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Barnett, Stephen M. (2013) *Comment on "Trouble with the Lorentz Law of Force: Incompatibility with Special Relativity and Momentum Conservation".* Physical Review Letters, 110 (8). ISSN 0031-9007

Comment on "Trouble with the Lorentz Law of Force: Incompatibility with Special Relativity and Momentum Conservation"

The recent claim that the Lorentz law of force is incompatible with the requirements of special relativity [1], essentially that the Lorentz force law is incompatible with the Lorentz transformations, was based on treating only the spatial part of the Lorentz force. Including the rates of change for both the energy and the momentum restores compatibility. We work throughout in the natural system of units: ε_0, μ_0, and c are all set equal to unity.

We first review, briefly, the argument given in Ref. [1]. The Lorentz force law describes the force exerted by the electric and magnetic fields on charges and currents. Expressed as a force density, it becomes

$$\mathbf{F} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}. \tag{1}$$

The objection under consideration deals with the action of a charge on a magnetic dipole. We were asked to restrict our attention to bound charges so as to rewrite the force density in terms of the polarization and magnetization:

$$\mathbf{F} = -(\nabla \cdot \mathbf{P})\mathbf{E} + (\dot{\mathbf{P}} + \nabla \times \mathbf{M}) \times \mathbf{B}. \tag{2}$$

Consider a charge q at the origin of a primed coordinate system and a magnetic dipole oriented in the x' direction placed a distance d from the charge along the z' axis. Naturally, the charge exerts neither a force nor a torque on the dipole in this frame, as may readily be verified using the magnetization

$$\mathbf{M} = m_0 \hat{\mathbf{x}}' \delta(x')\delta(y')\delta(z' - d) \tag{3}$$

and the force density (1). Consider now the situation as viewed from an unprimed frame in which the charge and the dipole are moving at speed V in the positive z direction. In this frame the charge is moving and so produces a magnetic field. The moving magnetic dipole, moreover, acquires some electric dipole character by virtue of its motion. It follows, therefore, that the force given in Eq. (2) will include electric and magnetic fields and also both a magnetization and a polarization centered on the magnetic dipole. The net force exerted by the charge on the dipole remains zero, but there is a torque, $\mathbf{T} = (Vqm_0/4\pi d^2)\hat{\mathbf{x}}$, suggesting that the charge coerces the dipole to spin about its axis and, moreover, that the torque inducing this is proportional to the velocity at which the charge and the dipole are seen to be moving. Were this to induce a rotation of the dipole, there would indeed be a conflict between the Lorentz force law and special relativity.

The problem with this analysis is that the Lorentz force law is not a three-vector but a *four-vector* [2]. The time component of this is $\mathbf{J} \cdot \mathbf{E}$, which gives the rate of change of the dipole's energy. The integral of $\mathbf{J} \cdot \mathbf{E}$ over space is zero and there is no net transfer of energy just as there is no net force. If we consider the torque, however, then the $\mathbf{J} \cdot \mathbf{E}$ term produces a change to the moment of inertia of the dipole, just as the force density (1) contributes to the torque. When combined, these two effects result in an angular acceleration:

$$\mathbf{a}_{\text{angular}} = \frac{1}{I} \int d^3 r \mathbf{r} \times [\rho\mathbf{E} + \mathbf{J} \times \mathbf{B} - \mathbf{V}(\mathbf{J} \cdot \mathbf{E})], \tag{4}$$

where I is the moment of inertia of the dipole. For the situation just described, this angular acceleration is *zero* as expected.

The $-\mathbf{V}(\mathbf{J} \cdot \mathbf{E})$ term in (4) is reminiscent of that found when calculating the acceleration of a particle with charge Q and rest mass m, moving with velocity $\mathbf{v}$ [3]:

$$\mathbf{a} = \frac{Q}{m\gamma}[\mathbf{E} + \mathbf{v} \times \mathbf{B} - \mathbf{v}(\mathbf{v} \cdot \mathbf{E})], \tag{5}$$

where $\gamma = (1 - v^2)^{-1/2}$ is the usual Lorentz factor. We can associate the additional $\mathbf{v} \cdot \mathbf{E}$ term with the change in energy of the particle caused by its change in speed. In Mansuripur's magnetic-dipole thought experiment there is no change in the velocity of the dipole because there is no net force acting on it, but there is a change in the moment of inertia and this balances exactly the torque derived from (1). This effect is simply the angular analog of Mansuripur's own argument for resolving the apparent conflict with relativity for the force on a current carrying wire [1].

Finally, we note that there are related treatments in the literature [4].

I am grateful to Colin Baxter, Alison Yao, and Masud Mansuripur for helpful comments and discussions and to all those who kindly brought to my attention the papers cited in Ref. [4]. This work was supported by the U.K. EPSRC, the Royal Society, and the Wolfson Foundation.

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Received 8 May 2012; published 20 February 2013
DOI: 10.1103/PhysRevLett.110.089402
PACS numbers: 41.20.−q

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