Some questions on singularities in 2 and 3 dimensions.

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This thesis has two parts, summarized below. The links between them are discussed at the end of this introduction.

Part 1 is concerned with the problem of giving necessary and sufficient conditions for a family of surfaces to have a simultaneous resolution; this property can be regarded as a very weak form of equisingularity (cf. [Te]). I conjecture that, roughly speaking, the plurigenera P_n of a family of singular surfaces of general type are upper semi-continuous and that simultaneous resolution is possible if and only if P_n is locally constant for some $n \geq 2$ (equivalently, for all $n \geq 2$). Two cases of this conjecture are proved, under different hypotheses on the special fibre. The techniques used are the use of adjunction ideals, suggested to me by Reid, and the results of Brieskorn, Tyurina and others on deformations of Du Val singularities (also known as rational double points, ...). A very similar approach was used by Lipman [Li] for the study of deformations of arbitrary rational singularities.

Part 2 is concerned with canonical singularities, as defined by Reid [R3]. We first prove that in dimensions ≤ 4 they are Cohen-Macaulay, and then deduce a corollary on the invariance of plurigenera in some special circumstances; this answers, in part, questions asked me by Reid. Since these results were proved, Elkik and Gabber have shown that canonical singularities are Cohen-Macaulay in all dimensions. We then consider some specific classes of singularities, and prove that they are canonical. The idea of using the techniques and results of Kulikov in this situation was suggested to me by Dave Morrison, and I subsequently learnt that Theorem 5 was already known to him and others, including Pinkham and Wahl. The point of this section is twofold; firstly it gives an analysis of what are the simplest canonical singularities, and secondly it shows quite explicitly that the contractibility of a given configuration of surfaces in a 3-fold is a much more delicate question than in the case of curves lying on a surface. The problem of contractibility underlies Chapter 1 as well; a sufficiently strong result would kill certain cohomology groups that are the obstruction to proving the conjecture.

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Notation, definitions and well known facts:

1) ω denotes a dualizing sheaf.

K : a canonical divisor on a normal variety, so that $\omega = \underset{\sim}{O}(K)$ and by definition, $\omega^{[n]} = \underset{\sim}{O}(nK)$ for $n \geq 1$ [R3 §1 Appendix].

Let $P \in X$ be a point of the normal variety X, and let $f:\tilde{X} \to X$ be a resolution.

2) P is rational if $R^i f_* \underset{\sim}{O}_{\tilde{X}} = 0$ for all $i > 0$; by a theorem of Kempf [Ke] this is equivalent to the following condition:

P is Cohen-Macaulay and $f_* \omega_{\tilde{X}} = \omega_X$.

3) If $\dim X = 2$, then P is rational and Gorenstein <=> P is rational and a hypersurface point <=> P is rational and a double point. Such singularities have many names; e.g. Du Val singularities, rational double points, Kleinian singularities. For more details, see [D].

4) If $\dim X = 2$, then P is minimally elliptic or elliptic Gorenstein if P is Gorenstein and $\dim R^1 f_* \underset{\sim}{O}_{\tilde{X}} = 1$; equivalently, P is Gorenstein and $f_* \omega_{\tilde{X}} = \underline{m}_P \cdot X$ (see ch.1, Cor. 3). Suppose that f is a minimal

resolution (i.e. that $f^{-1}(P)$ contains no exceptional curve of the first kind); then we can write $K_{\tilde{X}} = f^* K_X - Z$, where $Z > 0$ and $\mathrm{Supp}\, Z = f^{-1}(P)$. The <u>degree</u> deg P is defined by $\deg P = -Z^2$.
The following facts are proved in [La], [R2] and [Sal].

(i) If $\deg P \geq 3$, then $\deg P = \mathrm{mult}\, P = \mathrm{embdim}\, P$

(ii) If $\deg P \leq 2$, then $\mathrm{mult}\, P = 2$, $\mathrm{embdim}\, P = 3$.

(iii) If $\deg P \geq 3$, then the tangent cone $T_P(X)$ is projectively Gorenstein, and $Bl_P X$ is obtained from $\tilde{X}$ by contracting all the (-2) curves.

5) Of particular interest in the surface case are the <u>simple elliptic</u> singularities and the <u>cusps</u> for the Hilbert modular group. The former (resp. latter) are characterized by having a resolution whose exceptional locus is a smooth elliptic curve (resp. a cycle of smooth rational curves).

CHAPTER 1.

Simultaneous resolution of Gorenstein surfaces.

§0. In this chapter we consider the problem of simultaneous resolution of a family of normal surfaces. Our approach is via the sheaves $\omega^{\otimes n}$ of n-fold 2-forms, which limits us in general to Gorenstein surfaces. This technique was first used in this context by Lipman and Wahl ([Li], [Wa 2]) in their study of rational singularities.

The chapter is organized as follows. In §1 we define the relative canonical model of a singular surface and prove some results of a local nature, analogous to those of Bombieri [Bo]. The proofs, however, are very much simpler; indeed, they are valid in any characteristic. In §§ 2 and 3 we state a conjecture, and prove it for singularities satisfying a certain very simple and explicit condition on their minimal resolutions, and for double points. A corollary is a result of Wahl's on minimally elliptic singularities ([La], [R2], [Wa 1]).

§1. Throughout this section, $f:\tilde{X} \to X$ will denote a desingularization of the unique singular point P of the normal affine surface X, defined over an algebraically closed field of arbitrary characteristic.

Lemma 1: There is a natural inclusion $f_*\omega_{\tilde{X}}^{\otimes n} \to \omega_X^{[n]}$, independent of the resolution chosen.

<u>Proof:</u> This is well-known. In Zariski's notation, the sections of $f_*\omega_{\tilde{X}}^{\otimes n}$ are the <u>absolutely regular</u> n-fold 2-forms.

Henceforth we shall assume that f is the minimal resolution. Recall that the intersection matrix of the exceptional locus of f is negative definite [Mul], and that if E is an irreducible exceptional curve, then $E.K_{\tilde{X}} \geq 0$, with equality $\Leftrightarrow E^2 = -2$ and $p_a(E) = 0$. Such curves are called <u>Du Val curves</u> or <u>(-2) curves</u>.

<u>Definition:</u> Let $\underline{F}$ be a coherent $O_{\tilde{X}}$-module. Then we say that $\underline{F}$ <u>is generated by its sections</u> if the natural map $f^* f_* \underline{F} \to \underline{F}$ is surjective.

<u>Theorem 2:</u> $\omega_{\tilde{X}}^{\otimes n}$ is generated by its sections for all $n \geq 2$.

<u>Proof:</u> Let $Q \in \tilde{X}$ be a closed point; it is enough to prove that the natural inclusion $f_*(\underline{m}_Q \cdot \omega_{\tilde{X}}^{\otimes n}) \to f_*\omega_{\tilde{X}}^{\otimes n}$ is not an isomorphism for any $n \geq 2$.

Let $g: Y \to \tilde{X}$ be the blow-up of Q; put $\ell = g^{-1}(Q)$, $h = f \circ g: Y \to X$. By the Leray spectral sequence, it is enough to show that $R^1 h_*(O(-\ell) \otimes g^* \omega_{\tilde{X}}^{\otimes n}) = 0$ for all $n \geq 2$.

Suppose that $E_1, \ldots, E_r$ are the exceptional curves on $\tilde{X}$; let $\tilde{E}_i$ denote the strict transform of E_i on Y and $\tilde{E}_{r+1}$ denote ℓ.

By the holomorphic functions theorem, it is enough to show that

$H^1(\underline{O}_C(ng^* K_{\tilde{X}} - \ell)) = 0$ for all curves $C = \Sigma\, m_i \tilde{E}_i > 0$. We argue by induction on $\sum_{i=1}^{r+1} m_i = \gamma$, say.

Suppose $\gamma = 1$.

Case (i) : $C = \ell$. Then $h^1(\underline{O}_C(nK_Y - (n+1)\ell)) = h^0(\underline{O}_{\mathbb{P}^1}(-1)) = 0$

Case (ii): $C \neq \ell$. $h^1(\underline{O}_C(nK_Y - (n+1)\ell)) = h^0(\underline{O}_C(C + (n+1)\ell - (n-1)K_Y))$.

Say $C = \tilde{E}_1 : \tilde{E}_1 = g^* E_1 - \mu.\ell$, where $\mu = \text{mult}_Q(E_1) \geq 0$.

So $C.\ell = \mu$, $C^2 = E_1^2 - \mu^2$ and $C.K_Y = E_1.K_{\tilde{X}} + \mu$.

Hence $C.(C + (n+1)\ell - (n-1)K_Y) = E_1^2 - \mu^2 + (n+1)\mu - (n-1)(E_1.K_{\tilde{X}} + \mu)$

$= E_1^2 - (n-1)E_1.K_X - \mu^2 + 2\mu < -(\mu-1)^2$, since f is a minimal resolution and $n \geq 2$.

Hence $h^1(\underline{O}_C(nK_Y - (n+1)\ell)) = 0$ in this case.

Now assume that $\gamma > 1$ and that the result holds for all smaller curves. Let $\tilde{E}$ be a component of C, and set $C - \tilde{E} = C'$. Tensoring the exact sequence

$$0 \to \underline{O}_{\tilde{E}}(-C') \to \underline{O}_C \to \underline{O}_{C'} \to 0$$

with $\underline{O}(nK_Y-(n+1)\ell)$ and applying Serre duality, we see that it is enough to find a component $\tilde{E}$ of C with $\tilde{E}.(C + (n+1)\ell - (n-1)K_Y) < 0$. Then clearly we can assume that C is not a multiple of ℓ. So choose E_i

such that $C.g^*E_i < 0$; set $\mu = \mathrm{mult}_Q(E_i)$. Then

$$\tilde{E}_i.(C + (n+1)\ell - (n-1)K_Y) = (g^*E_i).C - \mu\ \ell.C + (n+1)\mu - (n-1)E_i.K_{\tilde{X}} - \mu(n-1) ;$$

hence we can assume that $\ell.C \leq 1$. Then $\ell.(C + (n+1)\ell - (n-1)K_Y) \leq -1$, and so we can assume that ℓ is not a component of C . If $\ell.C = 0$, then $C = g^*D$ for some $D > 0$ supported on the exceptional locus of $\tilde{X}$. If we choose E_i such that $E_i.D < 0$, then $\tilde{E}_i.(C + (n+1)\ell - (n-1)K_Y) < 0$.

So assume that $\ell.C = 1$. Then $C = g^*D - \ell$, some D as before. Say $Q \in E_\alpha$; then for all components E' of D with $E' \neq E_\alpha$, we can assume that $E'.D \geq 0$, for else $E'.(C + (n+1)\ell - (n-1)K_Y) < 0$ for some such E' .

Now $\tilde{E}_\alpha.(C + (n+1)\ell - (n-1)K_Y)$

$$= E_\alpha.D - \mu + \mu\ (n+1) - (n-1)E_\alpha.K_{\tilde{X}} - \mu(n-1)$$

$$= E_\alpha.D + 1 - (n-1)E_\alpha.K_{\tilde{X}} , \quad \text{since} \quad \mu = 1 .$$

Hence we are done unless $E_\alpha.D = -1$ and $E_\alpha.K_{\tilde{X}} = 0$, since $n \geq 2$. In this case, $D^2 = -1$ and $E'.D = 0$ for all components E' of D with $E' \neq E_\alpha$. Then D contains a non-Du Val component F , say; $F.K_{\tilde{X}} > 0$, and so $\tilde{F}.(C+(n+1)\ell-(n-1)K_Y) < 0$.

<u>Corollary 3</u> (<u>of proof</u>): $R^1f_*\omega_{\tilde{X}}^{\otimes n} = 0$ for all $n \geq 1$.

<u>Definition:</u> X is <u>canonically free</u> if $\omega_{\tilde{X}}$ is generated by its sections.

<u>Proposition 4</u>: X is canonically free if there does not exist an effective divisor C supported on the exceptional locus $\tilde{X}$ with $C^2 = -1$ and that is reduced at at least one of its generic points.

The proof is almost identical to that of the preceding theorem. In fact, we see that X is canonically free unless there is a cycle C as above that is generically reduced along some component F such that $C.F = -1$ and $C.E = 0$ for all other components E of C.

Recall from [A1] that we can contract the Du Val curves on $\tilde{X}$ to get a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{g} & \overline{X} \\ & \searrow^{f} & \downarrow h \\ & & X \end{array}$$

where $\overline{X}$ has only Du Val singularities and $K_{\tilde{X}} = g^* K_{\overline{X}}$. $\overline{X}$ is the <u>Du Val model</u> or the <u>RDP model</u> [Li] or the <u>relative canonical model</u> of X. By construction, for all effective Weil divisors D supported on the exceptional locus of $\overline{X}$, $D.K_{\overline{X}} > 0$. Hence $\omega_{\overline{X}}$ is ample relative to h, so that the graded O_X-algebra $\underline{S} = \bigoplus_{\underline{n \geq 0}} h_* \omega_{\overline{X}}^{\otimes n} = \bigoplus_{\underline{n \geq 0}} f_* \omega_{\tilde{X}}^{\otimes n}$ is of finite type and $\overline{X} \cong \operatorname{Proj} \underline{S}$ as schemes over X. We can improve this as follows.

<u>Corollary 5</u> (i) (conjectured by Reid [R1]): The algebra $\underline{S}$ is generated by its homogeneous components of degrees 1, 2, 3 and 4.

(ii) : If X is canonically free, then the components of degrees 1 and 2 suffice.

(iii): If X has only a rational singularity, then $\underline{S}$ is generated by $f_*\omega_{\tilde{X}}$.

Proof: Recall Mumford's form of Castelnuovo's lemma [Mu2]:

Let $\phi: Y \to S$ be a projective morphism of Noetherian schemes, $\underline{F}$ a coherent O_Y-module and $\underline{L}$ an invertible O_Y-module that is ample relative to ϕ and generated by its sections. Suppose also that $R^i\phi_*(\underline{F} \otimes \underline{L}^{-i}) = 0$ for all $i > 0$.

Then the natural map $\phi_*(\underline{F} \otimes \underline{L}^n) \otimes \phi_*\underline{L} \to \phi_*(\underline{F} \otimes \underline{L}^{n+1})$ is surjective for all $n \geq 0$. (Mumford in fact assumes that $S = \mathrm{Spec}\, k$ for a field k , but his proof carries over to the present context.)

The proof of (i) and (ii) is now immediate, by Theorem 2 and Corollary 3. Part (iii) follows similarly, once we know that rational singularities are canonically free. This is so because if C is a cycle as in the statement of Proposition 4, it must be the fundamental cycle of its support. Then by Artin's results [A2] it must be contractible to a smooth point, contradicting the minimality of the resolution.

Remark: Reid conjectured also that $f_*\omega_{\tilde{X}}^{\otimes 3} \otimes f_*\omega_{\tilde{X}}^{\otimes 3} \to f_*\omega_{\tilde{X}}^{\otimes 6}$ is surjective. I have been unable to settle this, however.

Corollary 6: Suppose that X is Gorenstein, so that we can write $f_*\omega_{\tilde{X}}^{\otimes n} = I_n.\omega_X^{\otimes n}$ as subsheaves of $\omega_X^{\otimes n}$, for some ideal I_n of O_X. I_n is the n th adjunction ideal of X [R1]. Then for all $n \geq 4$, $\tilde{X} \cong Bl_{I_n} X$, and if X is canonically free then $\overline{X} \cong Bl_{I_n} X$ for all $n \geq 2$. Moreover, $\overline{X}$ is the normalization of $Bl_{I_n} X$ for all $n \geq 2$ (resp. the normalization of $Bl_{I_n} X$ for all $n \geq 1$ if X is canonically free).

Proof: Immediate. In fact, $\bigoplus_{n\geq 0} I_n$ is finite over $\bigoplus_{n\geq 0} (I_r)^n$ for all $r \geq 2$ (and if X is canonically free then this holds for all $r \geq 1$), so that I_{rn} is the integral closure $\overline{(I_r)^n}$ of $(I_r)^n$ for all $r \geq 2$, $n \geq 1$ (resp. for all $r \geq 1$, $n \geq 1$ if X is canonically free). Note that in any case $\overline{(I_1)^n} \subseteq I_n$.

Lemma 7 (Laufer): $\dim R^1 f_* O_{\tilde{X}} = \dim(\omega_X/f_*\omega_{\tilde{X}})$.

Proof: This follows immediately from Grothendieck's duality theorem [Ha, p. 210]. In fact the two sheaves are dual under the functor $Ext_X^2(-,\omega_X)$. $\dim R^1 f_* O_{\tilde{X}}$ is the arithmetic genus of the singularity P.

Lemma 8 (Knöller): Suppose that X is Gorenstein, so that $K_{\tilde{X}} = f^* K_X - Z$, with $Z \geq 0$ and $Z = 0 \iff X$ has only a Du Val singularity. Then $I_{n+1} \subseteq I_n$, and $\dim(I_n/I_{n+1}) = -n.Z^2$.

Proof: Apply the Riemann-Roch theorem and Corollary 3 to the exact sequence.

$$0 \to \underline{O}(-(n+1)Z) \to \underline{O}(-nZ) \to \underline{O}_Z(-nZ) \to 0 .$$

Examples (i) : Suppose that $X \subseteq A^3_{\mathbb{C}}$ is defined by $x^2+y^3+z^6 = 0$ (a simple elliptic singularity of degree 1). Then the exceptional locus of the minimal resolution $\tilde{X}$ consists of a smooth elliptic curve E with $E^2 = -1$, so that $Z = E$. It is easy to see that $I_1 = (x,y,z)$, $I_2 = (x,y,z^2)$, $I_3 = (x,y^2,yz,z^3)$ and $I_4 = (x^2,xy,xz,y^2,yz^2,z^4)$, so that $I_4 \neq I_2^2$, although $I_4 = I_1 . I_3 + I_2^2$. X is not canonically free, for $\underline{O}_E(E) = \underline{O}(-Q)$ for some $Q \in E$: Q is then a base point of $\omega_{\tilde{X}}$.

(ii) : Suppose that $X \subseteq A^3_{\mathbb{C}}$ is defined by $x^2+y^4+z^4 = 0$ (a simple elliptic singularity of degree 2). The exceptional locus of $\tilde{X}$ is as above, except that $E^2 = -2$. This time $I_1 = (x,y,z)$, $I_2 = (x,y^2,yz,z^2)$, so that $I_2 \neq I_1^2$.

§2. Henceforth we work over the complex numbers $\mathbb{C}$.

Definition: Let $\underline{X} \to S$ be a flat family of surfaces, where S is a $\mathbb{C}$-scheme or analytic space. A base point $0 \in S$ is fixed; we assume that $\underline{X}_o$ is normal. Shrinking S around 0 if necessary, it follows that $\underline{X}$ is normal for all $\sigma \in S$. We say that the family $\underline{X} \to S$ admits a (minimal) simultaneous resolution if there is a finite surjective map $S' \to S$ and a proper map $\tilde{\underline{X}} \to X' = \underline{X} \times_S S'$ such that $\tilde{\underline{X}}$ is flat over S' for all $\sigma \in S'$, the induced map $\tilde{\underline{X}}_\sigma \to \underline{X}'_\sigma$ is a (minimal) resolution of the singularities of $\underline{X}'_\sigma$. We shall usually suppress mention of the base change.

From now on we shall consider only families where the base S is a smooth curve.

The next result shows that there is essentially no distinction between minimal and non-minimal simultaneous resolutions.

Proposition 9 (Wilson [Wil]): If the family $\underline{X} \to S$ admits a simultaneous resolution, then it admits a minimal simultaneous resolution.

Proof Suppose that there is a simultaneous resolution $f:\tilde{\underline{X}} \to \underline{X}$. By Kodaira's stability theorem [Ko] we can assume that $\tilde{\underline{X}}_o$ is minimal, so that $R^1 f_* \omega^{\otimes 2}_{\tilde{\underline{X}}_o} = 0$. Then the usual upper semi-continuity theorems show that $R^1 f_* \omega^{\otimes 2}_{\tilde{\underline{X}}_\sigma} = 0$ for all σ in a neighbourhood of 0 in S. Following Wilson, this implies that $\tilde{\underline{X}}_\sigma$ is minimal for all such σ.

Definition: Let $\underline{X} \to S$ be as above, and suppose that $\underline{X}_o$ is Gorenstein; then we can assume that $\underline{X}_\sigma$ is Gorenstein for all $\sigma \in S$. For each $\sigma \in S$, let $g_{(\sigma)}:Y_{(\sigma)} \to \underline{X}_\sigma$ be a minimal resolution, and write

$K_{Y_{(\sigma)}} = g^*_{(\sigma)} K_{\underline{X}} - Z_{(\sigma)}$. Define functions $r, d : S \to \mathbb{N}$ by

$$r(\sigma) = \dim R^1 g_{(\sigma)*} \underline{O}_{Y_{(\sigma)}} \quad \text{and}$$

$$d(\sigma) = -Z^2_{(\sigma)} \ .$$

Note that $r(\sigma) = 0 \iff d(\sigma) = 0 \iff \underline{X}_\sigma$ has only Du Val singularities.

<u>Proposition 10:</u> Let $\underline{X} \to S$ be as above, with $\underline{X}_o$ Gorenstein. Then if the family admits a simultaneous resolution, the functions r and d are constant near 0 .

<u>Proof:</u> Let $f:\underline{\tilde{X}} \to X$ be a simultaneous resolutions; by Prop. 9, we can assume that it is minimal. Consider the Cartesian diagram

$$\begin{array}{ccc} \underline{\tilde{X}}_\sigma & \hookrightarrow & \underline{\tilde{X}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \underline{X} & \hookrightarrow & \underline{X} \\ \downarrow & & \downarrow \\ \{\sigma\} & \hookrightarrow & S \end{array}$$

By the base change theorem [EGA III 6.9.8] there are two spectral sequences whose E_2 terms are

$${}'\underline{E}^2_{pq} = \mathrm{Tor}^S_p(R^{-q} f_* \omega^{\otimes n}_{\underline{\tilde{X}}/S} \, , \, \underline{O}_\sigma) \quad \text{and}$$

$${}''\underline{E}^2_{pq} = R^{-p} f_* \mathrm{Tor}^S_q(\omega^{\otimes n}_{\underline{\tilde{X}}/S} \, , \, \underline{O}_\sigma)$$ respectively, which have the same

abutment. By the minimality of f,

$''\underline{E}^2_{pq} = 0$ unless $p = q = 0$. So $'\underline{E}^2_{0,-1} = 0$, and so

$R^1 f_* \omega^{\otimes n}_{\underline{\tilde{X}}/S} = 0$ for all $n \geq 1$. Thus $'\underline{E}^2_{1,-1} = 0$ also, and so the natural

maps $(f_* \omega^{\otimes n}_{\underline{\tilde{X}}/S}) \otimes \underline{O}_\sigma \to f_* \omega^{\otimes n}_{\underline{\tilde{X}}_\sigma}$ are isomorphisms for all $\sigma \in S$ and for all $n \in N$.

We have a commutative diagram of natural maps

$$\begin{array}{ccc}
(f_* \omega^{\otimes n}_{\underline{\tilde{X}}/S}) \otimes \underline{O}_\sigma & \xrightarrow{\ \cong\ } & f_* \omega^{\otimes n}_{\underline{\tilde{X}}_\sigma} \\
\beta_\sigma \downarrow & & \downarrow \\
\omega^{\otimes n}_{\underline{\tilde{X}}/S} \otimes \underline{O}_\sigma & \xrightarrow{\ \cong\ } & \omega^{\otimes n}_{\underline{\tilde{X}}_\sigma}
\end{array}$$

and so β_σ is injective for all $\sigma \in S$, $n \in N$.

Thus $(\omega^{\otimes n}_{\underline{X}/S} / f_* \omega^{\otimes n}_{\underline{\tilde{X}}/S})$ is S- flat for all $n \geqslant 1$. The result now follows from Lemmas 7 and 8.

Elkik has proved that for an arbitrary family of normal surfaces r is upper semi-continuous, and that if there is a simultaneous resolution then r is locally constant [E].

<u>Conjecture 1</u>: The function d is also upper semi-continuous, and r and d are both constant near $0 \iff$ the family $\underline{X} \to S$ admits a simultaneous resolution.

The point is straightforward. We first make some notation; for each

$\sigma \in S$, let $I_{\sigma,n}$ denote the n th adjunction ideal of $\underline{X}_\sigma$.
Now consider the following assertion, where $n \in \mathbb{N}$:

A_n : There is an ideal $\underline{I}_n$ of $\underline{O}_X$ such that for a general point η of S $\underline{I}_n.\underline{O}_{\underline{X}_\eta} \subseteq I_{\eta,n}$, while $\underline{I}_n.\underline{O}_{\underline{X}_o} \supseteq I_{0,n}$.

I claim that if A_n holds for some $n \geq 2$, then Conjecture 1 is true. To establish this, we need a lemma.

Lemma 11: Suppose that $\underline{I}$, $\underline{J}$ are ideals of $\underline{O}_X$ such that $\underline{J}_o \subseteq (\overline{\underline{I}_o})$ (bars again denoting integral closure).
Then $\underline{J} \subseteq \overline{\underline{I}}$.

Proof: Clearly we can assume that $\underline{I} \subseteq \underline{J}$. Write $\underline{O}_X = \underline{O}$. Let $x \in \underline{J}$.
Consider the graded rings $\oplus(\underline{I}^m.T^m)[xT] \hookrightarrow \underline{O}[T]$, where T is an indeterminate of degree one and every element of $\underline{O}$ is of degree zero.
By hypothesis, $\oplus(\underline{I}_o^m T^m)[x_oT]$ is finite over $\oplus(\underline{I}_o^m T^m)$; ie. there exists $n_o \in \mathbb{N}$ such that $\bigoplus_{i=0}^{n_o} (\underline{I}_o^{m-i} \otimes \underline{H}_{i,o}) \to \underline{H}_{m,o}$ is surjective for all $m \geq n_o$, where $\underline{H}_i = \sum_{j=0}^{i} \underline{I}^j.x^{i-j}$. Then by Nakayama's lemma,
$\bigoplus_{i=0}^{n_o} (\underline{I}^{m-i} \otimes \underline{H}_i) \to \underline{H}_m$ is surjective for all $m \geq n_o$, and the lemma is proved.

Suppose then that $n \geq 2$ and that A_n holds, giving an ideal $\underline{I}_n$ as stated. For each $s \geq 1$, set $\underline{I}_{sn} = \overline{(\underline{I}_n^s)}$. Then by Lemma 11 and

Corollary 6, A_{sn} holds for all $s \geq 1$. Letting $I_{\sigma,m}$ denote the m th adjunction ideal of $\underline{X}_\sigma$, we see that $\dim(\underline{O}_{\underline{X}_0}/I_{0,sn}) \geq \dim(\underline{O}_{\underline{X}_\eta}/I_{\eta,sn})$, and the upper semi-continuity of d follows from Lemma 8 by letting $s \to \infty$. Conversely, suppose that r and d are locally constant. Then for all $s \geq 1$, $\dim(\underline{O}_{\underline{X}_0}/\underline{I}_{sn,0}) = \dim(\underline{O}_{\underline{X}_\eta}/\underline{I}_{sn,\eta})$, so that $\underline{O}_{\underline{X}}/\underline{I}_{sn}$ is S-flat and $\underline{I}_{sn,\sigma} = I_{\sigma,sn}$ for all $\sigma \in S$. Then $\underline{I}_{sn} \otimes \underline{O}_{\underline{X}_\sigma} \xrightarrow{\sim} \underline{I}_{sn}$, for all $\sigma \in S$, and so $\mathrm{Proj}(\bigoplus_{s \geq 0} \underline{I}_{sn}) \otimes k(\sigma) \cong \mathrm{Proj}(\bigoplus_{s \geq 0} \underline{I}_{sn,\sigma})$, which is just the relative canonical model of $\underline{X}_\sigma$. Hence we have constructed a simultaneous Du Val model; this can be simultaneously resolved by the results of Tyurina and Brieskorn [Ty], [Br 1-3] (cf. also Remark 5) at the end of Ch. 2).

It follows from the Riemann-Roch formula for the plurigenera of a surface of general type and (a slight extension of) Wilson's results on the "arithmetic plurigenera" [Wi 1,2] that for suitably compactifiable families of surfaces, Conjecture 1 can be written in the following form.

<u>Conjecture 2:</u> Suppose that $\underline{X} \to S$ is a proper morphism with $\underline{X}_0$ normal, Gorenstein and having a smooth model of general type. For each n $\mathbb{N}$, define $\pi_n : S \to \mathbb{N}$ by $\pi_n(\sigma) = P_n(\tilde{\underline{X}}_{(\sigma)})$, $\tilde{\underline{X}}_{(\sigma)}$ being a smooth model of $\tilde{\underline{X}}_\sigma$. Then for all $n \in \mathbb{N}$ π_n is upper semi-continuous, and if for some $n \geq 2$ π_n is locally constant, then the family has a simultaneous resolution.

Remark: Note that Conjecture 2 can be stated without the restriction that $\underline{X}_o$ be Gorenstein. However, the difficulty with families $\underline{X} \to S$ of arbitrary normal surfaces is that although it is possible to define sheaves $\omega_{\underline{X}/S}^{[n]}$ of relative n-fold 2-forms (see [Li]), their formation need not commute with base change. For example, consider the cone on the Veronese and the cone on a normal rational scroll of degree 4 as total spaces of 1-parameter smoothings of the cone on a normal rational quartic. Then (see [Pi]) the first example does not admit a simultaneous resolution, while the second does; if $\omega_{\underline{X}/S}^{[n]} \otimes \underline{O}_\sigma \cong \omega_{\underline{X}_\sigma}^{[n]}$ always, then the argument of Prop. 10 would imply the impossibility of the second example.

Theorem 12: Let $\underline{X} \to S$ be a family of normal Gorenstein surfaces, and assume that $\underline{X}_o$ is canonically free. Then Conjecture 1 holds.

Proof: We first show that in any case the assertion A_1 holds. This is proved by Elkik [E], but we give a proof both for completeness and because hers (which is adapted to a much more general situation) carries rather more notation.

By Hironaka's theorems [Hi] there is a desingularization $f:\tilde{\underline{X}} \to \underline{X}$ such that the strict transform Z of $\underline{X}_o$ is smooth; by generic smoothness, $\tilde{\underline{X}}_\sigma \to \underline{X}_\sigma$ is a desingularization for all $\sigma \in S - \{0\}$. By the Grauert-Riemenschneider vanishing theorem [G-R] $R^i f_* \omega_{\tilde{\underline{X}}/S} = 0$ for all $i > 0$, and so by the base change theorem the natural map $(f_*\omega_{\tilde{X}/S}) \otimes \underline{O}_\sigma \to f_*\omega_{\tilde{\underline{X}}_\sigma}$ is an isomorphism for all $\sigma \in S$. From the natural inclusion $\omega_Z \hookrightarrow \omega_{\tilde{\underline{X}}_o}$ we see that we can define a suitable $\underline{I}$ by $f_*\omega_{\tilde{\underline{X}}/S} = \underline{I}.\omega_{\underline{X}/S}$.

Now suppose that $\underline{X}_o$ is canonically free. As noted in the proof of Corollary 6, $I_{o,n} = \overline{(I^n_{o,1})}$. Moreover, $\overline{\underline{I}^n}.\underline{O}_{\underline{X}_\sigma} \subseteq I_{\sigma,n}$ for all $\sigma \in S - \{0\}$, and by Lemma 11 $\overline{\underline{I}^n}.\underline{O}_{\underline{X}_o} \supseteq I_{o,n}$. Hence assertion A_n holds for all $n \in \mathbb{N}$, and we are done.

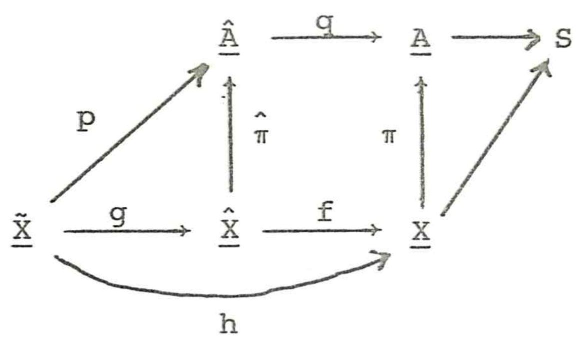
§3. In this section we consider families where the special fibre $\underline{X}_o$ is a double point.

<u>Theorem 13</u>: Conjecture 1 holds in this case.

<u>Proof</u>: We can assume that $\underline{X}_o \subseteq A^3_C$ is defined by $z^2 + f(x,y) = 0$; then considering a versal deformation of $\underline{X}_o$, we can assume that if t is a local parameter at $0 \in S$, $\underline{X} \to A^3_S$ is defined by $z^2+F(x,y,t) = 0$, where $F(x,y,0) = f(x,y)$. $f(x,y)$ is square-free, since $\underline{X}_o$ is normal, and so the family $F \to A^2_S \to S$ of curves defined by $F(x,y,t) = 0$ is reduced. We regard $\underline{X}$ as the double cover of A^2_S branched along F.

Write $\underline{A} = A^2_S$. Pulling back (over S) by a component of the normalization of the singular locus of F, we may assume that the projection $\underline{A} \to S$ has a section contained in the singular locus of F. Blow up $\underline{A}$ along this section. Continue this process until the reduced total inverse image of the pull-back of F has normal crossings over $t \neq 0$.

We have a commutative diagram



where q is the composite of the blow-ups and π is the double cover branched along F. g is the normalization, so that p is the double cover branched along $\tilde{F}$, where $\tilde{F}$ is obtained from q^*F by subtracting even multiples of exceptional divisors until we are left with something reduced. Blowing up some more if necessary, we can assume that $\tilde{F}$ is smooth (but probably disconnected) over $t \neq 0$. Then $\tilde{\underline{X}}_t$ is smooth for all $t \neq 0$, while $\tilde{\underline{X}}_o \to \underline{X}_o$ is proper and birational.

We have $\omega_{\tilde{\underline{X}}/S} \cong p^*\omega_{\hat{\underline{A}}/S} \otimes p^*\underline{O}(\tilde{D})$, where $\tilde{F} = 2\tilde{D}$ ($\tilde{F}$ is even, by construction). Writing $F = 2D$, we see that

$$\pi_* R^1 h_* \omega_{\tilde{X}/S}^{\otimes n} \cong \omega_{\underline{A}/S}^{\otimes n} \otimes \left[\underline{O}(nD) \otimes R^1 q_* \underline{O}(A) \oplus \underline{O}((n-1)D) \otimes R^1 q_* \underline{O}(B) \right],$$

where A and B are divisors on $\hat{\underline{A}}$ supported on the exceptional locus of q. Hence to prove that $R^1 h_* \omega_{\tilde{X}/S}^{\otimes n}$ is S-flat for all $n \geq 1$, it is enough to prove the following result.

<u>Lemma 14</u>: Let $\tilde{S} = S_r \xrightarrow{\pi_r} \cdots \xrightarrow{\pi_1} S_o = S$ be a sequence of blow-ups of points on smooth surfaces. Fix $a_1, \ldots, a_r \in \mathbb{Z}$. For $i = 0, \ldots, r$ define $\underline{L}_i \in \mathrm{Pic}\, S_i$ by $\underline{L}_o = \underline{O}_S$, $\underline{L}_i = \pi_i^* \underline{L}_{i-1} \otimes \underline{O}(a_i E_i)$, where E_i is the curve contracted by π_i. Set $\rho_i = \pi_1 \circ \ldots \circ \pi_i$, $\rho = \rho_r$, $\underline{L} = \underline{L}_r$.

Then $\dim(R^1 \rho_* \underline{L})$ depends only on $(a_1, \ldots, a_r)$ and not on the configuration of points blown up.

<u>Proof:</u> We shall show also that $\dim(\mathrm{coker}(\rho_*(\omega_{\tilde{S}} \otimes \underline{L}^{-1}) \to \omega_S))$ depends only on $(a_1, \ldots, a_r)$. We argue by induction on r.
$r = 1$: Obvious.

Assume that $r > 1$ and that the result holds for all shorter sequences of blow-ups.

<u>Case (i)</u> $a_r \geq 0$.

Leray : $E_2^{pq} = R^p \rho_{r-1*} R^q \pi_{r*} \underline{L} \Rightarrow R^{p+q} \rho_* \underline{L}$.

So $\dim R^1\rho_*\underline{L} = \dim R^1\pi_{r*}\underline{L} + \dim R^1\rho_{r-1*}\,\pi_{r*}\,\underline{L}$

$$= \dim R^1\pi_{r*}\,\underline{O}(a_rE_r) + \dim R^1\rho_{r-1*}\underline{L}_{r-1} \quad ,$$

and half the induction is complete in this case.

Case (ii) : $a_r < 0$.

The duality theorem gives two spectral sequences with the same abutment:

$$I_2^{pq} = R^p\rho_*\mathrm{Ext}^q_S(\underline{L},\omega_S^{\cdot})$$

$$II_2^{pq} = \mathrm{Ext}^p_S(R^{-q}\rho_*\underline{L},\ \omega_S) \quad .$$

Note that $I_2^{pq} = 0$ if $q \neq 0$.

There is another Leray spectral sequence:

$$'E_2^{pq} = R^p\rho_{r-1*}R^q\pi_{r*}(\pi_r^*(\omega_{S_{r-1}} \otimes \underline{L}_{r-1}^{-1}) \otimes \underline{O}((1-a_r)E_r))$$

$$\Rightarrow I_2^{p+q,0}$$

$$'E_2^{pq} = R^p\rho_{r-1*}(\omega_{S_{r-1}} \otimes \underline{L}_{r-1}^{-1} \otimes R^q\pi_{r*}\underline{O}((1-a_r)E_r)) \ .$$

Hence $\dim I_2^{1,0} = \dim R^1\rho_{r-1*}(\omega_{S_{r-1}} \otimes \underline{L}_{r-1}^{-1})$

$$+ \dim R^1\pi_{r*}\,\underline{O}((1-a_r)E_r) \quad ,$$

and so is a function of $(a_1,\dots,a_r)$ by the induction hypothesis. $I_2^{0,0} = \rho_{r-1*}(\omega_{S_{r-1}} \otimes \underline{L}_{r-1}^{-1}) \to \omega_S$ with cokernel $\underline{F}$, say, of finite length; by hypothesis, this is a function of $(a_1,\dots,a_{r-1})$. $II_2^{0,0} = \underline{Hom}(\rho_*\underline{L},\omega_S) \cong \omega_S$, and generally $II_2^{pq} = 0$ unless either $p = q = 0$ or $p = 2$ and $q = -1$.

Comparing the diagrams for I_2^{pq} and II_2^{pq}, we get an exact sequence

$$0 \to \underline{F} \to II_2^{2,-1} \to I_2^{1,0} \to 0 .$$

Now $II_2^{2,-1}$ is dual to $R^1\rho_*\underline{L}$, via $Ext_S^2(-,\omega_S)$, and so $\dim R^1\rho_*\underline{L}$ is a function of $(a_1,\dots,a_r)$.

To complete the induction, we must show that $\dim \operatorname{coker}(\rho_*(\omega_{\tilde{S}} \otimes \underline{L}^{-1}) \to \omega_S)$ is a function of $(a_1,\dots,a_r)$.

<u>Case (i)</u> : $a_r \leq 0$. Then $\rho_*(\omega_{\tilde{S}} \otimes \underline{L}^{-1}) = \rho_{r-1*}(\omega_{S_{r-1}} \otimes \underline{L}_{r-1}^{-1})$, and we are done, by the induction hypothesis.

<u>Case (ii)</u> : $a_r > 0$. Then $\pi_{r*}(\omega_{\tilde{S}} \otimes \underline{L}^{-1}) = m_Q^{a_r-1}.(\omega_{S_{r-1}} \otimes \underline{L}_{r-1}^{-1})$, where Q is the centre of π_r, and so we have an exact sequence

$$0 \to m_Q^{a_r-1}.(\omega_{S_{r-1}} \otimes \underline{L}_{r-1}^{-1}) \to \omega_{S_{r-1}} \otimes \underline{L}_{r-1}^{-1} \to \underline{O}_Q/\underline{m}_Q^{a_r-1} \to 0 .$$

There is a Leray spectral sequence

$$''E_2^{pq} = R^p\rho_{r-1*}R^q\pi_{r*}(\omega_{\tilde{S}} \otimes \underline{L}^{-1}) \Rightarrow R^{p+q}\,\rho_*(\omega_{\tilde{S}} \otimes \underline{L}^{-1}) \quad ,$$

from whence $\dim R^1\,\rho_{r-1*}(m_Q^{a_r-1}\cdot\omega_{S_{r-1}}\otimes\underline{L}_{r-1}^{-1}) = \dim R^1\rho_*(\omega_{\tilde{S}}\otimes\underline{L}^{-1})$,

which we know to be a function of $(a_1,\ldots,a_r)$.

Now the induction is complete and the lemma proved.

Returning to the proof of the theorem, we see that $R^1h_*\omega_{\tilde{\underline{X}}/S}^{\otimes n}$ is S-flat. Consideration of the ideal I_n defined by $h_*\omega_{\tilde{\underline{X}}/S}^{\otimes n} = \underline{I}_n\cdot\omega_{\underline{X}/S}^{\otimes n}$ shows that A_n holds.

<u>Corollary 15</u> (Wahl): Conjecture 1 holds for 1-parameter deformation of minimally elliptic singularities.

<u>Proof:</u> Such singularities are either canonically free or are double points. [La], [R2].

* CHAPTER 2.

Canonical singularities.

§1. Reid has introduced the following definition [R3]:

Definition: A variety X has canonical singularities if

(i) X is normal;

(ii) rK_X is locally principal for some $r \neq 0$ (the least positive such r is the index of X);

(iii) for some (and hence every) desingularization $f:\tilde{X} \to X$, $f_*\mathcal{O}(nK_{\tilde{X}}) = \mathcal{O}(nK_X)$ for all $n \in \mathbb{N}$.

Given (i) and (ii), (iii) is equivalent to (iii)' $f_*\mathcal{O}(rK_{\tilde{X}}) = \mathcal{O}(rK_X)$, where r is the index.

Reid asked (loc. cit.) whether canonical singularities are Cohen-Macaulay; in this section we prove the following result.

Theorem 1: Suppose that X has canonical singularities and that $\dim X \leq 4$. Then X is Cohen-Macaulay.

The proof hinges on the following two results, where $f:\tilde{X} \to X$ denotes a desingularization of the variety X.

*In this chapter, all varieties etc. are defined over the complex numbers.

Theorem 2 [G-R]: $R^i f_* \omega_{\tilde{X}} = 0$ for all $i > 0$.

Theorem 3 [Ke]: The following are equivalent:

(i) X is normal and $R^i f_* \mathcal{O}_{\tilde{X}} = 0$ for all $i > 0$;

(ii) X is Cohen-Macaulay and $f_* \omega_{\tilde{X}} = \omega_X$.

The proof of Theorem 1 goes as follows: for $\dim X \leq 2$, it is well known that X can have only Du Val singularities if $\dim X = 2$ ([R3], but this is also implicit in [Wal]), or that X must be smooth if $\dim X = 1$. We first prove the result for $\dim X = 3$; this enables us to apply the results of [R3 §2] to prove the 4-dimensional case.

Proof of Theorem 1: First note that by [R3 Cor. 1.9] and the fact that the quotient of a Cohen-Macaulay variety by a finite group is itself Cohen-Macaulay [Ho, Lemma 5], we may assume that X is of index 1, i.e. that ω_X is invertible.

Now suppose that $\dim X = 3$. Then by [R3 Prop. 5.4] there is a 0-minimal resolution $f:\tilde{X} \to X$, in the sense that $K_{\tilde{X}} = f^* K_X + Z$, where $Z \geq 0$ and $\dim f(Z) \leq 0$. In the notation of [R], Z is the discrepancy of the resolution. We give a proof of this, both for completeness and because it provides a clear picture of our approach to the 4-fold case.

Recall from [Hi] that we can construct a resolution $f:\tilde{X} \to X$ as a sequence of blowings-up $g_i : X_i \to X_{i-1}$ with a centre Y contained in

Sing X_{i-1} along which X_{i-1} is normally flat. By [R3. Cor. 1.14], apart perhaps from some finite set Σ, Sing X is a smooth curve C, say, of Du Val points (in the sense that for $P \in C-\Sigma$, a general hyperplane section of X through P has only Du Val singularities). In the first place $X-\Sigma$ is normally flat along $C-\Sigma$ (for example by the numerical criterion of [Si]), and secondly blowing up along C can give rise only to 0-minimal discrepancy (lying over Σ). Moreover, such a blow-up is normal (except possibly over Σ), by Du Val's characterization of these singularities as being absolutely isolated. Thus consideration of each step g_i as above shows that we can find a 0-minimal resolution.

Consider the short exact sequence

$$0 \to \underline{O}_{\tilde{X}} \to \underline{O}(Z) \to \underline{O}_Z(Z) \to 0 .$$

The corresponding cohomology sequence, together with Theorem 2 and the projection formula, gives isomorphisms

$$H^i(\underline{O}_Z(Z)) \xrightarrow{\sim} R^{i+1}f_*\underline{O}_{\tilde{X}} \quad \text{for all} \quad i \geq 0 .$$

Thus $H^2(\underline{O}_Z(Z)) = 0$. By the adjunction formula, $\omega_Z = \underline{O}_Z(2Z)$, so that Serre duality on Z gives $H^0(\underline{O}_Z(Z)) = 0$; hence $R^1f_*\underline{O}_X = 0$. That $R^2f_*\underline{O}_{\tilde{X}} = 0$ follows via the duality theorem [Ha. p. 210] from the hypotheses that $f_*\omega_{\tilde{X}} = \omega_X$ and that X be normal, exactly as in Kempf's proof of Theorem 3. Then by Theorem 3 X is Cohen-Macaulay.

We now give a proof in the case $\dim X = 4$. Since this is rather complicated and we are in any case more concerned with 3-folds, the reader might like to go straight to Corollary 4.

Suppose that $\dim X = 4$. We aim to construct a model $f:\tilde{X} \to X$ such that $\tilde{X}$ is a Gorenstein variety whose locus of non-rational singularities is at most one-dimensional and which is O-minimal over X. We in fact define a sequence $g_i:X_i \to X_{i-1}$ of models O-minimal over X (so that if $G_i:X_i \to X$ is the composite morphism, then there exists open $U \subseteq X$ such that $X-U$ is finite and if $V = G_i^{-1}(U)$, then $\omega_V = G_i^*\omega_U$. In particular, therefore, V is normal in codimension 1).

Define $X_o = X$. Suppose that $i > 0$ and that X_{i-1} has been constructed as required. Let S be an irreducible smooth Samuel stratum (see [Be]) lying in $\mathrm{Sing}(X_{i-1})$. There are several cases to consider:

(i) $\dim G_{i-1}(S) = 0$: then set $X_i = Bl_S X_{i-1}$.

(ii) $\dim G_{i-1}(S) > 0$. By the normality of V as above, $\dim S \leq 2$.

(a) $\dim S = 2$. Then a general surface section of X_{i-1} through a general point of S has only Du Val singularities [R3. Thm. 1.13], so that blowing up along S introduces no discrepancy. Set $X_i = Bl_S X_i$.

(b) $\dim S = 1$ and $\mathrm{mult}_p(X_{i-1}) \geq 3$ for a general point P of S. Again a general hyperplane section of X_{i-1} through P is canonical of index 1, and so Gorenstein, as already shown. Then by [R3. Thm. 2.11] blowing up along S introduces no discrepancy; set $X_i = Bl_S X_{i-1}$.

(c) $\dim S = 1$ and $\mathrm{mult}_p(X_{i-1}) = 2$, P as before. Then by hypothesis, every point of S has the same Hilbert-Samuel function (not just polynomial) as P , and so is a double point (of a hypersurface); in particular it is Gorenstein as a point of X_{i-1} . We may assume then that none of (i), (ii) (a) or (ii) (b) hold for any Samuel stratum and that every singularity of X_{i-1} is a double point. Moreover, since the normalization of a double point is a double point, we can take X_{i-1} to be normal. Consider the locus $NR(X_{i-1})$ of non-rational singularities of X_{i-1} ; it is clearly closed in the Zariski topology. By construction, $\exists$ dense open U X such that $X-U$ is finite and contains $NR(X)$, and if $V = G_{i-1}^{-1}(U)$, then $\omega_r = G_{i-1}^{*}\omega_u$, so that U and V are rational Gorenstein. So $G_{i-1}(NR(X_{i-1})) \subseteq X-U$, and so is finite. Suppose that $NR(X_{i-1})$ has a 2-dimensional component W , say. Then W is a component of Sing (X_{i-1}) , and so contains a smooth Samuel stratum; this contradicts our assumptions. So $\dim NR(X_{i-1}) \leq 1$: take $\tilde{X} = X_{i-1}$ in this case.

Note that there is clearly a resolution $g:\hat{X} \to \tilde{X}$ such that if $h = f\circ g:\hat{X} \to X$, then $\hat{X}$ is 1-minimal over X (i.e. $K_{\hat{X}} = h^{*}K_X + Y$, with $Y \geq 0$ and $\dim h(Y) \leq 1)$.

There is an exact sequence $0 \to g_*\omega_{\hat{X}} \to \omega_{\tilde{X}} \to \underline{H} \to 0$.
Theorem 2 and the Leray spectral sequence give $R^i f_*(g_{*}\,{}_X) = 0$ for all $i > 0$; $\dim(\mathrm{supp}\ \underline{H}) \leqslant 1$, and so $R^2 f_*\omega_{\tilde{X}} = R^3 f_*\omega_{\tilde{X}} = 0$.
Say $K_{\tilde{X}} = f^{*}K_X + Z$; the same argument as before shows that $R^2 f_*\underline{O}_{\tilde{X}} = R^3 f_*\underline{O}_{\tilde{X}} = 0$.

Say $K_{\hat{X}} = h^* K_X + Y$, $Y \geq 0$. From the exact sequence

(α) $\qquad 0 \to \underline{O}_{\hat{X}} \to \underline{O}(Y) \to \underline{O}_Y(Y) \to 0$

and the fact that $R^3 h_* \underline{O}_{\hat{X}} = 0$ (via duality as before), we get

(β) $\qquad R^2 h_* \underline{O}_Y(Y) = R^3 h_* \underline{O}_Y(Y) = 0$.

Say $X \to S$, S smooth of dimension s . Then the duality theorem gives a spectral sequence

$$E_2^{pq} = \underline{\mathrm{Ext}}_S^{s-3+p}(R^{-q} h_* \underline{O}_Y(Y), \omega_S) \Rightarrow R^{p+q} h_* \underline{O}_Y(Y) .$$

Consideration of this, together with (α) and (β), gives first that $h_* \underline{O}_Y(Y) = 0$ and then that $R^1 h_* \underline{O}_{\hat{X}} = 0$. By the Leray spectral sequence, $R^1 f_* \underline{O}_{\tilde{X}} = 0$. Then duality on the morphism $\tilde{X} \to X \to S$ gives an isomorphism

$$\mathrm{Ext}_S^{s-4+p}(\underline{O}_{\tilde{X}}, \omega_S) \cong R^p f_* \omega_{\tilde{X}} = 0 \quad \text{for all } p \geq 2 .$$

I.e. depth $X \geq 3$ everywhere. Then by a theorem of Hartshorne and Ogus [Ha-O Th.1.6] (pointed out to me by Wahl), X is Cohen-Macaulay.

<u>Corollary 4</u>: Suppose that X is a smooth complete 3-fold with the following properties:

(i) X is of general type;

(ii) the canonical ring $R = \bigoplus_{n \geq 0} H^0(\omega_X^{\otimes n})$ is finitely generated;

(iii) the birational map $\pi : X \dashrightarrow \overline{X} = \operatorname{Proj} R$ is a morphism;

(iv) the canonical model $\overline{X}$ is of index 1.

Then the plurigenera of X are deformation-invariant.

Proof: The point is a theorem of Wahl's [Wa 2], that deformations of a resolution of a rational singularity "blow down" to deformations of the singularity. The details are as follows.

Suppose $\underline{X} \to S$ is flat, with $o \in S$ and $X \cong \underline{X}_o$. Write $\overline{X} = \underline{X}_o$, $A_n = \underline{O}_{S,o}/\underline{m}_o^n$. Take an affine cover $\{U_i = \operatorname{Spec} B_i\}$ of $\overline{X}$, and set $V_i = \pi^{-1}(U_i)$. We have a Cartesian diagram

$$\begin{array}{ccc} V_j & \hookrightarrow & V_j^{(n)} \\ \pi \downarrow & & \\ U_j & & \downarrow \\ \downarrow & & \\ \operatorname{Spec} k & \hookrightarrow & \operatorname{Spec} A_n \end{array}$$

, where $V_j^{(n)}$ is the n'th infinitesimal neighbourhood of V_j in $\underline{X}$.

By Theorem 1, $R^i\pi_*\underline{O}_{V_j} = 0$ for all $i > 0$, and so by the base change theorem $\Gamma(\underline{O}_{V_j}(n)) = B_j^{(n)}$, say, is A_n-flat, and $B_j^{(n)} \otimes_{A_n} k \cong B_j$ naturally (for details, see [Wa 2 §1]).

I.e. $\operatorname{Spec} B_j^{(n)} = U_j^{(n)}$, say, is a deformation of U_j over A_n, and the diagram

$$\begin{array}{ccc}
V_j & \hookrightarrow & V_j^{(n)} \\
\downarrow & & \downarrow \\
U_j & \hookrightarrow & U_j^{(n)} \\
\downarrow & & \downarrow \\
\operatorname{Spec} k & \hookrightarrow & \operatorname{Spec} A_n
\end{array}$$

is Cartesian. Moreover, by [Wa2 Lemma 1.2] the $U_j^{(n)}$ are uniquely defined, and so can be glued together to give a deformation $\overline{X}^{(n)}$ of $\overline{X}$ over A_n. Set $\hat{\underline{X}} = \underline{X}_{/X}$, the completion of $\underline{X}$ along X, and $\hat{\overline{\underline{X}}} = \varinjlim \overline{X}^{(n)}$. By the results of [EGA III, §5], $\hat{\overline{\underline{X}}}$ is an algebraizable formal scheme (note that $\varprojlim \omega_{\overline{X}^{(n)}/A_n}$ gives a polarization), and so we have a Cartesian diagram

$$\begin{array}{ccc}
X & \hookrightarrow & \underline{X}' \\
\downarrow & & \downarrow \\
\overline{X} & \hookrightarrow & \overline{\underline{X}}' \\
\downarrow & & \downarrow \\
\operatorname{Spec} k & \hookrightarrow & \operatorname{Spec}(\hat{\underline{O}}_{S,0}) = S' .
\end{array}$$

Clearly we can assume that $S' = S = \operatorname{Spec} k[[t]] = \{0,\eta\}$, with 0 (resp. η) the closed (resp. generic) point. $\underline{X}' \cong \underline{X}$, $\overline{\underline{X}}' \cong \overline{\underline{X}}$, say. Put $Y = \underline{X}_\eta$, $\overline{Y} = \overline{\underline{X}}_\eta$. Since $\omega_{\overline{X}}$ is ample, $h^i(\omega_{\overline{X}}^{\otimes n}) = h^i(\omega_{\overline{Y}}^{\otimes n}) = 0$ for all $i > 0$, $n \geq 2$, and so P_n is deformation-invariant for all $n \geq 2$. $P_1 = p_g$ is invariant anyway, by Hodge theory.

<u>Remark</u>: (i) This also shows that properties (i)-(iv) above are together deformation-invariant.

(ii) Entirely similar remarks hold for 4-folds.

§2. In [R3, Thm. 2.11], Reid shows that if P is a rational Gorenstein singularity of the local 3-fold X, then a general hyperplane section Y of X through P has a Du Val or minimally elliptic singularity at P, and that if P is a Du Val point of Y, then it is a rational point of X. It is easy to see that $P \in Y$ can be minimally elliptic without $P \in X$ being rational, even if $X-\{P\}$ is smooth; what we prove here is a best possible converse, at least for isolated singularities. This has also been noted by several others, including Morrison, Pinkham and Wahl.

<u>Theorem 5</u>: Suppose that $X \to \Delta$ is a 1-parameter deformation of the cusp or simple elliptic singularity $P \in Y$, where the general fibre X_t has only Du Val singularities (we call such a deformation a <u>rationalization</u> of Y). Then X has only rational singularities. Moreover, if $P \in Y$ is a smoothable normal Gorenstein surface singularity which is neither simple elliptic nor a cusp, then there is a 1-parameter smoothing $X \to \Delta$ of Y having a non-rational singularity.

Proof: We shall give two proofs of this result. The first is only valid if $\mathrm{mult}_p Y > 2$ and the general fibre X_t is smooth, but has the advantage of being explicit. The second uses the ideas and techniques described in [PP].

First proof: We begin with some lemmas.

Lemma 6: Suppose that $T \subset \mathbb{P}^d$ is a non-degenerate (i.e. spanning $\mathbb{P}^d$) projectively Gorenstein surface of degree d having a hyperplane section that is either a smooth elliptic curve or a rational cycle (i.e. either a nodal rational curve or a cycle of at least two smooth rational curves, crossing normally). Then T is reduced and is of one of the following forms:

(i) T is irreducible: then T is either a Del Pezzo surface (possibly with Du Val singularities) or a normal quadric embedded by $\mathcal{O}(2)$ or a cone over a normal elliptic curve or a projection of a surface S of degree d in $\mathbb{P}^{d+1}$ from a point $Q \notin S$, but coplanar with a reduced conic in S;

(ii) T has 2 components X_1, X_2: then X_i is of degree d_i in $\mathbb{P}^{d_i+1}$ and X_1 and X_2 cross generically transversely in a reduced conic;

(iii) T has $n \geq 3$ components $X_1, \ldots, X_n$: then each X_i is of degree d_i in $\mathbb{P}^{d_i+1}$ and, re-ordering if necessary and writing $X_{n+1} = X_1$, the configuration satisfies

(a) $X_i \cap X_{i+1}$ is a line for all i ;

(b) $\bigcap_{i=1}^{n} X_i$ is a single point Q, say;

(c) $X_i \cap X_j = \{Q\}$ unless $X_j = X_i$ or $X_{i\pm1}$;

(d) no X_i is the Veronese. If X_i is a cone, then $X_i \cap X_{i\pm1}$ is a generator; if X_i is a scroll, then $X_i \cap X_{i-1}$ is a generator and $X_i \cap X_{i+1}$ is a directrix (or vice versa).

We call such a configuration a <u>tent</u>, and Q its <u>vertex</u>.

<u>Proof</u>: First note that a general hyperplane section of T must also be either a normal elliptic curve or a rational cycle. T is clearly reduced.

(i) This is classical.

(ii) Let $\langle Z \rangle$ denote the linear span of a subscheme Z of $\mathbb{P}^r$. Clearly $\dim \langle X_1 \cap X_2 \rangle \geq 2$. Say $\deg X_i = d_i$, $\langle X_i \rangle \cong \mathbb{P}^{r_i}$, so that $d_i \geq r_i - 1$. Then $\Sigma d_i = d \leq \Sigma r_i - 2$, so that $d_i = r_i - 1$; the final assertion is obvious.

(iii) A general hyperplane section $H \cap T$ is a rational cycle $(C_1, \ldots, C_n)$. Order $\{X_i\}$ so that $X_i \cap H = C_i$. Say $\langle X_i \cap X_{i+1} \rangle = M_i$, $\deg X_i = d_i$ and $\langle X_i \rangle \cong \mathbb{P}^{r_i}$. Suppose $1 < j < n$; then

$$\dim \langle X_1 \cup \ldots \cup X_j \rangle \leq r_1 + \ldots + r_j - (j-1) \quad \text{and}$$
$$\dim \langle X_{j+1} \cup \ldots \cup X_n \rangle \leq r_{j+1} + \ldots + r_n - (n-j-1) ,$$

so that $d \leq \Sigma\, r_i - j+1 - n+j+1 - 2 = \Sigma\, (r_i - 1)$. So $d_i = r_i - 1$ for all i, and each M_i is scheme-theoretically a line. Moreover,

$$(X_1 \cup \ldots \cup X_j) \cap (X_{j+1} \cup \ldots \cup X_n) \supseteq M_j \cap M_n \text{ , and so } M_j \cap M_n \neq \emptyset .$$

Finally, we must show that $\bigcap_{i=1}^{n} M_i \neq \emptyset$. Suppose that $M_i = \emptyset$; then without loss of generality $M_n \cap M_1 \cap M_2 = \emptyset$. Say $M_1 \cap M_2 = \{R\}$. $R \in X_1 \cap (X_3 \cup \ldots \cup X_n)$, and so $\dim \langle X_1 \quad (X_3 \ldots X_n) \rangle \geq 2$. However, the previous argument gives $\dim \langle X_1 \quad (X_3 \ldots X_n) \rangle \leq 1$, and we are done.

<u>Lemma 7</u>: Suppose that Δ' is an r-fold cyclic cover of Δ , ramified completely over 0 . Write $X' = X \times_\Delta \Delta'$. Then if X' has only rational singularities, so does X .

<u>Proof</u>: Let t (resp. s) be a local parameter on Δ (resp. Δ') . We can assume that $s^r = t$. Let ω be a local generator of ω_X ; then ω / s^{r-1} is a local generator of $\omega_{X'}$. Suppose that $R \to S$ is an extension of DVR's of the function fields $k(X)$ and $k(X')$ respectively, and let ρ, σ denote the corresponding valuations. We must show that $\rho(\omega) \geq 0$. Let x,y be local parameters in R,S respectively; $x = u.y^e$ with $u \in S$ a unit and e the ramification index. Let ψ be a generator of ω_R ; then $\chi = \psi / y^{e-1}$ is a generator of ω_S . Say $\omega = \psi / x^\alpha$, $\alpha \in Z$; then $\chi = \omega . x^\alpha / y^{e-1}$, so that $\sigma(y^{e-1} / x^\alpha . s^{r-1}) \geq 0$, by the rationality of X' . $\sigma(y) = 1$ and $e \leq r$, so that $\alpha \leq 0$, as required.

<u>Lemma 8</u>: Suppose that $X \to \Delta$ is flat, with X a local Gorenstein scheme. Suppose that X_o is <u>either</u> a double point <u>or</u> satisfies the condition (§) emb dim(X_o) = mult$(X_o$ + dim(X_o) - 2 (cf. [Sal]) . then there is a base change $\Delta' \to \Delta$ such that $X' = X \times_\Delta \Delta'$ satisfies the same condition as $X_o \cong X'_o$, so that the local parameter t on Δ' is superficial of order 1, in the sense of Samuel ([Sam. p.22] and [Z-S. p. 285 and proof of VIII Th. 22, p. 294]), in the local ring of X' .

<u>Proof</u>: Denote embdim, mult and dim by e, μ, d respectively. In any case, $\mu(X_o) \geq \mu(X)$, $e(X_o) = e(X)$ or $e(X)-1$, $d(X_o) = d(X)-1$. Also $e(X_o) = e(X) \Rightarrow \mu(X_o) \geq 2.\mu(X)$.

(i) $\mu(X_o) = 2$. Then $\mu(X) \leq 2$. If $\mu(X) = 2$, we are done; otherwise make a base change of arbitrary order ≥ 2 .

(ii) $\mu(X_o) > 2$ and $e(X_o) = \mu(X_o) + d(X_o) - 2$.

Let $\underline{m}$ denote the closed point of X . Making a base change of any order ≥ 2 , we can assume that $t \in \underline{m} - \underline{m}^2$. Then $e(X_o) = e(X) - 1$, and we get $e(X) \geq \mu(X) + d(X) - 2$. If X is not a double point, then $e(X) \leq \mu(X) + d(X) - 2$, by [Sal Cor. 3.2], so we are done in this case. Suppose that X is a double point, and X_o is defined by $f(x_1,..,x_n) = 0$, where $n = e(X_o)$. Then X is defined by $f(x_1,..x_n) + t.g(x_1,..,x_n,t) = 0$. deg $f = \mu(X_o) = \mu$, say; then a base change of order μ will give us what we want. The last statement follows from the results of [Sa1] and Theorem 1 of [Si].

Recall that the geometrical meaning of superficiality is that the (projective) tangent cone of the hypersurface defined by a superficial equation is a hyperplane section of the tangent cone of the variety.

Lemma 9: Suppose that $(A,\underline{m})$ is a Gorenstein local ring with $t \in \underline{m}$ a non-zero-divisor. Write $(\bar{A},\bar{\underline{m}}) = (A/tA, \underline{m}/tA)$ and suppose that $\bar{A}$ satisfies the condition (§) of Lemma 8. Say $\mathrm{mult}(A) = \mu$. Then $\mu > 2 \Rightarrow \mathrm{mult}(\bar{A}) = \mu$, while $\mu = 2 \Rightarrow \mathrm{mult}\ (\bar{A}) = 2,3$ or 4.

Proof: Immediate, via the results of [Sa1].

Lemma 10: Let $P \in X$ be a normal Gorenstein 3-fold singularity of multiplicity ≥ 3 such that some section of X through P is minimally elliptic. Let $g:X_1 \to X$ denote the blow-up at P . Then X_1 is normal and Gorenstein, and $K_{X_1} = g^* K_X$.

Proof: This is an easy converse to the argument in the middle of p.290. of [R3].

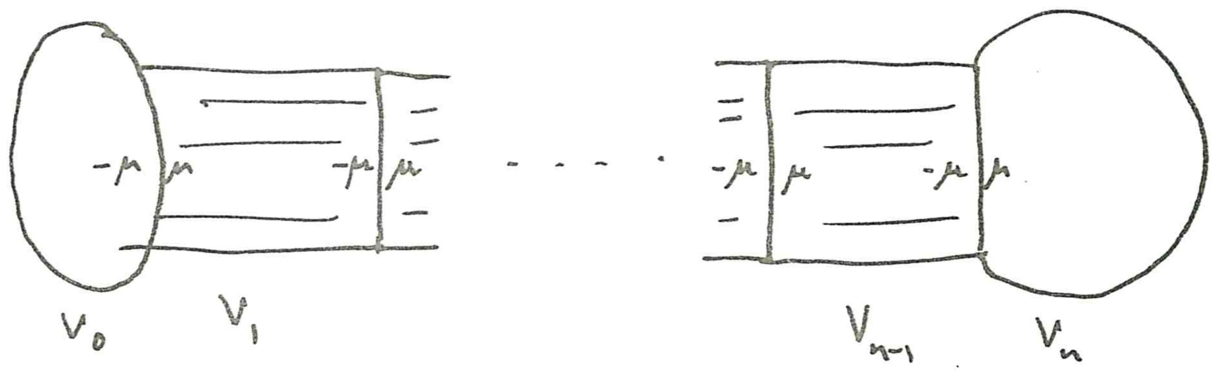
We now proceed with the proof of the theorem. In outline, we blow up points until we are left with only double lines and double points; we then blow up the lines, noting that for every blow-up $g_i:X^i \to X^{i-1}$,

X^i and X^{i-1} are normal and Gorenstein, and $K_{X^i} = g_i^* K_{X^{i-1}}$.
Finally, we analyze the remaining singularities. This process, described in detail below, gives a very explicit partial resolution $X' \to X$; X' has only compound-A_k singularities, and X'_o is reduced. (Recall that a point Q of a variety V is said to be of compound-A_k type (abbreviated to c-A_k) if Q is normal and there is a curve section of V through Q having only a node there; equivalently, $Q \in V$ is a hypersurface singularity defined locally by an equation of the form $xy = g(z_1,\ldots,z_r)$, where g does not vanish identically.)

Denote $\text{mult}_P(X_o)$ by μ ; $\mu \geq 3$. By Lemmas 7 and 8, we can assume that mult $(X) = \mu$ also. Then $T = T_P X \hookrightarrow \mathbb{P}^\mu$ of degree μ . Set $X^1 = Bl_P X$. $T_P Y$ consists either of a smooth elliptic curve or of a rational cycle, so that a general hyperplane section H of T is of the same form. There are two cases to consider:

(i) H is a smooth elliptic curve C .

Then T is either a Del Pezzo surface or a normal cone over C . Suppose the latter; call the vertex Q . Then $\text{mult}_Q T = \mu = \text{embdim}_Q T$. In turn $\text{embdim}_Q T = \text{embdim}_Q X_1 - \varepsilon$, where $\varepsilon = 0$ or 1 , and we see readily that $\text{mult}_Q X_1 = \mu = \text{embdim}_Q X_1 - 1$. Note that the equation $t = 0$ defines $T \subset X_1$ near Q , so that t is a superficial element of $\underline{O}_{X_1,Q}$ and a general hyperplane section of $T_Q X^1$ is also a smooth elliptic curve. Now blow up Q and continue in this way until we get a tangent cone that is a Del Pezzo surface (this must happen, for else the singularity $P \in X$ could not be resolved). Denote the composite of these blow-ups by $h:\tilde{X} \to X$. $\tilde{X}_0$ has the form shown in the diagram:

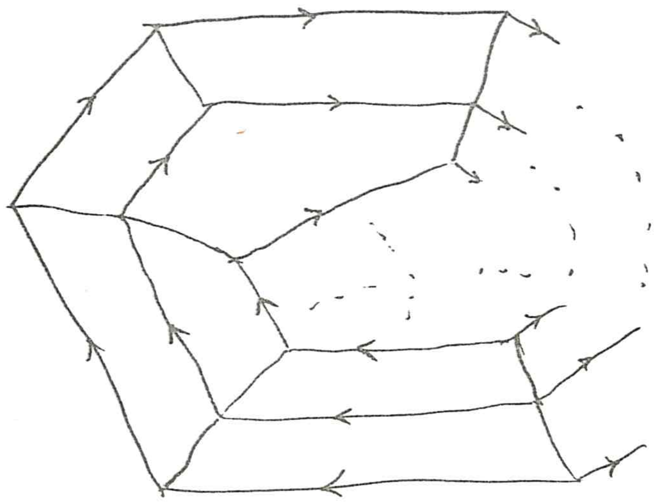


V_0 is the strict transform of X_0 (and its minimal desingularization), $V_1,\ldots,V_{n-1}$ are minimally ruled surfaces with the same elliptic base C and V_n is a Del Pezzo surface, possibly with Du Val singularities. These last are the only singularities of $\tilde{X}$; they do not lie on V_{n-1}, and so $\tilde{X}$ has only cDV singularities. By lemma 10 and the Grauert-Riemenschneider vanishing theorem, we are done in this case.

<u>Remark</u>: Since Del Pezzo surfaces have degree ≤ 9, we see that $\mu \leq 9$; this gives a local proof of a result of Pinkham [Pi], that elliptic cones of degree ≥ 10 are not smoothable, that avoids having first to lift deformations to the projective cone.

(ii) H is a rational cycle.

Then from Lemma 6, $X^1 = Bl_P X$ has at most one point Q, say, which is not $c\text{-}A_k$; if T has ≥ 3 components, Q is the vertex of the tent T . Say $\mathrm{mult}_Q X^1 = \mu^1$. It is clear from the geometry of the configurations described in Lemma 6 that $\mathrm{embdim}_Q T = \mathrm{mult}_Q T$. Then if $\mu^1 \geq 3$, by Lemmas 8 and 9 T is defined in X^1 at Q by a superficial element of $\underline{O}_{X^1,Q}$, namely t . Continue to blow up points of multiplicity ≥ 3 until we are left with only double points (not necessarily isolated). Denote the composite of these blow-ups by $h:\tilde{X} \to X$. Then the total exceptional divisor can be depicted as a set of concentric annuli:



Each annulus is the strict transform of a tangent cone to a point of multiplicity ≥ 3 ; the outermost annulus is the strict transform of $T_P(X)$, and so on. The following properties of $\tilde{X}$ are obvious:

(i) $\tilde{X}$ is normal and Gorenstein, and $K_{\tilde{X}} = h^* K_X$;

(ii) The special fibre $\tilde{X}_o$ of $\tilde{X} \to \Delta$ is reduced; it consists of the strict transform of X_o together with the exceptional divisor above;

(iii) $\tilde{X}$ is smooth along every arrowed curve in the above figure, except possibly where two of them meet;

(iv) Every irreducible curve of singularities of $\tilde{X}$ is a smooth rational curve;

(v) No annulus has more components than any annulus outside it;

(vi) Every component is rational. (A priori, an annulus might consist of an elliptic cone or its strict transform. Consideration of the first tangent cone corresponding to such an annulus gives an elliptic cone containing a hyperplane section that is a rational cycle, which is absurd.)

(vii) The central annulus (i.e. the tangent cone of the last blow-up) is of one of the following forms:

(a) a Del Pezzo surface;

(b) a projection S of a scroll or Veronese from a point coplanar with a reduced conic Γ ;

(c) 3 smooth components with a single common point Q ;

(d) 4 smooth components with a single common point Q ;

(e) 2 smooth components and a quadric cone, whose vertex is the unique common point Q ;

(f) One smooth component A , and a quadric cone whose vertex Q lies on A ;

(g) One smooth component A , and a cone over a twisted cubic whose vertex Q lies on A ;

(h) The cone over a twisted rational nodal quartic;

(i) The cone over a plane nodal cubic;

(j) 2 quadric cones intersecting transversely in a smooth conic disjoint from their vertices;

(k) One smooth component and a quadric cone meeting transversely in a smooth conic which does not contain the vertex of the cone;

(l) 2 smooth surfaces meeting generically transversely in a reduced conic Γ (where Q denotes the singular point of Γ, if such exists).

In case (b), S has a double line ℓ containing either 2 pinch points (when Γ is smooth) or a single "degenerate pinch point", given locally by the equation $w^2 = u^2(v^2+u)$ (when Γ is singular). (See [S-R. p.132] for the case when Γ is smooth; for Γ singular, the equation may be derived readily.)

(viii) In cases (a), (j), (k) above $\tilde{X}$ has only c-A_k singularities; in case (b) $\tilde{X}$ has c-A_k singularities away from the (degenerate) pinch points of S. In all other cases $\tilde{X}$ has c-A_k singularities away from Q.

Now blow up $\tilde{X}$ along the irreducible double curves, one by one, until we arrive at a model $g:X^* \to \tilde{X}$, say. The singularities of X^* consist of isolated double points, and the special fibre X_o^* of $X^* \to \Delta$ is reduced; this is because at the generic point of every double curve, $\tilde{X}$ is c-A_k, and blowing up along such a curve yields only c-A_k singularities. Finally, a base change of order 2 followed by further blow-ups may be necessary to ensure only c-A_k singularities. We give the details in case (ℓ); the others are at least as simple.

If Γ is smooth, there is nothing to prove, so suppose Γ to be singular at $Q \in \tilde{X}$. Then near Q, $\tilde{X}$ is defined as a subvariety of A^4 by $x^2+y^2z^2+t.g = 0$, $g = g(y,z,t)$, choosing suitable co-ordinates. Denote the order of g by $O(g)$; clearly we can assume $O(g) \geq 2$.

(a) $Q \in \tilde{X}$ is not isolated. Then we can assume that $\tilde{X}$ is singular along the line $\{x = y = t = 0\}$; i.e. no power of z apears in g. Say $g = yh_1 + th_2$. By the fact that $\tilde{X}$ is smooth over $\Delta^* = \Delta-\{0\}$ either some power of t, say t^{q-1}, or a term of the form $\lambda.t^{q-1}$, where λ is a linear function of (x,y,z), must appear in g.

Say $Z = Bl_{(x,y,t)} \tilde{X} \xrightarrow{\pi} \tilde{X}$. There are 3 affine charts to consider:

$$Z_x : 1 + y'^2z^2 + t'y'h_1(xyp,z,xt') + t'^2h_1(xy',z,xt') = 0$$

$$Z_y : x'^2 + z^2 + t'h_1 + t'^2h_1 = 0$$

$$Z_t : x'^2 + y'^2z^2 + y'h_1(y't,z,t) + h_2(y't,z,t) = 0 .$$

$Z_x \cap \pi^{-1}(Q) = \emptyset$ and Z_y is c-A_k along $\pi^{-1}(P)$. Clearly $\{x' = z = t = 0\}$ lies in Sing Z_k unless some h_i has a linear term in (y',z,t), but then Z_t has just c-A_k singularities.

Set $W = Bl_{(x',z,t)} Z_t$: the only chart that needs checking is

$$W_t : x''^2 + y'^2z'^2 + \frac{y'}{t^2} h_1(y't,z't,t) + \frac{1}{t^2} h_2(y't,z't,t) = 0 .$$

Either this has only c-A_k singularities or it is defined by an equation similar to that defining $\tilde{X}$, except that the exponent of t has dropped.

This reduces us to

(b) $Q \in \tilde{X}$ is isolated.

$$x^2 + y^2z^2 + t(y^n+z^p+t^{q-1}+zt.h_1 + yt.h_2 + yz.h_3) = 0$$

$$h_i \in k||y,z,t|| \ .$$

Now make a base change of order 2 , and call the new total space X^* .

$$X^* : x^2 + y^2z^2 + t^2(y^n + z^p + t^{2q-2} + zt^2h_1 + yt^2h_2 + yz.h_3) = 0$$

$$Bl_{(x,y,t)} X^* = Z \to X^* :$$

$$Z_t : x'^2 + y'^2z^2 + y'^nt^n + z^p + t^{2q-2} + zt^2h_1 + y't^3h_2 + y'zth_3 = 0$$

$$Bl_{(x',z,t)} Z_t = W :$$

$$W_t : x''^2 + y'^2z'^2 + y'^nt^{n-2} + z'^p + t^{2q-4} + z'th_1 + y'th_2 + y'z'h_3 = 0$$

$h_i = h_i(y't,z,t) = h_i(y't,z't,t)$, and so W_t is defined by an equation of the same form as that defining X^* . Consideration of the powers of t that occur shows that blowing up lines will eventually give a model with only $c\text{-}A_k$ singularities.

Finally we must check the effect of a base change of order 2 on the rest of $\tilde{X}$.

This is very easy; the worst that we have to consider is a point Q of $\tilde{X}$ such that near Q, $\tilde{X}_o$ is the union of two Cartier divisors A and B, each with a singularity at Q with an equation of the form $xy = z^{k+1}$ (resp. $xy = u^{q+1}$) (Possibly k or $q = \infty$). A and B intersect scheme-theoretically in the curve $\{xy = z = u = 0\}$. (A and B are components of the total transform of adjacent annuli.)

With the same notation as above, X^* is then defined in A^5, near Q, by equations of the form

$$xy + O(3) = 0$$

$$uz-t^2 + O(3) = 0 ,$$

unless $\tilde{X}$ had a node at Q; in this case $T_Q(X^*)$ is at worst a Del Pezzo surface of degree 4 with 4 nodes. Otherwise $T_Q(X^*)$ is of the form described in (vii)(j) above, so that blowing up Q leaves us with only smooth curves of $c\text{-}A_k$ points, and we are done.

<u>Second proof</u>: Making a base change if necessary, we have a resolution $\pi:\tilde{X} \to X'$ as above; moreover, it is easy to see that we can construct such a π with the property that for all $t \neq 0$, X_t is a minimal resolution of X'_t. Since all blow-ups are centred over the relative singular locus of X'/Δ', $\tilde{X}_o = \bigcup V_i$, say, where V_o is the strict transform (and a desingularization) of X'_o and for all $i > 0$, V_i is complete. We shall use the ideas and techniques described in [PP], [Ku] and [Mo] without further explicit reference; these include modifications of types I and II,

puncturing and slitting, generic contraction and patching.

Say $K_{\tilde{X}} = \Sigma r_i V_i$, $r_i \in \mathbb{Z}$; the r_i are not well-defined, but their differences are. It will be enough to show that $r_o \leq r_i$ for all i ; suppose then that $r_o > r_i$ for some i . Then following [PP], we can modify $\tilde{X}$ to get a quasi-degeneration, still denoted by $\tilde{X} \to \Delta'$, in which V_o is strictly maximal; i.e. $r_o \geq r_i$ for all i, and $r_o > r_j$ for some V_j meeting V_o . Note that V_o is unpunctured. V_o is a resolution of X'_o ; suppose that it contains an exceptional curve E of the first kind. Note that E meets at most two other complete curves in V_o . There are three cases to consider:

(i) E meets no double curve.

If E is not itself double, then by [Ko] it is stable and can be blown down. So assume that E is double; say $E = V_o \cap W$. Then clearly $r(W) = r_o+2$, which is absurd.

(ii) E meets one double curve $F = V_o \cap U$, say. If E is not double, then it can be transferred onto U by a modification of type I, so suppose E to be double. Say $E = V_o \cap T$. Then $E_T^2 = 0$, and T is generically contractible onto U . Then contract it generically, making any necessary punctures in U and slits in T .

(iii) E meets two double curves $F = V_o \cap U$ and $G = V_o \cap T$, say. Suppose that E is not a double curve. Then $V_o.E = -2$, while $K_X.E = r_u + r_T - 2r_{V_o}$. Then $K_V.E = r_u + r_T - 2r_{V_o} - 2 \leq -2$, by strict maximality; this is absurd. So suppose that E is double; then make a type II modification along it.

In this way we reduce to the case where V_o is a minimal resolution of X'_o. This is an obvious contradiction, and we are done.

Finally, suppose that $P \in Y$ is a smoothable normal Gorenstein surface singularity such that for every 1-parameter smoothing $X \to \Delta$ of Y, X has rational singularities. We want to show that P is a cusp or simple elliptic point.

Choose a smoothing $X \to \Delta$; as above, after a base change there is a semi-stable resolution $\pi : \tilde{X} \to X$. Say $\tilde{X}_o = \bigcup V_i$, as above. By hypothesis, $K_{\tilde{X}} = \pi^* K_X + \Sigma r_i V_i$, where $r_i \geq r_o$ for all i. Then in any quasi-degeneration bimeromorphic to $\tilde{X}$, V_o is never strictly maximal, so that we can use the methods referred to above to find a birational modification $\tilde{X} \dashrightarrow X^*$ such that the diagram

$$\begin{array}{ccc} \tilde{X} & \dashrightarrow & X^* \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\sigma} & \\ & X & \end{array}$$

commutes, X^*_o is reduced with locally normal crossings and $K_{X^*} = \sigma^* K_X$.

Say $X^*_o = \bigcup V^*_i$, with V^*_o the strict transform of $Y \cong X_o$, D its divisor of double curves. Then $K_{V^*_o} \sim -D$. As before, we can modify X^* so that V^*_o is a minimal resolution of X_o, and $r^*_i = r^*_o$ for all V^*_i meeting V^*_o. D still has locally normal crossings, and we see at once that $P \in Y$ must be a cusp or simple elliptic.

Remarks: 1) The partial resolution constructed in the proof of the preceding theorem suggests that a 1-parameter smoothing $X \to \Delta$ of a cusp has, after a base change, a resolution $\pi:\tilde{X} \to X$ such that $\tilde{X}_o$ is reduced with normal crossings, $\pi^* K_X = K_{\tilde{X}}$ and the dual complex of $\tilde{X}_o$ is a triangulation of the sphere. If the family $X \to \Delta$ can be compactified in such a way that X_o is of general type and if $\tilde{X}$ can be constructed so as to be Kähler, then this follows easily from [Per. Prop. 2.7.3.] and [Wi 1]. In turn, this implies that if $X \to C$ is a 3-fold of general type with $0 \in C$ such that

(i) X_o is reduced with normal crossings;

(ii) the dual complex of X_o is a triangulation of the torus, say;

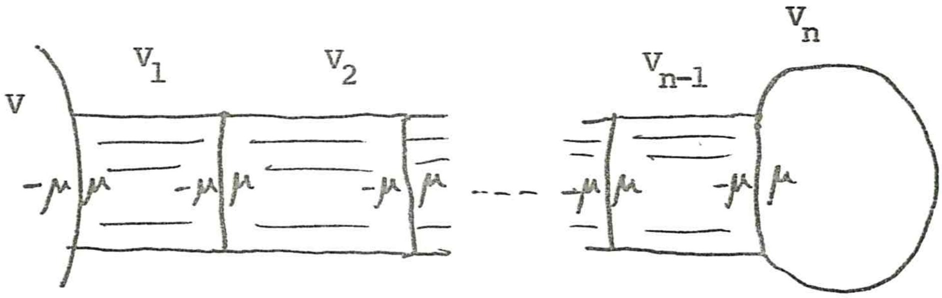
(iii) every component V_i of X_o, except one, say V_o, is rational, and its double curve $D_{V_i} \in |-K_{V_i}|$;

(iv) $K_X.E \geq 0$ for all curves E in X ;

then the canonical ring of X is not finitely generated. For, briefly, if it were, then $X_o - V_o$ would be contracted to a canonical singularity, and we would have a rationalization of a cusp; this contradicts the previous comments. I do not know whether such a 3-fold exists.

2) Conversely, one might ask whether all but one of the components in a semi-stable degeneration of K3 surfaces can be contracted. In the type II case, this is easy to see (subject to an obvious restriction, namely that $\mu > 0$, where μ is as defined below), via Grauert's criterion.

Suppose we have



$K_X|_{V_i} \sim 0$ for all $i \geq 0$, $V_1,\ldots,V_{n-1}$ are minimally ruled over the same elliptic base and $\mu > 0$. Write $V_i \cap V_{i+1} = E_i$. Then if C is an irreducible curve on V_n and $C.E_{n-1} = 0$, $C^2 = -2$ and $p_a(C) = 0$ by the index theorem; any configuration of such C's is negative definite. Then by [B-W §3], regarding X as a deformation of a resolution of Du Val singularities, we can blow down X so as to contract the (-2) curves in V_n. (This might also contract (-2) curves in X_t for $t \neq 0$.) Each V_i remains a Cartier divisor. Choose $N >> n$, and put

$$F = N.V_n + \ldots + (N-i^2).V_i + \ldots + (N-(n-1)^2).V_1 \quad .$$

Then clearly $-F|_{V_i}$ is ample for all $i = 1,\ldots,n$, and so $\bigcup_{i=1}^{n} V_i$ can be contracted [Gr. Satz 8].

Note that when $\mu \geq 3$ and V_n is smooth, such a configuration arises as the exceptional locus of a resolution of the singularity obtained by taking the cone over a Del Pezzo surface of degree μ isomorphic V_n, projecting it in a sufficiently general way onto a disc Δ and then making a base change of order n.

3) As pointed out above, Pinkham [Pi] has shown that simple elliptic singularities of degree ≥ 10 are not smoothable, while Wahl [Wa 4], [Wa 5] has shown that a cusp of degree m with r components in its minimal resolution is smoothable provided that $r > m^2-m$. In other words, only "some" simple elliptic singularities arise as hyperplane sections of rational Gorenstein 3-fold singularities, while "almost all' cusps so appear.

4) The explicit approach used in the first proof of Theorem 5 can also be applied to prove the following result.

Proposition 11: Suppose that $X \to \Delta$ is a 1-parameter deformation of the cusp $P \in X_o$, assumed to be the only singularity of X_o. Then if X_t for $t \neq 0$ contains a non-rational singularity Q_t, say, Q_t is the only non-rational singularity of X_t and is either simple elliptic or a cusp.

Proof: The first statement follows from the upper semi-continuity result of Elkik [E]. For the second, the results of Karras [Ka] show us that we can assume $\text{mult}_P X_o \geq 5$. If $\text{mult}_{Q_t} X_t = \text{mult}_P X_o$ for $t \neq 0$, then the result follows from Cor. 15 of Ch. 1. If the multiplicity drops, then blow up points of multiplicity > 2 lying over P; eventually we get to a model $\tilde{X} \to X$ such that $\tilde{X} \to \Delta$ has a section Δ_o, say, where $\Delta_o \cap \tilde{X}_t = \{Q_t\}$ for all $t \neq 0$ ($\tilde{X}_t \xrightarrow{\sim} X_t$ for all $t \neq 0$) and $\tilde{X}$ is normally flat along Δ_o. There are 2 cases to consider:

(i) $\mathrm{mult}_{Q_t} X_t \geq 3$ for $t \neq 0$. Then $\Delta_o \cap \tilde{X}_o$ is the vertex of some tent or the vertex of the cone in cases (vii),(h)-(i) in the first proof of Theorem 5. In any case, blowing up along Δ_o shows that for all $t \neq 0$, $T_{Q_t}(X_t)$ is a deformation of a rational cycle, and we are done. This also shows that in this case, #(non-(-2) curves in a minimal resolution of Q_t) $\leq$ #(non-(-2) curves in a minimal resolution of P).

(ii) $\mathrm{mult}_{Q_t} X_t = 2$ for $t \neq 0$. Say $\Delta_o \cap \tilde{X}_o = \{Q_o\}$. By normal flatness, $\mathrm{mult}_{Q_o} \tilde{X}_o = 2$, and we are in cases (vii)(b) or (vii)(l) of the first proof of Theorem 5. In (vii)(b), Q_o is a possibly degenerate pinch point on a projected scroll or Veronese, and so is defined locally by either $x^2 = y^2z$ or $x^2 = y^2(z^2+y)$. In case (vii)(l), Q_o is defined locally by $x^2 = y^2z^2$. It is clear that any singular normal deformation of these equations defines a cusp or a simple elliptic point or a Du Val singularity.

5) The ideas and techniques used in the second proof of Theorem 5 can also be applied to show that any 1-parameter family of Du Val singularities can be simultaneously resolved. Of course, the methods of Brieskorn and Tyurina are much more revealing in this context.

6) Consider again the conjecture on simultaneous resolution of a family $\tilde{\underline{X}} \to S$ of Gorenstein surfaces. Make a semi-stable resolution $\tilde{\underline{X}} \to S$. If it were possible to contract $\tilde{\underline{X}}$ to a model $\underline{X}^*$ dominating

$\underline{X}$, having only canonical singularities and with $\omega^{[r]}_{\underline{X}^*/S}$ invertible and ample relative to $g:\underline{X}^* \to \underline{X}$ for some r , then $R^i g_* \omega^{[m]}_{\underline{X}^*/S}$ would vanish for all $i > 0$ and for all $n >> 0$, and the conjecture would follow immediately. (In this case,

$$\omega^{[m]}_{\underline{X}^*_\sigma} \cong \omega^{[m]}_{\underline{X}^*} \otimes k(\sigma) \quad \text{for} \quad \sigma \in S \quad \text{since} \quad \omega^{[m]}_{\underline{X}^*/S} \quad \text{is invertible.)}$$

This is of course a very special case of the contraction problem for 3-folds of general type; the fact that there are only finitely many divisors involved might make it easier.

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