given by mainly using $\mathbb{Q}$-Gorenstein smoothing and unprojection techniques. We construct surfaces by explicitly giving generators and relations of the canonical rings of the surfaces. As we shall see, the construction is simplified by the use of key varieties as in Chapter 4. The idea of the "key variety technique" is to construct a large simple variety containing lots of interesting and complicated varieties, and our required variety is usually given as a linear or quadric section inside it. The key varieties that we construct and use here are Fano n-folds.

5.2 Godeaux Surfaces

A Godeaux surface T is a general type surface with $p_g = 0$ and $K^2 = 1$. By [Bom73, Lemma 14] such a surface is always regular. Bombieri also showed in [Bom73] that in the case of Godeaux surfaces, $|\operatorname{Tors} T| \leq 5$. Reid in [Rei78] proved that $\operatorname{Tors} T \cong \mathbb{Z}_m$ and also gave complete description of the cases $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$. That each choice of $\operatorname{Tors} T$ corresponds to an irreducible component of the Gieseker moduli space is a conjecture of Reid, still unproven.

Let T be a Godeaux surface with $\pi_1^{\mathrm{alg}} = \operatorname{Tors} T = \mathbb{Z}_4$ and $S \to T$ an etale Galois cover. Since any etale cover of Godeaux surface must be regular, again from Bombieri [Bom73], we have $H^1(\mathcal{O}_S) = 0$. The other numerical invariants of S are $p_g(S) = 3$ and $K^2 = 4$. It is known that S is a complete intersection $f_4 \cap g_4$ of two quartics in $\mathbb{P}(1^3, 2^2)$ [Rei78]; the quartics can be chosen to be general in their eigenspaces, that is, the canonical ring of S is given by

$$\begin{aligned} R(S, K_S) &= \bigoplus_{n \geq 0} H^0(S, \mathcal{O}(nK_S)) \\ &= \bigoplus_{n \geq 0, \sigma \in \operatorname{Tors} T} H^0(T, \mathcal{O}(nK_T + \sigma)) \\ &= k[x_1, x_2, x_3, u_1, u_3]/(f_4, g_4), \end{aligned}$$

where $x_i \in H^0(\mathcal{O}_T(K+i))$, $u_i \in H^0(\mathcal{O}_T(2K+i))$, and f_4 and g_4 are invariant and anti-invariant quartics. In other words the action of $\mathbb{Z}_4$ on the ambient space of S is given by

$$x_i \mapsto \varepsilon^i x_i, \quad u_i \mapsto \varepsilon^i u_i\,,$$

where ε is a primitive fourth root of unity.

5.3 Strategy for construction

Let us assume that X^1, X^2 and X^3 are canonical models of Campedelli surfaces with $\operatorname{Tors} X^1 = \mathbb{Z}_4$, $\operatorname{Tors} X^2 = \mathbb{Z}_8$ and $\operatorname{Tors} X^3 = \mathbb{Z}_4 \times \mathbb{Z}_2$. We need to give a construction of the etale Galois covers Y^i of the X^i, for $i = 1, 2, 3$, corresponding to $\operatorname{Tors} X^i$. We mainly use $\mathbb{Q}$-Gorenstein smoothing and unprojection technique to construct the canonical models. The construction of Y^1, Y^2 and Y^3 with a fixed point free action of $\mathbb{Z}_4$, $\mathbb{Z}_8$ and $\mathbb{Z}_4 \times \mathbb{Z}_2$ respectively is required.

Construction of Campedelli with π_1 containing $\mathbb{Z}_4$: In Subsection 5.8.1 We give a construction of etale Galois $\mathbb{Z}_4$-cover Y^1 of Campedelli surface X^1 together with fixed point free action of $\mathbb{Z}_4$. We hope that Y^1 is simply connected, but have so far been unable to prove it. So we get a Campedelli surface $U^1 = Y^1/\mathbb{Z}_4$ such that $\pi_1(U^1) \supseteq \mathbb{Z}_4$.

The surface Y^1 has invariants $p_g = 3$, $K^2_{Y^1} = 8$, and $q(Y^1) = 0$, since every etale Galois cover of Campedelli surface is regular [Reid]. Motivated by Example 4.2.3 we find an etale Galois $\mathbb{Z}_4$-cover S' of Godeaux surface containing $4 \times (-4)$-curves. From S', after contracting the (-4)-curves, we get a surface Y^1_0 containing $4 \times \frac{1}{4}(1,1)$ singularities with $p_g(Y^1_0) = 3$ and $K^2_{Y^1_0} = 8$. A $\mathbb{Q}$-Gorenstein smoothing Y^1_t of Y^1_0 together with a fixed point free action of $\mathbb{Z}_4$ on Y^1_t gives us the canonical model of Campedelli surface U^1 with $\pi_1(U^1) \supseteq \mathbb{Z}_4$. So far we have been unable to prove the isomorphism $\pi_1(U^1) \cong \mathbb{Z}_4$.

Campedelli with $\pi_1 \cong \mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2$: These surfaces are constructed by giving simply connected surfaces Y^i $(i = 2,3)$ with numerical invariants $p_g(Y^i) = 7$ and $K^2_{Y^i} = 16$, and $q(Y^i) = 0$ (for $i = 2,3$)[Reid] with the appropriate group action.

To construct Y^i $(i = 2,3)$ we adopt the same technique as in the previous case. We start from a construction of etale Galois $\mathbb{Z}_4$-covers S^i $(i = 2,3)$ of Godeaux surface containing $4 \times (-4)$-curves. In the next stage we consider the contraction $\Upsilon^i_0 \to S^i$ $(i = 2,3)$ of all of these (-4)-curves such that $p_g(\Upsilon^i_0) = 3$, $K^2_{\Upsilon^i_0} = 8$. Each of the surface Υ^i_0 $(i = 2,3)$ contains $4 \times \frac{1}{4}(1,1)$ singularities. In next step we find $\mathbb{Q}$-Gorenstein smoothing of Υ^i_0, let Υ^i_t $(i = 2,3)$ be a general fibre of the smoothing. The $\mathbb{Z}_4$-covers S^i of Godeaux surface are constructed in such a way that we get extra symmetry in Υ^i_t $(i = 2,3)$. Due to this symmetry we are able to find surfaces Y^i_t $(i = 2,3)$ such that $\Upsilon^i_t = Y^i_t/\mathbb{Z}_2$ $(i = 2,3)$. The surfaces Y^i_t $(i = 2,3)$ are simply connected with invariants $p_g(Y^i_t) = 7$ and $K^2_{Y^i_t} = 16$. A general element of the family Y^i_t (for $i = 2,3$) is our required surface Y^i (i=2,3). The difference in the constructions of S^2 and S^3 ensures that we get a fixed point free action of $\mathbb{Z}_8$ on Y^2, and of $\mathbb{Z}_4 \times \mathbb{Z}_2$ on Y^3 respectively.

Key varieties: To simplify the situation we use the key varieties technique at every stage of our construction. Our key varieties are Fano n-folds. In first stage taking appropriate quadric sections of a key variety gives us the required etale Galois $\mathbb{Z}_4$-covers of Godeaux surface for each construction. In fact in each case the $\mathbb{Z}_4$-covers of Godeaux surface are given as $(a-2)$ quadric sections of Fano n-folds.

In the second stage, instead of unprojecting $4 \times (-4)$-curves from the $\mathbb{Z}_4$-covers S', S^1, and S^2 we unproject suitable loci from the key varieties. As we shall see by taking appropriate quadric sections of the unprojected key varieties we get the surfaces Y_0^1, Υ_0^i (i=2,3). In the third stage we get the $\mathbb{Q}$-Gorenstein smoothings by taking different quadric sections of the unprojected key varieties. In the final step we then calculate the fundamental groups.

Up to projective transformation the n-fold key varieties $V^a \subset \mathbb{P}(1^3, 2^a)$, for $a \geq 3$, and with coordinates $x_1, x_2, x_3, u_1, u_2, \ldots, u_a$, may or may not contain $\mathbb{P}^2_{x_1,x_2,x_3}$. These two possibilities differentiate the Campedelli surfaces with algebraic fundamental group of order 8 and of order at least 4.

We start from the problem of constructing etale Galois $\mathbb{Z}_4$-covers of Godeaux surface containing $4 \times (-4)$-curves.

5.4 (-4)-curves on $\mathbb{Z}_4$-cover of Godeaux

Let Γ be a (-4)-curve on a surface S. This means $\Gamma \cong \mathbb{P}^1$ and $\Gamma^2 = -4$. By the adjunction formula we get $K_S \cdot \Gamma = 2$. We further assume that $\Gamma \subset \mathbb{P}(1^3, 2^2)$

is given by the intersection of three quadrics $\Gamma = \bigcap_{i=1}^{3} Q_i \subset \mathbb{P}(1^3, 2^2)$.

Our next aim is to find out how many general conics, defined as above, we can find in a general intersection of quartics in $\mathbb{P}(1^3, 2^2)$. Let $S_{4,4}\colon (D_4^1 = D_4^2 = 0) \subset \mathbb{P}(1^3, 2^2)$, where D_4^1 and D_4^2 are two general quartics.

Since $S_{4,4}$ is a complete intersection we first discuss the situation for a general quartic $D_4 \in \mathbb{P}(1^3, 2^2)$. There is one-to-one correspondence between such curves Γ in $\mathbb{P}(1^3, 2^2)$ and points of $\mathrm{Gr}(3, V)$, where $V = H^0(\mathbb{P}(1^3, 2^2), \mathcal{O}(2))$. By abuse of notation we also denote by Γ its image in $\mathrm{Gr}(3, V)$. Consider the incidence variety

$$I := \left\{(\Sigma, \Gamma) \;\middle|\; \Gamma \in \mathrm{Gr}(3, V),\, \Sigma \in H^0(\mathbb{P}(1^3, 2^2), \mathcal{O}(4)),\, \Gamma \subset \Sigma\right\} \qquad (5.4.1)$$

with two projections, $\pi_1\colon I \to \mathrm{Gr}(3, V)$ and $\pi_2\colon I \to H^0(\mathbb{P}(1^3, 2^2), \mathcal{O}(4))$. To determine the dimension of I we must work out the dimension of the fibres of π_1. For a fixed $\Gamma \in \mathrm{Gr}(3, V)$ the fibre of π_1 is made up of quartics vanishing at Γ, imposes 9 conditions on D_4. The fibre has projective dimension 20, so I is 35 dimensional. So we have:

Lemma 5.4.2 *There exist a quartic in $\mathbb{P}(1^3, 2^2)$ that contains a 6-dimensional family of conics.*

To find the number of general conics contained in a quartic we proceed as follows. Let $D_4 \subset \mathbb{P}(1^3, 2^2)$ be a general quartic and Γ a degree 2 curve in $\mathbb{P}(1^3, 2^2)$. The curve Γ can be given by a map

$$\phi : \mathbb{P}^1_{u,v} \to \mathbb{P}(1^3, 2^2),$$

defined by

$$p \mapsto (x_1(p) : x_2(p) : x_3(p) : u_1(p) : u_3(p)),$$

where $p = (u : v)$ and

$$\begin{aligned} x_1, x_2, x_3 &\in \operatorname{Sym}^2 \langle u, v \rangle, \\ u_1, u_3 &\in \operatorname{Sym}^4 \langle u, v \rangle. \end{aligned}$$

Here x_1, x_2, x_3, u_1, u_3 are homogeneous coordinates of $\mathbb{P}(1^3, 2^2)$ with weights $1, 1, 1, 2, 2$ respectively. To find the conditions on D_4 such that $\Gamma \subset D_4$, we restrict D_4 to $\operatorname{Im} \phi$ and this imposes 9 conditions on the coefficients of D_4. Hence if we fix 3 general conics, we can only expect to find a 3-dimensional family of D_4, and if we fix four general conics then we expect to get none.

Coming back to our case, since $H^0(\mathcal{O}_{\mathbb{P}}(4))$ is 30 dimensional so by abuse of notation we can have $S_{4,4} \in \operatorname{Gr}(2, 30)$. By the above computation imposing a general conic on $S_{4,4}$ implies that $S_{4,4} \in \operatorname{Gr}(2, 21)$. Hence we can only have three general conics in a general intersection of quartics $S_{4,4} \subset \mathbb{P}(1^3, 2^2)$.

So its impossible to find four general conics in an intersection of two quartics in $\mathbb{P}(1^3, 2^2)$. But if we specialize our conics then the situation becomes easy, as the following discussion shows.

Consider $D_4 \subset H^0(\mathbb{P}(1^3, 2^2), \mathcal{O}(4))$. We can write D_4 in the form

$$D_4 = \alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_3,$$

where $\alpha_i, \beta_i \in H^0(\mathbb{P}(1^3, 2^2), \mathcal{O}(2))$. The eight conics, given by intersection of three quadrics, contained in D_4 are

$$\Gamma_1\colon \alpha_1=\alpha_2=\alpha_3=0, \quad \Gamma_2\colon \alpha_1=\alpha_2=\beta_3=0,$$
$$\Gamma_3\colon \alpha_1=\beta_2=\alpha_3=0, \quad \Gamma_4\colon \alpha_1=\beta_2=\beta_3=0,$$
$$\Gamma_5\colon \beta_1=\alpha_2=\alpha_3=0, \quad \Gamma_6\colon \beta_1=\alpha_2=\beta_3=0,$$
$$\Gamma_7\colon \beta_1=\beta_2=\alpha_3=0, \quad \Gamma_8\colon \beta_1=\beta_2=\beta_3=0,$$

These eight conics form two 4-tuples $(\Gamma_1,\Gamma_4,\Gamma_6,\Gamma_7)$, $(\Gamma_2,\Gamma_3,\Gamma_5,\Gamma_8)$ in such a way that in each 4-tuple the conics are pairwise disjoint.

So there is a way of finding four pairwise disjoint conics in a general quartic in $\mathbb{P}(1^3,2^2)$. But in our construction of S, the two quartics f_4 and g_4 belong to different eigenspaces. The next section discusses a solution.

5.5 Canonical ring revisited

Let S be an n-fold with $G \cong \mathbb{Z}_r = \langle g\rangle$ acting on S. If $T = S/G$ then the canonical ring of S is

$$R(S,K_S) = \bigoplus_{n\geq 0} H^0(S,\mathcal{O}(nK_S)) = \bigoplus_{n\geq 0,\sigma\in G} H^0(T,\mathcal{O}(nK_T+\sigma)).$$

Thus any $f\in H^0(\mathcal{O}(nK_S))$ can be written as

$$f = f_{(0)} + f_{(1)} + \cdots + f_{(|G|-1)},$$

where

$$f_{(i)} \in H^0(\mathcal{O}(nK_T+i)), \quad \text{for} \quad i\in G.$$

For fixed n there exist a filtration

$$H^0(S,\mathcal{O}(nK_S))[1] \subset \cdots \subset H^0(S,\mathcal{O}(nK_S))[h] \subset \cdots \subset H^0(S,\mathcal{O}(nK_S))[|G|], \tag{5.5.1}$$

where h is a divisor of $|G|$ and $f \in H^0(S,\mathcal{O}(nK_S))[h]$ iff h is a positive integer such that

$$\operatorname{div}(g^h \cdot f) = \operatorname{div} f, \tag{5.5.2}$$

in other words f is fixed by $\mathbb{Z}_r/\mathbb{Z}_h = \langle g^h \rangle$. From this filtration we can write a filtration of the canonical ring $R(S,K_S)$

$$R(S,K_S)[1] \subset \cdots \subset R(S,K_S)[h] \subset \cdots \subset R(S,K_S)[|G|], \tag{5.5.3}$$

where h is a divisor of $|G|$ and

$$R(S,K_S)[h] = \bigsqcup_{n\geq 0} H^0(S,\mathcal{O}(nK_S))[h] \tag{5.5.4}$$

From the above filtration (5.5.1) we define the following for fixed n

$$H^0(S,\mathcal{O}(nK_S))^h = H^0(S,\mathcal{O}(nK_S))[h] \backslash \left(\bigcup_{r|G,\ r<h} H^0(S,\mathcal{O}(nK_S))[r] \right),$$

and from here we define

$$R(S,K_S)^h = \bigsqcup_{n\geq 0} H^0(S,\mathcal{O}(nK_S))^h \tag{5.5.5}$$

which extends to the canonical ring $R(S, K_S)$ as the following disjoint union

$$R(S, K_S) = \bigsqcup_{h|r} R(S, K_S)^h.$$

Observe that if $f \in R(S, K_S)^h$ then so are $g \cdot f, \dots, g^{h-1} \cdot f$, that is, elements in $R(S, K_S)^h$ exist in h-tuples. We denote $f_i := g^{i-1} \cdot f$ for $1 \leq i \leq h$ and the h-tuple containing f by $[f]_h := (f_1, \dots, f_h)$.

We use this way of writing canonical ring in the next section for finding $4 \times (-4)$-curves in $S_{4,4} \subset \mathbb{P}(1^3, 2^2)$. We discuss the situation in a general case.

5.5.6 Intersection of two quartics

Let V^a be the intersection of two quartics F_4^a and G_4^a in $\mathbb{P}(1^3, 2^a)$, for $a \geq 2$. Suppose that V^a has an action of $\mathbb{Z}_4 := \langle g \rangle$ and $W^a = V^a/\mathbb{Z}_4$. We take F_4^a to be invariant and G_4^a anti-invariant under the action of $\mathbb{Z}_4$. The aim of this section is to understand how to force V^a to contain one or more codimension three loci obtained as a complete intersection of three quadrics. If we take $a = 2$ then this gives us $4 \times (-4)$-curves in the $\mathbb{Z}_4$-cover S of the Godeaux surface with $\pi_1 = \mathbb{Z}_4$.

The idea is to first describe the codimension three loci, defined by the intersection of three quadrics, and then to find the quartics containing them. Since the loci are intersections of three quadrics we consider the space of quadrics in V^a, that is $H^0(\mathcal{O}_{V^a}(2))$, which according to Section 5.5 has the following filtration

$$H^0(\mathcal{O}_{V^a}(2))[1] \subset H^0(\mathcal{O}_{V^a}(2))[2] \subset H^0(\mathcal{O}_{V^a}(2))[4]. \tag{5.5.7}$$

From here we can explicitly write

$$
\begin{aligned}
H^0(\mathcal{O}_{V^a}(2))^1 &:= H^0(\mathcal{O}_{V^a}(2))[1], \\
H^0(\mathcal{O}_{V^a}(2))^2 &:= H^0(\mathcal{O}_{V^a}(2))[2] \backslash H^0(\mathcal{O}_{V^a}(2))^1, \\
H^0(\mathcal{O}_{V^a}(2))^4 &:= H^0(\mathcal{O}_{V^a}(2))[4] \backslash \left(H^0(\mathcal{O}_{V^a}(2))^1 \cup H^0(\mathcal{O}_{V^a}(2))^2\right).
\end{aligned}
$$

To understand more about the construction of $H^0(\mathcal{O}(2K_{V^a}))$ we consider the monomial basis B_i of ith eigenspace of $H^0(\mathcal{O}_{V^a}(2))$, for $i = 0, 1, 2, 3$. Then we have

$$
\begin{aligned}
H^0(\mathcal{O}_{V^a}(2))^1 = &\bigsqcup_{i=0}^{3} \langle B_i \rangle, \\
H^0(\mathcal{O}_{V^a}(2))^2 = &\bigsqcup_{i=0,1} \langle B_i, B_{i+2} \rangle \backslash H^0(\mathcal{O}_{V^a}(2))^1, \\
H^0(\mathcal{O}_{V^a}(2))^4 = &\langle B_0, \ldots, B_3 \rangle \backslash (H^0(\mathcal{O}_{V^a}(2))^1 \cup H^0(\mathcal{O}_{V^a}(2))^2).
\end{aligned}
$$

Further more $H^0(\mathcal{O}_{V^a}(2))^2$ can be written as

$$
H^0(\mathcal{O}_{V^a}(2))^2 = H^0(\mathcal{O}_{V^a}(2))^{2,0} \bigsqcup H^0(\mathcal{O}_{V^a}(2))^{2,1}, \tag{5.5.8}
$$

where

$$
H^0(\mathcal{O}_{V^a}(2))^{2,0} = \langle B_0, B_2 \rangle \backslash H^0(\mathcal{O}_{V^a}(2))^1, \tag{5.5.9}
$$

$$
H^0(\mathcal{O}_{V^a}(2))^{2,1} = \langle B_1, B_3 \rangle \backslash H^0(\mathcal{O}_{V^a}(2))^1. \tag{5.5.10}
$$

Now consider two general tuples $[Q]_4 \in H^0(\mathcal{O}_{V^a}(2))^4$ and $[R]_2 \in H^0(\mathcal{O}_{V^a}(2))^2$. In fact a general element $Q \in H^0(\mathcal{O}_{V^a}(2))^4$ is a general linear combination of elements of $H^0(\mathcal{O}_{V^a}(2))$ and

$$[Q]_4 = (Q_1, Q_2, Q_3, Q_4) = (Q,\, g \cdot Q,\, g^2 \cdot Q,\, g^3 \cdot Q).$$

Similarly

$$[R]_2 = (R_1, R_2) = (R,\, g \cdot R).$$

The two tuples $[Q]_4$ and $[R]_2$ give us four codimension 3 loci Γ_i^a in $\mathbb{P}(1^3, 2^a)$, for $1 \leq i \leq 4$. Each Γ_i^a is defined by the intersection of three quadrics, according to the following table:

curves	Q_1	Q_2	Q_3	Q_4	R_1	R_2
Γ_1	✓	✓				✓
Γ_2		✓	✓		✓	
Γ_3			✓	✓		✓
Γ_4	✓			✓	✓	

. (5.5.11)

Here the defining quadrics of Γ_i^a are represented by the ✓ in the above table, for example,

$$\Gamma_1^a \colon (Q_1 = Q_2 = R_2 = 0) \subset \mathbb{P}(1^3, 2^a).$$

The degree of the Γ_i^a is 2^{3-a}. Notice from the table that these Γ_i^a are permuted by the $\mathbb{Z}_4$ group action. Since any pair of the Γ_i^a has only one common defining equation and each of the Γ_i^a is defined by three equations so the Γ_i^a intersect with each other only in codimension ≥ 5. which means that in the case $a = 2$ the Γ_i^2 are pairwise disjoint.

Next we discuss a way to construct F_4^a and G_4^a in $\mathbb{P}(1^3, 2^a)$ containing the Γ_i^a, for $1 \leq i \leq 4$. We construct these quartics from $[Q]_4$ and $[R]_2$. According to (5.5.8) the tuple $[R]_2$ can belong to exactly one of the following spaces

$$H^0(\mathcal{O}_{V^a}(2))^{2,0}, \quad H^0(\mathcal{O}_{V^a}(2))^{2,1}, \tag{5.5.12}$$

which give rise to two cases depending on which space the tuple $[R]_2$ belong to. We discuss the situation below according to these two cases.

The case $H^0(\mathcal{O}_{V^a}(2))^{2,0}$

If $R_1, R_2 \in H^0(\mathcal{O}_{V^a}(2))^{2,0}$ then R_1R_2 is invariant. We can write every invariant quartic as

$$F_4^a := R_1R_2 - \frac{1}{2}(Q_1Q_3 + Q_2Q_4), \tag{5.5.13}$$

and we can choose anti-invariant G_4^a to be

$$G_4^a := Q_1Q_3 - Q_2Q_4, \tag{5.5.14}$$

an anti-invariant quartic.

The case $H^0(\mathcal{O}_{V^a}(2))^{2,1}$

In case of $R_1, R_2 \in H^0(\mathcal{O}_{V^a}(2))^{2,1}$ the quartic R_1R_2 is anti-invariant. For the sake of simplicity we change the tuple $[Q]_4$ by replacing one of its elements by negative multiple of it. In other words we take another tuple $[P]_4 \in H^0(\mathcal{O}_{V^a}(2))^4$ such that

$$Q_1 = P_1, \quad Q_2 = -P_2, \quad Q_3 = P_3, \quad Q_4 = P_4.$$

Thus, under the action of $\mathbb{Z}_4$ we have

$$\begin{aligned} Q_1 &\to -Q_2, \qquad & Q_2 &\to -Q_3, \\ Q_3 &\to \quad Q_4, \qquad & Q_4 &\to \quad Q_1\,. \end{aligned}$$

Under this action the polynomial

$$Q_1Q_3 + Q_2Q_4$$

is anti-invariant. In this case every anti-invariant G_4 can be written as

$$G_4^a := R_1R_2 - \frac{1}{2}(Q_1Q_3 + Q_2Q_4), \tag{5.5.15}$$

and F_4^a can be taken to be:

$$F_4^a := Q_1Q_3 - Q_2Q_4. \tag{5.5.16}$$

In the light of the above discussion we have:

Theorem 5.5.17 *There exist a nonsingular $\mathbb{Z}_4$-cover of a Godeaux surface with $\pi_1 = \mathbb{Z}_4$ containing a $\mathbb{Z}_4$ orbit of disjoint smooth conics.*

Proof Let T be a Godeaux surface with $\pi_1 = \mathbb{Z}_4$, and suppose $S \to T$ is the etale Galois $\mathbb{Z}_4$-cover of T. We have seen in Section 5.2 that S is given by the intersection of two quartics f_4 and g_4, where f_4 is invariant and g_4 is anti-invariant. We specialize S by taking it to be V^2, that is, we take f_4 and

g_4 to be F_4^2 and G_4^2 respectively, so we have

$$S = V^2 \colon (F_4^2 = G_4^2 = 0) \subset \mathbb{P}(1^3, 2^2).$$

In this case the Γ_i^2 are degree 2 curves. Since the embedding of V^2 in $\mathbb{P}(1^3, 2^2)$ is canonical, we have $K_{V^2} \cdot \Gamma_i^2 = 2$, for $i \in \{1, \dots, 4\}$. Hence the adjunction formula says that we have $4 \times (-4)$-curves on V^2. Now since each of the two Γ_i^2 intersect only in codim≥ 5 we have

$$\Gamma_i^2 \cdot \Gamma_j^2 = 0 \quad \text{for } i \neq j.$$

Moreover the Γ_i^2 are permuted by the group $\mathbb{Z}_4$. We have seen that there are two different constructions of V^a for $a \geq 2$. So we get two different constructions of a $\mathbb{Z}_4$-cover S of Godeaux each containing $4 \times (-4)$-curves. The smoothness of S and the Γ_i can be proved by a computer algebra system, for example, in Magma. $\square$

5.6 Gorenstein unprojection

The next object is to calculate the unprojection of the $\Gamma_i^a \subset V^a$, for $1 \leq i \leq 4$. We do this process in four steps, one step for each Γ_i^a.

For fixed i the unprojection $\Gamma_i^a \subset V^a$ is obtained by writing down functions on V^a with poles along Γ_i^a and then adjoin them to the coordinate ring of V^a. For this we first consider

$$0 \to \mathcal{I}_{\Gamma_i^a} \to \mathcal{O}_{V^a} \to \mathcal{O}_{\Gamma_i^a} \to 0,$$

by applying the derived functor of $\mathcal{H}om_{\mathcal{O}_{V^a}}(-, \omega_{V^a})$ we get

$$0 \to \omega_{V^a} \to \mathcal{H}om_{\mathcal{O}_{V^a}}(\mathcal{I}_{\Gamma_i^a}, \omega_{V^a}) \to \omega_{\Gamma_i^a} \to 0.$$

The above is obtained after using Grothendieck–Serre duality to calculate $\omega_{\Gamma_i^a}$, and also using $\mathcal{H}om_{\mathcal{O}_{V^a}}(\mathcal{O}_{\Gamma_i^a}, \omega_{V^a}) = 0$ since each Γ_i^a is a codimension 1 effective divisor on V^a. Since V^a is a complete intersection it is Gorenstein so we have $\omega_{V^a} \cong \mathcal{O}_{V^a}$ as an $\mathcal{O}_{V^a}$-module. Since $\mathcal{I}_{\Gamma_i^a} = \mathcal{O}_{V^a}(-\Gamma_i^a)$ so $\mathcal{H}om_{V^a}(\mathcal{I}_{\Gamma_i^a}, \omega_{V^a})$ can be seen as functions of V^a with poles along Γ_i^a. So to calculate unprojection we need to calculate the generators s_{ij} of $\mathcal{H}om_{V^a}(\mathcal{I}_{\Gamma_i^a}, \omega_{V^a})$ and the relations over $\mathcal{O}_{V^a}$ holding between them. The s_{ij} are called the unprojection variables and the unprojection $\Gamma_i^a \subset V^a$ is the ring $\mathcal{O}_{V^a}[s_{ij}]$. For more details on this theory see [PR04] and [Reib].

5.7 Calculating unprojection from Γ_i^a

Let V_i^a, for $1 \leq i \leq 4$, be the variety after unprojecting i times. It is interesting to see that the Gorenstein varieties V_i^a at each stage are given by the 4×4 Pfaffians of an $(4+i) \times (4+i)$ antisymmetric matrix J_i^a. Moreover J_i^a is submatrix of J_{i+1}^a, for $1 \leq i \leq 3$.

As we have seen that there are two different constructions for V^a. Due to our choices of coefficients, in both cases we have

$$F_4^a = G_4^a = 0 \iff Q_1Q_3 = Q_2Q_4 = R_1R_2. \tag{5.7.1}$$

From the above we may consider the following two quartics to give a common treatment of both cases

$$\begin{aligned} Q_1Q_3 - Q_2Q_4 &= 0, \\ Q_1Q_3 - R_1R_2 &= 0\,. \end{aligned} \tag{5.7.2}$$

In the case of $\Gamma_i^a \subset V^a$ the Γ_i^a are projectively Gorenstein so $\mathcal{H}om_{V^a}(\mathcal{I}_{\Gamma_i^a}, \omega_{V^a})$ has only one generator.

Unprojection from $\Gamma_1^a \subset V^a$

The unprojection $\Gamma_1^a \subset V^a$ is the ring $\mathcal{O}_{V^a}[s_1]$ so we want to find relations expressing s_1 as a rational function. Geometrically the rational section s_1 of $\mathcal{H}om(\mathcal{I}_{\Gamma_1^a}, \mathcal{O}_{V^a}(K_{V^a} - K_{\Gamma_1^a}))$ defines a rational map

$$V^a \dashrightarrow V_1^a \subset \operatorname{Proj} k[x_1, x_2, x_3, u_1, \ldots, u_a, s_1].$$

To find relations expressing s_1 we proceed as follows; we first write (5.7.2) as

$$AX = 0, \tag{5.7.3}$$

where

$$A = \begin{pmatrix} Q_3 & -Q_4 & 0 \\ Q_3 & 0 & R_1 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ R_2 \end{pmatrix}, \tag{5.7.4}$$

here the entries of the column matrix are the defining equations of Γ_1^a. Since V^a is a complete intersection, the equations of V_1^a can be described using Cramer's rule

$$s_1 = A_i/X_i, \quad \text{for } i = 1, 2, 3,$$

where the A_i are the 2×2 minors of A obtained by omitting the ith column. The variety V_1^a can be given as the $4{\times}4$ Pfaffians of the following antisymmetric matrix

$$J_1 = \begin{pmatrix} -s_1 & 0 & Q_4 & R_1 \\ & Q_3 & Q_4 & 0 \\ -\operatorname{Sym} & & R_2 & Q_2 \\ & & & Q_1 \end{pmatrix}, \tag{5.7.5}$$

and V_1^a is Gorenstein by [KM83].

Unprojection from $\Gamma_2^a \subset V_1^a$

After unprojecting from Γ_1^a the defining equations of $\Gamma_2^a \subset V_1^a$ are

$$\Gamma_2^a \colon Q_2 = Q_3 = R_2 = s_1 = 0,$$

and $\Gamma_2^a \subset V_1^a$ is again projectively Gorenstein. Using the same arguments as above we get a rational map

$$V_1^a \dashrightarrow V_2^a \subset \operatorname{Proj} k[x_1, x_2, x_3, u_1, \ldots, u_a, s_1, s_2].$$

The defining relations of V_2^a are given by the 4×4 Pfaffians of the matrix (5.7.5), the following matrix

$$J_{21} = \begin{pmatrix} -s_2 & 0 & Q_4 & R_2 \\ & Q_1 & Q_4 & 0 \\ -\operatorname{Sym} & & R_1 & Q_2 \\ & & & Q_3 \end{pmatrix}, \tag{5.7.6}$$

and a relation between two unprojection variable given by

$$s_2s_1 - Q_4^2. \tag{5.7.7}$$

The defining relations of the projectively Gorenstein variety V_2^a can also be obtained as the 4×4 Pfaffians of an antisymmetric matrix J_2^a, given by

$$J_2^a = \begin{pmatrix} Q_4 & 0 & R_2 & s_2 & 0 \\ & -s_1 & 0 & Q_4 & R_1 \\ & & Q_3 & Q_4 & 0 \\ & \text{-Sym} & & R_2 & Q_2 \\ & & & & Q_1 \end{pmatrix}. \tag{5.7.8}$$

Unprojection from Γ_3^a and Γ_4^a

Let the unprojection from $\Gamma_4^a \subset V_2^a$ be V_3^a and from $\Gamma_3^a \subset V_3^a$ be V_4^a, such that

$$V_2^a \dashrightarrow V_4^a \subset \operatorname{Proj} k[x_1, x_2, x_3, u_1, \ldots, u_a, s_1, \ldots, s_4]. \tag{5.7.9}$$

The relations of V_4^a are the 4×4 Pfaffians of matrix (5.7.8), the following two antisymmetric matrices

$$J_{31}^a = \begin{pmatrix} -s_3 & 0 & Q_2 & R_1 \\ & Q_3 & Q_2 & 0 \\ -\text{Sym} & & R_2 & Q_4 \\ & & & Q_3 \end{pmatrix}, \tag{5.7.10}$$

$$J_{41}^a = \begin{pmatrix} -s_4 & 0 & Q_2 & R_2 \\ & Q_3 & Q_2 & 0 \\ -\text{Sym} & & R_1 & Q_4 \\ & & & Q_1 \end{pmatrix}, \tag{5.7.11}$$

and the following relations between the unprojection variables $s_1, \dots, s_4$

$$
\begin{aligned}
&s_4 s_3 - Q_2^2\,, \quad s_4 s_2 - R_2^2\,,\\
&s_4 s_1 - Q_3^2\,, \quad s_3 s_2 - Q_1^2\,,\\
&s_3 s_1 - R_1^2\,, \quad s_2 s_1 - Q_4^2.
\end{aligned}
\tag{5.7.12}
$$

All of the defining relations of the projectively Gorenstein variety V_4^a, given above, can also be given by the 4×4 Pfaffians of the following matrix

$$
J_4^a = \begin{pmatrix}
-Q_1 & -Q_2 & -R_1 & -R_1 & -Q_2 & 0 & s_3 \\
 & 0 & Q_3 & 0 & s_4 & R_2 & 0 \\
 & & Q_4 & 0 & R_2 & s_2 & 0 \\
 & & & -s_1 & 0 & Q_4 & R_1 \\
 & & & & Q_3 & Q_4 & 0 \\
 & \text{-Sym} & & & & R_2 & Q_2 \\
 & & & & & & Q_1
\end{pmatrix}. \tag{5.7.13}
$$

Here the weight of each of the unprojection variables is 2. Moreover the group $\mathbb{Z}_4$ acts, in both cases, on the unprojection variables as follows

$$
s_1 \mapsto s_2 \mapsto s_3 \mapsto s_4 \mapsto s_1. \tag{5.7.14}
$$

The following is interesting in its own right and also help us in our future constructions.

Lemma 5.7.15 *The 4×4 Pfaffians of the matrix (5.7.13), and the 2×2 minors of*

$$
\begin{pmatrix}
s_4 & Q_2 & R_2 & Q_3 \\
 & s_3 & Q_1 & R_1 \\
\text{Sym} & & s_2 & Q_4 \\
 & & & s_1
\end{pmatrix}. \tag{5.7.16}
$$

generate the same ideal.

Proof A direct computation proves they are same. □

Remark 5.7.17 The motivation behind the above unprojection calculations and Lemma 5.7.15 are results of Chapter 3 and Chapter 4. In particular the construction of a surface is obtained as 4 quadric sections of the key variety V_8.

5.8 Constructing Campedelli surface

Now we are in a position to start our constructions, that is, the construction of canonical rings of Campedelli surfaces U^1, X^2, and X^3. See Section 5.3 for a general idea of the method used.

5.8.1 Campedelli with π_1 containing $\mathbb{Z}_4$

This section discusses the case when, up to projective transformation, the Fano n-fold $V^a\colon F_4^a \cap G_4^a \subset \mathbb{P}(1^3, 2^a)$, for $a \geq 2$, contains a proper subset of $\mathbb{P}(x_1, x_2, x_3) \subset \mathbb{P}(1^3, 2^a)$. We take the simplest of these cases when $a = 3$.

Consider a Campedelli surface X^1 with $\operatorname{Tors} X^1 = \mathbb{Z}_4$. We consider Y^1, where $Y^1 \to X^1$ is the etale Galois cover corresponding to $\operatorname{Tors} X^1$. We construct the canonical ring of Y^1 together with the action of $\mathbb{Z}_4$.

From [Reia] we have $H^1(\mathcal{O}(Y^1)) = 0$. The other invariants of Y^1 are $p_g(Y^1) = 3$ and $K^2_{Y^1} = 8$. We start from the etale Galois $\mathbb{Z}_4$-cover S' of a Godeaux surface with $\pi_1 = \mathbb{Z}_4$ containing $4 \times (-4)$-curves. Unprojecting these (-4)-curves on S gives us a singular surface Y^1_0 with $4 \times \frac{1}{4}(1,1)$ singularities.

The invariants of Y_0^1 are $p_g(Y_0^1) = 3$, and $K^2_{Y_0^1} = 8$. A $\mathbb{Q}$-Gorenstein smoothing of Y_0^1 gives us our required surface Y^1.

In fact our special $\mathbb{Z}_4$-cover S' of Godeaux surface, containing $4 \times (-4)$-curves, can also be given as a quadric section of the Fano 3-fold V^3, see Section 5.5 for the definition. Instead of unprojecting (-4)-curves on S' we unproject the $\Gamma_i \cong \mathbb{P}^2$ (5.5.11) on V^3. Taking appropriate quadric sections of the unprojected variety V_4^3 gives rise to a smoothing Y_t^1 of Y_0^1. We choose the $4 \times (-4)$-curves in such a way that the action of $\mathbb{Z}_4$ on S' extends to Y_t^1, which leads us to our construction of U^1.

The construction

An etale Galois cover S of Godeaux surface with $\pi_1 = \mathbb{Z}_4$ has already been discussed in 5.2 and is given as $f_4 \cap g_4 \subset \mathbb{P}(1^3, 2^2)$. We take special quartics so that the special cover, S', contains $4 \times (-4)$-curves. By Theorem (5.5.17) there are at least two possible solutions for this, each containing $4 \times (-4)$-curves. In fact we take $S' = V^2$ by taking f_4, g_4 to be F_4^2 and G_4^2 respectively. As we have seen in Section 5.5 that there are two cases of construction of V^2. The following discussion is independent of the choice of the cases. Up to projective transformation, we can write

$$S' = q_2 \cap V^3,$$

a quadratic section q_2 of the Fano 3-fold V^3 of index 1, given by

$$V^3 \colon (F_4^3 \cap G_4^3 = 0) \subset \mathbb{P}(1^3, 2^3)_{\langle x_1, x_2, x_3, u_1, u_2, u_3 \rangle},$$

such that $\mathrm{wt}\, x_i = 1$, $\mathrm{wt}\, u_i = 2$ and the group $\mathbb{Z}_4$ acts as

$$x_i \mapsto \varepsilon^i x_i \,, u_i \mapsto \varepsilon^i u_i \,, \forall\, i, \tag{5.8.2}$$

where ε is a primitive 4th root of unity. The quadric q_2 is a general anti-invariant quadric. In the case of V^3 we have $\Gamma_i \cong \mathbb{P}^2$. The four copies of $\mathbb{P}^2$ are given by table (5.5.11). So in this case S' contains the following $4 \times (-4)$-curves

$$\Gamma_i^3 \cap q_2, \quad \text{for } 1 \leq i \leq 4.$$

After a series of unprojections from $\Gamma_i^3 \subset V^3$ we get V_4^3. From Equation (5.7.1) it is clear that in both cases of V^3 we have the same unprojection variety V_4^3. Hence we have a 3-fold V_4^3 having 4 singularities given as

$$R(V_4^3, -K_{V_4^3}) = k[x_1, x_2, x_3, u_1, u_2, u_3, s_1, \ldots, s_4]/(R), \tag{5.8.3}$$

where R are 2×2 minors of the matrix (5.7.16), in this case $(Q_1, Q_2, Q_3, Q_4) = (Q,\, g \cdot Q,\, g^2 \cdot Q, g^3 \cdot Q)$, where Q is a general element of

$$\mathrm{Sym}^2 \langle x_1, x_2, x_3 \rangle \bigoplus \langle u_1, u_2, u_3 \rangle \,,$$

and $(R_1, R_2) = (R, g \cdot R)$, where the construction of R is given in (5.5.12). Using [CPR00], and [Bro] we can see that V_4^3 is a Fano 3-fold of index 1 with $A^3 = 4$, where $A = -K_{V_4^3}$, and genus 1.

The quadric section q_2 of V_4^3 is a singular surface Y_0^1 with $4 \times \frac{1}{4}(1,1)$ singularities. To get a smoothing of Y_0^1 we move the quadric section q_2 to avoid these singularities, that is, we take the quadric

$$q_2' = q_2 + s_4 - s_3 + s_2 - s_1. \tag{5.8.4}$$

Hence the canonical ring of Y^1 is

$$R(Y^1, K_{Y^1}) = k[x_1, x_2, x_3, u_1, u_2, u_3, s_1, \dots, s_4]/(R, q_2'),$$

where R and q_2' are given in (5.8.3) and (5.8.4) respectively, and the action of group $\mathbb{Z}_4$ is given in (5.8.2) and in the following

$$s_1 \mapsto s_2 \mapsto s_3 \mapsto s_4 \mapsto s_1. \tag{5.8.5}$$

The surface Y^1 may or may not be a simply connected surface. The task is left for future.

So far we have proved the following:

Theorem 5.8.5.1 *There exist a minimal surface U^1 with $p_g = 0$, $K^2 = 2$ and* $\operatorname{Tors} U^1 \supseteq \mathbb{Z}_4$.

Remark 5.8.6 The construction of V^3 is interesting in its own right being a codimension 6 three-fold defined by 4×4 Pfaffians of a skew symmetric matrix.

5.8.7 Campedelli with $\pi_1^{\text{alg}} \cong \mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_4$

The dichotomy that our key varieties $V^a \subset \mathbb{P}(1^3, 2^a)$, for $a \geq 3$, contain a proper or improper subset of $\mathbb{P}^2_{x_1,x_2,x_3} (\subset \mathbb{P}(1^3, 2^a))$ distinguishes this section from the previous section. Here we discuss the second possibility when, up to projective transformation, V^a contains $\mathbb{P}^2(x_1, x_2, x_3)$. This case only happens when $a \geq 6$, and we take the simplest of these cases that is $a = 6$. As we have seen in Section 5.5 there are two cases of construction namely $H^0(\mathcal{O}_{V^6}(2))^{2,0}$ and $H^0(\mathcal{O}_{V^6}(2))^{2,1}$. We shall see that these two cases differentiate the cases of construction for Campedelli surface with fundamental group $\mathbb{Z}_8$ and $\mathbb{Z}_2 \times \mathbb{Z}_4$.

Rewriting the $\mathbb{Z}_4$-cover of Godeaux

Consider the 6-fold key variety V^6

$$V^6 \colon (F_4^6 = G_4^6 = 0) \subset \mathbb{P}(1^3, 2^6)_{\langle x_1,x_2,x_3,u_0,u_1,u_2,u_3,w_{a_1},w_{a_2}\rangle},$$

where $\operatorname{wt} x_i = 1$, $\operatorname{wt} u_i = 2$ and $\operatorname{wt} w_{a_i} = 2$. The subscripts are taken according to the group action of $\mathbb{Z}_4$ on $\mathbb{P}(1^3, 2^6)$, that is the group acts as

$$x_i \mapsto \varepsilon^i x_i\,,\ u_j \mapsto \varepsilon^j u_j\,,\ w_k \mapsto \varepsilon^k w_k,$$

where ε is a primitive 4th root of unity. The values of a_1 and a_2 in two different cases are

$$\begin{aligned} H^0(\mathcal{O}_{V^6}(2))^{2,0} \quad &: \quad a_1 = 0, \quad a_2 = 2, \\ H^0(\mathcal{O}_{V^6}(2))^{2,1} \quad &: \quad a_1 = 1, \quad a_2 = 3. \end{aligned}$$

Let B_i be the monomial basis of the ith eigenspace of $H^0(\mathcal{O}_{V6}(2)$, for $0 \le i \le 3$. Then in the two cases B_i are

Monomial basis	$H^0(\mathcal{O}_{V^6}(2))^{2,0}$	$H^0(\mathcal{O}_{V^6}(2))^{2,1}$
B_0	x_2^2, x_1x_3, u_0, w_0	x_2^2, x_1x_3, u_0
B_1	x_2x_3, u_1	x_2x_3, u_1, w_1
B_2	x_1^2, x_2^2, u_2, w_2	x_1^2, x_2^2, u_2
B_3	x_1x_2, u_3	x_1x_2, u_3, w_3

We know that V^6 is constructed from $[Q]_4 \in H^0(\mathcal{O}_{V^6}(2))^4$ and $[R]_2 \in H^0(\mathcal{O}_{V^6}(2))^2$. So in the case of V^6 the two tuples $[Q]_4 = (Q_1, \dots, Q_4)$ and $[R]_2 = (R_1, R_2)$, up to projective transformation, can be written as

$$\begin{aligned} R_1 &= w_{a_1} + w_{a_2}, \\ R_2 &= \varepsilon^{a_1} w_{a_1} + \varepsilon^{a_2} w_{a_2}, \end{aligned} \tag{5.8.8}$$

and

$$\begin{aligned} Q_1 &= u_0 + u_1 + u_2 + u_3, \\ Q_2 &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3, \\ Q_3 &= u_0 + \varepsilon^2 u_1 + u_2 + \varepsilon^2 u_3, \\ Q_4 &= u_0 + \varepsilon^3 u_1 + \varepsilon^2 u_2 + \varepsilon u_3. \end{aligned} \tag{5.8.9}$$

Clearly with the above change of variables the 6-fold V^6 is cone over $\mathbb{P}(1^3)$. We can write a suitable $\mathbb{Z}_4$-cover S of a Godeaux surface, containing $4 \times (-4)$-curves, obtained as quadric sections of V^6, that is

$$S = q_0 \cap q_2 \cap p_{a_1} \cap p_{a_2} \cap V^6, \tag{5.8.10}$$

and the four quadric sections are

$$\begin{aligned} q_i &\in \langle B_i \rangle, \quad \text{for } i = 0, 2 \\ p_{a_i} &\in \langle B_{a_i} \rangle, \quad \text{for } i = 1, 2. \end{aligned}$$

Since in the case of V^6 the $\Gamma_i^6 \cong \mathbb{P}(1^3, 2^3)$, for $1 \le i \le 4$ so

$$\Gamma_i^6 \cap q_0 \cap p_{a_1} \cap q_2 \cap p_{a_2} \subset S, \quad \text{for } 1 \le i \le 4,$$

which gives $4 \times (-4)$-curves in S.

The main construction

Let X^2 and X^3 be Campedelli surfaces with fundamental group $\mathbb{Z}_8$ and $\mathbb{Z}_4 \times \mathbb{Z}_2$ respectively. Let $Y^i \to X^i$ be the etale Galois cover of X^i corresponding to $\operatorname{Tors} X^i$, for $i = 1, 2$. For the construction of the X^i we use the same technique as in our previous construction. We shall see that, due to extra symmetry in these cases we get above mentioned Campedelli surfaces.

The $\mathbb{Z}_8$ case

The aim is to construct $R(Y^2, K_{Y^2})$ with a fixed point free action of $\mathrm{Gal}(Y^2/X^2)$. In this case we take the construction of V^6 when $R_1, R_2 \in H^0(\mathcal{O}_{V^6}(2))^{2,1}$, so we take a_1 and a_2 to be 1 and 3 respectively. Recall that in this case we first take $[P] \in H^0(\mathcal{O}_{V^6}(2))^4$ and then define

$$Q_1 = P_1, Q_2 = -P_2, Q_3 = P_3, Q_4 = P_4.$$

Hence the invariant and anti-invariant quartics defining V^6 are

$$\begin{aligned} F_4^6 &= Q_1Q_3 - Q_2Q_4, \\ G_4^6 &= R_1R_2 + \frac{1}{2}(Q_1Q_3 + Q_2Q_4) \end{aligned}$$

respectively. After unprojection from the $\Gamma_i^6 \subset V^6$, for $i = 1, 2, 3, 4$, we get V_4^6 defined by the 2×2 minors of the following matrix

$$\begin{pmatrix} s_4 & Q_2 & R_2 & Q_3 \\ & s_3 & Q_1 & R_1 \\ \mathrm{Sym} & & s_2 & Q_4 \\ & & & s_1 \end{pmatrix}. \tag{5.8.11}$$

Hence the surface $\Upsilon_0^2 = q_0 \cap q_2 \cap p_1 \cap p_3 \cap V_4^6$ has $4 \times \frac{1}{4}(1,1)$ singularities with $p_g(\Upsilon_0^2) = 3$ and $K^2_{\Upsilon_0^2} = 8$. To get a smoothing Υ_t^2 of Υ_0^2 we move the quadric section and take the following quadrics

$$\begin{aligned}
q_0' &:= q_0 + \sum_{i=1}^4 s_i, \\
p_1' &:= p_1 + \sum_{i=1}^4 \varepsilon^{3i} s_i, \\
q_2' &:= q_2 + \sum_{i=1}^4 (-1)^i s_i, \\
p_3' &:= p_3 + \sum_{i=1}^4 \varepsilon^i s_i.
\end{aligned}$$

Hence for fix t ($\neq 0$) we have

$$R(\Upsilon_t^2, K_{\Upsilon_t^2}) = k[x_1, x_2, x_3, u_j, w_1, w_3, s_k]/(q_0', q_2', p_1', p_3', R),$$

where $1 \leq i \leq 3$, $0 \leq j \leq 3$, $1 \leq k \leq 4$ and R are 2×2 minors of the matrix (5.8.11). From (5.8.8) and (5.8.9) we have the following nonsingular matrix of coefficients

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
-1 & -\varepsilon & -\varepsilon^2 & -\varepsilon^3 & 0 & 0 \\
1 & \varepsilon^2 & \varepsilon^4 & \varepsilon^2 & 0 & 0 \\
1 & \varepsilon^3 & \varepsilon^2 & \varepsilon & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & \varepsilon & \varepsilon^3
\end{pmatrix}. \tag{5.8.12}$$

So we can express $u_0, u_1, u_2, u_3, w_1, w_3$ in terms of $Q_1, Q_2, Q_3, Q_4, R_1, R_2$, making these as our new variables. Let q_i'', p_i'' are quadrics under this change of coordinates. Then

$$\begin{aligned} R(\Upsilon_t^2, K_{\Upsilon_t^2}) &= k[x_1, x_2, x_3, s_j, Q_j, R_1, R_2]/(q_0'', q_2'', p_1'', p_3'', R) \\ &\cong \Big[k[x_1, x_2, x_3, , z_j]/(q_0'', q_2'', p_1'', p_3'')\Big]^{\mathbb{Z}_2}, \end{aligned}$$

where by abuse of notation we call $q_0'', q_2'', p_1'', p_3''$ the quadrics in our new variables and the action of $\mathbb{Z}_2$ is given by

$$x_i \mapsto x_i, \quad z_i \mapsto -z_j, \ \forall i, j \tag{5.8.13}$$

and $\mathbb{Z}_4$ acts on the new variables z_i as

$$\begin{aligned} z_0 \mapsto -z_3, \qquad z_3 \mapsto z_2, \\ z_2 \mapsto \ \ z_1, \qquad z_1 \mapsto z_0. \end{aligned} \tag{5.8.14}$$

Let

$$R(Y_t^2, K_{Y_t^2}) = k[x_1, x_2, x_3, , z_j]/(q_0'', q_2'', p_1'', p_3''),$$

and Y^2 be a general member of the family. The surface Y^2 has invariants $p_g = 7$ and $K^2 = 16$ and is given as intersection of four quadrics in $\mathbb{P}^6_{x_1,x_2,x_3,z_0,\dots,z_3}$. The surface Y^2 has a fixed point free action of the group $\mathbb{Z}_8$ given in (5.8.13) and (5.8.14). The surface Y^2 together with with action of $\mathbb{Z}_8$ gives a construction of Campedelli with $\mathbb{Z}_8$.

The $\mathbb{Z}_4 \times \mathbb{Z}_2$ case

This case corresponds to $H^0(\mathcal{O}_{V^6}(2))^{2,0}$. The calculations run very similar to the previous case. The quartics in this case are

$$\begin{aligned} F_4^6 &= \frac{1}{2}(Q_1Q_3 + Q_2Q_4) - R_1R_2, \\ G_4^6 &= Q_1Q_3 - Q_2Q_4. \end{aligned}$$

The unprojection from the $\Gamma_i^6 \subset V^6$ is V_4^6 and from Lemma (5.7.15) is defined by the 2×2 minors of the matrix (5.8.11). The canonical ring of Υ_t^3 is

$$R(\Upsilon_t^3, K_{\Upsilon_t^3}) = k[x_1, x_2, x_3, u_j, w_0, w_2, s_k]/(q_0', q_2', p_1', p_3', R),$$

for $0 \leq j \leq 3$, $1 \leq k \leq 4$, the relations R are 2×2 minors of the matrix (5.8.11) and the four quadrics are

$$\begin{aligned} q_0' &:= q_0 + \sum_{i=1}^{4} s_i, \\ q_2' &:= q_2 + \sum_{i=1}^{4} (-1)^i s_i, \\ p_0' &:= p_0 + \sum_{i=1}^{4} s_i, \\ p_2' &:= p_2 + \sum_{i=1}^{4} (-1)^i s_i. \end{aligned}$$

Notice that determinant of the following matrix is nonzero

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & \varepsilon & \varepsilon^2 & \varepsilon^3 & 0 & 0 \\ 1 & \varepsilon^2 & \varepsilon^4 & \varepsilon^2 & 0 & 0 \\ 1 & \varepsilon^3 & \varepsilon^2 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \varepsilon & \varepsilon^3 \end{pmatrix} \tag{5.8.15}$$

so we can take $Q_1, Q_2, Q_3, Q_4, R_1, R_2$ as our new variables. Let q_i'', p_i'' are quadrics under this change of coordinates. Then

$$\begin{aligned} R(\Upsilon_t^3, K_{\Upsilon^3}) &= k[x_1, x_2, x_3, s_j, Q_j, R_1, R_2]/(q_0'', q_2'', p_0'', p_2'', R) \\ &\cong \Big[k[x_i, z_j]/(q_0'', q_2'', p_0'', p_2'')\Big]^{\mathbb{Z}_2}, \end{aligned}$$

where the action of $\mathbb{Z}_2$ is given by

$$x_i \mapsto x_i, \quad z_i \mapsto -z_j, \ \forall i, j \tag{5.8.16}$$

and $\mathbb{Z}_4$ acts on new variables z_i as

$$\begin{aligned} z_0 \mapsto z_3, &\quad z_3 \mapsto z_2, \\ z_2 \mapsto z_1, &\quad z_1 \mapsto z_0. \end{aligned} \tag{5.8.17}$$

Hence we have a surface Y^3 with $p_g(Y^3) = 7$ and $K_{Y^3}^2 = 16$. The surface Y^3 together with an action of $\mathbb{Z}_4 \times \mathbb{Z}_2$ ((5.8.16), (5.8.17)) gives a Campedelli with $\pi^{\mathrm{alg}} = \mathbb{Z}_4 \times \mathbb{Z}_2$.

Remark 5.8.18 There are three Campedelli surfaces with Abelian fundamental group of order 8 namely $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The two cases are described above.

For the third case, suppose there exist a $\mathbb{Q}$-Gorenstein deformation of Campedelli with $\pi_1^{\text{alg}} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ such that the central fibre has a $\frac{1}{4}(1,1)$ singularity. Locally a $\frac{1}{4}(1,1)$ singularity is obtained by quotient of an ODP by $E := \mathbb{Z}_2$. If we take E to be subgroup of π_1^{alg} then this leads to construction of Godeaux surface with $\pi_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$, a contradiction.

We tried constructing Godeaux surface with $\pi_1 = \mathbb{Z}_3$ by using $\mathbb{Q}$-Gorenstein deformation of Campedelli surface with $\pi_1 = \mathbb{Z}_6$. We used the construction of Campedelli surface with $\pi_1 = \mathbb{Z}_6$ given by Papadakis and Neves [NP09]. So far our attempts have been frustrating.

The construction of Campedelli with $\pi_1 = \mathbb{Z}_4$ using $\mathbb{Q}$-Gorenstein smoothing of $\mathbb{Z}_4$ Godeaux surface has some difficulties, like calculating fundamental group, which need to be fixed in future.

Also the question of how to obtain the Campedelli surface with $|\pi_1| = 9$ using $\mathbb{Q}$-Gorenstein smoothing is worth answering.

Chapter 6

Finding exceptional T-divisors

6.1 Introduction

Finding new families of surfaces is important in the classification of surfaces. There are many techniques for constructing algebraic surfaces (for example see [Rei78], [God31]). In 2007 [LP07] Lee and Park introduced another technique, mainly using $\mathbb{Q}$-Gorenstein smoothing. This technique consists of first finding exceptional T-divisors on a surface X. Contracting these divisors gives us a singular surface Y, and then a $\mathbb{Q}$-Gorenstein smoothing of Y gives us a new family of nonsingular surfaces. Moreover instead of checking the obstruction for the $\mathbb{Q}$-Gorenstein smoothing on Y an equivalent condition on X was found.

The current chapter give construction of two surfaces which contain certain exceptional curves. Which could be a step forward in understanding relation between surfaces in particular to understand the boundary of the moduli space of certain surfaces. We find exceptional T-divisors on some surfaces, namely the (-4)-curves.

In the first section we construct a Godeaux surface with $\pi_1 \cong \mathbb{Z}_5$ containing a (-4)-curve. Contracting this exceptional divisor gives us a surface with $p_g = 0$ and $K^2 = 2$, the calculation of fundamental is still open and is left for future.

In the second section we find a (-4)-curve in a Godeaux–Reid surface. This could lead to a construction of surfaces of general type with $p_g = 0$, $K^2 = 3$, the fundamental group is aimed to be calculated in future.

Whether or not these constructions lead us to a solid technique for finding exceptional T-divisors on surfaces could be an open lead.

We start with some results on $\mathbb{Q}$-Gorenstein smoothing. This review is a continuation of Section (4.2).

6.2 $\mathbb{Q}$-Gorenstein smoothing II

First we define the following for quick reference.

Definition 6.2.1 An exceptional T-divisor is the resolution graph of a surface quotient singularity of type $\frac{1}{dr^2}(1, dra - 1)$, where $\gcd(a, r) = 1$.

If X is a nonsingular surface containing an exceptional T-divisor then contracting this divisor gives us a singular surface with invariants different to X. The numerical invariant K_X^2 is affected in all cases. For example, we have seen the cases of $\frac{1}{4}(1, 1)$ and $\frac{1}{9}(1, 2)$ in Example 4.2.3.

Let $X \dashrightarrow Y$ be the contraction of T-divisor. The next step is to find a $\mathbb{Q}$-Gorenstein smoothing of Y. In [KSB88] Kóllar and Shepherd-Barron proved the existence of a local $\mathbb{Q}$-Gorenstein smoothing for such singularities on Y. But the next problem is whether this local $\mathbb{Q}$-Gorenstein smoothing can be extended over the global surface Y or not. The answer lies in the obstruction map of the sheaves of deformation $T^i_X = Ext^i_X(\Omega_X, \mathcal{O}_X)$ for $i = 0, 1, 2$, as given by the following result:

Proposition 6.2.2 *[Wah81] Let X be a normal projective surface with quotient singularities. Then the obstruction lies in the global Ext 2-group $\mathbb{T}^2_X = Ext^2_X(\Omega_X, \mathcal{O}_X)$.*

Also we have:

Proposition 6.2.3 *[Man91] Let X be a normal projective surface with quotient singularities. If $H^2(T^0_X) = 0$ then every local deformation of the singularities may be globalized.*

The following result simplifies the situation. It is an essential part of our proposed construction.

Theorem 6.2.4 *[LP07] Let X be a normal projective surface with singularities of class T. Let $\pi : V \to X$ be the minimal resolution and let E be the reduced exceptional divisors. Suppose that $H^2(T_V(-\log E)) = 0$. Then there is a $\mathbb{Q}$-Gorenstein smoothing of X.*

6.3 Godeaux with $\pi_1 = \mathbb{Z}_5$

The Godeaux surfaces are one of the early examples in the geography of surfaces of general type. After 1914, the year when the classification of surfaces was completed (cf: see summary of Enriques work [Enr49]), geometers were looking for criteria for rationality and examples of surfaces with $p_g = q = 0$. Earlier at the end of nineteenth century Enriques discovered a sextic surface in $\mathbb{P}^3$ passing doubly through a tetrahedron, an example of a nonrational surface with $p_g = q = 0$. Another such example of nonrational surface was constructed by Godeaux in 1931. This example is now known as a Godeaux surface with $\pi_1 = \mathbb{Z}_5$.

We will follow the method of Reid [Rei78] for the construction of a Godeaux surface with $\pi_1 = \mathbb{Z}_5$. Let T be a Godeaux surface with $\operatorname{Tors} T = \mathbb{Z}_5$. We take $S \to T$ to be an etale Galois cover such that $H^1(\mathcal{O}_S) = 0$. The other numerical invariants of S are $p_g(S) = 4$ and $K^2 = 5$. Here S is defined as a quintic surface in $\mathbb{P}^3$ [Rei78]. Thus the canonical ring of S is given by

$$\begin{aligned} R(S, K_S) &= \bigoplus_{n \geq 0, \sigma \in \operatorname{Tors} T} H^0(T, \mathcal{O}(nK_T + \sigma)) \\ &= k[x_1, x_2, x_3, x_4]/(Q_5), \end{aligned}$$

where $x_i \in H^0(T, \mathcal{O}_T(K + i))$. In other words the action of $\mathbb{Z}_5$ is given by

$$x_i \mapsto \varepsilon^i x_i, \quad \text{for } 1 \leq i \leq 4,$$

where ε is a primitive 5th root of unity. Moreover Q_5 is an invariant quintic. Our next aim is to find a (-4)-curve on T. Instead we find $5 \times (-4)$-curves on S, such that these curves are permuted by the group action of $\mathbb{Z}_5$.

We use the filtration of the canonical ring introduced in Section (5.5), that is we may write

$$\begin{aligned} R(S, K_S) &= \bigoplus_{n \geq 0} H^0(S, \mathcal{O}(nK_S)) \\ &= \bigsqcup_{h|r} \left(R(S, K_S)^h\right), \end{aligned}$$

where

$$R(S, K_S)^h = \bigsqcup_{n \geq 0} H^0(S, \mathcal{O}(nK_S))^h,$$

and

$$H^0(S, \mathcal{O}(nK_S))^h = H^0(S, \mathcal{O}(nK_S))[h] \backslash \left(\bigcup_{r|G,\ r<h} H^0(S, \mathcal{O}(nK_S))[r] \right)$$

(see Section 5.5 for details). As we have seen before elements of $H^0(S, \mathcal{O}(nK_S))^h$ exist as h-tuples. For $f \in H^0(S, \mathcal{O}(nK_S))^h$ we denote the h-tuple containing f by $[f]_h = (f_1, \ldots, f_h)$, where $f_i = g^{i-1} \cdot f$ for $1 \leq i \leq h$. For $T = S/G$ we define the following to simplify the situation:

Definition 6.3.1 Let $f^1, \ldots, f^m \in H^0(S, \mathcal{O}(K_S))^r$, $g \in G$, and $1 \leq i_j \leq r$ for $1 \leq j \leq m$. The group closure of the product $f^1_{i_1} f^2_{i_2} \ldots f^m_{i_m}$ with respect to g, denoted by $\overline{f^1_{i_1} f^2_{i_2} \ldots f^m_{i_m}}^{\,g}$, is an element of $H^0(\mathcal{O}(mK_T + g))$.

Example 6.3.2 Let $X_6 \subset \mathbb{P}(x_0, \ldots, x_4)$ be a 3-fold with an action of $\mathbb{Z}_2 = \langle i \rangle$ given by

$$x_k \mapsto (-1)^k x_k \quad \text{for } 0 \leq k \leq 4.$$

Let f^1, f^2 be given by

$$\begin{aligned} f^1 &= x_0 + x_1, \\ f^2 &= x_2 + x_3 \end{aligned}$$

Then

$$f_1^1 = x_0 + x_1, \quad f_2^1 = x_0 - x_1,$$
$$f_1^2 = x_2 + x_3, \quad f_2^2 = x_2 - x_3\,.$$

Hence in this case

$$\overline{f_1^1 f_2^2}^I = f_1^1 f_2^2 + f_2^1 f_1^2,$$
$$\overline{f_1^1 f_2^2}^i = f_1^1 f_2^2 - f_2^1 f_1^2.$$

In the case of our Godeaux surface with $\pi_1 = \mathbb{Z}_5$ we have

$$R(S, K_S) = R(S, K_S)^1 \bigsqcup R(S, K_S)^5.$$

The defining quintic of S can also be written as

$$Q_5 = \overline{f^1 f^2 f^3 f^4 f^5}^I,$$

where $f^1, \ldots, f^5 \in H^0(S, \mathcal{O}(K_S))^5$. To find $5 \times (-4)$-curves in S we must choose suitable $f^1, \ldots, f^5$. For general $L, M \in H^0(S, \mathcal{O}_S(K_S))^5$, we consider Q_5 by taking

$$f^1 = L_1, \quad f^2 = L_2, \quad f^3 = L_3, \quad f^4 = M_4, \quad f^5 = L_5 + M_5.$$

The quintic Q_5 in this case becomes

$$Q_5 = \overline{L_1 L_2 L_3 M_4 (L_5 + M_5)}^I,$$

which explicitly means

$$Q_5 = \sum_{i=1}^{5} L_i L_{1+[i]} L_{1+[1+i]} M_{1+[2+i]} (L_{1+[3+i]} + M_{1+[3+i]}),$$

where $[k]$ is unique integer with $1 \leq [k] \leq 4$ and $[k] = k \mod 5$. With this choice of quintic the five (-4)-curves are given by

$$C_i \colon \left(L_i = L_{1+[i]} M_{1+[3+i]} + L_{1+[3+i]} \left(L_{1+[i]} + M_{1+[3+i]}\right) = 0\right) \subset \mathbb{P}^3,$$

for $i = 1, \ldots, 5$. Let $S \to Y_0$ be the contraction of these $5 \times (-4)$-curves. The invariants of Y_0 are $p_g(Y_0) = 4$ and $K_{Y_0^2} = 10$, as each (-4)-curve increases K_S^2 by 1. Let $\Psi \colon \mathcal{X} \to \Delta$ be $\mathbb{Q}$-Gorenstein smoothing of Y_0. We hope to find group action of $\mathbb{Z}_5$ on a general fibre $Y_t = \Psi^{-1}(t)$ which could lead to a construction of Campedelli with $\pi_1 = \mathbb{Z}_5$.

To check if the $\mathbb{Q}$-Gorenstein smoothing of Y_0 exists we use Theorem (6.2.4). Thus we calculate the space $H^2(T_{Q_5}(-\log D))$, where $D = \sum_{i=1}^{5} C_i$. For this we consider the following

$$0 \to T_{Q_5}(-\log D) \to T_{Q_5} \to \bigoplus_{i=1}^{5} N_{C_i|Q_5} \to \mathrm{Ext}^1_{\mathcal{O}_{Q_5}}(\Omega^1(\log C_i), \mathcal{O}_{Q_5}) \to \tag{6.3.3}$$

The long exact sequence related to (6.3.3) is

$$\begin{tikzcd}
0 \arrow[r] & H^0(T_{Q_5}(-\log D)) \arrow[r] & H^0(T_{Q_5}) \arrow[r] & H^0\left(\bigoplus_{i=1}^{5} N_{C_i|Q_5}\right) \arrow[dll] & \\
& H^1(T_{Q_5}(-\log D)) \arrow[r] & H^1(T_{Q_5}) \arrow[r] & H^1\left(\bigoplus_{i=1}^{5} N_{C_i|Q_5}\right) \arrow[dll] & \\
& H^2(T_{Q_5}(-\log D)) \arrow[r] & H^2(T_{Q_5}) \arrow[r] & H^2\left(\bigoplus_{i=1}^{5} N_{C_i|Q_5}\right) \arrow[r] & 0.
\end{tikzcd}$$

In this case we have

$$h^0\left(\oplus_{i=1}^5 N_{C_i|Q_5}\right)=0,\quad h^0(T_{Q_5}(-\log D))=0,\quad h^0(T_{Q_5})=0,\quad h^2(T_{Q_5})=0.$$

Hence the above long exact sequence becomes

$$H^1(T_{Q_5}(-\log D))\to H^1(T_{Q_5})\xrightarrow{\alpha} H^1\left(\bigoplus_{i=1}^5 N_{C_i|Q_5}\right)\to H^2(T_{Q_5}(-\log D))\to 0. \tag{6.3.4}$$

In this sequence we have

$$h^1\left(\oplus_{i=1}^5 N_{C_i|Q_5}\right)=15,\quad h^1(T_{Q_5})=40.$$

Thus $H^2(T_{Q_5}(-\log D))$ vanishes if and only if α is surjective. We could not give a mathematical proof of why α is surjective. However we do have a code in Magma that checks that $H^2(T_{Q_5}(-\log D))$ is zero.

Magma Code for Campedelli with $\pi_1=\mathbb{Z}_5$

Please note that the code only works in the latest version of Magma, that is, in V2.17.

```
FF:=FiniteField(131);
ep5:=FF! 2^26;
RR<x1,x2,x3,x4>:=PolynomialRing(FF,4);
PP:=Proj(RR);
a:=[Random(FF): i in [1..4]];
L:= [&+[ep5^(i*j)*RR.i : i in [1..4]] : j in [0..4]];
LP := [&+[a[1]*ep5^j*x1+ a[2]*ep5^(j*2)*x2+a[3]*ep5^(j*3)*x3+
```

```
              [4]*ep5^(j*4)*x4: i in [1..4]] : j in [0..4]];
//define quintic by taking group closure of a quintic monomial
// and related machinery

F:=[ L[i+1]*L[(i+1) mod 5+1]*L[(i+2) mod 5+1]*LP[(i+3) mod 5+1]*
            (L[(i+4) mod 5+1]+LP[(i+4) mod 5+1]): i in [0..4] ];
Q5:=&+[F[i]: i in [1..5]];
S:=Scheme(PP,[Q5]);
IS:=Ideal([Q5]);
TS:=TangentSheaf(S);
MT:=Module(TS);
//The equations of conics
q:=[L[(i+1) mod 5+1]*LP[(i+4) mod 5+1]+L[(i+4) mod 5+1]*
    (L[(i+1) mod 5+1]+LP[(i+1) mod 5+1]) : i in [0..4]];
CL:=[ Scheme(PP,[L[i],q[i]]) : i in [1..5]];
IIC:=[Ideal([L[i],q[i]]): i in [1..5]];
// the conics don't intersect and are inside surface
[[i,j]  : j in [i+1..5], i in [1..4] | not IsEmpty(Intersection(C[i],
C[j])) where C is [CL[i]: i in [1..5]]];
[CL[i] subset S: i in [1..5]];
//to define normal sheaf of each conic and then its module
M1:=[GradedModule(IIC[i]^2+IS): i in [1..5]];
M2:=[sub<M1[i] | [ [f]: f in Basis(IIC[i])]>: i in [1..5]];
M3:=[GradedModule(IIC[i]): i in [1..5]];
H:=[Hom(M2[i],M3[i]): i in [1..5]];
NC:=[Sheaf(H[i],S): i in [1..5]];
```

```
MNC:=[Module(NC[i]): i in [1..5]];
//Direct sum to take module of normal sheaf of five conics
MNCi:=DirectSum(MNC); //the related sheaf has H^1=15;


//Hom
H1 ,f:=Hom(MT,MNCi);
fB:=f(Basis(H1));
fBS:=&+[ fB[i] : i in [1..5]];
//using short exact sequence
KfBS:=Kernel(fBS);
DesS:=Sheaf(KfBS,S);
//the final result
CHS2:=CohomologyDimension(DesS,2,0);
```

6.4 Godeaux–Reid surface

Let T be the surface in the title given by the invariants $p_g = 0$, $K^2 = 2$ and $\pi_1 = \mathbb{Z}_8$ this is also known as Campedelli surface with $\pi_1 = \mathbb{Z}_8$. We take $S \to T$ to be an etale Galois cover of order 8 such that $H^1(\mathcal{O}_S) = 0$. The other numerical invariants of S are $p_g(S) = 7$ and $K^2 = 16$. In [Reid] Reid gave a way of constructing S as follows

$$
\begin{aligned}
R(S, K_S) &= \bigoplus_{n\geq 0, \sigma\in \operatorname{Tors} T} H^0(T, \mathcal{O}(nK_T + \sigma)) \\
&= k[x_1, \dots, x_7]/(Q_0, Q_2, Q_4, Q_6),
\end{aligned}
$$

where $x_i \in H^0(\mathcal{O}_T(K+i))$. Explicitly the action of $\mathbb{Z}_8$ on $\mathbb{P}^6$ is given by

$$x_i \mapsto \varepsilon^i x_i, \quad \text{for } 1 \le i \le 7,$$

where ε is a primitive 8th root of unity. Moreover the four quadrics come from the linear relations between

$$\begin{aligned} H^0(T, \mathcal{O}(2K_T)) &: \quad x_1x_7, x_2x_6, x_3x_5, x_4^2, \\ H^0(T, \mathcal{O}(2K_T+2)) &: \quad x_3x_7, x_4x_6, x_5^2, x_1^2, \\ H^0(T, \mathcal{O}(2K_T+4)) &: \quad x_1x_3, x_5x_7, x_2^2, x_6^2, \\ H^0(T, \mathcal{O}(2K_T+6)) &: \quad x_1x_5, x_2x_4, x_3^2, x_7^2. \end{aligned}$$

Our next aim is to find a Godeaux surface with a (-4)-curve. Instead we find $8 \times (-4)$-curves on S, in such a way that these curves are permuted by the group $\mathbb{Z}_8$. We choose suitable quadrics so that S contain $8 \times (-4)$-curves. Take $\alpha \in H^0(S, \mathcal{O}(K_S))^8$ to be a general element. Now consider

$$Q_0 = \overline{\alpha_1\alpha_5}^{I}, \quad Q_2 = \overline{\alpha_1\alpha_5}^{g^2}, \quad Q_4 = \overline{\alpha_1\alpha_5}^{g^4},$$

where g is a generator of $\mathbb{Z}_8$, and a general quadric Q_6. With these choice of quadrics the intersection $Q_0 \cap Q_2 \cap Q_4$ contains 8 copies of $\mathbb{P}^2$. The $\mathbb{P}^2$ are defined by linear equations given by the ✓ marks in the following table

$\mathbb{P}^2$	Linear forms							
	α_8	α_2	α_3	α_5	α_4	α_6	α_7	α_1
Π_1	✓	✓	✓	✓				
Π_2		✓		✓	✓		✓	
Π_3			✓	✓	✓	✓		
Π_4	✓			✓		✓	✓	
Π_5					✓	✓	✓	✓
Π_6	✓		✓			✓		✓
Π_7	✓	✓					✓	✓
Π_8		✓	✓		✓			✓

. (6.4.1)

From here we take the (-4)-curves on S to be

$$C_i := Q_6 \cap \Pi_i, \quad \text{for } i = 1, \dots, 8$$

Let

$$D = \sum_{i=1}^{8} C_i\,;$$

then by Theorem 6.2.4 we need to check whether or not $H^2(T_S(-\log D))$ vanishes. A similar code can be used to calculate this space. Please be aware that the running time of the code on a normal machine is not practical in this case.

```
FF := FiniteField(97);

ep := FF!5^12;

RR<a,b,c,d,e,f,g> := PolynomialRing(FF,7);

PP:=Proj(RR);

// cycle of 8 linear forms making an orbit under ZZ/8

L := [&+[ep^(i*j)*RR.i : i in [1..7]] : j in [0..7]];

//special choice of quadrics in 0,2,4

Q0 := L[1]*L[5]+L[2]*L[6]+L[3]*L[7]+L[4]*L[8];
```

```
Q2 := L[1]*L[5]+ep^-2*L[2]*L[6]+ep^-4*L[3]*L[7]+ep^-6*L[4]*L[8];
Q4 := L[1]*L[5]-L[2]*L[6]+L[3]*L[7]-L[4]*L[8];
Q6:=Random(FF)*a*e + Random(FF)*b*d + Random(FF)*c^2 + Random(FF)*g^2;
S := Scheme(PP,[Q0,Q2,Q4,Q6]);
IS:=Ideal([Q0,Q2,Q4,Q6]);
TS:=TangentSheaf(S);
MT:=Module(TS);
L1:=[L[8],L[2],L[3],L[5]];
L2:=[L[2],L[5],L[4],L[7]];
L3:=[L[3],L[5],L[4],L[6]];
L4:=[L[8],L[5],L[6],L[7]];
L5:=[L[4],L[6],L[7],L[1]];
L6:=[L[8],L[3],L[6],L[1]];
L7:=[L[8],L[2],L[7],L[1]];
L8:=[L[2],L[3],L[4],L[1]];
P:=[L1,L2,L3,L4,L5,L6,L7,L8];
CL:=[Scheme(PP, P[i] cat [Q6]) : i in [1..8]];
IIC:=[Ideal(P[i] cat [Q6]): i in [1..8]];
// the conics don't intersect and are inside surface
[[i,j]  : j in [i+1..8], i in [1..8] | not IsEmpty(Intersection(C[i],
C[j])) where C is [CL[i]: i in [1..8]]];
[CL[i] subset S: i in [1..8]];
//to define normal sheaf of each conic and then its module
M1:=[GradedModule(IIC[i]^2+IS): i in [1..8]];
M2:=[sub<M1[i] | [ [f]: f in Basis(IIC[i])]>: i in [1..8]];
M3:=[GradedModule(IIC[i]): i in [1..8]];
```

```
H:=[Hom(M2[i],M3[i]): i in [1..8]];

NC:=[Sheaf(H[i],S): i in [1..8]];

MNC:=[Module(NC[i]): i in [1..8]];

//Direct sum to take module of normal sheaf of five conics

MNCi:=DirectSum(MNC); //the related sheaf has H^1=15;

H1 ,f:=Hom(MT,MNCi); //Hom

fB:=f(Basis(H1));

fBS:=&+[ fB[i] : i in [1..8]];

//using short exact sequence

KfBS:=Kernel(fBS);

DesS:=Sheaf(KfBS,S);

//the final result

CHS2:=CohomologyDimension(DesS,2,0);
```

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