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Interaction of Two Charges in a Uniform Magnetic Field

by

Diogo Pinheiro

Thesis

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Declarations

I declare that, to the best of my knowledge and unless otherwise stated, all the work in this thesis is original. I confirm that this thesis has not been submitted for a degree at another university.

Several sections of chapter 2 have been published has a paper [32] jointly written with Robert MacKay. Chapter 3 has been taken from a paper jointly written with Robert MacKay, recently submitted for publication.

Summary

The thesis starts with a short introduction to smooth dynamical systems and Hamiltonian dynamical systems. The aim of the introductory chapter is to collect basic results and concepts used in the thesis to make it self–contained.

The second chapter of the thesis deals with the interaction of two charges moving in $\mathbb{R}^2$ in a magnetic field $\boldsymbol{B}$. This problem can be formulated as a Hamiltonian system with four degrees of freedom. Assuming that the magnetic field is uniform and the interaction potential has rotational symmetry we reduce this Hamiltonian system to one with two degrees of freedom; for certain values of the conserved quantities and choices of parameters, we obtain an integrable system. Furthermore, when the interaction potential is of Coulomb type, we prove that, for suitable regime of parameters, there are invariant subsets on which this system contains a suspension of a subshift of finite type. This implies non–integrability for this system with a Coulomb type interaction. Explicit knowledge of the reconstruction map and a dynamical analysis of the reduced Hamiltonian systems are the tools we use in order to give a description for the various types of dynamical behaviours in this system: from periodic to quasiperiodic and chaotic orbits, from bounded to unbounded motion.

In the third chapter of the thesis we study the interaction of two charges moving in $\mathbb{R}^3$ in a magnetic field $\boldsymbol{B}$. This problem can also be formulated as a Hamiltonian system, but one with six degrees of freedom. We keep the assumption that the magnetic field is uniform and the interaction potential has rotational symmetry and reduce this Hamiltonian system to one with three degrees of freedom; for certain values of the conserved quantities and choices of parameters, we obtain a system with two degrees of freedom. Furthermore, when the interaction potential is chosen to be Coulomb we prove the existence of an invariant submanifold where the system can be reduced by a further degree of freedom. The reductions simplify the analysis of some properties of this system: we use the reconstruction map to obtain a classification for the dynamics in terms of boundedness of the motion and the existence of collisions. Moreover, we study the scattering map associated with this system in the limit of widely separated trajectories. In this limit we prove that the norms of the gyroradii of the particles are conserved during an interaction and that the interaction of the two particles is responsible for a rotation of the guiding centres around a fixed centre in the case of two charges whose sum is not zero and a drift of the guiding centres in the case of two charges whose sum is zero.

Chapter 1

Introduction

A dynamical system is a rule describing the evolution with time of a point in a given set. This rule might be specified by very different means like ordinary differential equations, iterated maps, partial differential equations or cellular automata. In this thesis we will be most concerned with a very special case of differential equations: Hamilton's equations. Indeed, some famous examples of dynamical systems are mechanical systems that can be written in terms of Hamilton's equations like the harmonic oscillator, the pendulum and double pendulum and the N-body problem.

The mathematical theory of dynamical systems has its roots in classical mechanics, which started to be developed in the XVI and XVII centuries by Galileo and Newton, respectively. In 1686, with the publication of the Principia, Newton laid down the mathematical principles of classical mechanics with three laws governing the motion of bodies under the presence of external forces and described the universal law of gravity. This inspired the work of mathematicians like Euler, Lagrange, Hamilton and Poincaré that built on the shoulders of Newton!

The work of Poincaré was a great influence to the present state of the subject since it led to a change in the motivation from the quantitative to the qualitative and geometrical study of such mechanical systems and more general systems of nonlinear differential equations. This change was a key step for the development of the modern theory of dynamical systems during the XX century. This qualitative way of looking at nonlinear dynamical systems was further developed by Birkhoff in the first half of the XX century. At the same time the subject was flourishing in the Soviet Union with the works of Lyapunov, Andronov, Pontryagin and others. A new wave of development came by around 1960 with the influential works of Smale and Moser in the United States, Kolmogorov, Arnold and Sinai in the Soviet Union and Peixoto in Brazil.

1.1 A short introduction to smooth dynamical systems

In a more precise way, a dynamical system is a triple $\left(M, \phi^t, K\right)$ where M is called the *phase space* and is usually a smooth manifold, $\phi^t : M \times K \to M$, called the *evolution*, is a smooth action of K in M and $K \ni t$ is either a subset of $\mathbb{R}$ in the case of a *continuous time dynamical system* or a subset of $\mathbb{Z}$ in the case of a *discrete time dynamical system*.

Throughout this thesis we deal mainly with dynamical systems determined by a special type of ordinary differential equations called Hamilton's equations. In this section we will introduce basic concepts of the theory of dynamical systems that will be used in this thesis. We will concentrate only on introducing concepts related with the analysis of dynamical systems defined by ordinary differential equations and iterated maps. All the concepts, statements and its proofs in this section can be found in [16, 20, 31, 34] and references therein.

1.1.1 Basic definitions and results: maps

In this section we introduce a class of dynamical systems with an evolution rule of the form

$$\boldsymbol{x}_t = f(\boldsymbol{x}_{t-1}) \ , \qquad t \in K \ , \tag{1.1.1}$$

where, for simplicity of exposition, we assume that $f : M \to M$ is a C^k diffeomorphism with $k \geq 1$. For convenience, we will assume that M is a connected Riemannian manifold (with a metric $d : M \times M \to \mathbb{R}$).

The *forward orbit* of a point $\boldsymbol{p}$ is the subset of M defined by

$$\mathcal{O}^+(\boldsymbol{p}) = \{f^n(\boldsymbol{p}) : n \geq 0\} \ ,$$

the *backward orbit* of a point $\boldsymbol{p}$ is the subset of M defined by

$$\mathcal{O}^-(\boldsymbol{p}) = \{f^n(\boldsymbol{p}) : n < 0\}$$

and the (whole) *orbit* of a point $\boldsymbol{p}$ is the set

$$\mathcal{O}(\boldsymbol{p}) = \mathcal{O}^+(\boldsymbol{p}) \cup \mathcal{O}^-(\boldsymbol{p}) \ .$$

Definition 1.1.1. *We say that a point* $\boldsymbol{p} \in M$ *is a* fixed point of f *if* $f(\boldsymbol{p}) = \boldsymbol{p}$*. A point* $\boldsymbol{p} \in M$ *is a* periodic point of period N *if there exists* $N \in \mathbb{N}$ *such that* $f^N(\boldsymbol{p}) = \boldsymbol{p}$ *and* $f^j(\boldsymbol{p}) \neq \boldsymbol{p}$ *for every* $0 < j < N$*.*

If $\boldsymbol{p}$ is a periodic point of period N then $\mathcal{O}(\boldsymbol{p}) = \left\{\boldsymbol{p}, f(\boldsymbol{p}), ..., f^{N-1}(\boldsymbol{p})\right\}$. If $N = 1$ then $\boldsymbol{p}$ is a fixed point and $\mathcal{O}(\boldsymbol{p}) = \{\boldsymbol{p}\}$.

Next we introduce the concept of stability of points. Such concepts are of crucial importance for applications of dynamical systems.

Definition 1.1.2. *A point* $\boldsymbol{p}$ *is* Lyapunov stable *if for any* $\epsilon > 0$ *there is* $\delta > 0$ *such that if* $d(\boldsymbol{q}, \boldsymbol{p}) < \delta$ *then* $d\left(f^j(\boldsymbol{q}), f^j(\boldsymbol{p})\right) < \epsilon$ *for every* $j \geq 0$*. A point* $\boldsymbol{p}$ *is* asymptotically stable *if it is Lyapunov stable and there exists a neighbourhood* V *of* $\boldsymbol{p}$ *such that for every* $\boldsymbol{q} \in V$*,* $d(f^j(\boldsymbol{q}), f^j(\boldsymbol{p}))$ *tends to zero as* j *tends to infinity.*

Intuitively, if $\boldsymbol{p}$ is a Lyapunov stable point, then for every point $\boldsymbol{q}$ close enough to $\boldsymbol{p}$ its orbit stays close to the orbit of $\boldsymbol{p}$. If $\boldsymbol{p}$ is asymptotically stable, then for every point $\boldsymbol{q}$ close enough to $\boldsymbol{p}$ the forward orbit of $\boldsymbol{q}$ will converge to the forward orbit of $\boldsymbol{p}$.

Some subsets of M have a property that makes them special when studying a dynamical system: they are invariant under the dynamics. The orbits $\mathcal{O}(\boldsymbol{p})$ of points of M under iteration by f are examples of invariant subsets of M. Below we introduce the notion of invariance.

Definition 1.1.3. *A subset* $S \subset M$ *is*

- positively invariant *if* $f(S) \subset S$.
- negatively invariant *if* $f^{-1}(S) \subset S$.
- invariant *if* $f(S) = S$.

Local stability of periodic points

The stability of periodic points (and fixed points) is, under certain conditions, determined by a linear system associated with (1.1.1). We will now discuss such conditions.

Let $\boldsymbol{p}$ be a periodic point of (1.1.1) with least period $N \in \mathbb{N}$. Since $f^N(\boldsymbol{p}) = \boldsymbol{p}$ we obtain that $\boldsymbol{p}$ is a fixed point of the dynamical system

$$\boldsymbol{x} \mapsto f^N(\boldsymbol{x}) \ . \tag{1.1.2}$$

Expanding (1.1.2) in Taylor series about $\boldsymbol{p}$ we obtain

$$\boldsymbol{x} \mapsto \boldsymbol{p} + Df^N_{\boldsymbol{p}}(\boldsymbol{x} - \boldsymbol{p}) + O\left(|\boldsymbol{x} - \boldsymbol{p}|^2\right) \ ,$$

where $Df^N_{\boldsymbol{p}}$ denotes the Jacobian matrix of f^N at $\boldsymbol{p}$. Introducing the variable $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{p} \in \mathbb{R}^m$ in a neighbourhood of $\boldsymbol{p}$, we obtain

$$\boldsymbol{y} \mapsto Df^N_{\boldsymbol{p}}\boldsymbol{y} + O\left(|\boldsymbol{y}|^2\right) \ .$$

The linearized system associated with (1.1.2) is then given by

$$\boldsymbol{y} \mapsto Df^N_{\boldsymbol{p}}\boldsymbol{y} \ .$$

We define the *stable eigenspace* $\mathbb{E}^s$, *unstable eigenspace* $\mathbb{E}^u$ and *centre eigenspace* $\mathbb{E}^c$ by

$$\begin{aligned}
\mathbb{E}^s = \mathrm{span}\{\boldsymbol{v}^s \in \mathbb{R}^m : \quad & \boldsymbol{v}^s \quad \text{is a generalized eigenvector for} \\
& \text{an eigenvalue } \lambda_s \text{ of } Df^N_{\boldsymbol{p}} \text{ with } |\lambda_s| < 1\} \\
\mathbb{E}^u = \mathrm{span}\{\boldsymbol{v}^u \in \mathbb{R}^m : \quad & \boldsymbol{v}^u \quad \text{is a generalized eigenvector for} \\
& \text{an eigenvalue } \lambda_u \text{ of } Df^N_{\boldsymbol{p}} \text{ with } |\lambda_u| > 1\} \\
\mathbb{E}^c = \mathrm{span}\{\boldsymbol{v}^c \in \mathbb{R}^m : \quad & \boldsymbol{v}^c \quad \text{is a generalized eigenvector for} \\
& \text{an eigenvalue } \lambda_c \text{ of } Df^N_{\boldsymbol{p}} \text{ with } |\lambda_c| = 1\} \ .
\end{aligned}$$

Definition 1.1.4. *Let $\boldsymbol{p}$ be a periodic point of* (1.1.1) *with least period $N \in \mathbb{N}$. We say that*

- $\boldsymbol{p}$ *is* hyperbolic *if* $\mathbb{E}^c = \{\mathbf{0}\}$.
- $\boldsymbol{p}$ *is* elliptic *if all eigenvalues λ_c of $Df^N_{\boldsymbol{p}}$ are such that $|\lambda_c| = 1$ and $\lambda_c \neq \pm 1$. If $\boldsymbol{p}$ is elliptic we have that $\mathbb{E}^s = \mathbb{E}^u = \{\mathbf{0}\}$.*

Definition 1.1.5. *Let $\boldsymbol{p}$ be a hyperbolic periodic point of* (1.1.1) *with least period $N \in \mathbb{N}$. We say that*

- $\boldsymbol{p}$ *is a* sink *if* $\mathbb{E}^u = \{\mathbf{0}\}$.
- $\boldsymbol{p}$ *is a* source *if* $\mathbb{E}^s = \{\mathbf{0}\}$.
- $\boldsymbol{p}$ *is a* saddle *if* $\mathbb{E}^s \neq \{\mathbf{0}\}$ *and* $\mathbb{E}^u \neq \{\mathbf{0}\}$.

It is clear that the stability of periodic points of (1.1.1) can be studied by analysing the stability of fixed points of (1.1.2). From now on, we will restrict our attention to fixed points of (1.1.1).

The next theorem states that if $\boldsymbol{p}$ is a hyperbolic fixed point of (1.1.1) then the linear part of $Df_{\boldsymbol{p}}$ completely determines the stability of $\boldsymbol{p}$. More specifically, the theorem ensures the existence (in a neighbourhood of $\boldsymbol{p}$) of a conjugacy between (1.1.1) and its linearization.

Theorem 1.1.6 (Hartman–Grobman Theorem)**.** *Let $f : M \to M$ be a C^r diffeomorphism ($r \geq 1$) with a hyperbolic fixed point $\boldsymbol{p}$. Then there exist neighbourhoods U of $\boldsymbol{p}$ and V of $\mathbf{0}$ and a homeomorphism $h : V \to U$ such that $f(h(\boldsymbol{x})) = h(A\boldsymbol{x})$ for all $\boldsymbol{x} \in V$, where $A = Df_{\boldsymbol{p}}$.*

The stability of a hyperbolic fixed point (or periodic point) follows from the Hartman–Grobman theorem.

Corollary 1.1.7. *Let $f : M \to M$ be a C^r diffeomorphism ($r \geq 1$) with a hyperbolic fixed point $\boldsymbol{p}$. If $\boldsymbol{p}$ is a source or a saddle then the fixed point $\boldsymbol{p}$ is not Liapunov stable. If $\boldsymbol{p}$ is a sink then it is asymptotically stable.*

Definition 1.1.8. *Let $\boldsymbol{p}$ be a hyperbolic fixed point of* (1.1.1) *and U a neighbourhood of $\boldsymbol{p}$. The* local stable manifold *for $\boldsymbol{p}$ in the neighbourhood U is the set*

$$W^s(\boldsymbol{p}, U) = \left\{\boldsymbol{q} \in U : f^j(\boldsymbol{q}) \in U \text{ for } j > 0 \text{ and } d(f^j(\boldsymbol{q}), \boldsymbol{p}) \to 0 \text{ as } j \to \infty\right\} .$$

The local unstable manifold *for $\boldsymbol{p}$ in the neighbourhood U is the set*

$$W^u(\boldsymbol{p}, U) = \left\{\boldsymbol{q} \in U : f^{-j}(\boldsymbol{q}) \in U \text{ for } j > 0 \text{ and } d(f^{-j}(\boldsymbol{q}), \boldsymbol{p}) \to 0 \text{ as } j \to \infty\right\} .$$

To simplify notation we also denote $W^s(\boldsymbol{p}, U)$ and $W^u(\boldsymbol{p}, U)$ by W^s_{loc} and W^u_{loc}, respectively, provided this does not causes any ambiguity regarding the point $\boldsymbol{p}$. Furthermore, it is usual to take $U = B(\boldsymbol{p}, \epsilon) = \{\boldsymbol{q} \in M : d(\boldsymbol{q}, \boldsymbol{p}) < \epsilon\}$. In this case we denote the stable and unstable manifolds for $\boldsymbol{p}$ in $B(\boldsymbol{p}, \epsilon)$ by $W^s_\epsilon(\boldsymbol{p})$ and $W^u_\epsilon(\boldsymbol{p})$, respectively.

It follows from Hartman–Grobman theorem that the local stable and unstable manifolds are topological disks. The next theorem states that the local stable and unstable manifolds are C^k embedded manifolds which can be represented as the graph of a map from one of the spaces $\mathbb{E}^s$ or $\mathbb{E}^u$ to the other. Before stating the theorem we need to introduce the notion of *exponential map* at a point $\boldsymbol{p} \in M$.

Note that there exists a neighbourhood V_p of $\boldsymbol{0} \in T_{\boldsymbol{p}}M$ such that all geodesics through $\boldsymbol{p} \in M$ are defined for $t \in [0, 1]$. For a tangent vector $\boldsymbol{v} \in T_{\boldsymbol{p}}M$, let $\gamma_{\boldsymbol{v}}(t)$ be the unique geodesic with $\gamma_{\boldsymbol{v}}(0) = \boldsymbol{p}$ and $\dot{\gamma}_{\boldsymbol{v}}(0) = \boldsymbol{v}$. We define the *exponential map* at $\boldsymbol{p} \in M$, $\exp_{\boldsymbol{p}} : V_{\boldsymbol{p}} \to M$, by

$$\exp_{\boldsymbol{p}}(\boldsymbol{v}) = \gamma_{\boldsymbol{v}}(1) .$$

Theorem 1.1.9 (Stable Manifold Theorem)**.** *Let $f : M \to M$ be a C^k diffeomorphism ($k \geq 1$) with a hyperbolic fixed point $\boldsymbol{p}$. Then, there is $\epsilon > 0$ such that $W^s_\epsilon(\boldsymbol{p})$ and $W^u_\epsilon(\boldsymbol{p})$ are each C^k embedded disks which are tangent to $\mathbb{E}^s$ and $\mathbb{E}^u$, respectively. Let $D^s(\epsilon)$ and $D^u(\epsilon)$ denote disks of radius ϵ contained in $\mathbb{E}^s$ and $\mathbb{E}^u$, respectively. The local stable manifold $W^s_\epsilon(\boldsymbol{p})$ can be represented as the graph of a C^k function $\sigma^s : D^s(\epsilon) \to D^u(\epsilon)$ with $\sigma^s(0) = 0$ and $D(\sigma^s)_0 = 0$:*

$$W^s_\epsilon(\boldsymbol{p}) = \exp_{\boldsymbol{p}}\left(\{(\sigma^s(\boldsymbol{v}), \boldsymbol{v}) : \boldsymbol{v} \in D^s(\epsilon)\}\right) .$$

Similarly, the local unstable manifold $W^u_\epsilon(\boldsymbol{p})$ can be represented as the graph of a C^k function $\sigma^u : D^u(\epsilon) \to D^s(\epsilon)$ with $\sigma^u(0) = 0$ and $D(\sigma^u)_0 = 0$:

$$W^u_\epsilon(\boldsymbol{p}) = \exp_{\boldsymbol{p}}\left(\{(\boldsymbol{v}, \sigma^u(\boldsymbol{v})) : \boldsymbol{v} \in D^u(\epsilon)\}\right) \ .$$

The *global stable and unstable manifolds*, denoted by $W^s(\boldsymbol{p})$ and $W^u(\boldsymbol{p})$ respectively, are obtained from the local stable and unstable manifolds by the relations

$$\begin{aligned} W^s(\boldsymbol{p}) &= \bigcup_{n \geq 0} f^{-n}\left(W^s(\boldsymbol{p}, U)\right) \\ W^u(\boldsymbol{p}) &= \bigcup_{n \geq 0} f^{n}\left(W^u(\boldsymbol{p}, U)\right) \ . \end{aligned}$$

1.1.2 Basic definitions and results: flows

In this section we introduce a class of dynamical systems for which the evolution rule is determined by an ordinary differential equation of the form

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) \ , \tag{1.1.3}$$

where $f : M \to TM$ is a C^r vector field $(r \geq 1)$ on the manifold M. For convenience we will also assume throughout this section that M is a connected Riemannian manifold (with a metric $d : M \times M \to \mathbb{R}$).

We now state a fundamental result for the study of solutions of ordinary differential equations that will guarantee the existence of an evolution rule (flow) associated with (1.1.3).

Theorem 1.1.10 (Existence and Uniqueness of solutions of ordinary differential equations)**.** *Let M be a smooth manifold and $f : M \to TM$ a Lipschitz map. Let $\boldsymbol{x}_0 \in M$ and $t_0 \in \mathbb{R}$. Then, there exists an $\alpha > 0$ and a unique solution $\boldsymbol{x}(t)$ of $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ defined for $t_0 - \alpha < t < t_0 + \alpha$ such that $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. Furthermore, the solution $\boldsymbol{x}(t)$ depends continuously on the initial condition $\boldsymbol{x}_0$.*

Let $\boldsymbol{x}(t)$ be a solution of (1.1.3) with initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. Since (1.1.3) is independent of t, any translate of the solution, $\boldsymbol{x}(t - \tau)$, is also a solution. Hence, from now on we will only consider initial conditions of the form $\boldsymbol{x}(0) = \boldsymbol{x}_0$.

We define the *maximal domain of definition* of a solution $\boldsymbol{x}(t)$ of (1.1.3) with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ as the largest open interval (t^-, t^+) for which the solution is defined. The interval must be open by the existence of solutions on a short interval guaranteed by theorem 1.1.10. We say that the vector field f is *complete* if the maximal domain of definition of all solutions of (1.1.3) is $\mathbb{R}$. We next state a condition that ensures the existence of solutions of (1.1.3) for all $t \in \mathbb{R}$.

Theorem 1.1.11. *Let M be a compact manifold with no boundary and $f : M \to TM$ a C^1 vector field. Then f is a complete vector field.*

The last theorem does not includes the fundamental case where M is a more general subset of $\mathbb{R}^m$ (or the whole $\mathbb{R}^m$) for some $m \in \mathbb{N}$. We will now see how to modify a differential equation so that its solutions are defined for all $t \in \mathbb{R}$.

Assume that $f : U \subset \mathbb{R}^m \to \mathbb{R}^m$ is a vector field defined on a open set U of $\mathbb{R}^m$ and let $g : U \to \mathbb{R}$ be a smooth positive function. Then a *reparametrization* of the solutions of (1.1.3) is defined by

$$\mathrm{d}t = g(\boldsymbol{x})\mathrm{d}\tau \ .$$

Let a prime denote the derivative with respect to τ, then (1.1.3) becomes

$$\boldsymbol{x}' = f(\boldsymbol{x})g(\boldsymbol{x}) \ .$$

Theorem 1.1.12. *Let $f : U \subset \mathbb{R}^m \to \mathbb{R}^m$ be a C^1 vector field defined on a open set U of $\mathbb{R}^m$. There exists a reparametrization of* (1.1.3) *such that all solutions are defined for all t.*

For simplicity of exposition, from now on we will assume that the vector field in (1.1.3) is complete, that is, either M is a compact manifold with no boundary or M is a open subset of $\mathbb{R}^m$ and (1.1.3) has been conveniently reparametrized.

For the case of dynamical systems defined by differential equations, the time parameter is continuous $t \in \mathbb{R}$ and the evolution ϕ^t is the *flow of the vector field $f(\boldsymbol{x})$*, i.e. $\phi^t : M \times \mathbb{R} \to M$ satisfies each of the following properties:

i) $\frac{\mathrm{d}}{\mathrm{d}t}\phi^t(\boldsymbol{x}) = f \circ \phi^t(\boldsymbol{x})$ for every $t \in \mathbb{R}$ and $\boldsymbol{x} \in M$,

ii) $\phi^0(\boldsymbol{x}) = \boldsymbol{x}$ for every $\boldsymbol{x} \in M$,

iii) $\phi^t \circ \phi^s(\boldsymbol{x}) = \phi^{t+s}(\boldsymbol{x})$ for every $t, s \in \mathbb{R}$ and $\boldsymbol{x} \in M$,

iv) for fixed t, ϕ^t is a homeomorphism on its domain of definition.

Given a point $\boldsymbol{p} \in M$, the *orbit of* $\boldsymbol{p}$ is the subset of M defined by

$$\mathcal{O}(\boldsymbol{p}) = \left\{\phi^t(\boldsymbol{p}) : t \in \mathbb{R}\right\} \ .$$

Definition 1.1.13. *We say that a point $\boldsymbol{p} \in M$ is an* equilibrium *for the flow of* (1.1.3) *if $\phi^t(\boldsymbol{p}) = \boldsymbol{p}$ for all $t \in \mathbb{R}$. A point $\boldsymbol{p} \in M$ is a* periodic point of period T *if there exists positive $T \in \mathbb{R}$ such that $\phi^T(\boldsymbol{p}) = \boldsymbol{p}$ and $\phi^t(\boldsymbol{p}) \neq \boldsymbol{p}$ for every $0 < t < T$. The orbit $\mathcal{O}(\boldsymbol{p})$ of a periodic point is called a* periodic orbit.

If $\boldsymbol{p}$ is a periodic point of period T then $\mathcal{O}(\boldsymbol{p}) = \left\{\phi^t(\boldsymbol{p}) : 0 \leq t < T\right\}$ is called a *periodic orbit.*

Since the flows we consider here are solutions of differential equations we obtain that an equilibrium $\boldsymbol{p}$ for the flow ϕ^t of the differential equation (1.1.3) must satisfy $f(\boldsymbol{p}) = \boldsymbol{0}$.

We next give the definitions of Lyapunov stability and asymptotic stability in the context of flows.

Definition 1.1.14. *The orbit of a point $\boldsymbol{p}$ is* Lyapunov stable *for a flow ϕ^t if for any $\epsilon > 0$ there is $\delta > 0$ such that if $d(\boldsymbol{q}, \boldsymbol{p}) < \delta$, then $d\left(\phi^t(\boldsymbol{q}), \phi^t(\boldsymbol{p})\right) < \epsilon$ for all $t \geq 0$. The orbit of a point $\boldsymbol{p}$ is* asymptotically stable *if it is Lyapunov stable and there exists a neighbourhood V of $\boldsymbol{p}$ such that for every $\boldsymbol{q} \in V$, $d\left(\phi^t(\boldsymbol{q}), \phi^t(\boldsymbol{p})\right)$ tends to zero as t tends to infinity.*

The notion of an invariant set also generalizes for the case of flows.

Definition 1.1.15. *A subset $S \subset M$ is* invariant *if $\phi^t(S) = S$ for all $t \in \mathbb{R}$.*

Local stability of equilibria

The stability of equilibria is, under certain conditions, determined by a linear system associated with (1.1.3). We will now discuss such conditions.

Let $\boldsymbol{p}$ be an equilibrium point of (1.1.3). Since $f(\boldsymbol{p}) = \boldsymbol{0}$, expanding (1.1.3) in Taylor series about $\boldsymbol{p}$ we obtain

$$\dot{\boldsymbol{x}} = Df_{\boldsymbol{p}}(\boldsymbol{x} - \boldsymbol{p}) + O\left(|\boldsymbol{x} - \boldsymbol{p}|^2\right) \ ,$$

where $Df_{\boldsymbol{p}}$ denotes the Jacobian matrix of f at $\boldsymbol{p}$. The linearized system at $\boldsymbol{p}$ associated with (1.1.3) is then given by

$$\dot{\boldsymbol{x}} = Df_{\boldsymbol{p}}(\boldsymbol{x} - \boldsymbol{p}) \ . \tag{1.1.4}$$

We define the *stable eigenspace* $\mathbb{E}^s$, *unstable eigenspace* $\mathbb{E}^u$ and *centre eigenspace* $\mathbb{E}^c$ by

$$\begin{aligned}
\mathbb{E}^s = \mathrm{span}\{\boldsymbol{v}^s \in \mathbb{R}^m : \quad \boldsymbol{v}^s \quad & \text{is a generalized eigenvector for} \\
& \text{an eigenvalue } \lambda_s \text{ of } Df_{\boldsymbol{p}} \text{ with } \mathrm{Re}(\lambda_s) < 0\} \\
\mathbb{E}^u = \mathrm{span}\{\boldsymbol{v}^u \in \mathbb{R}^m : \quad \boldsymbol{v}^u \quad & \text{is a generalized eigenvector for} \\
& \text{an eigenvalue } \lambda_u \text{ of } Df_{\boldsymbol{p}} \text{ with } \mathrm{Re}(\lambda_u) > 0\} \\
\mathbb{E}^c = \mathrm{span}\{\boldsymbol{v}^c \in \mathbb{R}^m : \quad \boldsymbol{v}^c \quad & \text{is a generalized eigenvector for} \\
& \text{an eigenvalue } \lambda_c \text{ of } Df_{\boldsymbol{p}} \text{ with } \mathrm{Re}(\lambda_c) = 0\} \ .
\end{aligned}$$

Definition 1.1.16. *Let $\boldsymbol{p}$ be an equilibrium for the flow of* (1.1.3). *We say that*

- $\boldsymbol{p}$ *is* hyperbolic *if* $\mathbb{E}^c = \{\boldsymbol{0}\}$.
- $\boldsymbol{p}$ *is* elliptic *if* $\mathbb{E}^s = \mathbb{E}^u = \{\boldsymbol{0}\}$.

Definition 1.1.17. *Let $\boldsymbol{p}$ be a hyperbolic equilibrium for the flow of* (1.1.3). *We say that*

- $\boldsymbol{p}$ *is a* sink *if* $\mathbb{E}^u = \{\boldsymbol{0}\}$.
- $\boldsymbol{p}$ *is a* source *if* $\mathbb{E}^s = \{\boldsymbol{0}\}$.

- $\boldsymbol{p}$ *is a* saddle *if* $\mathbb{E}^s \neq \{\mathbf{0}\}$ *and* $\mathbb{E}^u \neq \{\mathbf{0}\}$.

The next theorem states that if $\boldsymbol{p}$ is a hyperbolic equilibrium point for the flow of (1.1.3) then the linear part of $Df_{\boldsymbol{p}}$ completely determines the stability of $\boldsymbol{p}$. More specifically, the theorem ensures the existence (in a neighbourhood of $\boldsymbol{p}$) of a conjugacy between (1.1.3) and its linearization (1.1.4).

Theorem 1.1.18 (Hartman–Grobman Theorem)**.** *Let $\boldsymbol{p}$ be a hyperbolic equilibrium for the flow of $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$. Then, the flow ϕ^t of f is conjugate in a neighbourhood of $\boldsymbol{p}$ to the affine flow $\boldsymbol{p} + \mathrm{e}^{At}(\boldsymbol{y} - \boldsymbol{p})$, where $A = Df_{\boldsymbol{p}}$. More precisely, there exist a neighbourhood U of $\boldsymbol{p}$ and a homeomorphism $h : U \to U$ such that $\phi^t(h(\boldsymbol{x})) = h\left(\boldsymbol{p} + \mathrm{e}^{At}(\boldsymbol{y} - \boldsymbol{p})\right)$ as long as $\boldsymbol{p} + \mathrm{e}^{At}(\boldsymbol{y} - \boldsymbol{p}) \in U$.*

The stability of a hyperbolic equilibrium follows from the Hartman–Grobman theorem.

Corollary 1.1.19. *Let $\boldsymbol{p}$ be a hyperbolic equilibrium for the flow of $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$. If $\boldsymbol{p}$ is a source or a saddle, then $\boldsymbol{p}$ is not Liapunov stable. If $\boldsymbol{p}$ is a sink, then it is asymptotically stable.*

Remark Let $\boldsymbol{p}$ be a hyperbolic equilibrium for the flow of $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ and let $\lambda_1, ..., \lambda_m$ be the eigenvalues of the matrix $Df_{\boldsymbol{p}}$. Assume a non–degeneracy condition of the form

$$\lambda_j \neq \sum_{i=1}^{m} m_i \lambda_i \ , \qquad j \in \{1, ..., m\}$$

for any choice of $m_i \in \mathbb{Z}$ such that $m_i \geq 0$ for every $i \in \{1, ..., m\}$ and $\sum_{i=1}^{m} m_i \geq 2$. Then, if the vector field f is smooth enough, the homeomorphism h in the Hartman–Grobman theorem 1.1.18 can be shown to be a diffeomorphism. See [3, 19] and references therein for more details on smooth linearization of vector fields.

Definition 1.1.20. *Let $\boldsymbol{p}$ be an equilibrium point for the flow of $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ and U a neighbourhood of $\boldsymbol{p}$. The* local stable manifold *for $\boldsymbol{p}$ in the neighbourhood U is the set*

$$W^s(\boldsymbol{p}, U) = \left\{\boldsymbol{q} \in U : \phi^t(\boldsymbol{q}) \in U \text{ for } t > 0 \text{ and } d(\phi^t(\boldsymbol{q}), \boldsymbol{p}) \to 0 \text{ as } t \to \infty\right\} \ .$$

The local unstable manifold *for $\boldsymbol{p}$ in the neighbourhood U is the set*

$$W^u(\boldsymbol{p}, U) = \left\{\boldsymbol{q} \in U : \phi^{-t}(\boldsymbol{q}) \in U \text{ for } t > 0 \text{ and } d(\phi^{-t}(\boldsymbol{q}), \boldsymbol{p}) \to 0 \text{ as } t \to \infty\right\} \ .$$

To simplify notation we also denote $W^s(\boldsymbol{p}, U)$ and $W^u(\boldsymbol{p}, U)$ by W^s_{loc} and W^u_{loc}, respectively, provided this does not causes any ambiguity regarding the point $\boldsymbol{p}$. Furthermore, it is usual to take $U = B(\boldsymbol{p}, \epsilon) = \{\boldsymbol{q} \in M : d(\boldsymbol{q}, \boldsymbol{p}) < \epsilon\}$. In this case we denote the stable and unstable manifolds for $\boldsymbol{p}$ in $B(\boldsymbol{p}, \epsilon)$ by $W^s_\epsilon(\boldsymbol{p})$ and $W^u_\epsilon(\boldsymbol{p})$, respectively.

It follows from Hartman–Grobman theorem that the local stable and unstable manifolds are topological disks. The next theorem states that the local stable and unstable manifolds are C^k embedded manifolds which can be represented as the graph of a map from one of the spaces $\mathbb{E}^s$ or $\mathbb{E}^u$ to the other.

Theorem 1.1.21 (Stable Manifold Theorem)**.** *Let $\boldsymbol{p}$ be a hyperbolic equilibrium for the flow of $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$, where $f : M \to TM$ is a C^k vector field ($k \geq 1$). Then, there is $\epsilon > 0$ such that $W^s_\epsilon(\boldsymbol{p})$ and $W^u_\epsilon(\boldsymbol{p})$ are each C^k embedded disks which are tangent to $\mathbb{E}^s$ and $\mathbb{E}^u$, respectively. Let $D^s(\epsilon)$ and $D^u(\epsilon)$ denote disks of radius ϵ contained in $\mathbb{E}^s$ and $\mathbb{E}^u$, respectively. The local stable manifold $W^s_\epsilon(\boldsymbol{p})$ can be represented as the graph of a C^k function $\sigma^s : D^s(\epsilon) \to D^u(\epsilon)$ with $\sigma^s(0) = 0$ and $D(\sigma^s)_0 = 0$:*

$$W^s_\epsilon(\boldsymbol{p}) = \exp_{\boldsymbol{p}}\left(\{(\sigma^s(\boldsymbol{v}), \boldsymbol{v}) : \boldsymbol{v} \in D^s(\epsilon)\}\right) \ .$$

Similarly, the local unstable manifold $W^u_\epsilon(\boldsymbol{p})$ can be represented as the graph of a C^k function $\sigma^u : D^u(\epsilon) \to D^s(\epsilon)$ with $\sigma^u(0) = 0$ and $D(\sigma^u)_0 = 0$:

$$W^u_\epsilon(\boldsymbol{p}) = \exp_{\boldsymbol{p}}\left(\{(\boldsymbol{v}, \sigma^u(\boldsymbol{v})) : \boldsymbol{v} \in D^u(\epsilon)\}\right) \ .$$

The *global stable and unstable manifolds*, denoted by $W^s(\boldsymbol{p})$ and $W^u(\boldsymbol{p})$ respectively, are obtained from the local stable and unstable manifolds by the relations

$$\begin{aligned} W^s(\boldsymbol{p}) &= \bigcup_{t \leq 0} \phi^t\left(W^s(\boldsymbol{p}, U)\right) \\ W^u(\boldsymbol{p}) &= \bigcup_{t \geq 0} \phi^t\left(W^u(\boldsymbol{p}, U)\right) \ . \end{aligned}$$

1.1.3 The Poincaré map and the suspension of a map

In this section we will look at two constructions that connect the two kinds of dynamical systems introduced in the previous sections.

The Poincaré map

We consider again the differential equation (1.1.3) and let γ denote a periodic orbit of the flow of (1.1.3) with period T and $\boldsymbol{p} \in \gamma$. Then, for some k, the k^{th} coordinate of the vector field f must be non–zero at $\boldsymbol{p}$, $f_k(\boldsymbol{p}) \neq 0$. We take the hyperplane given by

$$\Sigma = \{\boldsymbol{x} : x_k = p_k\} \ .$$

The hyperplane Σ is called a *cross section* at $\boldsymbol{p}$. For some $\boldsymbol{x} \in \Sigma$ near $\boldsymbol{p}$, the flow $\phi^t(\boldsymbol{x})$ returns to Σ in time $\tau(\boldsymbol{x})$ close to T. We call $\tau(\boldsymbol{x})$ the *first return time*.

Definition 1.1.22. *Let $V \subset \Sigma$ be an open set in Σ on which $\tau(\boldsymbol{x})$ is a differentiable function. The* Poincaré map, *$P : V \to \Sigma$, is defined by*

$$P(\boldsymbol{x}) = \phi^{\tau(\boldsymbol{x})}(\boldsymbol{x}) \ . \tag{1.1.5}$$

Thus, the Poincaré map reduces the analysis of a continuous time dynamical system to the analysis of a discrete time dynamical system. This is very useful for the analysis of the behaviour of periodic orbits of flows since such orbits are fixed points of the Poincaré map. We list below some properties of the Poincaré map of a flow near a periodic orbit.

Theorem 1.1.23. *Let ϕ^t be a C^r flow ($r \geq 1$) of $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$.*

i) If $\boldsymbol{p}$ is on a periodic orbit of period T and Σ is transversal at $\boldsymbol{p}$, then the first return time $\tau(\boldsymbol{x})$ is defined in a neighbourhood V of $\boldsymbol{p}$ and $\tau : V \to \mathbb{R}$ is C^r.

ii) The Poincaré map (1.1.5) *is C^r.*

iii) If γ is a periodic orbit of period T and $\boldsymbol{p} \in \gamma$, then $D\phi^T_{\boldsymbol{p}}$ has 1 as an eigenvalue with eigenvector $f(\boldsymbol{p})$.

iv) If γ is a periodic orbit of period T and $\boldsymbol{p}, \boldsymbol{q} \in \gamma$, then the derivatives $D\phi^T_{\boldsymbol{p}}$ and $D\phi^T_{\boldsymbol{q}}$ are linearly conjugate and so have the same eigenvalues.

We will now use the Poincaré map to study the stability of periodic orbits of the dynamical system defined by (1.1.3).

Definition 1.1.24. *Let γ be a periodic orbit of period T for the flow of* (1.1.3) *with $\boldsymbol{p} \in \gamma$ and let $1, \lambda_1, ..., \lambda_{m-1}$ be the eigenvalues of $D\phi^T_{\boldsymbol{p}}$. The $m-1$ eigenvalues $\lambda_1, ..., \lambda_{m-1}$ are called the* characteristic multipliers *of the periodic orbit γ.*

Definition 1.1.25. *Let γ be a periodic orbit of period T for the flow of* (1.1.3) *with characteristic multipliers $\lambda_1, ..., \lambda_{m-1}$. We say that*

- *γ is* hyperbolic *if $|\lambda_j| \neq 1$ for all $j \in \{1, ..., m-1\}$.*
- *γ is* elliptic *if $|\lambda_j| = 1$ for all $j \in \{1, ..., m-1\}$.*

Definition 1.1.26. *Let γ be a hyperbolic periodic orbit of period T for the flow of* (1.1.3) *with characteristic multipliers $\lambda_1, ..., \lambda_{m-1}$. We say that*

- *γ is a* periodic sink *if $|\lambda_j| < 1$ for all $j \in \{1, ..., m-1\}$.*
- *γ is a* periodic source *if $|\lambda_j| > 1$ for all $j \in \{1, ..., m-1\}$.*
- *γ is a* saddle periodic orbit *if γ is neither a periodic sink nor a periodic source.*

The next result establishes the relation between the characteristic multipliers and the Poincaré map near a periodic orbit.

Theorem 1.1.27. *Let $\boldsymbol{p}$ be a point on a periodic orbit γ of period T for the flow of $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$. Then, the characteristic multipliers of the periodic orbit are the same as the eigenvalues of the derivative of the Poincaré map at $\boldsymbol{p}$.*

The next theorem regarding the stability of periodic orbits of flows follows the theorem above and the analysis of the stability of fixed points of maps.

Theorem 1.1.28. *Let γ be a periodic orbit of period T for the flow of $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$.*

i) If γ is a periodic sink, then γ is asymptotically stable.

ii) If γ has at least one characteristic multiplier λ_k such that $|\lambda_k| > 1$, then γ is not Lyapunov stable.

The suspension of a map

We now describe a construction that takes a C^r diffeomorphism on a given space to a C^r flow on a space of one higher dimension. The flow we obtain is called the *suspension of the map*.

Given a map $f : X \to X$, consider the space $X \times \mathbb{R}$ with the equivalence relation $\sim$ given by

$$(\boldsymbol{x}, s+1) \sim (f(\boldsymbol{x}), s) \ ,$$

and consider the quotient space of $X \times \mathbb{R}$ under $\sim$

$$\widetilde{X} = X \times \mathbb{R} / \sim \ .$$

It is enough to consider $0 \leq s \leq 1$, but including the other points makes it clear that the quotient space has a C^r structure provided f is C^r. Consider the equations on $X \times \mathbb{R}$ given by

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{0} \\ \dot{s} &= 1 \ . \end{aligned}$$

This induces a flow ϕ^t on $X \times \mathbb{R}$ which passes to a flow $\widetilde{\phi}^t$ on the quotient space $\widetilde{X}$. The flow $\widetilde{\phi}^t$ is the suspension flow of the map f. Noting that

$$\phi^1(\boldsymbol{x}, 0) = (\boldsymbol{x}, 1) \sim (f(\boldsymbol{x}), 0) \ ,$$

we obtain that the flow on $\widetilde{X}$ has f as its Poincaré map.

1.1.4 The λ-Lemma

In this section we state two versions of the λ-Lemma and a result that follows from it that will be a key ingredient for section 2.5. Although the λ-Lemma holds in other settings, we will restrict our attention to its formulation for continuous time dynamical systems of the form (1.1.3).

The standard version of the λ-Lemma

Let $\boldsymbol{p}$ be a hyperbolic equilibrium for the flow ϕ^t of (1.1.3) and let W^s_{loc} and W^u_{loc} denote, respectively, the local stable and unstable manifolds for the point $\boldsymbol{p}$. Let B^s be a disc embedded in W^s_{loc} and B^u a disc contained in W^u_{loc}. Take $V = B^u \times B^s$ for a neighbourhood of $\boldsymbol{p}$.

We consider a point $\boldsymbol{q} \in W^s_{\text{loc}}$ and a disc D^u of dimension $u = \dim W^u_{\text{loc}}$ transversal to W^s_{loc} at $\boldsymbol{q}$.

Theorem 1.1.29 (λ-Lemma). *Let $V = B^u \times B^s$, $q \in W^s_{\text{loc}} \backslash \{\boldsymbol{p}\}$ and D^u be as above. Let D^u_t denote the connected component of $\phi^t(D^u) \cap V$ that contains $\phi^t(\boldsymbol{q})$. Given $\epsilon > 0$ there exists $t_0 > 0$ such that, if $t > t_0$, then D^u_t is ϵ C^1–close to W^u_{loc}.*

The strong λ-Lemma

Theorem 1.1.29 was improved in [10]. We will now introduce the appropriate setting for this stronger version of the λ-Lemma. Let $m, n \in \mathbb{N}$ and $d = m + n$. Assume that $\boldsymbol{x} \in \mathbb{R}^m$, $\boldsymbol{y} \in \mathbb{R}^n$, A is a real $m \times m$ matrix and B is a real $n \times n$ matrix and let $|\cdot|$ denote the sup norm for vectors in Euclidean space. Let $U \subset \mathbb{R}^d$ be a neighbourhood of the origin, $f \in C^{k+1}(U, \mathbb{R}^m)$ and $g \in C^{k+1}(U, \mathbb{R}^n)$ with $k \geq 1$. Consider the following hypothesis:

H1: There exist constants $\lambda < 0 < \mu$ and $C > 1$ satisfying

$$\left|e^{At}\right| \leq Ce^{\lambda t} \text{ for all } t \geq 0$$

and

$$\left|e^{Bt}\right| \leq Ce^{\mu t} \text{ for all } t \leq 0 \ .$$

H2: The maps f and g are such that

$$f(\mathbf{0}, \boldsymbol{y}) \quad = \quad \mathbf{0} \qquad \text{for all} \qquad (\mathbf{0}, \boldsymbol{y}) \in U$$

$$g(\boldsymbol{x},\mathbf{0}) \;=\; \mathbf{0} \qquad \text{for all} \qquad (\boldsymbol{x},\mathbf{0}) \in U$$

and

$$Df(\mathbf{0},\mathbf{0}) = \mathbf{0}\ , \qquad Dg(\mathbf{0},\mathbf{0}) = \mathbf{0}\ ,$$

where D is the differentiation operator with respect to the variables $(\boldsymbol{x},\boldsymbol{y})$.

Consider a system of autonomous ordinary differential equations

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + f(\boldsymbol{x},\boldsymbol{y}) \\ \dot{\boldsymbol{y}} &= B\boldsymbol{y} + g(\boldsymbol{x},\boldsymbol{y})\ , \end{aligned} \tag{1.1.6}$$

with $(\boldsymbol{x},\boldsymbol{y}) \in U$ and A, B, f and g satisfying H1 and H2. Note that every autonomous ordinary differential equation $\dot{\boldsymbol{z}} = F(\boldsymbol{z})$ with $\boldsymbol{z} \in \mathbb{R}^d$ and $F \in C^{k+2}$ is C^{k+2} locally conjugate to (1.1.6) in a neighbourhood of its hyperbolic equilibrium.

Let $B^d(\mathbf{0},\delta)$ denote the closed δ-ball in $\mathbb{R}^d$ with its centre at the origin. Assume that D^n (an n–dimensional disc) is the graph of a smooth function h of $y \in B^n(\mathbf{0},\delta)$ taking values in $B^m(\mathbf{0},\delta)$. We say that D^n is C^k if h is C^k.

Denote by $\phi^t(\boldsymbol{z}_0)$ the solution of (1.1.6) with initial data $\phi^0(\boldsymbol{z}_0) = \boldsymbol{z}_0$ and let

$$D^n_t = \phi^t(D^n) \cap B^d(\mathbf{0},\delta)\ .$$

Denote by W^u_{loc} the local unstable manifold of the equilibrium $(\mathbf{0},\mathbf{0})$. We say that D^n_t is C^k–close to W^u_{loc} by ϵ if D^n_t is the graph of a C^k function h_t of $\boldsymbol{y} \in B^n(\mathbf{0},\delta)$ such that all the derivatives of h_t in $\boldsymbol{y}$ up to the order k are bounded by ϵ for all $\boldsymbol{y} \in B^n(\mathbf{0},\delta)$.

We can now state the strong λ-Lemma.

Theorem 1.1.30 (Strong λ-Lemma)**.** *Assuming the conditions above we have that for every given n–dimensional disc D^n of class C^k there exist constants $t_0 > 0$ and $K > 0$ such that D^n_t is C^k exponentially close to W^u_{loc} by $Ke^{\lambda t}$ for all $t \geq t_0$.*

An application of the strong λ-Lemma

The next result and its proof can be found in [6]. It follows from theorem 1.1.30 and it is an improvement of Shilnikov's Lemma [35].

We consider the differential equation

$$\dot{\boldsymbol{z}} = F(\boldsymbol{z}) \ , \tag{1.1.7}$$

where F is a C^3 vector field in a neighbourhood of $\mathbf{0} \in \mathbb{R}^m$. Let ϕ^t, $t \in \mathbb{R}$, be the flow of (1.1.7). Suppose that $F(\mathbf{0}) = \mathbf{0}$ and the matrix $D_{\boldsymbol{z}}F(\mathbf{0})$ has no eigenvalues on the imaginary axis. Furthermore, assume that there are eigenvalues both with positive and negative real parts and let

$$\lambda = \min \left|\mathrm{Re}\left(\mathrm{Spec}\ D_{\boldsymbol{z}}F(\mathbf{0})\right)\right| \ .$$

Denote by W^s_{loc} and W^u_{loc} the local stable and unstable manifolds of the equilibrium $\mathbf{0}$. The following result holds.

Theorem 1.1.31. *Let X, Y be manifolds in $\mathbb{R}^m$ intersecting the manifolds W^s_{loc} and W^u_{loc}, respectively, transversally at some points $\boldsymbol{x}_0$ and $\boldsymbol{y}_0$. Then for sufficiently large $T > 0$ and $\tau \geq T$ there exists a solution $\boldsymbol{z}(t) = \phi^t(\boldsymbol{z}(0))$, $0 \leq t \leq \tau$, such that:*

- *$\boldsymbol{z}(0) \in X$ and $\boldsymbol{z}(\tau) \in Y$,*
- *there is a representation*

  $$\boldsymbol{z}(t) = \phi^t(\boldsymbol{x}_0) + \phi^{t-\tau}(\boldsymbol{y}_0) + e^{-\lambda\tau}\psi\left(\tau, t\right) \ ,$$

  where ψ is C^2 uniformly bounded on D_T, i.e. there exists a constant $C > 0$ such that

  $$\|\psi\|_{C^2(D_T,\mathbb{R}^m)} \leq C \ , \qquad D_T = \{(\tau, t) : \tau \geq T, 0 \leq t \leq \tau\} \ .$$

- *if the manifolds X and Y depend smoothly on a parameter c taking values in a compact manifold Z, then $\psi\left(\tau, t, c\right)$ is a C^2 function of $(\tau, t, c) \in D_T \times Z$, and $\|\psi\|_{C^2(D_T \times Z,\mathbb{R}^m)} \leq C$.*

1.1.5 Chaotic dynamical systems

In this section we will give two possible definitions for a chaotic dynamical system. The first one is based on sensitive dependence on initial conditions and the second one on the notion of topological entropy. It will be useful for section 2.5 that we exemplify the behaviour of a chaotic dynamical system by introducing the notion of a subshift of finite type.

We choose to introduce these concepts in the setting of discrete time dynamical systems since this is the most appropriated for section 2.5.

Chaos

Let (X, d) be a metric space. In what follows we will consider discrete time dynamical systems determined by maps $f : X \to X$. Before defining what we mean by saying that the dynamical system defined by f is chaotic we need to introduce some auxiliary notions.

Definition 1.1.32. *We say that a map $f : X \to X$ is* transitive *on a invariant set Y if the forward orbit of some point $\boldsymbol{x} \in X$ is dense in Y.*

Definition 1.1.33. *We say that a map $f : X \to X$ has* sensitive dependence on initial conditions *if there exists $\delta > 0$ such that for each point $\boldsymbol{x} \in X$ and for each $\epsilon > 0$ there is a point $\boldsymbol{y} \in X$ such that $d(\boldsymbol{x}, \boldsymbol{y}) < \epsilon$ and $k \geq 0$ such that $d\left(f^k(\boldsymbol{x}), f^k(\boldsymbol{y})\right) \geq \delta$.*

We now state a possible definition for a chaotic dynamical system.

Definition 1.1.34. *We say that a map f on a metric space (X, d) is* chaotic on an invariant set *Y if it satisfies the following properties:*

i) f is transitive on Y.

ii) f has sensitive dependence on initial conditions.

iii) the set of periodic points of f is dense in Y.

Some other definitions of chaos in dynamical systems are available. For example, another possible definition would be: *a map f on a metric space (X, d) is chaotic on an invariant set Y if the restriction of f to the invariant set Y has positive topological* *entropy.* We should remark, however, that the two definitions given above are not equivalent.

We will now discuss the notion of topological entropy.

Topological entropy

As in the preceding section let (X, d) be a metric space. In what follows we will consider the map $f : X \to X$ to be continuous.

The topological entropy of the map f is a quantity that describes the amount of chaos a dynamical system has. We start by giving an intuitive description of topological entropy before giving a precise definition of it.

Suppose that it is not possible to distinguish between points in X which are closer together by less than a given distance ϵ. Then, their orbits of length n can be distinguished provided there is some iterate between 0 and n for which they are at a distance greater than ϵ. Let $r(n, \epsilon, f)$ be the number of such orbits of length n that can be distinguished. The entropy for a given ϵ, $h(\epsilon, f)$, is the growth rate of $r(n, \epsilon, f)$ as n goes to infinity. The limit of $h(\epsilon, f)$ as ϵ goes to 0 is the entropy of f, $h(f)$.

We now make the notion rigorous. We define the distance $d_{n,f} : X \times X \to \mathbb{R}$ given by

$$d_{n,f}\left(\boldsymbol{x}, \boldsymbol{y}\right) = \sup_{0 \leq i \leq n} d\left(f^i(\boldsymbol{x}), f^i(\boldsymbol{y})\right)$$

and say that a subset S of X is (n, ϵ) *separated for* f if $d_{n,f}\left(\boldsymbol{x}, \boldsymbol{y}\right) > \epsilon$ for every pair of distinct points $\boldsymbol{x}, \boldsymbol{y} \in S$.

The number of different orbits of length n is defined by

$$r(n, \epsilon, f) = \max\left\{\operatorname{card}(S) : S \subset X \text{ is a } (n, \epsilon) \text{ separated set for } f\right\} ,$$

where $\operatorname{card}(S)$ denotes the cardinal of a set S. To measure the growth rate of $r(n, \epsilon, f)$ as n increases, we define

$$h(\epsilon, f) = \limsup_{n \to \infty} \frac{\log\left(r(n, \epsilon, f)\right)}{n} .$$

Definition 1.1.35. *Let $f : X \to X$ be a continuous map of a metric space (X, d). The* topological entropy of f *is given by*

$$h(f) = \lim_{\epsilon \to 0, \epsilon > 0} h(\epsilon, f) \ .$$

Since $h(\epsilon, f)$ is a decreasing monotone function of ϵ, it follows that the limit of $h(\epsilon, f)$ as $\epsilon \to 0$ exists or is equal to ∞.

Subshifts of finite type

In this section we introduce the shift map and subshifts of finite type. We also state some results that ensure the existence of chaotic behaviour in this systems.

Let n be an integer such that $n \geq 2$ and let Σ_n^+ be the space of functions from $\mathbb{N}$ to the set $\{1, 2, ..., n\}$. We define a metric on Σ_n^+ by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{k=0}^{\infty} \frac{\delta(x_k, y_k)}{3^k}$$

for $\boldsymbol{x} = (x_0, x_1, ...)$ and $\boldsymbol{y} = (y_0, y_1, ...)$, where

$$\delta(i, j) = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases} .$$

Definition 1.1.36. *We define the* full shift map *on Σ_n^+ by $\sigma(\boldsymbol{x}) = \boldsymbol{y}$ where $y_k = x_{k+1}$. The space Σ_n^+ with the shift map σ, (Σ_n^+, σ), is called the* symbol space on n symbols.

The shift map has many interesting invariant sets. We describe one of those invariant sets below.

A *transition matrix* is an $n \times n$ matrix $A = (a_{ij})$ such that

i) $a_{ij} \in \{0, 1\}$ for all $i, j \in \{1, ..., n\}$.

ii) $\sum_{j=1}^{n} a_{ij} \geq 1$ for all $i \in \{1, ..., n\}$ and $\sum_{i=1}^{n} a_{ij} \geq 1$ for all $j \in \{1, ..., n\}$.

The transition matrix restricts the number of sequences in Σ_n^+ to a set of "allowable" sequences: if we let $\boldsymbol{x}$ be a sequence in Σ_n^+ given by $\boldsymbol{x} = (x_0, x_1, ...)$ then if $a_{x_i x_{i+1}} = 1$ we allow the symbol x_{i+1} to follow the symbol x_i in the sequence $\boldsymbol{x}$ whereas if $a_{x_i x_{i+1}} = 0$ it is not possible for the symbol x_{i+1} to follow the symbol x_i in the sequence $\boldsymbol{x}$. This subset of "allowed" sequences of Σ_n^+ is invariant for the shift map.

Definition 1.1.37. *Let (Σ_n^+, σ) be the symbol space on n symbols. Given an $n \times n$ transition matrix A, let*

$$\Sigma_A^+ = \left\{ \boldsymbol{x} \in \Sigma_n^+ : a_{x_i x_{i+1}} = 1 \text{ for } i \geq 0 \right\} .$$

The space Σ_A^+ is made of the allowed sequences of Σ_n^+ for A. Let $\sigma_A = \sigma_{|_{\Sigma_A^+}}$. The map $\sigma_A : \Sigma_A^+ \to \Sigma_A^+$ is called the subshift of finite type for the matrix A.

The shift map is a very good example of what a (strongly) chaotic dynamical system looks like. We state below some results in this direction.

Theorem 1.1.38. *Let (Σ_n^+, σ) be the symbol space on n symbols. Then*

i) σ has periodic points of all periods.

ii) The set of periodic points of σ is dense in Σ_n^+.

iii) σ is transitive on Σ_n^+.

iv) σ has sensitive dependence on initial conditions.

From the statement above we obtain that σ defines a chaotic dynamical system on Σ_n^+. We will now look at the topological entropy of the shift map and subshifts of finite type. It is clear that σ must have positive topological entropy but this need not be the case for subshifts of finite type.

Theorem 1.1.39. *Let (Σ_n^+, σ) be the symbol space on n symbols. Then, the topological entropy of σ is given by*

$$h(\sigma) = \log(n) .$$

Theorem 1.1.40. *Let A be a transition matrix (on n symbols) and $\sigma_A : \Sigma_A^+ \to \Sigma_A^+$ be the associated subshift of finite type. Then, $h(\sigma_A) = \log(\lambda_1)$, where λ_1 is the real eigenvalue of A such that $\lambda_1 \geq |\lambda_j|$ for all the other eigenvalues λ_j of A.*

Since the matrix A has all its entries equal to either 0 or 1, and every line and column of A have at least one entry equal to 1, then its largest eigenvalue is always real.

1.2 A short introduction to Hamiltonian systems, symmetries and symplectic reduction

Hamiltonian mechanics are one of the possible formulations for classical mechanics. Since its invention in 1833 by William Rowan Hamilton it has been one of the most useful tools for the mathematical analysis of physical systems and it is still a flourishing field as a mathematical theory.

Its appeal possibly comes from two major qualities in its modern formulation. First of all, we can say that the main concept in Hamiltonian mechanics is the Hamiltonian function which, in the physical setting has the interpretation of energy, giving the mathematical theory a clear interpretation for some physical problems. Another major quality for this theory is the mathematical nature of the phase spaces in which Hamiltonian systems are defined - symplectic manifolds - which are, in some sense, the perfect setting to deal with symmetries and conservation of quantities in such mechanical systems.

In this section we shortly revise some of the basic concepts on the theory of Hamiltonian systems. All the concepts, statements and its proofs in this section can be found in [1, 4, 9, 24] and references therein. For simplicity of exposition we deal only with the finite dimensional case here.

1.2.1 Basic definitions and results

Definition 1.2.1. *Let M be a smooth manifold and TM its tangent bundle. A* symplectic form *on M is a map $\omega : TM \times TM \to \mathbb{R}$ satisfying the following conditions:*

i) ω *is bilinear.*

ii) ω *is skew-symmetric:* $\omega(\boldsymbol{u}, \boldsymbol{v}) = -\omega(\boldsymbol{v}, \boldsymbol{u})\ \forall \boldsymbol{u}, \boldsymbol{v} \in TM$.

iii) ω *is closed:* $\mathrm{d}\omega = 0$.

iv) ω *is non–degenerate:* $\omega(\boldsymbol{v}, \boldsymbol{w}) = 0\ \forall \boldsymbol{w} \in TM \Rightarrow \boldsymbol{v} = 0$.

Definition 1.2.2. *A* symplectic manifold *is a pair (M, ω) where M is a smooth manifold and ω is a symplectic form on M.*

It is worth noting that non–degeneracy of the symplectic form implies that every symplectic manifold has even dimension. The most immediate example of a symplectic manifold is $\mathbb{R}^{2n}$ with the *canonical symplectic form*

$$\omega = \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i \ .$$

There is another reason for the importance of this last example: we next state Darboux's theorem which guarantees that in small neighbourhoods of symplectic manifolds we can choose coordinates such that the symplectic form is the canonical one.

Theorem 1.2.3 (Darboux's Theorem)**.** *Let (M, ω) be a symplectic manifold of dimension $2n$. Then in a neighbourhood of $\boldsymbol{z} \in M$ there are local coordinates*

$$(q_1, ..., q_n, p_1, ..., p_n)$$

such that

$$\omega = \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i \ .$$

Definition 1.2.4. *Let* (M, ω) *be a symplectic manifold and* $H : M \to \mathbb{R}$ *be a* C^r *map* ($r \geq 1$)*, to which we call* Hamiltonian function*. The* Hamiltonian vector field $X_H : M \to TM$ *associated with the Hamiltonian function* H *is defined by*

$$\omega\left(X_H, \xi\right) = \mathrm{d}H(\xi) \ , \ \forall \xi \in TM \ . \tag{1.2.1}$$

We call Hamiltonian system *the pair* $((M, \omega), H)$ *and* Hamiltonian dynamical system *the dynamical system determined by* Hamilton's equations

$$\dot{\boldsymbol{z}} = X_H(\boldsymbol{z}) \ .$$

Non–degeneracy of the symplectic form implies that the Hamiltonian vector field X_H is uniquely defined by (1.2.1).

We now introduce a geometrical structure that generalizes symplectic manifolds: Poisson manifolds.

Definition 1.2.5. *Let* M *be a smooth manifold and* $C^\infty(M)$ *be the algebra of smooth functions over* M*. A Poisson bracket on* M*,* $\{.,.\} : C^\infty(M) \times C^\infty(M) \longrightarrow \mathbb{R}$*, is a bilinear, skew-symmetric operator satisfying Jacobi's identity*

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \ \ \forall F, G, H \in C^\infty(M) \ .$$

Definition 1.2.6. *A Poisson manifold is a pair* $(M, \{.,.\})$ *where* M *is a smooth manifold and* $\{.,.\}$ *is a Poisson bracket on* M*.*

A Poisson manifold is a more general setting than a symplectic manifold to study Hamiltonian systems. We have that:

1) every symplectic manifold is a Poisson manifold with Poisson bracket given by

$$\{F, G\} = \omega\left(X_F, X_G\right) \ ,$$

where X_F and X_G are the Hamiltonian vector fields of F and G, respectively.

2) Hamiltonian dynamical systems are well–defined in a Poisson manifold by the following differential equations:

$$\dot{z} = \{H, z\} \ .$$

In this thesis we will only deal with Hamiltonian systems defined on symplectic manifolds.

Definition 1.2.7. *We say that the Hamiltonian system* $((M, \omega), H)$ *has* n *degrees of freedom if* M *is* $2n$*–dimensional. We will often refer to* M *as the phase space for the associated Hamiltonian dynamical system.*

Note that in Definition 1.2.4 we have defined the Hamiltonian function in a way that it does not depend explicit on the time. This will be the case for the particular class of Hamiltonian systems to be studied in this thesis. One of the fundamental properties of time–independent Hamiltonian systems is the preservation of phase space volume. This is a classical result and we state it below.

Theorem 1.2.8 (Liouville's Theorem)**.** *Let* $((M, \omega), H)$ *be a Hamiltonian system and let* $\psi^t(\boldsymbol{x})$ *denote the flow of Hamilton's equations. The flow* $\psi^t(\boldsymbol{x})$ *preserves the volume in phase space. For any region* $D \subset M$ *we have*

$$\mathrm{Vol}\left(\psi^t(D)\right) = \mathrm{Vol}\,(D) \ ,$$

for every t *in the maximal domain of definition of* $\psi^t(\boldsymbol{x})$, $\boldsymbol{x} \in D$.

Symmetries and conserved quantities

We will now look at the relation between symmetries and conserved quantities in Hamiltonian systems and its consequences for the analysis of Hamiltonian dynamical systems.

Definition 1.2.9. *Let* $((M, \omega), H)$ *be a Hamiltonian system and let* $\psi^t(\boldsymbol{x})$ *denote the flow of Hamilton's equations. A* conserved quantity *(or* first integral*) of the Hamiltonian system is a map* $J : M \to \mathbb{R}$ *such that* $J\left(\psi^t(\boldsymbol{x})\right)$ *is a constant function of* t.

Definition 1.2.10. *We say that a Hamiltonian system* $((M, \omega), H)$ *has (continuous) symmetry if there exists a one–parameter group of transformations of the phase space* M *of the Hamiltonian system,* $\phi_\lambda : M \times \mathbb{R} \to M$, $\lambda \in \mathbb{R}$, *that preserves both the Hamiltonian function and the symplectic form, i.e. the following equalities are satisfied:*

$$\begin{aligned} {\phi_\lambda}^* H &= H \\ {\phi_\lambda}^* \omega &= \omega \ . \end{aligned}$$

The next theorem provides the relation between symmetries and conserved quantities in Hamiltonian systems.

Theorem 1.2.11 (Noether's Theorem)**.** *If the Hamiltonian system $((M,\omega),H)$ has a one–parameter group of symmetries ϕ_λ, $\lambda \in \mathbb{R}$, then Hamilton's equations have a conserved quantity $J : M \to \mathbb{R}$. Furthermore, the following identity holds*

$$\omega\left(\frac{\partial\phi_\lambda}{\partial\lambda},\xi\right) = \mathrm{d}J\left(\xi\right)\ \forall\,\xi \in TM\ . \tag{1.2.2}$$

Thus, associated to each one–parameter group of symmetries there is one conserved quantity of the Hamiltonian system. The existence of symmetries and conserved quantities in a Hamiltonian system enables the reduction of the dimension of its phase space which might lead to a simplification on the analysis of the dynamical behaviour of Hamilton's equations. We will explore this relation between symmetries, conserved quantities and reduction of the phase space dimension in the analysis we do for the problem of two interacting charges in a uniform magnetic field.

Under certain "mild" conditions on the conserved quantities, for each conserved quantity of a Hamiltonian system we are able to reduce the dimension of its phase space by two dimensions. We will now provide the setting for the particular and fundamental case of a Liouville (or completely) integrable Hamiltonian system: n degrees of freedom with n conserved quantities (independent and in involution).

Definition 1.2.12. *Let $((M,\omega),H)$ be a Hamiltonian system with conserved quantities $F_1,...,F_n$. We say that the conserved quantities are in* involution *if the following set of equalities is satisfied*

$$\{F_i,F_j\} = 0\ ,\qquad i \neq j\ .$$

Definition 1.2.13. *Let $((M,\omega),H)$ be a Hamiltonian system with conserved quantities $F_1,...,F_n$ and consider the following level set of the conserved quantities $F_1,...,F_n$:*

$$C_{\boldsymbol{a}} = \{\boldsymbol{x} \in M : F_i\left(\boldsymbol{x}\right) = a_i\ ,\ i \in \{1,...,n\}\}\ , \tag{1.2.3}$$

where $\boldsymbol{a} = (a_1,...,a_n) \in \mathbb{R}^n$. We say that the conserved quantities are independent on the set $C_{\boldsymbol{a}} \subset M$ *if the* n *1-forms* $\mathrm{d}F_i$, $i \in \{1, ..., n\}$, *are linearly independent at each point of* $C_{\boldsymbol{a}}$.

Remark An equivalent condition to independence of the conserved quantities $F_1, ..., F_n$ on $C_{\boldsymbol{a}}$ is that the $(n \times n)$ matrix

$$A = (\nabla F_1(\boldsymbol{x}), ..., \nabla F_n(\boldsymbol{x}))$$

has rank n for every $\boldsymbol{x} \in C_{\boldsymbol{a}}$. If such a point $\boldsymbol{x}$ exists, then there is a neighbourhood V of $\boldsymbol{x}$ such that for all points in V the matrix A has rank n.

Theorem 1.2.14 (Arnold–Liouville Theorem)**.** *Let* $((M, \omega), H)$ *be a Hamiltonian system with* n *degrees of freedom and assume that*

i) $((M, \omega), H)$ *has* n *analytic conserved quantities* F_1*,...,* F_n *in involution.*

ii) the conserved quantities F_1*,...,* F_n *are independent on the level set* $C_{\boldsymbol{a}}$ *(defined in* (1.2.3)*).*

Then the Hamiltonian system is completely integrable and $C_{\boldsymbol{a}}$ *is a smooth manifold invariant under the phase flow of the Hamiltonian dynamical system determined by* $((M, \omega), H)$*. If, in addition, we have that the Hamiltonian vector fields are complete on* $C_{\boldsymbol{a}}$ *, then*

i) each connected component of $C_{\boldsymbol{a}}$ *is diffeomorphic to the product of a* k*–dimensional torus* $\mathbb{T}^k$ *with an* $(n-k)$*–dimensional Euclidean space* $\mathbb{R}^{n-k}$ *for some* k*. If moreover,* $C_{\boldsymbol{a}}$ *is compact, then* $k = n$ *and* $C_{\boldsymbol{a}}$ *is diffeomorphic to a torus* $\mathbb{T}^n$*.*

ii) on $\mathbb{T}^k \times \mathbb{R}^{n-k}$*, there exist coordinates* $\varphi_1, ..., \varphi_k$ *and* $z_1, ..., z_{n-k}$ *such that Hamilton's equations on* $C_{\boldsymbol{a}}$ *are*

$$\begin{aligned} \dot{\varphi}_i &= \omega_i \ , \qquad 1 \le i \le k \\ \dot{z}_j &= c_j \ , \qquad 1 \le j \le n-k \ , \end{aligned}$$

where $\omega_i = \omega_i(\boldsymbol{a})$ *and* $c_j = c_j(\boldsymbol{a})$ *are constants.*

Remark The theorem above still holds if the condition of independence of the conserved quantities on $C_{\boldsymbol{a}}$ is replaced by independence on a subset of full measure of $C_{\boldsymbol{a}}$.

If a Hamiltonian system does have less conserved quantities than degrees of freedom a weaker version of Arnold–Liouville theorem still applies.

Theorem 1.2.15. *Let $((M,\omega),H)$ be a Hamiltonian system with n degrees of freedom and let k be a positive integer such that $k \leq n$. Assume that*

i) $((M,\omega),H)$ has k conserved quantities F_1,..., F_k in involution.

ii) the conserved quantities F_1,..., F_k are independent on the level set $C_{\boldsymbol{a}}$ given by

$$C_{\boldsymbol{a}} = \{\boldsymbol{x} \in M : F_i(\boldsymbol{x}) = a_i \ , \ i \in \{1,...,k\}\} \ ,$$

where $\boldsymbol{a} = (a_1,...,a_k) \in \mathbb{R}^k$.

Then the Hamiltonian system $((M,\omega),H)$ induces a Hamiltonian system $((M_{\boldsymbol{a}},\omega_{\boldsymbol{a}}),H_{\boldsymbol{a}})$ with $n-k$ degrees of freedom.

The explicit construction of the reduced Hamiltonian system $((M_{\boldsymbol{a}},\omega_{\boldsymbol{a}}),H_{\boldsymbol{a}})$ is given by Marsden–Weinstein theorem that we will state in the next section.

1.2.2 Symplectic reduction

A more formal and general approach to the question of reduction of the dimension of the phase space of Hamiltonian systems is the Marsden–Weinstein symplectic reduction theory which uses techniques from Lie groups Theory and Symplectic Geometry.

Lie groups and Lie algebras

In this section we introduce the basic concepts concerning Lie groups Theory.

Definition 1.2.16. *A* Lie group *G is a smooth manifold that has a group structure consistent with its manifold structure in the sense that the group operation $\mu : G\times G \to G$ given by*

$$\mu(g,h) = gh$$

is a smooth map.

Let e denote the identity element of G under the group operation and g^{-1} the inverse element of g, for each $g \in G$. We define the *left and right translation maps* $L_g : G \to G$ and $R_g : G \to G$ by

$$\begin{aligned} L_g(h) &= gh \\ R_g(h) &= hg \ , \end{aligned}$$

and the *inner automorphism* $I_g : G \longrightarrow G$ by

$$I_g(h) = L_g \circ R_{g^{-1}}(h) = ghg^{-1}$$

Definition 1.2.17. *A* Lie Algebra *is a vector space together with a bilinear, antisymmetric bracket* $[\xi, \eta]$ *satisfying Jacobi's identity*

$$[[\xi, \eta], \zeta] + [[\zeta, \xi], \eta] + [[\eta, \zeta], \xi] = 0 \ .$$

Let $f : M \to \mathbb{R}$ be a (smooth) real valued function on a manifold M and $\boldsymbol{X}$ a vector field on M. The *Lie derivative of f along $\boldsymbol{X}$* is the directional derivative

$$\mathcal{L}_{\boldsymbol{X}} = \boldsymbol{X}[f] = \mathrm{d}f.\boldsymbol{X} \ .$$

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be vector fields on M. The *Lie derivative of $\boldsymbol{Y}$ along $\boldsymbol{X}$*

$$[\boldsymbol{X}, \boldsymbol{Y}] = \mathcal{L}_{\boldsymbol{X}} \boldsymbol{Y} \tag{1.2.4}$$

is the unique vector field such that

$$\mathcal{L}_{[\boldsymbol{X}, \boldsymbol{Y}]} = [\mathcal{L}_{\boldsymbol{X}}, \mathcal{L}_{\boldsymbol{Y}}] \ .$$

The bracket defined by (1.2.4) is called *Jacobi–Lie bracket*. In local coordinates such that $\boldsymbol{X} = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$ and $\boldsymbol{Y} = \sum_{i=1}^n Y_i \frac{\partial}{\partial x_i}$, the Jacobi–Lie bracket is given by

$$[\boldsymbol{X}, \boldsymbol{Y}] = \sum_{j=1}^n \left(\sum_{i=1}^n X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} \ .$$

Lemma 1.2.18. *Let M be a smooth manifold and $\mathcal{X}(M)$ be the set of vector fields on M. Then, $\mathcal{X}(M)$ with the Jacobi–Lie bracket is a Lie Algebra.*

We say that a vector field $\boldsymbol{X}$ on a Lie group G is *left invariant* if for every $g \in G$ we have that $L_g{}^*\boldsymbol{X} = \boldsymbol{X}$, i.e. the following equality holds

$$(T_h L_g)\, \boldsymbol{X}(h) = \boldsymbol{X}(gh) \ .$$

Let $\mathcal{X}_L(G)$ denote the set of left invariant vector fields on G. If $g \in G$ and $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{X}_L(G)$, then

$$L_g{}^* [\boldsymbol{X}, \boldsymbol{Y}] = [L_g{}^*\boldsymbol{X}, L_g{}^*\boldsymbol{Y}] = [\boldsymbol{X}, \boldsymbol{Y}] \ ,$$

so $[\boldsymbol{X}, \boldsymbol{Y}] \in \mathcal{X}_L(G)$. Hence, $\mathcal{X}_L(G)$ is a Lie subalgebra of $\mathcal{X}(G)$. For each $\xi \in T_eG$, we define a vector field X_ξ on G by

$$X_\xi(g) = T_e L_g(\xi) \ .$$

We can now define the *Lie bracket* in T_eG by the relation

$$[\xi, \eta] = [X_\xi, X_\eta]\,(e) \tag{1.2.5}$$

where $\xi, \eta \in T_eG$ and $[X_\xi, X_\eta]$ is the Jacobi–Lie bracket of vector fields.

Lemma 1.2.19. *The vector space T_eG with the Lie bracket* (1.2.5) *is a Lie Algebra. We call it the* Lie Algebra of G *and denote it by $\mathcal{G}$.*

Actions of Lie groups on manifolds

In this section we study actions of Lie groups on manifolds. For simplicity of exposition we fix the following notation: G is a Lie group and M is a smooth manifold.

Definition 1.2.20. *An* action ϕ of G on M *is a smooth map $\phi : G \times M \to M$, which we denote by*

$$\phi(g, m) = \phi_g(m) \ ,$$

and satisfies the following conditions *i)* $\phi_e(\boldsymbol{x}) = \boldsymbol{x}$ *for every* $\boldsymbol{x} \in M$*.*

ii) $\phi_{gh}(\boldsymbol{x}) = \phi_g(\phi_h(\boldsymbol{x}))$ *for every* $g, h \in G$ *and every* $\boldsymbol{x} \in M$*.*

Let $\boldsymbol{x} \in M$. The *orbit of* $\boldsymbol{x}$ is the subset of M defined by

$$O_{\boldsymbol{x}} = \{\phi_g(\boldsymbol{x}) : g \in G\} \ .$$

An action ϕ of G on M defines an equivalence relation $\sim$ on M given by

$$\boldsymbol{y} \sim \boldsymbol{x} \text{ if there exists some } g \in G \text{ such that } \boldsymbol{y} \in O_{\boldsymbol{x}} \ .$$

Let M/G be the set of the equivalence classes of $\sim$, to which we call *orbit space*, define the projection $\pi : M \to M/G$ as

$$\pi(\boldsymbol{x}) = O_{\boldsymbol{x}}$$

and give M/G the quotient topology, i.e. $A \subset M/G$ is open if and only if $\pi^{-1}(A)$ is open in M. If we want to make M/G a smooth manifold we have to put some more conditions on the action ϕ of G on M.

Definition 1.2.21. *We say that the action* ϕ *of* G *on* M *is* proper *if the map* ψ : $G \times M \to M \times M$*, given by*

$$\psi(g, \boldsymbol{x}) = (\boldsymbol{x}, \phi_g(\boldsymbol{x}))$$

is a proper map.

Since G and M are manifolds, the condition given above for ϕ to be proper is equivalent to require that the preimage $\psi^{-1}(K)$ of a compact set $K \subset M \times M$ under the map ψ is still a compact set. If the Lie group G is compact this condition is automatically satisfied.

We define the *isotropy group of* ϕ *at* $\boldsymbol{x} \in M$ by

$$G_{\boldsymbol{x}} = \{g \in G : \phi_g(\boldsymbol{x}) = \boldsymbol{x}\} \ .$$

Noting that the isotropy group $G_{\boldsymbol{x}}$ is a closed subgroup of G we obtain that $G_{\boldsymbol{x}}$ is also a Lie group and hence a Lie subgroup of G.

Definition 1.2.22. *We say that the action ϕ of G on M is* free *when each point $\boldsymbol{x} \in M$ has a trivial isotropy group, i.e. $G_{\boldsymbol{x}} = \{e\}$ for all $\boldsymbol{x} \in M$.*

Remark If ϕ is an action of G on M which is proper and free then the quotient M/G is a smooth manifold.

We now look at actions of G on its Lie Algebra $\mathcal{G}$ and the dual of its Lie Algebra $\mathcal{G}^*$.

Definition 1.2.23. *The* adjoint action of G on $\mathcal{G}$, *$Ad : G \times \mathcal{G} \longrightarrow \mathcal{G}$, is given by*

$$Ad\,(g, \xi) = T_e I_g \xi$$

and the adjoint representation of G on $\mathcal{G}$, *$Ad_g : \mathcal{G} \longrightarrow \mathcal{G}$, is given by*

$$Ad_g(\xi) = Ad\,(g, \xi) \ .$$

Denote by $Ad_g^* : \mathcal{G}^* \longrightarrow \mathcal{G}^*$ the dual of Ad_g, defined by

$$\left\langle Ad_g^* \mu, \xi \right\rangle = \langle \mu, Ad_g \xi \rangle \ ,$$

where $\mu \in \mathcal{G}^*$, $\xi \in \mathcal{G}$ and $\langle \cdot, \cdot \rangle$ is the natural pairing between $\mathcal{G}^*$ and $\mathcal{G}$.

Definition 1.2.24. *The* coadjoint action of G on $\mathcal{G}^*$, *$Ad^* : G \times \mathcal{G}^* \longrightarrow \mathcal{G}^*$, is given by*

$$Ad^*(g, \mu) = Ad^*_{g^{-1}}(\mu) \ .$$

The *isotropy group of the coadjoint action of G on $\mathcal{G}^*$ at $\mu \in \mathcal{G}^*$* is defined by

$$G_\mu = \left\{ g \in G : Ad^*_{g^{-1}} \mu = \mu \right\} \ .$$

Momentum map

Throughout this section let $((M, \omega), H)$ be a Hamiltonian system and G a Lie group acting on M through ϕ.

For every $\xi \in \mathcal{G}$ we define the vector field X^ξ on M by

$$X^\xi(\boldsymbol{x}) = \frac{d}{dt}_{|_{t=0}} \phi_{\exp(t\xi)}(\boldsymbol{x}) \ .$$

We call X^ξ the *infinitesimal generator of ϕ in the direction ξ*.

Definition 1.2.25. *We say that an action ϕ of G on M is a* Hamiltonian action *if for every $\xi \in \mathcal{G}$ the infinitesimal generator X^ξ is a Hamiltonian vector field on the symplectic manifold (M, ω).*

Equivalently, one could have defined a Hamiltonian action by requiring that for every $\xi \in \mathcal{G}$ there exists a smooth function $J^\xi : M \to \mathbb{R}$ such that X^ξ is the Hamiltonian vector field X_{J^ξ} of J^ξ.

Remark If G is a connected Lie group and ϕ is a Hamiltonian action then, for every $g \in G$, ϕ_g is a symplectomorphism. From now on we will assume that G is connected so that all Hamiltonian actions are also symplectomorphisms.

Definition 1.2.26. *We say that the map $J : M \to \mathcal{G}^*$ is a* momentum map *for the action ϕ of G on M if for every $\xi \in \mathcal{G}$*

$$\omega\left(X^\xi, \zeta\right) = \mathrm{d}J^\xi(\zeta) \ ,$$

where $J^\xi : M \to \mathbb{R}$ is defined by

$$J^\xi(\boldsymbol{x}) = J(\boldsymbol{x}).\xi$$

and X^ξ is the infinitesimal generator of ϕ in the direction ξ.

In other words, the definition above says that J is a momentum map if X^ξ is the Hamiltonian vector field of J^ξ.

If the Hamiltonian action ϕ is a group of symmetries of the Hamiltonian system $((M, \omega), H)$, i.e for every $g \in G$ we have that

$$\begin{aligned} {\phi_g}^* H &= H \\ {\phi_g}^* \omega &= \omega \ , \end{aligned}$$

then for every $\xi \in \mathcal{G}$ the function $J^\xi(\boldsymbol{x})$ is a conserved quantity of the Hamiltonian system $((M, \omega), H)$.

Definition 1.2.27. *We say that the momentum map $J : M \longrightarrow \mathcal{G}^*$ is* coadjoint equivariant *if for every $g \in G$ the following equality holds:*

$$J\left(\phi_g(\boldsymbol{x})\right) = Ad^*_{g^{-1}}\left(J(\boldsymbol{x})\right) \ .$$

We now look at the case of (possibly) non coadjoint equivariant momentum maps. We will see that it is always possible to make a (constant) correction to the coadjoint action of G on $\mathcal{G}^*$ such that the new coadjoint action makes the momentum map coadjoint equivariant.

Proposition 1.2.28. *Let (M, ω) be a symplectic manifold, ϕ a Hamiltonian action of a Lie group G on M, Ad^* the coadjoint action of G on $\mathcal{G}^*$ and J a momentum map for the action ϕ. Define, for every $g \in G$ and $\xi \in \mathcal{G}$, the map $\psi_{g,\xi} : M \to \mathbb{R}$, given by*

$$\psi_{g,\xi}(\boldsymbol{x}) = J^{\xi}\left(\phi_g(\boldsymbol{x})\right) - J^{Ad^*_{g^{-1}}\xi}(\boldsymbol{x}) \ .$$

Then $\psi_{g,\xi}$ is constant on M. Let $\sigma : G \to \mathcal{G}^*$ be defined by $\sigma(g).\xi = \psi_{g,\xi}(\boldsymbol{x})$. We call σ the coadjoint cocycle *associated to J.*

Proposition 1.2.29. *Let (M, ω) be a symplectic manifold, ϕ a Hamiltonian action of a Lie group G on M, Ad^* the coadjoint action of G on $\mathcal{G}^*$ and J a momentum map for the action ϕ, with cocycle σ. Then:*

i) the map $\widetilde{Ad}^* : G \times \mathcal{G}^* \to \mathcal{G}^*$ defined by

$$\widetilde{Ad}^*(g, \mu) = Ad^*_{g^{-1}}(\mu) + \sigma(g)$$

is a (coadjoint) action of G on $\mathcal{G}^*$.

ii) the momentum map J is coadjoint equivariant with respect to the coadjoint action $\widetilde{Ad}^*$.

Marsden–Weinstein Theorem

The first step towards the next theorem was done by Smale [36]. This was later generalized by Meyer [27] and Marsden and Weinstein [25].

Theorem 1.2.30 (Marsden–Weinstein Reduction)**.** *Let ϕ be a free proper action of the Lie group G on the symplectic manifold (M, ω), which has a coadjoint equivariant momentum map $J : M \longrightarrow \mathcal{G}^*$. Suppose that $\mu \in \mathcal{G}^*$ is a regular value of J. Then the reduced space $M_\mu = J^{-1}(\mu)/G_\mu$ is a smooth symplectic manifold with symplectic form ω_μ defined by $\pi_\mu^* \omega_\mu = i^* \omega$ with dimension*

$$\dim M_\mu = \dim J^{-1}(\mu) - \dim G_\mu \ .$$

Here $\pi_\mu : J^{-1}(\mu) \longrightarrow M_\mu$ is the orbit map (or reduction map) of the action $\phi_{|_{G_\mu \times J^{-1}(\mu)}}$ of G_μ on $J^{-1}(\mu)$ and $i : J^{-1}(\mu) \longrightarrow M$ is the inclusion.

One of the main applications of the last theorem is to remove symmetries from a Hamiltonian system defined on (M, ω). Thus, if the Hamiltonian function $H : M \longrightarrow \mathbb{R}$ is invariant under the action ϕ given in the theorem with coadjoint equivariant momentum map $J : M \longrightarrow \mathcal{G}^*$ then, for every regular value μ of J, $H_{|_{J^{-1}(\mu)}}$ induces a smooth function H_μ on the reduced space M_μ, called the reduced Hamiltonian, satisfying

$$\pi_\mu^* H_\mu = i^* H \ .$$

We call the pair $((M_\mu, \omega_\mu), H_\mu)$ *reduced Hamiltonian system*. It is relevant to note that the Hamiltonian vector field associated with H in $J^{-1}(\mu)$ is related with the Hamiltonian vector field associated with H_μ in M_μ by

$$T\pi_\mu \circ X_H = X_{H_\mu} \circ \pi_\mu \ .$$

Remark We note that:

i) by Sard's theorem almost every $\mu \in \mathcal{G}^*$ is a regular value of J.

ii) the assumption that $\mu \in \mathcal{G}^*$ is a regular value of J can be relaxed in theorem 1.2.30. Indeed, it is enough to assume that $\mu \in \mathcal{G}^*$ is a *weakly regular value* of J, i.e. $J^{-1}(\mu)$ is a submanifold of M and $T_pJ^{-1}(\mu) = \ker T_pJ$.

iii) if μ is a regular value of J, the action of G_μ is locally free. Even if the action is not globally free and proper the construction of the reduced space in theorem 1.2.30 can be done locally.

When the group of symmetries of a Hamiltonian system is of the form $G \times K$ it is useful to use symplectic reduction by stages. The next theorem assures that this is possible. See [26] for more details and an extension of the next result to semidirect products of Lie groups.

Theorem 1.2.31 (Commuting Reduction Theorem)**.** *Let (M, ω) be a symplectic manifold, G a Lie group (with Lie algebra $\mathcal{G}$) acting symplectically on M and having an equivariant momentum map $J_G : M \to \mathcal{G}^*$. Assume that $\mu \in \mathcal{G}^*$ is a regular value of J_G and that the action of G_μ is free and proper, so that the symplectic reduced space $M_\mu = J_G^{-1}(\mu)/G_\mu$ is a smooth manifold. Let K be another group (with Lie algebra $\mathcal{K}$) acting on M with an equivariant momentum map $J_K : M \to \mathcal{K}^*$. Assume that ν is a regular value for the K-action. Suppose that the actions of G and K on M commute. Then $J_G \times J_K$ is a momentum map for the action of $G \times K$ on M and*

(i) if J_G is K-invariant and G is connected, then J_K is G-invariant and $J_G \times J_K$ is equivariant. Moreover, K induces a symplectic action on M_μ, and the map $J_\mu : M_\mu \to \mathcal{K}^*$ induced by J_K is an equivariant momentum map for this action.

(ii) the (symplectic) reduced space for the action of K on M_μ at ν is symplectically diffeomorphic to the reduction of M at the point (μ, ν) by the action of $G \times K$.

Chapter 2

The planar problem

Understanding the interaction of two charges in a magnetic field is important to plasma physics but this problem seems to have been given little attention. What attention it has received has tended to be in some limiting regimes such as very strong magnetic field or plasmas with all the particles of the same kind (see [2, 11, 12, 33]) or with one heavy particle idealized as fixed (the diamagnetic Kepler Problem, see [18, 37]). In this chapter we will study the dynamics of two charged particles in a uniform magnetic field without making any restrictions on the sizes of the magnetic field, the charges or the masses. We will assume that the particles behave classically and that their velocities and accelerations are small enough that we can neglect any relativistic and radiation effects. Although it is well known that non–uniformity of the magnetic field introduces further significant effects, we believe that there is value in establishing firm results for the uniform case. The ultimate goal is to treat the three–dimensional case but we limit our attention here to the two–dimensional case, which will form an important part of the three–dimensional case.

The motion of one particle in a uniform magnetic field is the well known gyromotion. The particle moves in a circle of fixed centre - the guiding centre, and radius - gyroradius, with constant angular velocity - gyrofrequency. Orienting the magnetic field upwards, the motion in the circle is clockwise if the charge is positive and anticlockwise otherwise. We sign the gyrofrequency according to the direction of rotation. This problem can be formulated as a two degrees of freedom Hamiltonian system. It has three–dimensional Euclidean symmetry (translation and rotation). These symmetries induce conserved quantities for this system and it is easily seen to be integrable.

On the other hand, the interaction of two charges in the absence of a magnetic field is also a well known problem. It is a standard two-body problem with four degrees of freedom. If the interaction potential is chosen to depend only on the distance between the two particles then the problem is integrable and for the particular case of a Coulomb potential the classical description obtained by Newton for the dynamics of a planet orbiting the Sun completely describes the dynamics of this problem too.

In this chapter, we study the interaction of two particles with non–zero charge, with an interaction potential depending on the distance between the particles, under the action of a uniform magnetic field. It is then a mixture of the two problems briefly described above. In contrast to those problems this one presents much greater complexity - there is a rich variety of dynamical behaviour. The trajectories of the two particles no longer look like circles or ellipses and for some regimes of parameters the trajectories can look extremely complicated. Indeed, we prove that whenever the charges have opposite signs of charge (except for the case where the gyrofrequencies sum to zero) chaotic orbits exist for this system. This last statement implies that with opposite signs of charge (except for the case where the gyrofrequencies sum to zero) this problem is non–integrable. However, we also identify regimes of parameters where there is extra symmetry in the system or an invariant subsystem so that it can be proven to be integrable.

In section 2.1 we recall two possible formulations for the problem of one particle moving under the action of a magnetic field. We choose a non–canonical formulation (as in [22]), that makes easier to identify the system symmetries. Based on this information, we proceed to formulate the problem of the interaction of two charges in a magnetic field in a similar fashion. We identify translational and rotational symmetries of the system and the corresponding conserved quantities. Furthermore, we prove the existence of an exceptional conserved quantity when the two particles have the same gyrofrequency.

In sections 2.2 and 2.3 we use two different approaches to the reduction of the Hamiltonian system in consideration by its symmetries. We prove that the problem of the interaction of two particles in a magnetic field can be reduced to one with 2 degrees of freedom. Furthermore, when the two particles have the same gyrofrequency we use the exceptional conserved quantity to prove integrability of the Hamiltonian system in this case. We also prove that if the sum of the two charges is zero the dynamics in the zero sets of the linear momenta are also integrable. In 2.2 we use symplectic reduction and in 2.3 we do this by constructing a set of coordinates on which the system exhibits a reduction to two degrees freedom, and integrability when it applies. We should remark that a similar reduction is obtained in [15] for the problem of two interacting vortices with mass moving in a plane - in that paper the analogy between that problem and the one we treat here is also given. However, one key point of the present chapter is that the total change of coordinates that exhibits the reduction is computed. This change of coordinates is just the $SE(2)$ lift that, given the base dynamics of the reduced Hamiltonian systems, enables us to describe the full eight–dimensional dynamics.

In section 2.4, we specialize our analysis of the problem by choosing a specific interaction potential. The natural choice for the potential V is the Coulomb potential

$$V\left(r\right) = \frac{e_1 e_2}{4\pi\epsilon_0}\frac{1}{r} \ , \tag{2.0.1}$$

where r denotes the distance between the two particles, e_1 and e_2 denote the values of the charges and ϵ_0 denotes the permittivity of the vacuum. Depending on the problem other potentials would be plausible as, for example, in [15] a logarithmic potential is chosen for the interaction of two vortices. In fact, our results are valid for a class of potential functions (described in section 2.4) that includes both the Coulomb potential and the screened Coulomb potential. We give a brief description of the reduced Hamiltonian system obtained in section 2.3 with the generic potential V replaced by the Coulomb potential, including:

(1) boundedness of some of the variables on the reduced space. In particular, the distance between the two particles is always bounded;

(2) existence of regimes of parameters where close approaches between the particles are possible.

In conjunction with the explicit knowledge of the reconstruction map, point **(1)** gives

- boundedness of the trajectories of the two particles when the sum of the two charges is non–zero;
- unboundedness (typically) of the trajectories of the two particles when the sum of the two charges is zero and certain restrictions on the level sets of the linear momenta are satisfied.

Point **(2)** is crucial for the proof of existence of chaotic orbits later in the chapter.

In section 2.5 we prove the existence of periodic and chaotic trajectories shadowing sequences of collision orbits. In particular we obtain large subshifts of solutions of this type. The method used here was developed in [8] for a proof of the existence of chaotic orbits of the second species for the circular restricted 3-body problem. To apply it to our problem we choose appropriate coordinates for our system - relative positions and corresponding canonically conjugate momenta - and generalize the result in [8] to include our kind of system. The main ingredients are the construction of a set of collision orbits satisfying some non–degeneracy conditions, the implicit function theorem and Levi-Civita regularization. By a result of Moser, the existence of chaotic orbits, and more precisely, the existence of an invariant subset on a energy level on which the system contains a suspension of a subshift of finite type with positive entropy, implies that the system is not integrable in the sense of Liouville, i.e. apart from the conserved quantities exhibited in section 2.1.2 and the Hamiltonian function there are no independent analytic conserved quantities - it is not possible to find a set of four conserved quantities independent and in involution for all regimes of parameters.

2.1 Problem formulation, symmetries and conserved quantities

2.1.1 One charged particle in a magnetic field

For pedagogical reasons we start by considering the well understood case of one particle moving in a uniform magnetic field $\boldsymbol{B}$ of norm $B \neq 0$, orthogonal to the plane of the motion and pointing upwards. A particle of mass $m > 0$ and charge e moving in $\mathbb{R}^2$ under the action of such a field is subject to a Lorentz force $\boldsymbol{F_L}$ of the form

$$\boldsymbol{F_L} = \frac{eB}{c}\mathbf{J}\boldsymbol{v} \ ,$$

where $\boldsymbol{v} = (v_x, v_y) \in \mathbb{R}^2$ is the particle velocity and $\mathbf{J}$ is the standard symplectic matrix in $\mathbb{R}^2$, given by

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{2.1.1}$$

The motion of the particle is then described by Newton's second law

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F_L} \ , \tag{2.1.2}$$

and introducing the equation $\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}$ in (2.1.2) we obtain the system

$$\begin{aligned} m\frac{d\boldsymbol{v}}{dt} &= \frac{e}{c}\boldsymbol{v} \times \boldsymbol{B} \\ \frac{d\boldsymbol{x}}{dt} &= \boldsymbol{v} \ , \end{aligned} \tag{2.1.3}$$

which is known to be Hamiltonian with Hamiltonian function and (non–canonical) symplectic form, given by

$$\begin{aligned} H(\boldsymbol{x}, \boldsymbol{v}) &= \frac{1}{2}m\,|\boldsymbol{v}|^2 \\ \omega &= m\mathrm{d}x \wedge \mathrm{d}v_x + m\mathrm{d}y \wedge \mathrm{d}v_y - \frac{eB}{c}\mathrm{d}x \wedge \mathrm{d}y \ , \end{aligned} \tag{2.1.4}$$

where $\boldsymbol{x} = (x, y) \in \mathbb{R}^2$ denotes the particle position (see [22]). To put the Hamiltonian system given by (2.1.4) into canonical form it is common to introduce the canonical coordinates $\boldsymbol{q} = (q_x, q_y) \in \mathbb{R}^2$ and $\boldsymbol{p} = (p_x, p_y) \in \mathbb{R}^2$, given by

$$\begin{aligned} \boldsymbol{q} &= \boldsymbol{x} \\ \boldsymbol{p} &= m\boldsymbol{v} + \frac{e}{c}\boldsymbol{A}(\boldsymbol{x}) \ , \end{aligned} \tag{2.1.5}$$

where $\boldsymbol{A}(\boldsymbol{x}) = (A_x(\boldsymbol{x}), A_y(\boldsymbol{x})) \in \mathbb{R}^2$ is a vector potential for $\boldsymbol{B}$. The new Hamiltonian system (with phase space $M = \mathbb{R}^4$) is then given by

$$\begin{aligned} H(\boldsymbol{q},\boldsymbol{p}) &= \frac{1}{2m}\left|\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{q})\right|^2 \\ \omega &= \mathrm{d}q_x \wedge \mathrm{d}p_x + \mathrm{d}q_y \wedge \mathrm{d}p_y - \frac{e}{c}\left(\frac{\partial A_x}{\partial q_y} - \frac{\partial A_y}{\partial q_x} + B\right)\mathrm{d}q_x \wedge \mathrm{d}q_y \ . \end{aligned}$$

Hence, for the system to be canonical the vector field $\boldsymbol{A}(\boldsymbol{x})$ must be chosen to verify the equation

$$\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} + B = 0 \ ,$$

which is indeed the condition for $\boldsymbol{A}(\boldsymbol{x})$ to be a vector potential for $\boldsymbol{B}$. If needed, we make the choice $\boldsymbol{A}(\boldsymbol{x}) = -\frac{B}{2}\mathbf{J}\boldsymbol{x}$. We consider it better, however, to use the formulation (2.1.4) because translation symmetry is more transparent, so instead of the change of variables (2.1.5) we just make the change of variables given by

$$\begin{aligned} \boldsymbol{q} &= \boldsymbol{x} \\ \boldsymbol{p} &= m\boldsymbol{v} \end{aligned}$$

obtaining the Hamiltonian system

$$\begin{aligned} H(\boldsymbol{q},\boldsymbol{p}) &= \frac{1}{2m}|\boldsymbol{p}|^2 \\ \omega &= \mathrm{d}q_x \wedge \mathrm{d}p_x + \mathrm{d}q_y \wedge \mathrm{d}p_y + k\,\mathrm{d}q_x \wedge \mathrm{d}q_y \ , \end{aligned} \tag{2.1.6}$$

where

$$k = -\frac{eB}{c} \ .$$

The symplectic form in (2.1.6) defines a Poisson bracket $\{.,.\} : C^\infty\left(\mathbb{R}^4\right) \times C^\infty\left(\mathbb{R}^4\right) \to C^\infty\left(\mathbb{R}^4\right)$ given by

$$\{F,G\} = \frac{\partial F}{\partial q_x}\frac{\partial G}{\partial p_x} - \frac{\partial G}{\partial q_x}\frac{\partial F}{\partial p_x} + \frac{\partial F}{\partial q_y}\frac{\partial G}{\partial p_y} - \frac{\partial G}{\partial q_y}\frac{\partial F}{\partial p_y} - k\left(\frac{\partial F}{\partial p_x}\frac{\partial G}{\partial p_y} - \frac{\partial G}{\partial p_x}\frac{\partial F}{\partial p_y}\right) \ .$$

In the formulation (2.1.6) the Lorentz force effect can not be seen in the Hamiltonian function but it is present in the $k\, dq_x \wedge dq_y$ term of the symplectic form and equivalent term in the Poisson bracket.

There is an obvious $SE(2)$ symmetry of the system (2.1.6) generated by the following transformations of the phase space $M = \mathbb{R}^4$

$$\begin{aligned} \phi_{\boldsymbol{v}}\left(\boldsymbol{q},\boldsymbol{p}\right) &= \left(\boldsymbol{q}+\boldsymbol{v},\boldsymbol{p}\right) \\ \phi_{\theta}\left(\boldsymbol{q},\boldsymbol{p}\right) &= \left(R_\theta \boldsymbol{q}, R_\theta \boldsymbol{p}\right)\ , \end{aligned} \tag{2.1.7}$$

where $\boldsymbol{v} = (v_x, v_y)$ is an $\mathbb{R}^2$ vector and R_θ is the rotation matrix defined by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\ . \tag{2.1.8}$$

Associated to the symmetries in (2.1.7) we have, by Noether's theorem 1.2.11, three conserved quantities, which are given below.

Proposition 2.1.1. *The Hamiltonian system (2.1.6) has the following conserved quantities:*

- *Linear momentum* $\boldsymbol{P} = (P_x, P_y) = \boldsymbol{p} + k\mathbf{J}\boldsymbol{q}$.
- *Angular momentum* $L = \boldsymbol{q}.\mathbf{J}\boldsymbol{p} - \frac{k}{2}\left|\boldsymbol{q}\right|^2$.

Furthermore, the following commutation relations hold

$$\{P_x, P_y\} = k\ , \qquad \{L, P_x\} = P_y\ , \qquad \{L, P_y\} = -P_x\ .$$

To finish this section we just note that the dynamical behaviour associated with this Hamiltonian system is trivial: the particle moves in circles of fixed centre and radius and constant angular velocity, which is the well known gyromotion: a particle with initial conditions $\boldsymbol{q}_0 = (q_{x_0}, q_{y_0})$ and $\boldsymbol{p}_0 = (p_{x_0}, p_{y_0})$ moves in circles of fixed centre $\left(x_0 - \frac{c}{eB}p_{y0}, y_0 + \frac{c}{eB}p_{x0}\right)$ - the guiding centre - and radius $\frac{c}{|e|B}\left|\boldsymbol{p}_0\right|$ - the gyroradius, with constant angular velocity and period $2\pi\frac{cm}{|e|B}$. Furthermore, the motion in the circle is clockwise if the charge is positive and anticlockwise otherwise.

2.1.2 Two charged particles in a magnetic field

We consider two particles with positive masses m_1 and m_2 and non–zero charges e_1 and e_2, respectively, in the same magnetic field as described in section 2.1.1 (uniform of norm $B \neq 0$, orthogonal to the plane of the motion and pointing upwards). Furthermore, we assume that the interaction of the two particles is determined by a potential $V(r)$ depending on the distance r between the two particles.

The phase space M for this problem is $\mathbb{R}^8$ with the singular points of the interaction potential removed (six–dimensional planes if V is the Coulomb potential (2.0.1)). Let $\boldsymbol{q}_i = (q_{x_i}, q_{y_i}) \in \mathbb{R}^2$ denote the vector position of the i-th particle and $\boldsymbol{p}_i = (p_{x_i}, p_{y_i}) \in \mathbb{R}^2$ denote its (non–conjugate) momentum

$$\boldsymbol{p}_i = m\boldsymbol{v}_i \ , \qquad i \in \{1,2\} \ ,$$

where $\boldsymbol{v}_i$ is the velocity of the i-th particle. The motion of the two particles can still be described by a Hamiltonian system, with Hamiltonian function $H : M \longrightarrow \mathbb{R}$ and non–canonical symplectic form $\omega : TM \times TM \to \mathbb{R}$, given by

$$\begin{aligned} H &= \frac{1}{2m_1}\left|\boldsymbol{p}_1\right|^2 + \frac{1}{2m_2}\left|\boldsymbol{p}_2\right|^2 + V\left(\left|\boldsymbol{q}_1 - \boldsymbol{q}_2\right|\right) \\ \omega &= \sum_{i=1,2} \mathrm{d}q_{x_i} \wedge \mathrm{d}p_{x_i} + \mathrm{d}q_{y_i} \wedge \mathrm{d}p_{y_i} + k_i \mathrm{d}q_{x_i} \wedge \mathrm{d}q_{y_i} \ , \end{aligned} \tag{2.1.9}$$

where, for simplicity of notation, we introduce the constants

$$k_i = -\frac{e_i B}{c} \ , \qquad i \in \{1,2\} \ .$$

The Poisson bracket associated with the symplectic form given in (2.1.9), $\{.,.\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, is given by

$$\begin{aligned} \{F,G\} = \sum_{i=1,2} \quad & \frac{\partial F}{\partial q_{x_i}}\frac{\partial G}{\partial p_{x_i}} - \frac{\partial G}{\partial q_{x_i}}\frac{\partial F}{\partial p_{x_i}} + \frac{\partial F}{\partial q_{y_i}}\frac{\partial G}{\partial p_{y_i}} - \frac{\partial G}{\partial q_{y_i}}\frac{\partial F}{\partial p_{y_i}} \\ & -k_i\left(\frac{\partial F}{\partial p_{x_i}}\frac{\partial G}{\partial p_{y_i}} - \frac{\partial G}{\partial p_{x_i}}\frac{\partial F}{\partial p_{y_i}}\right) \ . \end{aligned} \tag{2.1.10}$$

Hamilton's equations corresponding to the Hamiltonian system (2.1.9) are given by

$$\begin{aligned}
\dot{q}_{x_i} &= \frac{\partial H}{\partial p_{x_i}} = \frac{1}{m_i} p_{x_i} \\
\dot{q}_{y_i} &= \frac{\partial H}{\partial p_{y_i}} = \frac{1}{m_i} p_{y_i} \\
\dot{p}_{x_i} &= -\frac{\partial H}{\partial q_{x_i}} - k_i \frac{\partial H}{\partial p_{y_i}} = (-1)^i \frac{V'(r)}{r} (q_{x_1} - q_{x_2}) - \frac{k_i}{m_i} p_{y_i} \\
\dot{p}_{y_i} &= -\frac{\partial H}{\partial q_{y_i}} + k_i \frac{\partial H}{\partial p_{x_i}} = (-1)^i \frac{V'(r)}{r} (q_{y_1} - q_{y_2}) + \frac{k_i}{m_i} p_{x_i} \ .
\end{aligned} \tag{2.1.11}$$

The Hamiltonian system defined by (2.1.9) is invariant under the group generated by the following families of symmetries

$$\begin{aligned}
\phi_{\boldsymbol{v}} (\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{p}_1, \boldsymbol{p}_2) &= (\boldsymbol{q}_1 + \boldsymbol{v}, \boldsymbol{q}_2 + \boldsymbol{v}, \boldsymbol{p}_1, \boldsymbol{p}_2) \\
\phi_\theta (\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{p}_1, \boldsymbol{p}_2) &= (R_\theta \boldsymbol{q}_1, R_\theta \boldsymbol{q}_2, R_\theta \boldsymbol{p}_1, R_\theta \boldsymbol{p}_2) \ ,
\end{aligned} \tag{2.1.12}$$

where $\boldsymbol{v} = (v_x, v_y) \in \mathbb{R}^2$ is a translation vector and R_θ is the rotation matrix in $\mathbb{R}^2$ given by (2.1.8). We define the (signed) gyrofrequency Ω_i of each particle as

$$\Omega_i = \frac{k_i}{m_i} \ , \qquad i \in \{1, 2\} \ .$$

Proposition 2.1.2. *The Hamiltonian system (2.1.9) has the following conserved quantities:*

- *Linear momentum* $\boldsymbol{P} = (P_x, P_y) = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \mathbf{J} (k_1 \boldsymbol{q}_1 + k_2 \boldsymbol{q}_2)$.
- *Angular momentum* $L = \sum_{i=1,2} \boldsymbol{q}_i . \mathbf{J} \boldsymbol{p}_i - \frac{k_i}{2} |\boldsymbol{q}_i|^2$.

Furthermore, if the particles have equal gyrofrequencies $\Omega_1 = \Omega_2$*, there exists another conserved quantity* W*, given by*

$$W = |\boldsymbol{p}_1 + \boldsymbol{p}_2|^2 \ .$$

The following commutation relations between the conserved quantities given above hold:

$$\begin{aligned}
\{P_x, P_y\} &= k_1 + k_2 \ , & \{L, P_x\} &= P_y \ , & \{L, P_y\} &= -P_x \ , \\
\{W, L\} &= 0 \ , & \{W, P_x\} &= 0 \ , & \{W, P_y\} &= 0 \ .
\end{aligned}$$

Proof. The existence of a one–parameter group of symmetries $\phi_\lambda : M \to M$ (with parameter λ) of a Hamiltonian system $((M,\omega),H)$ implies, by Noether's theorem 1.2.11, the existence of a conserved quantity $J : M \to \mathbb{R}$ determined, up to an additive constant, by

$$\omega\left(\frac{\partial \phi_\lambda}{\partial \lambda}, \xi\right) = \mathrm{d}J\left(\xi\right) \qquad \forall \xi \in TM \ . \tag{2.1.13}$$

Using the symmetry groups (2.1.12) and (2.1.13) we obtain the linear momentum $\boldsymbol{P} = (P_x, P_y)$ and the angular momentum L.

From Hamilton's equations (2.1.11), summing up the derivatives of the momenta of the two particles, we get

$$\dot{\boldsymbol{p}}_1 + \dot{\boldsymbol{p}}_2 = -\mathbf{J}\left(\frac{k_1}{m_1}\boldsymbol{p}_1 + \frac{k_2}{m_2}\boldsymbol{p}_2\right) \ . \tag{2.1.14}$$

Using (2.1.14), we obtain

$$\frac{d}{dt}\left|\boldsymbol{p}_1 + \boldsymbol{p}_2\right|^2 = 2\left(\frac{k_1}{m_1} - \frac{k_2}{m_2}\right)\boldsymbol{p}_1.\mathbf{J}\boldsymbol{p}_2 \ . \tag{2.1.15}$$

Hence, from (2.1.15) we obtain that $W = \left|\boldsymbol{p}_1 + \boldsymbol{p}_2\right|^2$ is conserved provided $\Omega_1 = \Omega_2$.

The commutation relations can be obtained by inserting the conserved quantities L, P_x, P_y and W in the Poisson bracket (2.1.10). □

Remarks We note that:

i) the conserved quantities $\boldsymbol{P}$ and L are, respectively, the usual linear and angular momenta for the two body problem with extra terms representing the presence of the magnetic field and hence the effect of the Lorentz force on the particles.

ii) combining P_x and P_y into the conserved quantity

$$P = |\boldsymbol{P}|^2 = {P_x}^2 + {P_y}^2 \tag{2.1.16}$$

we obtain the following commutation relations

$$\{L, P\} = 0 \ , \quad \{L, W\} = 0 \ , \quad \{P, W\} = 0 \ , \tag{2.1.17}$$

which show L, P and W to be in involution.

Lemma 2.1.3. *The conserved quantity W corresponds to the one–parameter group of symmetries $\phi_\lambda : \mathbb{R} \times M \longrightarrow M$ of the Hamiltonian system* (2.1.9)*, given by*

$$
\begin{aligned}
\boldsymbol{q}_1 &\rightarrow \boldsymbol{q}_1 + \frac{1}{k_1+k_2}\left[R_{2(k_1+k_2)\lambda} - \mathbf{Id}_{2\times 2}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2) \\
\boldsymbol{q}_2 &\rightarrow \boldsymbol{q}_2 + \frac{1}{k_1+k_2}\left[R_{2(k_1+k_2)\lambda} - \mathbf{Id}_{2\times 2}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2) \\
\boldsymbol{p}_1 &\rightarrow \boldsymbol{p}_1 + \frac{k_1}{k_1+k_2}\left[R_{2(k_1+k_2)\lambda} - \mathbf{Id}_{2\times 2}\right](\boldsymbol{p}_1+\boldsymbol{p}_2) \\
\boldsymbol{p}_2 &\rightarrow \boldsymbol{p}_2 + \frac{k_2}{k_1+k_2}\left[R_{2(k_1+k_2)\lambda} - \mathbf{Id}_{2\times 2}\right](\boldsymbol{p}_1+\boldsymbol{p}_2)\ ,
\end{aligned}
\tag{2.1.18}
$$

where $\lambda \in \mathbb{R}$, $R_{2(k_1+k_2)\lambda}$ is defined by (2.1.8) substituting θ by $2(k_1+k_2)\lambda$ and $\mathbf{Id}_{2\times 2}$ is the 2×2 identity matrix.

Proof. The one–parameter family of symmetries of the Hamiltonian system (2.1.9) associated with the conserved quantity W is the flow of the Hamiltonian vector field of W with respect to the symplectic form in (2.1.9), i.e. by Noether's theorem 1.2.11, ϕ_λ is determined by

$$\omega\left(\frac{\partial \phi_\lambda}{\partial \lambda}, \xi\right) = \mathrm{d}W(\xi)\ \forall\, \xi \in TM\ ,$$

which turns out to be a system of 8 linear differential equations, which is easily integrated, with solution given by (2.1.18). □

If the interaction potential in (2.1.9) is chosen to be the Coulomb potential (2.0.1) (as we will do in section 2.4) then the scaling transformation given by

$$
\begin{aligned}
\overline{\boldsymbol{q}}_i &= \lambda \boldsymbol{q}_i \\
\bar{t} &= \lambda^{3/2}\, t \\
\overline{B} &= \lambda^{-3/2} B\ ,
\end{aligned}
$$

where $\lambda > 0$, transforms the Hamiltonian function and symplectic form (2.1.9) to $\overline{H} = \lambda^{-1} H$ and $\overline{\omega} = \lambda^{1/2}\omega$. We could then choose λ so that $\overline{B} = 1$ by a rescaling of the level sets of the Hamiltonian function in (2.1.9). Furthermore, choosing e_1 and m_1 to be units of charge and mass, respectively, we could further reduce the number of parameters of (2.1.9) by two. The Hamiltonian system (2.1.9) would then depend only on the charge e_2, mass m_2 and physical constants c and ϵ_0.

In section 2.3 we will use the symmetries and conserved quantities discussed above in order to derive reduced Hamiltonian systems and respective reconstruction maps. The explicit knowledge of the reconstruction map will enable us to recover the full dynamics from the reduced dynamics.

2.2 Reduction: Marsden–Weinstein symplectic reduction

In this section we look at the symmetries of the Hamiltonian system (2.1.9) as an action of a Lie group acting on the phase space M of (2.1.9) and use Marsden–Weinstein symplectic reduction to obtain the dimensions of the reduced symplectic manifolds. The results we obtain here completely agree with the results in section 2.3. However, this procedure gives more detailed information about the global structure of the reduced phase spaces.

2.2.1 Symmetry group $SE(2)$

We consider the symmetries (2.1.12) of the Hamiltonian system (2.1.9) as an action of the Lie group G whose elements are the composition of the symmetries in (2.1.12). Let $(R_\theta, \boldsymbol{v})$ denote an element of G, where $\theta \in S^1$, R_θ is the rotation by the angle θ and $\boldsymbol{v} = (v_x, v_y) \in \mathbb{R}^2$. We note that G is a Lie group of dimension 3 and it is isomorphic to the special Euclidean group of the plane $SE(2) = S^1 \ltimes \mathbb{R}^2$ where $\ltimes$ denotes the semidirect product, given by the group operation

$$(R_{\theta_1}, \boldsymbol{v}_1).(R_{\theta_2}, \boldsymbol{v}_2) = (R_{\theta_1}R_{\theta_2}, R_{\theta_1}\boldsymbol{v}_2 + \boldsymbol{v}_1) \ .$$

The identity element of G is given by

$$0_G = (\mathbf{Id}_{2\times 2}, \mathbf{0}_{\mathbb{R}^2})$$

and the inverse of an element of G is given by

$$(R_\theta, \boldsymbol{v})^{-1} = (R_{-\theta}, -R_{-\theta}\boldsymbol{v}) \ .$$

The action of G on M

The action of the Lie group G on M, $\phi_{(R_\theta,\boldsymbol{v})} : G \times M \longrightarrow M$, is given by

$$\phi_{(R_\theta,\boldsymbol{v})}\left(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{p}_1, \boldsymbol{p}_2\right) = \left(R_\theta \boldsymbol{q}_1 + \boldsymbol{v}, R_\theta \boldsymbol{q}_2 + \boldsymbol{v}, R_\theta \boldsymbol{p}_1, R_\theta \boldsymbol{p}_2\right) \ . \tag{2.2.1}$$

We have already seen in section 2.1.2 that the Hamiltonian system (2.1.9) is invariant under this action. Hence, G is a group of symmetries for the Hamiltonian system (2.1.9).

Lemma 2.2.1. *The action $\phi_{(R_\theta,\boldsymbol{v})}$ acts freely and properly on M.*

Proof. To prove that $\phi_{(R_\theta,\boldsymbol{v})}$ acts freely on M we compute its isotropy group at $\boldsymbol{m} = (\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{p}_1, \boldsymbol{p}_2) \in M$, defined by

$$G_{\boldsymbol{m}} = \left\{(R_\theta, \boldsymbol{v}) \in G : \phi_{(R_\theta,\boldsymbol{v})}\left(\boldsymbol{m}\right) = \boldsymbol{m}\right\} \ .$$

Solving the system of equations $\phi_{(R_\theta,\boldsymbol{v})}\left(\boldsymbol{m}\right) = \boldsymbol{m}$ with respect to $(R_\theta, \boldsymbol{v})$ we obtain

$$G_{\boldsymbol{m}} = \left\{(\mathbf{Id}_{2\times 2}, \mathbf{0}_{\mathbb{R}^2})\right\} \ .$$

Hence, $\phi_{(R_\theta,\boldsymbol{v})}$ acts freely on M.

We will now see that $\phi_{(R_\theta,\boldsymbol{v})}$ acts properly on M. Let $\psi : G \times M \to M \times M$, be given by

$$\psi((R_\theta, \boldsymbol{v}), \boldsymbol{m}) = (\boldsymbol{m}, \phi_{(R_\theta,\boldsymbol{v})}(\boldsymbol{m})) \ .$$

The action $\phi_{(R_\theta,\boldsymbol{v})}$ acts properly on M if and only if the map ψ defined above is proper. Equivalently, we need to check that for every compact set $K \in M \times M$ its preimage $\psi^{-1}(K)$ is also compact.

Let $K \subset M \times M$ be a compact set and let $(\boldsymbol{y}^n)_{n\in\mathbb{N}}$ be a sequence in $\psi^{-1}(K) \subset G \times M$. The sequence $(\boldsymbol{y}^n)_{n\in\mathbb{N}}$ is of the form

$$\boldsymbol{y}^n = (R_{\theta^n}, \boldsymbol{v}^n, \boldsymbol{q}_1^n, \boldsymbol{q}_2^n, \boldsymbol{p}_1^n, \boldsymbol{p}_2^n) \ .$$

Consider the sequence $(\boldsymbol{x}^n)_{n\in\mathbb{N}} = (\psi(\boldsymbol{y}^n))_{n\in\mathbb{N}}$ in K. Then, by definition of ψ, $(\boldsymbol{x}^n)_{n\in\mathbb{N}}$ is of the form

$$\boldsymbol{x}^n = (\boldsymbol{q}_1^n, \boldsymbol{q}_2^n, \boldsymbol{p}_1^n, \boldsymbol{p}_2^n, \overline{\boldsymbol{q}}_1^n, \overline{\boldsymbol{q}}_2^n, \overline{\boldsymbol{p}}_1^n, \overline{\boldsymbol{p}}_2^n) \ ,$$

where

$$\overline{\boldsymbol{q}}_i^n = R_{\theta^n}\boldsymbol{q}_i^n + \boldsymbol{v}^n \ , \qquad \overline{\boldsymbol{p}}_i^n = R_{\theta^n}\boldsymbol{p}_i^n \ , \qquad i \in \{1,2\} \ . \tag{2.2.2}$$

Since K is compact there exists a subsequence $(\boldsymbol{x}^{n_k})_{k\in\mathbb{N}}$ of $(\boldsymbol{x}^n)_{n\in\mathbb{N}}$ which converges to a point $\boldsymbol{x}^* = (\boldsymbol{q}_1^*, \boldsymbol{q}_2^*, \boldsymbol{p}_1^*, \boldsymbol{p}_2^*, \overline{\boldsymbol{q}}_1^*, \overline{\boldsymbol{q}}_2^*, \overline{\boldsymbol{p}}_1^*, \overline{\boldsymbol{p}}_2^*) \in K$. It is clear that each one of the sequences $\boldsymbol{q}_1^n$, $\boldsymbol{q}_2^n$, $\boldsymbol{p}_1^n$,$\boldsymbol{p}_2^n$, $\overline{\boldsymbol{q}}_1^n$, $\overline{\boldsymbol{q}}_2^n$, $\overline{\boldsymbol{p}}_1^n$ and $\overline{\boldsymbol{p}}_2^n$ have a convergent subsequence.

To prove that $\psi^{-1}(K)$ is compact we need only to check that the sequences $(\theta^n)_{n\in\mathbb{N}}$ and $(\boldsymbol{v}^n)_{n\in\mathbb{N}}$ have convergent subsequences. Using the second equality in (2.2.2) we obtain

$$\theta^{n_k} = \arctan\left(\frac{\boldsymbol{p}_i^{n_k}.\mathbf{J}\overline{\boldsymbol{p}}_i^{n_k}}{\boldsymbol{p}_i^{n_k}.\overline{\boldsymbol{p}}_i^{n_k}}\right) \ .$$

Since for each $i \in \{1,2\}$ the sequences $(\boldsymbol{p}_i^{n_k})_{k\in\mathbb{N}}$ and $(\overline{\boldsymbol{p}}_i^{n_k})_{k\in\mathbb{N}}$ converge and $\arctan(x)$ is a continuous function then the sequence $(\theta^{n_k})_{k\in\mathbb{N}}$ converges to a point θ^* satisfying $\overline{\boldsymbol{p}}_i^* = R_{\theta^*}\boldsymbol{p}_i^*$. Similarly, using the first second equality in (2.2.2), we obtain

$$\boldsymbol{v}^{n_k} = \overline{\boldsymbol{q}}_i^{n_k} - R_{\theta^{n_k}}\boldsymbol{q}_i^{n_k} \ .$$

Convergence of $(\boldsymbol{p}_i^{n_k})_{k\in\mathbb{N}}$, $(\overline{\boldsymbol{p}}_i^{n_k})_{k\in\mathbb{N}}$ and $(\theta^{n_k})_{k\in\mathbb{N}}$ implies convergence of $(\boldsymbol{v}^{n_k})_{n_k\in\mathbb{N}}$ to a limit point $\boldsymbol{v}^*$ satisfying $\overline{\boldsymbol{q}}_i^* = R_{\theta^*}\boldsymbol{q}_i^* + \boldsymbol{v}^*$.

Thus, the sequence $(\boldsymbol{y}^n)_{n\in\mathbb{N}}$ has a subsequence $(\boldsymbol{y}^{n_k})_{k\in\mathbb{N}}$ converging to a point $\boldsymbol{y}^* = (R_{\theta^*}, \boldsymbol{v}^*, \boldsymbol{q}_1^*, \boldsymbol{q}_2^*, \boldsymbol{p}_1^*, \boldsymbol{p}_2^*) \in \psi^{-1}(K)$, which ends the proof. $\square$

The actions of G on $\mathcal{G}$ and $\mathcal{G}^*$

As G is isomorphic to $SE(2)$, we use the matricial representation for elements of G given by 3×3 matrices of the form

$$\begin{pmatrix} R_\theta & \boldsymbol{v} \\ 0 & 1 \end{pmatrix} \ .$$

The Lie Algebra $\mathcal{G}$ of G also admits a representation in terms of 3×3 matrices of the form

$$\begin{pmatrix} -\omega\mathbf{J} & \boldsymbol{a} \\ 0 & 0 \end{pmatrix} \ ,$$

where $\omega \in \mathbb{R}$ and $\boldsymbol{a} \in \mathbb{R}^2$. The bracket in $\mathcal{G}$ is given by the commutator bracket $[A, B] = AB - BA$. We identify $\mathcal{G}$ with $\mathbb{R}^3$ by the isomorphism

$$\begin{pmatrix} -\omega\mathbf{J} & \boldsymbol{a} \\ 0 & 0 \end{pmatrix} \in \mathcal{G} \mapsto (\omega, \boldsymbol{a}) \in \mathbb{R}^3 ,$$

so that the expression for the Lie Algebra bracket reduces to

$$[(\omega_1, \boldsymbol{a}_1), (\omega_2, \boldsymbol{a}_2)] = \left(0, \omega_1 \mathbf{J}^T \boldsymbol{a}_2 - \omega_2 \mathbf{J}^T \boldsymbol{a}_1\right) .$$

The adjoint action of G on $\mathcal{G}$ is then given by the conjugation

$$\begin{pmatrix} R_\theta & \boldsymbol{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\omega\mathbf{J} & \boldsymbol{a} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R_{-\theta} & -R_{-\theta}\boldsymbol{v} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\omega\mathbf{J} & \omega\mathbf{J}\boldsymbol{v} + R_\theta \boldsymbol{a} \\ 0 & 0 \end{pmatrix} ,$$

which is, using the identification with $\mathbb{R}^3$, equal to

$$Ad_{(R_\theta, \boldsymbol{v})}(\omega, \boldsymbol{a}) = (\omega, \omega\mathbf{J}\boldsymbol{v} + R_\theta \boldsymbol{a}) .$$

Using the pairing between $\mathcal{G}$ and $\mathcal{G}^*$ given by the trace of the product, we obtain a representation for $\mathcal{G}^*$ in terms of matrices of the form

$$\begin{pmatrix} \frac{\mu}{2}\mathbf{J} & 0 \\ \boldsymbol{\alpha} & 0 \end{pmatrix} ,$$

where $\mu \in \mathbb{R}$ and $\boldsymbol{\alpha} \in \mathbb{R}^2$. As we did with $\mathcal{G}$, we identify $\mathcal{G}^*$ with $\mathbb{R}^3$ by

$$\begin{pmatrix} \frac{\mu}{2}\mathbf{J} & 0 \\ \boldsymbol{\alpha} & 0 \end{pmatrix} \in \mathcal{G}^* \mapsto (\mu, \boldsymbol{\alpha}) \in \mathbb{R}^3$$

so that the pairing becomes $\langle(\mu, \boldsymbol{\alpha}), (\omega, \boldsymbol{a})\rangle = \mu\omega + \boldsymbol{\alpha}.\boldsymbol{a}$. To obtain the coadjoint action of G on $\mathcal{G}^*$ we compute

$$\begin{aligned}
\left\langle Ad^*_{(R_\theta, \boldsymbol{v})^{-1}} (\mu, \boldsymbol{\alpha}), (\omega, \boldsymbol{a}) \right\rangle &= \left\langle (\mu, \boldsymbol{\alpha}), Ad_{(R_{-\theta}, -R_{-\theta}\boldsymbol{v})} (\omega, \boldsymbol{a}) \right\rangle \\
&= \langle (\mu, \boldsymbol{\alpha}), (\omega, -\omega\mathbf{J}R_{-\theta}\boldsymbol{v} + R_{-\theta}\boldsymbol{a}) \rangle \\
&= \mu\omega - \omega\boldsymbol{\alpha}.\mathbf{J}R_{-\theta}\boldsymbol{v} + \boldsymbol{\alpha}.R_{-\theta}\boldsymbol{a} \\
&= (\mu - \boldsymbol{\alpha}.R_{-\theta}\mathbf{J}\boldsymbol{v})\,\omega + R_\theta\boldsymbol{\alpha}.\boldsymbol{a} \\
&= \langle (\mu - R_\theta\boldsymbol{\alpha}.\mathbf{J}\boldsymbol{v}, R_\theta\boldsymbol{\alpha}), (\omega, \boldsymbol{a}) \rangle .
\end{aligned}$$

Hence, we can write the coadjoint action of G on $\mathcal{G}^*$ as

$$Ad^*_{(R_\theta,\boldsymbol{v})^{-1}}(\mu,\boldsymbol{\alpha}) = (\mu - R_\theta\boldsymbol{\alpha}.\mathbf{J}\boldsymbol{v}, R_\theta\boldsymbol{\alpha}) \ . \tag{2.2.3}$$

The momentum map

By proposition 2.1.2 the action $\phi_{(R_\theta,\boldsymbol{v})}$ of G on M is Hamiltonian with momentum map $J_G : M \to \mathcal{G}^*$ given by

$$J_G\left(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{p}_1, \boldsymbol{p}_2\right) = (L, \boldsymbol{P}) \ . \tag{2.2.4}$$

To simplify notation, we set

$$\boldsymbol{m} = (\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{p}_1, \boldsymbol{p}_2) \in M \ ,$$

throughout this section.

Remark Let J_G be the momentum map defined in (2.2.4) and $(\mu,\boldsymbol{\alpha}) \in \mathcal{G}^*$.

i) If $k_1+k_2 \neq 0$ then every $(\mu,\boldsymbol{\alpha})$ such that $2(k_1+k_2)\mu+|\boldsymbol{\alpha}|^2 \neq 0$ is a regular value of J_G. If $2(k_1+k_2)\mu+|\boldsymbol{\alpha}|^2 = 0$ then $J_G^{-1}(\mu,\boldsymbol{\alpha})$ is still a smooth submanifold of M provided we remove the points of the form $\boldsymbol{q}_1 = \boldsymbol{q}_2$, $\boldsymbol{p}_1 = \boldsymbol{p}_2 = \mathbf{0}$.

ii) If $k_1+k_2 = 0$ then every $(\mu,\boldsymbol{\alpha})$ such that $\mu \neq 0$ is a regular value of J_G. If $\mu = 0$ then $J_G^{-1}(0,\boldsymbol{\alpha})$ is still a smooth submanifold of M provided we remove the points of the form $\boldsymbol{q}_1 = \boldsymbol{q}_2$, $\boldsymbol{p}_1 = \boldsymbol{p}_2 = \mathbf{0}$.

Lemma 2.2.2. *The momentum map J_G is not equivariant with respect to the coadjoint action $Ad^*_{(R_\theta,v)^{-1}}$. Furthermore, the following equalities are satisfied*

$$\begin{aligned} J_G\left(\phi_{(R_\theta,\boldsymbol{v})}\left(\boldsymbol{m}\right)\right) &= \left(L - R_\theta\boldsymbol{P}.\mathbf{J}\boldsymbol{v} - \frac{k_1+k_2}{2}|\boldsymbol{v}|^2, R_\theta\boldsymbol{P} + (k_1+k_2)\,\mathbf{J}\boldsymbol{v}\right) \\ Ad^*_{(R_\theta,\boldsymbol{v})^{-1}}\left(J_G\left(\boldsymbol{m}\right)\right) &= (L - R_\theta\boldsymbol{P}.\mathbf{J}\boldsymbol{v}, R_\theta\boldsymbol{P}) \ . \end{aligned} \tag{2.2.5}$$

Proof. The fact that the momentum map (2.2.4) is not equivariant with respect to the coadjoint action $Ad^*_{(R_\theta,\boldsymbol{v})^{-1}}$ follows from the equalities (2.2.5). The second equality in (2.2.5) follows trivially from the expression for the coadjoint action (2.2.3).

We now prove the first equality in (2.2.5). To compute $J_G\left(\phi_{(R_\theta,\boldsymbol{v})}(\boldsymbol{m})\right)$, we note that

$$J_G\left(\phi_{(R_\theta,\boldsymbol{v})}(\boldsymbol{m})\right)=\left(L\circ\phi_{(R_\theta,\boldsymbol{v})}(\boldsymbol{m}),\boldsymbol{P}\circ\phi_{(R_\theta,\boldsymbol{v})}(\boldsymbol{m})\right)\ .$$

Recall from proposition 2.1.2 that

$$\boldsymbol{P}=\boldsymbol{p}_1+\boldsymbol{p}_2+\mathbf{J}\left(k_1\boldsymbol{q}_1+k_2\boldsymbol{q}_2\right)\ . \tag{2.2.6}$$

Using (2.2.1), we get

$$\begin{aligned}(\boldsymbol{p}_1+\boldsymbol{p}_2)\circ\phi_{(R_\theta,\boldsymbol{v})}(\boldsymbol{m}) &= R_\theta\left(\boldsymbol{p}_1+\boldsymbol{p}_2\right)\\ (k_1\boldsymbol{q}_1+k_2\boldsymbol{q}_2)\circ\phi_{(R_\theta,\boldsymbol{v})}(\boldsymbol{m}) &= R_\theta\left(k_1\boldsymbol{q}_1+k_2\boldsymbol{q}_2\right)+(k_1+k_2)\boldsymbol{v}\ .\end{aligned}$$

From the two equalities above and (2.2.6), we obtain

$$\boldsymbol{P}\circ\phi_{(R_\theta,\boldsymbol{v})}(\boldsymbol{m})=R_\theta\boldsymbol{P}+(k_1+k_2)\mathbf{J}\boldsymbol{v}\ .$$

Recall from proposition 2.1.2 that

$$L=\sum_{i=1,2}\boldsymbol{q}_i.\mathbf{J}\boldsymbol{p}_i-\frac{k_i}{2}\left|\boldsymbol{q}_i\right|^2\ . \tag{2.2.7}$$

Using (2.2.1), for each $i\in\{1,2\}$ we get

$$\begin{aligned}\boldsymbol{q}_i.\mathbf{J}\boldsymbol{p}_i\circ\phi_{(R_\theta,\boldsymbol{v})}(\boldsymbol{m}) &= \left(R_\theta\boldsymbol{q}_i+\boldsymbol{v}\right).\mathbf{J}R_\theta\boldsymbol{p}_i\\ &= R_\theta\boldsymbol{q}_i.R_\theta\mathbf{J}\boldsymbol{p}_i+R_\theta\mathbf{J}\boldsymbol{p}_i.\boldsymbol{v}\\ &= \boldsymbol{q}_i.\mathbf{J}\boldsymbol{p}_i+R_\theta\mathbf{J}\boldsymbol{p}_i.\boldsymbol{v} \qquad (2.2.8)\end{aligned}$$

and

$$\begin{aligned}\left|\boldsymbol{q}_i\right|^2\circ\phi_{(R_\theta,\boldsymbol{v})}(\boldsymbol{m}) &= \left|R_\theta\boldsymbol{q}_i+\boldsymbol{v}\right|^2\\ &= \left|\boldsymbol{q}_i\right|^2+2R_\theta\boldsymbol{q}_i.\boldsymbol{v}+\left|\boldsymbol{v}\right|^2\ . \qquad (2.2.9)\end{aligned}$$

From (2.2.8) and (2.2.9), we obtain

$$\begin{aligned}\left(\boldsymbol{q}_i.\mathbf{J}\boldsymbol{p}_i - \frac{k_i}{2}|\boldsymbol{q}_i|^2\right) \circ \phi_{(R_\theta,\boldsymbol{v})}(\boldsymbol{m}) &= \\ &= \boldsymbol{q}_i.\mathbf{J}\boldsymbol{p}_i - \frac{k_i}{2}|\boldsymbol{q}_i|^2 + R_\theta\mathbf{J}\boldsymbol{p}_i.\boldsymbol{v} - k_i R_\theta \boldsymbol{q}_i.\boldsymbol{v} - \frac{k_i}{2}|\boldsymbol{v}|^2 \\ &= \boldsymbol{q}_i.\mathbf{J}\boldsymbol{p}_i - \frac{k_i}{2}|\boldsymbol{q}_i|^2 - R_\theta\boldsymbol{p}_i.\mathbf{J}\boldsymbol{v} - k_i R_\theta \mathbf{J}\boldsymbol{q}_i.\mathbf{J}\boldsymbol{v} - \frac{k_i}{2}|\boldsymbol{v}|^2 \\ &= \boldsymbol{q}_i.\mathbf{J}\boldsymbol{p}_i - \frac{k_i}{2}|\boldsymbol{q}_i|^2 - R_\theta\left(\boldsymbol{p}_i - k_i\mathbf{J}\boldsymbol{q}_i\right).\mathbf{J}\boldsymbol{v} - \frac{k_i}{2}|\boldsymbol{v}|^2 \ . \end{aligned} \tag{2.2.10}$$

Putting together (2.2.6), (2.2.7) and (2.2.10), we obtain

$$L \circ \phi_{(R_\theta,\boldsymbol{v})}(\boldsymbol{m}) = L - R_\theta \boldsymbol{P}.\mathbf{J}\boldsymbol{v} - \frac{k_1+k_2}{2}|\boldsymbol{v}|^2 \ ,$$

as required. □

Lemma 2.2.3. *The momentum map J_G is equivariant with respect to the coadjoint action $\widetilde{Ad}^*_{(R_\theta,v)^{-1}} : G \times \mathcal{G}^* \to \mathcal{G}^*$ given by*

$$\widetilde{Ad}^*_{(R_\theta,v)^{-1}}(\mu,\boldsymbol{\alpha}) = \left(\mu - R_\theta\boldsymbol{\alpha}.\mathbf{J}\boldsymbol{v} - \frac{k_1+k_2}{2}|\boldsymbol{v}|^2, R_\theta\boldsymbol{\alpha} + (k_1+k_2)\mathbf{J}\boldsymbol{v}\right) \ .$$

Proof. We use propositions 1.2.28 and 1.2.29. We introduce the coadjoint cocycle associated to J_G, measuring the lack of equivariance of J_G, given by

$$\begin{aligned}\sigma(R_\theta,\boldsymbol{v}) &= J_G\left(\phi_{(R_\theta,\boldsymbol{v})}(\boldsymbol{m})\right) - Ad^*_{(R_\theta,\boldsymbol{v})^{-1}}\left(J_G(\boldsymbol{m})\right) \\ &= \left(-\frac{k_1+k_2}{2}|\boldsymbol{v}|^2, (k_1+k_2)\mathbf{J}\boldsymbol{v}\right) \ .\end{aligned}$$

We then define a new coadjoint action of G on $\mathcal{G}^*$, given by

$$\begin{aligned}\widetilde{Ad}^*_{(R_\theta,\boldsymbol{v})^{-1}}(\mu,\boldsymbol{\alpha}) &= Ad^*_{(R_\theta,\boldsymbol{v})^{-1}}(\mu,\boldsymbol{\alpha}) + \sigma(R_\theta,\boldsymbol{v}) \\ &= \left(\mu - R_\theta\boldsymbol{\alpha}.\mathbf{J}\boldsymbol{v} - \frac{k_1+k_2}{2}|\boldsymbol{v}|^2, R_\theta\boldsymbol{\alpha} + (k_1+k_2)\mathbf{J}\boldsymbol{v}\right) \ .\end{aligned}$$

This new coadjoint action of G on $\mathcal{G}^*$ makes the momentum map J_G equivariant. □

Application of Marsden–Weinstein Theorem

Lemma 2.2.4. *Let $G_{(\mu,\boldsymbol{\alpha})}$ denote the isotropy group of the coadjoint action $\widetilde{Ad}^*_{(R_\theta,\boldsymbol{v})^{-1}}$ of G on $\mathcal{G}^*$.*

i) If $k_1 + k_2 = 0$ *and* $\boldsymbol{\alpha} = \mathbf{0}_{\mathbb{R}^2}$ *then*

$$G_{(\mu,\boldsymbol{\alpha})} = G \ .$$

ii) If $k_1 + k_2 = 0$ *and* $\boldsymbol{\alpha} \neq \mathbf{0}_{\mathbb{R}^2}$ *then*

$$G_{(\mu,\boldsymbol{\alpha})} = \{(\mathbf{Id}_{2\times 2}, \boldsymbol{v}) \in G : \boldsymbol{\alpha}.\mathbf{J}\boldsymbol{v} = 0\} \ .$$

iii) If $k_1 + k_2 \neq 0$ *then*

$$G_{(\mu,\boldsymbol{\alpha})} = \left\{(R_\theta, \boldsymbol{v}(\theta)) \in G : \boldsymbol{v}(\theta) = \frac{1}{k_1 + k_2}\mathbf{J}^{-1}\left(\mathbf{Id} - R_\theta\right)\boldsymbol{\alpha}\right\} \ .$$

Moreover, the action $\phi_{(R_\theta,\boldsymbol{v})}$ *of* G *on* M *induces an action of* $G_{(\mu,\boldsymbol{\alpha})}$ *on* $J_G^{-1}\left(\mu, \boldsymbol{\alpha}\right)$ *which is free and proper.*

Proof. To prove the first part of the lemma recall that the isotropy group of the coadjoint action $\widetilde{Ad}^*_{(R_\theta,\boldsymbol{v})^{-1}}$ of G on $\mathcal{G}^*$ is defined by

$$G_{(\mu,\boldsymbol{\alpha})} = \left\{(R_\theta, \boldsymbol{v}) \in G : \widetilde{Ad}^*_{(R_\theta,\boldsymbol{v})^{-1}}\left(\mu, \boldsymbol{\alpha}\right) = \left(\mu, \boldsymbol{\alpha}\right)\right\} \ ,$$

where $\widetilde{Ad}^*_{(R_\theta,\boldsymbol{v})^{-1}}$ is given in lemma 2.2.3. Points i), ii) and iii) are easily obtained solving the equation $\widetilde{Ad}^*_{(R_\theta,\boldsymbol{v})^{-1}}\left(\mu, \boldsymbol{\alpha}\right) = \left(\mu, \boldsymbol{\alpha}\right)$ with respect to $(R_\theta, \boldsymbol{v})$.

For the second part of the lemma note that for every $(\mu, \boldsymbol{\alpha}) \in \mathcal{G}^*$ the isotropy group $G_{(\mu,\boldsymbol{\alpha})}$ is a closed subgroup of the Lie group G. Hence, $G_{(\mu,\boldsymbol{\alpha})}$ is itself a Lie group.

Denote by $\phi_{(\mu,\boldsymbol{\alpha})}$ the restriction of $\phi_{(R_\theta,\boldsymbol{v})}$ to $G_{(\mu,\boldsymbol{\alpha})} \times J_G^{-1}(\mu, \boldsymbol{\alpha})$. By lemma 2.2.1 the action $\phi_{(R_\theta,\boldsymbol{v})}$ acts freely and properly on M. Thus, its restriction $\phi_{(\mu,\boldsymbol{\alpha})}$ is a free and proper action of $G_{(\mu,\boldsymbol{\alpha})}$ on $J_G^{-1}(\mu, \boldsymbol{\alpha})$, which ends the proof. □

We now gather all the information obtained in this section:

1) The action $\phi_{(R_\theta,\boldsymbol{v})}$ acts freely and properly on M by lemma 2.2.1.

2) The momentum map J_G is coadjoint equivariant by lemma 2.2.3 (for a corrected coadjoint action $\widetilde{Ad}^*_{(R_\theta,\boldsymbol{v})^{-1}}$ of G on $\mathcal{G}^*$).

3) If $k_1 + k_2 \neq 0$ then for all $(\mu, \boldsymbol{\alpha}) \in \mathcal{G}^*$ such that $2(k_1 + k_2)\mu + |\boldsymbol{\alpha}|^2 \neq 0$, $J_G^{-1}(\mu, \boldsymbol{\alpha})$ is a smooth submanifold of M (if $2(k_1 + k_2)\mu + |\boldsymbol{\alpha}|^2 = 0$ we remove the points of the form $\boldsymbol{q}_1 = \boldsymbol{q}_2$, $\boldsymbol{p}_1 = \boldsymbol{p}_2 = \mathbf{0}$ to make it a smooth manifold). Similarly, if $k_1 + k_2 = 0$ then for all $(\mu, \boldsymbol{\alpha}) \in \mathcal{G}^*$ such that $\mu \neq 0$, $J_G^{-1}(\mu, \boldsymbol{\alpha})$ is a smooth submanifold of M (if $\mu = 0$ we remove the points of the form $\boldsymbol{q}_1 = \boldsymbol{q}_2$, $\boldsymbol{p}_1 = \boldsymbol{p}_2 = \mathbf{0}$ to make it a smooth manifold).

4) By lemma 2.2.4 the isotropy group of the coadjoint action $\widetilde{Ad}^*_{(R_\theta, \boldsymbol{v})^{-1}}$ of G on $\mathcal{G}^*$ has dimension given by

$$\dim G_{(\mu,\boldsymbol{\alpha})} = \begin{cases} 3 & \text{if } k_1 + k_2 = 0 \text{ and } \boldsymbol{\alpha} = \mathbf{0}_{\mathbb{R}^2} \\ 1 & \text{if } k_1 + k_2 = 0 \text{ and } \boldsymbol{\alpha} \neq \mathbf{0}_{\mathbb{R}^2} \\ 1 & \text{if } k_1 + k_2 \neq 0 \end{cases} .$$

From points 1), 2), 3) and 4) above and theorem 1.2.30 (Marsden–Weinstein reduction) we obtain:

a) $M_{(\mu,\boldsymbol{\alpha})} = J_G^{-1}(\mu, \boldsymbol{\alpha})/G_{(\mu,\boldsymbol{\alpha})}$ is a smooth submanifold of M.

b) $M_{(\mu,\boldsymbol{\alpha})}$ is a symplectic manifold of dimension given by

$$\begin{aligned} \dim M_{(\mu,\boldsymbol{\alpha})} &= \dim J_G^{-1}(\mu, \boldsymbol{\alpha}) - \dim G_{(\mu,\boldsymbol{\alpha})} \\ &= \begin{cases} 2 & \text{if } \boldsymbol{\alpha} = \mathbf{0}_{\mathbb{R}^2} \text{ and } k_1 + k_2 = 0 \\ 4 & \text{otherwise} \end{cases} . \end{aligned}$$

We have proven the following result.

Proposition 2.2.5. *The Hamiltonian system (2.1.9) always reduces to one with two degrees of freedom. Furthermore, if the sum of the two charges is zero, i.e* $k_1 + k_2 = 0$*, the dynamics in the zero sets of the linear momenta are integrable.*

2.2.2 Extended symmetry group for the case $\Omega_1 = \Omega_2$

We proved in proposition 2.1.2 that for the case of equal gyrofrequencies $\Omega_1 = \Omega_2$ the Hamiltonian system (2.1.9) has an extra conserved quantity $W = |\boldsymbol{p}_1 + \boldsymbol{p}_2|^2$. In lemma

2.1.3 we described the group of symmetries of (2.1.9) associated with the conservation of W. In this section we discuss the symplectic reduction of the phase space of (2.1.9) by the extended group of symmetries that includes the symmetry associated with W. We will use the Commuting reduction theorem 1.2.31 and the results in the preceding section to obtain the reduction of the Hamiltonian system (2.1.9) for the case where $\Omega_1 = \Omega_2$.

Let $K = \mathbb{R}$ and consider the action ϕ_λ of K on M given by (2.1.18). The group operation in K is given by

$$\lambda_1 . \lambda_2 = \lambda_1 + \lambda_2 \ , \tag{2.2.11}$$

the identity element of K is $0 \in \mathbb{R}$ and the inverse of an element of $\lambda \in K$ is given by $-\lambda$. Let $\mathcal{K}$ denote the Lie Algebra of K and $\mathcal{K}^*$ its dual. Since $K = \mathbb{R}$ and K is abelian (with respect to the group operation (2.2.11)), we obtain that

1) $\mathcal{K} \approx \mathbb{R}$ and the adjoint action of K on $\mathcal{K}$, $Ad : K \times \mathcal{K} \to \mathcal{K}$, is given by

$$Ad_\lambda(\xi) = \xi \ , \qquad \xi \in \mathcal{K} \ .$$

2) $\mathcal{K}^* \approx \mathbb{R}$ and the coadjoint action of K on $\mathcal{K}^*$, $Ad^* : K \times \mathcal{K}^* \to \mathcal{K}^*$, is given by

$$Ad^*_{\lambda^{-1}}(\zeta) = \zeta \ , \qquad \zeta \in \mathcal{K}^* \ . \tag{2.2.12}$$

By lemma 2.1.3 the action of K on M is Hamiltonian with momentum map $J_K : M \to \mathcal{K}^*$ given by

$$J_K(\boldsymbol{m}) = W \ , \tag{2.2.13}$$

where $W = |\boldsymbol{p}_1 + \boldsymbol{p}_2|^2$ and $\boldsymbol{m} = (\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{p}_1, \boldsymbol{q}_2)$ denotes a point of M.

Remark Let J_K be the momentum map defined in (2.2.13) and $\xi \in \mathcal{K}^*$. Every $\xi \neq 0$ is a regular value of J_K. In this section we avoid the case $\xi = 0$ since it needs singular reduction to be treated. We deal with this case in section 2.3.

Lemma 2.2.6. *The momentum map J_K is equivariant with respect to the coadjoint action $Ad^*_{\lambda^{-1}} : K \times \mathcal{K}^* \to \mathcal{K}^*$ given by* (2.2.12).

Proof. Putting together (2.2.12) and (2.2.13), we obtain

$$Ad^*_{\lambda^{-1}}\left(J_K(\boldsymbol{m})\right) = W \ .$$

We now compute $J_K(\phi_\lambda(\boldsymbol{m}))$. Adding the two last equations in (2.1.18), we get

$$(\boldsymbol{p}_1 + \boldsymbol{p}_2) \circ \phi_\lambda(\boldsymbol{m}) = R_{2(k_1+k_2)\lambda}(\boldsymbol{p}_1 + \boldsymbol{p}_2) \ .$$

Thus, we obtain that

$$\begin{aligned} W \circ \phi_\lambda(\boldsymbol{m}) &= |\boldsymbol{p}_1 + \boldsymbol{p}_2|^2 \circ \phi_\lambda(\boldsymbol{m}) \\ &= \left|R_{2(k_1+k_2)\lambda}(\boldsymbol{p}_1 + \boldsymbol{p}_2)\right|^2 \\ &= |\boldsymbol{p}_1 + \boldsymbol{p}_2|^2 \\ &= W \ . \end{aligned}$$

Hence, we obtain that

$$J_K(\phi_\lambda(\boldsymbol{m})) = Ad^*_{\lambda^{-1}}\left(J_K(\boldsymbol{m})\right) \ ,$$

for every $\boldsymbol{m} \in M$, which concludes the proof. □

Lemma 2.2.7. *The actions $\phi_{(R_\theta,\boldsymbol{v})}$ of G on M and ϕ_λ of K on M commute.*

Proof. It is an easy but tedious computation to check that

$$\phi_{(R_\theta,\boldsymbol{v})} \circ \phi_\lambda(\boldsymbol{m}) = \phi_\lambda \circ \phi_{(R_\theta,\boldsymbol{v})}(\boldsymbol{m}) \ . \tag{2.2.14}$$

□

Furthermore, the equality (2.2.14) defines an action of $G \times K$ on M, $\phi_{(R_\theta,\boldsymbol{v},\lambda)}$, given by

$$\begin{aligned} \boldsymbol{q}_1 &\rightarrow R_\theta\left[\boldsymbol{q}_1 + \frac{1}{k_1+k_2}\left(R_\nu - \mathbf{Id}_{2\times 2}\right)\mathbf{J}(\boldsymbol{p}_1 + \boldsymbol{p}_2)\right] + \boldsymbol{v} \\ \boldsymbol{q}_2 &\rightarrow R_\theta\left[\boldsymbol{q}_2 + \frac{1}{k_1+k_2}\left(R_\nu - \mathbf{Id}_{2\times 2}\right)\mathbf{J}(\boldsymbol{p}_1 + \boldsymbol{p}_2)\right] + \boldsymbol{v} \\ \boldsymbol{p}_1 &\rightarrow R_\theta\left[\boldsymbol{p}_1 + \frac{k_1}{k_1+k_2}\left(R_\nu - \mathbf{Id}_{2\times 2}\right)(\boldsymbol{p}_1 + \boldsymbol{p}_2)\right] \\ \boldsymbol{p}_2 &\rightarrow R_\theta\left[\boldsymbol{p}_2 + \frac{k_2}{k_1+k_2}\left(R_\nu - \mathbf{Id}_{2\times 2}\right)(\boldsymbol{p}_1 + \boldsymbol{p}_2)\right] \ , \end{aligned} \tag{2.2.15}$$

where $\nu = 2(k_1 + k_2)\lambda$. The action $\phi_{(R_\theta, \boldsymbol{v}, \lambda)}$ defined above is a group of symmetries of the Hamiltonian system (2.1.9) provided $\Omega_1 = \Omega_2$.

Lemma 2.2.8. *The momentum map J_G given by* (2.2.4) *is invariant under the action ϕ_λ of K on M.*

Proof. Throughout the proof we use the notation $\nu = 2(k_1 + k_2)\lambda$. We need to check that

$$J_G\left(\phi_\lambda(\boldsymbol{m})\right) = \left(L \circ \phi_\lambda(\boldsymbol{m}), \boldsymbol{P} \circ \phi_\lambda(\boldsymbol{m})\right) = \left(L, \boldsymbol{P}\right) .$$

To compute $\boldsymbol{P} \circ \phi_\lambda\left(\boldsymbol{m}\right)$ recall from proposition 2.1.2 that

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \mathbf{J}\left(k_1 \boldsymbol{q}_1 + k_2 \boldsymbol{q}_2\right) . \tag{2.2.16}$$

Summing the two last entries in (2.1.18), we get

$$\left(\boldsymbol{p}_1 + \boldsymbol{p}_2\right) \circ \phi_\lambda\left(\boldsymbol{m}\right) = R_\nu\left(\boldsymbol{p}_1 + \boldsymbol{p}_2\right) . \tag{2.2.17}$$

Similarly, using the first two entries of (2.1.18), we obtain

$$\begin{aligned}
&\left(k_1 \boldsymbol{q}_1 + k_2 \boldsymbol{q}_2\right) \circ \phi_\lambda\left(\boldsymbol{m}\right) = \\
&= k_1 \boldsymbol{q}_1 + k_2 \boldsymbol{q}_2 - \mathbf{J}\left(\boldsymbol{p}_1 + \boldsymbol{p}_2\right) + \mathbf{J} R_\nu\left(\boldsymbol{p}_1 + \boldsymbol{p}_2\right) \\
&= -\mathbf{J}\left(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \mathbf{J}\left(k_1 \boldsymbol{q}_1 + k_2 \boldsymbol{q}_2\right)\right) + \mathbf{J} R_\nu\left(\boldsymbol{p}_1 + \boldsymbol{p}_2\right) \\
&= -\mathbf{J}\boldsymbol{P} + \mathbf{J} R_\nu\left(\boldsymbol{p}_1 + \boldsymbol{p}_2\right) .
\end{aligned} \tag{2.2.18}$$

Combining (2.2.16), (2.2.17) and (2.2.18) gives

$$\boldsymbol{P} \circ \phi_\lambda\left(\boldsymbol{m}\right) = \boldsymbol{P} . \tag{2.2.19}$$

To compute $L \circ \phi_\lambda\left(\boldsymbol{m}\right)$, recall from proposition 2.1.2 that

$$L = \sum_{i=1,2} \boldsymbol{q}_i . \mathbf{J} \boldsymbol{p}_i - \frac{k_i}{2}\left|\boldsymbol{q}_i\right|^2 . \tag{2.2.20}$$

Let $\mathbf{Id}_{2\times 2}$ be denoted by $\mathbf{Id}$. Using (2.1.18) we obtain

$$\begin{aligned} &\boldsymbol{q}_i\mathbf{J}\boldsymbol{p}_i \circ \phi_\lambda(\boldsymbol{m}) = \\ &= \left(\frac{1}{k_1+k_2}\left[R_\nu - \mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2)+\boldsymbol{q}_i\right)\left(\frac{k_i}{k_1+k_2}\left[R_\nu - \mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2)+\mathbf{J}\boldsymbol{p}_i\right) \\ &= \boldsymbol{q}_i\mathbf{J}\boldsymbol{p}_i + \frac{k_i}{k_1+k_2}\boldsymbol{q}_i\left[R_\nu - \mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2) + \frac{1}{k_1+k_2}\mathbf{J}\boldsymbol{p}_i\left[R_\nu - \mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2) \\ &\quad + \frac{k_i}{(k_1+k_2)^2}\left|\left[R_\nu - \mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2)\right|^2 \ . \end{aligned} \tag{2.2.21}$$

Similarly, we obtain

$$\begin{aligned} \frac{k_i}{2}\left|\boldsymbol{q}_i\right|^2 \circ \phi_\lambda(\boldsymbol{m}) &= \frac{k_i}{2}\left|\frac{1}{k_1+k_2}\left[R_\nu - \mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2)+\boldsymbol{q}_i\right|^2 \\ &= \frac{k_i}{2}\left|\boldsymbol{q}_i\right|^2 + \frac{k_i}{k_1+k_2}\boldsymbol{q}_i\left[R_\nu - \mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2) \\ &\quad + \frac{k_i}{2(k_1+k_2)^2}\left|\left[R_\nu - \mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2)\right|^2 \ . \end{aligned} \tag{2.2.22}$$

Putting together (2.2.21) and (2.2.22), we get

$$\begin{aligned} \left(\boldsymbol{q}_i\mathbf{J}\boldsymbol{p}_i - \frac{k_i}{2}\left|\boldsymbol{q}_i\right|^2\right)\circ \phi_\lambda(\boldsymbol{m}) &= \boldsymbol{q}_i\mathbf{J}\boldsymbol{p}_i - \frac{k_i}{2}\left|\boldsymbol{q}_i\right|^2 \\ &\quad + \frac{1}{k_1+k_2}\mathbf{J}\boldsymbol{p}_i\left[R_\nu - \mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2) \\ &\quad + \frac{k_i}{2(k_1+k_2)^2}\left|\left[R_\nu - \mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2)\right|^2 \ , \end{aligned}$$

or equivalently

$$\begin{aligned} L\circ \phi_\lambda(\boldsymbol{m}) &= L + \frac{1}{k_1+k_2}\mathbf{J}\left(\boldsymbol{p}_1+\boldsymbol{p}_2\right)\left[R_\nu - \mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2) \\ &\quad + \frac{1}{2(k_1+k_2)}\left|\left[R_\nu - \mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2)\right|^2 \ . \end{aligned} \tag{2.2.23}$$

Noting that

$$
\begin{aligned}
&\mathbf{J}\left(\boldsymbol{p}_1+\boldsymbol{p}_2\right)\left[R_\nu-\mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2)+\frac{1}{2}\left|\left[R_\nu-\mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2)\right|^2 = \\
&= \mathbf{J}\left(\boldsymbol{p}_1+\boldsymbol{p}_2\right).\left[R_\nu-\mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2) \\
&\quad +\frac{1}{2}\left[R_\nu-\mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2).\left[R_\nu-\mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2) \\
&= \left(\mathbf{J}\left(\boldsymbol{p}_1+\boldsymbol{p}_2\right)+\frac{1}{2}\left[R_\nu-\mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2)\right).\left[R_\nu-\mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2) \\
&= \frac{1}{2}\left[R_\nu+\mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2).\left[R_\nu-\mathbf{Id}\right]\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2) \\
&= \frac{1}{2}\left(\left|R_\nu\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2)\right|^2-\left|\mathbf{J}(\boldsymbol{p}_1+\boldsymbol{p}_2)\right|^2\right) \\
&= 0
\end{aligned}
$$

we obtain that the last two terms on the right hand side of equation (2.2.23) sum up to zero. Thus, we obtain

$$L\circ\phi_\lambda\left(\boldsymbol{m}\right)=L\ ,$$

as required. □

We can now apply the Commuting Reduction theorem 1.2.31. From the previous section 2.2.1, we have that:

i) G acts symplectically on M and has a momentum map $J_G : M \to \mathcal{G}^*$ which is coadjoint equivariant with respect to the coadjoint action of G on $\mathcal{G}^*$ given in lemma 2.2.3.

ii) for all $(\mu,\boldsymbol{\alpha})\in\mathcal{G}^*$, $J_G^{-1}\left(\mu,\boldsymbol{\alpha}\right)$ is a smooth submanifold of M.

iii) the action of the isotropy group $G_{(\mu,\boldsymbol{\alpha})}$ of the coadjoint action $\widetilde{Ad}^*_{(R_\theta,\boldsymbol{v})^{-1}}$ of G on $\mathcal{G}^*$ acts freely and properly on $J_G^{-1}\left(\mu,\boldsymbol{\alpha}\right)$ by lemma 2.2.4 (or lemma 2.2.1).

iv) applying theorem 1.2.30 we obtained that $M_{(\mu,\boldsymbol{\alpha})}=J_G^{-1}\left(\mu,\boldsymbol{\alpha}\right)/G_{(\mu,\boldsymbol{\alpha})}$ is a symplectic smooth manifold. Furthermore, since $\Omega_1=\Omega_2$ we have that k_1 and k_2 are of the same sign and hence $k_1+k_2\neq 0$. Thus, we have that $\dim(M_{(\mu,\boldsymbol{\alpha})})=4$.

Moreover, we have that:

1) K acts symplectically on M by lemma 2.1.3.

2) by lemma 2.2.6, the momentum map J_K is equivariant with respect to the coadjoint action $Ad^*_{\lambda^{-1}} : K \times \mathcal{K}^* \to \mathcal{K}^*$ given by (2.2.12).

3) for all non–zero $\xi \in \mathcal{K}^*$, ξ is a regular value of J_K.

4) by lemma 2.2.7 the actions of G and K on M commute.

5) by lemma 2.2.8 the momentum map J_G is invariant under the action of K on M.

Statements i) – iv) and 1) – 5) above, together with theorem 1.2.31, imply that

a) the momentum map J_K is invariant under the action of G on M.

b) the momentum map $J_G \times J_K$ is equivariant with respect to the coadjoint action $\widetilde{Ad}^*_{(R_\theta,\boldsymbol{v})^{-1}} \times Ad^*_{\lambda^{-1}}$ of $G \times K$ on the dual of its Lie Algebra.

c) K induces a symplectic action on $M_{(\mu,\boldsymbol{\alpha})}$ with an equivariant momentum map $J_{(\mu,\boldsymbol{\alpha})} : M_{(\mu,\boldsymbol{\alpha})} \to \mathcal{K}^*$ induced by the momentum map J_K. Hence, for all $\xi \neq 0$, we have that

$$M_{(\mu,\boldsymbol{\alpha},\xi)} = J^{-1}_{(\mu,\boldsymbol{\alpha})}(\xi)/K_\xi \ .$$

is a symplectic manifold with dimension

$$\begin{aligned} \dim\left(M_{(\mu,\boldsymbol{\alpha},\xi)}\right) &= \dim\left(J^{-1}_{(\mu,\boldsymbol{\alpha})}(\xi)\right) - \dim(K_\xi) \\ &= \dim\left(M_{(\mu,\boldsymbol{\alpha})}\right) - \dim J_K^{-1}(\xi) - \dim(K) \\ &= 2 \ , \end{aligned}$$

where K_ξ is the isotropy group of the coadjoint action $Ad^*_{\lambda^{-1}}$ of K on $\mathcal{K}^*$ and $\dim(K_\xi) = \dim(K)$.

d) the symplectic reduced space for the action of K on $M_{(\mu,\boldsymbol{\alpha})}$ at $\xi \neq 0$ is symplectically diffeomorphic to the reduction of M at the point $(\mu, \boldsymbol{\alpha}, \xi)$ by the action of $G \times K$.

Thus, we have proved the following result.

Proposition 2.2.9. *Let* $\Omega_1 = \Omega_2$ *and assume that the dynamics of the Hamiltonian system* (2.1.9) *do not lie on the invariant subset* $W = 0$. *Then, the Hamiltonian system* (2.1.9) *is integrable.*

In the next section we will construct an appropriate change of coordinates, to which we will call reconstruction map, to obtain the reduced Hamiltonian system in local coordinates, for each case studied in this section and in the previous one. Furthermore, we will see that proposition 2.2.9 holds even if the dynamics of (2.1.9) lie on the invariant subset $W = 0$.

2.3 Reduction: cyclic variables

In this section we provide local coordinates that exhibit the reduction of the Hamiltonian system (2.1.9) to two degrees of freedom. Moreover, we identify regimes of parameters and invariant subsets of $\mathbb{R}^8$ where the system can be proved to be integrable. To simplify notation we define the combinations

$$\begin{aligned} M &= m_1 + m_2 \\ m &= \frac{m_1 m_2}{m_1 + m_2} \,. \end{aligned}$$

We separate our analysis into two cases: $k_1 + k_2 \neq 0$ and $k_1 + k_2 = 0$.

2.3.1 Case $k_1 + k_2 \neq 0$

We start by noting that since $k_1 + k_2 \neq 0$ the following combinations are well–defined:

$$\begin{aligned} \mu &= k_1 + k_2 \\ e &= \frac{k_1 k_2}{k_1 + k_2} \,. \end{aligned}$$

We make a change of coordinates given by

$$\begin{aligned}
\boldsymbol{q} &= \boldsymbol{q}_1 - \boldsymbol{q}_2 \\
\boldsymbol{p} &= e\left(\frac{1}{k_1}\boldsymbol{p}_1 - \frac{1}{k_2}\boldsymbol{p}_2 + \frac{1}{2}\mathbf{J}\left(\boldsymbol{q}_1 - \boldsymbol{q}_2\right)\right) \\
\boldsymbol{f} &= \boldsymbol{p}_1 + \boldsymbol{p}_2 \\
\boldsymbol{P} &= \boldsymbol{p}_1 + \boldsymbol{p}_2 + \mathbf{J}\left(k_1\boldsymbol{q}_1 + k_2\boldsymbol{q}_2\right) \ ,
\end{aligned} \tag{2.3.1}$$

where $\boldsymbol{q} = (q_x, q_y) \in \mathbb{R}^2$ is the relative position of the two particles, $\boldsymbol{p} = (p_x, p_y) \in \mathbb{R}^2$ a conjugate momentum, $\boldsymbol{f} = (f_x, f_y) \in \mathbb{R}^2$ and $\boldsymbol{P} = (P_x, P_y) \in \mathbb{R}^2$. Inverting (2.3.1) we obtain

$$\begin{aligned}
\boldsymbol{q}_1 &= \frac{1}{\mu}\mathbf{J}\left(\boldsymbol{f} - \boldsymbol{P}\right) + \frac{k_2}{\mu}\boldsymbol{q} \\
\boldsymbol{p}_1 &= \frac{k_1}{\mu}\boldsymbol{f} - \frac{e}{2}\mathbf{J}\boldsymbol{q} + \boldsymbol{p} \\
\boldsymbol{q}_2 &= \frac{1}{\mu}\mathbf{J}\left(\boldsymbol{f} - \boldsymbol{P}\right) - \frac{k_1}{\mu}\boldsymbol{q} \\
\boldsymbol{p}_2 &= \frac{k_2}{\mu}\boldsymbol{f} + \frac{e}{2}\mathbf{J}\boldsymbol{q} - \boldsymbol{p} \ .
\end{aligned} \tag{2.3.2}$$

Combining (2.3.2) with (2.1.9), we obtain

$$\begin{aligned}
H &= \frac{1}{2m}\left|\boldsymbol{p}\right|^2 + \frac{e^2}{8m}\left|\boldsymbol{q}\right|^2 + \frac{e}{2m}\boldsymbol{q}.\mathbf{J}\boldsymbol{p} + \frac{k_1\Omega_1 + k_2\Omega_2}{2\mu^2}\left|\boldsymbol{f}\right|^2 \\
&\quad + \epsilon\left(2\boldsymbol{p} - e\mathbf{J}\boldsymbol{q}\right).\boldsymbol{f} + V\left(\left|\boldsymbol{q}\right|\right) \\
\omega &= \mathrm{d}q_x \wedge \mathrm{d}p_x + \mathrm{d}q_y \wedge \mathrm{d}p_y + \frac{1}{\mu}\left(\mathrm{d}P_x \wedge \mathrm{d}P_y - \mathrm{d}f_x \wedge \mathrm{d}f_y\right) \ ,
\end{aligned} \tag{2.3.3}$$

where

$$\epsilon = \frac{\Omega_1 - \Omega_2}{2\mu} \tag{2.3.4}$$

measures the displacement from the set of parameters satisfying $\Omega_1 = \Omega_2$. The quantities L and W are now given by

$$\begin{aligned}
L &= \boldsymbol{q}.\mathbf{J}\boldsymbol{p} + \frac{1}{2\mu}\left(\left|\boldsymbol{f}\right|^2 - \left|\boldsymbol{P}\right|^2\right) \\
W &= \left|\boldsymbol{f}\right|^2 \ .
\end{aligned}$$

Since $\boldsymbol{P}$ is conserved we remove the $-|\boldsymbol{P}|^2/(2\mu)$ term from the angular momentum, corresponding to a change in the level set of the angular momentum, defining the following conserved quantity

$$p_\theta = \boldsymbol{q}.\mathbf{J}\boldsymbol{p} + \frac{1}{2\mu}|\boldsymbol{f}|^2 \ .$$

A final change of coordinates makes the system canonical and exhibits the reduction to two degrees of freedom. It is given by writing

$$\begin{array}{rclrcl} \boldsymbol{q} & = & r\boldsymbol{e}_r & \boldsymbol{p} & = & p_r\boldsymbol{e}_r + \frac{2\mu p_\theta - p_\phi}{2\mu r}\boldsymbol{e}_\theta \\ \boldsymbol{f} & = & p_\phi^{1/2}\boldsymbol{e}_{2\mu\phi+\theta} & & & \\ P_x & = & \mu\Pi_x & P_y & = & \Pi_y \ , \end{array} \tag{2.3.5}$$

where θ is the direction of $\boldsymbol{q}$, i.e.

$$\begin{array}{rcl} \boldsymbol{e_r} & = & \cos\theta\boldsymbol{e}_x + \sin\theta\boldsymbol{e}_y \\ \boldsymbol{e}_\theta & = & -\sin\theta\boldsymbol{e}_x + \cos\theta\boldsymbol{e}_y \ , \end{array} \tag{2.3.6}$$

with $\boldsymbol{e}_x = (1,0) \in \mathbb{R}^2$ and $\boldsymbol{e}_y = (0,1) \in \mathbb{R}^2$. The vector $\boldsymbol{e}_{2\mu\phi+\theta}$ is defined in the same way as $\boldsymbol{e}_\theta$ with θ replaced by $2\mu\phi+\theta$. The coordinate change given in (2.3.5) is singular at $p_\phi = 0$ since ϕ is undefined in this case. There exists another coordinate singularity at $r = 0$, which is not included in the phase space in the case of interaction of Coulomb type.

We obtain the following result.

Theorem 2.3.1. *Let $k_1 + k_2 \neq 0$. Then, under the change of coordinates given by*

$$\begin{array}{rcl} \boldsymbol{q}_1 & = & -\dfrac{\mathbf{J}\boldsymbol{P}}{\mu} + \dfrac{1}{\mu}\left(k_2 r\boldsymbol{e}_r + p_\phi^{1/2}\mathbf{J}\boldsymbol{e}_{2\mu\phi+\theta}\right) \\ \boldsymbol{q}_2 & = & -\dfrac{\mathbf{J}\boldsymbol{P}}{\mu} - \dfrac{1}{\mu}\left(k_1 r\boldsymbol{e}_r - p_\phi^{1/2}\mathbf{J}\boldsymbol{e}_{2\mu\phi+\theta}\right) \\ \boldsymbol{p}_1 & = & \dfrac{k_1}{\mu}p_\phi^{1/2}\boldsymbol{e}_{2\mu\phi+\theta} + \left(p_r\boldsymbol{e}_r + \left(\dfrac{e}{2}r + \dfrac{2\mu p_\theta - p_\phi}{2\mu r}\right)\boldsymbol{e}_\theta\right) \\ \boldsymbol{p}_2 & = & \dfrac{k_2}{\mu}p_\phi^{1/2}\boldsymbol{e}_{2\mu\phi+\theta} - \left(p_r\boldsymbol{e}_r + \left(\dfrac{e}{2}r + \dfrac{2\mu p_\theta - p_\phi}{2\mu r}\right)\boldsymbol{e}_\theta\right) \ , \end{array} \tag{2.3.7}$$

where

$$\begin{aligned} p_\theta &= L + \frac{1}{2\mu}P \\ (\mu\Pi_x, \Pi_y) &= \boldsymbol{P} \\ p_\phi &= W \ , \end{aligned}$$

the Hamiltonian system (2.1.9) reduces to one with two degrees of freedom in the variables (r, p_r, ϕ, p_ϕ)*, given by*

$$\begin{aligned} H &= H_0(r, p_r, p_\theta, p_\phi) + \epsilon H_1 \left(r, p_r, p_\theta, \phi, p_\phi\right) \\ \omega &= \mathrm{d}r \wedge \mathrm{d}p_r + \mathrm{d}\phi \wedge \mathrm{d}p_\phi + \mathrm{d}\theta \wedge \mathrm{d}p_\theta + \mathrm{d}\Pi_x \wedge \mathrm{d}\Pi_y \ , \end{aligned} \tag{2.3.8}$$

where $H_0(r, p_r, p_\theta, p_\phi)$ *is given by*

$$H_0 = \frac{1}{2m}{p_r}^2 + \frac{1}{2m}\left(\frac{2\mu p_\theta - p_\phi}{2\mu r}\right)^2 + \frac{e^2}{8m}r^2 + \frac{e}{2m}\left(p_\theta + \frac{p_\phi}{2\mu}\right) + V\left(r\right)$$

and $H_1\left(r, p_r, p_\theta, \phi, p_\phi\right)$ *is given by*

$$H_1 = {p_\phi}^{1/2}\left(\left(er + \frac{2\mu p_\theta - p_\phi}{\mu r}\right)\cos\left(2\mu\phi\right) - 2p_r \sin\left(2\mu\phi\right)\right) + \frac{k_1 - k_2}{\mu}p_\phi \ .$$

The reduced phase space for the Hamiltonian system (2.3.8) is the symplectic blow up of $\mathbb{C}^2$ (see [17] for more details).

If the gyrofrequencies of the two particles are equal, i.e. $\Omega_1 = \Omega_2$, we have that $\epsilon = 0$. Applying theorem 2.3.1 we see that ϕ is a cyclic variable and so we obtain the following result.

Corollary 2.3.2. *If* $\Omega_1 = \Omega_2$*, using the change of coordinates (2.3.7) given in theorem 2.3.1 the Hamiltonian system (2.1.9) reduces to one with one degree of freedom in the variables* (r, p_r)*, given by*

$$\begin{aligned} H &= H_0(r, p_r, p_\theta, p_\phi) \\ \omega &= \mathrm{d}r \wedge \mathrm{d}p_r + \mathrm{d}\phi \wedge \mathrm{d}p_\phi + \mathrm{d}\theta \wedge \mathrm{d}p_\theta + \mathrm{d}\Pi_x \wedge \mathrm{d}\Pi_y \ , \end{aligned}$$

where H_0 *is as given in theorem 2.3.1.*

2.3.2 Case $k_1 + k_2 = 0$

We now treat the case where the charges sum to zero. To simplify notation, we define

$$\kappa = k_1 = -k_2 \ .$$

We make the change of coordinates given by

$$\begin{aligned} \boldsymbol{q} &= \boldsymbol{q}_1 - \boldsymbol{q}_2 \\ \boldsymbol{p} &= \frac{1}{2}\left(\boldsymbol{p}_1 - \boldsymbol{p}_2\right) \\ \boldsymbol{C} &= -\frac{1}{2}\mathbf{J}\left(\boldsymbol{q}_1 + \boldsymbol{q}_2\right) \\ \boldsymbol{\Pi} &= \kappa\left(\boldsymbol{q}_1 - \boldsymbol{q}_2\right) - \mathbf{J}\left(\boldsymbol{p}_1 + \boldsymbol{p}_2\right) \ , \end{aligned} \tag{2.3.9}$$

where $\boldsymbol{q} = (q_x, q_y) \in \mathbb{R}^2$ is the relative position of the two particles, $\boldsymbol{p} = (p_x, p_y) \in \mathbb{R}^2$ a conjugate momentum, $\boldsymbol{C} = (C_x, C_y) \in \mathbb{R}^2$ and $\boldsymbol{\Pi} = (\Pi_x, \Pi_y) \in \mathbb{R}^2$. Inverting (2.3.9) we obtain

$$\begin{aligned} \boldsymbol{q}_1 &= \mathbf{J}\boldsymbol{C} + \frac{1}{2}\boldsymbol{q} \\ \boldsymbol{q}_2 &= \mathbf{J}\boldsymbol{C} - \frac{1}{2}\boldsymbol{q} \\ \boldsymbol{p}_1 &= \frac{1}{2}\mathbf{J}\boldsymbol{\Pi} - \frac{\kappa}{2}\mathbf{J}\boldsymbol{q} + \boldsymbol{p} \\ \boldsymbol{p}_2 &= \frac{1}{2}\mathbf{J}\boldsymbol{\Pi} - \frac{\kappa}{2}\mathbf{J}\boldsymbol{q} - \boldsymbol{p} \ . \end{aligned} \tag{2.3.10}$$

From (2.3.10) and (2.1.9), we get the Hamiltonian system

$$\begin{aligned} H &= \frac{1}{2m}\left|\boldsymbol{p}\right|^2 + \frac{\kappa^2}{8m}\left|\boldsymbol{q}\right|^2 + \frac{(m_2 - m_1)\,\kappa}{2m_1 m_2}\boldsymbol{q}.\mathbf{J}\boldsymbol{p} + V\left(\left|\boldsymbol{q}\right|\right) \\ &\quad - \left(\frac{\kappa}{4m}\boldsymbol{q} + \frac{m_2 - m_1}{2m_1 m_2}\mathbf{J}\boldsymbol{p}\right).\boldsymbol{\Pi} + \frac{1}{8m}\left|\boldsymbol{\Pi}\right|^2 \\ \omega &= \mathrm{d}q_x \wedge \mathrm{d}p_x + \mathrm{d}q_y \wedge \mathrm{d}p_y + \mathrm{d}C_x \wedge \mathrm{d}\Pi_x + \mathrm{d}C_y \wedge \mathrm{d}\Pi_y \ , \end{aligned} \tag{2.3.11}$$

with the conserved quantities

$$\begin{aligned} \boldsymbol{P} &= \mathbf{J}\boldsymbol{\Pi} \\ L &= \boldsymbol{q}.\mathbf{J}\boldsymbol{p} + \boldsymbol{C}.\mathbf{J}\boldsymbol{\Pi} \ . \end{aligned} \tag{2.3.12}$$

The Hamiltonian system (2.3.11) is already reduced to two degrees of freedom by the conservation of $\boldsymbol{\Pi}$ and the elimination of $\boldsymbol{C}$. Unless $\boldsymbol{\Pi} = \mathbf{0}$ (or equivalently $\boldsymbol{P} = \mathbf{0}$), it is not possible to use the angular momentum L to further reduce (2.3.11) since L depends on the cyclic variables $\boldsymbol{C}$ and hence it is not a function defined on the reduced space. We make a final change of coordinates, given by

$$\begin{aligned} \boldsymbol{q} &= r\boldsymbol{e}_r \\ \boldsymbol{p} &= p_r\boldsymbol{e}_r + \frac{p_\theta}{r}\boldsymbol{e}_\theta \ , \end{aligned} \tag{2.3.13}$$

where $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are as given in (2.3.6). We obtain the following result.

Theorem 2.3.3. *Let $k_1 + k_2 = 0$. Then, under the change of coordinates given by*

$$\begin{aligned} \boldsymbol{q}_1 &= \mathbf{J}\boldsymbol{C} + \frac{1}{2}r\boldsymbol{e}_r \\ \boldsymbol{q}_2 &= \mathbf{J}\boldsymbol{C} - \frac{1}{2}r\boldsymbol{e}_r \\ \boldsymbol{p}_1 &= \frac{1}{2}\boldsymbol{P} + p_r\boldsymbol{e}_r + \left(\frac{p_\theta}{r} + \frac{\kappa r}{2}\right)\boldsymbol{e}_\theta \\ \boldsymbol{p}_2 &= \frac{1}{2}\boldsymbol{P} - p_r\boldsymbol{e}_r - \left(\frac{p_\theta}{r} - \frac{\kappa r}{2}\right)\boldsymbol{e}_\theta \ , \end{aligned} \tag{2.3.14}$$

where

$$\begin{aligned} p_\theta &= L - \boldsymbol{C}.\mathbf{J}\boldsymbol{\Pi} \\ \boldsymbol{\Pi} &= -\mathbf{J}\boldsymbol{P} \ , \end{aligned}$$

the Hamiltonian system (2.1.9) reduces to one with two degrees of freedom in the variables $(r, p_r, \theta, p_\theta)$, given by

$$\begin{aligned} H &= H_0(r, p_r, p_\theta) + H_1\left(r, p_r, \theta, p_\theta, \Pi_x, \Pi_y\right) \\ \omega &= \mathrm{d}r \wedge \mathrm{d}p_r + \mathrm{d}\theta \wedge \mathrm{d}p_\theta + \mathrm{d}C_x \wedge \mathrm{d}\Pi_x + \mathrm{d}C_y \wedge \mathrm{d}\Pi_y \ , \end{aligned} \tag{2.3.15}$$

where $H_0(r, p_r, p_\theta)$ is given by

$$H_0 = \frac{1}{2m}{p_r}^2 + \frac{1}{2m}\left(\frac{p_\theta}{r}\right)^2 + \frac{\kappa^2}{8m}r^2 + \frac{(m_2 - m_1)\,\kappa}{2m_1 m_2}p_\theta + V\left(r\right)$$

and $H_1(r, p_r, \theta, p_\theta, \Pi_x, \Pi_y)$ *is given by*

$$H_1 = -\left(\left(\frac{\kappa}{4m}r + \frac{m_2 - m_1}{2m_1 m_2}\frac{p_\theta}{r}\right)\boldsymbol{e}_r - \frac{m_2 - m_1}{2m_1 m_2}p_r \boldsymbol{e}_\theta\right).\boldsymbol{\Pi} + \frac{1}{8m}|\boldsymbol{\Pi}|^2 \ .$$

If $\boldsymbol{P} = \mathbf{0}$ then $\boldsymbol{\Pi} = \mathbf{0}$ and hence H_1, as given in the statement of theorem 2.3.3, is identically zero. From theorem 2.3.3, we obtain the following result.

Corollary 2.3.4. *If $k_1 + k_2 = 0$ and $\boldsymbol{P} = \mathbf{0}$ then, using the change of coordinates (2.3.14) given in theorem 2.3.3 the Hamiltonian system (2.1.9) reduces to one with one degree of freedom in the variables (r, p_r), given by*

$$\begin{aligned} H &= H_0(r, p_r, p_\theta) \\ \omega &= \mathrm{d}r \wedge \mathrm{d}p_r + \mathrm{d}\theta \wedge \mathrm{d}p_\theta + \mathrm{d}C_x \wedge \mathrm{d}\Pi_x + \mathrm{d}C_y \wedge \mathrm{d}\Pi_y \ , \end{aligned}$$

where H_0 is as given in theorem 2.3.3.

2.4 Reconstructed dynamics for a Coulomb potential

In this section we use the reduced Hamiltonian systems and the corresponding reconstruction maps obtained in section 2.3 to provide a qualitative description of the possible types of dynamics in the full eight–dimensional phase space in terms of the properties of the dynamics of the reduced systems. Throughout this section we consider the interaction potential to be Coulomb

$$V(r) = \frac{e_1 e_2}{4\pi\epsilon_0}\frac{1}{r} \ ,$$

where r is the distance between the particles and ϵ_0 is the permittivity of the vacuum. We should remark, however, that the description given below still holds for a class of Coulomb–type potentials of the form

$$W(r) = \frac{e_1 e_2}{4\pi\epsilon_0}\frac{f(r)}{r} \ ,$$

where $f(r)$ is a positive bounded smooth function. A physically interesting particular case is the screened Coulomb potential where $f(r) = \mathrm{e}^{-r/r_D}$ and r_D is the Debye length.

The next two lemmas follow from an analysis of the form of the Hamiltonian functions given in theorems 2.3.1 and 2.3.3, respectively. We skip their proof.

Lemma 2.4.1. *Let $k_1 + k_2 \neq 0$ and consider the reduced Hamiltonian system given in theorem 2.3.1. For every level set of the Hamiltonian function the dynamics of r and p_ϕ are bounded for all time.*

Lemma 2.4.2. *Let $k_1 + k_2 = 0$ and consider the reduced Hamiltonian system given in theorem 2.3.3. For every level set of the Hamiltonian function the dynamics of r and p_θ are bounded for all time.*

In the next lemma we provide a complete description for the orbits in the two–dimensional phase space corresponding to the integrable reduced Hamiltonian systems given in corollaries 2.3.2 and 2.3.4.

Lemma 2.4.3.

- *Let $\Omega_1 = \Omega_2$. The reduced Hamiltonian system given in corollary 2.3.2 has a unique equilibrium. The equilibrium is elliptic and the rest of the reduced phase space is filled by periodic orbits.*
- *Let $k_1 + k_2 = 0$, $\boldsymbol{P} = \mathbf{0}$ and $p_\theta \neq 0$. The reduced Hamiltonian system given in corollary 2.3.4 has a unique equilibrium. The equilibrium is elliptic and the rest of the reduced phase space is filled by periodic orbits.*
- *Let $k_1 + k_2 = 0$, $\boldsymbol{P} = \mathbf{0}$ and $p_\theta = 0$. The phase space of reduced Hamiltonian system given in corollary 2.3.4 is filled by orbits doubly asymptotic to a collision.*

Proof. From corollaries 2.3.2 and 2.3.4 we have that in the integrable regimes the Hamiltonian system (2.1.9) reduces to one of one degree of freedom of the form

$$\begin{aligned} H &= A{p_r}^2 + \left(\frac{B}{r}\right)^2 + Cr^2 + \frac{D}{r} \\ \omega &= \mathrm{d}r \wedge \mathrm{d}p_r \ , \end{aligned} \tag{2.4.1}$$

where A and C are positive and D is non–zero. Let $U(r)$ denote the effective potential

$$U(r) = Cr^2 + \left(\frac{B}{r}\right)^2 + \frac{D}{r} \ .$$

Differentiating with respect to r we obtain

$$\begin{aligned} U'(r) &= 2Cr - \frac{2B^2}{r^3} - \frac{D}{r^2} \\ U''(r) &= 2C + \frac{6B^2}{r^4} + \frac{2D}{r^3} \ . \end{aligned} \tag{2.4.2}$$

We separate our analysis into three cases.

1) Assume that $D > 0$. Using (2.4.2) we obtain that $U''(r) > 0$ for every $r > 0$ and hence $U'(r)$ is strictly increasing in that range. Since we also have that $\lim_{r\to 0^+} U'(r) = -\infty$ and $\lim_{r\to+\infty} U'(r) = +\infty$ we obtain that $U'(r)$ has a unique zero on $(0, +\infty)$ corresponding to an elliptic equilibrium of (2.4.1). Apart from the equilibrium, all the level sets of H are regular and closed, so the orbits of (2.4.1) are periodic.

2) Assume that $D < 0$ and $B \neq 0$ and rewrite $U'(r)$ as

$$U'(r) = \frac{1}{r^3} g(r) \ ,$$

where $g(r) = 2Cr^4 - 2B^2 - Dr$. Noting that $g(r)$ is strictly increasing in $[0, +\infty)$, $g(0) < 0$ and $\lim_{r\to+\infty} g(r) = +\infty$ we obtain that $U'(r)$ is also increasing and since $\lim_{r\to 0^+} U'(r) = -\infty$ and $\lim_{r\to+\infty} U'(r) = +\infty$ we obtain that the unique zero of $U'(r)$ on $(0, +\infty)$ corresponds to an elliptic equilibrium of (2.4.1). Apart from the equilibrium, all the level sets of H are regular and closed, so the orbits of (2.4.1) are periodic.

3) Assume that $D < 0$ and $B = 0$. From (2.4.2) we obtain that $U'(r) > 0$ for every $r > 0$ and hence $U(r)$ is strictly increasing in that range. Furthermore, we have that $\lim_{r\to 0^+} U(r) = -\infty$ and $\lim_{r\to+\infty} U(r) = +\infty$. In this case (2.4.1) does not have any equilibria and all the orbits in the reduced phase space are doubly asymptotic to a collision.

□

By conservation of the linear momenta $\boldsymbol{P}$ and putting together lemma 2.4.1 and the reconstruction map (2.3.7) given in the statement of theorem 2.3.1, we obtain the following result.

Corollary 2.4.4. *Let $k_1 + k_2 \neq 0$. Then the positions of the two particles are bounded for all time.*

For any function v of time we define the average value by

$$\langle v \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T v(t)\mathrm{d}t$$

if the limit exists. By Birkhoff's ergodic theorem, if v is the value of a continuous function on the state space evaluated along an orbit of a volume–preserving system of finite volume, the limit exists for the orbit of almost every point.

Corollary 2.4.5. *Let $k_1 + k_2 = 0$. We have that:*

i) the relative position of the two particles $\boldsymbol{q}_1 - \boldsymbol{q}_2$ is bounded.

ii) for small non–zero values of the conserved quantity $\boldsymbol{P}$ the motion of $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ is typically unbounded: they drift with a non–zero average velocity.

Proof. Item i) follows from lemma 2.4.2 and the reconstruction map (2.3.14) given in the statement of theorem 2.3.3.

To prove item ii) we use (2.3.15) to compute

$$\dot{\boldsymbol{C}} = \left(\frac{\partial H}{\partial \Pi_x}, \frac{\partial H}{\partial \Pi_y} \right) = \frac{1}{4m}\boldsymbol{\Pi} - R_{\theta(t)}\ \boldsymbol{v}\left(r(t), p_r(t), p_\theta(t)\right)\ , \tag{2.4.3}$$

where the evolution of θ is determined by

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{1}{m}\frac{p_\theta}{r^2} + \frac{m_2 - m_1}{2m_1 m_2}\left(\kappa - \frac{\boldsymbol{e}_r.\boldsymbol{\Pi}}{r} \right)\ , \tag{2.4.4}$$

$R_{\theta(t)}$ denotes the rotation by the angle $\theta(t)$ and $\boldsymbol{v}\left(r, p_r, p_\theta\right)$ is the vector in $\mathbb{R}^2$ given by

$$\boldsymbol{v}\left(r, p_r, p_\theta\right) = \left(\frac{\kappa}{4m} r + \frac{m_2 - m_1}{2m_1 m_2}\frac{p_\theta}{r}, -\frac{m_2 - m_1}{2m_1 m_2} p_r \right)\ .$$

From (2.4.3) we obtain that

$$\langle \dot{\boldsymbol{C}} \rangle = \frac{1}{4m}\boldsymbol{\Pi} - \lim_{T\to\infty}\frac{1}{T}\int_0^T R_{\theta(t)}\ \boldsymbol{v}\left(r(t), p_r(t), p_\theta(t)\right)\mathrm{d}t \ , \tag{2.4.5}$$

and hence we obtain that for those orbits with $\langle \dot{\boldsymbol{C}} \rangle \neq \mathbf{0}$, $\boldsymbol{C}(t)$ grows like $\langle \dot{\boldsymbol{C}} \rangle t$.

Evaluating the integral of $\dot{\boldsymbol{C}}$ over the level sets of the energy and conserved quantities, with respect to the invariant measure induced from Liouville measure, we obtain that it is zero when we restrict to level sets with $\boldsymbol{P} = \mathbf{0}$. Differentiating that integral with respect to Π_x and Π_y we obtain that at least one of these partial derivatives is non–zero when evaluated at $\boldsymbol{P} = \mathbf{0}$. It follows that in a small neighbourhood of the level sets $\boldsymbol{P} = \mathbf{0}$ the integral of $\dot{\boldsymbol{C}}$ over the level sets of the energy and conserved quantities is non–zero. Hence, the drift velocity $\langle \dot{\boldsymbol{C}} \rangle$ must be non–zero for a subset of positive measure.

From lemma 2.4.2 and the reconstruction map (2.3.14) we would get that the motion is unbounded with average velocity $\mathbf{J}\langle \dot{\boldsymbol{C}} \rangle$ □

Remarks We note that:

i) if $k_1 + k_2 = 0$, $\boldsymbol{P} = \mathbf{0}$ and $p_\theta \neq 0$, then $\boldsymbol{\Pi}/(4m) = \mathbf{0}$, the reduced motion is periodic with period T depending on the values of p_θ and H and the equation for the evolution of $\dot{\theta}$ does not contain θ, so the second term in (2.4.5) vanishes if

$$\alpha = \int_0^T \frac{1}{m}\frac{p_\theta}{r^2(t)} + \frac{(m_2 - m_1)\,\kappa}{2m_1 m_2}\mathrm{d}t \notin 2\pi\mathbb{Z} \ .$$

Now α is an analytic function of the value h of H (above its minimum) and is not identically $2\pi N$ for any $N \in \mathbb{Z}$ (as h tends to infinity, for every non–zero p_θ the period T tends to $4\pi m/|\kappa|$ and α approaches the value $\mathrm{sign}(\kappa)2\pi(m_2 - m_1)/M$ in a non–constant way), so there are at most isolated values of h (given p_θ) for which $\alpha \in 2\pi\mathbb{Z}$. If these orbits are avoided then $\langle \dot{\boldsymbol{C}} \rangle = \mathbf{0}$ and the positions of the two particles are bounded for all time.

ii) We believe that for large $\boldsymbol{P} \neq \mathbf{0}$, item ii) of corollary 2.4.5 still holds, i.e. the motion of $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ is typically unbounded. In fact, the third term of (2.4.4) induces a preference for θ to be in the direction of $\boldsymbol{\Pi}$, so it would be an unlikely accident for the second term of (2.4.5) to exactly cancel the first..

One of the key steps for the proof of the existence of chaotic orbits is proving the existence of an abundant number of trajectories connecting two close approaches between the particles. The next two lemmas describe the set of parameters and level sets of the conserved quantities where such trajectories might exist. We skip the proofs of these lemmas, which follow from an analysis of the Hamiltonian functions given in theorems 2.3.1 and 2.3.3.

Lemma 2.4.6. *Let $k_1 + k_2 \neq 0$. Then*

- *if $k_1 k_2 > 0$, or $k_1 k_2 < 0$ and the value of the conserved quantity p_θ is fixed so that $\mu p_\theta < 0$, the distance between the two particles is bounded away from zero, i.e. there exists $d > 0$ such that $r(t) > d$ for all $t \in \mathbb{R}$.*

- *if $k_1 k_2 < 0$ and the value of the conserved quantity p_θ is fixed so that μp_θ is positive, the distance r between the two particles can be arbitrarily close to 0. Furthermore, $p_r \to \infty$ and $p_\phi \to 2\mu p_\theta$ as $r \to 0$.*

Lemma 2.4.7. *Let $k_1 + k_2 = 0$. Then*

- *if $\boldsymbol{P} = \mathbf{0}$ and $p_\theta \neq 0$, the distance between the two particles is bounded away from zero, i.e. there exists $d > 0$ such that $r(t) > d$ for all $t \in \mathbb{R}$.*

- *if $\boldsymbol{P} = \mathbf{0}$ and $p_\theta = 0$, the distance r between the two particles can be arbitrarily close to 0. Furthermore, $p_r \to \infty$ as $r \to 0$.*

- *if $\boldsymbol{P} \neq \mathbf{0}$, the distance r between the two particles can be arbitrarily close to 0. Furthermore, $p_r \to \infty$ and $p_\theta \to 0$ as $r \to 0$.*

The dynamics of the Hamiltonian system (2.1.9) are completely characterized by the dynamics of the reduced Hamiltonian systems and their cyclic variables (θ and ϕ in the case $k_1 + k_2 \neq 0$, θ and $\boldsymbol{C}$ if $k_1 + k_2 = 0$) given in theorems 2.3.1 and 2.3.3 and the respective reconstruction maps. The full dynamics correspond to Euclidean extensions, given by the reconstruction maps, of the reduced dynamics. Extensions of dynamical systems by Lie groups have been extensively studied in [5, 13, 14, 29]. For the case of extensions of dynamical systems by the Special Euclidean group of the plane it was proven in [29] that:

i) extensions of Anosov base dynamics are generically unbounded;

ii) extensions of quasiperiodic base dynamics are typically bounded in a probabilistic sense, but there is a dense set of base rotations for which extensions are typically unbounded in a topological sense.

The boundedness of the trajectories of the Hamiltonian system (2.1.9) obtained in corollary 2.4.4 for the case $k_1 + k_2 \neq 0$ is due to the very special form of the $SE(2)$ extension provided by the reconstruction map (2.3.7) and the existence of conserved quantities for this system.

The reduced Hamiltonian systems exhibit a rich dynamical behaviour:

- In the integrable regimes the energy levels are foliated by periodic orbits.
- Close to the integrable regimes most of the periodic orbits cease to exist but almost all orbits in the energy levels are quasiperiodic and hence the dynamics still look regular.
- As we will prove in the next section, for opposite signs of charge (except for the case $\Omega_1 + \Omega_2 = 0$) there is chaotic dynamics which implies non–integrability for this system.

Using the reconstruction maps we obtain the following.

1) If $k_1+k_2 \neq 0$ periodic and quasiperiodic base dynamics lift to quasiperiodic dynamics under the reconstruction map (figures 2.4.1a–2.4.1c). In this case the dynamics are, generically, quasiperiodic with three rationally independent frequencies. The particles rotate with these three frequencies about a fixed centre determined by the linear momenta.

2) If $k_1 + k_2 = 0$ periodic and quasiperiodic base dynamics lift to possibly unbounded motion corresponding to a combination of a drift and quasiperiodic dynamics. The quasiperiodic dynamics have, generically, two rationally independent frequencies.

3) Chaotic dynamics lift to chaotic dynamics under the reconstruction maps. The motion is always bounded if $k_1 + k_2 \neq 0$ (figures 2.4.1d–2.4.1f) and typically unbounded otherwise (figures 2.4.1g–2.4.1i).

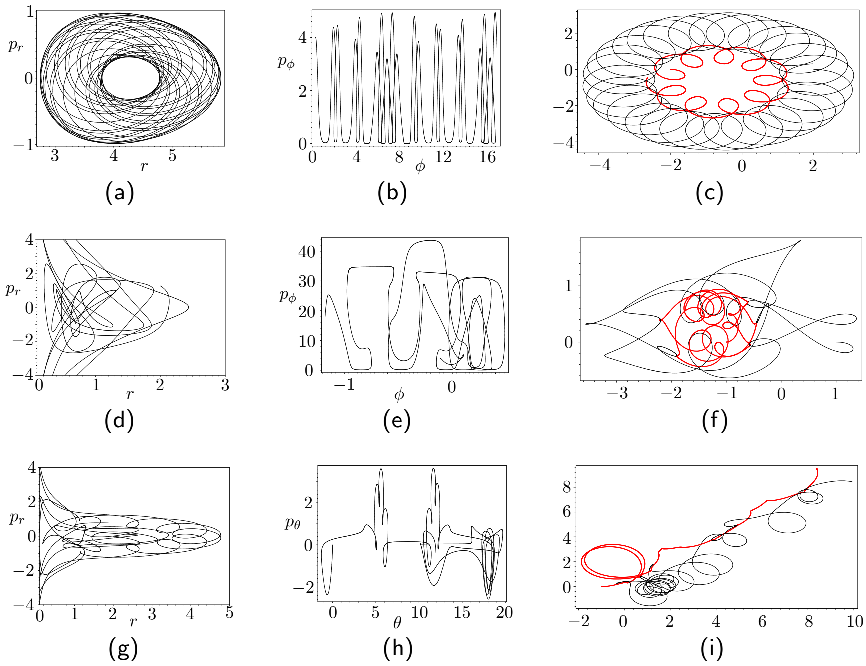


Figure 2.4.1: Three distinct dynamical behaviours. For all the figures we fix the parameters $e_1 = m_1 = 1$, $B = c = 1$ and $\epsilon_0 = 0.1$ and initial conditions $q_{y_1}(0) = q_{y_2}(0) = p_{y_1}(0) = p_{y_2}(0) = 0$ and $p_{x_1}(0) = p_{x_2}(0) = 1$. On the left and centre columns we have plots of projections of the reduced dynamics on the $r - p_r$ and $\phi - p_\phi$ planes respectively ($r - p_r$ and $\theta - p_\theta$ on the bottom line) and on the right column the respective reconstructed dynamics, i.e. trajectories of the two particles in $\mathbb{R}^2$, where the black trajectory corresponds to the first particle and the red trajectory corresponds to the second particle. On the top figures $e_2 = 2$, $m_2 = 6$ and $q_{x_1}(0) = -q_{x_2}(0) = 2$, on the centre figures $e_2 = -8$, $m_2 = \pi$ and $q_{x_1}(0) = -q_{x_2}(0) = 1$ and on the bottom figures $e_2 = -1$, $m_2 = 5$ and $q_{x_1}(0) = -q_{x_2}(0) = 1$.

2.5 Non–integrability of (2.1.9) with a Coulomb–type potential and opposite signs of charge

2.5.1 Motivation and main result of the section

In this section we will prove that the Hamiltonian system (2.1.9) is, for opposite signs of charge (except for the case $\Omega_1 + \Omega_2 = 0$), not integrable. We use a method developed in [8] to prove that there exist regimes of parameters and energy for which there is an invariant subset where the system contains a suspension of a subshift of finite type and has positive entropy. Roughly, this corresponds to the existence of a horseshoe in the dynamics and hence, from a result in [28], we obtain that, for the two degree of freedom Hamiltonian systems in theorems 2.3.1 and 2.3.3, there is no other analytic conserved quantity independent of the Hamiltonian function.

By lemma 2.4.3, the integrable case $\Omega_1 = \Omega_2$ described in corollary 2.3.2 does not have any saddle point in its reduced phase space, so there are no possibilities for a simple use of Melnikov method to obtain chaos for nearby $\Omega_1 \neq \Omega_2$.

The condition of opposite signs for the charges is needed to guarantee, by lemmas 2.4.6 and 2.4.7, arbitrarily close approaches on the level sets of the conserved quantities of (2.1.9). The construction of a large set of collision orbits will form an important part in the proof of existence of chaotic orbits.

Let $Q = \mathbb{R}^2$ and consider a two degrees of freedom canonical Hamiltonian system with phase space $M = T^*(Q\backslash\{\mathbf{0}\})$ and Hamiltonian function $H_\delta : M \to \mathbb{R}$, given by

$$H_\delta = H + \delta V(\boldsymbol{q}) \ , \qquad \delta \in \mathbb{R} \ . \tag{2.5.1}$$

We assume that H is C^4 on M and has the form

$$H = \frac{1}{2}|\boldsymbol{p}|^2 + W(\boldsymbol{q}, \boldsymbol{p}) \ , \tag{2.5.2}$$

where $W : M \to \mathbb{R}$ is a C^4 function of M, such that

$$W(\boldsymbol{q}, \boldsymbol{p}) = W_1(\boldsymbol{q}) + W_2(\boldsymbol{q}.\mathbf{J}\boldsymbol{p}) + (\boldsymbol{a}.\boldsymbol{q} + \boldsymbol{b}.\boldsymbol{p})\, W_3(\boldsymbol{q}.\mathbf{J}\boldsymbol{p}) \ , \tag{2.5.3}$$

where $W_1 : Q \to \mathbb{R}$ and $W_2, W_3 : \mathbb{R} \to \mathbb{R}$ are at least C^4 functions and $\boldsymbol{a}, \boldsymbol{b}$ are constant vectors in $\mathbb{R}^2$. Furthermore, we assume that $V : Q \backslash \{\mathbf{0}\} \to \mathbb{R}$ is of the form

$$V(\boldsymbol{q}) = -\frac{f(\boldsymbol{q})}{|\boldsymbol{q}|} \ ,$$

where $f : Q \to \mathbb{R}$ is a C^4 function with $f(\mathbf{0}) \neq 0$. We will study the Hamiltonian system (2.5.1), for small $\delta > 0$, as a perturbation of the canonical Hamiltonian system with Hamiltonian function (2.5.2). The main example we are concerned with is the interaction of two charges in a uniform magnetic field. In lemma 2.5.2 we will prove that the Hamiltonian system (2.1.9) can be reduced to a canonical Hamiltonian system of the form (2.5.1).

We consider energies E satisfying the following:

i) the domain $D = \{\boldsymbol{q} \in Q : W(\boldsymbol{q}, \mathbf{0}) < E\}$ contains $\mathbf{0}$,

ii) $E > W(\mathbf{0}, \mathbf{0}) - (|\boldsymbol{b}| W_3(0))^2 / 2$,

and study the system (2.5.1) on the energy level $\{H_\delta = E\} \subset M$. We say that a trajectory $\gamma : [0, \tau] \to D$ is a *collision trajectory* of the Hamiltonian system (2.5.2) if $\gamma(0) = \mathbf{0}$, $\gamma(\tau) = \mathbf{0}$ and $\gamma(t) \neq \mathbf{0}$ for every $t \in (0, \tau)$. Let $\boldsymbol{q}(\lambda, t)$, $\boldsymbol{p}(\lambda, t)$ represent the general solution of the Hamiltonian system (2.5.2), where λ is a parameter of dimension $2 \dim Q$. Then $H(\boldsymbol{q}(\lambda, t), \boldsymbol{p}(\lambda, t)) = h(\lambda)$ for some function h. Collision orbits with energy E correspond to solutions of the system of equations

$$\begin{aligned} \boldsymbol{q}(\lambda, 0) &= \mathbf{0} \\ \boldsymbol{q}(\lambda, \tau) &= \mathbf{0} \\ h(\lambda) &= E \ , \end{aligned} \tag{2.5.4}$$

in the variables λ, τ. A solution of (2.5.4) is *non–degenerate* if the rank of the derivative of (2.5.4) at the solution is maximal, i.e. equals $2 \dim Q + 1$. The definition of non–degeneracy given above is suitable for verification of non–degeneracy on concrete examples. For completeness we give below two equivalent formulations of non–degeneracy that will be useful later in this section. See [8] and references therein for more details.

i) To any solution $(\gamma(t), \xi(t))$ of Hamilton's equations associated with the canonical Hamiltonian system with Hamiltonian H, there corresponds the variational equation

$$\begin{aligned} \dot{\boldsymbol{v}} &= H_{\boldsymbol{qp}}(\gamma(t), \xi(t))\, \boldsymbol{v} + H_{\boldsymbol{pp}}(\gamma(t), \xi(t))\, \boldsymbol{u} \\ \dot{\boldsymbol{u}} &= -H_{\boldsymbol{qq}}(\gamma(t), \xi(t))\, \boldsymbol{v} - H_{\boldsymbol{qp}}(\gamma(t), \xi(t))\, \boldsymbol{u} \,. \end{aligned} \tag{2.5.5}$$

We say that the points $\gamma(\tau_1)$ and $\gamma(\tau_2)$ are *conjugate along* γ if there exists a non–identically zero solution $(\boldsymbol{v}(t), \boldsymbol{u}(t))$ of (2.5.5) such that $\boldsymbol{v}(\tau_1) = \boldsymbol{v}(\tau_2) = \mathbf{0}$.

We say that a collision orbit $\gamma : [0, \tau] \to D$ is non–degenerate if the points $\gamma(0)$ and $\gamma(\tau)$ are not conjugate along γ.

ii) Let Ω be the space of $W^{1,2}$ curves $u : [0, 1] \to Q$ such that $u(0) = u(1) = \mathbf{0}$. Any point $(u, \tau) \in \Omega \times \mathbb{R}^+$ defines a curve $\gamma : [0, \tau] \to Q$ by $\gamma(t) = u(t/\tau)$. Let $(\gamma(t), \xi(t))$ be the orbit in phase space corresponding to the trajectory $\gamma(t)$ and define its action as

$$F(u, \tau) = \int_0^\tau \xi(t).\frac{\partial H}{\partial \boldsymbol{p}}(\gamma(t), \xi(t))\,\mathrm{d}t \,. \tag{2.5.6}$$

Then F is a C^2 functional on $\Omega \times \mathbb{R}^+$ and its critical points correspond to trajectories of energy $E = H(\gamma(t), \xi(t))$ connecting two collisions. A collision orbit $\gamma : [0, \tau] \to D$ is non–degenerate if (u, τ), where $u(t) = \gamma(t\tau)$ is a non–degenerate critical point for F.

Let $\mathcal{K}$ be a finite set of non–degenerate collision trajectories of (2.5.2). Denote such trajectories by $\gamma_k : [0, \tau_k] \to D$, $k \in K$. A sequence $(\gamma_{k_i})_{i\in\mathbb{Z}}$ of non–degenerate collision trajectories is called a *chain* if $\dot{\gamma}_{k_i}(\tau_{k_i}) \neq \pm\dot{\gamma}_{k_{i+1}}(0)$ for all $i \in \mathbb{Z}$. Let W_k be a small neighbourhood of $\gamma_k([0, \tau_k])$. We say that a trajectory $\gamma : \mathbb{R} \to D\backslash\{\mathbf{0}\}$ *shadows* the chain $(\gamma_{k_i})_{i\in\mathbb{Z}}$ if there exists an increasing sequence $(t_i)_{i\in\mathbb{Z}}$ such that $\gamma([t_i, t_{i+1}]) \subset W_{k_i}$.

Theorem 2.5.1. *Given a finite set $\mathcal{K}$ of non–degenerate collision orbits, there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0]$ and any chain $(\gamma_{k_i})_{i\in\mathbb{Z}}$, $k_i \in K$, the following statements hold.*

- *There exists a trajectory $\gamma : \mathbb{R} \to D\backslash\{\mathbf{0}\}$ of energy E for the canonical Hamiltonian system determined by (2.5.1) shadowing the chain $(\gamma_{k_i})_{i\in\mathbb{Z}}$, and it is unique (up to a time shift) if the neighbourhoods W_{k_i} of $\gamma_{k_i}(0,\tau_{k_i})$ in D are chosen small enough.*

- *The orbit γ converges to the chain of collision orbits as $\delta \to 0$, i.e. there exists an increasing sequence $(t_i)_{i\in\mathbb{Z}}$ such that*

$$\max_{t_i\le t\le t_{i+1}} \operatorname{dist}(\gamma(t), \gamma_{k_i}([0,\tau_{k_i}])) \le C\delta \ ,$$

where the constant $C > 0$ depends only on the set $\mathcal{K}$ of collision orbits.

- *the orbit γ avoids collision by a distance of order δ, i.e. there exists a constant $c \in (0, C)$, depending only on $\mathcal{K}$ such that*

$$c\delta \le \min_{t_i\le t\le t_{i+1}} \operatorname{dist}(\gamma(t), \mathbf{0}) \ .$$

A more precise version of theorem 2.5.1 is given in theorem 2.5.12 of section 2.5.5. Theorem 2.5.1 implies that there is an invariant subset in $\{H_\delta = E\}$ on which the system contains a suspension of a subshift of finite type (see [20, 21]). The topological entropy is positive provided the graph with the set of vertices K and the set of edges

$$G = \left\{(k,l) \in K^2 : \dot\gamma_k(\tau_k) \ne \pm\dot\gamma_l(0)\right\} \tag{2.5.7}$$

has a connected branched subgraph. In this last case theorem 2.5.1 implies non–integrability, i.e. the Hamiltonian system determined by (2.5.1) does not have any more analytic first integrals apart from the Hamiltonian function (see [28]). Theorem 2.5.1 generalizes the main theorem in [8] where function W in (2.5.3) was allowed to depend on $\boldsymbol{p}$ through only a linear term in $\boldsymbol{p}$. This theorem still holds for Hamiltonian systems of the form (2.5.1) with n degrees of freedom and for potentials V with several Newtonian singularities and for kinetic energy given by a general Riemannian metric (see [8] for more details). For simplicity of exposition however, we choose not to deal with such a general system here. The proof of theorem 2.5.1 occupies sections 2.5.4–2.5.6

and follows the technique developed in [8] up to some minor modifications that are due to the chosen dependence of the function W on the momenta $\boldsymbol{p}$.

2.5.2 Application to the problem of the interaction of two charges in a uniform magnetic field

In this section we start by proving that the Hamiltonian system (2.1.9) with a Coulomb interaction potential can be reduced to one of the form (2.5.1). Then, we use the collision orbits constructed in section 2.5.3 and theorem 2.5.1 to prove the existence of chaotic orbits in the interaction of two charges in a uniform magnetic field.

Lemma 2.5.2. *The Hamiltonian system (2.1.9) can always be reduced to a two degrees of freedom canonical Hamiltonian system of the form (2.5.1).*

Proof. As in section 2.3 we separate the proof into two cases. If $k_1 + k_2 = 0$, apply the change of coordinates given by

$$\begin{aligned} \boldsymbol{q} &\mapsto \frac{1}{m^{1/2}}\boldsymbol{q} \\ \boldsymbol{p} &\mapsto m^{1/2}\boldsymbol{p} \end{aligned} \tag{2.5.8}$$

to the Hamiltonian system given by (2.3.11) to obtain one in the form (2.5.1) with

$$\begin{aligned} W_1(\boldsymbol{x}) &= \frac{\kappa^2}{8m^2}|\boldsymbol{x}|^2 + \frac{1}{8m}|\boldsymbol{\Pi}|^2 \\ W_2(\ell) &= \frac{(m_2 - m_1)\,\kappa}{2m_1 m_2}\ell \\ W_3(\ell) &= 1\ , \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{a} &= -\frac{\kappa}{4m^{3/2}}\boldsymbol{\Pi} \\ \boldsymbol{b} &= \frac{(m_2 - m_1)}{2m^{1/2}M}\mathbf{J}\boldsymbol{\Pi} \\ \delta &= \frac{|e_1 e_2| m^{1/2}}{4\pi\epsilon_0} \\ f(\boldsymbol{q}) &= 1\ . \end{aligned}$$

If $k_1 + k_2 \neq 0$, we consider the three degrees of freedom Hamiltonian system given by (2.3.3). This system has the following symmetry

$$\phi_\theta (\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{f}) = (R_\theta \boldsymbol{q}, R_\theta \boldsymbol{p}, R_\theta \boldsymbol{f}) \ , \tag{2.5.9}$$

with an associated conserved quantity $L = \boldsymbol{q}.\mathbf{J}\boldsymbol{p} + |\boldsymbol{f}|^2 /(2\mu)$. To quotient by the symmetry group (2.5.9), we use the equivalence relation between elements of the phase space (already reduced by translations), given by

$$(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{f}) \sim (\boldsymbol{q}', \boldsymbol{p}', \boldsymbol{f}')$$

if and only if there exist $\theta \in S^1$ such that $(\boldsymbol{q}', \boldsymbol{p}', \boldsymbol{f}') = (R_\theta \boldsymbol{q}, R_\theta \boldsymbol{p}, R_\theta \boldsymbol{f})$, and choose for representative elements of the equivalence classes of the above relation elements satisfying $f_y = 0$, $f_x \geq 0$, and use conservation of L to obtain

$$f_x = \sqrt{2\mu (L - \boldsymbol{q}.\mathbf{J}\boldsymbol{p})} \ .$$

Applying the above reduction to the Hamiltonian system (2.3.3), we get the following two degrees of freedom canonical Hamiltonian system in the variables $(\boldsymbol{q}, \boldsymbol{p})$:

$$\begin{aligned} H = & \frac{1}{2m} |\boldsymbol{p}|^2 + \frac{e^2}{8m} |\boldsymbol{q}|^2 + \Lambda L + \left(\frac{e}{2m} - \Lambda \right) \boldsymbol{q}.\mathbf{J}\boldsymbol{p} \\ & + \epsilon (2p_x - eq_y) \sqrt{2\mu (L - \boldsymbol{q}.\mathbf{J}\boldsymbol{p})} + V(|\boldsymbol{q}|) \ , \end{aligned} \tag{2.5.10}$$

where

$$\Lambda = \frac{k_1 \Omega_1 + k_2 \Omega_2}{\mu} \ .$$

Apply the change of coordinates (2.5.8) to the Hamiltonian system determined by (2.5.10) to obtain one of the form (2.5.1) with

$$\begin{aligned} W_1 (\boldsymbol{x}) &= \frac{e^2}{8m^2} |\boldsymbol{x}|^2 + \Lambda L \\ W_2 (\ell) &= \left(\frac{e}{2m} - \Lambda \right) \ell \\ W_3 (\ell) &= \epsilon \sqrt{2\mu (L - \ell)} \ , \end{aligned}$$

where

$$\begin{aligned}\boldsymbol{a} &= \Big(0, -\frac{e}{m^{1/2}}\Big)\\ \boldsymbol{b} &= \Big(2m^{1/2}, 0\Big)\\ \delta &= \frac{|e_1e_2|m^{1/2}}{4\pi\epsilon_0}\\ f(\boldsymbol{q}) &= 1\ .\end{aligned}$$

□

Introducing the rescaling given by

$$\boldsymbol{q}_i \mapsto \frac{1}{\lambda}\boldsymbol{q}_i\ , \qquad \boldsymbol{p}_i \mapsto \frac{1}{\lambda}\boldsymbol{p}_i\ ,$$

the Hamiltonian system (2.1.9) is transformed to the Hamiltonian system determined by $\overline{H}/\lambda^2$ and ω/λ^2, where $\overline{H}$ is given by

$$\overline{H} = \frac{1}{2m_1}|\boldsymbol{p}_1|^2 + \frac{1}{2m_2}|\boldsymbol{p}_2|^2 + \lambda^3\frac{G}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|}$$

and $G = e_1e_2/(4\pi\epsilon_0)$. Moreover, noting that Hamilton's equations associated with the Hamiltonian system defined by $\overline{H}/\lambda^2$ and ω/λ^2 are the same as for the Hamiltonian system defined by $\overline{H}$ and ω we obtain that the existence of chaotic orbits for small values of λ for the Hamiltonian system defined by $\overline{H}$ and ω implies the existence of chaotic orbits for the Hamiltonian system (2.1.9) on level sets of high energy.

Theorem 2.5.3. *Let e_1 and e_2 be non–zero and have opposite signs. Furthermore, assume that $e_1 + e_2$ is non–zero and fix values $\ell \in \mathbb{R}$ of L and $h > 0$ of H such that*

$$\xi = \frac{(k_1 + k_2)\ell}{h} \in (0, m_1 + m_2)\ . \tag{2.5.11}$$

Then,

- *if Ω_1 and Ω_2 are rationally independent then for every $\xi \in (0, m_1 + m_2)$ there are infinitely many non–degenerate collision trajectories of energy h and for any finite set $\mathcal{K}$ of them there exists $\delta_0 > 0$ such that for every chain $(\gamma_{k_i})_{i\in\mathbb{Z}}$, $k_i \in K$, and $\delta \in (0, \delta_0)$ there is a unique trajectory of energy h near the collision chain and converging to the chain as $\delta \to 0$.*

- *Let m' be given by*

$$m' = \frac{(k_1+k_2)^2}{k_1\Omega_1 + k_2\Omega_2} \ .$$

If $|\Omega_1/\Omega_2|$ is rational, say N_1/N_2 in lowest terms, then

(i) if $\min\{m_1, m_2\} \geq m'$ and $N_1 > 2$ (respectively $N_2 > 2$) there is a subinterval (m_1, m^*) (respectively (m_2, m^*)) of $(0, m_1+m_2)$ such that for all $\xi \in (m_1, m^*)$ (respectively $\xi \in (m_2, m^*)$) there are at least four non–degenerate collision trajectories of energy h, and the set of chains formed from them has positive entropy. Furthermore, if $N_2 - 2 < N_1$ or $N_1 - 2 < N_2$ there is a subinterval (m'', m') of $(0, m_1+m_2)$ such that for all $\xi \in (m'', m')$ there are at least four non–degenerate collision trajectories of energy h, and the set of chains formed from them has positive entropy.

(ii) if $m_2 < m' < m_1$ (respectively $m_1 < m' < m_2$) and $N_1 > 2$ (respectively $N_2 > 2$) there is a subinterval (m_1, m^*) (respectively (m_2, m^*)) of $(0, m_1 + m_2)$ such that for all $\xi \in (m_1, m^*)$ (respectively $\xi \in (m_2, m^*)$) there are at least four non–degenerate collision trajectories of energy h, and the set of chains formed from them has positive entropy.

(iii) if $m' < \min\{m_1, m_2\}$ there is a subinterval $(m', \min\{m_1, m_2\})$ of $(0, m_1 + m_2)$ with $2(N_1 + N_2 - 1)$ non–degenerate collision trajectories of energy h.

Given a finite set $\mathcal{K}$ of non–degenerate collision trajectories, there exists $\delta_0 > 0$ such that for every chain $(\gamma_{k_i})_{i\in\mathbb{Z}}$, $k_i \in K$, and $\delta \in (0, \delta_0)$ there is a unique trajectory of energy h near the collision chain and converging to the chain as $\delta \to 0$.

Proof. From lemma 2.5.2 and a rescalling we obtain that the Hamiltonian system (2.1.9) can be reduced to one of the form (2.5.1). In section 2.5.3 we prove the existence of chains for a Hamiltonian system of the form (2.5.2): in lemma 2.5.6 we prove the existence of collision orbits for such system, in lemma 2.5.8 we check non–degeneracy of such collision orbits and in lemma 2.5.9 we prove that we can build sets of such non–

degenerate collision orbits satisfying the direction change condition. The result then follows by theorem 2.5.1 above, which guarantees that each chain of collision orbits of the system (2.5.2) is shadowed by an orbit of the system (2.5.1). □

Theorem 2.5.4. *Let e_1 and e_2 be non–zero and assume that $e_1 + e_2 = 0$. Fix the values $\boldsymbol{p} \in \mathbb{R}^2$ of $\boldsymbol{P}$ and $h > 0$ of H such that*

$$\xi = \frac{|\boldsymbol{p}|^2}{2h} \in (0, m_1 + m_2) \ . \tag{2.5.12}$$

Then,

- *if Ω_1 and Ω_2 are rationally independent then for every $\xi \in (0, m_1 + m_2)$ there are infinitely many non–degenerate collision trajectories of energy h, and for any finite set $\mathcal{K}$ of them there exists $\delta_0 > 0$ such that for every chain $(\gamma_{k_i})_{i \in \mathbb{Z}}$, $k_i \in K$, and $\delta \in (0, \delta_0)$ there is a unique trajectory of energy h near the collision chain and converging to the chain as $\delta \to 0$.*

- *If $|\Omega_1/\Omega_2|$ is rational and not equal to 1, say N_1/N_2 in lowest terms, for all $\xi \in (0, m_1 + m_2)$ there is at least one chain and for $\xi \in (0, \min\{m_1, m_2\})$ there is a set of chains with entropy at least $\log(N_1 + N_2 - 1)$. For each finite set $\mathcal{K}$ of non–degenerate collision trajectories there exists $\delta_0 > 0$ such that for every chain $(\gamma_{k_i})_{i \in \mathbb{Z}}$, $k_i \in K$, and $\delta \in (0, \delta_0)$ there is a unique trajectory of energy h near the collision chain and converging to the chain as $\delta \to 0$.*

Proof. From lemma 2.5.2 and a rescalling we obtain that the Hamiltonian system (2.1.9) can be reduced to one of the form (2.5.1). In section 2.5.3 we prove the existence of chains for a Hamiltonian system of the form (2.5.2): in lemma 2.5.7 we prove the existence of collision orbits for such system, in lemma 2.5.8 we check non–degeneracy of such collision orbits and in lemma 2.5.9 we prove that we can build sets of such non–degenerate collision orbits satisfying the direction change condition. The result then follows by theorem 2.5.1 above, which guarantees that each chain of collision orbits of the system (2.5.2) is shadowed by an orbit of the system (2.5.1). □

Remark Note that in the case $\Omega_1 = -\Omega_2$, theorem 2.5.3 for $k_1 + k_2 \neq 0$ produces only one orbit (in fact it is periodic with two near collisions per period) and theorem 2.5.4 for $k_1 + k_2 = 0$ produces none (because we will see in the proof of lemma 2.5.9 that in this case the direction change condition can not be satisfied). For all other negative frequency ratios, theorems 2.5.3 and 2.5.4 produce chaos.

2.5.3 Construction of collision orbits

In this section we prove the existence of a countably infinite subset of collision orbits for the Hamiltonian system (2.1.9) after reduction to the form (2.5.1), as given in lemma 2.5.2, for $\delta = 0$. Furthermore, we prove that the collision orbits are non–degenerate and satisfy the direction change condition on the reduced space. This construction combined with theorem 2.5.1 implies theorems 2.5.3 and 2.5.4 .

The general solution for the Hamiltonian system (2.1.9) with zero interaction potential can be written as

$$\begin{aligned} \boldsymbol{q}_i(t) &= \boldsymbol{R}_i + \rho_i \mathbf{J} \boldsymbol{e}_{\Omega_i t + \phi_i} \\ \boldsymbol{p}_i(t) &= k_i \rho_i \boldsymbol{e}_{\Omega_i t + \phi_i} \ , \end{aligned} \tag{2.5.13}$$

for $i \in \{1, 2\}$, where $\boldsymbol{R}_i = (R_{x_i}, R_{y_i}) \in \mathbb{R}^2$ are the guiding centres of the particles, $\rho_i \geq 0$ their gyroradii, $\Omega_i \in \mathbb{R}$ their gyrofrequencies, $\phi_i \in S^1$ their initial phases and $\boldsymbol{e}_{\Omega_i t + \phi_i} = (-\sin(\Omega_i t + \phi_i), \cos(\Omega_i t + \phi_i))$. Substituting (2.5.13) in the expressions for the Hamiltonian function given in (2.1.9) and the linear and angular momenta given in proposition 2.1.2, we obtain the conserved quantities of the Hamiltonian system (2.1.9) as functions of the parameters introduced above

$$H = \frac{k_1 \Omega_1}{2} {\rho_1}^2 + \frac{k_2 \Omega_2}{2} {\rho_2}^2 \tag{2.5.14}$$

$$\boldsymbol{P} = \mathbf{J} \left(k_1 \boldsymbol{R}_1 + k_2 \boldsymbol{R}_2 \right) \tag{2.5.15}$$

$$L = \sum_{i=1,2} \frac{k_i}{2} \left(\rho_i^2 - |\boldsymbol{R}_i|^2 \right) \ . \tag{2.5.16}$$

By lemma 2.5.2, on level sets $\{H = h, L = \ell, \boldsymbol{P} = \boldsymbol{p}\}$ of the Hamiltonian and the conserved quantities, to each orbit of the Hamiltonian system (2.1.9), satisfying the conditions

$$\begin{aligned} \boldsymbol{q}_1(0) &= \boldsymbol{q}_2(0) \\ \boldsymbol{q}_1(\tau) &= \boldsymbol{q}_2(\tau) \ , \end{aligned} \tag{2.5.17}$$

for some $\tau > 0$ and such that $\boldsymbol{q}_1(t) \neq \boldsymbol{q}_2(t)$ for every $0 < t < \tau$, there exists a collision orbit

$$\boldsymbol{q}(t) = \frac{1}{m^{1/2}} \left(\boldsymbol{q}_1(t) - \boldsymbol{q}_2(t) \right) \ ,$$

of the corresponding reduced Hamiltonian system, given in lemma 2.5.2, on the level set with energy h and fixed parameters $L = \ell$ and $\boldsymbol{P} = \boldsymbol{p}$ such that

$$\begin{aligned} \boldsymbol{q}(0) &= \mathbf{0} \\ \boldsymbol{q}(\tau) &= \mathbf{0} \ , \end{aligned}$$

for some $\tau > 0$ and $\boldsymbol{q}(t) \neq \mathbf{0}$ for every $0 < t < \tau$.

The next lemma follows by some simple geometric arguments. We skip its proof.

Lemma 2.5.5. *Assume that the trajectories of the two particles are given by (2.5.13). Then*

- *the images of the trajectories intersect in two distinct points if and only if the inequalities*

  $$|\rho_1 - \rho_2| < |\boldsymbol{R}_1 - \boldsymbol{R}_2| < \rho_1 + \rho_2 \ .$$

  are satisfied. Furthermore, the intersections are transversal and if the angles ϕ_1 and ϕ_2 are fixed by the condition $\boldsymbol{q}(0) = \mathbf{0}$ they satisfy $\phi_2 - \phi_1 \neq 0 \pmod{\pi}$.

- *if Ω_1 and Ω_2 are rationally independent then the two particles collide at most once at each of the intersection points.*

We now prove the existence of a large set of collision orbits of (2.1.9) on fixed level sets of the conserved quantities. In the construction we choose these collision orbits to connect distinct collision points since collision orbits connecting a point to itself are possible only for rationally dependent gyrofrequencies. Moreover, such orbits turn out to be degenerate. As before, we separate the analysis into two cases: $k_1 + k_2 \neq 0$ and $k_1 + k_2 = 0$.

Case $k_1 + k_2 \neq 0$

Lemma 2.5.6. *Let e_1 and e_2 be such that $e_1 + e_2 \neq 0$ and $e_1 e_2 < 0$ and fix $\ell \in \mathbb{R}$ and $h > 0$ such that*

$$\xi = \frac{(k_1 + k_2)\ell}{h} \in (0, m_1 + m_2) \ . \tag{2.5.18}$$

Then,

- *if Ω_1 and Ω_2 are rationally independent then for every $\xi \in (0, m_1 + m_2)$ there are infinitely many $SE(2)$ equivalence classes of orbits of the Hamiltonian system (2.1.9) with zero interaction on the level set*

$$\left\{(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{p}_1, \boldsymbol{p}_2) \in \mathbb{R}^8 : H = h \ , L = \ell \ , \boldsymbol{P} = \boldsymbol{p}\right\} \tag{2.5.19}$$

  satisfying (2.5.17) with no early collisions.

- *If $|\Omega_1/\Omega_2|$ is rational, say N_1/N_2 in lowest terms, then the following holds.*

  (i) If $\min\{m_1, m_2\} \geq m'$ and $N_1 > 2$ (respectively $N_2 > 2$) there is a subinterval (m_1, m^*) (respectively (m_2, m^*)) of $(0, m_1 + m_2)$ such that for all $\xi \in (m_1, m^*)$ (respectively $\xi \in (m_2, m^*)$) there are at least four $SE(2)$ equivalence classes of orbits of the Hamiltonian system (2.1.9) with zero interaction on the level set (2.5.19) satisfying (2.5.17) with no early collisions. Furthermore, if $N_2 - 2 < N_1$ or $N_1 - 2 < N_2$ there is a subinterval (m'', m') of $(0, m_1 + m_2)$ such that for all $\xi \in (m'', m')$ there are at least four of such equivalence classes.

  (ii) If $m_2 < m' < m_1$ (respectively $m_1 < m' < m_2$) and $N_1 > 2$ (respectively $N_2 > 2$) there is a subinterval (m_1, m^*) (respectively (m_2, m^*)) of $(0, m_1 + m_2)$ such that for all $\xi \in (m_1, m^*)$ (respectively $\xi \in (m_2, m^*)$) there are at least four of such equivalence classes.

(iii) If $m' < \min\{m_1, m_2\}$ there is a subinterval $(m', \min\{m_1, m_2\})$ of $(0, m_1 + m_2)$ with $2(N_1 + N_2 - 1)$ of such equivalence classes.

Proof. We fix the values of the masses $m_1 > 0$ and $m_2 > 0$ and charges e_1 and e_2 such that $e_1 e_2 < 0$ and $e_1 + e_2 \neq 0$. We assume that $k_1 + k_2 > 0$, as the case $k_1 + k_2 < 0$ can be transformed to this by time reversal. Without loss of generality we assume that $k_1 > 0$ and $k_2 < 0$. With this choice we also have $\Omega_1 = k_1/m_1 > 0$ and $\Omega_2 = k_2/m_2 < 0$.

From (2.5.15) we get that the centre of charge of the guiding centres

$$\frac{k_1 \boldsymbol{R}_1 + k_2 \boldsymbol{R}_2}{k_1 + k_2} = \frac{-\mathbf{J}\boldsymbol{P}}{k_1 + k_2}$$

is a constant. So by a translation we can assume it and $\boldsymbol{P}$ are $\mathbf{0}$. This implies the relation

$$\boldsymbol{R}_2 = -\frac{k_1}{k_2}\boldsymbol{R}_1 \ . \tag{2.5.20}$$

We remove the symmetry associated with the conservation of angular momentum using a rotation that makes the guiding centre of the first particle $\boldsymbol{R}_1 = (R_{x_1}, R_{y_1})$ a horizontal vector, i.e.

$$R_{y_1} = 0 \ , \qquad R_{x_1} > 0 \ . \tag{2.5.21}$$

Let us treat the case where at time $t = 0$ the two charges are at the intersection point of the two circles above the horizontal axis. We will treat the other case similarly. The situation is pictured in figure 2.5.1.

From the sine rule we obtain

$$\begin{aligned} \rho_1 &= \frac{r \sin\phi_2}{\sin(\phi_1 + \phi_2)} \\ \rho_2 &= \frac{r \sin\phi_1}{\sin(\phi_1 + \phi_2)} \ , \end{aligned} \tag{2.5.22}$$

where ϕ_1 and ϕ_2 belong to the set

$$S = \left\{(\phi_1, \phi_2) \in S^1 \times S^1 : 0 < \phi_1 < \pi, 0 < \phi_2 < \pi - \phi_1\right\} \tag{2.5.23}$$

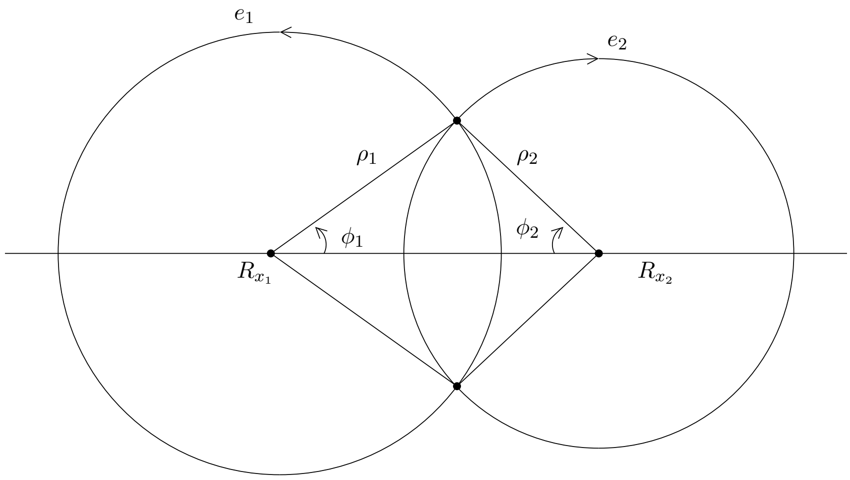


Figure 2.5.1: Trajectories of the two charges between collision points (for $k_1 \geq |k_2|$).

and r denotes the distance between the guiding centres of the particles and satisfies

$$R_{x_1} = -\frac{k_2}{k_1 + k_2} r \ . \tag{2.5.24}$$

From (2.5.14) and (2.5.22) we obtain

$$H = \frac{r^2}{2\sin^2(\phi_1 + \phi_2)} \left(k_1 \Omega_1 \sin^2 \phi_2 + k_2 \Omega_2 \sin^2 \phi_1 \right) \ . \tag{2.5.25}$$

Similarly, using (2.5.16), (2.5.20), (2.5.22) and (2.5.24) we obtain that

$$L = \frac{r^2}{2\sin^2(\phi_1 + \phi_2)} \left(k_1 \sin^2 \phi_2 + k_2 \sin^2 \phi_1 - \frac{k_1 k_2}{k_1 + k_2} \sin^2(\phi_1 + \phi_2) \right) \ . \tag{2.5.26}$$

We define the function

$$\Xi = \frac{(k_1 + k_2)L}{H} = \frac{(k_1 + k_2)\left(k_1 \sin^2 \phi_2 + k_2 \sin^2 \phi_1 \right) - k_1 k_2 \sin^2(\phi_1 + \phi_2)}{k_1 \Omega_1 \sin^2 \phi_2 + k_2 \Omega_2 \sin^2 \phi_1} \tag{2.5.27}$$

and note that Ξ has range $(0, m_1 + m_2)$ and takes the values m_1, m_2 and

$$m' = \frac{(k_1 + k_2)^2}{k_1 \Omega_1 + k_2 \Omega_2} \tag{2.5.28}$$

along the boundaries $\phi_1 = 0$, $\phi_2 = 0$ and $\phi_1 + \phi_2 = \pi$, respectively. Note that $m' < \max\{m_1, m_2\}$ (in particular, since we are assuming $k_1 + k_2 > 0$ then $m' < m_1$) and if $m_1 = m_2$ then $m' < m_1$. The supremum $m_1 + m_2$ of Ξ is approached

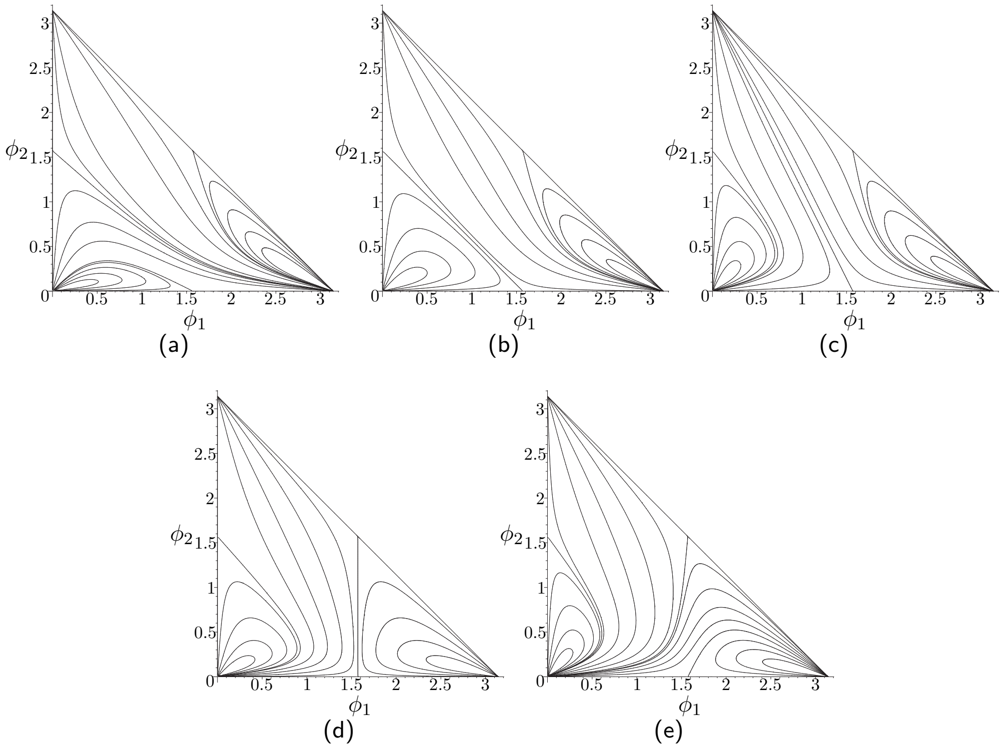


Figure 2.5.2: Contour plots of Ξ in the set S for $k_1 > |k_2|$. From left to right, on the top line are the cases $m' < m_1 < m_2$ (2.5.2a), $m' < m_1 = m_2$ (2.5.2b) and $m' < m_2 < m_1$ (2.5.2c) and on the bottom line are the cases $m' = m_2 < m_1$ (2.5.2d) and $m_2 < m' < m_1$ (2.5.2e). The contour plots are qualitatively the same for all parameters in these ranges; the values used in the figures are $k_1 = m_1 = 1$ and $(k_2, m_2, m') = (-1/2, 2, 2/9)$ (2.5.2a), $(-1/2, 1, 1/5)$ (2.5.2b), $(-1/2, 1/2, 1/6)$ (2.5.2c), $(-1/4, 1/2, 1/2)$ (2.5.2d), $(-1/4, 1/4, 9/20)$ (2.5.2e).

along the line $\phi_2/\phi_1 = |\Omega_2|/\Omega_1$ to $(0,0)$. Its infimum 0 is approached along the line $\phi_2/(\pi - \phi_1) = |k_2|/k_1$ to $(\pi, 0)$. See figure 2.5.2 for plots of the level sets of Ξ.

From an analysis of figure 2.5.1 we get that the time $\tau > 0$ in (2.5.17) for the particles to collide at the intersection point lying below the horizontal line must satisfy the conditions

$$\tau = \frac{2\pi n_1 - 2\phi_1}{\Omega_1} = \frac{2\pi n_2 - 2\phi_2}{|\Omega_2|} \, , \qquad (2.5.29)$$

for some integers $n_1, n_2 \geq 1$.

If the gyrofrequencies Ω_1 and Ω_2 are rationally independent then, by lemma

2.5.5, there are no collisions for times $t \in (0, \tau)$. On the other hand, if Ω_1 and Ω_2 are rationally dependent, say $\Omega_1/|\Omega_2| = N_1/N_2$ in lowest terms, then early collisions might occur, i.e. there might exist $0 < t^* < \tau$ such that $\boldsymbol{q}_1(t^*) = \boldsymbol{q}_2(t^*)$. It suffices, however, to reduce (n_1, n_2) by the first integer multiple of (N_1, N_2) to make $n_1 \leq N_1$ or $n_2 \leq N_2$ (maintaining $n_1, n_2 \geq 1$) to obtain a collision trajectory with the same start and end as before with no early collisions.

Also from (2.5.29) we get the relation

$$\phi_2 = |\Omega_2| \left(C(n_1, n_2) + \frac{\phi_1}{\Omega_1} \right) , \tag{2.5.30}$$

where $C(n_1, n_2)$ is given by

$$C(n_1, n_2) = \frac{\pi n_2}{|\Omega_2|} - \frac{\pi n_1}{\Omega_1} .$$

The case where at $t = 0$ the two particles are at the lower intersection of figure 2.5.1 is similar, but

$$\tau = \frac{2\phi_1 + 2\pi n_1'}{\Omega_1} = \frac{2\phi_2 + 2\pi n_2'}{|\Omega_2|}$$

for some integers $n_1', n_2' \geq 0$. So the two cases can be combined by allowing (n_1, n_2) in (2.5.29) to range over $\mathcal{N} = \left\{(n_1, n_2) \in \mathbb{Z}^2 : n_1, n_2 \geq 1 \text{ or } n_1, n_2 \leq 0\right\}$ (reduced suitably by multiples of (N_1, N_2) in the rational case to avoid early collisions).

We fix a level set of $\Xi = \xi$, where $\xi \in (0, m_1 + m_2)$. Collision orbits correspond to intersections of that level set with the set of lines in S given by (2.5.30) for $n_1, n_2 \in \mathcal{N}$. Now we separate the analysis into two cases.

Claim 1: If Ω_1 and Ω_2 are rationally independent there are infinitely many transverse intersections in S for any given $\xi \in (0, m_1 + m_2)$.

Claim 2: If Ω_1 and Ω_2 are rationally dependent, say $\Omega_1/|\Omega_2| = N_1/N_2$ in lowest terms, there are transverse intersections for all $\xi \in (m', m_1 + m_2)$. Furthermore, we have that the following holds.

(i) If $\min\{m_1, m_2\} \geq m'$ and $N_1 > 2$ (respectively $N_2 > 2$) there are at least two transverse intersections for all $\xi \in (m_1, m^*)$ (respectively $\xi \in (m_2, m^*)$)

for some $m^* \in (m_1, m_1+m_2)$ (respectively $m^* \in (m_2, m_1+m_2)$). Moreover, if $N_2-2 < N_1$ or $N_1-2 < N_2$ there are at least two transverse intersections for all $\xi \in (m'', m')$ for some $m'' \in (0, m')$.

(ii) If $m_2 < m' < m_1$ and $N_1 > 2$ there are at least two transverse intersections for all $\xi \in (m_1, m^*)$ for some $m^* \in (m_1, m_1 + m_2)$.

(iii) If $m' < \min\{m_1, m_2\}$ there is a subinterval $(m', \min\{m_1, m_2\})$ with at least $(N_1 + N_2 - 1)$ transverse intersections.

(iv) Each intersection defines two collision trajectories, one from the upper point to the lower point, the other from the lower to the upper.

Claim 1 is trivial since rational independence of Ω_1 and Ω_2 implies that the lines (2.5.30) densely fill the set S. Since there are no level sets of Ξ parallel to the lines (2.5.30) then we have infinitely many intersections with each level set $\xi \in (0, m_1+m_2)$ of Ξ. Infinitely many of them are transverse.

We now prove Claim 2. Equation (2.5.30) defines $N_1 + N_2 - 1$ lines of slope N_2/N_1, from the origin, the points $n\pi/N_1$ on the ϕ_2-axis and the points $n\pi/N_2$ on the ϕ_1-axis. On the line from $(0,0)$ one of the two situations happen: either Ξ decreases at non–zero rate from $m_1 + m_2$ to a minimum and then rises at non–zero rate to m' or Ξ decreases at non–zero rate to m'. Thus, transverse intersections exist for all $\xi \in (m', m_1 + m_2)$.

Now if $N_1 > 2$ there is a line starting from π/N_1 on the ϕ_2-axis. On this line Ξ rises at non–zero rate from m_1 to a maximum value and then either Ξ decreases to m' at non–zero rate or Ξ decreases to a minimum below m' to rise again to m'. Take m^* to be the maximum value in this line. If $N_2 > 2$ there is a line starting from π/N_2 on the ϕ_1-axis. On this line Ξ rises at non–zero rate from m_2 to a maximum value and then either Ξ decreases to m' at non–zero rate or Ξ decreases to a minimum below m' to rise again to m'. Take m^* to be the maximum value of Ξ on this line. To finish the proof of (i) note that the condition $N_2 - 2 < N_1$ implies that on the line from π/N_2 the map Ξ has a maximum above m' and a minimum m'' below m'. The condition

$N_1 - 2 < N_2$ implies the same conclusion when the line from π/N_1 is considered. To prove (ii) note that if $m_2 < m' < m_1$ then on the line from π/N_1 on the ϕ_2-axis Ξ rises at non–zero rate from m_1 to a maximum value and then either Ξ decreases to m' at non–zero rate or Ξ decreases to a minimum below m' to rise again to m'. For (iii), note that on all the $N_1 + N_2 - 1$ lines, Ξ connects one of $m_1 + m_2$ (at $(0, 0)$), m_1 (at $\phi_1 = 0$) or m_2 (at $\phi_2 = 0$) to m' at $\phi_1 + \phi_2 = \pi$. Statement (iv) is a result of the rational frequency ratio. This finishes the proof of Claim 2.

To finish the proof of lemma 2.5.6 we note that given $(\phi_1, \phi_2) \in S$ and fixing $H = h$ in (2.5.25) we determine r. Having determined r we obtain R_1 and R_2 by (2.5.20) and (2.5.24). The values of ρ_1 and ρ_2 are determined by (2.5.22) once r, ϕ_1 and ϕ_2 are known. The results for $k_1 + k_2 < 0$ are obtained by time reversal, the only effect being that the case $m_1 < m' < m_2$ of item (ii) of the lemma applies. □

Case $k_1 + k_2 = 0$

Lemma 2.5.7. *Let e_1 and e_2 be such that $e_1 + e_2 = 0$ and fix $\boldsymbol{p} \in \mathbb{R}^2$ non–zero and $h > 0$ such that*

$$\xi = \frac{|\boldsymbol{p}|^2}{2h} \in (0, m_1 + m_2) \; . \tag{2.5.31}$$

Then,

- *if Ω_1 and Ω_2 are rationally independent then for every $\xi \in (0, m_1 + m_2)$ there exist infinitely many $SE(2)$ equivalence classes of orbits of the Hamiltonian system (2.1.9) with zero interaction on the level set*

$$\left\{(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{p}_1, \boldsymbol{p}_2) \in \mathbb{R}^8 : H = h \; , L = \ell \; , \boldsymbol{P} = \boldsymbol{p}\right\}$$

  satisfying (2.5.17) with no early collisions.

- *If $|\Omega_1/\Omega_2|$ is rational, say N_1/N_2 in lowest terms, and $\xi \in (0, m_1 + m_2)$ there are at least two $SE(2)$ equivalence classes of orbits of the Hamiltonian system (2.1.9) with zero interaction on the level set*

$$\left\{(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{p}_1, \boldsymbol{p}_2) \in \mathbb{R}^8 : H = h \; , L = \ell \; , \boldsymbol{P} = \boldsymbol{p}\right\}$$

satisfying (2.5.17) with no early collisions. If $\xi \in (0, \min\{m_1, m_2\})$ *there are at least* $2(N_1 + N_2 - 1)$ *of them.*

Proof. Without loss of generality, we fix the values of the masses $m_1 > 0$ and $m_2 > 0$ and charges $e_1 < 0$ and $e_2 > 0$ so that $k_1 > 0$ and $k_2 < 0$. Since $e_1 + e_2 = 0$ we have $k_1 + k_2 = 0$, so throughout this proof we set $k_2 = -k_1$.

We fix a non–zero value $\boldsymbol{p} = (p_x, p_y) \in \mathbb{R}^2$ for the level set $\{\boldsymbol{P} = \boldsymbol{p}\}$ (we have seen already that if $k_1 + k_2 = 0$ and $\boldsymbol{P} = \boldsymbol{0}$ then the system is integrable). Without loss of generality we apply a rotation to make $\boldsymbol{p}$ vertical, i.e.

$$\boldsymbol{p} = (p_x, p_y) \ , \qquad p_x = 0 \ , \qquad p_y > 0 \ . \tag{2.5.32}$$

Hence, from (2.5.15) and (2.5.32), we get that the vector

$$\boldsymbol{R}_2 - \boldsymbol{R}_1 = \frac{1}{k_1}\mathbf{J}\boldsymbol{p} \ , \tag{2.5.33}$$

is horizontal, oriented to the right, and has norm

$$r = \frac{|\boldsymbol{p}|}{k_1} \ . \tag{2.5.34}$$

Then the situation is as in figure 2.5.1 again.

Let us first treat the case where at $t = 0$ the two charges are at the intersection point of the two circles above the horizontal axis.

From the sine-rule we obtain ρ_1 and ρ_2 as given in (2.5.22) with r given by (2.5.34) and ϕ_1 and ϕ_2 belong to the set S defined in (2.5.23). From (2.5.14) and (2.5.22) we obtain H as given in (2.5.25). We define the function

$$\Xi = \frac{k_1^2 r^2}{2H} = \frac{k_1 \sin^2(\phi_1 + \phi_2)}{\Omega_1 \sin^2\phi_2 - \Omega_2 \sin^2\phi_1} = \frac{m_1 m_2 \sin^2(\phi_1 + \phi_2)}{m_1 \sin^2\phi_1 + m_2 \sin^2\phi_2} \tag{2.5.35}$$

and note that Ξ again has range $(0, m_1 + m_2)$ and takes values m_1, m_2, $m' = 0$ on $\phi_1 = 0$, $\phi_2 = 0$, $\phi_1 + \phi_2 = \pi$, respectively. See figure 2.5.3 for plots of the level sets of Ξ.

From an analysis of figure 2.5.1 we get that the time $\tau > 0$ in (2.5.17) for which a collision occurs must satisfy the conditions (2.5.29) for some $n_1, n_2 \in \mathbb{N}$.

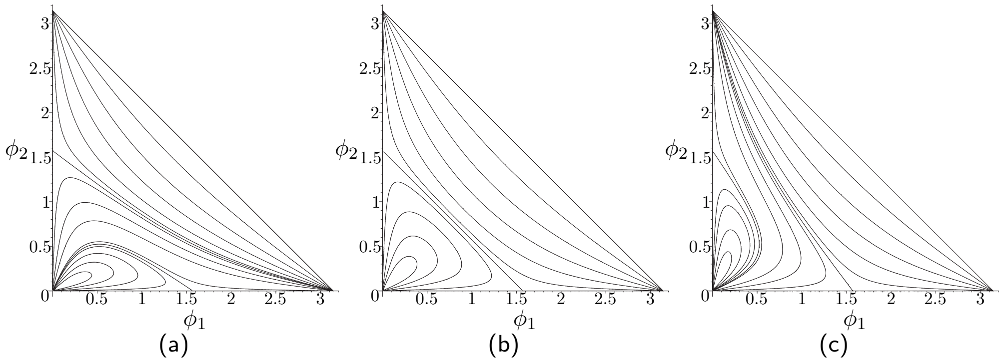


Figure 2.5.3: Contour plots of Ξ in the set S for the case $k_2 + k_1 = 0$. From left to right are the cases $m_1 < m_2$ (2.5.3a), $m_1 = m_2$ (2.5.3b) and $m_1 > m_2$ (2.5.3c). The figures are drawn for $k_1 = 1$, $k_2 = -1$ and $(m_1, m_2) = (1, 2)$ (2.5.3a), $(1, 1)$ (2.5.3b), $(1, 1/2)$ (2.5.3c), but all other parameter choices satisfying the given conditions give qualitatively equivalent pictures.

If the gyrofrequencies Ω_1 and Ω_2 are rationally independent then, by lemma 2.5.5, there are no collisions for times $t \in (0, \tau)$. On the other hand, if Ω_1 and Ω_2 are rationally dependent, say $\Omega_1/|\Omega_2| = N_1/N_2$ in lowest terms, then early collisions might occur, i.e. there might exist $0 < t^* < \tau$ such that $\boldsymbol{q}_1(t^*) = \boldsymbol{q}_2(t^*)$. As before, reducing (n_1, n_2) by a multiple of (N_1, N_2) removes any early collisions. As before, we obtain the relation (2.5.30), and the case where the particles start at the lower intersection in figure 2.5.1 can be incorporated by allowing (n_1, n_2) to range over $\mathcal{N}$.

We fix a level set of $\Xi = \xi$, where $\xi \in (0, m_1 + m_2)$. Collision orbits correspond to intersections of that level set with the set of lines in S given by (2.5.30) for $(n_1, n_2) \in \mathcal{N}$. Now we separate the analysis into two cases.

Claim 1: If Ω_1 and Ω_2 are rationally independent there are infinitely many transverse intersections in S for any given $\xi \in (0, m_1 + m_2)$.

Claim 2: If Ω_1 and Ω_2 are rationally dependent, say $\Omega_1/|\Omega_2| = N_1/N_2$ in lowest terms, there are transverse intersections for all $\xi \in (0, m_1 + m_2)$. For $\xi \in (0, \min\{m_1, m_2\})$ there are at least $N_1 + N_2 - 1$ transverse intersections. Each intersection gives two collision orbits.

The proofs of Claims 1 and 2 are analogous (though simpler) to those given in the proof of the previous lemma 2.5.6.

To finish the proof we note that given $(\phi_1, \phi_2) \in S$ and fixing $H = h$ in (2.5.25) we determine r, from which we obtain $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ (up to a translation) by (2.5.33). The values of ρ_1 and ρ_2 are determined by (2.5.22) once r, ϕ_1 and ϕ_2 are known. □

Analysis of the sets of collision trajectories

Lemma 2.5.8. *Consider the collision orbits at $\delta = 0$ constructed in lemmas 2.5.6 and 2.5.7.*

- *If Ω_1 and Ω_2 are rationally independent there are infinitely many non–degenerate collision orbits for any given $\xi \in (0, m_1 + m_2)$.*
- *Suppose $|\Omega_1/\Omega_2|$ is rational, say N_1/N_2 in lowest terms.*
  - *If $k_1 + k_2 \neq 0$ then*
    - *(i) if $\min\{m_1, m_2\} \geq m'$ and $N_1 > 2$ (respectively $N_2 > 2$) for all $\xi \in (m_1, m^*)$ (respectively $\xi \in (m_2, m^*)$) there are at least four non–degenerate collision orbits. Furthermore, if $N_2 - 2 < N_1$ or $N_1 - 2 < N_2$ for all $\xi \in (m'', m')$ there are at least four.*
    - *(ii) if $m_2 < m' < m_1$ (respectively $m_1 < m' < m_2$) and $N_1 > 2$ (respectively $N_2 > 2$) for all $\xi \in (m_1, m^*)$ (respectively $\xi \in (m_2, m^*)$) there are at least four non–degenerate collision orbits.*
    - *(iii) if $m' < \min\{m_1, m_2\}$ for all $\xi \in (m', \min\{m_1, m_2\})$ there are $2(N_1 + N_2 - 1)$ non–degenerate collision orbits.*
  - *If $k_1 + k_2 = 0$ then for all $\xi \in (0, m_1 + m_2)$ there are at least two non–degenerate collision orbits and for $\xi \in (0, \min\{m_1, m_2\})$ there are at least $2(N_1 + N_2 - 2)$.*

Proof. In the proofs of lemmas 2.5.6 and 2.5.7 we have constructed sets of collision orbits by removing the symmetries and fixing the conserved quantities of (2.1.9) and enforcing the collision conditions (2.5.17). Collision orbits correspond to intersections of the lines (2.5.30) with level sets of the quantity Ξ (defined in (2.5.27) for the case $k_1 + k_2 \neq 0$ and (2.5.35) for the case $k_1 + k_2 = 0$). To prove non–degeneracy we must check that the derivative of (2.5.4) has full rank when evaluated on a collision orbit. We prove this for the case $k_1 + k_2 = 0$. The case $k_1 + k_2 \neq 0$ is analogous. Substituting (2.5.13) in (2.5.4) and using (2.5.14) and (2.5.33), we obtain

$$\begin{aligned} \rho_1 \mathbf{J}\boldsymbol{e}_{\phi_1} - \rho_2 \mathbf{J}\boldsymbol{e}_{\phi_2} &= \frac{1}{k_1}\mathbf{J}\boldsymbol{p} \\ \rho_1 \mathbf{J}\boldsymbol{e}_{\Omega_1\tau+\phi_1} - \rho_2 \mathbf{J}\boldsymbol{e}_{\Omega_2\tau+\phi_2} &= \frac{1}{k_1}\mathbf{J}\boldsymbol{p} \\ \frac{1}{2m_1}{\rho_1}^2 + \frac{1}{2m_2}{\rho_2}^2 &= E \ . \end{aligned} \tag{2.5.36}$$

Note that the right hand side of (2.5.36) does not contain any of the five parameters ρ_1, ρ_2, ϕ_1, ϕ_2 and τ. Differentiating (2.5.36) with respect to these five parameters, computing the determinant of the resulting matrix, substituting ρ_1 and ρ_2 by the expressions given in (2.5.22) and noting that

$$\cos(\Omega_i\tau + \phi_i) = \cos(\phi_i) \ , \qquad \sin(\Omega_i\tau + \phi_i) = -\sin(\phi_i) \ , \qquad i \in \{1, 2\} \ ,$$

we obtain

$$D(\phi_1, \phi_2) = \frac{2r^4 k_1 \sin(\phi_1)\sin(\phi_2)}{m_1 m_2 \sin^4(\phi_1 + \phi_2)} \left(F_1(\phi_1, \phi_2) + \frac{m_1}{m_2} F_1(\phi_1, \phi_2) + \frac{m_2}{m_1} F_3(\phi_1, \phi_2) \right) \ ,$$

where

$$\begin{aligned} F_1(\phi_1, \phi_2) &= \cos^3(\phi_1)\sin(\phi_1)\sin^2(\phi_2) - \cos^2(\phi_1)\sin^2(\phi_1)\sin(\phi_2)\cos(\phi_2) \\ &\quad + \sin^3(\phi_1)\sin^2(\phi_2)\cos(\phi_1) - \sin(\phi_2)\sin^2(\phi_1)\cos^3(\phi_2) \\ &\quad + \sin^2(\phi_2)\cos(\phi_1)\sin(\phi_1)\cos^2(\phi_2) - \sin^4(\phi_1)\sin(\phi_2)\cos(\phi_2) \\ &\quad - \sin^3(\phi_2)\sin^2(\phi_1)\cos(\phi_2) + \sin^4(\phi_2)\cos(\phi_1)\sin(\phi_1) \\ F_2(\phi_1, \phi_2) &= -\cos(\phi_1)\sin^3(\phi_1)\cos^2(\phi_2) + \cos^2(\phi_1)\sin^2(\phi_1)\cos(\phi_2)\sin(\phi_2) \\ &\quad - \sin^4(\phi_1)\sin(\phi_2)\cos(\phi_2) + \sin^3(\phi_1)\cos(\phi_1)\sin^2(\phi_2) \\ F_3(\phi_1, \phi_2) &= \sin^3(\phi_2)\cos(\phi_2)\cos^2(\phi_1) - \sin^2(\phi_2)\cos(\phi_1)\cos^2(\phi_2)\sin(\phi_1) \\ &\quad + \sin^4(\phi_2)\sin(\phi_1)\cos(\phi_1) - \sin^3(\phi_2)\sin^2(\phi_1)\cos(\phi_2) \ , \end{aligned}$$

and r is as given in (2.5.34). To obtain non–degeneracy it is enough to check that the level sets of D do not coincide with the level sets of Ξ. Changing variables in (2.5.35) to $x = \cos(\phi_1)$ and $y = \cos(\phi_2)$, where

$$(x, y) \in \left\{(u, v) \in \mathbb{R}^2 : -1 \leq u, v \leq 1 \ , u + v \geq 0\right\} \ .$$

we obtain

$$\Xi = \frac{m_1 m_2 \left(x^2 + y^2 - 2x^2y^2\right)}{m_1 + m_2 - m_1 x^2 - m_2 y^2} \ ,$$

Applying the same change to D, we obtain

$$D(x, y) = \frac{-2k_1 r^4 \sqrt{(1 - x^2)(1 - y^2)}}{{m_1}^2 {m_2}^2 \left(x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\right)^4} G(x, y) \ ,$$

where

$$\begin{aligned} G(x, y) &= \left({m_1}^2 \left(1 - x^2\right) + {m_2}^2 \left(1 - y^2\right)\right) G_1(x, y) + 2m_1 m_2 G_2(x, y) \\ G_1(x, y) &= y \left(1 - 2x^2\right) \sqrt{1 - y^2} - x \left(1 - 2y^2\right) \sqrt{1 - x^2} \\ G_2(x, y) &= y \left(1 - x^2\right) \sqrt{1 - y^2} - x \left(1 - y^2\right) \sqrt{1 - x^2} \ . \end{aligned}$$

Since for the variables (x, y) the level sets of Ξ are quartic curves in $\mathbb{R}^2$ while the level sets of D are non–algebraic curves in $\mathbb{R}^2$ we obtain that the level sets of Ξ and D do not coincide. $\square$

Let $\Gamma \subset \mathbb{N}$ be a set of labels for the non–degenerate collision orbits constructed in lemmas 2.5.6 and 2.5.7. The set Γ is countably infinite if Ω_1 and Ω_2 are rationally independent and finite otherwise. Let $n \in \Gamma$ and denote by $\boldsymbol{q}^n(t)$ a collision orbit with given momenta $\mathbf{P}$, L and energy h, given by

$$\boldsymbol{q}^n(t) = \frac{1}{m^{1/2}} \left(\boldsymbol{q}_1^n(t) - \boldsymbol{q}_2^n(t)\right) \ ,$$

where

$$\begin{aligned} \boldsymbol{q}_1^n(t) &= \boldsymbol{R}_1^n + \rho_1^n \mathbf{J} \boldsymbol{e}_{\Omega_1 t + \sigma^n \phi_1^n} \\ \boldsymbol{q}_2^n(t) &= \boldsymbol{R}_2^n + \rho_2^n \mathbf{J} \boldsymbol{e}_{\Omega_2 t + \pi - \sigma^n \phi_2^n} \ , \end{aligned}$$

$\boldsymbol{R}_i^n$, ρ_i^n and ϕ_i^n for $i \in \{1,2\}$ are as constructed in lemmas 2.5.6 and 2.5.7 and $\sigma^n \in \pm$ corresponds to starting at the upper or lower intersection in figure 2.5.1.

Lemma 2.5.9. *Let $n \in \Gamma$ and $\boldsymbol{q}^n(t)$ be a collision orbit with given momenta $\mathbf{P}$, L and energy H. Then,*

- *if Ω_1 and Ω_2 are rationally independent there exist infinitely many non–degenerate collision orbits that leave $\mathbf{0}$ with the same $\mathbf{P}$, L and H in neither the same nor the opposite direction as $\boldsymbol{q}^n(\tau^n)$.*
- *If $|\Omega_1/\Omega_2|$ is rational, say N_1/N_2 in lowest terms, then*
  - *if $k_1 + k_2 \neq 0$:*
    - *(i) if $\min\{m_1, m_2\} \geq m'$ and $N_1 > 2$ (respectively $N_2 > 2$) for $\xi \in (m_1, m^*)$ (respectively $\xi \in (m_2, m^*)$) there is a set of chains with entropy $\log 2$. Furthermore, if $N_2 - 2 < N_1$ or $N_1 - 2 < N_2$ for $\xi \in (m'', m')$ there is also a set of chains with entropy $\log 2$.*
    - *(ii) if $m_2 < m' < m_1$ (respectively $m_1 < m' < m_2$) and $N_1 > 2$ (respectively $N_2 > 2$) for $\xi \in (m_1, m^*)$ (respectively $\xi \in (m_2, m^*)$) there is a set of chains with entropy $\log 2$.*
    - *(iii) if $m' < \min\{m_1, m_2\}$ for $\xi \in (m', \min\{m_1, m_2\})$ there is a set of chains with entropy $\log(N_1 + N_2 - 1)$.*
  - *If $k_1 + k_2 = 0$ and $N_1/N_2 \neq 1$ for all $\xi \in (0, m_1 + m_2)$ there is a chain and for all $\xi \in (0, \min\{m_1, m_2\})$ there is a set of chains with entropy at least $\log(N_1 + N_2 - 1)$.*

Proof. Using (2.5.13), we get

$$\dot{\boldsymbol{q}}^n(t) = \frac{1}{m^{1/2}} \left(\Omega_1 \rho_1^n \boldsymbol{e}_{\Omega_1 t + \sigma^n \phi_1^n} - \Omega_2 \rho_2^n \boldsymbol{e}_{\Omega_2 t + \pi - \sigma^n \phi_2^n} \right) \ .$$

Let $\theta^n(t)$ denote the angle between $\dot{\boldsymbol{q}}^n(t)$ and the horizontal axis of $\mathbb{R}^2$. We have the following expression for the tangent of $\theta^n(t)$:

$$\tan \theta^n(t) = \frac{\Omega_1 \rho_1^n \cos(\Omega_1 t + \sigma^n \phi_1^n) - \Omega_2 \rho_2^n \cos(\Omega_2 t + \pi - \sigma^n \phi_2^n)}{\Omega_2 \rho_2^n \sin(\Omega_2 t + \pi - \sigma^n \phi_2^n) - \Omega_1 \rho_1^n \sin(\Omega_1 t + \sigma^n \phi_1^n)} \ , \tag{2.5.37}$$

which implies

$$\begin{aligned}
\tan\theta^n(0) &= \sigma^n \frac{\Omega_1\rho_1^n\cos\phi_1^n + \Omega_2\rho_2^n\cos\phi_2^n}{\Omega_2\rho_2^n\sin\phi_2^n - \Omega_1\rho_1^n\sin\phi_1^n} \\
\tan\theta^n(\tau^n) &= \sigma^n \frac{\Omega_1\rho_1^n\cos\phi_1^n + \Omega_2\rho_2^n\cos\phi_2^n}{\Omega_1\rho_1^n\sin\phi_1^n - \Omega_2\rho_2^n\sin\phi_2^n} \ ,
\end{aligned} \tag{2.5.38}$$

Substituting in (2.5.38) the expressions for ρ_1^n and ρ_2^n given in (2.5.22), we get

$$\tan\theta^n(0) = \frac{\sigma^n}{\Omega_2 - \Omega_1}\left(\Omega_1\cot\phi_1^n + \Omega_2\cot\phi_2^n\right) = -\tan\theta^n(\tau^n) \ ,$$

(note that $\Omega_1 > 0$ and $\Omega_2 < 0$ so the denominator is negative).

If $k_1 + k_2 = 0$ then all the collision orbits can be treated in a common frame where $\boldsymbol{P}$ is vertical, so the change of direction condition is that the next collision orbit m must satisfy

$$\tan\theta^n(\tau^n) \neq \tan\theta^m(0) \ ,$$

i.e.

$$\sigma^m\left(\Omega_1\cot\phi_1^m + \Omega_2\cot\phi_2^m\right) \neq -\sigma^n\left(\Omega_1\cot\phi_1^n + \Omega_2\cot\phi_2^n\right) \ .$$

Assume Ω_1/Ω_2 is irrational. For the condition above to be satisfied it is enough to check that the level sets of

$$F\left(\phi_1, \phi_2\right) = \Omega_1\cot\phi_1 + \Omega_2\cot\phi_2$$

do not coincide with the level sets of Ξ (given in (2.5.35)). We change variables to $x = \cot(\phi_1)$ and $y = \cot(\phi_2)$ so that the level sets of F are just lines of the form $\Omega_1 x + \Omega_2 y = const.$. With this choice of variables the function Ξ has the form

$$\Xi(x, y) = \frac{k_1\left(y^2 - x^2\right)}{(\Omega_1 - \Omega_2) + \Omega_1 y^2 - \Omega_2 x^2} \ .$$

Since Ω_1 and Ω_2 are rational independent none of the level sets $\Xi = \xi$, with $\xi \in (0, m_1 + m_2)$, is a line of the form $\Omega_1 x + \Omega_2 y = const.$. Hence, there are infinitely many choices for m and all but finitely many satisfy this condition; thus one can make sets of chains with arbitrarily large entropy. If $\Omega_1/|\Omega_2| \neq 1$ is rational, say N_1/N_2 in lowest terms, then the choice of $\sigma^m \in \pm$ is free so the condition can always be satisfied;

thus for all $\xi \in (0, m_1 + m_2)$ one can make a chain and for $\xi \in (0, \min\{m_1, m_2\})$ one can make a set of chains with entropy at least $\log(N_1 + N_2 - 1)$. If $\Omega_1/|\Omega_2| = 1$ or equivalently $m_1 = m_2$ then $\phi_1^m = \phi_2^m$ and hence $\Omega_1 \cot\phi_1^m + \Omega_2 \cot\phi_2^m = 0$ so the condition can not be satisfied.

If $k_1+k_2 \neq 0$ then the analysis for the next collision orbit m in a chain needs to be rotated by some angle ψ about $-\mathbf{J}\boldsymbol{P}/(k_1+k_2)$ (which we choose to be at $\mathbf{0}$), determined to superimpose the collisions $\boldsymbol{q}_j^n(\tau)$ and $\boldsymbol{q}_j^m(0)$, $j \in \{1,2\}$: $\psi = -(\psi^n + \psi^m)$, where $-\psi^n$ is the angle that $(k_1 q_1^n(\tau) + k_2 q_2^n(\tau))/(k_1+k_2)$ makes with the positive horizontal axis of $\mathbb{R}^2$ and is given by

$$\cot(\psi^n) = \frac{\sigma^n}{k_1 + k_2}\left(|k_2| \cot\phi_2^n + k_1 \cot\phi_1^n\right)$$

and ψ^m is defined in a similar way. Then the direction change condition is

$$\tan(\theta^n(\tau) - \theta^m(0)) \neq \tan(\psi)\ .$$

This is some quadratic inequality in $\cot(\phi_j^n)$ and $\cot(\phi_j^m)$, $j \in \{1,2\}$. If Ω_1/Ω_2 is irrational there are infinitely many choices for m satisfying this condition; thus one can make sets of chains with arbitrarily large entropy. If $\Omega_1/|\Omega_2|$ is rational, say N_1/N_2 in lowest terms, then the choice of $\sigma^m \in \pm$ is free so the condition can be satisfied. □

2.5.4 Boundary value problem

In this section we state a result which is a particular case of one proved in [8] and which is a key ingredient for the proof of theorem 2.5.1: the existence for small $\delta > 0$, of orbits with energy $E > 0$ connecting two points $\boldsymbol{a}, \boldsymbol{b}$ in a small ball U centered at $\mathbf{0}$.

For any $\boldsymbol{a} \in U$, there is a unique trajectory $\gamma_{\boldsymbol{a}}^+ : [0, \tau^+(\boldsymbol{a})] \to U$ of energy E for the canonical Hamiltonian system with Hamiltonian function (2.5.2) connecting $\boldsymbol{a}$ to $\mathbf{0}$. Similarly, for any $\boldsymbol{b} \in U$, there is a unique trajectory $\gamma_{\boldsymbol{b}}^- : [\tau^-(\boldsymbol{b}), 0] \to U$ of energy E connecting $\mathbf{0}$ to $\boldsymbol{b}$. Denote

$$\begin{aligned} S^+(\boldsymbol{a}) &= \textstyle\int_0^{\tau^+(\boldsymbol{a})} \boldsymbol{p}.\mathrm{d}\boldsymbol{q} &&= \displaystyle\int_0^{\tau^+(\boldsymbol{a})} \boldsymbol{p}(t).\frac{\partial H}{\partial \boldsymbol{p}}(\boldsymbol{q}(t), \boldsymbol{p}(t))\,\mathrm{d}t \\ S^-(\boldsymbol{b}) &= \textstyle\int_{\tau^-(\boldsymbol{b})}^0 \boldsymbol{p}.\mathrm{d}\boldsymbol{q} &&= \displaystyle\int_{\tau^-(\boldsymbol{b})}^0 \boldsymbol{p}(t).\frac{\partial H}{\partial \boldsymbol{p}}(\boldsymbol{q}(t), \boldsymbol{p}(t))\,\mathrm{d}t\ . \end{aligned}$$

Then $S^{\pm}$ are continuous functions on U and C^3 on $U\backslash\{\mathbf{0}\}$.

Denote by

$$\begin{aligned} u^+(\boldsymbol{a}) &= \dot{\gamma}^+_{\boldsymbol{a}}\left(\tau^+(\boldsymbol{a})\right) \\ u^-(\boldsymbol{b}) &= \dot{\gamma}^-_{\boldsymbol{b}}\left(\tau^-(\boldsymbol{b})\right) \ , \end{aligned}$$

the tangent vectors to $\gamma^+_{\boldsymbol{a}}, \gamma^-_{\boldsymbol{b}}$ at the point $\mathbf{0}$. Let $\Sigma = \partial U$. Fix small $\xi > 0$ and let

$$X = \left\{(\boldsymbol{a}, \boldsymbol{b}) \in \Sigma^2 : \left\|u^+(\boldsymbol{a}) - u^-(\boldsymbol{b})\right\| \geq \xi\right\} \ .$$

Equivalently, a pair of points $(\boldsymbol{a}, \boldsymbol{b}) \in \Sigma$ belongs to X if the solution of the system with Hamiltonian function (2.5.2) with energy E connecting $\boldsymbol{a}$ to $\boldsymbol{b}$ does not pass too close to the centre $\mathbf{0}$. Let

$$Y = \left\{(\boldsymbol{a}, \boldsymbol{b}) \in X : \left\|u^+(\boldsymbol{a}) + u^-(\boldsymbol{b})\right\| \geq \xi\right\} \ . \tag{2.5.39}$$

Lemma 2.5.10. *There exists $\delta_0 > 0$ such that:*

- *for any $\delta \in (0, \delta_0]$ and $(\boldsymbol{a}, \boldsymbol{b}) \in X$, there exists a unique trajectory $\gamma = \gamma^\delta_{\boldsymbol{a},\boldsymbol{b}} : [0, \tau] \to U$ of energy E for the canonical Hamiltonian system with Hamiltonian function (2.5.1) connecting $\boldsymbol{a}$ to $\boldsymbol{b}$, i.e. $\gamma^\delta_{\boldsymbol{a},\boldsymbol{b}}(0) = \boldsymbol{a}$ and $\gamma^\delta_{\boldsymbol{a},\boldsymbol{b}}(\tau) = \boldsymbol{b}$.*

- *$\tau = \tau(\boldsymbol{a}, \boldsymbol{b}, \delta)$ is a C^2 function on $X \times (0, \delta_0]$ and $\tau(\boldsymbol{a}, \boldsymbol{b}, \delta) \to \tau^+(\boldsymbol{a}) + \tau^-(\boldsymbol{b})$ uniformly as $\delta \to 0$.*

- *$\gamma^\delta_{\boldsymbol{a},\boldsymbol{b}}|_{[0,\tau^+(\boldsymbol{a})]}(t) \to \gamma^+_{\boldsymbol{a}}(t)$ and $\gamma^\delta_{\boldsymbol{a},\boldsymbol{b}}|_{[\tau^-(\boldsymbol{b}),0]}(t + \tau) \to \gamma^-_{\boldsymbol{b}}(t)$ uniformly as $\delta \to 0$.*

- *the action of the trajectory γ*

$$S(\boldsymbol{a}, \boldsymbol{b}, \delta) = \int_0^\tau \boldsymbol{p}.\mathrm{d}\boldsymbol{q} = \int_\tau^0 \boldsymbol{p}(t).\frac{\partial H_\delta}{\partial \boldsymbol{p}}(\boldsymbol{q}(t), \boldsymbol{p}(t))\,\mathrm{d}t \tag{2.5.40}$$

  is a C^2 function on $X \times (0, \delta]$ and

$$S(\boldsymbol{a}, \boldsymbol{b}, \delta) = S^+(\boldsymbol{a}) + S^-(\boldsymbol{b}) + \delta s(\boldsymbol{a}, \boldsymbol{b}, \delta) \ , \tag{2.5.41}$$

  where s is uniformly C^2 bounded on X as $\delta \to 0$.

- *if, additionally, $(\boldsymbol{a},\boldsymbol{b}) \in Y$, then the trajectory $\gamma^{\delta}_{\boldsymbol{a},\boldsymbol{b}}$ does not pass too close to $\mathbf{0}$. The following inequality holds:*

$$\min_{0\le t\le \tau} \operatorname{dist}\left(\gamma^{\delta}_{\boldsymbol{a},\boldsymbol{b}}, \mathbf{0}\right) \ge c\delta \ , \qquad c>0 \ . \tag{2.5.42}$$

Without the condition $(\boldsymbol{a},\boldsymbol{b}) \in Y$ the lemma above still holds except for the last statement and the trajectory may pass through or close to $\mathbf{0}$.

The proof of lemma 2.5.10 is given in section 2.5.7. The main ingredients are the Levi-Civita regularization for binary collisions and the λ-Lemma 1.1.30.

2.5.5 Shadowing collision orbits

For any $k \in K$, let $\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k \in \Sigma$ be the two intersection points of γ_k with Σ. Then $\gamma_k(t) = \gamma^{-}_{\boldsymbol{\alpha}_k}\left(t-\tau^{-}\left(\boldsymbol{\alpha}_k\right)\right)$ for $0 \le t \le -\tau^{-}\left(\boldsymbol{\alpha}_k\right)$, $\gamma_k(t) = \gamma^{+}_{\boldsymbol{\beta}_k}\left(t+\tau_k-\tau^{+}\left(\boldsymbol{\beta}_k\right)\right)$ for $\tau_k - \tau^{+}\left(\boldsymbol{\beta}_k\right) \le t \le \tau_k$. Without loss of generality we assume that the points $\boldsymbol{\alpha}_k$ and $\boldsymbol{\beta}_k$ are not conjugated on the fixed energy level along γ_k for all $k \in K$. If not, we change the radius of the ball U a little to make the new intersection points non–conjugate.

Let $A_k \subset \Sigma$ be a small neighbourhood of $\boldsymbol{\alpha}_k$, $B_k \subset \Sigma$ a small neighbourhood of $\boldsymbol{\beta}_k$ and W_k a small neighbourhood of $\gamma_k\left([0,\tau_k]\right)$. We may assume that $W_k \cap \Sigma = A_k \cup B_k$ with A_k and B_k disjoint sets, making W_k smaller if necessary. If the neighbourhoods A_k, B_k and W_k are small enough, by the non–conjugacy of $\boldsymbol{\alpha}_k$ and $\boldsymbol{\beta}_k$ along γ_k and the implicit function theorem, for any $\boldsymbol{u} \in A_k$ and $\boldsymbol{v} \in B_k$, there exists a unique solution $\sigma = \sigma^{\delta}_{\boldsymbol{u},\boldsymbol{v}} : [0,\tau] \to W_k$, $\tau = \tau^{\delta}_{\boldsymbol{u},\boldsymbol{v}}$, of energy E for the system with Hamiltonian (2.5.1), such that $\sigma(0) = \boldsymbol{u}$ and $\sigma(\tau) = \boldsymbol{v}$, which is close to $\gamma_k\left(t-\tau^{-}\left(\boldsymbol{\alpha}_k\right)\right)$ for $0 \le t \le \tau$. This solution is a C^3 function of $\boldsymbol{u},\boldsymbol{v}$. Let $(\sigma(t),\psi(t))$ denote the path in phase space corresponding to the trajectory $\sigma(t)$ and define the action of the trajectory $\sigma^{\delta}_{\boldsymbol{u},\boldsymbol{v}}$ as

$$f^{\delta}\left(\boldsymbol{u},\boldsymbol{v}\right) = \int_0^{\tau} \psi(t).\frac{\partial H_\delta}{\partial \boldsymbol{p}}\left(\sigma(t),\psi(t)\right)\mathrm{d}t \ .$$

Then f^{δ} is a C^3 function on $A_k \times B_k$.

Define the following C^3 function on $A_k \times B_k$:

$$g^{\delta}_k\left(\boldsymbol{u},\boldsymbol{v}\right) = f^{\delta}\left(\boldsymbol{u},\boldsymbol{v}\right) + S^{-}\left(\boldsymbol{u}\right) + S^{+}\left(\boldsymbol{v}\right) \ . \tag{2.5.43}$$

Lemma 2.5.11. *The function $g_k^0(\boldsymbol{u},\boldsymbol{v}) = f^0(\boldsymbol{u},\boldsymbol{v}) + S^-(\boldsymbol{u}) + S^+(\boldsymbol{v})$ on $A_k \times B_k$ has a non–degenerate critical point at $z_k = (\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k)$*

Proof. The existence of a non–degenerate critical point of g_k^0 follows from non–degeneracy of γ_k. In fact, $g_k^0(\boldsymbol{u},\boldsymbol{v})$ is the action of the piecewise smooth trajectory of the Hamiltonian system determined by (2.5.2) obtained by gluing together the trajectories $\gamma_{\boldsymbol{u}}^-$, $\sigma_{\boldsymbol{u},\boldsymbol{v}}^0$ and $\gamma_{\boldsymbol{v}}^+$ with appropriate shifts of time parametrization and hence, g_k^0 is the restriction of the action functional (2.5.6) to a finite–dimensional submanifold consisting of broken trajectories, with break points $\boldsymbol{u}$ and $\boldsymbol{v}$, connecting the collision point $\mathbf{0}$ to itself. □

Choose coordinates $\boldsymbol{u},\boldsymbol{v}$ in $A_k \times B_k$. Lemma 2.5.11 implies that if the neighbourhoods A_k and B_k are small enough, there exists $C > 0$ such that

$$\left\|(g_k^{0\prime\prime}(s))^{-1}\right\| \leq C \ , \tag{2.5.44}$$

where $s = (\boldsymbol{u},\boldsymbol{v}) \in A_k \times B_k$ and $\|\cdot\|$ is the max norm in $A_k \times B_k$. Note the abuse of notation: we denote by $\boldsymbol{u},\boldsymbol{v}$ both the points in $A_k \times B_k$ and the corresponding coordinates that parametrize $A_k \times B_k$.

Let $G \subset K^2$ be the set (2.5.7). Taking the neighbourhoods A_k, B_k small enough, it can be assumed that for all $(k,l) \in G$, $B_k \times A_l \subset Y$ where Y is defined in (2.5.39).

The next result is a precise formulation of theorem 2.5.1. Assume that the neighbourhoods W_k are sufficiently small.

Theorem 2.5.12. *There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$ and any chain $(\gamma_{k_i})_{i\in\mathbb{Z}}$ of collision orbits there exists, up to a time shift, a unique trajectory $\gamma : \mathbb{R} \to (\cup_{k\in K} W_k) \setminus \{\mathbf{0}\}$ of energy E for the Hamiltonian system determined by (2.5.1) and a sequence*

$$... < a_i < b_i < a_{i+1} < b_{i+1} < ....$$

such that for all $i \in \mathbb{Z}$:

- $\gamma([a_i, b_i]) \subset W_{k_i}$, $\gamma(a_i) \in A_{k_i}$, $\gamma(b_i) \in B_{k_i}$;
- $\gamma([b_i, a_{i+1}]) \subset U$.

The asymptotic behaviour of this trajectory as $\delta \to 0$ is as follows:

- $b_i - a_i \to \tau_{k_i} - \tau^-(\boldsymbol{\alpha}_{k_i}) - \tau^+(\boldsymbol{\beta}_{k_i})$ *as* $\delta \to 0$;
- $\gamma(t)$ *is* $O(\delta)$*–close to* $\gamma_{k_i}([\tau^-(\boldsymbol{\alpha}_{k_i}), \tau - \tau^+(\boldsymbol{\beta}_{k_i})])$ *for* $a_i \le t \le b_i$;
- $\gamma(t) = \gamma^\delta_{\boldsymbol{q}(b_i),\boldsymbol{q}(a_{i+1})}(t - b_i)$ *for all* $t \in [b_i, a_{i+1}]$.

The constant δ_0 depends only on the set $\{\gamma_k\}_{k \in K}$ of collision orbits and is independent of the sequence $(k_i \in K)$. Thus, $\gamma(t)$ is $O(\delta)$–close to a chain of collision orbits. Furthermore, by inequality (2.5.42) in lemma 2.5.10 the trajectory avoids $\mathbf{0}$ by a distance of order δ.

Proof. This proof follows, up to some minor changes, the proof of a similar theorem in [8]. We give it here for completeness. The strategy is continuation from the case $\delta = 0$.

Given a sequence $(k_i)_{i\in\mathbb{Z}}$ with $(k_i, k_{i+1}) \in G$ for all $i \in \mathbb{Z}$, let

$$Y = \prod_{i\in\mathbb{Z}} A_{k_i} \times B_{k_i} \ ,$$

with supremum norm in the chosen charts on A_k and B_k. Then Y is a ball in the Banach space $Z = l_\infty$. Choose δ_0 as in lemma 2.5.10 and let $\delta \in (0, \delta_0]$. Then trajectories of the system (2.5.1) of energy E near the chain $(\gamma_{k_i})_{i\in\mathbb{Z}}$ correspond to critical points of the formal functional

$$F_\delta(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i\in\mathbb{Z}} f^\delta(\boldsymbol{u}_i, \boldsymbol{v}_i) + S(v_i, u_{i+1}, \delta) \ ,$$

over sequences $(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}_i, \boldsymbol{v}_i)_{i\in\mathbb{Z}} \in Y$, where S is defined in (2.5.40). Hence, $F_\delta(\boldsymbol{u}, \boldsymbol{v})$ is the action for the concatenation of trajectories $\sigma^\delta_{\boldsymbol{u}_i,\boldsymbol{v}_i}$ connecting $\boldsymbol{u}_i$ to $\boldsymbol{v}_i$ and trajectories $\gamma^\delta_{\boldsymbol{v}_i,\boldsymbol{u}_{i+1}}$ connecting $\boldsymbol{v}_i$ to $\boldsymbol{u}_{i+1}$ inside U. By (2.5.41) and (2.5.43), we obtain

$$F_\delta(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i\in\mathbb{Z}} g^\delta_{k_i}(\boldsymbol{u}_i, \boldsymbol{v}_i) + \delta s(v_i, u_{i+1}, \delta) \ .$$

Thus, trajectories of (2.5.1) of energy E correspond to zeros of the map $\phi_\delta = \nabla F_\delta : Y \to Z$ defined by

$$\begin{aligned} U_i &= \frac{\partial g^\delta_{k_i}}{\partial \boldsymbol{u}} (\boldsymbol{u}_i, \boldsymbol{v}_i) + \delta \frac{\partial s}{\partial \boldsymbol{u}} (\boldsymbol{v}_{i-1}, \boldsymbol{u}_i, \delta) \\ V_i &= \frac{\partial g^\delta_{k_i}}{\partial \boldsymbol{v}} (\boldsymbol{u}_i, \boldsymbol{v}_i) + \delta \frac{\partial s}{\partial \boldsymbol{v}} (\boldsymbol{v}_i, \boldsymbol{u}_{i+1}, \delta) \ . \end{aligned}$$

The sequences $(U, V) = (U_i, V_i)_{i \in \mathbb{Z}} \in Z$ represent jumps, measured in the supremum norm, in the components of momentum tangent to Σ at the points $\boldsymbol{u}_i, \boldsymbol{v}_i$, between the trajectories outside and inside the ball U. Since all the trajectories $\sigma^\delta_{\boldsymbol{u}_i, \boldsymbol{v}_i}$ and $\gamma^\delta_{\boldsymbol{v}_i, \boldsymbol{u}_{i+1}}$ have the same energy E, zero jumps in the tangential component of the momentum implies that the concatenation of these trajectories is a smooth curve and hence a solution of (2.5.1).

Since the function s is uniformly C^2 bounded as $\delta \to 0$, and the second derivative matrix of g^0_k is uniformly invertible, the chain of collision trajectories is a non–degenerate zero of ϕ_0 and the implicit function theorem gives a locally unique continuation for a range of δ independent of the sequence $(k_i)_{i \in \mathbb{Z}}$. More precisely, ϕ_δ has a unique zero $(\boldsymbol{u}, \boldsymbol{v}) \in Y$ provided that

$$\delta_0^{-1} > C \max_{(k,l) \in G} \max_{B_k \times A_l} \left\| s'' \right\| \ ,$$

where C is the constant in (2.5.44). □

2.5.6 Regularization of collisions

Without loss of generality one can replace W by $W - E$ and assume $E = 0$. We make $f(\mathbf{0}) = 1/4$ by rescaling δ. Let $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\boldsymbol{y} = (y_1, y_2) \in \mathbb{R}^2$ and consider the transformation g from $\mathbb{R}^4 \{\boldsymbol{x}, \boldsymbol{y}\}$ to $\mathbb{R}^4 \{\boldsymbol{q}, \boldsymbol{p}\}$, given by

$$\begin{aligned} q_x &= {x_1}^2 - {x_2}^2 \\ q_y &= 2 x_1 x_2 \\ p_x &= \frac{x_1 y_1 - x_2 y_2}{2 \left({x_1}^2 + {x_2}^2\right)} \\ p_y &= \frac{x_2 y_1 + x_1 y_2}{2 \left({x_1}^2 + {x_2}^2\right)} \ . \end{aligned} \tag{2.5.45}$$

Lemma 2.5.13 (Levi-Civita regularization)**.** *There exists a C^4 Hamiltonian on $\mathbb{R}^4\left\{\boldsymbol{x},\boldsymbol{y}\right\}$ given by*

$$\mathcal{H}\left(\boldsymbol{x},\boldsymbol{y}\right)=\frac{1}{2}\left|\boldsymbol{y}-\boldsymbol{B}\left(\boldsymbol{x}\right)\right|^2-\frac{\lambda}{2}\left|\boldsymbol{x}\right|^2+O_4\left(\boldsymbol{x},\boldsymbol{y}\right)\ , \tag{2.5.46}$$

where $\boldsymbol{B}\left(\boldsymbol{x}\right)=2W_3(0)\left(-\boldsymbol{b}.\boldsymbol{x},\boldsymbol{b}.\mathbf{J}\boldsymbol{x}\right)$ and $\lambda=4\left|\boldsymbol{b}\right|^2\left(W_3(0)\right)^2-8W(\mathbf{0},\mathbf{0})$, such that for $\boldsymbol{x}\neq\mathbf{0}$ the transformation g given in (2.5.45) takes trajectories of the canonical Hamiltonian system with Hamiltonian function $\mathcal{H}$ on the energy level $\mathcal{H}=\delta$ to trajectories of the canonical Hamiltonian system with Hamiltonian function H_δ on the energy level $H_\delta=0$.

Proof. Using (2.5.45), we obtain the following estimates

$$\begin{aligned}
g^*W_1\left(\boldsymbol{x}\right) &= W_1(\mathbf{0})+O_2\left(\boldsymbol{x}\right)\\
g^*W_i\left(\boldsymbol{x}\mathbf{J}\boldsymbol{y}\right) &= W_i\left(\frac{1}{2}\boldsymbol{x}\mathbf{J}\boldsymbol{y}\right)\\
&= W_i(0)+\boldsymbol{y}.O_1(\boldsymbol{x})\ ,\ i\in\{2,3\} \qquad (2.5.47)\\
g^*\left[\boldsymbol{a}.\boldsymbol{q}+\boldsymbol{b}.\boldsymbol{p}\right]\left(\boldsymbol{x},\boldsymbol{y}\right) &= \frac{1}{2\left|\boldsymbol{x}\right|^2}\boldsymbol{b}.\left(x_1y_1-x_2y_2,x_2y_1+x_1y_2\right)+O_2\left(\boldsymbol{x}\right)\ .
\end{aligned}$$

Using (2.5.45) and (2.5.47) in (2.5.1), we compute $\tilde{H}=g^*H_\delta$, which is given by

$$\begin{aligned}
\tilde{H} &= \frac{1}{8\left|\boldsymbol{x}\right|^2}\Big(\left|\boldsymbol{y}\right|^2+4\boldsymbol{b}.\left(x_1y_1-x_2y_2,x_2y_1+x_1y_2\right)W_3(0)\Big)\\
&\quad +W\left(\mathbf{0},\mathbf{0}\right)-\delta\frac{f\left(h(\boldsymbol{x})\right)}{\left|\boldsymbol{x}\right|^2}+O_2\left(\boldsymbol{x}\right)+\boldsymbol{y}.O_1(\boldsymbol{x})\ ,
\end{aligned} \tag{2.5.48}$$

where $h:\mathbb{R}^2\to\mathbb{R}^2$ is the squaring map given by

$$h(\boldsymbol{x})=\left({x_1}^2-{x_2}^2,2x_1x_2\right)\ .$$

Introducing the vector potential $\boldsymbol{B}\left(\boldsymbol{x}\right)$, given by

$$\boldsymbol{B}\left(\boldsymbol{x}\right)=2W_3(0)\left(-\boldsymbol{b}.\boldsymbol{x},\boldsymbol{b}.\mathbf{J}\boldsymbol{x}\right)$$

we get the equalities

$$\begin{aligned}
&\left|\boldsymbol{y}\right|^2+4\boldsymbol{b}.\left(x_1y_1-x_2y_2,x_2y_1+x_1y_2\right)W_3(0)=\\
&= \left|\boldsymbol{y}-\boldsymbol{B}\left(\boldsymbol{x}\right)\right|^2-\left|\boldsymbol{B}\left(\boldsymbol{x}\right)\right|^2\\
&= \left|\boldsymbol{y}-\boldsymbol{B}\left(\boldsymbol{x}\right)\right|^2-4(W_3(0))^2\left|\boldsymbol{b}\right|^2\left|\boldsymbol{x}\right|^2\ .
\end{aligned} \tag{2.5.49}$$

Multiply (2.5.48) by $|\boldsymbol{x}|^2 / f(h(\boldsymbol{x}))$ and use (2.5.49) and $f(h(\boldsymbol{x})) = f(\mathbf{0}) + O_2(\boldsymbol{x})$ to obtain a Hamiltonian function defined on $\mathbb{R}^4$, given by

$$\mathcal{H} = \frac{|\boldsymbol{x}|^2}{f(h(\boldsymbol{x}))}\tilde{H} + \delta = \frac{1}{2}|\boldsymbol{y} - \boldsymbol{B}(\boldsymbol{x})|^2 - \frac{\lambda}{2}|\boldsymbol{x}|^2 + O_4(\boldsymbol{x}, \boldsymbol{y}) \ ,$$

where $\lambda = 4|\boldsymbol{b}|^2 (W_3(0))^2 - 8W(\mathbf{0}, \mathbf{0})$, as given in the statement.

By construction of $\mathcal{H}$, we have that the energy levels $\{\mathcal{H} = \delta\}$ and $\{H_\delta = 0\}$ coincide, which finishes the proof. □

Standard formulations of lemma 2.5.13 and proofs can be found in [9, 23, 38].

The transformation g defined by (2.5.45) does not preserve the time parametrization of the solutions, but it preserves the actions

$$\int_\gamma \langle \boldsymbol{y}, \mathrm{d}\boldsymbol{x} \rangle = \int_{g(\gamma)} \langle \boldsymbol{p}, \mathrm{d}\boldsymbol{q} \rangle \ .$$

The Hamiltonian system (2.5.46), has an equilibrium point at the origin, with eigenvalues $\pm\sqrt{\lambda}$, and hence it is hyperbolic if and only if $\lambda > 0$. From now on, we assume that $\lambda > 0$. In this case the hyperbolic equilibrium $\mathbf{0}$ has two–dimensional stable and unstable manifolds $W^\pm_{\mathrm{loc}}$ [31]. Since $W^\pm_{\mathrm{loc}}$ are Lagrangian manifolds and project diffeomorphically to $\mathbb{R}^2\{\boldsymbol{x}\}$, they are defined by C^4 generating functions $s^\pm$ on a small ball U with centre $\mathbf{0} \in \mathbb{R}^2$. We have that

$$W^\pm_{\mathrm{loc}} = \left\{(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{y} = \mp\nabla s^\pm(\boldsymbol{x}) \ , \ \boldsymbol{x} \in U\right\} \ . \tag{2.5.50}$$

The functions $s^\pm$ have a non–degenerate minimum 0 at the point $\mathbf{0}$.

By the definition of $W^\pm_{\mathrm{loc}}$, for any point $\boldsymbol{a} \in U$ there exists a unique trajectory $\omega^+_{\boldsymbol{a}} : [0, +\infty) \to U$ such that $\lim_{t\to\infty} \omega^+_{\boldsymbol{a}}(t) = \mathbf{0}$ and $\omega^+_{\boldsymbol{a}}(0) = \boldsymbol{a}$. Similarly, there exists a unique trajectory $\omega^-_{\boldsymbol{a}} : (-\infty, 0] \to U$ such that $\lim_{t\to-\infty} \omega^-_{\boldsymbol{a}}(t) = \mathbf{0}$ and $\omega^-_{\boldsymbol{a}}(0) = \boldsymbol{a}$. By (2.5.50) the actions of these trajectories equal

$$\int_{\omega^\pm_{\boldsymbol{a}}} \langle \boldsymbol{y}, \mathrm{d}\boldsymbol{x} \rangle = s^\pm(\boldsymbol{a}) \ .$$

For the trajectories $\omega^\pm_{\boldsymbol{a}}(t)$, let $z^\pm_{\boldsymbol{a}}(t) \in W^\pm_{\mathrm{loc}}$ be the corresponding orbits in the phase space.

Lemma 2.5.14. *Let $T > 0$ be sufficiently large. Then for any points $\boldsymbol{a}, \boldsymbol{b} \in U$ and $\tau \geq T$:*

- *there exists a unique trajectory*

  $$\boldsymbol{z}(t) = (\boldsymbol{x}(t), \boldsymbol{y}(t)) = \boldsymbol{f}\left(\boldsymbol{a}, \boldsymbol{b}, \tau, t\right) \ , \ \ (\tau, t) \in D_T = \{(\tau, t) : \tau \geq T \ , 0 \leq t \leq \tau\} \ ,$$

  such that $\boldsymbol{x}(0) = \boldsymbol{a}$ and $\boldsymbol{x}(\tau) = \boldsymbol{b}$.

- *the map $\boldsymbol{f}$ is C^2 on $U^2 \times D_T$ and*

  $$\boldsymbol{f}\left(\boldsymbol{a}, \boldsymbol{b}, \tau, t\right) = z_{\boldsymbol{a}}^{+}(t) + z_{\boldsymbol{b}}^{-}(t - \tau) + e^{-\sqrt{\lambda}\tau} \phi\left(\boldsymbol{a}, \boldsymbol{b}, \tau, t\right) \ , \qquad (2.5.51)$$

  where ϕ is uniformly C^2 bounded on $U^2 \times D_T$.

- *the action*

  $$S\left(\boldsymbol{a}, \boldsymbol{b}, \tau\right) = \int_0^\tau \boldsymbol{y}.\mathrm{d}\boldsymbol{x}$$

  of the trajectory $\boldsymbol{z}(t)$ is C^2 on $U^2 \times [T, \infty)$ and

  $$S\left(\boldsymbol{a}, \boldsymbol{b}, \tau\right) = s^{+}\left(\boldsymbol{a}\right) + s^{-}\left(\boldsymbol{b}\right) + e^{-\sqrt{\lambda}\tau} R\left(\boldsymbol{a}, \boldsymbol{b}, \tau\right) + \tau h\left(\boldsymbol{a}, \boldsymbol{b}, \tau\right) \ , \qquad (2.5.52)$$

  where R is uniformly C^2 bounded as $\tau \to \infty$ and $h\left(\boldsymbol{a}, \boldsymbol{b}, \tau\right)$ is the energy of $\boldsymbol{z}$.

This result follows from lemma 1.1.31 in section 1.1.4. See [6] for the proof of (2.5.51) and [8] for the proof of (2.5.52).

The next result gives a useful representation for the energy function $h\left(\boldsymbol{a}, \boldsymbol{b}, \tau\right)$ used in (2.5.52). Similar statements can be found in [6, 7, 8].

Let $v^{\pm}\left(\boldsymbol{a}\right)$ denote the tangent vectors at $\mathbf{0}$ to the asymptotic trajectories $\omega_{\boldsymbol{a}}^{\pm}$.

Lemma 2.5.15. *The energy $h\left(\boldsymbol{a}, \boldsymbol{b}, \tau\right)$ of the trajectory $\boldsymbol{z}(t)$ is a C^2 function on $U^2 \times [T, +\infty)$ and has the form*

$$h\left(\boldsymbol{a}, \boldsymbol{b}, \tau\right) = e^{-\sqrt{\lambda}\tau} \left(h_0\left(\boldsymbol{a}, \boldsymbol{b}\right) + h_1\left(\boldsymbol{a}, \boldsymbol{b}, \tau\right)\right) \ ,$$

where

$$h_0\left(\boldsymbol{a}, \boldsymbol{b}\right) = 2 v^{+}\left(\boldsymbol{a}\right).v^{-}\left(\boldsymbol{b}\right) \ , \qquad v^{\pm}\left(\boldsymbol{a}\right) = \lim_{t \to \pm\infty} e^{\pm\sqrt{\lambda}t} \dot{\omega}_{\boldsymbol{a}}^{\pm}(t) \ , \qquad (2.5.53)$$

and $\|h_1\|_{C^2(U^2 \times [\tau, +\infty))} \to 0$ as $\tau \to +\infty$.

Proof. There exist local coordinates $(\boldsymbol{u},\boldsymbol{v}) \in \mathbb{R}^4$ in a neighbourhood of the equilibrium $\mathbf{0}$ such that $W^-_{\mathrm{loc}} = \{v=0\}$, $W^+_{\mathrm{loc}} = \{u=0\}$, and

$$\mathcal{H}(\boldsymbol{u},\boldsymbol{v}) = \sqrt{\lambda}\boldsymbol{u}.\boldsymbol{v}\,(1+O(\boldsymbol{u},\boldsymbol{v})) \ . \tag{2.5.54}$$

The symplectic transformation $(\boldsymbol{x},\boldsymbol{y}) \mapsto (\boldsymbol{u},\boldsymbol{v})$ is given by

$$\begin{aligned} \boldsymbol{x} &= \frac{\boldsymbol{u}-\sqrt{\lambda}\boldsymbol{v}}{\sqrt{2\lambda}} \\ \boldsymbol{y} &= \frac{\boldsymbol{u}+\sqrt{\lambda}\boldsymbol{v}}{\sqrt{2}} + \boldsymbol{B}\left(\frac{\boldsymbol{u}-\sqrt{\lambda}\boldsymbol{v}}{\sqrt{2\lambda}}\right) \ , \end{aligned} \tag{2.5.55}$$

where $\boldsymbol{B}(\boldsymbol{x})$ is as given in lemma 2.5.13. The Hamiltonian vector field on the unstable manifold W^-_{loc} takes the form

$$\dot{\boldsymbol{u}} = \sqrt{\lambda}\boldsymbol{u} + O_2(\boldsymbol{u}) \ ,$$

where the right hand side is of class C^3. This equation can be transformed [3] to a linear equation $\dot{\xi} = \sqrt{\lambda}\xi$ by a C^2 change of variables $\xi = f_-(u)$. Hence the phase flow on W^-_{loc} takes the form

$$g_{-t}(\boldsymbol{u},0) = \left(f_-^{-1}\left(e^{-\sqrt{\lambda}t} f_-(\boldsymbol{u})\right),0\right) = e^{-\sqrt{\lambda}t}\left(f_-(\boldsymbol{u}) + G(\boldsymbol{u},t),0\right) \ , \tag{2.5.56}$$

where $\|G\|_{C^2(V\times[T,+\infty))} \to 0$ uniformly on W^-_{loc} as $t \to +\infty$ and V is such that the preimage of U^2 under (2.5.55) is contained in V^2 and $V^2 \subset B_R(\mathbf{0}) \subset \mathbb{R}^4$, for some finite $R>0$.

A similar representation holds for the flow on the stable manifold W^+_{loc},

$$g_t(0,\boldsymbol{v}) = e^{-\sqrt{\lambda}t}\left(0, f_+(\boldsymbol{v}) + E(\boldsymbol{v},t)\right) \ , \tag{2.5.57}$$

where $\|E\|_{C^2(V\times[T,+\infty))} \to 0$ uniformly on W^+_{loc} as $t \to +\infty$. Furthermore, note that

$$\begin{aligned} (f_-(\boldsymbol{u}),0) &= \lim_{t\to+\infty} e^{\sqrt{\lambda}t} g_{-t}(\boldsymbol{u},0) \\ (0,f_+(\boldsymbol{v})) &= \lim_{t\to+\infty} e^{\sqrt{\lambda}t} g_t(0,\boldsymbol{v}) \ . \end{aligned} \tag{2.5.58}$$

Put $t=\tau/2$ in (2.5.51). By (2.5.56) and (2.5.57), we get

$$\boldsymbol{z}(\tau/2) = e^{-\sqrt{\lambda}\tau/2}\left(f_-(\boldsymbol{u}), f_+(\boldsymbol{v})\right) + e^{-\sqrt{\lambda}\tau/2} F(\boldsymbol{u},\boldsymbol{v},\tau) \ , \tag{2.5.59}$$

where $\boldsymbol{u} = \boldsymbol{u}(\tau)$ and $\boldsymbol{v} = \boldsymbol{v}(0)$, and $\|F\|_{C^2(V^2\times[T,+\infty))} \to 0$ as $\tau \to \infty$. Since $\mathcal{H}$ is a conserved quantity, substituting (2.5.59) into (2.5.54), we get the following estimate for the energy

$$\begin{aligned} h\left(\boldsymbol{a},\boldsymbol{b},\tau\right) &= \mathcal{H}\left(\boldsymbol{z}(\tau/2)\right) \\ &= \sqrt{\lambda}e^{-\sqrt{\lambda}\tau}\left(f_-(\boldsymbol{u}).f_+(\boldsymbol{v}) + h_1\left(\boldsymbol{u},\boldsymbol{v},\tau\right)\right) \ , \end{aligned} \tag{2.5.60}$$

where $\|h_1\|_{C^2(V^2\times[T,+\infty))} \to 0$ as $t \to +\infty$. Passing to the variables $\boldsymbol{x}, \boldsymbol{y}$ and using (2.5.58), we get

$$\begin{aligned} \lim_{t\to+\infty} e^{\sqrt{\lambda}t}\dot{\omega}_a^+(t) = \sqrt{\frac{\lambda}{2}}f_+(\boldsymbol{v}) &= v^+(\boldsymbol{a}) \\ \lim_{t\to-\infty} e^{-\sqrt{\lambda}t}\dot{\omega}_b^-(t) = \sqrt{\frac{\lambda}{2}}f_-(\boldsymbol{u}) &= v^-(\boldsymbol{b}) \ . \end{aligned} \tag{2.5.61}$$

Putting together (2.5.60) and (2.5.61) we get the required result. □

Take $\nu > 0$ and let $B = \left\{(\boldsymbol{a},\boldsymbol{b}) \in U^2 : h_0\left(\boldsymbol{a},\boldsymbol{b}\right) > \nu\right\}$. Then, for $(\boldsymbol{a},\boldsymbol{b}) \in B$, the function $h_0\left(\boldsymbol{a},\boldsymbol{b}\right)$ is bounded away from zero. Thus $h\left(\boldsymbol{a},\boldsymbol{b},\tau\right)$ is monotone in τ for sufficiently large τ. For small $\delta > 0$, solving the equation $h\left(\boldsymbol{a},\boldsymbol{b},\tau\right) = \delta$ for τ yields a C^2 function $\tau = \tau_\delta\left(\boldsymbol{a},\boldsymbol{b}\right)$. This, combined with the implicit function theorem and lemmas 2.5.14 and 2.5.15 gives the following result.

Proposition 2.5.16. *There exists $\delta_0 > 0$ such that for all $\delta \in (0,\delta_0]$ the following statements hold.*

- *For any $(\boldsymbol{a},\boldsymbol{b}) \in B$, there exists a unique trajectory $\boldsymbol{z}^\delta_{\boldsymbol{a},\boldsymbol{b}} = \left(\boldsymbol{x}^\delta_{\boldsymbol{a},\boldsymbol{b}}, \boldsymbol{y}^\delta_{\boldsymbol{a},\boldsymbol{b}}\right) : [0,\tau] \to U \times \mathbb{R}^2$ of energy δ connecting the points $\boldsymbol{a}$ and $\boldsymbol{b}$.*

- *the time $\tau = \tau_\delta\left(\boldsymbol{a},\boldsymbol{b}\right)$ is a C^2 function on B and*

$$\tau_\delta\left(\boldsymbol{a},\boldsymbol{b}\right) = -\frac{\log\delta}{\sqrt{\lambda}} + \mu\left(\boldsymbol{a},\boldsymbol{b},\delta\right) \ , \tag{2.5.62}$$

  where the function μ is uniformly C^2 bounded on B as $\delta \to 0$.

- *We have*

$$\boldsymbol{z}^\delta_{\boldsymbol{a},\boldsymbol{b}}(t) = \boldsymbol{z}^+_{\boldsymbol{a}}(t) + \boldsymbol{z}^-_{\boldsymbol{b}}(t-\tau) + \delta\zeta\left(\boldsymbol{a},\boldsymbol{b},\delta\right) \ , \tag{2.5.63}$$

where the function ζ is uniformly C^2 bounded as $\delta \to 0$.

- *The action $f_\delta(\boldsymbol{a},\boldsymbol{b}) = S(\boldsymbol{a},\boldsymbol{b},\tau_\delta(\boldsymbol{a},\boldsymbol{b},\delta))$ of the trajectory $\boldsymbol{z}^\delta_{\boldsymbol{a},\boldsymbol{b}}$ is a C^2 function on B and*

$$f_\delta(\boldsymbol{a},\boldsymbol{b}) = s^+(\boldsymbol{a}) + s^-(\boldsymbol{b}) + +\delta r(\boldsymbol{a},\boldsymbol{b},\delta) - \frac{\delta\log\delta}{\sqrt{\lambda}} , \tag{2.5.64}$$

where r is uniformly C^2 bounded on B as $\delta \to 0$.

See [6, 7, 8] for results similar to the proposition 2.5.16 above.

Lemma 2.5.17. *For any $(\boldsymbol{a},\boldsymbol{b}) \in B$, the trajectory $x^\delta_{\boldsymbol{a},\boldsymbol{b}}(t)$ avoids $\mathbf{0}$ provided $v^+(\boldsymbol{a}) \neq v^-(\boldsymbol{b})$. More precisely, the following equality holds.*

$$\min_{0\le t\le\tau}\left|x^\delta_{\boldsymbol{a},\boldsymbol{b}}(t)\right|^2 = \frac{2\delta}{\lambda}\left(\left|v^+(\boldsymbol{a})\right|\left|v^-(\boldsymbol{b})\right| - v^+(\boldsymbol{a}).v^-(\boldsymbol{b})\right)u(\boldsymbol{a},\boldsymbol{b},\delta) + o(\delta) , \tag{2.5.65}$$

where u is a positive function and uniformly C^2 bounded on B as $\delta \to 0$.

Proof. Let $(\boldsymbol{a},\boldsymbol{b}) \in B$. To leading order in δ, we have that

$$\begin{aligned} \omega^+_{\boldsymbol{a}}(t) &= \boldsymbol{a}\mathrm{e}^{-\sqrt{\lambda}t} \\ \omega^-_{\boldsymbol{b}}(t) &= \boldsymbol{b}\mathrm{e}^{\sqrt{\lambda}t} . \end{aligned}$$

By (2.5.53) we get

$$\begin{aligned} v^+(\boldsymbol{a}) &= -\sqrt{\lambda}\boldsymbol{a} \\ v^-(\boldsymbol{b}) &= \sqrt{\lambda}\boldsymbol{b} , \end{aligned} \tag{2.5.66}$$

Using (2.5.63), we get

$$x^\delta_{\boldsymbol{a},\boldsymbol{b}}(t) = \boldsymbol{a}\mathrm{e}^{-\sqrt{\lambda}t} + \boldsymbol{b}\mathrm{e}^{\sqrt{\lambda}(t-\tau)} + \delta\varsigma(\boldsymbol{a},\boldsymbol{b},\delta) , \tag{2.5.67}$$

where the function ς is uniformly C^2 bounded as $\delta \to 0$. From (2.5.67), we obtain

$$\begin{aligned} \left|x^\delta_{\boldsymbol{a},\boldsymbol{b}}(t)\right|^2 &= |\boldsymbol{a}|^2\mathrm{e}^{-2\sqrt{\lambda}t} + |\boldsymbol{b}|^2\mathrm{e}^{2\sqrt{\lambda}(t-\tau)} + 2\boldsymbol{a}.\boldsymbol{b}\mathrm{e}^{-\sqrt{\lambda}\tau} \\ &\quad +2\delta\left(\boldsymbol{a}\mathrm{e}^{-\sqrt{\lambda}t} + \boldsymbol{b}\mathrm{e}^{\sqrt{\lambda}(t-\tau)}\right).\varsigma(\boldsymbol{a},\boldsymbol{b},\delta) + O\left(\delta^2\right) . \end{aligned} \tag{2.5.68}$$

To leading order in δ, $\left|x^{\delta}_{\boldsymbol{a},\boldsymbol{b}}(t)\right|^2$ attains its minimum at

$$t' = \frac{\tau}{2} + \frac{1}{4\sqrt{\lambda}} \log \frac{|\boldsymbol{a}|^2}{|\boldsymbol{b}|^2} \ .$$

Thus, we get

$$\begin{aligned}\left|x^{\delta}_{\boldsymbol{a},\boldsymbol{b}}(t')\right|^2 &= 2\mathrm{e}^{-\sqrt{\lambda}\tau} \left(|\boldsymbol{a}|\,|\boldsymbol{b}| + \boldsymbol{a}.\boldsymbol{b}\right) + \\ &\quad +2\delta \mathrm{e}^{-\sqrt{\lambda}\tau/2} \sqrt{|\boldsymbol{a}|\,|\boldsymbol{b}|} \left(\frac{\boldsymbol{a}}{|\boldsymbol{a}|} + \frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right) .\varsigma(\boldsymbol{a},\boldsymbol{b},\delta) + O\left(\delta^2\right) \ .\end{aligned}$$

By (2.5.62), we obtain

$$\left|x^{\delta}_{\boldsymbol{a},\boldsymbol{b}}(t')\right|^2 = 2\delta \left(|\boldsymbol{a}|\,|\boldsymbol{b}| + \boldsymbol{a}.\boldsymbol{b}\right) u(\boldsymbol{a},\boldsymbol{b},\delta) + o\left(\delta\right) \ , \qquad (2.5.69)$$

where u is a positive function and uniformly C^2 bounded on B as $\delta \to 0$. Combining (2.5.66) and (2.5.69) we obtain the result. □

2.5.7 Proof of lemma 2.5.10

This next proof is based on proposition 2.5.16 and lemma 2.5.13 and follows the proof of the same result given in [8].

Proof. Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be the squaring map

$$h\left(x_1, x_2\right) = \left({x_1}^2 - {x_2}^2, 2x_1x_2\right) \ ,$$

corresponding to the first two arguments in the Levi-Civita transformation (2.5.45).

Collision solutions $\gamma^+_{h(\boldsymbol{a})} : [0, \tau^+\left(h\left(\boldsymbol{a}\right)\right)] \to U$ for the system (2.5.1) correspond, up to time reparametrization, to asymptotic orbits $\omega^+_{\boldsymbol{a}} : [0, +\infty) \to h^{-1}(U)$ to the equilibrium $\mathbf{0}$ for the system with Hamiltonian $\mathcal{H}$, given by (2.5.46), on the level set $\mathcal{H} = 0$. The map h take the trajectory $\omega^+_{\boldsymbol{a}} : [0, +\infty) \to h^{-1}(U)$ to the trajectory $\gamma^+_{h(\boldsymbol{a})}$: $[0, \tau^+\left(h\left(\boldsymbol{a}\right)\right)] \to U$, with changed time parametrization, i.e. $\gamma^+_{h(\boldsymbol{a})}(t) = \omega^+_{\boldsymbol{a}}\left(t_+(t)\right)$, where $t_+ : [0, \tau^+\left(h\left(\boldsymbol{a}\right)\right)] \to [0, +\infty)$. Similarly, $\gamma^-_{h(\boldsymbol{a})}(t) = \omega^-_{\boldsymbol{a}}\left(t_-(t)\right)$, where t_- : $[\tau^-\left(h\left(\boldsymbol{a}\right)\right), 0] \to (-\infty, 0]$. From the definition of the squaring map h, we obtain that for any $\boldsymbol{a} \in U$,

$$h\left(v^{\pm}(\boldsymbol{a})\right) = \mp\kappa u^{\pm}\left(h\left(\boldsymbol{a}\right)\right) \ ,$$

where $\kappa = |v^{\pm}(\boldsymbol{a})|^2/2$. By conservation of the action, we obtain $S^{\pm}(h(\boldsymbol{a})) = s^{\pm}(\boldsymbol{a})$.

Noting that $h(\boldsymbol{y}) = \alpha h(\boldsymbol{x})$, where $\alpha < 0$, is equivalent to $\boldsymbol{y} \perp \boldsymbol{x}$, we obtain that $u^{+}(h(\boldsymbol{a})) \neq u^{-}(h(\boldsymbol{b}))$ if and only if $v^{+}(\boldsymbol{a})$ and $v^{+}(\boldsymbol{b})$ are not orthogonal.

Take two points $\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}} \in \Sigma$ such that $\left| u^{+}(\tilde{\boldsymbol{a}}) - u^{-}(\tilde{\boldsymbol{b}}) \right| \geq \xi$. Then, there exist $\boldsymbol{a}, \boldsymbol{b} \in U$ such that $\tilde{\boldsymbol{a}} = h(\boldsymbol{a})$, $\tilde{\boldsymbol{b}} = h(\boldsymbol{b})$ and

$$h_0(\boldsymbol{a}, \boldsymbol{b}) = 2v^{+}(\boldsymbol{a}).v^{-}(\boldsymbol{b}) \geq \nu \ , \tag{2.5.70}$$

provided $\nu > 0$ is sufficiently small. By proposition 2.5.16 we can connect $\boldsymbol{a}$ to $\boldsymbol{b}$ by a trajectory $\boldsymbol{z}^{\delta}_{\boldsymbol{a},\boldsymbol{b}}$ of energy δ for the system (2.5.1). Under the squaring map h and an appropriate time reparametrization, $h\left(\boldsymbol{x}^{\delta}_{\boldsymbol{a},\boldsymbol{b}}\right)$ gives a trajectory of energy 0 for the regularized system with Hamiltonian $\mathcal{H}$ connecting $\tilde{\boldsymbol{a}} = h(\boldsymbol{a})$ to $\tilde{\boldsymbol{b}} = h(\boldsymbol{b})$. Since the action is invariant under the transformation h, (2.5.64) implies (2.5.41) with $s(h(\boldsymbol{a}), h(\boldsymbol{b}), \delta) = r(\boldsymbol{a}, \boldsymbol{b}, \delta)$. Condition (2.5.42) follows from lemma 2.5.17 and $|h(x)| = |x|^2$. □

2.6 Conclusions

We have proved that the Hamiltonian system (2.1.9) can always be reduced to one with two degrees of freedom. Moreover, we have proven that, for an interaction between the two charged particles determined by a Coulomb potential, with opposite sign charges (except for the case $\Omega_1 + \Omega_2 = 0$), the system contains a suspension of a non–trivial subshift of finite type for level sets of high energy. Thus, the system can not be reduced further in such regime. On the other hand the system is integrable for the special case of same sign charges when the particles have equal gyrofrequencies (equal ratio of charge to mass). The system is also integrable if the two charges sum to zero and the dynamics lie on the zero sets of the linear momenta. Furthermore, we explicitly computed the reduced Hamiltonian systems and corresponding reconstruction maps for the reduced dynamics, enabling us to lift the dynamics from the reduced spaces and hence obtain a description for the dynamics on the initial phase space. In particular we determined that the motion is bounded if the charges do not sum to zero but if they sum to zero there is an average drift velocity which depends on the energy and momenta.

It would be interesting to establish what happens when $\Omega_1 + \Omega_2 = 0$ and whether there is chaos for unequal gyrofrequencies of the same sign: presumably there is.

An interesting future work would be to prove an analogous result of non–integrability for the system (2.1.9) but with a logarithmic potential. This would have applications to the interaction of two vortices with masses, as was remarked in [15], where a proof in the limiting regime where the masses tend to zero was given with the help of some numerical computations.

Chapter 3

The spatial problem

In this chapter we analyse the interaction of two particles with non–zero charges moving in three–dimensional space under the action of a uniform magnetic field and an interaction potential depending on the distance between the particles. This problem looks important to plasma physics but it seems to have been given little attention so far. What attention it has received has tended to be in some limiting regimes such as very strong magnetic field or plasmas with all the particles of the same kind (see [2, 11, 12, 33]) or with one heavy particle idealized as fixed (the diamagnetic Kepler Problem, see [18, 37]) or the planar case of chapter 2. We will study the dynamics of two charged particles in a uniform magnetic field without making restrictions on the sizes of the magnetic field, the charges or the masses, except that we will assume that the particles behave classically and that their velocities and accelerations are small enough that we can neglect any relativistic and radiation effects. Although it is well known that non–uniformity of the magnetic field introduces further significant effects, we believe that there is value in establishing firm results for the uniform case.

In chapter 2 we made a detailed study of the problem of the interaction of two particles with non–zero charge moving in a plane under the effect of a uniform magnetic field. We assumed that the interaction between the particles was given by a potential depending on the distance between the two particles and that the magnetic field was orthogonal to the plane of motion. That problem can be formulated as a Hamiltonian system with four degrees of freedom. We made extensive use of the symmetries in that Hamiltonian system to obtain a reduction in the dimension of the problem to two degrees of freedom. In the special case of same sign charges with equal gyrofrequencies (equal ratio of charge to mass) or on some special submanifolds we proved that this system is integrable. We then specialized our analysis to the more physically interesting case of a Coulomb–like potential. Analysing the reduced systems and the associated reconstruction maps we provided a detailed description for the regimes of parameters and level sets of the conserved quantities where bounded and unbounded motion are possible and we identified the cases where close approaches between the two particles are possible. Furthermore, we identified regimes where the system is non–integrable and contains chaos by proving the existence of invariant subsets containing a suspension of a non–trivial subshift.

The motion of one particle moving in three–dimensional space under the action of a uniform magnetic field is simple. It is the composition of two motions: a drift with constant velocity in the direction of the magnetic field and a uniform rotation of fixed centre - the guiding centre, and radius - gyroradius, with constant angular velocity - gyrofrequency, in a plane orthogonal to the magnetic field. Choosing the magnetic field to be vertical and oriented upwards, the motion in the circle is clockwise if the charge is positive and anticlockwise otherwise. We sign the gyrofrequency according to the direction of rotation. This problem can be formulated as a three degrees of freedom Hamiltonian system. It has a four–dimensional subgroup of the Special Euclidean group of $\mathbb{R}^3$ as its symmetry group (three–dimensional translations and a one–dimensional rotation). These symmetries induce conserved quantities for this system which is easily seen to be integrable.

One of the main goals of this chapter is to study the scattering problem associated with the interaction of the two charges in the presence of a magnetic field and a

Coulomb interaction potential:

$$V(R) = \frac{e_1 e_2}{4\pi\epsilon_0}\frac{1}{R}\ , \tag{3.0.1}$$

where R denotes the distance between the two particles, e_1 and e_2 denote the values of the charges and ϵ_0 denotes the permittivity of the vacuum. If there is a large distance between the particles then the interaction is negligible and in this case the two particles move freely as described above. If the distance between the two particles is small then the strength of the interaction can not be neglected anymore and the particles interact. We will be looking for a situation where the particles have initially a large separation and both move freely towards each other so that the particles start interacting when they get closer and then start moving apart until both particles move again like free particles. The goal is to describe the changes in the particles trajectories due to this interaction. For more details on scattering in classical mechanics see the review [30] and references therein.

In section 3.1 we formulate our problem as a Hamiltonian system with a non–canonical symplectic form (see [22]), that makes easier to identify the system symmetries. We identify translational and rotational symmetries of the system and the corresponding conserved quantities. Furthermore, we prove the existence of an exceptional conserved quantity when the two particles have the same gyrofrequency.

We start section 3.2 by proving that the problem of the interaction of two particles in a magnetic field can be reduced to one with three degrees of freedom. Furthermore, when the two particles have the same gyrofrequency we use the exceptional conserved quantity to prove that the system reduces to two degrees of freedom in this case. We also prove that if the sum of the two charges is zero the dynamics in the zero sets of the linear momenta are described by a two degrees of freedom Hamiltonian system. We do these by constructing a set of coordinates on which the system exhibits a reduction to three degrees of freedom, and two degrees of freedom when it applies. We should remark that this reduction is an extension to the three–dimensional space of similar reductions obtained for the planar case in chapter 2 (and for a similar problem in [15]). The total change of coordinates that exhibits the reduction is computed. This change of coordinates is just the lift of a $SE(3)$ subgroup that, given the base dynamics of the reduced Hamiltonian systems, enables us to describe the full twelve–dimensional dynamics.

In section 3.3, we specialize our analysis of the problem by choosing a specific interaction potential. The natural choice for the potential V is the Coulomb potential (3.0.1). We give a description of the reduced Hamiltonian systems obtained in section 3.2 with the generic potential V replaced by the Coulomb potential, including:

- existence of an invariant plane where the reduced dynamics are just the reduced dynamics associated with the interaction of two particles moving in a plane under the action of a uniform magnetic field (orthogonal to the plane of motion).
- boundedness of some of the variables on the reduced space.
- existence of regimes of parameters where close approaches between the particles are possible.

Using this information we obtain a classification of the various distinct types of behaviour in this system: "planar", "molecule–like", "bouncing–back" and "unbounded".

We start section 3.4 with an analysis of the scattering map associated with this problem in the limit where the two particles trajectories are widely separated. We obtain that the magnetic moment of the particles is conserved and that the guiding centres have the following dynamics:

i) in the case of two charges whose sum is not zero, the guiding centres rotate about a fixed centre during an interaction,

ii) in the case of two charges which sum to zero, the guiding centres drift in a direction determined by the conserved quantities.

The results obtained in this limit agree with the more general qualitative description provided in section 3.3. Furthermore, we prove that in the case of "bouncing–back" behaviour there is a transfer of vertical kinetic energy between the particles. We finish this section with a numerical study of the scattering map without using the assumption that the two particles trajectories are widely separated. We observe regular behaviour for large energies and chaotic scattering for small positive energies.

3.1 Problem formulation

3.1.1 One charged particle in a magnetic field

We start by considering the well understood case of one particle moving in $\mathbb{R}^3$ under the action of a uniform magnetic field $\boldsymbol{B} = (0,0,B)$. A particle of mass $m > 0$ and non–zero charge e under the action of such a field is subject to a Lorentz force of the form $\boldsymbol{F_L} = eBc^{-1}\boldsymbol{v}\times\boldsymbol{B}$ where $\boldsymbol{v} = (v_x, v_y, v_z) \in \mathbb{R}^3$ is the particle velocity and $\times$ denotes the exterior product between vectors of $\mathbb{R}^3$. This system is known to be Hamiltonian with Hamiltonian function and (non–canonical) symplectic form, given by

$$\begin{aligned} H &= \frac{1}{2}m\left|\boldsymbol{v}\right|^2 \\ \omega &= m\;\mathrm{d}x\wedge\mathrm{d}v_x + m\;\mathrm{d}y\wedge\mathrm{d}v_y + m\;\mathrm{d}z\wedge\mathrm{d}v_z - \frac{eB}{c}\;\mathrm{d}x\wedge\mathrm{d}y\;. \end{aligned} \tag{3.1.1}$$

where $\boldsymbol{x} = (x,y,z) \in \mathbb{R}^3$ denotes the particle position (see [22]). To put the Hamiltonian system given by (3.1.1) into canonical form it is common to introduce the canonical coordinates $\boldsymbol{q} = (q_x, q_y, q_z) \in \mathbb{R}^3$ and $\boldsymbol{p} = (p_x, p_y, p_z) \in \mathbb{R}^3$, given by

$$\begin{aligned} \boldsymbol{q} &= \boldsymbol{x} \\ \boldsymbol{p} &= m\boldsymbol{v} + \frac{e}{c}\boldsymbol{A}\left(\boldsymbol{x}\right)\;, \end{aligned} \tag{3.1.2}$$

where $\boldsymbol{A}\left(\boldsymbol{x}\right) = \left(A_x\left(\boldsymbol{x}\right), A_y\left(\boldsymbol{x}\right), 0\right) \in \mathbb{R}^3$ is a vector potential for $\boldsymbol{B}$. The new Hamiltonian system is then given by

$$\begin{aligned} H &= \frac{1}{2m}\left|\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\left(\boldsymbol{q}\right)\right|^2 \\ \omega &= \mathrm{d}q_x\wedge\mathrm{d}p_x + \mathrm{d}q_y\wedge\mathrm{d}p_y + \mathrm{d}q_z\wedge\mathrm{d}p_z - \frac{e}{c}\left(\frac{\partial A_x}{\partial q_y} - \frac{\partial A_y}{\partial q_x} + B\right)\mathrm{d}q_x\wedge\mathrm{d}q_y\;. \end{aligned}$$

Hence, for the system to be canonical the vector field $\boldsymbol{A}(\boldsymbol{x})$ must be chosen to verify the equation

$$\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} + B = 0 \ ,$$

which is indeed the condition for $\boldsymbol{A}(\boldsymbol{x})$ to be a vector potential for $\boldsymbol{B}$. If needed, we make the choice $\boldsymbol{A}(\boldsymbol{x}) = -\frac{B}{2}(y, -x, 0)$. We consider it better, however, to use the formulation (3.1.1) because translation symmetry is more transparent, so instead of the change of variables (3.1.2) we just make the change of variables given by

$$\begin{aligned} \boldsymbol{q} &= \boldsymbol{x} \\ \boldsymbol{p} &= m\boldsymbol{v} \end{aligned} \tag{3.1.3}$$

obtaining the Hamiltonian system

$$\begin{aligned} H &= \frac{1}{2m}|\boldsymbol{p}|^2 \\ \omega &= \mathrm{d}q_x \wedge \mathrm{d}p_x + \mathrm{d}q_y \wedge \mathrm{d}p_y + \mathrm{d}q_z \wedge \mathrm{d}p_z + k\ \mathrm{d}q_x \wedge \mathrm{d}q_y \ , \end{aligned} \tag{3.1.4}$$

where

$$k = -\frac{eB}{c} \ .$$

The symplectic form in (3.1.4) defines a Poisson bracket $\{.,.\} : C^\infty(\mathbb{R}^6) \times C^\infty(\mathbb{R}^6) \to C^\infty(\mathbb{R}^6)$ given by

$$\begin{aligned} \{F, G\} &= \frac{\partial F}{\partial q_x}\frac{\partial G}{\partial p_x} - \frac{\partial G}{\partial q_x}\frac{\partial F}{\partial p_x} + \frac{\partial F}{\partial q_y}\frac{\partial G}{\partial p_y} - \frac{\partial G}{\partial q_y}\frac{\partial F}{\partial p_y} + \frac{\partial F}{\partial q_z}\frac{\partial G}{\partial p_z} - \frac{\partial G}{\partial q_z}\frac{\partial F}{\partial p_z} \\ &\quad -k\left(\frac{\partial F}{\partial p_x}\frac{\partial G}{\partial p_y} - \frac{\partial G}{\partial p_x}\frac{\partial F}{\partial p_y}\right) \ . \end{aligned}$$

In the formulation (3.1.4) the Lorentz force effect can not be seen in the Hamiltonian function but it is present in the $k\ dq_x \wedge dq_y$ term of the symplectic form and equivalent term in the Poisson bracket.

3.1.2 Two charged particles in a magnetic field

We consider two particles with positive masses m_1 and m_2 and non–zero charges e_1 and e_2, respectively, moving in $\mathbb{R}^3$ under the action of a uniform magnetic field $\boldsymbol{B} = (0, 0, B)$.

Each one of the particles moving under the action of such a field is subject to a Lorentz force of the form $\boldsymbol{F_L} = e_i B c^{-1} \boldsymbol{v}_i \times \boldsymbol{B}$ where $\boldsymbol{v}_i = (v_{x_i}, v_{y_i}, v_{z_i}) \in \mathbb{R}^3$ is the i-th particle velocity ($i \in \{1,2\}$) and $\times$ denotes the exterior product between vectors of $\mathbb{R}^3$. Furthermore, we assume that the two particles interaction is determined by a potential $V(r)$ depending on the distance r between the two particles.

The phase space M for this problem is $\mathbb{R}^{12}$ with the singular points of the interaction potential removed (nine–dimensional planes if V is the Coulomb potential (3.0.1)). Let $\boldsymbol{q_i} = (q_{x_i}, q_{y_i}, q_{z_i}) \in \mathbb{R}^3$ denote the vector position of the i-th particle and $\boldsymbol{p_i} = (p_{x_i}, p_{y_i}, p_{z_i}) \in \mathbb{R}^3$ denote its (non–conjugated) momentum

$$\boldsymbol{p}_i = m\boldsymbol{v}_i \ , \qquad i \in \{1,2\} \ .$$

The motion of the two particles can be described by a Hamiltonian system, with Hamiltonian function $H : M \longrightarrow \mathbb{R}$ and non–canonical symplectic form ω (see [22]), given by

$$\begin{aligned} H &= \frac{1}{2m_1} |\boldsymbol{p}_1|^2 + \frac{1}{2m_2} |\boldsymbol{p}_2|^2 + V(|\boldsymbol{q}_1 - \boldsymbol{q}_2|) \\ \omega &= \sum_{i=1,2} \mathrm{d}q_{x_i} \wedge \mathrm{d}p_{x_i} + \mathrm{d}q_{y_i} \wedge \mathrm{d}p_{y_i} + \mathrm{d}q_{z_i} \wedge \mathrm{d}p_{z_i} + k_i \ \mathrm{d}q_{x_i} \wedge \mathrm{d}q_{y_i} \ , \end{aligned} \tag{3.1.5}$$

where, for simplicity of notation, we introduce the constants

$$k_i = -\frac{e_i B}{c} \ , \qquad i \in \{1,2\} \ .$$

The Poisson bracket associated with this symplectic form, $\{.,.\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, is given by

$$\begin{aligned} \{F,G\} \ = \ & \sum_{i=1,2} \frac{\partial F}{\partial q_{x_i}} \frac{\partial G}{\partial p_{x_i}} - \frac{\partial G}{\partial q_{x_i}} \frac{\partial F}{\partial p_{x_i}} + \frac{\partial F}{\partial q_{y_i}} \frac{\partial G}{\partial p_{y_i}} - \frac{\partial G}{\partial q_{y_i}} \frac{\partial F}{\partial p_{y_i}} \\ & + \frac{\partial F}{\partial q_{z_i}} \frac{\partial G}{\partial p_{z_i}} - \frac{\partial G}{\partial q_{z_i}} \frac{\partial F}{\partial p_{z_i}} - k_i \left(\frac{\partial F}{\partial p_{x_i}} \frac{\partial G}{\partial p_{y_i}} - \frac{\partial G}{\partial p_{x_i}} \frac{\partial F}{\partial p_{y_i}} \right) \ . \end{aligned}$$

The Hamiltonian system defined by (3.1.5) is invariant under the following families of symmetries

$$\begin{aligned} \phi_{\boldsymbol{v}}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{p}_1, \boldsymbol{p}_2) &= (\boldsymbol{q}_1 + \boldsymbol{v}, \boldsymbol{q}_2 + \boldsymbol{v}, \boldsymbol{p}_1, \boldsymbol{p}_2) \\ \phi_\theta(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{p}_1, \boldsymbol{p}_2) &= (R_\theta \boldsymbol{q}_1, R_\theta \boldsymbol{q}_2, R_\theta \boldsymbol{p}_1, R_\theta \boldsymbol{p}_2) \ , \end{aligned}$$

where $\boldsymbol{v} = (v_x, v_y, v_z) \in \mathbb{R}^3$ is a translation vector and R_θ is the matrix representing a rotation in $\mathbb{R}^3$ of angle θ about the z axis and is given by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} . \qquad (3.1.6)$$

We define the (signed) gyrofrequency Ω_i of each particle as

$$\Omega_i = \frac{k_i}{m_i} , \qquad i \in \{1, 2\} ,$$

and introduce the notation $\overline{\mathbf{J}}$ and $\mathbf{I}$ for the 3×3 matrices given by

$$\overline{\mathbf{J}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Proposition 3.1.1. *The Hamiltonian system* (3.1.5) *has the following conserved quantities:*

- *Linear momentum, given by*

$$\boldsymbol{P}_3 = (P_x, P_y, P_z) = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \overline{\mathbf{J}} \left(k_1 \boldsymbol{q}_1 + k_2 \boldsymbol{q}_2 \right) .$$

- *Angular momentum, given by*

$$L = \sum_{i=1,2} \boldsymbol{q}_i . \overline{\mathbf{J}} \boldsymbol{p}_i - \frac{k_i}{2} \left| \mathbf{I} \, \boldsymbol{q}_i \right|^2 .$$

Furthermore, if the particles have equal gyrofrequencies $\Omega_1 = \Omega_2$, there exists another conserved quantity W, given by

$$W = \left| \mathbf{I} \left(\boldsymbol{p}_1 + \boldsymbol{p}_2 \right) \right|^2 .$$

The following commutation relations between the conserved quantities given above hold:

$$\begin{aligned} &\{P_x, P_y\} = k_1 + k_2 , && \{P_x, P_z\} = 0 , && \{P_y, P_z\} = 0 , \\ &\{L, P_x\} = P_y , && \{L, P_y\} = -P_x , && \{L, P_z\} = 0 , \\ &\{W, P_x\} = 0 , && \{W, P_y\} = 0 , && \{W, P_z\} = 0 , \quad \{W, L\} = 0 . \end{aligned}$$

For a proof of a similar statement (with the two particles moving on a plane) see proposition 2.1.2.

We will use the notation

$$\boldsymbol{P} = (P_x, P_y) \in \mathbb{R}^2$$

for the (x, y)–components of the linear momentum $\boldsymbol{P}_3$ given in proposition 3.1.1 and will use the notation P_z when referring to its z–component.

Remarks We note that:

i) the conserved quantities $\boldsymbol{P}_3$ and L are, respectively, the usual linear and angular momenta for the two body problem with extra terms for the magnetic field representing the presence of the magnetic field and hence the effect of the Lorentz force on the particles.

ii) combining P_x and P_y into the conserved quantity

$$P = |\boldsymbol{P}|^2 = P_x^2 + P_y^2$$

we obtain the following commutation relations

$$\{P, L\} = 0 \ , \quad \{P, P_z\} = 0 \ , \quad \{P, W\} = 0 \ ,$$

which together with the commutation relations in proposition 3.1.1 show L, P, P_z and W to be in involution.

iii) corresponding to W there is a "hidden" symmetry in the case of equal gyrofrequencies $\Omega_1 = \Omega_2$, given by

$$\begin{aligned}
\boldsymbol{q}_1 &\rightarrow \boldsymbol{q}_1 + \frac{1}{k_1 + k_2} \left[R_{2(k_1+k_2)\phi} - \mathbf{Id}_{3\times 3}\right] \overline{\mathbf{J}}(\boldsymbol{p}_1 + \boldsymbol{p}_2) \\
\boldsymbol{q}_2 &\rightarrow \boldsymbol{q}_2 + \frac{1}{k_1 + k_2} \left[R_{2(k_1+k_2)\phi} - \mathbf{Id}_{3\times 3}\right] \overline{\mathbf{J}}(\boldsymbol{p}_1 + \boldsymbol{p}_2) \\
\boldsymbol{p}_1 &\rightarrow \boldsymbol{p}_1 + \frac{k_1}{k_1 + k_2} \left[R_{2(k_1+k_2)\phi} - \mathbf{Id}_{3\times 3}\right] (\boldsymbol{p}_1 + \boldsymbol{p}_2) \\
\boldsymbol{p}_2 &\rightarrow \boldsymbol{p}_2 + \frac{k_2}{k_1 + k_2} \left[R_{2(k_1+k_2)\phi} - \mathbf{Id}_{3\times 3}\right] (\boldsymbol{p}_1 + \boldsymbol{p}_2) \ ,
\end{aligned}$$

where $\phi \in \mathbb{R}$, $\mathbf{Id}_{3\times 3}$ is the identity matrix in $\mathbb{R}^3$ and $R_{2(k_1+k_2)\phi}$ is of the form (3.1.6) with θ replaced by $2(k_1 + k_2)\phi$.

iv) if the interaction potential in (3.1.5) is chosen to be the Coulomb potential (3.0.1) (as we will do in section 3.3) then the scaling transformation given by

$$\begin{aligned} \overline{\boldsymbol{q}}_i &= \lambda \boldsymbol{q}_i \\ \bar{t} &= \lambda^{3/2} t \\ \overline{B} &= \lambda^{-3/2} B \ , \end{aligned}$$

where $\lambda > 0$, transforms the Hamiltonian function and symplectic form (3.1.5) to $\overline{H} = \lambda^{-1} H$ and $\overline{\omega} = \lambda^{1/2}\omega$. We could then choose λ so that $\overline{B} = 1$ by a rescaling of the level sets of the Hamiltonian function in (3.1.5). Furthermore, choosing e_1 and m_1 to be units of charge and mass, respectively, we could further reduce the number of parameters of (3.1.5) by two. The Hamiltonian system (3.1.5) would then depend only on the charge e_2, mass m_2 and physical constants c and ϵ_0.

v) if the particles were also under the action of an electric field $\boldsymbol{E}$ perpendicular to the magnetic field $\boldsymbol{B}$ then the particles would drift with constant velocity given by

$$\boldsymbol{u} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{|\boldsymbol{B}|^2} \ .$$

Changing to a moving frame with velocity $\boldsymbol{u}$, we reduce this problem to one of the form (3.1.5).

3.2 Reduction

In this section we provide local coordinates that exhibit the reduction of the Hamiltonian system (3.1.5) to three degrees of freedom and identify the regime of parameters and invariant subsets of $\mathbb{R}^{12}$ where the system can be reduced to two degrees of freedom. This reduction is valid for all potentials V which depend only on the distance between the particles. For simplicity of notation we introduce the combinations

$$\begin{aligned} M &= m_1 + m_2 \\ m &= \frac{m_1 m_2}{m_1 + m_2} \ . \end{aligned}$$

We change coordinates from positions $\boldsymbol{q}_i$ and momentum $\boldsymbol{p}_i$ to guiding centre $\boldsymbol{R}_i = (R_{x_i}, R_{y_i}) \in \mathbb{R}^2$, gyroradius $\boldsymbol{\rho}_i = (\rho_{x_i}, \rho_{y_i}) \in \mathbb{R}^2$, relative vertical position $q_z \in \mathbb{R}$ and a conjugate momentum $p_z \in \mathbb{R}$, vertical centre of mass $C_z \in \mathbb{R}$ and vertical linear momentum $P_z \in \mathbb{R}$ by making

$$\begin{aligned} \boldsymbol{\rho}_i &= \tfrac{1}{k_i}\mathbf{J}\left(p_{x_i}, p_{y_i}\right) & \boldsymbol{R}_i &= \left(q_{x_i}, q_{y_i}\right) - \boldsymbol{\rho}_i \\ q_z &= q_{z_1} - q_{z_2} & p_z &= \left(m_2 p_{z_1} - m_1 p_{z_2}\right)/M \\ C_z &= \left(m_1 q_{z_1} + m_2 q_{z_2}\right)/M & P_z &= p_{z_1} + p_{z_2} \ , \end{aligned} \tag{3.2.1}$$

where $\mathbf{J}$ is the standard symplectic matrix in $\mathbb{R}^2$, given by

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ .$$

Then, the Hamiltonian system (3.1.5) transforms to

$$\begin{aligned} H &= \frac{k_1\Omega_1}{2}\left|\boldsymbol{\rho}_1\right|^2 + \frac{k_2\Omega_2}{2}\left|\boldsymbol{\rho}_2\right|^2 + V(R) + \frac{{p_z}^2}{2m} + \frac{{P_z}^2}{2M} \\ \omega &= \sum_{i=1,2} k_i\left(\mathrm{d}R_{x_i}\wedge\mathrm{d}R_{y_i} - \mathrm{d}\rho_{x_i}\wedge\mathrm{d}\rho_{y_i}\right) + \mathrm{d}q_z\wedge\mathrm{d}p_z + \mathrm{d}C_z\wedge\mathrm{d}P_z \ , \end{aligned} \tag{3.2.2}$$

where $R = \left(\left|\boldsymbol{R}_1 - \boldsymbol{R}_2 + \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2\right|^2 + {q_z}^2\right)^{1/2}$. This coordinate change reduces (3.1.5) by one degree of freedom by conservation of P_z and elimination of C_z. The quantities $\boldsymbol{P}$, L and W are now given by

$$\begin{aligned} \boldsymbol{P} &= \mathbf{J}\left(k_1\boldsymbol{R}_1 + k_2\boldsymbol{R}_2\right) \\ L &= \sum_{i=1,2}\frac{k_i}{2}\left(\left|\boldsymbol{\rho}_i\right|^2 - \left|\boldsymbol{R}_i\right|^2\right) \\ W &= \left|k_1\boldsymbol{\rho}_1 + k_2\boldsymbol{\rho}_2\right|^2 \ . \end{aligned}$$

We separate our analysis into two cases: $k_1 + k_2 \neq 0$ and $k_1 + k_2 = 0$.

3.2.1 Case $k_1 + k_2 \neq 0$

We introduce the combinations

$$\begin{aligned} \mu &= k_1 + k_2 \\ e &= \frac{k_1 k_2}{k_1 + k_2} \ , \end{aligned}$$

and note that since $k_1 + k_2 \neq 0$ then μ is non–zero and e is well–defined. We introduce the planar relative positions $\boldsymbol{q} = (q_x, q_y) \in \mathbb{R}^2$ and a conjugate momentum $\boldsymbol{p} = (p_x, p_y) \in \mathbb{R}^2$, by making the change of coordinates

$$\begin{aligned} \boldsymbol{q} &= \boldsymbol{R}_1 - \boldsymbol{R}_2 + \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2 \\ \boldsymbol{p} &= \frac{e}{2}\mathbf{J}\left(\boldsymbol{R}_1 - \boldsymbol{R}_2 - \boldsymbol{\rho}_1 + \boldsymbol{\rho}_2\right) \\ \boldsymbol{P} &= \mathbf{J}\left(k_1 \boldsymbol{R}_1 + k_2 \boldsymbol{R}_2\right) \\ \boldsymbol{f} &= -\mathbf{J}\left(k_1 \boldsymbol{\rho}_1 + k_2 \boldsymbol{\rho}_2\right) \ , \end{aligned}$$

where $\boldsymbol{f} = (f_x, f_y) \in \mathbb{R}^2$ and $\boldsymbol{P} = (P_x, P_y) \in \mathbb{R}^2$. The inverse transformation is given by

$$\begin{aligned} \boldsymbol{R}_1 &= \frac{1}{\mu}\left[\frac{k_2}{2}\left(\boldsymbol{q} - \frac{2}{e}\mathbf{J}\boldsymbol{p}\right) - \mathbf{J}\boldsymbol{P}\right] \\ \boldsymbol{R}_2 &= \frac{1}{\mu}\left[-\frac{k_1}{2}\left(\boldsymbol{q} - \frac{2}{e}\mathbf{J}\boldsymbol{p}\right) - \mathbf{J}\boldsymbol{P}\right] \\ \boldsymbol{\rho}_1 &= \frac{1}{\mu}\left[\frac{k_2}{2}\left(\boldsymbol{q} + \frac{2}{e}\mathbf{J}\boldsymbol{p}\right) + \mathbf{J}\boldsymbol{f}\right] \\ \boldsymbol{\rho}_2 &= \frac{1}{\mu}\left[-\frac{k_1}{2}\left(\boldsymbol{q} + \frac{2}{e}\mathbf{J}\boldsymbol{p}\right) + \mathbf{J}\boldsymbol{f}\right] \ . \end{aligned} \tag{3.2.3}$$

Combining (3.2.2) and (3.2.3) we obtain

$$\begin{aligned} H &= \frac{1}{2m}\left(|\boldsymbol{p}|^2 + {p_z}^2\right) + \frac{e^2}{8m}|\boldsymbol{q}|^2 + \frac{e}{2m}\boldsymbol{q}.\mathbf{J}\boldsymbol{p} + V(R) \\ &+ \epsilon\left(2\boldsymbol{p} - e\mathbf{J}\boldsymbol{q}\right).\boldsymbol{f} + \frac{k_1\Omega_1 + k_2\Omega_2}{2\mu^2}|\boldsymbol{f}|^2 + \frac{{P_z}^2}{2M} \ , \end{aligned} \tag{3.2.4}$$

and

$$\begin{aligned} \omega &= \mathrm{d}q_x \wedge \mathrm{d}p_x + \mathrm{d}q_y \wedge \mathrm{d}p_y + \mathrm{d}q_z \wedge \mathrm{d}p_z + \mathrm{d}C_z \wedge \mathrm{d}P_z \\ &\quad + \frac{1}{\mu}\left(\mathrm{d}P_x \wedge \mathrm{d}P_y - \mathrm{d}f_x \wedge \mathrm{d}f_y\right) \ , \end{aligned}$$

where $R = \left(|\boldsymbol{q}|^2 + {q_z}^2\right)^{1/2}$ and

$$\epsilon = \frac{\Omega_1 - \Omega_2}{2\mu}$$

measures the displacement from the set of parameters satisfying $\Omega_1 = \Omega_2$. This coordinate change reduces (3.1.5) by a further degree of freedom by conservation (and elimination) of P_x and P_y. The quantities L and W are now given by

$$\begin{aligned} L &= \boldsymbol{q}.\mathbf{J}\boldsymbol{p} + \frac{1}{2\mu}\left(|\boldsymbol{f}|^2 - |\boldsymbol{P}|^2\right) \\ W &= |\boldsymbol{f}|^2 \ . \end{aligned}$$

Since $\boldsymbol{P}$ is conserved we remove the $-|\boldsymbol{P}|^2/(2\mu)$ term from the angular momentum, corresponding to a change in the level set of the angular momentum, defining the following conserved quantity

$$p_\theta = \boldsymbol{q}.\mathbf{J}\boldsymbol{p} + \frac{1}{2\mu}|\boldsymbol{f}|^2 \ .$$

A final change of coordinates makes the system canonical and exhibits the reduction to three degrees of freedom. It is given by writing

$$\begin{array}{lllllll} \boldsymbol{q} &=& r\boldsymbol{e}_r & \qquad \boldsymbol{p} &=& p_r\boldsymbol{e}_r + \frac{2\mu p_\theta - p_\phi}{2\mu r}\boldsymbol{e}_\theta \\ \boldsymbol{f} &=& p_\phi^{1/2}\boldsymbol{e}_{2\mu\phi+\theta} & & & \\ P_x &=& \mu\Pi_x & \qquad P_y &=& \Pi_y \ , \end{array} \tag{3.2.5}$$

where θ is the direction of $\boldsymbol{q}$, i.e.

$$\boldsymbol{e_r} = \cos\theta\boldsymbol{e}_x + \sin\theta\boldsymbol{e}_y \ , \qquad \boldsymbol{e}_\theta = -\sin\theta\boldsymbol{e}_x + \cos\theta\boldsymbol{e}_y \ , \tag{3.2.6}$$

with $\boldsymbol{e}_x = (1,0) \in \mathbb{R}^2$ and $\boldsymbol{e}_y = (0,1) \in \mathbb{R}^2$. The vector $\boldsymbol{e}_{2\mu\phi+\theta}$ is defined in the same way as $\boldsymbol{e}_\theta$ with θ replaced by $2\mu\phi+\theta$. The coordinate change given in (3.2.5) is singular at $p_\phi = 0$ since ϕ is undefined in this case. There exists another coordinate singularity at $r = 0$ that corresponds to collisions when $q_z = 0$.

We obtain the following result.

Theorem 3.2.1. *Let $k_1 + k_2 \neq 0$. Then the Hamiltonian system* (3.1.5) *reduces to one with three degrees of freedom in the variables $(r, p_r, \phi, p_\phi, q_z, p_z)$, given by*

$$\begin{aligned} H &= H_0 + \epsilon H_1 \qquad (3.2.7) \\ \omega &= \mathrm{d}r \wedge \mathrm{d}p_r + \mathrm{d}\phi \wedge \mathrm{d}p_\phi + \mathrm{d}q_z \wedge \mathrm{d}p_z + \mathrm{d}\theta \wedge \mathrm{d}p_\theta + \mathrm{d}\Pi_x \wedge \mathrm{d}\Pi_y + \mathrm{d}C_z \wedge \mathrm{d}P_z \ , \end{aligned}$$

where $H_0 = H_0(r, p_r, p_\phi, q_z, p_z, p_\theta, P_z)$ is given by

$$H_0 = \frac{1}{2m}\left({p_r}^2 + {p_z}^2\right) + \frac{1}{2m}\left(\frac{2\mu p_\theta - p_\phi}{2\mu r}\right)^2 + \frac{e^2}{8m}r^2 + \frac{e}{2m}\left(p_\theta + \frac{p_\phi}{2\mu}\right) + \frac{{P_z}^2}{2M} + V(R) \ ,$$

$H_1 = H_1(r, p_r, p_\theta, \phi, p_\phi)$ is given by

$$H_1 = {p_\phi}^{1/2}\left(\left(er + \frac{2\mu p_\theta - p_\phi}{\mu r}\right)\cos(2\mu\phi) - 2p_r\sin(2\mu\phi)\right) + \frac{k_1 - k_2}{2\mu}p_\phi$$

and

$$R = \left(r^2 + {q_z}^2\right)^{1/2} \ , \qquad p_\theta = L + \frac{1}{2\mu}P \ , \qquad p_\phi = W \ , \qquad (\mu\Pi_x, \Pi_y) = \boldsymbol{P} \ .$$

The reconstruction map is given by

$$\begin{aligned} \boldsymbol{R}_1 &= -\frac{\mathbf{J}\boldsymbol{P}}{\mu} + \frac{k_2}{2\mu}\left(\left(r - \frac{2\mu p_\theta - p_\phi}{k_1 k_2 r}\right)\boldsymbol{e}_r + \frac{2}{e}p_r\boldsymbol{e}_\theta\right) \\ \boldsymbol{R}_2 &= -\frac{\mathbf{J}\boldsymbol{P}}{\mu} - \frac{k_1}{2\mu}\left(\left(r - \frac{2\mu p_\theta - p_\phi}{k_1 k_2 r}\right)\boldsymbol{e}_r + \frac{2}{e}p_r\boldsymbol{e}_\theta\right) \\ \boldsymbol{\rho}_1 &= \frac{p_\phi^{1/2}}{\mu}\mathbf{J}\boldsymbol{e}_{2\mu\phi+\theta} + \frac{k_2}{2\mu}\left(\left(r + \frac{2\mu p_\theta - p_\phi}{k_1 k_2 r}\right)\boldsymbol{e}_r - \frac{2}{e}p_r\boldsymbol{e}_\theta\right) \\ \boldsymbol{\rho}_2 &= \frac{p_\phi^{1/2}}{\mu}\mathbf{J}\boldsymbol{e}_{2\mu\phi+\theta} - \frac{k_1}{2\mu}\left(\left(r + \frac{2\mu p_\theta - p_\phi}{k_1 k_2 r}\right)\boldsymbol{e}_r - \frac{2}{e}p_r\boldsymbol{e}_\theta\right) \\ q_{z_1} &= C_z + \frac{m_2}{M}q_z \ , \qquad q_{z_2} = C_z - \frac{m_1}{M}q_z \\ p_{z_1} &= \frac{m_1}{M}P_z + p_z \ , \qquad p_{z_2} = \frac{m_2}{M}P_z - p_z \ . \end{aligned}$$

If the gyrofrequencies of the two particles are equal, i.e. $\Omega_1 = \Omega_2$, we have that $\epsilon = 0$. Applying theorem 3.2.1 we see that ϕ is a cyclic variable and so we obtain the following result.

Corollary 3.2.2. *If $\Omega_1 = \Omega_2$ the Hamiltonian system* (3.1.5) *reduces to one with two degree of freedom in the variables (r, p_r, q_z, p_z), given by*

$$\begin{aligned} H &= H_0(r, p_r, p_\phi, q_z, p_z, p_\theta, P_z) \\ \omega &= \mathrm{d}r \wedge \mathrm{d}p_r + \mathrm{d}q_z \wedge \mathrm{d}p_z + \mathrm{d}\phi \wedge \mathrm{d}p_\phi + \mathrm{d}C_z \wedge \mathrm{d}P_z + \mathrm{d}\theta \wedge \mathrm{d}p_\theta + \mathrm{d}\Pi_x \wedge \mathrm{d}\Pi_y \ , \end{aligned}$$

where H_0 is as given in theorem 3.2.1.

3.2.2 Case $k_1 + k_2 = 0$

We now treat the case where the charges sum to zero. For simplicity of notation let

$$\kappa = k_1 = -k_2 \ .$$

We make the change of coordinates given by

$$\begin{aligned} \boldsymbol{q} &= \boldsymbol{R}_1 - \boldsymbol{R}_2 + \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2 \\ \boldsymbol{p} &= -\frac{\kappa}{2}\mathbf{J}\left(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2\right) \\ \boldsymbol{C} &= -\frac{1}{2}\mathbf{J}\left(\boldsymbol{R}_1 + \boldsymbol{R}_2 + \boldsymbol{\rho}_1 + \boldsymbol{\rho}_2\right) \\ \boldsymbol{\Pi} &= \kappa\left(\boldsymbol{R}_1 - \boldsymbol{R}_2\right) \ , \end{aligned} \tag{3.2.8}$$

where $\boldsymbol{q} = (q_x, q_y) \in \mathbb{R}^2$ is the relative position of the two particles, $\boldsymbol{p} = (p_x, p_y) \in \mathbb{R}^2$ a conjugate momentum, $\boldsymbol{C} = (C_x, C_y) \in \mathbb{R}^2$ and $\boldsymbol{\Pi} = (\Pi_x, \Pi_y) \in \mathbb{R}^2$. Inverting (3.2.8) we obtain

$$\begin{aligned} \boldsymbol{R}_1 &= \frac{1}{2\kappa}\boldsymbol{\Pi} + \mathbf{J}\boldsymbol{C} - \frac{1}{\kappa}\mathbf{J}\boldsymbol{p} \\ \boldsymbol{R}_2 &= -\frac{1}{2\kappa}\boldsymbol{\Pi} + \mathbf{J}\boldsymbol{C} - \frac{1}{\kappa}\mathbf{J}\boldsymbol{p} \\ \boldsymbol{\rho}_1 &= -\frac{1}{2\kappa}\boldsymbol{\Pi} + \frac{1}{2}\left(\boldsymbol{q} + \frac{2}{\kappa}\mathbf{J}\boldsymbol{p}\right) \\ \boldsymbol{\rho}_2 &= \frac{1}{2\kappa}\boldsymbol{\Pi} - \frac{1}{2}\left(\boldsymbol{q} - \frac{2}{\kappa}\mathbf{J}\boldsymbol{p}\right) \ . \end{aligned} \tag{3.2.9}$$

From (3.2.2) and (3.2.9), we get the Hamiltonian system determined by the Hamiltonian function

$$\begin{aligned} H &= \frac{1}{2m}|\boldsymbol{p}|^2 + \frac{\kappa^2}{8m}|\boldsymbol{q}|^2 + \frac{(m_2-m_1)\,\kappa}{2m_1m_2}\boldsymbol{q}.\mathbf{J}\boldsymbol{p} + V(R) \\ &\quad - \left(\frac{\kappa}{4m}\boldsymbol{q} + \frac{m_2-m_1}{2m_1m_2}\mathbf{J}\boldsymbol{p}\right).\boldsymbol{\Pi} + \frac{1}{8m}|\boldsymbol{\Pi}|^2 + \frac{1}{2m}{p_z}^2 + \frac{1}{2M}{P_z}^2 \ , \end{aligned} \tag{3.2.10}$$

where $R = \left(|\boldsymbol{q}|^2 + {q_z}^2\right)^{1/2}$, and symplectic form

$$\omega = \mathrm{d}q_x \wedge \mathrm{d}p_x + \mathrm{d}q_y \wedge \mathrm{d}p_y + \mathrm{d}q_z \wedge \mathrm{d}p_z + \mathrm{d}C_x \wedge \mathrm{d}\Pi_x + \mathrm{d}C_y \wedge \mathrm{d}\Pi_y + \mathrm{d}C_z \wedge \mathrm{d}P_z \ ,$$

with the conserved quantities

$$\begin{aligned} \boldsymbol{P} &= \mathbf{J}\boldsymbol{\Pi} \\ L &= \boldsymbol{q}.\mathbf{J}\boldsymbol{p} + \boldsymbol{C}.\mathbf{J}\boldsymbol{\Pi} \ . \end{aligned}$$

The Hamiltonian system (3.2.10) is already reduced to three degrees of freedom by conservation of $\boldsymbol{\Pi}$ and P_z and elimination of $\boldsymbol{C}$ and C_z. Unless $\boldsymbol{\Pi} = \mathbf{0}$ (or equivalently $\boldsymbol{P} = \mathbf{0}$), it is not possible to use the angular momentum L to reduce further (3.2.10) since L depends on the cyclic variables $\boldsymbol{C}$ and hence it is not a function defined on the reduced space. We make a final change of coordinates, given by

$$\begin{aligned} \boldsymbol{q} &= r\boldsymbol{e}_r \\ \boldsymbol{p} &= p_r\boldsymbol{e}_r + \frac{p_\theta}{r}\boldsymbol{e}_\theta \ , \end{aligned}$$

where $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are as given in (3.2.6). We obtain the following result.

Theorem 3.2.3. *Let $k_1 + k_2 = 0$. Then the Hamiltonian system* (3.1.5) *reduces to one with three degrees of freedom in the variables $(r, p_r, \theta, p_\theta, q_z, p_z)$, given by*

$$\begin{aligned} H &= H_0 + H_1 \\ \omega &= \mathrm{d}r \wedge \mathrm{d}p_r + \mathrm{d}\theta \wedge \mathrm{d}p_\theta + \mathrm{d}q_z \wedge \mathrm{d}p_z + \mathrm{d}C_x \wedge \mathrm{d}\Pi_x + \mathrm{d}C_y \wedge \mathrm{d}\Pi_y + \mathrm{d}C_z \wedge \mathrm{d}P_z \ , \end{aligned} \tag{3.2.11}$$

where $H_0 = H_0(r, p_r, p_\theta, q_z, p_z, P_z)$ is given by

$$H_0 = \frac{1}{2m}\left({p_r}^2 + {p_z}^2\right) + \frac{1}{2m}\left(\frac{p_\theta}{r}\right)^2 + \frac{\kappa^2}{8m}r^2 + \frac{(m_2-m_1)\,\kappa}{2m_1m_2}p_\theta + \frac{{P_z}^2}{2M} + V(R) \ ,$$

$H_1 = H_1\left(r, p_r, \theta, p_\theta, \Pi_x, \Pi_y\right)$ *is given by*

$$H_1 = -\left(\left(\frac{\kappa}{4m}r + \frac{m_2 - m_1}{2m_1m_2}\frac{p_\theta}{r}\right)\boldsymbol{e}_r - \frac{m_2 - m_1}{2m_1m_2}p_r\boldsymbol{e}_\theta\right).\boldsymbol{\Pi} + \frac{1}{8m}\left|\boldsymbol{\Pi}\right|^2$$

and

$$R = \left(r^2 + {q_z}^2\right)^{1/2} \ , \qquad p_\theta = L - \boldsymbol{C}.\mathbf{J}\boldsymbol{\Pi} \ , \qquad \boldsymbol{\Pi} = -\mathbf{J}\boldsymbol{P} \ .$$

The reconstruction map is given by

$$\begin{aligned}
\boldsymbol{R}_1 &= \frac{1}{2\kappa}\boldsymbol{\Pi} + \mathbf{J}\boldsymbol{C} - \frac{1}{\kappa}\frac{p_\theta}{r}\boldsymbol{e}_r + \frac{1}{\kappa}p_r\boldsymbol{e}_\theta \\
\boldsymbol{R}_2 &= -\frac{1}{2\kappa}\boldsymbol{\Pi} + \mathbf{J}\boldsymbol{C} - \frac{1}{\kappa}\frac{p_\theta}{r}\boldsymbol{e}_r + \frac{1}{\kappa}p_r\boldsymbol{e}_\theta \\
\boldsymbol{\rho}_1 &= -\frac{1}{2\kappa}\boldsymbol{\Pi} + \frac{1}{2}\left(r + \frac{2}{\kappa}\frac{p_\theta}{r}\right)\boldsymbol{e}_r - \frac{1}{\kappa}p_r\boldsymbol{e}_\theta \\
\boldsymbol{\rho}_2 &= \frac{1}{2\kappa}\boldsymbol{\Pi} - \frac{1}{2}\left(r - \frac{2}{\kappa}\frac{p_\theta}{r}\right)\boldsymbol{e}_r - \frac{1}{\kappa}p_r\boldsymbol{e}_\theta \\
q_{z_1} &= C_z + \frac{m_2}{M}q_z \ , \qquad q_{z_2} = C_z - \frac{m_1}{M}q_z \\
p_{z_1} &= \frac{m_1}{M}P_z + p_z \ , \qquad p_{z_2} = \frac{m_2}{M}P_z - p_z \ .
\end{aligned}$$

If $\boldsymbol{P} = \mathbf{0}$ then $\boldsymbol{\Pi} = \mathbf{0}$ and hence H_1, as given in the statement of theorem 3.2.3, is identically zero. From theorem 3.2.3, we obtain the following result.

Corollary 3.2.4. *If* $k_1 + k_2 = 0$ *and* $\boldsymbol{P} = \mathbf{0}$ *then the Hamiltonian system* (3.1.5) *reduces to one with two degrees of freedom in the variables* (r, p_r, q_z, p_z)*, given by*

$$\begin{aligned}
H &= H_0(r, p_r, q_z, p_z, p_\theta, P_z) \\
\omega &= \mathrm{d}r \wedge \mathrm{d}p_r + \mathrm{d}q_z \wedge \mathrm{d}p_z + \mathrm{d}\theta \wedge \mathrm{d}p_\theta + \mathrm{d}C_x \wedge \mathrm{d}\Pi_x + \mathrm{d}C_y \wedge \mathrm{d}\Pi_y + \mathrm{d}C_z \wedge \mathrm{d}\Pi_z \ ,
\end{aligned}$$

where H_0 *is as given in theorem 3.2.3.*

3.3 Reconstructed dynamics for a Coulomb potential

In this section we use the reduced Hamiltonian systems and the corresponding reconstruction maps obtained in section 3.2 to provide a qualitative description of the possible types of dynamics in the full twelve–dimensional phase space in terms of the properties of the dynamics of the reduced systems. Throughout this section we consider the interaction potential to be Coulomb

$$V(R) = \frac{e_1 e_2}{4\pi\epsilon_0}\frac{1}{R} ,$$

where R is the distance between the particles and ϵ_0 is the permittivity of the vacuum.

From theorems 3.2.1 and 3.2.3 we obtain that the vertical z–component of the centre of mass of the two particles moves with constant velocity $\dot{C}_z = P_z/M$. Hence, by a translation we can assume that $\dot{C}_z$ and P_z are 0. Furthermore, without loss of generality we will assume that $C_z = 0$. This corresponds to considering the system as moving with the centre of mass of the z–component. From now on, we will assume this is the case. To extend the results in the sections below to non–zero P_z it is enough to add a drift $P_z t/M$ to the vertical positions q_{z_1} and q_{z_2} of the particles (by the reconstruction maps of theorems 3.2.1 and 3.2.3).

3.3.1 Dynamics on the invariant plane $q_z = 0$, $p_z = 0$

In this section we prove the existence of an invariant plane for the dynamics of the reduced Hamiltonian systems given in theorems 3.2.1 and 3.2.3 and relate it with an invariant plane of the Hamiltonian system (3.1.5). We proceed by giving a brief description of both the reduced and reconstructed dynamics contained in this invariant plane.

Lemma 3.3.1. *The reduced Hamiltonian systems* (3.2.7) *and* (3.2.11) *given, respectively, in theorems 3.2.1 and 3.2.3 have an invariant plane determined by the conditions* $q_z = 0$ *and* $p_z = 0$.

Proof. From (3.2.7) and (3.2.11) we obtain

$$\begin{aligned} \dot{q}_z &= \frac{1}{m} p_z \\ \dot{p}_z &= \frac{e_1 e_2}{4\pi\epsilon_0}\frac{q_z}{(r^2 + {q_z}^2)^{3/2}} . \end{aligned} \tag{3.3.1}$$

From (3.3.1) we obtain $\dot{q}_z = \dot{p}_z = 0$ if and only if $q_z = p_z = 0$. Since the right hand side of (3.3.1) is locally Lipschitz away from the singularity, it follows that q_z and p_z remain equal to zero if they both start at zero. □

Invariance of the plane $q_z = 0$, $p_z = 0$ under the dynamics of (3.2.7) and (3.2.11) corresponds to invariance of the (ten–dimensional) plane

$$\Lambda = \left\{(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{p}_1, \boldsymbol{p}_2) \in \mathbb{R}^{12} : q_{z_1} = q_{z_2} \ , m_2 p_{z_1} = m_1 p_{z_2}\right\}$$

under the dynamics of (3.1.5). Furthermore, on the invariant plane Λ the dynamics of the Hamiltonian system (3.1.5) reduce by a further degree of freedom compared to the reduced systems in theorems 3.2.1 and 3.2.3. In fact, by setting $q_z = 0$, $p_z = 0$ in the Hamiltonian systems (3.2.7) and (3.2.11) we obtain reduced dynamical systems for the problem of the interaction of two charges moving in a plane under the action of a magnetic field (with a shift on the energy level sets by ${P_z}^2/(2M)$). This system was extensively studied in chapter 2. The next two results follow from theorems 3.2.1 and 3.2.3, lemma 3.3.1 and the results in chapter 2. We skip their proof.

Corollary 3.3.2. *Let $k_1 + k_2 \neq 0$ and assume that the dynamics of* (3.1.5) *are contained in Λ. Then*

- *the Hamiltonian system* (3.1.5) *reduces to one with two degrees of freedom in the variables (r, p_r, ϕ, p_ϕ), given by the restriction of* (3.2.7) *to $q_z = 0$, $p_z = 0$.*
- *if $\Omega_1 = \Omega_2$ the Hamiltonian system* (3.1.5) *reduces to one with one degree of freedom in the variables (r, p_r), given by the restriction of* (3.2.7) *to $q_z = 0$, $p_z = 0$ (and $\epsilon = 0$).*
- *if e_1 and e_2 have opposite signs and $\Omega_1 + \Omega_2 \neq 0$ the Hamiltonian system* (3.1.5) *contains a suspension of a non–trivial subshift of finite type on level sets of high energy.*

Corollary 3.3.3. *Let $k_1 + k_2 = 0$ and assume that the dynamics of* (3.1.5) *are contained in Λ. Then*

- *the Hamiltonian system* (3.1.5) *reduces to one with two degrees of freedom in the variables* $(r, p_r, \theta, p_\theta)$*, given by the restriction of* (3.2.11) *to* $q_z = 0$, $p_z = 0$.

- *if* $\boldsymbol{P} = \mathbf{0}$ *the Hamiltonian system* (3.1.5) *reduces to one with one degree of freedom in the variables* (r, p_r)*, given by the restriction of* (3.2.11) *to* $q_z = 0$, $p_z = 0$ *(and* $\boldsymbol{P} = \mathbf{0}$*).*

- *if* $\boldsymbol{P} \neq \mathbf{0}$ *and* $\Omega_1 + \Omega_2 \neq 0$ *the Hamiltonian system* (3.1.5) *contains a suspension of a non–trivial subshift of finite type on level sets of high energy.*

As was observed in chapter 2 the reduced dynamics exhibit the following types of dynamical behaviour:

- In the integrable regimes the energy levels are foliated by periodic orbits.

- Close to the integrable regimes most of the periodic orbits cease to exist but almost all orbits in the energy levels are quasiperiodic and hence the dynamics still look regular.

- For opposite signs of charge (except for the case $\Omega_1 + \Omega_2 = 0$) there is chaotic dynamics on level sets of high energy, which implies non–integrability in this regime.

The full dynamics in Λ correspond to a drift of the two particles with constant and equal velocities (equal to P_z/M) in the z–direction. The dynamics in the (x, y) plane are as described in chapter 2:

1) If $k_1 + k_2 \neq 0$ the dynamics in the (x, y) plane are, generically, quasiperiodic with three rationally independent frequencies. The particles rotate with these three frequencies about a fixed centre determined by the linear momenta.

2) If $k_1 + k_2 = 0$ periodic and quasiperiodic base dynamics lift to possibly unbounded motion in the (x, y) plane corresponding to a combination of a drift and quasiperiodic dynamics. The quasiperiodic dynamics have, generically, two rationally independent frequencies.

3) Chaotic dynamics lift to chaotic dynamics in the (x, y) plane. The motion (in the (x, y) plane) is always bounded if $k_1 + k_2 \neq 0$ and typically unbounded otherwise.

3.3.2 Existence of close approaches and bounded motion

In this section we describe the regime of parameters where close approaches are possible and discuss the existence of bounded and unbounded motion for some of the variables in the reduced systems given in theorems 3.2.1 and 3.2.3. This will gives us important insight about distinct types of dynamical behaviour of (3.1.5) which will be of significance for the study of the scattering problem associated with this system.

The next two lemmas describe the set of parameters and level sets of the conserved quantities where trajectories with close approaches might exist. We skip the proofs of these lemmas, which follow from an analysis of the Hamiltonian functions given in theorems 3.2.1 and 3.2.3. In chapter 2 we give similar results for the case where the two particles move in a plane.

Lemma 3.3.4. *Let $k_1 + k_2 \neq 0$. Then*

- *if $k_1 k_2 > 0$, or $k_1 k_2 < 0$ and the value of the conserved quantity p_θ is fixed so that $\mu p_\theta < 0$, the planar distance r between the two particles is bounded away from zero, i.e. there exists $d > 0$ such that $r(t) > d$ for all $t \in \mathbb{R}$.*
- *if $k_1 k_2 < 0$ and the value of the conserved quantity p_θ is fixed so that μp_θ is positive, the distance $r^2 + {q_z}^2$ between the two particles can be arbitrarily close to 0. Furthermore, ${p_r}^2 + {p_z}^2 \to \infty$ and $p_\phi \to 2\mu p_\theta$ as $r^2 + {q_z}^2 \to 0$.*

Lemma 3.3.5. *Let $k_1 + k_2 = 0$. Then*

- *if $\boldsymbol{P} = \mathbf{0}$ and $p_\theta \neq 0$, the planar distance r between the two particles is bounded away from zero, i.e. there exists $d > 0$ such that $r(t) > d$ for all $t \in \mathbb{R}$.*
- *if $\boldsymbol{P} = \mathbf{0}$ and $p_\theta = 0$, the distance $r^2 + {q_z}^2$ between the two particles can be arbitrarily close to 0. Furthermore, ${p_r}^2 + {p_z}^2 \to \infty$ as $r^2 + {q_z}^2 \to 0$.*

- *if $\boldsymbol{P} \neq \mathbf{0}$, the distance ${r}^2 + {q_z}^2$ between the two particles can be arbitrarily close to 0. Furthermore, ${p_r}^2 + {p_z}^2 \to \infty$ and $p_\theta \to 0$ as $r^2 + {q_z}^2 \to 0$.*

In the next two results we identify the set of parameters and level sets of the conserved quantities for which the distance between the two particles remains bounded.

Lemma 3.3.6. *Let $k_1 + k_2 \neq 0$ and consider the reduced Hamiltonian system given in theorem 3.2.1. Then*

(i) for every level set of the Hamiltonian function the dynamics of r and p_ϕ are bounded for all time.

(ii) if e_1 and e_2 have equal signs, there exists $E' \in \mathbb{R}$ such that for every level set $E < E'$ of the Hamiltonian function the projection of the level set $\{H = E\}$ onto the q_z direction is unbounded but bounded away from $q_z = 0$, and for $E > E'$ the projection of the level set $\{H = E\}$ onto the q_z direction is unbounded. Furthermore, for every level set of the Hamiltonian function the dynamics of p_r and p_z are bounded for all time.

(iii) if e_1 and e_2 have opposite signs and the value of the conserved quantity p_θ is fixed so that $2\mu p_\theta < 0$, there exists $E'' \in \mathbb{R}$ such that for every level set $E < E''$ of the Hamiltonian function the dynamics of q_z are bounded and for every level set $E > E''$ the projection of the level set $\{H = E\}$ onto the q_z direction is unbounded. Furthermore, for every level set of the Hamiltonian function the dynamics of p_r and p_z are bounded for all time.

(iv) if e_1 and e_2 have opposite signs and the value of the conserved quantity p_θ is fixed so that $2\mu p_\theta \geq 0$, there exists $E'' \in \mathbb{R}$ such that for every level set $E < E''$ of the Hamiltonian function the dynamics of q_z are bounded and for every level set $E > E''$ the projection of the level set $\{H = E\}$ onto the q_z direction is unbounded.

Proof. Item (i) follows trivially from the fact that as $r \to \infty$ then $H \to \infty$ (and similarly for p_ϕ).

For the proof of items (iii)–(iv) we define the Hamiltonian function $\mathcal{H}$ by

$$\mathcal{H} = H - G\left(r^2 + {q_z}^2\right)^{-1/2} ,$$

where H is given in theorem 3.2.1 with V replaced by the Coulomb potential (3.0.1) and $G = e_1 e_2/(4\pi\epsilon_0)$. It is crucial to point out that the function $\mathcal{H}$ does not depend on q_z and is always bounded below. For simplicity of notation we consider H and $\mathcal{H}$ just as functions of the variables in the reduced phase spaces and do not make their dependence on the conserved quantities P_z and p_θ explicit.

We now prove item (ii). For the case of same sign charges the Hamiltonian function H is bounded below. Denote its lower bound by

$$E^- = \inf H(r, p_r, \phi, p_\phi, q_z, p_z) .$$

By lemma 3.3.4 we obtain that r is bounded away from 0 and hence $G\left(r^2 + {q_z}^2\right)^{-1/2}$ is bounded. On lines where r is constant $G\left(r^2 + {q_z}^2\right)^{-1/2}$ increases at non–zero rate from 0 at the limit $q_z \to -\infty$ to attains its (positive) maximum at $q_z = 0$ to decrease again to 0 at non–zero rate at the limit $q_z \to +\infty$. We now define

$$E' = \inf H(r, p_r, \phi, p_\phi, q_z = 0, p_z) ,$$

and note that $E^- < E'$. From a simple analysis of the Hamiltonian function H we obtain that for every level set $E^- \leq E < E'$ of the Hamiltonian function the projection of the level set $\{H = E\}$ onto the q_z direction is bounded away from 0 and for $E > E'$ $q_z = 0$ belongs to the level sets of the Hamiltonian function. Furthermore, we obtain that for every level set of the Hamiltonian function the projection of the level set $\{H = E\}$ onto the q_z direction is unbounded. Boundedness of the dynamics of p_r and p_z follow from the fact that r is always bounded away from 0 (lemma 3.3.4).

For the proof of item (iii) we start by defining

$$E'' = \inf \mathcal{H}(r, p_r, \phi, p_\phi, p_z) .$$

By lemma 3.3.4 we get that r is bounded away from 0 and hence $G\left(r^2 + {q_z}^2\right)^{-1/2}$ is bounded. On lines where r is constant $G\left(r^2 + {q_z}^2\right)^{-1/2}$ decreases at non–zero rate from 0 at the limit $q_z \to -\infty$ to its (negative) minimum at $q_z = 0$ to increase again to 0 at non–zero rate at the limit $q_z \to +\infty$. We note that in this case the function H is also bounded below and define

$$\begin{aligned} E^- &= \inf H(r, p_r, \phi, p_\phi, q_z = 0, p_z) \\ &= \inf \left(\mathcal{H}(r, p_r, \phi, p_\phi, p_z) + \frac{G}{r} \right) , \end{aligned}$$

and note that by item (i) in this lemma, r is bounded and hence G/r is strictly negative implying that $E^- < E''$. Using the fact that $\mathcal{H}$ does not depend on q_z a simple analysis of the Hamiltonian function H shows that for every level set $E^- \leq E < E''$ of H the dynamics of q_z are bounded for all time and for every level set $E > E''$ the projection of the level set $\{H = E\}$ onto the q_z direction is unbounded. Boundedness of the dynamics of p_r and p_z follow from the fact that r is always bounded away from 0 (lemma 3.3.4).

To prove (iv) we note that H is not bounded below at points satisfying $p_\phi = 2\mu p_\theta$ and define

$$E'' = \inf \mathcal{H}(r, p_r, \phi, p_\phi, p_z) \ .$$

Since $G \left(r^2 + {q_z}^2\right)^{-1/2} < 0$ (possibly unbounded) we obtain that $H \leq \mathcal{H}$. A simple analysis of the Hamiltonian function H shows that for every level set $E < E''$ of H the dynamics of q_z are bounded for all time and for every level set $E > E''$ the projection of the level set $\{H = E\}$ onto the q_z direction is unbounded. □

We skip the proof of the next lemma which is analogous (though simpler) to the proof of the previous lemma 3.3.6.

Lemma 3.3.7. *Let $k_1 + k_2 = 0$ and consider the reduced Hamiltonian system given in theorem 3.2.3. Then*

(i) for every level set of the Hamiltonian function the dynamics of r and p_θ are bounded for all time.

(ii) if $\boldsymbol{P} = 0$ and $p_\theta \neq 0$, there exists $E'' \in \mathbb{R}$ such that for every level set $E < E''$ of the Hamiltonian function the dynamics of q_z are bounded and for every level

set $E > E''$ the projection of the level set $\{H = E\}$ onto the q_z direction is unbounded. Furthermore, for every level set of the Hamiltonian function the dynamics of p_r and p_z are bounded for all time.

(iii) if $\boldsymbol{P} \neq 0$ or $\boldsymbol{P} = 0$ and $p_\theta = 0$, there exists $E'' \in \mathbb{R}$ such that for every level set $E < E''$ of the Hamiltonian function the dynamics of q_z are bounded and for every level set $E > E''$ the projection of the level set $\{H = E\}$ onto the q_z direction is unbounded.

3.3.3 Asymptotic properties of the dynamics of the reconstructed system

In this section we combine the results given in the two previous subsections to provide a partial characterization for the dynamics from an "asymptotic point of view". We should remark that this characterization might be incomplete and that the existence of extra conserved quantities or invariant hypersurfaces for the Hamiltonian system (3.1.5) may introduce other types of dynamical behaviour.

In the absence of extra conserved quantities or invariant hypersurfaces for the Hamiltonian system (3.1.5), the following (asymptotic) dynamical behaviours are possible:

(1) "planar" behaviour: in the invariant plane Λ the two particles drift with equal and constant velocity in the vertical z–direction while the dynamics in the (x, y) plane correspond to the interaction of the two particles in a plane.

(2) "molecule–like" behaviour: if the charges have opposite signs, then there exists $E_1 \in \mathbb{R}$ such that for every level set $E < E_1$ of the Hamiltonian function the relative position $\boldsymbol{q}_1 - \boldsymbol{q}_2$ of the two particles is bounded for all time. However, each particle position may be unbounded:

- if $P_z \neq 0$ the particles drift in the vertical direction with non–zero velocity.

– if $k_1 + k_2 \neq 0$ and $P_z = 0$ using (3.2.1) and the reconstruction map in theorem 3.2.1, we obtain

$$(q_{x_i}, q_{y_i}) = -\frac{\mathbf{J}\boldsymbol{P}}{\mu} + (-1)^{i+1}\frac{1}{\mu}\left(k_2 r \boldsymbol{e}_r + (-1)^{i+1} p_\phi^{1/2} \mathbf{J}\boldsymbol{e}_{2\mu\phi+\theta}\right) .$$

By lemma 3.3.6 we obtain that r and p_ϕ are bounded and hence $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are bounded.

– if $k_1 + k_2 = 0$ and $\boldsymbol{P} \neq \mathbf{0}$ we have that the trajectories of the two particles are typically unbounded in the (x, y) plane: they drift with a non–zero average velocity (see chapter 2 for more details).

(3) "unbounded" behaviour: there exists $E_2 \in \mathbb{R}$ such that for every level set $E > E_2$ of the Hamiltonian function the dynamics in the vertical z–direction are unbounded. More precisely, the asymptotic behaviour satisfies:

a) $\lim_{t\to\pm\infty} q_{z_1}(t) = -\lim_{t\to\pm\infty} q_{z_2}(t)$.

b) $\lim_{t\to\pm\infty} |q_{z_1}(t)| = \lim_{t\to\pm\infty} |q_{z_2}(t)| = \infty$.

We identify two distinct classes of "unbounded" behaviour to which we call "bouncing–back" behaviour and "pass–through" behaviour.

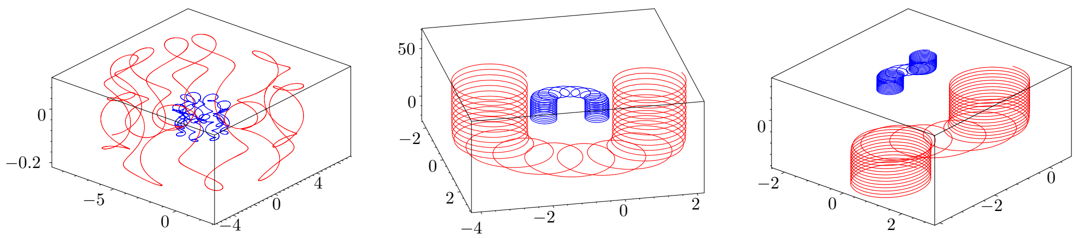
(3.1) "bouncing–back" behaviour: we call "bouncing–back" behaviour to an "unbounded" behaviour satisfying the additional condition that $q_{z_1}(t) - q_{z_2}(t)$ is bounded away from zero for all $t \in \mathbb{R}$.

In particular, if the charges have equal signs, then there exists $E_3 \in \mathbb{R}$ such that for every level set $E < E_3$ of the Hamiltonian function the dynamics exhibit "bouncing–back" behaviour, i.e. the dynamics are unbounded in the vertical z–direction but bounded away from the plane $q_{z_1} = q_{z_2}$.

(3.2) "pass–through" behaviour: we call "pass–through" behaviour to an "unbounded" behaviour satisfying the following additional condition:

$$\lim_{t\to+\infty} q_{z_i}(t) = -\lim_{t\to-\infty} q_{z_i}(t) , \qquad i \in \{1, 2\} .$$

In particular, there exists $E_4 \in \mathbb{R}$ such that for every level set $E > E_4$ of the Hamiltonian function the dynamics always exhibit "pass–through" behaviour.



(a) (b) (c)

Figure 3.3.1: Three distinct dynamical behaviours. For all the figures we fix the parameters $e_1 = m_1 = 1$, $B = c = 1$ and $\epsilon_0 = 0.1$ and initial conditions $q_{x_1}(0) = -q_{x_2}(0) = 2$, $p_{x_1}(0) = p_{x_2}(0) = 1$ and $q_{y_1}(0) = q_{y_2}(0) = p_{y_1}(0) = p_{y_2}(0) = 0$. On the left column we set $e_2 = -4$, $m_2 = 3$ and $q_{z_1}(0) = -q_{z_2}(0) = 0.1$, $p_{z_1}(0) = -p_{z_2}(0) = -0.01$ and obtain a "molecule–like" behaviour. On the centre column we set $e_2 = m_2 = 3$ and $q_{z_1}(0) = -q_{z_2}(0) = 0.1$, $p_{z_1}(0) = -p_{z_2}(0) = -0.01$ and obtain a "bouncing–back" behaviour. On the right column we set $e_2 = m_2 = 3$ and $q_{z_1}(0) = -q_{z_2}(0) = 10$, $p_{z_1}(0) = -p_{z_2}(0) = -0.85$ and obtain a "pass–through" behaviour.

Remark As we will see in the next section, for the case of opposite sign charges, the regime of energies corresponding to "unbounded" behaviour is the suitable one for the existence of chaotic scattering in this system: if the energy is high enough, then the particles always exhibit "pass–through" behaviour while for small positive energies there is a mixing of "pass–through" behaviour and "bounce–back" behaviour.

3.4 The scattering map

In this section we introduce the scattering map associated with the Hamiltonian system (3.1.5) and derive relevant properties of this map in some suitable regimes. As in section 3.3, throughout this section we will consider the interaction potential to be Coulomb $V(R) = G/R$, where R is the distance between the particles and $G = e_1 e_2 / 4\pi\epsilon_0$; as before, we will also set $C_z = 0$ and $P_z = 0$ without loss of generality.

In the case of zero interaction, the general solution of (3.1.5) can be written as

$$\begin{aligned} \boldsymbol{q}_i(t) &= (\boldsymbol{R}_i + \boldsymbol{\rho}_i(t), q_{z_i}(t)) \\ \boldsymbol{p}_i(t) &= (-k_i \mathbf{J} \boldsymbol{\rho_i}(t), p_{z_i}) \ , \end{aligned} \tag{3.4.1}$$

for $i \in \{1, 2\}$, where $\boldsymbol{R}_i = (R_{x_i}, R_{y_i}) \in \mathbb{R}^2$ are the guiding centres of the particles, $\boldsymbol{\rho}_i(t) \in \mathbb{R}^2$ their gyroradii vectors, $q_{z_i}(t) \in \mathbb{R}$ their vertical positions and $p_{z_i} \in \mathbb{R}$ their vertical momenta. For zero interaction, $\boldsymbol{R}_i$, $\rho_i = |\boldsymbol{\rho}_i(t)|$ and p_{z_i} are conserved, while $\arg(\boldsymbol{\rho}_i(t))$ rotates with gyrofrequency Ω_i and the vertical positions $q_{z_i}(t)$ evolve linearly with time.

As a consequence of theorems 3.2.1 and 3.2.3, for non–zero interactions the general solution of (3.1.5) can still be written in the form given in (3.4.1), with the variables $\boldsymbol{R}_i$, $\boldsymbol{\rho}_i$, q_{z_i} and p_{z_i} $(i \in \{1, 2\})$ evolving with time accordingly with the reconstruction maps and the reduced Hamiltonian systems given in the statements of theorems 3.2.1 and 3.2.3.

We will be considering the cases in section 3.3.3 where the dynamics in the vertical direction are unbounded: in the limits of $t \to \pm\infty$ we have that $|q_z(t)| = |q_{z_1}(t) - q_{z_2}(t)| \to \infty$. In such cases, the typical situation is the following: initially the particles have a large vertical separation and their trajectories are just helices of the form (3.4.1), as described for the zero interaction case. At some instant of time their vertical separation is small so the particles interact and their paths are no longer helices. The two particles will eventually separate again and their paths approach helices again. Due to the interaction, the helices in which the particles move before interacting and after interacting are different. The scattering map describes such asymptotic changes to the helices.

We will now rigorously introduce the scattering map. We assume that as $|t| \to \infty$ the two particles have infinite vertical separation and the particles move in helices. We would like the scattering map to map the main asymptotic properties of such helices, i.e. the guiding centres and gyroradius as well as the vertical position and momentum as the particles approach $t = -\infty$ to the asymptotic properties of the helices at $t = +\infty$. We proceed as follows. First of all, we note that since $C_z = P_z = 0$ from the reconstruction maps in theorems 3.2.1 and 3.2.3, we have that

$$\begin{aligned} q_{z_1} &= \frac{m_2}{M} q_z \qquad q_{z_2} = -\frac{m_1}{M} q_z \\ p_{z_1} &= p_z \qquad\quad p_{z_2} = -p_z \ , \end{aligned}$$

and hence it is enough to look at the asymptotics of q_z and p_z. There are, however, some difficulties: in the limit of large vertical separation, the gyroradii of the two particles rotate uniformly with angular velocity given by the gyrofrequency of each particle and hence do not have a well–defined limit as $|t| \to \infty$. To overcome this difficulty we consider the Hamiltonian system (3.2.2) and introduce the following change of coordinates

$$\boldsymbol{\rho}_i = (2\rho_i)^{1/2} \, \mathbf{J} \boldsymbol{e}_{\theta_i} \ , \qquad i \in \{1, 2\} \ , \tag{3.4.2}$$

where $\boldsymbol{e}_{\theta_i}$ is defined in the same way as $\boldsymbol{e}_\theta$ in (3.2.6) with θ replaced by θ_i. Note that $\rho_i \in \mathbb{R}$ is proportional to the gyroradii squared norm $|\boldsymbol{\rho}_i|^2$ and θ_i is the i-th particle gyrophase. Using (3.4.2), we obtain the Hamiltonian system

$$\begin{aligned} H &= \frac{{p_z}^2}{2m} + k_1 \Omega_1 \rho_1 + k_2 \Omega_2 \rho_2 + V(R) \\ \omega &= \sum_{i=1,2} k_i \left(\mathrm{d}R_{x_i} \wedge \mathrm{d}R_{y_i} + \mathrm{d}\theta_i \wedge \mathrm{d}\rho_i \right) + \mathrm{d}q_z \wedge \mathrm{d}p_z \ , \end{aligned} \tag{3.4.3}$$

where

$$R = \left(\left| \boldsymbol{R}_1 - \boldsymbol{R}_2 + (2\rho_1)^{1/2} \, \mathbf{J} \boldsymbol{e}_{\theta_1} - (2\rho_2)^{1/2} \, \mathbf{J} \boldsymbol{e}_{\theta_2} \right|^2 + {q_z}^2 \right)^{1/2} \ .$$

We should point out that the transformation (3.4.2) introduces two coordinate singularities at $\rho_1 = 0$ and $\rho_2 = 0$. We note that the gyrophases θ_i evolve by the differential equation

$$\dot{\theta}_i = \Omega_i + \frac{1}{k_i} \frac{\partial}{\partial \rho_i} V(R)$$

and hence do not have well–defined limits $\theta_i^{\pm} = \lim_{t \to \pm\infty} \theta_i(t)$. To avoid this inconvenience we introduce modified gyrophases by

$$\phi_i = \theta_i - \Omega_i t \ , \qquad i \in \{1, 2\} \ , \tag{3.4.4}$$

measuring the displacement between the gyrophases in the non–zero interaction and zero interaction settings. Going to the extended phase space by introducing the conjugated variables energy E and time t and making the change of coordinates (3.4.4) we obtain the following Hamiltonian system

$$\begin{aligned} \mathcal{H} &= \frac{{p_z}^2}{2m} + k_1\Omega_1\rho_1 + k_2\Omega_2\rho_2 + V(R) - E \\ \omega &= \sum_{i=1,2} k_i\left(\mathrm{d}R_{x_i}\wedge \mathrm{d}R_{y_i} + \mathrm{d}\phi_i \wedge \mathrm{d}\rho_i\right) + \mathrm{d}q_z\wedge\mathrm{d}p_z \\ &\quad +\mathrm{d}(E - k_1\Omega_1\rho_1 - k_2\Omega_2\rho_2)\wedge \mathrm{d}t \ , \end{aligned} \tag{3.4.5}$$

where

$$R = \left(\left|\boldsymbol{R}_1 - \boldsymbol{R}_2 + (2\rho_1)^{1/2}\,\mathbf{J}\boldsymbol{e}_{\phi_1+\Omega_1 t} - (2\rho_2)^{1/2}\,\mathbf{J}\boldsymbol{e}_{\phi_2+\Omega_2 t}\right|^2 + {q_z}^2\right)^{1/2} \ . \tag{3.4.6}$$

Note that the dynamics on level sets $\{H = E\}$ of (3.4.3) correspond to the dynamics of the level set $\{\mathcal{H} = 0\}$ of (3.4.5). Introducing the asymptotic vertical energy

$$E_z = E - k_1\Omega_1\rho_1 - k_2\Omega_2\rho_2 \ ,$$

we obtain

$$\begin{aligned} \mathcal{H} &= \frac{{p_z}^2}{2m} + V(R) - E_z \\ \omega &= \sum_{i=1,2} k_i\left(\mathrm{d}R_{x_i}\wedge \mathrm{d}R_{y_i} + \mathrm{d}\phi_i \wedge \mathrm{d}\rho_i\right) + \mathrm{d}q_z\wedge\mathrm{d}p_z + \mathrm{d}E_z\wedge\mathrm{d}t \ , \end{aligned} \tag{3.4.7}$$

where R is given by (3.4.6).

Let $i \in \{1,2\}$ and let us denote by $\boldsymbol{R}_i^+$, ρ_i^+, ϕ_i^+ and ${p_z}^+$ (respectively $\boldsymbol{R}_i^-$, ρ_i^-, ϕ_i^- and ${p_z}^-$) the limits of the quantities $\boldsymbol{R}_i(t)$, $\rho_i(t)$, $\phi_i(t)$ and $p_z(t)$ as $t \to +\infty$ (respectively $t \to -\infty$). Furthermore, let us denote by ${q_z}^+$ the sign of $q_z(t)$ as $t \to +\infty$ and by ${q_z}^-$ the sign of $q_z(t)$ as $t \to -\infty$. The scattering map is the map $S : \mathbb{R}^9 \times \{+,-\} \to \mathbb{R}^9 \times \{+,-\}$ given by

$$\begin{aligned} &S\left({\boldsymbol{R}_1}^-, {\boldsymbol{R}_2}^-, {\rho_1}^-, {\rho_2}^-, {\phi_1}^-, {\phi_2}^-, {p_z}^-, {q_z}^-\right) = \\ &\quad = \left({\boldsymbol{R}_1}^+, {\boldsymbol{R}_2}^+, {\rho_1}^+, {\rho_2}^+, {\phi_1}^+, {\phi_2}^+, {p_z}^+, {q_z}^+\right) \ . \end{aligned} \tag{3.4.8}$$

Remarks We note that:

i) the scattering map is well–defined provided $|q_z(t)| \to \infty$ as $|t| \to \infty$ and the particles do not go through close approaches in their orbits.

ii) a complete study of the scattering map should take into account the asymptotic time difference for the particles to travel between two points with and without an interaction. We skip an analytic study of this asymptotic time difference since the Coulomb potential leads to unbounded time difference due to a logarithmic term. However, we do a numerical study of the time difference in section 3.4.3.

3.4.1 Scattering map in the limit of widely separated trajectories

Throughout this section we will consider that the projections of the helices on the horizontal plane in the limits $|t| \to \infty$ are two widely separated circles, i.e. the following condition is satisfied

$$|\boldsymbol{\rho}_1|^2 + |\boldsymbol{\rho}_2|^2 << |\boldsymbol{R}_1 - \boldsymbol{R}_2|^2 \ . \tag{3.4.9}$$

Expanding (3.4.6) we obtain

$$\begin{aligned} 1/R &= \left(|\boldsymbol{R}_1 - \boldsymbol{R}_2|^2 + 2(\boldsymbol{R}_1 - \boldsymbol{R}_2).\boldsymbol{w} + |\boldsymbol{w}|^2 + {q_z}^2\right)^{-1/2} \qquad (3.4.10) \\ &= \left(|\boldsymbol{R}_1 - \boldsymbol{R}_2|^2 + {q_z}^2\right)^{-1/2} \left(1 + \frac{2(\boldsymbol{R}_1 - \boldsymbol{R}_2).\boldsymbol{w} + |\boldsymbol{w}|^2}{|\boldsymbol{R}_1 - \boldsymbol{R}_2|^2 + {q_z}^2}\right)^{-1/2} \ , \end{aligned}$$

where

$$\boldsymbol{w} = (2\rho_1)^{1/2}\,\mathbf{J}\boldsymbol{e}_{\phi_1+\Omega_1 t} - (2\rho_2)^{1/2}\,\mathbf{J}\boldsymbol{e}_{\phi_2+\Omega_2 t} \ . \tag{3.4.11}$$

From (3.4.9), we obtain the inequality

$$\left|\frac{2(\boldsymbol{R}_1 - \boldsymbol{R}_2).\boldsymbol{w} + |\boldsymbol{w}|^2}{|\boldsymbol{R}_1 - \boldsymbol{R}_2|^2 + {q_z}^2}\right| < 1$$

and hence, using Taylor series, we obtain

$$\begin{aligned} \left(1 + \frac{2(\boldsymbol{R}_1 - \boldsymbol{R}_2).\boldsymbol{w} + |\boldsymbol{w}|^2}{|\boldsymbol{R}_1 - \boldsymbol{R}_2|^2 + {q_z}^2}\right)^{-1/2} &= 1 - \frac{1}{2}\frac{2(\boldsymbol{R}_1 - \boldsymbol{R}_2).\boldsymbol{w} + |\boldsymbol{w}|^2}{|\boldsymbol{R}_1 - \boldsymbol{R}_2|^2 + {q_z}^2} \qquad (3.4.12) \\ &\quad + O\left(\frac{|\boldsymbol{R}_1 - \boldsymbol{R}_2|^2}{\left(|\boldsymbol{R}_1 - \boldsymbol{R}_2|^2 + {q_z}^2\right)^2}\right) \ . \end{aligned}$$

Putting together (3.4.10) and (3.4.12), we get

$$V(R) = \frac{G}{\left(|\boldsymbol{R}_1-\boldsymbol{R}_2|^2+{q_z}^2\right)^{1/2}} - \frac{G}{2}\frac{2(\boldsymbol{R}_1-\boldsymbol{R}_2).\boldsymbol{w}+|\boldsymbol{w}|^2}{(|\boldsymbol{R}_1-\boldsymbol{R}_2|^2+{q_z}^2)^{3/2}} + O\left(\frac{|\boldsymbol{R}_1-\boldsymbol{R}_2|^2}{\left(|\boldsymbol{R}_1-\boldsymbol{R}_2|^2+{q_z}^2\right)^{5/2}}\right) .$$

Neglecting terms of order $O\left(|\boldsymbol{R}_1-\boldsymbol{R}_2|^2/(|\boldsymbol{R}_1-\boldsymbol{R}_2|^2+{q_z}^2)^{5/2}\right)$ we obtain an effective Hamiltonian system for the interaction of two widely separated charges in a uniform magnetic field, which is given by

$$\begin{aligned}
\mathcal{H}^{eff} &= \frac{{p_z}^2}{2m} + \mathcal{V}^{eff} - E_z \\
\omega &= \sum_{i=1,2} k_i\left(\mathrm{d}R_{x_i}\wedge \mathrm{d}R_{y_i} + \mathrm{d}\phi_i \wedge \mathrm{d}\rho_i\right) + \mathrm{d}q_z\wedge\mathrm{d}p_z + \mathrm{d}E_z\wedge\mathrm{d}t \\
\mathcal{V}^{eff} &= \frac{G}{\left(|\boldsymbol{R}_1-\boldsymbol{R}_2|^2+{q_z}^2\right)^{1/2}} - \frac{G}{2}\frac{2(\boldsymbol{R}_1-\boldsymbol{R}_2).\boldsymbol{w}+|\boldsymbol{w}|^2}{(|\boldsymbol{R}_1-\boldsymbol{R}_2|^2+{q_z}^2)^{3/2}} , \qquad (3.4.13)
\end{aligned}$$

where $\boldsymbol{w}$ is given in (3.4.11).

Averaging

Since the two helices are widely separated, the values of the particles gyrophases have a negligible effect in the particles distance and hence on the strength of the Coulomb interaction. This enable us to average (3.4.13) with respect to the gyrophases ϕ_1 and ϕ_2. We obtain the Hamiltonian system

$$\begin{aligned}
\overline{\mathcal{H}^{eff}} &= \frac{{p_z}^2}{2m} + \overline{\mathcal{V}^{eff}} - E_z \qquad (3.4.14) \\
\omega &= \sum_{i=1,2} k_i\left(\mathrm{d}R_{x_i}\wedge \mathrm{d}R_{y_i} + \mathrm{d}\phi_i \wedge \mathrm{d}\rho_i\right) + \mathrm{d}q_z\wedge\mathrm{d}p_z + \mathrm{d}E_z\wedge\mathrm{d}t ,
\end{aligned}$$

where

$$\begin{aligned}
\overline{\mathcal{V}^{eff}} &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \mathcal{V}^{eff}\mathrm{d}\phi_1\mathrm{d}\phi_2 \\
&= \frac{G}{\left(|\boldsymbol{R}_1-\boldsymbol{R}_2|^2+{q_z}^2\right)^{1/2}} - \frac{G(\rho_1+\rho_2)}{(|\boldsymbol{R}_1-\boldsymbol{R}_2|^2+{q_z}^2)^{3/2}} .
\end{aligned}$$

Note that averaging with respect to ϕ_1 and ϕ_2 eliminates t and implies conservation of the quantities ρ_1, ρ_2 and E_z. Furthermore, the system (3.4.14) has a three–dimensional group of symmetries (2 translations and 1 rotation) to which correspond the following conserved quantities:

$$\begin{aligned} \boldsymbol{P} &= \mathbf{J}\left(k_1\boldsymbol{R}_1 + k_2\boldsymbol{R}_2\right) \\ L - k_1\rho_1 - k_2\rho_2 &= -\frac{k_1}{2}\left|\boldsymbol{R}_1\right|^2 - \frac{k_2}{2}\left|\boldsymbol{R}_2\right|^2 \ . \end{aligned} \tag{3.4.15}$$

We will use these symmetries and conserved quantities to reduce the Hamiltonian system (3.4.14). We divide our analysis into two cases: $k_1 + k_2 \neq 0$ and $k_1 + k_2 = 0$.

Case $k_1 + k_2 \neq 0$

From (3.4.15) we get that the centre of charge of the guiding centres

$$\frac{k_1\boldsymbol{R}_1 + k_2\boldsymbol{R}_2}{k_1 + k_2} = \frac{-J\boldsymbol{P}}{k_1 + k_2}$$

is a constant. So by a translation we can assume it and $\boldsymbol{P}$ are $\mathbf{0}$. This implies the relation

$$\boldsymbol{R}_2 = -\frac{k_1}{k_2}\boldsymbol{R}_1 \ . \tag{3.4.16}$$

Restricting the Hamiltonian system to the set determined by (3.4.16) gives

$$\begin{aligned} \overline{\mathcal{H}^{eff}} &= \frac{{p_z}^2}{2m} + \overline{\mathcal{V}^{eff}} - E_z \\ \omega &= \frac{{k_1}^2}{e}\mathrm{d}R_{x_1} \wedge \mathrm{d}R_{y_1} + k_1\mathrm{d}\phi_1 \wedge \mathrm{d}\rho_1 + k_2\mathrm{d}\phi_2 \wedge \mathrm{d}\rho_2 + \mathrm{d}q_z \wedge \mathrm{d}p_z + \mathrm{d}E_z \wedge \mathrm{d}t \\ \overline{\mathcal{V}^{eff}} &= \frac{G}{\left({k_1}^2\left|\boldsymbol{R}_1\right|^2/e^2 + {q_z}^2\right)^{1/2}} - \frac{G\left(\rho_1 + \rho_2\right)}{\left({k_1}^2\left|\boldsymbol{R}_1\right|^2/e^2 + {q_z}^2\right)^{3/2}} \ . \end{aligned}$$

The angular momentum given in (3.4.15) is now given by

$$L - k_1\rho_1 - k_2\rho_2 = -\frac{{k_1}^2}{2e}\left|\boldsymbol{R}_1\right|^2 \ . \tag{3.4.17}$$

We introduce a corrected angular momentum

$$p_{\theta_R} = L - k_1\rho_1 - k_2\rho_2 \ , \tag{3.4.18}$$

and note that:

- $-2ep_{\theta_R} > 0$,
- conservation of p_{θ_R} implies that the guiding centres $\boldsymbol{R}_i$ move in circles of radius

$$\sqrt{-2ep_{\theta_R}/{k_i}^2}\ , \qquad i \in \{1,2\}\ .$$

We reduce the system by angular momentum introducing polar coordinates

$$\boldsymbol{R}_1 = \sqrt{-2ep_{\theta_R}/{k_1}^2}\boldsymbol{e}_{\theta_R}\ ,$$

where $\boldsymbol{e}_{\theta_R}$ is defined in the same way as $\boldsymbol{e}_\theta$ in (3.2.6) with θ replaced by θ_R. We note that there is a coordinate singularity when $p_{\theta_R} = 0$ corresponding to the case when $\boldsymbol{R}_1 = \boldsymbol{R}_2 = 0$, which is not a problem since we are dealing with the case of large $|\boldsymbol{R}_1 - \boldsymbol{R}_2|$. To obtain a canonical Hamiltonian system we also introduce signed gyroradius p_{ϕ_i} defined by

$$p_{\phi_i} = k_i \rho_i$$

and reduce by the (extra) degree of freedom $\mathrm{d}E_z \wedge \mathrm{d}t$ corresponding to the coordinates in the extended phase space. We obtain the following one degree of freedom effective Hamiltonian system

$$\begin{aligned}
\overline{H^{eff}} &= \frac{{p_z}^2}{2m} + \overline{V^{eff}} \\
\overline{V^{eff}} &= \frac{G}{\left(-2p_{\theta_R}/e + {q_z}^2\right)^{1/2}} - \frac{G\left(p_{\phi_1}/k_1 + p_{\phi_2}/k_2\right)}{\left(-2p_{\theta_R}/e + {q_z}^2\right)^{3/2}} \\
\omega &= \mathrm{d}q_z \wedge \mathrm{d}p_z + \mathrm{d}\theta_R \wedge \mathrm{d}p_{\theta_R} + \mathrm{d}\phi_1 \wedge \mathrm{d}p_{\phi_1} + \mathrm{d}\phi_2 \wedge \mathrm{d}p_{\phi_2}\ .
\end{aligned} \tag{3.4.19}$$

Without any effect on the preceeding results, we can substitute (3.4.9) by the following (weaker) condition

$$\frac{3}{2}\left(|\boldsymbol{\rho}_1|^2 + |\boldsymbol{\rho}_2|^2\right) < |\boldsymbol{R}_1 - \boldsymbol{R}_2|^2\ . \tag{3.4.20}$$

Noting that condition (3.4.20) is equivalent to $3\left(p_{\phi_1}/k_1 + p_{\phi_2}/k_2\right) < -2p_{\theta_R}/e$, we obtain that the effective potential in (3.4.19) satisfies:

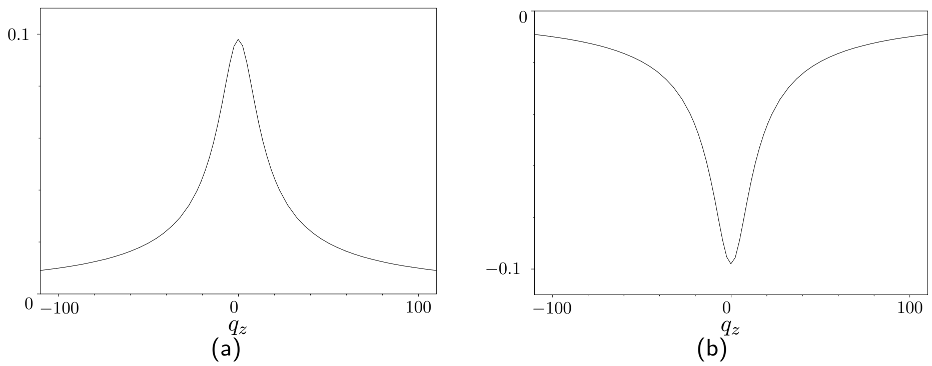


Figure 3.4.1: Plots of $\overline{V^{eff}}$. For both figures $p_{\phi_1}/k_1 = p_{\phi_2}/k_2 = 1$ and $-2p_{\theta_R}/e = 100$. In the left figure $G = 1$ while on the right figure $G = -1$.

- if $G > 0$, then $\overline{V^{eff}}$ is positive for all $q_z \in \mathbb{R}$ (see figure (3.4.1a)). It is increasing for $q_z < 0$, decreasing for $q_z > 0$ and has a maximum E' at $q_z = 0$, given by

$$E' = \frac{G}{(-2p_{\theta_R}/e)^{1/2}} - \frac{G\,(p_{\phi_1}/k_1 + p_{\phi_2}/k_2)}{(-2p_{\theta_R}/e)^{3/2}} \ . \qquad (3.4.21)$$

Furthermore $\lim_{t\to\pm\infty} \overline{V^{eff}} = 0$.

- if $G < 0$, then $\overline{V^{eff}}$ is negative for all $q_z \in \mathbb{R}$ (see figure (3.4.1b)). It is decreasing for $q_z < 0$, increasing for $q_z > 0$ and has global minimum E' (given by (3.4.21)) at $q_z = 0$. Furthermore $\lim_{t\to\pm\infty} \overline{V^{eff}} = 0$.

The next results summarize some of the most relevant properties of the scattering map (3.4.8) for the case of two widely separated charges (satisfying $k_1 + k_2 \neq 0$) moving under the action of a uniform magnetic field and a Coulomb interaction. Before stating the results we need to introduce some notation. We define the *asymptotic change for* $k_1 + k_2 \neq 0$, $\Gamma_{\gamma_0,\gamma_1,\gamma_2,\gamma_3}$, by

$$\Gamma_{\gamma_0,\gamma_1,\gamma_2,\gamma_3} = \gamma_1 \int_{\gamma_0}^{+\infty} \frac{1}{(1+x^2)^{3/2}\,(1 - U_{\gamma_2,\gamma_3}(x))^{1/2}}\, \mathrm{d}x \ ,$$

where $U_{\gamma_2,\gamma_3}(x)$ is given by

$$U_{\gamma_2,\gamma_3}(x) = \frac{\gamma_2}{(1+x^2)^{1/2}} - \frac{\gamma_3}{(1+x^2)^{3/2}} \ ,$$

γ_1, γ_2 and γ_3 are given by

$$\begin{aligned}\gamma_1 &= -\frac{Ge}{2p_{\theta_R}}\sqrt{\frac{m}{2E}}\\ \gamma_2 &= \frac{G}{E}\sqrt{-\frac{e}{2p_{\theta_R}}}\\ \gamma_3 &= \frac{G}{E}\left(\sqrt{-\frac{e}{2p_{\theta_R}}}\right)^3\left(\frac{p_{\phi_1}}{k_1}+\frac{p_{\phi_2}}{k_2}\right)\end{aligned}$$

and γ_0 is defined by

$$\gamma_0 = \begin{cases} -\infty & \text{if } k_1k_2 > 0 \text{ and } E > E' \\ {q_z}^*\sqrt{-e/(2p_{\theta_R})} & \text{if } k_1k_2 > 0 \text{ and } 0 < E < E' \\ -\infty & \text{if } k_1k_2 < 0 \text{ and } E > 0 \end{cases}, \qquad (3.4.22)$$

where ${q_z}^*$ is the only positive root of $\overline{V^{eff}}(q_z) = E$ and E' is given by (3.4.21).

Theorem 3.4.1. *Let $k_1 + k_2 \neq 0$ and $k_1k_2 > 0$ and assume that inequality* (3.4.20) *is satisfied. Then, for every level set $\{\overline{H^{eff}} = E\}$ of* (3.4.19) *such that $E > 0$, the scattering map S is well–defined. Furthermore, there exists $E' \in \mathbb{R}$ given by* (3.4.21) *such that:*

i) for every $E > E'$ we have that ${p_z}^+ = {p_z}^-$, $|{p_z}^\pm| = (2mE)^{1/2}$ and ${q_z}^+ = -{q_z}^-$.

ii) for every $0 < E < E'$ we have that ${p_z}^+ = -{p_z}^-$, $|{p_z}^\pm| = (2mE)^{1/2}$ and ${q_z}^+ = {q_z}^-$.

Let $i \in \{1,2\}$. Whenever the scattering map is well–defined, it also has the following properties:

a) there is no transfer of magnetic moment between the two particles, i.e. the norm of the gyroradius $|\boldsymbol{\rho}_i|^2 = 2(p_{\phi_i}/k_i)^2$ is conserved and hence $\rho_i^+ = \rho_i^-$.

b) the asymptotic gyrophases ϕ_i^+ and ϕ_i^- are related by

$$\Delta\phi_i = \phi_i^+ - \phi_i^- = -\frac{1}{k_i}\Gamma_{\gamma_0,\gamma_1,\gamma_2,\gamma_3}\ .$$

c) *the guiding centres $\boldsymbol{R}_i$ move in circles of radius $\sqrt{-2ep_{\theta_R}/{k_i}^2}$ about $-\mathbf{J}\boldsymbol{P}/\mu$ and rotate (about the centre $-\mathbf{J}\boldsymbol{P}/\mu$) by an angle $\Delta\theta_R$ which, to leading order, is given by*

$$\Delta\theta_R = \frac{1}{e}\Gamma_{\gamma_0,\gamma_1,\gamma_2,\gamma_3} \ .$$

Proof. The first part of the theorem and item i) of the second part follow trivially from an analysis of the Hamiltonian system (3.4.19). To prove item b) of the second part note that

$$\Delta\phi_i = \phi_i^+ - \phi_i^- = \int_{-\infty}^{+\infty} \frac{\partial \overline{H^{eff}}}{\partial p_{\phi_i}} \left(q_z(t), p_z(t)\right) \mathrm{d}t \ .$$

If $E > E'$, then

$$\Delta\phi_i = \int_{-\infty}^{+\infty} \frac{\partial \overline{H^{eff}}}{\partial p_{\phi_i}} \frac{\mathrm{d}t}{\mathrm{d}q_z} \mathrm{d}q_z \ . \tag{3.4.23}$$

From (3.4.19), we obtain

$$\begin{aligned} \frac{\partial \overline{H^{eff}}}{\partial p_{\phi_i}} &= -\frac{G}{k_i\left(-2p_{\theta_R}/e + {q_z}^2\right)^{3/2}} \\ \frac{\mathrm{d}q_z}{\mathrm{d}t} &= \pm\left(\frac{2}{m}\left(E - \overline{V^{eff}}(q_z)\right)\right)^{1/2} \ . \end{aligned} \tag{3.4.24}$$

From (3.4.23) and (3.4.24), we get

$$\Delta\phi_i = -\frac{G}{k_i}\int_{-\infty}^{+\infty} \frac{1}{\left(-2p_{\theta_R}/e + {q_z}^2\right)^{3/2}\left(\frac{2}{m}\left(E - \overline{V^{eff}}(q_z)\right)\right)^{1/2}} \, \mathrm{d}q_z \ . \tag{3.4.25}$$

Making the change of variable $q_z = \left(-2p_{\theta_R}/e\right)^{1/2} x$ in the integral on the right hand side of (3.4.25), we obtain

$$\Delta\phi_i = -\frac{\gamma_1}{k_i}\int_{-\infty}^{+\infty} \frac{1}{\left(1+x^2\right)^{3/2}\left(1 - \gamma_2/\left(1+x^2\right)^{1/2} + \gamma_3/\left(1+x^2\right)^{3/2}\right)^{1/2}} \, \mathrm{d}x \ ,$$

where

$$\begin{aligned} \gamma_1 &= -\frac{Ge}{2p_{\theta_R}}\sqrt{\frac{m}{2E}} \\ \gamma_2 &= \frac{G}{E}\sqrt{-\frac{e}{2p_{\theta_R}}} \\ \gamma_3 &= \frac{G}{E}\left(\sqrt{-\frac{e}{2p_{\theta_R}}}\right)^3\left(\frac{p_{\phi_1}}{k_1} + \frac{p_{\phi_2}}{k_2}\right) \ . \end{aligned} \tag{3.4.26}$$

With a similar procedure, if $0 < E < E'$, we obtain

$$\Delta\phi_i \;=\; \gamma_1 \int_{\gamma_0}^{+\infty} \frac{1}{\left(1+x^2\right)^{3/2}\left(1-\gamma_2/\left(1+x^2\right)^{1/2}+\gamma_3/\left(1+x^2\right)^{3/2}\right)^{1/2}}\,\mathrm{d}x \;,$$

where γ_i ($i \in \{1,2,3\}$) are still given by (3.4.26) and γ_0 is given by

$$\gamma_0 = {q_z}^* \sqrt{-\frac{e}{2p_{\theta_R}}} \;.$$

The first part of item c) follows from (3.4.16), (3.4.17) and (3.4.18) and the reconstruction map in theorem 3.2.1. To complete the proof we just need to evaluate

$$\Delta\theta_R = \lim_{t\to+\infty}\theta_R(t) - \lim_{t\to-\infty}\theta_R(t) = \int_{-\infty}^{+\infty} \frac{\partial\overline{H^{eff}}}{\partial p_{\theta_R}}\left(q_z(t),p_z(t)\right)\mathrm{d}t \;.$$

Noting that

$$\frac{\partial\overline{H^{eff}}}{\partial p_{\theta_R}} = \frac{G}{e\left(-2p_{\theta_R}/e+{q_z}^2\right)^{3/2}} - \frac{3G(p_{\phi_1}/k_1+p_{\phi_2}/k_2)}{e\left(-2p_{\theta_R}/e+{q_z}^2\right)^{5/2}} \;,$$

neglecting the term of order $O\left(\left(-2p_{\theta_R}/e+{q_z}^2\right)^{5/2}\right)$ and proceeding in the same way as we did for the proof of item b) we obtain the required result. □

Theorem 3.4.2. *Let $k_1+k_2 \neq 0$ and $k_1k_2 < 0$ and assume that inequality* (3.4.20) *is satisfied. Then, for every level set $\{\overline{H^{eff}} = E\}$ of* (3.4.19) *such that $E > 0$ the scattering map S is well–defined and we have that $p_z^+ = p_z^-$, $|{p_z}^{\pm}| = (2mE)^{1/2}$ and $q_z^+ = -q_z^-$. Let $i \in \{1,2\}$. Whenever the scattering map is well–defined, it also has the following properties:*

a) there is no transfer of magnetic moment between the two particles, i.e. the norm of the gyroradius $|\boldsymbol{\rho}_i|^2 = 2(p_{\phi_i}/k_i)^2$ is conserved and hence $\rho_i^+ = \rho_i^-$.

b) the asymptotic gyrophases ϕ_i^+ and ϕ_i^- are related by

$$\Delta\phi_i = \phi_i^+ - \phi_i^- = -\frac{1}{k_i}\Gamma_{\gamma_0,\gamma_1,\gamma_2,\gamma_3} \;.$$

c) *the guiding centres $\boldsymbol{R}_i$ move in circles of radius $\sqrt{-2ep_{\theta_R}/{k_i}^2}$ about $-\mathbf{J}\boldsymbol{P}/\mu$ and rotate (about the centre $-\mathbf{J}\boldsymbol{P}/\mu$) by an angle $\Delta\theta_R$ which, to leading order, is given by*

$$\Delta\theta_R = \frac{1}{e}\Gamma_{\gamma_0,\gamma_1,\gamma_2,\gamma_3} \ .$$

The proof of theorem 3.4.2 is identical to the proof of theorem 3.4.1 and so we skip it.

Remarks We note that:

i) item i) in theorem 3.4.1 corresponds to "pass–through" behaviour (high energy) while item ii) corresponds to "bounce–back" behaviour (small energy).

ii) in the case of two charges satisfying $k_1 + k_2 \neq 0$ and $k_1 k_2 < 0$ (as in theorem 3.4.2) the initial conditions that correspond to the scattering problem have energy $E > 0$. To level sets with negative energies (bounded below by E' given in (3.4.21)) correspond bounded motions and hence, the scattering map is not well–defined. In this case we have "molecule–like" behaviour in agreement with section 3.3.3.

iii) if $E > |E'|$ then $|\gamma_2|, |\gamma_3| < 1$ and we can make the following estimate:

$$\Gamma_{\gamma_0,\gamma_1,\gamma_2,\gamma_3} = \gamma_1 \int_{-\infty}^{+\infty} \frac{1}{(1+x^2)^{3/2}}\, \mathrm{d}x + O\left(\gamma_1\gamma_2\right) \ .$$

Evaluating the integral above, we obtain

$$\Gamma_{\gamma_0,\gamma_1,\gamma_2,\gamma_3} = 2\gamma_1 + O\left(\gamma_1\gamma_2\right) = -\frac{Ge}{p_{\theta_R}}\sqrt{\frac{m}{2E}} + O\left(G^2\left(\frac{m}{2E^3}\right)^{1/2}\left(-\frac{e}{2p_{\theta_R}}\right)^{3/2}\right) \ .$$

Case $k_1 + k_2 = 0$

Let us consider again the Hamiltonian system (3.4.14). We change coordinates to

$$\begin{aligned} \boldsymbol{C} &= \frac{1}{2}\left(\boldsymbol{R}_1 + \boldsymbol{R}_2\right) \\ \boldsymbol{P} &= \frac{1}{2\kappa}\mathbf{J}\left(\boldsymbol{R}_1 - \boldsymbol{R}_2\right) \ , \end{aligned} \qquad (3.4.27)$$

introduce signed gyroradius p_{ϕ_i} defined by

$$p_{\phi_i} = k_i \rho_i$$

and reduce by the (extra) degree of freedom $\mathrm{d}E_z \wedge \mathrm{d}t$ corresponding to the coordinates in the extended phase space obtaining the following one degree of freedom effective Hamiltonian system

$$\begin{aligned} \overline{H^{eff}} &= \frac{{p_z}^2}{2m} + \frac{G}{(4\kappa^2|\boldsymbol{P}|^2 + {q_z}^2)^{1/2}} - \frac{G\,(p_{\phi_1}/k_1 + p_{\phi_2}/k_2)}{(4\kappa^2|\boldsymbol{P}|^2 + {q_z}^2)^{3/2}} \\ \omega &= \mathrm{d}q_z \wedge \mathrm{d}p_z + \mathrm{d}C_x \wedge \mathrm{d}P_x + \mathrm{d}C_y \wedge \mathrm{d}P_y + \mathrm{d}\phi_1 \wedge \mathrm{d}p_{\phi_1} + \mathrm{d}\phi_2 \wedge \mathrm{d}p_{\phi_2}\ . \end{aligned} \qquad (3.4.28)$$

Using the weaker form (3.4.20) of condition (3.4.9) we obtain that $3\,(p_{\phi_1}/k_1 + p_{\phi_2}/k_2) < 4\kappa^2|\boldsymbol{P}|^2$ from where we get that the effective potential in (3.4.28) satisfies:

- $\overline{V^{eff}}$ is negative for all $q_z \in \mathbb{R}$ (see figure (3.4.1b)). It is decreasing for $q_z < 0$, increasing for $q_z > 0$ and has global minimum E' given by

  $$E' = \frac{G}{2|\kappa||\boldsymbol{P}|} - \frac{G\,(p_{\phi_1}/k_1 + p_{\phi_2}/k_2)}{(2|\kappa||\boldsymbol{P}|)^3} \qquad (3.4.29)$$

  at $q_z = 0$. Furthermore $\lim_{t\to\pm\infty} \overline{V^{eff}} = 0$.

The next result summarizes some of the most relevant properties of the scattering map (3.4.8) for the case of two widely separated charges (satisfying $k_1 + k_2 = 0$) moving under the action of a uniform magnetic field and a Coulomb interaction. Before stating the results we need to introduce some notation. We define the *asymptotic change for* $k_1 + k_2 = 0$, $\Gamma^0_{\gamma_0,\gamma_1,\gamma_2,\gamma_3}$, by

$$\Gamma^0_{\gamma_1^0,\gamma_2^0,\gamma_3^0} = \gamma_1^0 \int_{-\infty}^{+\infty} \frac{1}{(1+x^2)^{3/2}\left(1 - U_{\gamma_2^0,\gamma_3^0}(x)\right)^{1/2}}\ \mathrm{d}x\ ,$$

where $U_{\gamma_2^0,\gamma_3^0}(x)$ is given by

$$U_{\gamma_2^0,\gamma_3^0}(x) = \frac{\gamma_2^0}{(1+x^2)^{1/2}} - \frac{\gamma_3^0}{(1+x^2)^{3/2}}$$

and γ_1^0, γ_2^0 and γ_3^0 are given by

$$\begin{aligned}
\gamma_1^0 &= \frac{G}{4\kappa^2|\boldsymbol{P}|^2}\sqrt{\frac{m}{2E}} \\
\gamma_2^0 &= \frac{G}{2|\kappa||\boldsymbol{P}|E} \\
\gamma_3^0 &= \frac{G}{8|\kappa|^3|\boldsymbol{P}|^3E}\left(\frac{p_{\phi_1}}{k_1}+\frac{p_{\phi_2}}{k_2}\right) \ .
\end{aligned}$$

Theorem 3.4.3. *Let* $k_1 + k_2 = 0$ *and assume that inequality* (3.4.20) *is satisfied. Then, for every level set* $\{\overline{H^{eff}} = E\}$ *of* (3.4.28) *such that* $E > 0$ *the scattering map* S *is well–defined and we have that* $p_z^+ = p_z^-$, $|{p_z}^{\pm}| = (2mE)^{1/2}$ *and* $q_z^+ = -q_z^-$. *Let* $i \in \{1, 2\}$. *Whenever the scattering map is well–defined, it also has the following properties:*

a) there is no transfer of magnetic moment between the two particles, i.e. the norm of the gyroradius $|\boldsymbol{\rho}_i|^2 = 2(p_{\phi_i}/k_i)^2$ *is conserved and hence* $\rho_i^+ = \rho_i^-$.

b) the asymptotic gyrophases ϕ_i^+ *and* ϕ_i^- *are related by*

$$\Delta\phi_i = \phi_i^+ - \phi_i^- = -\frac{1}{k_i}\Gamma^0_{\gamma_1^0,\gamma_2^0,\gamma_3^0} \ .$$

c) the distance between the guiding centres is conserved:

$$\left|{\boldsymbol{R}_1}^+ - {\boldsymbol{R}_2}^+\right| = \left|{\boldsymbol{R}_1}^- - {\boldsymbol{R}_2}^-\right| = 2|\kappa|\,|\boldsymbol{P}| \ .$$

d) to leading order, the asymptotic guiding centres ${\boldsymbol{R}_i}^+$ *and* ${\boldsymbol{R}_i}^-$ *are related by*

$${\boldsymbol{R}_i}^+ - {\boldsymbol{R}_i}^- = 4\kappa^2\Gamma^0_{\gamma_1^0,\gamma_2^0,\gamma_3^0}\boldsymbol{P} \ .$$

Proof. The first part of the theorem and item i) of the second part follow trivially from an analysis of the Hamiltonian system (3.4.28). The proof of item b) is analogous to the proof of item b) of theorem 3.4.1. Item c) follows from (3.4.27). To prove item d) we invert (3.4.27) to obtain

$$\boldsymbol{R}_1 = \boldsymbol{C} - \kappa\mathbf{J}\boldsymbol{P} \ , \qquad \boldsymbol{R}_2 = \boldsymbol{C} + \kappa\mathbf{J}\boldsymbol{P} \ .$$

Noting that

$$\boldsymbol{R}_i{}^+ - \boldsymbol{R}_i{}^- = \lim_{t\to+\infty} \boldsymbol{C}(t) - \lim_{t\to-\infty} \boldsymbol{C}(t) = \int_{-\infty}^{+\infty} \frac{\partial \overline{H^{eff}}}{\partial \boldsymbol{P}} \left(q_z(t), p_z(t)\right) \mathrm{d}t$$

and proceeding in the same way as we did for the proof of item c) of theorem 3.4.1 we obtain the result. □

Remark We note that:

i) in the setting of theorem 3.4.3, to level sets with negative energies (bounded below by E' given in (3.4.29)) correspond bounded motions and the scattering map is not well–defined. In such energy regimes the dynamics exhibit "molecule–like" behaviour in agreement with section 3.3.3. Otherwise, in level sets with positive energies the dynamics exhibit "unbounded" behaviour (more precisely, "pass–trough" behaviour).

ii) if $E > |E'|$ then $|\gamma_2|, |\gamma_3| < 1$ and we can make the following estimate:

$$\Gamma^0_{\gamma_1,\gamma_2,\gamma_3} = \gamma_1 \int_{-\infty}^{+\infty} \frac{1}{(1+x^2)^{3/2}}\, \mathrm{d}x + O\left(\gamma_1\gamma_2\right) \; .$$

Evaluating the integral above, we obtain

$$\Gamma_{\gamma_0,\gamma_1,\gamma_2,\gamma_3} = 2\gamma_1 + O\left(\gamma_1\gamma_2\right) = \frac{G}{2\kappa^2|\boldsymbol{P}|^2}\sqrt{\frac{m}{2E}} + O\left(\frac{G^2}{8|\kappa|^3|\boldsymbol{P}|^3}\left(\frac{m}{2E^3}\right)^{1/2}\right) \; .$$

We should note that theorems 3.4.1, 3.4.2 and 3.4.3 apply only to the averaged systems (3.4.19) and (3.4.28). Furthermore, the error of approximation caused by the derivation of such Hamiltonian systems has not been analysed.

3.4.2 Transfer of energy

In the previous section we have done our analysis in a frame moving with the vertical centre of mass and did not observe any transfer of energy between the two particles. In this section we will see that in the original fixed frame there is a transfer of energy between the particles in the case of "bouncing–back" behaviour.

Assume the conditions of theorem 3.4.1 and recall that there exists $E' > 0$ such that for every level set $\{\overline{H^{eff}} = E\}$ of (3.4.19) with $0 < E < E'$ we have that ${p_z}^+ = -{p_z}^-$, $|{p_z}^\pm| = (2mE)^{1/2}$ and ${q_z}^+ = {q_z}^-$, i.e. the particles "bounce–back" during an interaction.

Let $i \in \{1, 2\}$ and let us denote by $p^+_{z_i}$ (respectively $p^-_{z_i}$) the limit of $p_{z_i}(t)$ as $t \to +\infty$ (respectively $t \to -\infty$). By the reconstruction map in theorem 3.2.1 we obtain that

$$p^\pm_{z_1} = \frac{m_1}{M}P_z + {p_z}^\pm \ , \qquad p^\pm_{z_2} = \frac{m_2}{M}P_z - {p_z}^\pm \ .$$

Thus, the change on the vertical kinetic energy of the first particle is given by

$$\begin{aligned} \Delta KE_1 &= \frac{1}{2m_1}\left({p^+_{z_1}}^2 - {p^-_{z_1}}^2\right) \\ &= \frac{2}{M}{p_z}^+ P_z \ . \end{aligned}$$

Similarly, we obtain that the change on the vertical kinetic energy of the second particle is given by

$$\begin{aligned} \Delta KE_2 &= \frac{1}{2m_2}\left({p^+_{z_2}}^2 - {p^-_{z_2}}^2\right) \\ &= -\frac{2}{M}{p_z}^+ P_z \ . \end{aligned} \tag{3.4.30}$$

Hence, if the vertical z–component of the centre of mass of the two particles moves with non–zero velocity $\dot{C}_z = P_z/M$ there is a transfer of vertical kinetic energy when the two particles "bounce–back".

3.4.3 Chaotic scattering: some numerical results

In this section we provide some numerical results that give strong evidence in favour of the existence of chaotic scattering for the problem of two charges moving in $\mathbb{R}^3$ under the action of a uniform magnetic field and a Coulomb interaction when the averaging assumptions of the previous subsection are not satisfied.

Procedure

We numerically integrate Hamilton's equations associated with the reduced Hamiltonian system given in theorem 3.2.1 under the following conditions: we fix the values of the parameters

$$e_1 = 1 \ , \qquad m_1 = 1 \ , \qquad e_2 = -3 \ , \qquad m_2 = 5 \ ,$$

$$c = 1 \ , \qquad B = 1 \ , \qquad \epsilon_0 = 0.1 \ ,$$

the initial conditions

$$p_r(0) = 0 \ , \qquad \phi(0) = 0 \ , \qquad p_\phi(0) = 1$$

and the level sets of the conserved quantities

$$p_\theta = 1 \ , \qquad P_z = 0 \ .$$

Furthermore, if the energy $E > 0$ and the initial conditions $r(0) = R$ and $q_z(0) = h > 0$ are fixed, we obtain the value of the remaining initial condition $p_z(0)$ as a function of E, R and h:

$$p_z(0) = \pm \left(\frac{3E}{2} - \frac{49}{64R^2} - \frac{25R^2}{64} - \frac{5}{8}\left(R - \frac{7}{5R} \right) + \frac{75}{4\pi\left(R^2 + h^2\right)^{1/2}} + \frac{21}{32} \right)^{1/2} .$$

For a given $h > 0$ choosing $p_z(0) \leq 0$ will make the particles move towards each other, interact and then move apart again.

We now fix the values of the energy E and initial relative vertical position $q_z(0) = h$ and let $T_{E,h}(R)$ be the bigger instant of time needed for the particles starting with horizontal distance $r(0) = R$ (and corresponding $p_z(0)$) to reach a vertical distance of absolute value h before escaping to ∞ when subject to a Coulomb interaction and let $T^0_{E,h}(R)$ be the time needed for the particles starting with the same initial conditions to reach a vertical distance $q_z\left(T^0_{E,h}(R)\right) = -h$ when moving freely in a uniform magnetic field. In the case of zero interaction the particles move with constant velocity in the vertical direction (equal to $(2E/m)^{1/2}$) and hence we obtain

$$T^0_{E,h}(R) = 2h\sqrt{\frac{m}{2E}} \ .$$

We define the *time difference map* $\tau_{E,h} : \mathbb{R}^+ \to \mathbb{R}$ by

$$\tau_{E,h}(R) = T_{E,h}(R) - T^0_{E,h}(R) \ ,$$

and define the *sign map* $\sigma_{E,h} : \mathbb{R}^+ \to \{+,-\}$ by

$$\sigma_{E,h}(R) = \mathrm{sign}\left(q_z\left(T_{E,h}(R)\right)\right) \ ,$$

where $\mathrm{sign}(x)$ denotes the sign of $x \in \mathbb{R}$. The sign map $\sigma_{E,h}(R)$ associates to each orbit the sign $+$ if the particles "bounce–back" during the interaction and the sign $-$ if the particles "pass–through".

Note that the definition given for the time difference map avoids the problems related with the unboundedness of the logarithmic term associated with the general time difference map (i.e. $h = \infty$) by making h finite. However, we still have that $\tau_{E,h} \to \infty$ as $h \to \infty$.

Simulations

The plots for the maps $\sigma_{E,h}(R)$ and $\tau_{E,h}(R)$ that we show below were made for $h = 500$ and values of energy $E = 20$, $E = 10$, $E = 5$ and $E = 1$.

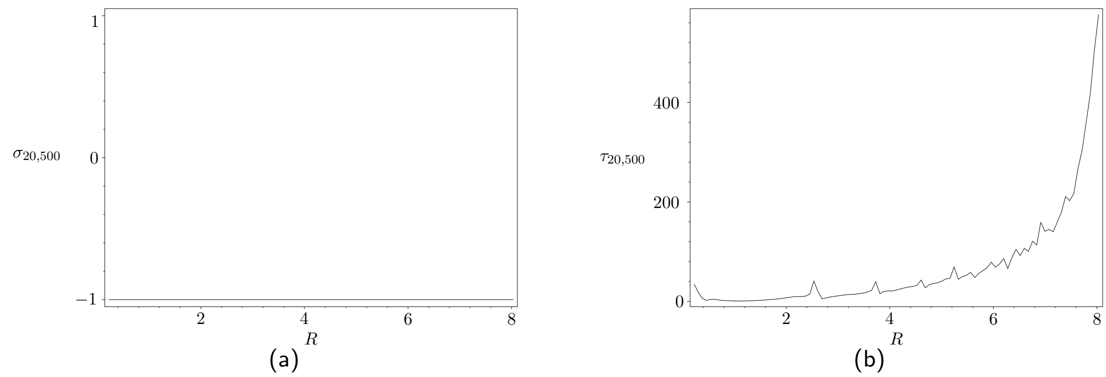


Figure 3.4.2: Plots of $\sigma_{E,h}(R)$ and $\tau_{E,h}(R)$ for $E = 20$ and $h = 500$. The motion is regular: the only type of dynamical behaviour observed is "pass–through" behaviour and the time difference map is reasonably smooth.

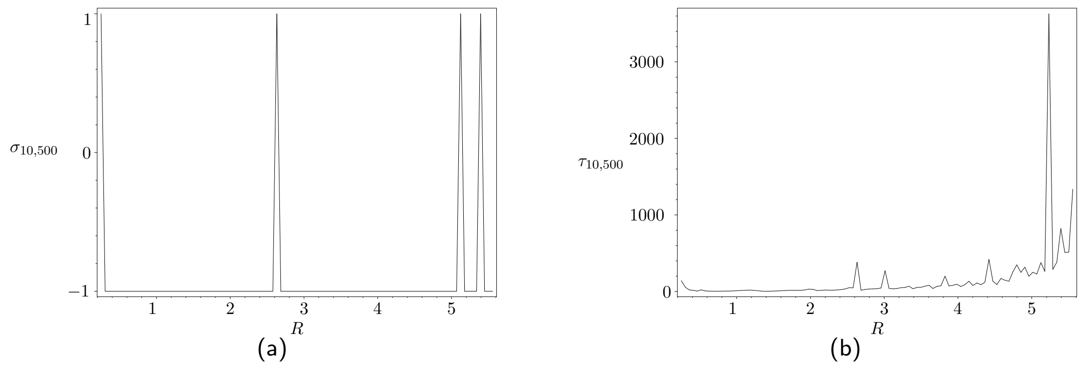


Figure 3.4.3: Plots of $\sigma_{E,h}(R)$ and $\tau_{E,h}(R)$ for $E = 10$ and $h = 500$. The motion is still mostly regular but there is already a small number of choices of initial horizontal distance that lead to a "bouncing–back" behaviour. The time difference map is still very regular.

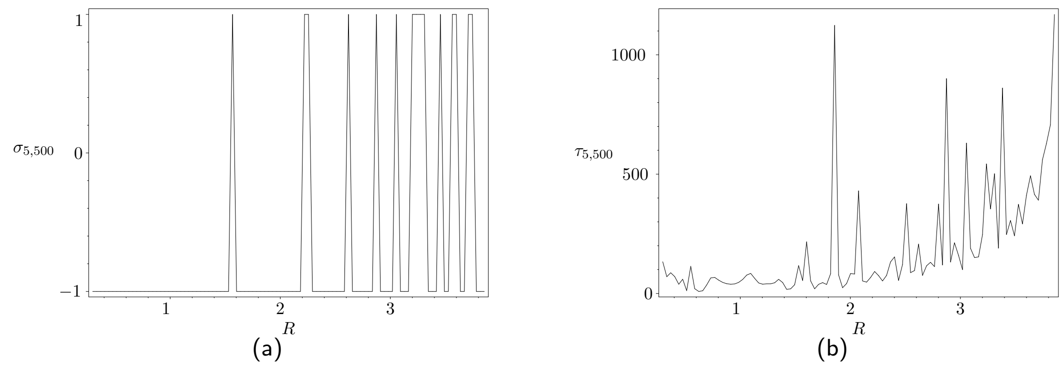


Figure 3.4.4: Plots of $\sigma_{E,h}(R)$ and $\tau_{E,h}(R)$ for $E = 5$ and $h = 500$. Although the dominant dynamical behaviour is "pass–through" there is already a large amount of "bouncing–back" behaviour. The time difference map loses much of its regularity.

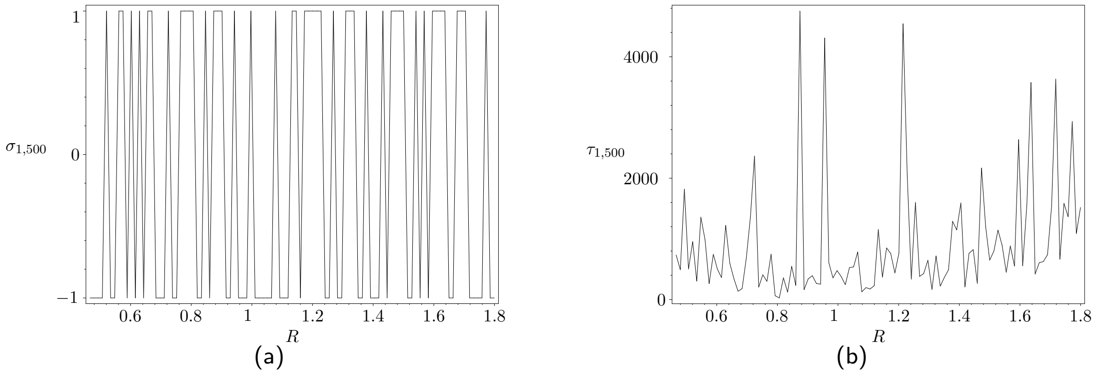


Figure 3.4.5: Plots of $\sigma_{E,h}(R)$ and $\tau_{E,h}(R)$ for $E = 1$ and $h = 500$. Chaotic scattering: both dynamical behaviours are observable and small changes in the initial horizontal distance might lead to each one of the observed behaviours. The time difference map is very irregular.

Note that large energies lead to either a large vertical relative velocity or a large horizontal displacement between the two particles. For large vertical relative velocities the interaction time between the two particles is small while for large horizontal displacements the interaction strength is small. This is the main reason for the regularity of the dynamics for large energies and the lack of it for small energies.

3.5 Conclusions

We have proved that the Hamiltonian system (3.1.5) can always be reduced to one with three degrees of freedom. Moreover, we have proven that it can be reduced to one with two degrees of freedom for the special case of same sign charges when the particles have equal gyrofrequencies (equal ratio of charge to mass) and on some special submanifolds. Furthermore, we explicitly computed the reduced Hamiltonian systems and corresponding reconstruction maps for the reduced dynamics, enabling us to lift the dynamics from the reduced spaces and hence obtain a description for the dynamics on the initial phase space.

Assuming that the interaction potential is Coulomb we have identified an invariant submanifold where the reduced dynamics are just the reduced dynamics associated with the interaction of two particles moving in a plane under the action of a uniform magnetic field (orthogonal to the plane of motion). In this invariant plane we obtain that the system always reduces by a further degree of freedom which leads to an integrable system in the case of two particles with equal gyrofrequency (and some special submanifolds). As a consequence of the results in chapter 2 we obtain that in the case of particles with opposite sign and non–zero gyrofrequencies sum, the system contains a suspension of a non–trivial subshift of finite type on level sets of high energy and hence it is non–integrable.

We have studied the scattering map associated with this problem in the limit where the two particles trajectories are widely separated. We have obtained that the magnetic moment is conserved and that the guiding centres either rotate about a fixed centre during an interaction in the case of two charges whose sum is not zero or drift in a direction determined by the linear momentum if the two charges sum is zero. We give explicit formulas for rotations and drifts in this regime. In the frame moving with the vertical centre of mass we showed there is no transfer of energy between the two particles in this regime. In a fixed frame, however, energy is transferred between the particles in the case of "bounce-back", unless the vertical centre of mass speed is zero. This extends the results in [2, 11, 12] from the case of infinite to finite magnetic field. To compute energy transport in a temperature or density gradient, however, one would also need to take into account the effects of the interaction–induced horizontal rotation (or drift) of the guiding centres that we have found, which move energy and particles horizontally in all cases of scattering.

We have also made a numerical study of the scattering map without using the assumption that the two particles trajectories are widely separated. We observed regular behaviour for large energies and chaotic scattering for small positive energies.

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