A thermo-mechanically consistent Burnett regime continuum flow equations without Chapman-Enskog expansion

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Chapman-Enskog expansion is the orthodox approach to derive continuum flow models from Boltzmann's kinetic equation for dilute gases. Beyond the Navier-Stokes-Fourier order, these models known as Burnett hydrodynamic-regime equations violate a number of fundamental mechanical and thermodynamic principles in their original forms. This has generated a widely investigated problem in the kinetic theory of gases. In this short article, we derive a Burnett hydrodynamic-regime continuum model that is systematically consistent with all known mechanical and thermodynamic principles without using any series' expansion. Close comparison with the conventional Burnett hydrodynamic set of equations is considered and their linear stabilities around an equilibrium point under small perturbations are presented.

Key words:

1. Introduction

Chapman-Enskog expansion is the commonly used method when solving Boltzmann's kinetic equation for dilute gases in order to obtain continuum flow models. This method is an asymptotic series expansion in which the unknown molecular distribution function and differential operators appearing therein are expanded in series in terms of a small parameter, taken to be the Knudsen number (Chapman & Cowling 1970; Karlin & Gorban 2002). At zeroth order this expansion method yields Euler's hydrodynamic equations, followed by Navier-Stokes-Fourier's continuum flow model at first order. The second and higher order terms are termed the Burnett and super-Burnett equations respectively. As Navier-Stokes-Fourier's model becomes invalid, Burnett and super-Burnett models are thought to improve flow predictions. Instead, however, these models become subject to mechanical, thermodynamic, and instability problems (Woods 1983; Comeaux *et al.* 1995; Karlin & Gorban 2002; Bobylev 2006).

In the attempt to resolve inconsistencies related to Burnett's hydrodynamic-regime equations, various set of equations, consisting essentially of modifying or adjusting the originals have been proposed (see, for example, Garcia-Colin *et al.* 2008). Among these is the class of models proposed to remedy particularly the problem of instability under small perturbations in time or space. The proposal of Bobylev & Windfall (2012) is perhaps the latest in this line.

A modification resolving one inconsistency does not necessarily resolve the others. As a

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result, a variety of modified Burnett equations are now found in the literature, depending upon the particular inconsistency considered by the author. Woods (1983) focused on the mechanical inconsistencies. Indeed, Burnett equations are known to depend upon the observer's reference frame. Woods therefore proposed a modification oriented toward resolving this particular issue, which differs from modifications focused on instabilities. In contrast, some authors simply assumed that it was, instead, this mechanical principle that was not valid in kinetic theory (Söderholm 1976).

The comprehensive review by Garcia-Colin *et al.* (2008) describes the successes and failures of Burnett equations. In that review, standing problems related to Burnett's equations are listed as follows: *(i) the boundary conditions are in general unknown; (ii) the convergence properties of the Chapman-Enskog series are not known; (iii) there is no proof about the Burnett equations exhibiting a positive definite entropy production; (iv) the Burnett equations are frame dependent in rotating coordinates; (v) the equilibrium state is linearly unstable under longitudinal perturbations.*

Developing an adequate Burnett-regime hydrodynamic set of equations is not only concerned with dilute gas flow predictions. Rather, this set of equations is also used to predict various phenomena pertinent to granular flows (Sela & Goldhirsch 1998) as well as other relativistic flows (Denicol *et al.* 2010). An appropriate resolution of problems related to Burnett's hydrodynamic-regime equations constitutes an utmost necessity.

2. Structure of Burnett's equation as obtained using Chapman-Enskog expansion

Starting with the Boltzmann kinetic equation for dilute gases, the zeroth, first, and second moments of the distribution function follow evolution equations associated with conservation of mass, momentum, and energy, written as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho U] = 0, \tag{2.1a}$$

$$\frac{\partial \rho U}{\partial t} + \nabla \cdot \, [\rho U U] + \nabla \cdot \, [p\mathbf{1} + \mathbf{\Pi}] = 0, \tag{2.1b}$$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho U^2 + \rho e_{in}\right] + \nabla \cdot \, \left[\frac{1}{2}\rho U^2 U + \rho e_{in} U\right] + \nabla \cdot \, [(p\mathbf{1} + \mathbf{\Pi}) \cdot U] + \nabla \cdot \mathbf{q} = 0. \tag{2.1c}$$

In the set of equations (2.1a) to (2.1c), ρ represents the fluid's macroscopic mass-density; U is the flow (unique) mass velocity vector; p is the hydrostatic pressure; and e_{in} is the fluid's specific internal energy. These hydrodynamic field variables are all functions of the time variable t and spatial variable X. Variable $\mathbf{\Pi}$ represents the shear stress tensor and $\mathbf{q}$ is the heat flux vector, both of whose constitutive equations are primarily unknown. To obtain expressions for $\mathbf{\Pi}$ and $\mathbf{q}$, so as to close the system of equations (2.1a) to (2.1c), a solution is sought by expanding the distribution function and differential operators in Boltzmann's kinetic equation as power series in the Knudsen number, restricted by the assumption that the Knudsen number is much less than unity. To the second order, these

expressions are written as (Chapman & Cowling 1970; Woods 1993):

$$\mathbf{\Pi} = -2\mu\overline{\overset{\circ}{\nabla U}} + \frac{\mu^2}{p}\,[\,\tilde{\omega}_1\nabla\cdot U\overline{\overset{\circ}{\nabla U}} + \tilde{\omega}_2\,(\mathrm{D}\overline{\overset{\circ}{\nabla U}} - 2\,\overline{\nabla U\cdot\overline{\overset{\circ}{\nabla U}}}^{\circ}) + \overbrace{\tilde{\omega}_3 R\,\overline{\nabla\nabla T}}^{(a)}$$
$$+\frac{\tilde{\omega}_4}{\rho T}\overbrace{\overline{\nabla p\nabla T}^{\circ}}^{(b)} + \tilde{\omega}_5\frac{R}{T}\,\overline{\nabla T\nabla T}^{\circ} + \tilde{\omega}_6\,\overline{\overline{\overset{\circ}{\nabla U}\cdot\overline{\overset{\circ}{\nabla U}}}^{\circ}}\,], \tag{2.2}$$

$$\mathbf{q} = -\kappa\nabla T + R\frac{\mu^2}{p}\,[\,\theta_1\nabla\cdot U\nabla T + \theta_2\overbrace{(\mathrm{D}\nabla T - \nabla U\cdot\nabla T)}^{(c)} + \theta_3\frac{T}{p}\overbrace{\nabla p\cdot\overline{\overset{\circ}{\nabla U}}}^{(d)}$$
$$+\theta_4\overbrace{T\nabla\cdot\overline{\overset{\circ}{\nabla U}}}^{(e)} + 3\theta_5\nabla T\cdot\overline{\overset{\circ}{\nabla U}}\,], \tag{2.3}$$

where the fluid's material derivative is defined as

$$\mathrm{D} = \frac{\partial}{\partial t} + U\cdot\nabla. \tag{2.4}$$

The symmetric traceless tensor operator defined with tensor ∇U reads

$$\overline{\overset{\circ}{\nabla U}} = \frac{1}{2}\left(\nabla U + \widetilde{\nabla U}\right) - \frac{1}{3}\mathbf{1}\nabla\cdot U, \tag{2.5}$$

with $\widetilde{\nabla U}$ denoting the transpose of ∇U, and $\mathbf{1}$ the unit matrix. Coefficients μ and κ are, respectively, the dynamic viscosity and heat conductivity, whereas R is the specific gas constant and T is the temperature. Coefficients $\tilde{\omega}_1$ to $\tilde{\omega}_6$ and θ_1 to θ_5 are the Burnett-order dimensionless constant transport coefficients, whose exact values depend upon the interaction potential. The set of equations (2.1a) to (2.1c), closed with (2.2) and (2.3), corresponds to the original full set of Burnett equations.

In the shear stress (2.2) and heat flux (2.3) the material derivative D appearing in terms (a) and (c) owes its presence directly to Chapman-Enskog's expansion. Its presence renders these shear stress and heat flux contributions dependent upon the observer's reference frame in rotating systems (Woods 1983). As such, these Burnett shear stress and heat flux terms are, in principle, mechanically inconsistent. As the above Burnett continuum flow equations are also known to be linearly unstable (Karlin & Gorban 2002), the methods used to correct all of these defects vary from modifications of the material derivative or its alternative forms, to ad hoc additions of selected terms from higher-order Chapman-Enskog contributions (Jin & Slemrod 2001; Woods 1983; Söderholm 2007; Bobylev & Windfall 2012). Furthermore, not all terms appearing in equations (2.2) and (2.3) have an established physical significance. For example, term (e) predicts in a neutral gas a heat flux generated by fluid shear alone. This was called a "strange result" by Woods (1993). These overall inconsistencies lead this author, including others, to claim that the Boltzmann kinetic equation may not be valid in the Burnett regime.

In the following we demonstrate a methodological derivation, originating from a kinetic equation and leading, ultimately, to a systematically consistent set of equations without any adjustment.

3. A non-conventional approach to Burnett hydrodynamics

3.1. *The kinetic model and macroscopic equations*

Our new starting point is a single-particle probability density function, $\mathfrak{f}(t, X, \xi, c)$. The latter is a function of the time and spatial variables, which are not considered as random variables. Spatial variable X specifies a given position in the inertial frame of reference. Variable ξ is the molecular-velocity random variable, identifying molecules associated with velocity ξ. Variable c is a scalar random variable whose dimensions are those of concentration (number of molecules per unit of gas volume). This additional scalar variable is incorporated into the single-particle probability density function to account for the local number of molecules and their spatial distribution. This formulation distinguishes an element of measurable gas volume from a volume element in the inertial reference frame that may be associated with a differential of X or any differential volume element in the phase space.

Following Liouville's theorem of conservation of probability density, a Boltzmann-like kinetic evolution equation may be written for $\mathfrak{f}(t, X, \xi, c)$ as (Dadzie *et al.* 2008)

$$\frac{\partial \mathfrak{f}}{\partial t} + (\xi \cdot \nabla)\mathfrak{f} + \mathfrak{W}\frac{\partial \mathfrak{f}}{\partial \mathbf{c}} = \mathbf{I}\left(\mathfrak{f}, \mathfrak{f}\right). \tag{3.1}$$

Appearing on the right-hand side of equation (3.1) is the Boltzmann hard-sphere molecular collision integral. The concentration production term is denoted as $\mathfrak{W}$ (which may also be viewed as a gas volume production term). Body forces such as gravity are neglected in equation (3.1) for conciseness.

With M denoting the gaseous molecular mass, we may define, as is done conventionally, the three macroscopic hydrodynamic variables ρ, ρU, ρe_{in}, as moments of $\mathfrak{f}$ associated, respectively, with the microscopic molecular variables M, $M\xi$, $0.5M\xi^2$. Subsequently, an additional macroscopic variable may also be associated to the moment given by c, the additional random variable. This additional scalar moment, denoted as $\bar{\rho}$, has the dimensions of a mass-density, as it is precisely associated with the microscopic element Mc.

Upon integrating kinetic equation (3.1) so as to derive evolution equations governing the four moments, ρ, $\bar{\rho}$, ρU, ρe_{in}, we obtain a new set of four macroscopic conservation equations, namely:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho U] = 0, \tag{3.2a}$$

$$\frac{\partial \bar{\rho}}{\partial t} + \nabla \cdot [\bar{\rho} U] + \nabla \cdot [\bar{\rho} \mathbf{J}_c] = \rho \mathfrak{W}, \tag{3.2b}$$

$$\frac{\partial \rho U}{\partial t} + \nabla \cdot \left[\rho U U\right] = -\nabla \cdot [\mathbf{P}_v - \rho \mathbf{J}_c \mathbf{J}_c], \tag{3.2c}$$

$$\begin{aligned} \frac{\partial}{\partial t}\left[\frac{1}{2}\rho U^2 - \frac{1}{2}\rho \mathbf{J}_c^2\right] + \frac{\partial}{\partial t}\left[\rho e_{in}\right] + \nabla \cdot \left[\left(\frac{1}{2}\rho U^2 - \frac{1}{2}\rho \mathbf{J}_c^2\right) U + \rho e_{in} U\right] = \\ -\nabla \cdot \left[\mathbf{P}_v \cdot U_v\right] - \nabla \cdot \left[\mathbf{q}_v + \rho e_{in} \mathbf{J}_c\right] + \nabla \cdot \left[\rho \mathbf{J}_c^2 \mathbf{J}_c + \rho \mathbf{J}_c \mathbf{J}_c \cdot U\right]. \end{aligned} \tag{3.2d}$$

In order to form a closed system, the above set of macroscopic equations requires constitutive expressions for the new pressure tensor $\mathbf{P}_v$, the heat flux $\mathbf{q}_v$, the diffusive flux $\mathbf{J}_c$, and $\mathfrak{W}$. The conventional method in kinetic theory for obtaining these expressions consists of returning to solve equation (3.1) for the probability density function using Chapman-Enskog expansion. In contrast with this conventional route, we propose to use thermodynamic laws and irreversible thermodynamic principles to obtain these expres-

sions, so that the final set of equations is systematically consistent with these fundamental principles.

3.2. *Constitutive equations and compatibility with thermodynamics*

First, we note that flux $\mathbf{J}_c$ appearing in equation (3.2*b*) is a consequence of distinguishing between the pure mass continuity equation and an equation describing the true molecular spatial distribution involving an evolution of the gas volume containing these molecules (Dadzie *et al.* 2008). As a consequence, this flux is only attributable to a molecular diffusive flux with respect to the mass-average velocity U. Accordingly, the most natural form of its expression is the Fickian diffusion law:

$$\bar{\rho}\mathbf{J}_c = -\kappa_m \nabla \bar{\rho}, \tag{3.3}$$

where κ_m is the molecular diffusivity coefficient (actually a volume diffusivity coefficient).

One of the major problems with Burnett's equations as noted in the Introduction is its violation of the second law. Our derivation of an entropy equation for the system (3.2) began with energy equation (3.2*d*) transformed, using continuity and momentum equations (3.2*a*) and (3.2*c*), into the expression

$$\rho \mathrm{D}\left[e_{in} - \frac{1}{2}\mathbf{J}_c^2\right] - \frac{p}{\rho}\mathrm{D}\rho + \nabla\cdot\left[\rho\left(e_{in} - \mathbf{J}_c^2\right)\mathbf{J}_c\right] = -\nabla\cdot\left[\mathbf{q}_v + \mathbf{P}_v\cdot\mathbf{J}_c\right] - \left(\boldsymbol{\Pi}_v - \rho\mathbf{J}_c\mathbf{J}_c\right) : \nabla U, \tag{3.4}$$

where $\boldsymbol{\Pi}_v = \mathbf{P}_v - p\mathbf{1}$, and in which D is the material derivative defined in equation (2.4). Using continuity equation (3.2*a*), equation (3.2*b*) yields

$$-\frac{p}{\rho}\mathrm{D}\rho = -\frac{p}{\bar{\rho}}\mathrm{D}\bar{\rho} - \frac{p}{\bar{\rho}}\nabla\cdot[\bar{\rho}\mathbf{J}_c] + \frac{\rho p}{\bar{\rho}}\mathfrak{W}. \tag{3.5}$$

Equation (3.5) may be substituted into energy equation (3.4) to obtain

$$\begin{aligned}\rho \mathrm{D}\left[e_{in} - \frac{1}{2}\mathbf{J}_c^2\right] + \bar{\rho}p\mathrm{D}\bar{\rho}^{-1} =& \frac{p}{\bar{\rho}}\nabla\cdot[\bar{\rho}\mathbf{J}_c] - \frac{\rho p}{\bar{\rho}}\mathfrak{W} - \nabla\cdot\left[p\mathbf{J_c} - \rho\mathbf{J_c^2}\mathbf{J_c}\right] \\ &- \boldsymbol{\Pi}_v : \nabla\mathbf{J}_c + \rho\mathbf{J}_c\mathbf{J}_c : \nabla U - \nabla\cdot\left[\boldsymbol{\Pi}_v\cdot\mathbf{J}_c\right] \\ &- \nabla\cdot\left[\mathbf{q}_v + \rho e_{in}\mathbf{J}_c\right] - \boldsymbol{\Pi}_v : \nabla U + \boldsymbol{\Pi}_v : \nabla\mathbf{J}_c.\end{aligned} \tag{3.6}$$

From this last equation, an expression for $\mathfrak{W}$ is extracted as

$$\begin{aligned}\frac{\rho p}{\bar{\rho}}\mathfrak{W} =& \frac{p}{\bar{\rho}}\nabla\cdot[\bar{\rho}\mathbf{J}_c] - \nabla\cdot\left[p\mathbf{J}_c - \rho\mathbf{J}_c^2\mathbf{J}_c\right] \\ &- \boldsymbol{\Pi}_v : \nabla\mathbf{J}_c + \rho\mathbf{J}_c\mathbf{J}_c : \nabla U - \nabla\cdot\left[\boldsymbol{\Pi}_v\cdot\mathbf{J}_c\right],\end{aligned} \tag{3.7}$$

such that equation (3.6) takes the form

$$\rho \mathrm{D}\left[e_{in} - \frac{1}{2}\mathbf{J}_c^2\right] + \bar{\rho}p\mathrm{D}\bar{\rho}^{-1} = -\nabla\cdot\left[\mathbf{q}_v + \rho e_{in}\mathbf{J}_c\right] - \boldsymbol{\Pi}_v : \nabla\left[U - \mathbf{J}_c\right]. \tag{3.8}$$

Considering the structure of equation (3.8) with regard to the fundamental Gibbs equation, a specific entropy quantity $\bar{s}$ is defined through the expression

$$\rho \mathrm{D}\left[e_{in} - \frac{1}{2}\mathbf{J}_c^2\right] + \bar{\rho}p\mathrm{D}\bar{\rho}^{-1} = \rho T \mathrm{D}\bar{s}, \tag{3.9}$$

so that, by substituting equation (3.9) into (3.8), an evolution equation for the entropy reads

$$\rho T \mathrm{D}\bar{s} = -\nabla\cdot\left[\mathbf{q}_v + \rho e_{in}\mathbf{J}_c\right] - \boldsymbol{\Pi}_v : \nabla\left[U - \mathbf{J}_c\right]. \tag{3.10}$$

From the entropy equation (3.10), the nonnegative production rate and bilinear structure requirements associated with linear irreversible thermodynamic principles impose the following constitutive equation upon the shear stress:

$$\mathbf{\Pi}_v = -2\mu\overset{\circ}{\overline{\nabla\,[U - \mathbf{J}_c]}}. \tag{3.11}$$

Furthermore, from equation (3.10) we also deduce that Fourier's law applies to the entropic heat flux $\mathbf{q}_{ve}$ as

$$\mathbf{q}_{ve} = \mathbf{q}_v + \rho e_{in}\mathbf{J}_c = -\kappa\nabla T, \tag{3.12}$$

where κ is the heat conductivity coefficient.

Using the expression for the concentration production term, $\mathfrak{W}$, given in equation (3.7), the volume (or concentration) diffusion equation (3.2b) takes the form

$$\begin{aligned}\frac{\partial\bar{\rho}}{\partial t} + \nabla\cdot[\bar{\rho}U] + \frac{\bar{\rho}}{p}\nabla\cdot[p\mathbf{J}_c] = \quad & -\tfrac{\bar{\rho}}{p}\mathbf{\Pi}_v : \nabla\mathbf{J}_c + \tfrac{\rho\bar{\rho}}{p}\mathbf{J}_c\mathbf{J}_c : \nabla U \\ & -\tfrac{\bar{\rho}}{p}\nabla\cdot[\mathbf{\Pi}_v\cdot\mathbf{J}_c] + \tfrac{\bar{\rho}}{p}\nabla\cdot\left[\rho\mathbf{J}_c^2\mathbf{J}_c\right],\end{aligned} \tag{3.13}$$

or, alternatively, in terms of the material derivative, as

$$\begin{aligned}\bar{\rho}\mathrm{D}\left(\ln\frac{\bar{\rho}}{\rho}\right) + \frac{\bar{\rho}}{p}\nabla\cdot[p\mathbf{J}_c] = \quad & -\tfrac{\bar{\rho}}{p}\mathbf{\Pi}_v : \nabla\mathbf{J}_c + \tfrac{\rho\bar{\rho}}{p}\mathbf{J}_c\mathbf{J}_c : \nabla U \\ & -\tfrac{\bar{\rho}}{p}\nabla\cdot[\mathbf{\Pi}_v\cdot\mathbf{J}_c] + \tfrac{\bar{\rho}}{p}\nabla\cdot\left[\rho\mathbf{J}_c^2\mathbf{J}_c\right].\end{aligned} \tag{3.14}$$

In equations (3.13) and (3.14) the right-hand sides consist only of high order non-linear terms involving the flux $\mathbf{J}_c$. If these terms are negligible compared with terms on the left-hand side we obtain the following equation for the concentration:

$$\frac{\partial\bar{\rho}}{\partial t} + \nabla\cdot[\bar{\rho}U] + \frac{\bar{\rho}}{p}\nabla\cdot[p\mathbf{J}_c] = 0. \tag{3.15}$$

3.3. *The new set of Burnett-regime continuum model equations*

From the preceding section, the set of four equations (3.2) may now be written using the material derivative and the entropy equation in place of the energy equation as:

$$\mathrm{D}\rho = -\rho\nabla\cdot U, \tag{3.16a}$$

$$\bar{\rho}\mathrm{D}\left(\ln\frac{\bar{\rho}}{\rho}\right) = -\frac{\bar{\rho}}{p}\nabla\cdot[p\mathbf{J}_c], \tag{3.16b}$$

$$\rho\mathrm{D}U = -\nabla\cdot[\mathbf{P}_v - \rho\mathbf{J}_c\mathbf{J}_c], \tag{3.16c}$$

$$\rho T\mathrm{D}\bar{s} = -\nabla\cdot[\mathbf{q}_{ve}] - \mathbf{\Pi}_v : \nabla[U - \mathbf{J}_c], \tag{3.16d}$$

which is then closed with (3.11) and (3.12).

The structure of the set (3.16) is such that its satisfaction of the following mechanical principles can be verified straightforwardly: galilean invariance, integrability, angular momentum conservation, center-of-mass position (See Appendix of Dadzie & Brenner 2012). These are due specifically to the material derivative being associated with the mass velocity, U, appearing exclusively and simultaneously on the left-hand side of the continuity and momentum equations. The set's satisfaction of the positive-definite entropy production requirement is seen explicitly in the entropy equation (3.16d).

In order to compare the set of equations (3.16) with the original Burnett set of equations, the new momentum equation (3.16c) is rewritten, by substituting into it $\mathbf{J}_c$ from

its equation (3.3) and using $p = \bar{\rho}RT$, as

$$\rho \mathrm{D}U = -\nabla\cdot\left[p\mathbf{1} - 2\mu\,\overset{\circ}{\overline{\nabla U}} - 2\mu\,\overset{\circ}{\overline{\nabla\left[\kappa_m\nabla\ln p\right]}} + 2\mu\,\overset{\circ}{\overline{\nabla\left[\kappa_m\nabla\ln T\right]}}\right] \tag{3.17}$$
$$-\nabla\cdot\left[\rho\kappa_m^2\left[-\nabla\ln p\nabla\ln p + \nabla\ln T\nabla\ln p + \nabla\ln p\nabla\ln T - \nabla\ln T\nabla\ln T\right]\right].$$

In equation (3.17) the last terms on the first and second lines involving the temperature gradient represent the linear and non-linear thermal stresses respectively. These terms are the most physically well-established among Burnett's shear stress terms appearing in equation (2.2), as they predict various flows driven by temperature gradients. The third term on the right-hand side of the first line, involving the pressure gradient, also appears in the conventional Burnett stress after one eliminates (without any concrete physical argument) the material derivative by use of Euler's or Navier-Stokes's equations. The derivation of our continuum set of equations (3.16) naturally disposes of, precisely, the frame-dependent terms involving the fluid's material derivative previously seen in the Burnett shear stress and heat flux terms. The cross-effect term between the pressure and temperature gradients in equation (2.2) is also observed in (3.17), but in a different format.

Despite the close similarity between the new and the standard Burnett models, there are also some noticeable differences between these two continuum equations. For example, equation (3.17) does not contain any higher-order term involving the rate of fluid strain, other than the Navier-Stokes shear stress. Importing constitutive equation (3.12) into the energy equation (3.2*d*) and rearranging, enables an accompanying energetic heat flux to be derived as:

$$\mathbf{J}_u = \mathbf{q}_{ve} + p\mathbf{J}_c = -\kappa\nabla T + \kappa_m p\nabla\ln T - \kappa_m p\nabla\ln p. \tag{3.18}$$

This expression has a component not driven by temperature gradient. This is a feature of the original Burnett equation also, but now in a different form.

4. Linear stability analysis and sound dispersion

We consider in a one-dimensional flow configuration, a perturbation from the equilibrium state defined by,

$$\rho = \rho^0(1+\rho^*),\ \bar{\rho} = \rho^0(1+\bar{\rho}^*),\ T = T^0(1+T^*),\ U = U^*\sqrt{RT^0},\ p = p^0(1+p^*), \tag{4.1}$$

where the asterisked variables represent dimensionless quantities, and the superscript '0' denotes an equilibrium flow parameter. The dimensionless space and time variables are specified using a characteristic length L and characteristic time τ by the expressions

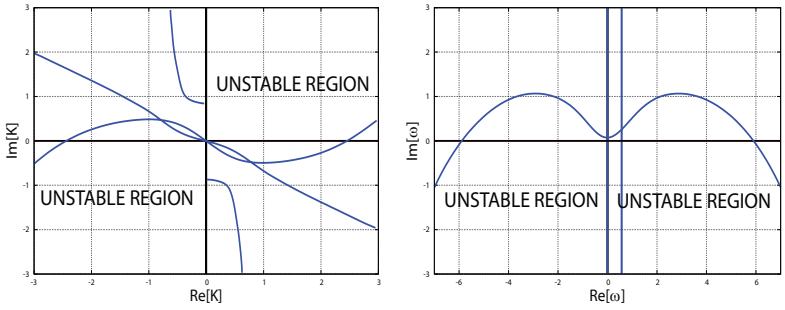
$$x = Lx^*,\quad t = \frac{L}{\sqrt{RT^0}}t^* = \tau t^*,\quad \tau = \frac{L}{\sqrt{RT^0}}, \tag{4.2}$$

with $p^0 = R\rho^0T^0$ and $U^0 = 0$. The one-dimensional linearized version of the closed set of equations (3.16) follows as:

$$\frac{\partial\rho^*}{\partial t^*} + \frac{\partial U^*}{\partial x^*} = 0, \tag{4.3a}$$

$$\frac{\partial\bar{\rho}^*}{\partial t^*} - \frac{\partial\rho^*}{\partial t^*} - \kappa_m^*\frac{\partial^2\bar{\rho}^*}{\partial x^{*2}} = 0, \tag{4.3b}$$

$$\frac{\partial U^*}{\partial t^*} - \frac{4}{3}\mu^*\frac{\partial^2 U^*}{\partial x^{*2}} + \frac{\partial\rho^*}{\partial x^*} + \frac{\partial T^*}{\partial x^*} - \frac{4}{3}\mu^*\kappa_m^*\frac{\partial^3\bar{\rho}^*}{\partial x^{*3}} = 0, \tag{4.3c}$$



(a) (b)

FIGURE 1. Stability analysis of the conventional Burnett equations: (a) spatial (b) temporal

$$\frac{\partial T^*}{\partial t^*} - \frac{2}{3}\frac{\partial \bar{\rho}^*}{\partial t^*} - \frac{2}{3}\kappa^* \frac{\partial^2 T^*}{\partial x^{*2}} = 0, \tag{4.3d}$$

where the different dimensionless transport coefficients, are given by

$$\mu^* = \frac{\mu\sqrt{RT^0}}{p^0 L} = Kn, \quad \kappa_m^* = \frac{\kappa_m \bar{\rho^0}}{\mu^0}, \quad \kappa^* = \frac{\kappa}{R\mu^0}, \tag{4.4}$$

where $\mu^0 = \rho^0 L\sqrt{RT^0}$ is a reference viscosity coefficient chosen such that the Knudsen number, Kn, is set equal to unity. In obtaining the momentum equation (4.3c) for the pressure we substituted $p = \rho RT$ (and not $p = \bar{\rho}RT$). This allows transferring the actual mass from the continuity equation through the pressure into $\bar{\rho}$ of equation (4.3b) representing gas volume evolution.

If the volume diffusivity coefficient is associated with the thermal diffusivity coefficient, i.e. $\kappa_m = \kappa/(\rho c_p)$, then $\kappa_m^* = Pr^{-1}$ whereas $\kappa^* = \gamma(Pr(\gamma-1))^{-1}$, where Pr is the Prandtl number and γ is the adiabatic exponent. But, from equation (3.18), because of additional non-Fourier heat flux terms, the entropic heat flux is no longer identical to the energetic heat flux (Brenner 2012). Consequently, the thermal conductivity coefficient κ within the entropic heat flux differs from the comparable conductivity κ_h conventionally associated with the energetic heat flux. This difference may be expressed as $\kappa_h\gamma = \kappa$ if the volume diffusivity coefficient equals the thermal diffusivity coefficient. More generally, the present distinction between energetic and entropic heat fluxes may imply some correcting coefficients between transport properties associated with the new and standard approaches. Although exact values of the correcting factor between transport coefficients have not been fully developed in the present work, we found it convenient to choose for the monatomic case, $\kappa_m^* = \bar{Pr}^{-1} = 2/5$ and $\kappa^* = \bar{\gamma}(\bar{Pr}(\bar{\gamma}-1))^{-1} = 9/4$, if $\bar{Pr}$ and $\bar{\gamma}$ respectively represent the Prandtl number and adiabatic exponent within the new entropic heat flux formalism.

For the sake of completeness, the one-dimensional linearized version of the conventional Burnett equations are, here, also set down as (Uribe *et al.* 2000):

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial U^*}{\partial x^*} = 0, \tag{4.5a}$$

$$\frac{\partial U^*}{\partial t^*} - \frac{4}{3}\mu^* \frac{\partial^2 U^*}{\partial x^{*2}} + \frac{\partial \rho^*}{\partial x^*} + \frac{\partial T^*}{\partial x^*} - \frac{2\tilde{\omega}_2}{3}\frac{\partial^3 \rho^*}{\partial x^{*3}} + \frac{2}{3}(\tilde{\omega}_3 - \tilde{\omega}_2)\frac{\partial^3 T^*}{\partial x^{*3}} = 0, \tag{4.5b}$$

$$\frac{\partial T^*}{\partial t^*} + \frac{2}{3}\frac{\partial U^*}{\partial x^*} - \frac{5}{2}\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{4}{9}(\theta_4 - \theta_2)\frac{\partial^3 U^*}{\partial x^{*3}} = 0, \tag{4.5c}$$

where the Burnett coefficients for Maxwellian molecules are: $\theta_2 = 45/8$, $\theta_4 = 3$, $\tilde{\omega}_2 = 2$,

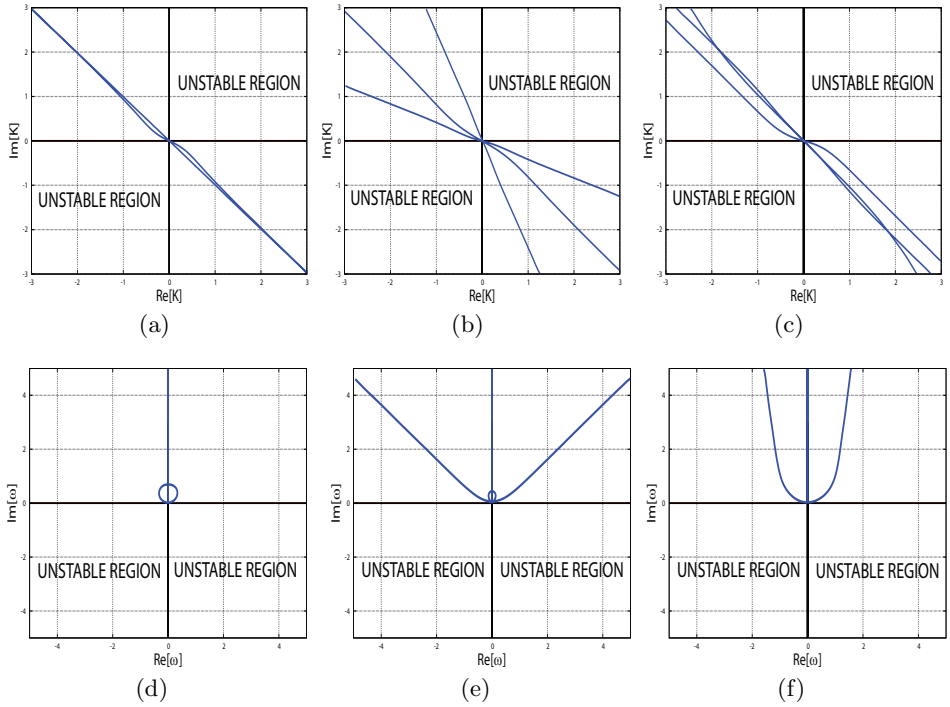


FIGURE 2. Stability analysis of our new Burnett equations: (a) spatial with $\kappa_m^* = 0.01$; (b) spatial with $\kappa_m^* = 4/3$; (c) spatial with $\kappa_m^* = 100$; (d) temporal with $\kappa_m^* = 0.01$; (e) temporal with $\kappa_m^* = 4/3$; (f) temporal with $\kappa_m^* = 100$;

$\tilde{\omega}_3 = 3$. We assume the disturbances ρ^*, $\bar{\rho}^*$, T^* and U^* to be wave functions of the form

$$\phi^* = \phi_a^* \exp\left[i\left(\omega t^* - K x^*\right)\right], \tag{4.6}$$

where ω is the complex wave frequency, K is the complex wave number, and ϕ_a^* is the complex amplitude. Then, substituting expression (4.6) into the two linearized sets of equations above, we deduce their stability or instability, as well as their predictions of sound dispersion, following the procedure described in (Dadzie & Reese 2010) or (Struchtrup & Torrilhon 2003).

Figures 1 present stability results for the conventional Burnett equations. The Burnett equations are linearly unstable in both space and time, as some of its modes enter unstable regions for certain values of the wave frequency (figure 1(a)) and for certain values of the wave number (figure 1(b)). We note that varying the Burnett transport coefficients, i.e. varying the molecular interaction potential, does not resolve these instabilities (Uribe *et al.* 2000). Figures 2 present the stability results for our new Burnett continuum model. The three different pairs of figures correspond, respectively, to: a vanishing; a finite; and a large value of κ_m^*. Remarkably, the new model is unconditionally stable in both space and time for all values of its transport coefficients.

The dimensionless inverse of sound speed and damping coefficient may be defined, respectively, as

$$\sqrt{\frac{5}{3}}\frac{Re[K]}{\omega} \quad \text{and} \quad -\sqrt{\frac{5}{3}}\frac{Im[K]}{\omega}. \tag{4.7}$$

These are plotted in figure 3 alongside experimental data by Meyer & Sessler (1957),

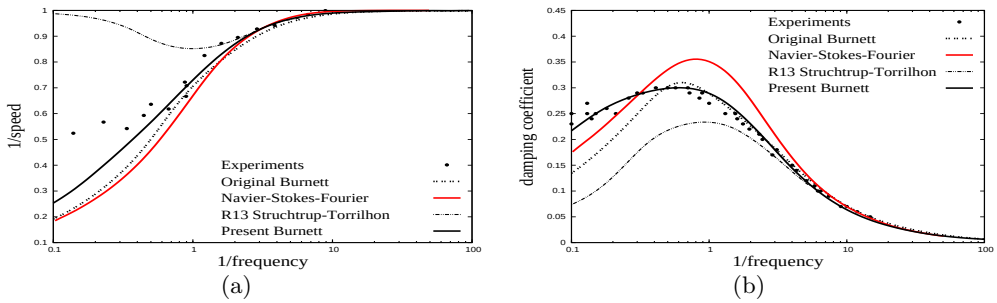


FIGURE 3. Sound dispersion compared with experiments: (a) inverse dimensionless sound speed; (b) dimensionless damping coefficient;

and predictions by a regularized moment model of Struchtrup & Torrilhon (2003). Taking together the sound speed and damping coefficient, the new Burnett-equation model achieved better agreement with experiments than all other models involved in figure 3. Moreover, the agreement is clearly well beyond the Knudsen number (dimensionless frequency) of unity.

Disregarding the concentration equation (3.16b) and the non-linear diffusive terms in the pressure tensor, the three remaining equations consisting of (3.16a), (3.16c) and (3.16d), have, remarkably, a structure identical to the Navier-Stokes-Fourier model in which we observe that the velocity in the Rayleigh dissipation function is replaced by $U_v = U - \mathbf{J}_c$ (the "volume velocity"). This is the three-set volume diffusion or bi-velocity hydrodynamic model presented in (Brenner 2012), although a petty difference exists with respect to the non-linear diffusive terms in the energy equation, depending on whether it is the energy or entropy equation that is used to introduce the volume-velocity. Beyond this difference, the representation of the fluid continuum model as a dual velocity model eliminates the last problem listed. Indeed, the original Burnett equations contain explicit spatial derivatives of order higher than those within Navier-Stokes-Fourier equations, leading to difficulties in setting accompanying physical boundary conditions. However, through use of the dual velocity method, the continuum model equations remain virtually of the same differential order as those of Navier-Stokes-Fourier. Setting additional boundary conditions in this case consists simply of imposing appropriate non-slip or slip boundary conditions on the volume velocity, without directly dealing with the explicit additional high-order of the differential equations (Dadzie & Brenner 2012).

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