

DAMTP-2004-90

hep-th/0501090

# Higher-spin conserved currents in supersymmetric sigma models on symmetric spaces

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## ABSTRACT

Local higher-spin conserved currents are constructed in the supersymmetric sigma models with target manifolds symmetric spaces  $G/H$ . One class of currents is based on generators of the de Rham cohomology ring of  $G/H$ ; a second class of currents are higher-spin generalizations of the (super)energy-momentum tensor. A comprehensive analysis of the invariant tensors required to construct these currents is given from two complimentary points of view, and sets of primitive currents are identified from which all others can be constructed as differential polynomials. The Poisson bracket algebra of the top component charges of the primitive currents is calculated. It is shown that one can choose the primitive currents so that the bosonic charges all Poisson-commute, while the fermionic charges obey an algebra which is a form of higher-spin generalization of supersymmetry. Brief comments are made on some implications for the quantized theories.

## 1 Introduction

Non-linear sigma models in 1+1 dimensions provide a fascinating set of highly non-trivial quantum field theories which have been intensively studied for many years [1]-[22]. Of particular note are those whose target manifolds are symmetric spaces, since these are known to be integrable classically [6, 12] and, in some cases, quantum-mechanically [7, 8, 10, 11, 13, 14]. As well as being of much intrinsic interest and displaying rich mathematical structures, these, or closely related models, are currently subjects of active study in fields

arXiv:hep-th/0501090v1 12 Jan 2005

as diverse as string theory [37, 38, 39, 40] and condensed matter physics (see e.g. [1, 8, 9] and references therein).

It was shown in [16, 18] that bosonic principal chiral models (PCMs), i.e. sigma models with target spaces Lie groups, contain classical, commuting charges with spins equal to the exponents of the Lie algebra modulo its Coxeter number. The corresponding currents are local functions of the underlying fields and are constructed using totally symmetric invariant tensors on the Lie algebra. Some of these conserved charges are known to survive in the quantum theory [13] and it is reasonable to conjecture that they all do, since their existence provides an explanation of common features shared by S-matrices for PCMs and affine Toda theories (which are, physically, quite different models see e.g. [31]). In so doing, they also fit together with known properties of non-local charges in a highly non-trivial, and rather intriguing, fashion. Analogous results have been established for the wider class of bosonic sigma models whose target manifolds are symmetric spaces. Not all such models are quantum integrable, and exactly when this happens has recently been clarified, in terms of both local and non-local quantum charges [11, 14].

Supersymmetric extensions of the PCMs were considered in [19] (which also contains extensive references to the earlier literature). Local, higher-spin conserved currents were shown to appear in two families: in terms of an underlying fermionic superfield current  $J^a$  taking values in the Lie algebra (more detailed conventions follow) these currents take the form

$$\Omega_{a_1 a_2 \dots a_p} J^{a_1} J^{a_2} \dots J^{a_p} \quad \text{and} \quad \Lambda_{a_1 \dots a_{q-1} a_q} J^{a_1} \dots J^{a_{q-1}} D J^{a_q} \quad (1)$$

(with  $D$  a superspace derivative). The tensors  $\Omega$  are invariant and totally antisymmetric and can be identified with cohomology generators of the Lie group. The tensors  $\Lambda$  are less familiar mathematically, but the first member of this sequence is nothing but the super-energy-momentum tensor. Each of these families gives rise to bosonic conserved charges whose spins are, once again, the exponents of the Lie algebra, but now with no repetition modulo the Coxeter number. This strong similarity with the bosonic PCMs is actually rather surprising, because the  $\Omega$  and  $\Lambda$  currents above are *not* merely super-extensions of the currents in the bosonic theory, rather they are intrinsic to the supersymmetric PCMs, in the sense that they vanish when the fermions are set to zero. It was also shown in [19] that the currents can be chosen so that the resulting bosonic conserved charges all Poisson-commute.

In this paper we will extend many of the results above and set them in a more general context by constructing and studying local conserved quantities in supersymmetric sigma models with target manifolds a compact symmetric space  $G/H$ . We will concentrate on

the cases in which  $G$  and  $H$  are classical groups, namely [24]:

$$SO(p+q)/SO(p)\times SO(q) , \quad SU(p+q)/S(U(p)\times U(q)) , \quad Sp(p+q)/Sp(p)\times Sp(q) , \\ SU(n)/SO(n) , \quad SU(2n)/Sp(n) , \quad SO(2n)/U(n) , \quad Sp(n)/U(n) \quad (2)$$

although some of our methods could be used to treat the remaining examples, involving exceptional groups, too. The first task is to construct generalizations of the  $\Omega$  and  $\Lambda$  currents above, which involves some interesting mathematical questions—in particular, how the cohomology of  $G/H$  enters. After introducing the models and taking care of other preliminary matters in section 2, we carry out the construction in sections 3 and 4. We summarize our results in section 5, identifying a set of ‘primitive’ currents from which all others can be found. A corollary of our construction is a rather concrete description of generators for the cohomology ring of  $G/H$ , which may be useful in other contexts.

In the remainder of the paper we consider the current algebra of these conserved quantities, starting by setting up the canonical formalism for the models in section 6. Although the results we obtain are all classical, a strong motivation for our work is the possibility of eventually extending them to the quantum level (all the supersymmetric models on symmetric spaces are known to be quantum integrable [22]) and we make a number of comments on the likely implications for the quantum theory throughout the course of the paper. In particular, the ‘top component’ charges of the conserved currents are the ones most likely to survive in the quantum theory, for reasons explained in section 5. We investigate their classical Poisson bracket algebra in section 7, and prove the existence of mutually commuting sets of bosonic charges, thereby generalizing the work in [19]. A completely new feature of certain symmetric space models is the existence of top component fermionic charges whose algebra closes in a kind of higher-spin generalization of supersymmetry. We conclude with some suggestions for future research in section 8.

## 2 The $G/H$ sigma model in superspace

### 2.1 Setting up the theory

To formulate the theory in a manifestly supersymmetric fashion, we will work in superspace with coordinates  $\{x^\pm, \theta^\pm\}$ , where  $x^\pm = \frac{1}{2}(x^0 \pm x^1)$  are the usual light-cone coordinates on two dimensional Minkowski space, and the additional fermionic coordinates  $\theta^\pm$  are real Grassmann numbers. The supersymmetry generators are

$$Q_\pm = \partial_{\theta^\pm} + i\theta^\pm \partial_\pm , \quad (3)$$

and the supercovariant derivatives are

$$D_{\pm} = \partial_{\theta^{\pm}} - i\theta^{\pm}\partial_{\pm} . \quad (4)$$

These obey

$$Q_{\pm}^2 = i\partial_{\pm} , \quad D_{\pm}^2 = -i\partial_{\pm} , \quad (5)$$

with all other anti-commutators vanishing. Note that a Lorentz boost of rapidity  $\lambda$  acts by  $x^{\pm} \mapsto e^{\pm\lambda}x^{\pm}$  and  $\theta^{\pm} \mapsto e^{\pm\lambda/2}\theta^{\pm}$  on superspace coordinates, but by  $\partial_{\pm} \mapsto e^{\mp\lambda}\partial_{\pm}$  and  $D_{\pm} \mapsto e^{\mp\lambda/2}D_{\pm}$  on derivatives. In general, the *spin*, or *Lorentz weight*, of any quantity can be read-off by counting  $\pm$  indices appropriately.

We begin by recalling the superspace formulation of the supersymmetric principal chiral model (SPCM) with target space a Lie group. Let  $\mathcal{G}(x, \theta)$  be a superfield taking values in a compact Lie group  $G$ , with Lie algebra  $\mathfrak{g}$ , from which we define a fermionic superfield current

$$\mathcal{J}_{\pm} = \mathcal{G}^{-1}D_{\pm}\mathcal{G} , \quad \text{with} \quad i\mathcal{J} \in \mathfrak{g} \quad (6)$$

(the factor of  $i$  here may seem strange but is typical of the sorts of reality conditions that arise when there are underlying Grassmann quantities). The superspace lagrangian for the SPCM on  $G$  is then

$$\mathcal{L}_G = -\frac{1}{2} \text{Tr} (\mathcal{J}_+\mathcal{J}_-) \quad (7)$$

which is invariant under a global symmetry  $G_L \times G_R$  acting on the superfield  $\mathcal{G}$  by left and right multiplication (for a fuller discussion, see e.g. [19]).

Now consider a subgroup  $H$  of  $G$ ; let  $\mathfrak{h}$  be its Lie algebra and  $\mathfrak{m}$  the orthogonal complement of this in  $\mathfrak{g}$ . The condition for  $G/H$  to be a symmetric space is

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \quad \text{where} \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} , \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} , \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} . \quad (8)$$

To fix notation: we will often make use of an orthonormal basis  $\{t_a\}$  of generators of  $\mathfrak{g}$  (anti-hermitian matrices in the defining representation) obeying

$$[t_a, t_b] = f_{abc} t_c , \quad \text{Tr}(t_a t_b) = -\delta_{ab} . \quad (9)$$

We may choose  $\{t_a\}$  to be the disjoint union of a basis  $\{t_{\hat{\alpha}}\}$  of  $\mathfrak{h}$  and a basis  $\{t_{\alpha}\}$  of  $\mathfrak{m}$ . Any quantity in the Lie algebra  $X \in \mathfrak{g}$  can be written

$$X = X^a t_a = X^{\hat{\alpha}} t_{\hat{\alpha}} + X^{\alpha} t_{\alpha} , \quad (10)$$

thereby decomposing it into parts belonging to  $\mathfrak{h}$  and  $\mathfrak{m}$ . (There is no distinction between upper or lower Lie algebra indices.) The symmetric space conditions (8) imply that

$$f_{\hat{\alpha}\hat{\beta}\gamma} = f_{\alpha\beta\gamma} = 0 \quad (11)$$

so that  $f_{\hat{\alpha}\hat{\beta}\hat{\gamma}}$  and  $f_{\alpha\beta\hat{\gamma}}$  are the only non-vanishing structure constants, up to permutations of indices. From the Jacobi identity, these satisfy

$$f^{\hat{\alpha}}_{\alpha\gamma} f^{\hat{\beta}}_{\gamma\beta} - f^{\hat{\beta}}_{\alpha\gamma} f^{\hat{\alpha}}_{\gamma\beta} = f^{\hat{\alpha}\hat{\beta}\hat{\gamma}} f^{\hat{\gamma}}_{\alpha\beta} , \quad f^{\hat{\alpha}}_{\alpha[\beta} f^{\hat{\alpha}}_{\gamma\delta]} = 0 . \quad (12)$$

To define the supersymmetric  $G/H$  sigma model, let  $\mathcal{G}(x, \theta)$  be a superfield taking values in  $G$ , as before, with  $\mathcal{J}_{\pm}$  defined by (6), and let

$$\mathcal{J}_{\pm} = \mathcal{K}_{\pm} + \mathcal{A}_{\pm} \quad \text{where} \quad i\mathcal{K} \in \mathfrak{m}, \quad i\mathcal{A} \in \mathfrak{h} \quad (13)$$

The superspace lagrangian for the  $G/H$  model is then

$$\mathcal{L}_{G/H} = -\frac{1}{2} \text{Tr}(\mathcal{K}_+ \mathcal{K}_-) . \quad (14)$$

This is invariant under a global  $G$  symmetry  $\mathcal{G} \mapsto U\mathcal{G}$  with  $U \in G$  and a local  $H$  gauge symmetry  $\mathcal{G} \mapsto \mathcal{G}\mathcal{H}$  for any superfield  $\mathcal{H}(x, \theta) \in H$ . It is useful to define a superspace derivative which is covariant with respect to this gauge symmetry:

$$\mathcal{D}_{\pm} = D_{\pm} + \mathcal{A}_{\pm} , \quad (15)$$

$$\mathcal{A}_{\pm} \mapsto \mathcal{H}^{-1} \mathcal{A}_{\pm} \mathcal{H} + \mathcal{H}^{-1} D_{\pm} \mathcal{H} . \quad (16)$$

and to note that

$$\mathcal{K}_{\pm} = \mathcal{G}^{-1} \mathcal{D}_{\pm} \mathcal{G} = \mathcal{G}^{-1} D_{\pm} \mathcal{G} - \mathcal{A}_{\pm} , \quad (17)$$

$$\mapsto \mathcal{H}^{-1} \mathcal{K}_{\pm} \mathcal{H} . \quad (18)$$

The lagrangian is clearly gauge-invariant, with the physical degrees of freedom confined to the coset space  $G/H$ , as desired.<sup>1</sup>

The superspace equations of motion following from the lagrangian can be written

$$\mathcal{D}_+ \mathcal{K}_- - \mathcal{D}_- \mathcal{K}_+ = 0 \quad (19)$$

and in addition we have, identically,

$$D_+ \mathcal{J}_- + D_- \mathcal{J}_+ + [\mathcal{J}_+, \mathcal{J}_-] = 0 . \quad (20)$$

Here, and throughout the paper, Lie algebra brackets of bosonic or fermionic quantities will always be understood to be graded appropriately (so, for example,  $F^2 = \frac{1}{2}[F, F]$  for a fermionic Lie algebra-valued quantity). Using the symmetric space property (8) we find

$$\mathcal{D}_{\mp} \mathcal{K}_{\pm} = D_{\mp} \mathcal{K}_{\pm} + [\mathcal{A}_{\mp}, \mathcal{K}_{\pm}] = 0 \quad (21)$$

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<sup>1</sup>The superfields  $\mathcal{A}_{\pm}$  could, alternatively, be introduced as independent variables, but they are then non-dynamical, with algebraic equations of motion given by (13).

and

$$[\mathcal{K}_+, \mathcal{K}_-] = D_+ \mathcal{A}_- + D_- \mathcal{A}_+ + [\mathcal{A}_+, \mathcal{A}_-] = 0. \quad (22)$$

These equations are of a rather special (Lax) form which implies the classical integrability of the model. One way to establish this is to construct *non-local* conserved quantities, as described in [21, 19]. Our aim in this paper is to investigate other exotic conserved quantities which, by contrast, are *local* in the sigma model fields.

## 2.2 Local conserved currents; $\Omega$ and $\Lambda$ tensors

We will be concerned with superfield currents  $\mathcal{C}$  which obey conservation equations of *super holomorphic* type, meaning

$$D_- \mathcal{C} = 0 \quad \text{or} \quad \mathcal{C} = r + \theta^+ s \quad \text{with} \quad \partial_- r = \partial_- s = 0. \quad (23)$$

Note that such a current  $\mathcal{C}$  has a component expansion of simplified form. The standard conserved charges arising from the component currents  $r$  and  $s$  are

$$R = \int dx r \quad \text{and} \quad S = \int dx s, \quad (24)$$

one of which is bosonic and the other fermionic (which is which depends on the grading of  $\mathcal{C}$ ). We shall refer to  $s$  and  $S$  as the *top component* current and charge, and to  $r$  and  $R$  as the *bottom component* current and charge. We could, of course, equally well consider *anti-holomorphic* currents, which are annihilated by  $D_+$ .

Given a set of superholomorphic quantities, we can take arbitrary polynomials in members of this set and their  $D_+$  derivatives to obtain new superholomorphic expressions. Note, however, that if  $\mathcal{C}$  obeys (23) then

$$D_+ \mathcal{C} = s - i\theta^+ \partial_+ r. \quad (25)$$

Although this expression is certainly superholomorphic, its top component charge vanishes, while its bottom component charge is just  $S$  again. Differential polynomials which are *total* derivatives can therefore be disregarded because they yield nothing new. More generally, we shall refer to a superholomorphic quantity as *composite* if it can be written as a non-trivial differential polynomial in other super-holomorphic quantities, and as *primitive* if it cannot. It is then natural to seek a set of primitive *generators*, in terms of which all other superholomorphic currents can be expressed as differential polynomials.

To construct superholomorphic quantities  $\mathcal{C}$  from the gauge-covariant currents  $\mathcal{K}_+$ , requires knowledge of  $H$ -invariant tensors on  $\mathfrak{m}$ . By definition, such a tensor  $T$  obeys

$$T(X, Y, \dots, Z) = T(hXh^{-1}, hYh^{-1}, \dots, hZh^{-1}) \quad (26)$$

for all  $X, Y, \dots, Z \in \mathfrak{m}$  and all  $h \in H$ ; or equivalently

$$T([X, W], Y, \dots, Z) + T(X, [Y, W], \dots, Z) + \dots + T(X, Y, \dots, [Z, W]) = 0 \quad (27)$$

for all  $X, Y, \dots, Z \in \mathfrak{m}$  and  $W \in \mathfrak{h}$ . With respect to a basis as in (9) and (11), we have components

$$T(X, Y, \dots, Z) = T_{\alpha_1 \alpha_2 \dots \alpha_p} X^{\alpha_1} Y^{\alpha_2} \dots Z^{\alpha_p} \quad \text{or} \quad T_{\alpha_1 \dots \alpha_p} = T(t^{\alpha_1}, \dots, t^{\alpha_p}), \quad (28)$$

and the condition for invariance is

$$T_{\beta \alpha_2 \dots \alpha_p} f^\beta_{\alpha_1 \hat{\gamma}} + T_{\alpha_1 \beta \dots \alpha_p} f^\beta_{\alpha_2 \hat{\gamma}} + \dots + T_{\alpha_1 \alpha_2 \dots \beta} f^\beta_{\alpha_p \hat{\gamma}} = 0. \quad (29)$$

Consider first the possibility of currents multilinear in  $\mathcal{K}_+$ . Let

$$\mathcal{C}_\Omega = \Omega(\mathcal{K}_+, \mathcal{K}_+, \dots, \mathcal{K}_+) = \Omega_{\alpha_1 \alpha_2 \dots \alpha_p} \mathcal{K}_+^{\alpha_1} \mathcal{K}_+^{\alpha_2} \dots \mathcal{K}_+^{\alpha_p}, \quad (30)$$

where  $\Omega$  is an invariant tensor on  $\mathfrak{m}$  which we may take to be totally antisymmetric (since  $\mathcal{K}_+$  is fermionic). For future reference we note that the invariance of  $\Omega$  can then be written

$$\Omega_{\beta[\alpha_2 \dots \alpha_p} f^\beta_{\alpha_1] \hat{\gamma}} = 0. \quad (31)$$

Invariance of  $\Omega$  implies, immediately, that  $\mathcal{C}_\Omega$  is superholomorphic from the equations of motion (21): because  $\mathcal{C}_\Omega$  is gauge-invariant, the derivatives  $D_-$  and  $\mathcal{D}_-$  agree on it, and so

$$D_- \mathcal{C}_\Omega = \mathcal{D}_- \mathcal{C}_\Omega \propto \Omega_{\alpha_1 \dots \alpha_p} \mathcal{K}_+^{\alpha_1} \dots \mathcal{K}_+^{\alpha_{p-1}} \mathcal{D}_- \mathcal{K}_+^{\alpha_p} = 0. \quad (32)$$

The resulting top and bottom component currents have spins  $(p-1)/2$  and  $(p-2)/2$ , respectively.

A second class of superconformal currents arise as higher-spin generalizations of the super-energy-momentum tensor. The general formula is

$$\mathcal{C}_\Lambda = \Lambda(\mathcal{K}_+, \dots, \mathcal{K}_+, ; \mathcal{D}_+ \mathcal{K}_+) = \Lambda_{\alpha_1 \dots \alpha_q; \beta} \mathcal{K}_+^{\alpha_1} \dots \mathcal{K}_+^{\alpha_q} \mathcal{D}_+ \mathcal{K}_+^\beta \quad (33)$$

where  $\Lambda$  is an  $H$ -invariant tensor on  $\mathfrak{m}$  which is antisymmetric on all its indices except the last (which is separated from the others by a semi-colon for this reason). The super-energy-momentum tensor  $\mathcal{K}_+^\alpha \mathcal{D}_+ \mathcal{K}_+^\alpha$  is obtained for  $\Lambda_{\alpha\beta} = \delta_{\alpha\beta}$ . Notice that  $\Lambda$  must never be *totally* antisymmetric, or else the expression above will be a total derivative. Now  $H$ -invariance alone is not enough to ensure that  $\mathcal{C}_\Lambda$  is superholomorphic:

$$D_- \mathcal{C}_\Lambda = \mathcal{D}_- \mathcal{C}_\Lambda = \Lambda_{\alpha_1 \dots \alpha_q; \beta} \mathcal{K}_+^{\alpha_1} \dots \mathcal{K}_+^{\alpha_q} \mathcal{D}_- (\mathcal{D}_+ \mathcal{K}_+^\beta) \quad (34)$$

using (21), but then

$$\mathcal{D}_- (\mathcal{D}_+ \mathcal{K}_+) = -[[\mathcal{K}_-, \mathcal{K}_+], \mathcal{K}_+] \quad (35)$$

using (22). So  $\mathcal{C}_\Lambda$  is superholomorphic iff

$$\Lambda_{[\alpha_1 \dots \alpha_q; \beta f^{\beta\hat{\gamma}} \delta f^{\hat{\gamma}\epsilon} \lambda]} = 0. \quad (36)$$

When this holds, the resulting top and bottom component currents have spins  $(q+1)/2$  and  $q/2$ , respectively.

Our task is to construct tensors  $\Omega$  and  $\Lambda$  with these properties for each classical symmetric space  $G/H$ . We will do this in sections 3 and 4, using two complimentary approaches. The first approach is rather general and could be applied to examples involving exceptional groups too (though we do not consider such cases in any detail in this paper). It involves starting from symmetric  $G$ -invariant tensors on  $\mathfrak{g}$ , or  $H$ -invariant tensors on  $\mathfrak{h}$ , and using these to build  $H$ -invariant tensors on  $\mathfrak{m}$  in a systematic way. The second approach is more case-specific and involves writing down  $H$ -invariants on  $\mathfrak{m}$  directly, using knowledge of the particular representation of  $H$  on  $\mathfrak{m}$  for each classical symmetric space.

The combination of these approaches will allow us to identify generating sets of primitive currents, as defined above. This will depend, in part, on understanding whether the invariant tensors used to define the currents are *primitive* as tensors, meaning that they cannot be written as tensor products (appropriately symmetrized or antisymmetrized) of invariants of lower degree, or whether they are *compound*, meaning that they *can* be decomposed in such a fashion. The relationship between the notions of primitive tensors and primitive superholomorphic currents is quite intricate, however, as we shall see.

## 3 General approach

### 3.1 Lie groups—review

We begin by recalling some details concerning invariants for Lie groups and Lie algebras. This will be essential for understanding the generalization to symmetric spaces and many of the details will also be needed at other points throughout the paper.

Note that  $G$ -invariant tensors on  $\mathfrak{g}$  are defined by the equations (26) to (29) given earlier, with  $\mathfrak{m}$  replaced by  $\mathfrak{g}$  and  $H$  replaced by  $G$ . For a simple Lie algebra  $\mathfrak{g}$  there are exactly  $\text{rank}(\mathfrak{g})$  independent primitive symmetric  $G$ -invariant tensors  $d$  (see e.g. [30]) The choice of these primitive invariants is certainly not unique, because we always have the freedom to add on products of invariants of lower degrees, but once we have chosen a particular set, then *any* other symmetric invariant can be expressed in terms of them in a unique way. Furthermore, the degrees  $p$  of the tensors  $d$  are the same for each primitive set; writing



$p = s + 1$ , the integers  $s$  are the *exponents* of  $\mathfrak{g}$ , and for the classical Lie algebras we have

$$\begin{aligned}
\mathfrak{a}_n &= \mathfrak{su}(n+1) & : & \quad s = 1, 2, \dots, n \\
\mathfrak{b}_n &= \mathfrak{so}(2n+1) & : & \quad s = 1, 3, \dots, 2n-1 \\
\mathfrak{c}_n &= \mathfrak{sp}(n) & : & \quad s = 1, 3, \dots, 2n-1 \\
\mathfrak{d}_n &= \mathfrak{so}(2n) & : & \quad s = 1, 3, \dots, 2n-3; n-1
\end{aligned} \tag{37}$$

A standard way to construct symmetric invariant tensors is as symmetrized traces in some representation; for the classical groups or algebras we can use the defining representations:

$$\mathrm{Tr}(X^p) = d_{a_1 a_2 \dots a_p} X^{a_1} X^{a_2} \dots X^{a_p} \quad \text{or} \quad d_{a_1 a_2 \dots a_p} = \mathrm{Tr}(t_{(a_1} t_{a_2} \dots t_{a_p)}). \tag{38}$$

One can always choose a set of primitive invariants from amongst these, with one exception. For  $\mathfrak{g} = \mathfrak{so}(2n)$  there is an invariant of degree  $n$ , called the Pfaffian, which is not of this form; it is defined by

$$\mathrm{Pf}(X) = d_{a_1 a_2 \dots a_n} X^{a_1} X^{a_2} \dots X^{a_n} = \frac{1}{2^n n!} \varepsilon_{i_1 j_1 i_2 j_2 \dots i_n j_n} X_{i_1 j_1} X_{i_2 j_2} \dots X_{i_n j_n}. \tag{39}$$

(This is related to a trace in a spinor representation.)

The values of  $s$  in (37) can be understood using the following general fact. Given any  $m \times m$  complex matrix,  $X$ , the trace-powers

$$\mathrm{Tr}X, \quad \mathrm{Tr}X^2, \quad \dots, \quad \mathrm{Tr}X^m \tag{40}$$

are independent, in general, but traces of all higher powers can always be expressed in terms of them. This follows from the identity  $\mathrm{Det}(1 - \lambda X) = \exp \mathrm{Tr} \log(1 - \lambda X)$ ; the left-hand side is a polynomial in  $\lambda$  of degree  $m$  but the right-hand side can be expanded in a power series (for suitable  $\lambda$ ) and equating coefficients yields the desired relations. When, in addition,  $X$  has specific properties by virtue of belonging to some Lie algebra, then some of the traces in (40) can vanish, and it is easy to check the details and recover (37). Once again, the Pfaffian in  $\mathfrak{so}(2n)$  is something of a special case, but  $\mathrm{Pf}(X)^2 = \mathrm{Det}X$ , which can be expressed in terms of the trace-powers (40) for  $m \leq 2n$ , and this relation then means that  $\mathrm{Tr}X^{2n}$  is compound, consistent with (37).

With these preparatory remarks in mind, we recall how to construct  $\Omega$  and  $\Lambda$  tensors for groups, i.e. for the SPCMs [19]. Our formulation of the general  $G/H$  model in section 2.1 can, of course, be specialized to the  $G$  SPCM, with lagrangian (7), by taking  $H$  to be the trivial subgroup (we will comment below on another way of viewing groups as symmetric spaces—see section 3.2). Both the  $\Omega$  and  $\Lambda$  tensors for groups are defined using a primitive symmetric invariant tensor  $d$  on  $\mathfrak{g}$ . If  $d$  has degree  $s+1$ , the resulting top component currents (or charges) have spins  $s+1$  (or  $s$ ), with  $s$  listed in (37).

To define  $\Omega$  tensors, we take a primitive symmetric invariant  $d$ , fill all but one of the slots with a Lie bracket, and antisymmetrize; so in components

$$\begin{aligned}\Omega_{a_1 \dots a_{2s+1}} &= \frac{1}{2^s} f^{b_1}_{[a_1 a_2} \dots f^{b_s}_{a_{2s-1} a_{2s}} d_{a_{2s+1}] b_1 \dots b_s} \\ &= \frac{1}{2^s} f^{b_1}_{a_1 [a_2} \dots f^{b_s}_{a_{2s-1} a_{2s}]} d_{a_{2s+1} b_1 \dots b_s} ,\end{aligned}\tag{41}$$

where the second line follows from the invariance of  $d$ . We will often write  $\Omega^{(d)}$  to indicate the underlying symmetric invariant  $d$  from which  $\Omega$  is built. Note that if we attempt to build a totally antisymmetric invariant tensor by putting Lie brackets in *all* of the slots, we get zero—because invariance of  $d$  and the Jacobi identity imply

$$f^{b_1}_{[a_1 a_2} \dots f^{b_{s+1}}_{a_{2s+1} a_{2s+2}]} d_{b_1 \dots b_{s+1}} = 0 .\tag{42}$$

This will prove important later. The tensors  $\Omega^{(d)}$  provide a set of generators for the algebra of  $G$ -invariant forms on  $\mathfrak{g}$ , and they can consequently be identified with the generators for the de Rham cohomology ring of  $G$  (see e.g. [30]).

For the  $\Lambda$  family, we start once again from a symmetric invariant  $d$  and define

$$\Lambda_{a_1 \dots a_{2s-1}; a_{2s}} = \frac{1}{2^{s-1}} f^{b_1}_{[a_1 a_2} \dots f^{b_{s-1}}_{a_{2s-3} a_{2s-2}} d_{a_{2s-1}] a_{2s} b_1 \dots b_{s-1}}\tag{43}$$

For groups, the key identity (36) becomes

$$\Lambda_{[a_1 \dots a_q; } {}^b f^{bc}{}_d f^{ce}{}_f] = 0\tag{44}$$

which can be derived from (42). Once again, we will often write  $\Lambda^{(d)}$  to indicate the dependence on  $d$ .

## 3.2 Symmetric spaces

It is not immediately clear how one should generalize the definitions (41) and (43) from groups to symmetric spaces. One possibility is to start with tensors  $\Omega$  and  $\Lambda$  on  $\mathfrak{g}$  and simply restrict them to  $\mathfrak{m}$ , which will certainly give  $H$ -invariant results. We must then determine when these restrictions are non-zero, or independent, however. Furthermore, the work in [19] suggests that the association of  $\Omega$  and  $\Lambda$  with the underlying  $d$  tensor is best kept as clear as possible. We therefore proceed as follows.

Recall that we may define the symmetric space  $G/H$  by means of an automorphism  $\sigma$  of  $\mathfrak{g}$ , with  $\sigma^2 = 1$ , the subspaces  $\mathfrak{h}$  and  $\mathfrak{m}$  being the eigenspaces of  $\sigma$  with eigenvalues  $\pm 1$ . Now consider how a given symmetric  $G$ -invariant tensor  $d$  on  $\mathfrak{g}$  behaves when its entries are acted on by  $\sigma$ . There are two possibilities:

$$d(\sigma(X), \sigma(Y), \dots, \sigma(Z)) = \eta d(X, Y, \dots, Z) , \quad \eta = \pm 1 ,\tag{45}$$

for all  $X, Y, \dots, Z \in \mathfrak{g}$ . The first possibility,  $\eta = 1$ , obviously holds whenever  $\sigma$  is an inner automorphism, but if  $\sigma$  is not inner then  $\eta = -1$  may occur for certain  $d$  (the relevant automorphisms are complex conjugation for  $SU(n)$  or  $E_6$ , and reflection for  $SO(2n)$  [24]). It is an easy matter to determine which tensors have  $\eta = \pm 1$  for each symmetric space and it follows from the behaviour of  $\mathfrak{h}$  and  $\mathfrak{m}$  under  $\sigma$  that the components of  $d$  have the following properties:

$$\eta = +1 \quad \Rightarrow \quad d_{\hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_s \hat{\alpha}_{s+1}} \neq 0, \quad d_{\hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_s \alpha_{s+1}} = 0, \quad d_{\hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_{s-1} \alpha_s \alpha_{s+1}} \neq 0, \quad \dots \quad (46)$$

$$\eta = -1 \quad \Rightarrow \quad d_{\hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_s \hat{\alpha}_{s+1}} = 0, \quad d_{\hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_s \alpha_{s+1}} \neq 0, \quad \dots \quad (47)$$

In these equations, we mean that the relevant components need not be identically zero by virtue of (45); some specific components may vanish.

Given any  $d$  with  $\eta = +1$ , the properties (46) imply that we can build a  $\Lambda$  tensor in much the same way that we did for groups:

$$\Lambda_{\alpha_1 \dots \alpha_{2s-1}; \beta}^{(d)} = \frac{1}{2^{s-1}} f^{\hat{\beta}_1}_{[\alpha_1 \alpha_2 \dots} f^{\hat{\beta}_{s-1}}_{\alpha_{2s-3} \alpha_{2s-2}} d_{\alpha_{2s-1}] \beta \hat{\beta}_1 \dots \hat{\beta}_{s-1}} \cdot \quad (48)$$

This is certainly  $H$ -invariant and, by careful use of the invariance of  $d$ , one can check that it also satisfies the additional property (36). These tensors are, indeed, just the restrictions of the  $\Lambda$  tensors (43) from  $\mathfrak{g}$  to  $\mathfrak{m}$ , but the relevant properties (46) of the underlying  $d$  tensor are now transparent. Note also that with this choice the definition (33) can be written

$$\mathcal{C}_\Lambda = d(\mathcal{K}_+^2, \dots, \mathcal{K}_+^2, \mathcal{K}_+, \mathcal{D}_+ \mathcal{K}_+) \cdot \quad (49)$$

Given a tensor  $d$  with  $\eta = -1$ , we cannot construct a  $\Lambda$  tensor (the expression above vanishes). But we can, instead, construct an  $\Omega$  tensor:

$$\Omega_{\alpha_1 \dots \alpha_{2s+1}}^{(d)} = \frac{1}{2^s} f^{\hat{\beta}_1}_{[\alpha_1 \alpha_2 \dots} f^{\hat{\beta}_s}_{\alpha_{2s-1} \alpha_{2s}} d_{\alpha_{2s+1}] \hat{\beta}_1 \dots \hat{\beta}_s} \cdot \quad (50)$$

With this choice, (30) becomes:

$$\mathcal{C}_\Omega = d(\mathcal{K}_+^2, \dots, \mathcal{K}_+^2, \mathcal{K}_+) \quad (51)$$

Once again, this  $\Omega$  tensor is the restriction to  $\mathfrak{m}$  of a  $G$ -invariant tensor (41) on  $\mathfrak{g}$ . These are not the only  $\Omega$  tensors however.

The problem of finding a set of generators for the algebra of  $H$ -invariant antisymmetric tensors, or forms, on  $\mathfrak{m}$  has been considered by mathematicians; it corresponds to finding a set of generators for the de Rham cohomology ring of  $G/H$  [26, 27, 25, 28] (see also [29] for an account in the physics literature). We have just learnt that there is one class of antisymmetric tensors, or cohomology representatives, which are based on symmetric

$G$ -invariant tensors  $d$  on  $\mathfrak{g}$  which vanish when restricted to  $\mathfrak{h}$ . For Lie groups, a complete set of cohomology generators arises in this way, and they are all of odd degree. But for symmetric spaces there may be additional  $\Omega$  tensors of even degree. These are of the form

$$\Omega_{\alpha_1 \dots \alpha_{2k}}^{(e)} = \frac{1}{2^k} f^{\hat{\beta}_1}_{[\alpha_1 \alpha_2} \dots f^{\hat{\beta}_k}_{\alpha_{2k-1} \alpha_{2k}] } e_{\hat{\beta}_1 \dots \hat{\beta}_k} \quad (52)$$

where  $e_{\hat{\beta}_1 \dots \hat{\beta}_k}$  is a symmetric  $H$ -invariant tensor on  $\mathfrak{h}$  which is *not* the restriction of a  $G$ -invariant tensor on  $\mathfrak{g}$ .<sup>2</sup> The general formula (30) then reads

$$\mathcal{C}_\Omega = e(\mathcal{K}_+^2, \mathcal{K}_+^2, \dots, \mathcal{K}_+^2) . \quad (53)$$

To understand why  $e$  must *not* be the restriction of a  $G$ -invariant tensor  $d$  on  $\mathfrak{g}$ , note that (42), together with the symmetric space conditions (11), implies that

$$f^{\hat{\beta}_1}_{[\alpha_1 \alpha_2} \dots f^{\hat{\beta}_{s+1}}_{\alpha_{2s+1} \alpha_{2s+2}] } d_{\hat{\beta}_1 \dots \hat{\beta}_{s+1}} = 0 \quad (54)$$

so that the expression in (52) then vanishes. Moreover, this identity implies that  $\Omega^{(e)}$  will also vanish if  $e$  is a symmetrized tensor product  $d \cdot e'$  where  $d$  is the restriction of any  $G$ -invariant tensor on  $\mathfrak{g}$  (of strictly positive degree, of course). We must therefore choose a maximal set of  $e$  tensors which are independent modulo such tensor products in order to obtain a full set of even-degree forms.

Let us now return briefly to Lie groups as special examples of symmetric spaces. For the formulation of the corresponding sigma models, it is convenient to think of the SPCMs as having target manifolds  $G/H$  with  $H$  taken to be the trivial subgroup. But we may instead regard  $G = G_L \times G_R / G$ , which has the advantage that the numerator, consisting of a direct product of two copies of  $G$ , is the isometry group of the manifold (as it should be for a symmetric space  $G/H$ ). If we apply our general construction of  $\Omega$  and  $\Lambda$  tensors to this case, we start with independent symmetric invariants  $d_L$  and  $d_R$  on the Lie algebras  $\mathfrak{g}_L$  and  $\mathfrak{g}_R$  for each factor. The automorphism which defines the symmetric space simply exchanges the  $L$  and  $R$  factors, so that the denominator  $G$  is the diagonal subgroup of  $G_L \times G_R$ . But this means that we always have combinations  $d_L \pm d_R$  on  $\mathfrak{g}_L \oplus \mathfrak{g}_R$  which are even/odd under the automorphism. Our general construction therefore gives a nice additional insight into why there are *both*  $\Omega$  and  $\Lambda$  tensors associated with *each* primitive invariant  $d$  for the case of groups.

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<sup>2</sup>Regarded as invariant forms on  $G/H$ , the  $\Omega^{(e)}$  correspond to characteristic classes. Thinking of  $G \rightarrow G/H$  as a principal  $H$ -bundle with a connection, the Chern-Weil homomorphism (see for example [25, 28]) is a map from the ring of invariant polynomials of the structure group to the de Rham cohomology of the base space; the definition (52) is essentially this map.

### 3.3 Choices of tensors and currents

To summarize, we have found that for each symmetric space  $G/H$  there are  $\Omega$  and  $\Lambda$  tensors which arise from symmetric  $d$  and  $e$  tensors as follows:

- $d$ ,  $G$ -invariant on  $\mathfrak{g}$  and even under the automorphism  $\sigma$   
 $\longrightarrow \Lambda^{(d)}$ , tensor of even degree
- $d$ ,  $G$ -invariant on  $\mathfrak{g}$  and odd under the automorphism  $\sigma$   
 $\longrightarrow \Omega^{(d)}$ , form of odd degree
- $e$ ,  $H$ -invariant on  $\mathfrak{h}$  and not the restriction of a  $G$ -invariant on  $\mathfrak{g}$   
 $\longrightarrow \Omega^{(e)}$ , form of even degree

In order that  $\mathcal{C}_\Omega$  or  $\mathcal{C}_\Lambda$  be primitive as superholomorphic currents it is necessary that  $d$  or  $e$  be primitive as symmetric tensors, although this is not usually a sufficient condition, as we shall explain below. It is also still important to understand how the construction works if the underlying symmetric tensors  $d$  or  $e$  are compound, or how the results are modified if they are primitive but changed by the addition of compound terms

$$d_{a_1 \dots a_p} \rightarrow d_{a_1 \dots a_p} + d'_{(a_1 \dots a_j} d''_{a_{j+1} \dots a_p)} \quad (55)$$

$$e_{\hat{a}_1 \dots \hat{a}_k} \rightarrow e_{\hat{a}_1 \dots \hat{a}_k} + e'_{(\hat{a}_1 \dots \hat{a}_j} e''_{\hat{a}_{j+1} \dots \hat{a}_k)} \quad (56)$$

The modification of  $d$  must, of course, have the same behaviour under  $\sigma$  in order to allow the construction of either a  $\Lambda$  or an  $\Omega$  tensor.

It is an immediate consequence of (54) that  $\Omega^{(d)}$  vanishes unless  $d$  is primitive (and  $d$  must be odd under  $\sigma$ ). For the same reason,  $\Omega^{(d)}$  and  $\mathcal{C}_\Omega$  are independent of any change (55). The  $\Omega^{(e)}$  tensors *are* modified by (56), but only by wedge products of forms of lower degree

$$\Omega^{(e)} \rightarrow \Omega^{(e)} + \Omega^{(e')} \wedge \Omega^{(e'')} . \quad (57)$$

This in turn implies that  $\mathcal{C}_\Omega$  changes by an expression  $\mathcal{C}_{\Omega'} \mathcal{C}_{\Omega''}$ .

The tensors  $\Lambda^{(d)}$ , for which  $d$  must be even under  $\sigma$ , behave in rather more complicated ways. If there exist  $d'$  and  $d''$  of the correct degrees which are each odd under  $\sigma$ , then the modification (55) results in a change in  $\Lambda^{(d)}$  proportional to

$$\Omega_{[\alpha_1 \dots \alpha_{2m-1}}^{(d')} \Omega_{\alpha_{2m} \dots \alpha_{2p-3}] \beta}^{(d'')} + \Omega_{[\alpha_1 \dots \alpha_{2n-1}}^{(d'')} \Omega_{\alpha_{2n} \dots \alpha_{2p-3}] \beta}^{(d')} \quad (58)$$

(this is discussed in some detail in [19] for the case of groups) and the current  $\mathcal{C}_\Lambda$  acquires extra terms of the form  $\mathcal{C}_{\Omega'} D_+ \mathcal{C}_{\Omega''}$ . If  $d'$  and  $d''$  are even under  $\sigma$ , however, then (46)

and (54) imply that  $\Lambda^{(d)}$  is unaffected. Furthermore, if the symmetrized product of *three* or more tensors is added to  $d$  then  $\Lambda^{(d)}$  is always unchanged, again by (54). Even if  $d$  is primitive, however,  $\Lambda^{(d)}$  need not be: it may decompose into a product of the form  $\Omega \dots \Omega' \Lambda'$ , implying that the conserved current is composite. We explain this in detail in the next section.

The observations above enable us to keep track of the effects of using different choices of primitive symmetric invariants  $d$  and  $e$  to define conserved currents, and there are two sets in particular that will prove useful. The simplest possibilities for many purposes are the symmetrized traces (38), or trace-powers (40), and we will deal mainly with these when determining the pattern of primitive  $\Omega$  and  $\Lambda$  tensors in the next section. On the other hand, it was shown in [16, 18] that for any Lie group  $G$  there exists another choice of primitive symmetric tensors on  $\mathfrak{g}$  such that any pair  $d_{a_1 a_2 \dots a_p}$  and  $\tilde{d}_{b_1 b_2 \dots b_q}$ , say, satisfy

$$d_{c(a_1 \dots a_{p-1} \tilde{d}_{b_1 \dots b_{q-2}}) b_{q-1} c} = d_{c(a_1 \dots a_{p-1} \tilde{d}_{b_1 \dots b_{q-2} b_{q-1}}) c} . \quad (59)$$

This property proved crucial to the construction of commuting sets of conserved charges in both the bosonic and supersymmetric PCMs [16, 19]. The same invariants will be equally important in our treatment of Poisson brackets in section 7.

## 4 Case-by-case construction

In this second approach we find conserved currents directly, for each classical symmetric space. We are able to do this because the representations of  $H$  on  $\mathfrak{m}$  are very familiar, involving defining representations or their tensor products, so that invariants can be found comparatively easily. We will be able to understand which of them are independent and primitive by using a variation on the result following (40).

Let  $X$  and  $Y$  be any  $m \times m$  complex matrices; then

$$\text{Tr} Y , \text{Tr} XY , \text{Tr} X^2 Y , \dots , \text{Tr} X^{m-1} Y \quad (60)$$

are in general independent, but all expressions  $\text{Tr} X^r Y$  for  $r \geq m$  can be expressed in terms of the quantities in (60). This follows from the corresponding result for (40) by considering  $X + \lambda Y$  and expanding to first order in  $\lambda$ . Note that the results for (40) and (60) also hold if  $X$  and  $Y$  are constructed from Grassmann quantities, provided  $X$  is bosonic, or even graded. The matrix  $Y$  in (60) is allowed to be fermionic, if desired, because in the proof indicated above the parameter  $\lambda$  can also be fermionic—no difficulties arise because we are expanding only to first order in  $\lambda$  (but higher powers of fermionic matrices could not be treated in this way.)

## 4.1 Complex Grassmannians

The complex Grassmannians are  $SU(p+q)/S(U(p) \times U(q))$ , with  $p \leq q$ , say. The current  $\mathcal{K}_+$  and its derivative  $\mathcal{D}_+\mathcal{K}_+$  belong to  $\mathfrak{m}$  and so take the block forms

$$\mathcal{K}_+ = \begin{pmatrix} 0 & K \\ -K^\dagger & 0 \end{pmatrix}, \quad \mathcal{D}_+\mathcal{K}_+ = \begin{pmatrix} 0 & L \\ -L^\dagger & 0 \end{pmatrix}, \quad (61)$$

where  $K$  (fermionic) and  $L$  (bosonic) are complex  $p \times q$  matrices. An element

$$h = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in H, \quad (62)$$

where  $H = S(U(p) \times U(q))$ , acts on these matrices according to

$$K \mapsto AKB^\dagger, \quad L \mapsto ALB^\dagger. \quad (63)$$

It is clear that to form  $H$ -invariants we must take traces of products of matrices such as  $KK^\dagger$  and  $KL^\dagger$ .

If we use just the matrices  $K$ , we can construct currents

$$\mathcal{C} = \text{Tr} [(KK^\dagger)^r] \quad (64)$$

and, considering (40) with  $X = KK^\dagger$ , they are independent for  $1 \leq r \leq p$ . These currents correspond to primitive forms  $\Omega^{(e)}$  of even degree. The invariants  $e$  are trace-powers on the Lie algebra of  $\mathfrak{u}(p)$ , but with no contribution from  $\mathfrak{u}(q)$ , which is why these  $e$  tensors are not restrictions to  $\mathfrak{h}$  of  $G$ -invariants on  $\mathfrak{g}$ . Note also that one could take traces of powers of the  $q \times q$  matrix  $K^\dagger K$  instead, and obtain the same currents (by cyclicity of the trace). But it is then no longer manifest that these invariants cease to be primitive beyond the  $p$ th power.

The currents based on  $\Lambda^{(d)}$  tensors involve  $K$ 's and a single  $L$ , for example

$$\mathcal{C} = \text{Tr} [(KK^\dagger)^{s-1}(KL^\dagger + LK^\dagger)] . \quad (65)$$

Note that if a relative sign is introduced in the last factor in the trace, then this expression becomes a total derivative, so we discount this possibility. Considering (60) with  $X = KK^\dagger$  and  $Y = (KL^\dagger + LK^\dagger)$ , these currents are primitive for  $1 \leq s \leq p$ . For  $s > p$ , the trace will factorize into a sum of traces of lower powers; in terms of tensors we have a compound  $\Lambda$  factorizing into products of primitive tensors of the form  $\Omega \dots \Omega \Lambda$ . To compare with our general construction of the last section, all the invariants on  $\mathfrak{g} = \mathfrak{su}(p+q)$  are unchanged under the automorphism  $\sigma$  defining the symmetric space. Hence there are no tensors  $\Omega^{(d)}$ —no cohomology generators of odd degree—for these spaces. The currents above correspond to tensors  $\Lambda^{(d)}$  in which each  $d$  is a trace-power on  $\mathfrak{g}$ .

## 4.2 Real and quaterionic Grassmannians

The real Grassmannians  $SO(p+q)/SO(p)\times SO(q)$ , with  $p \leq q$ , are of course similar in many ways to the complex family above. We have the same block forms (61) and (62) but with  $K$  and  $L$  now  $p\times q$  real matrices and  $H = SO(p)\times SO(q)$  acting as in (63). However, new features arise because the matrix  $KK^T$  is antisymmetric.

The currents

$$\mathcal{C} = \text{Tr} [(KK^T)^r] \quad (66)$$

are non-zero only if  $r$  is an even integer (the trace of any odd power of an antisymmetric matrix vanishes) and they are primitive when  $r \leq p$ , as before. These are the currents based on  $\Omega^{(e)}$  tensors with  $e$  a trace-power invariant on  $\mathfrak{so}(p)$ . But there are additional currents, in some cases, which can be constructed using  $\varepsilon$  tensors.

If  $p$  or  $q$  is even, we have

$$\mathcal{C} = \varepsilon_{i_1 i_2 \dots i_{p-1} i_p} (KK^T)_{i_1 i_2} \dots (KK^T)_{i_{p-1} i_p} , \quad (67)$$

$$\text{or } \mathcal{C} = \varepsilon_{j_1 j_2 \dots j_{q-1} j_q} (K^T K)_{j_1 j_2} \dots (K^T K)_{j_{q-1} j_q} , \quad (68)$$

which correspond to forms  $\Omega^{(e)}$  with  $e$  the Pfaffian invariant on  $\mathfrak{so}(p)$  or  $\mathfrak{so}(q)$ , respectively. These complete the set of primitive cohomology generators of even degrees, but there is one more generator of odd degree which occurs iff both  $p$  and  $q$  are odd. The current in question is

$$\begin{aligned} \mathcal{C} = & \varepsilon_{i_1 i_2 \dots i_{p-2} i_{p-1} i_p} \varepsilon_{j_1 j_2 \dots j_{q-2} j_{q-1} j_q} \\ & \times (KK^T)_{i_1 i_2} \dots (KK^T)_{i_{p-2} i_{p-1}} (K^T K)_{j_1 j_2} \dots (K^T K)_{j_{q-2} j_{q-1}} K_{i_p j_q} , \end{aligned} \quad (69)$$

which is based on the form  $\Omega^{(d)}$  with  $d$  the Pfaffian on  $\mathfrak{so}(p+q)$ . It is not difficult to see that this  $d$  tensor is odd under the automorphism  $\sigma$  defining the Grassmannian iff  $p$  and  $q$  are both odd, consistent with our general construction in the last section.

Turning now to the currents involving derivatives, we have

$$\mathcal{C} = \text{Tr} [(KK^T)^{s-1}(KL^T)] . \quad (70)$$

However,  $KL^T + LK^T$  being symmetric implies that this expression vanishes if  $s$  is even. In addition, the non-vanishing currents are primitive for  $s \leq p$ . These currents are based on tensors  $\Lambda^{(d)}$  with  $d$  a trace-power on  $\mathfrak{g} = \mathfrak{so}(p+q)$ . The trace-type  $d$  tensors are always invariant under  $\sigma$ , and the fact that  $d$  must be the trace of an even power for an orthogonal algebra fits precisely with the fact that the currents above are non-vanishing only when  $s$  is odd. Finally, if  $p$  and  $q$  are both even we can form one additional independent current

$$\begin{aligned} \mathcal{C} = & \varepsilon_{i_1 i_2 \dots i_{p-3} i_{p-2} i_{p-1} i_p} \varepsilon_{j_1 j_2 \dots j_{q-3} j_{q-2} j_{q-1} j_q} \\ & \times (KK^T)_{i_1 i_2} \dots (KK^T)_{i_{p-2} i_{p-1}} (K^T K)_{j_1 j_2} \dots (K^T K)_{j_{q-3} j_{q-2}} K_{i_{p-1} j_{q-1}} L_{i_p j_q} . \end{aligned} \quad (71)$$



This is based on  $\Lambda^{(d)}$  with  $d$  the Pfaffian on  $\mathfrak{so}(p+q)$ . We stated above that when  $p$  and  $q$  are both odd this  $d$  tensor changes sign under  $\sigma$  and so gives rise to an  $\Omega$  tensor, but when  $p$  and  $q$  are both even  $d$  is unchanged by  $\sigma$  and so gives rise to a  $\Lambda$  tensor instead.

The quaternionic Grassmannians  $Sp(p+q)/Sp(p)\times Sp(q)$  behave similarly to the real cases, but without any of the complications arising from Pfaffians. The block forms (61) and (62) hold with  $p$  and  $q$  replaced by  $2p$  and  $2q$ , and with a symplectic reality condition  $K^* = J_p K J_q^{-1}$  where  $J_p$  and  $J_q$  are symplectic structures of size  $2p\times 2p$  and  $2q\times 2q$  respectively. The currents (64) are non-vanishing for  $r$  even, while the currents (65) are non-vanishing for  $s$  odd, and each set is primitive for  $r, s \leq 2p$ . The cohomology generators  $\Omega$  are all of even degree, based on  $e$  tensors; the  $d$  tensors on  $\mathfrak{g} = \mathfrak{sp}(p+q)$  are unchanged by the automorphism defining the Grassmannian and so always give rise to  $\Lambda$  type tensors.

### 4.3 $SU(n)/SO(n)$ and $SU(2n)/Sp(n)$

Consider first  $SU(n)/SO(n)$ , which is defined by taking the automorphism  $\sigma$  of  $\mathfrak{g} = \mathfrak{su}(n)$  to be complex conjugation. This clearly implies that  $\mathfrak{h} = \mathfrak{so}(n)$ , while  $\mathfrak{m}$  consists of symmetric, traceless, imaginary  $n\times n$  matrices. Furthermore,  $\mathcal{K}_+ \in \mathfrak{m}$  transforms under  $h \in SO(n)$  according to  $\mathcal{K}_+ \mapsto h\mathcal{K}_+h^{-1}$ . The currents

$$\mathcal{C} = \text{Tr} [ (\mathcal{K}_+^2)^s \mathcal{K}_+ ] , \quad (72)$$

are non-vanishing only for  $s$  even (consider transposing the matrices) and they are independent for  $s+1 \leq n$ . Traces of even powers of  $\mathcal{K}_+$  vanish by cyclicity. Similar considerations imply that the currents

$$\mathcal{C} = \text{Tr} [ (\mathcal{K}_+^2)^{s-1} \mathcal{K}_+ \mathcal{D}_+ \mathcal{K}_+ ] \quad (73)$$

are non-vanishing only when  $s$  is odd and (60) suggests that they are independent for  $s \leq n$ , but there is a subtlety here (see below) and the case  $s = n$  can be ignored.

Comparing to section 3, the currents above arise from  $d$  tensors on  $\mathfrak{g} = \mathfrak{su}(n)$  corresponding to the trace-powers  $\text{Tr} X^{s+1}$ . These are inert under  $\sigma$  (complex conjugation) when  $s$  is odd, giving a tensor  $\Lambda^{(d)}$ , but they change sign when  $s$  is even, giving a tensor  $\Omega^{(d)}$ . Note that  $d$  must itself be primitive, which requires  $s+1 \leq n$  and that this is a slightly stronger condition than the one suggested by (60) for the currents (73). The reason is that (60) reveals whether  $\Lambda$  can be decomposed into products  $\Omega \dots \Omega \Lambda$ , but for  $s = n$ ,  $d$  is compound and  $\Lambda^{(d)}$  decomposes as in (58).

When  $n$  is even, there is one final current

$$\mathcal{C} = \varepsilon_{i_1 i_2 \dots i_{n-1} i_n} (\mathcal{K}_+^2)_{i_1 i_2} \dots (\mathcal{K}_+^2)_{i_{n-1} i_n} . \quad (74)$$

This corresponds to an even degree form  $\Omega^{(e)}$ , where  $e$  is the Pfaffian on  $\mathfrak{h} = \mathfrak{so}(n)$ .

Now consider the family  $SU(2n)/Sp(n)$ . This is very similar, because the automorphism  $\sigma$  is defined by complex conjugation together with conjugation by a symplectic structure. The currents and invariants are therefore just like those of  $SU(2n)/SO(2n)$ , but without the Pfaffian, for all  $n > 2$ . (The case  $n = 2$  deviates slightly from the general pattern because of some special properties of low-dimensional representations; but this case has already been dealt with above as one of the real Grassmannians, since  $SU(4)/Sp(2) = SO(6)/SO(5)$ ).

#### 4.4 $SO(2n)/U(n)$ and $Sp(n)/U(n)$

Consider first the family  $SO(2n)/U(n)$ . The embedding of  $\mathfrak{h} = \mathfrak{u}(n)$  in  $\mathfrak{g} = \mathfrak{so}(2n)$  is defined by taking  $A + iB$ , where the real  $n \times n$  matrices  $A$  and  $B$  are antisymmetric and symmetric respectively, and mapping it to  $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ . Similarly, elements of  $\mathfrak{m}$ , which are of the form  $\begin{pmatrix} C & D \\ D & -C \end{pmatrix}$ , where the  $n \times n$  matrices  $C$  and  $D$  are both real and antisymmetric, can be represented in the complex combination  $C + iD$ . Thus the current  $\mathcal{K}_+ \in \mathfrak{m}$  can be identified with an  $n \times n$ , complex, antisymmetric matrix  $K$ , and it can be checked that under  $h \in U(n)$  this transforms  $K \mapsto hKh^T$ . In addition,  $\mathcal{D}_+\mathcal{K}_+$  is identified with a similar matrix  $L$ .

Now (just as for the complex Grassmannians) one obvious class of currents is

$$\mathcal{C} = \text{Tr} [(KK^\dagger)^s] , \quad (75)$$

and a second class is

$$\mathcal{C} = \text{Tr} [(KK^\dagger)^{s-1}(KL^\dagger + LK^\dagger)] \quad (76)$$

They are non-zero only when  $s$  is odd (by flipping indices on all matrices, then re-arranging). The corresponding tensors are  $\Omega^{(e)}$  and  $\Lambda^{(d)}$ , respectively, where  $d$  and  $e$  are both trace-powers (on  $\mathfrak{g}$  and  $\mathfrak{h}$ ). These currents are primitive for  $s \leq n$ , provided  $n > 3$ . (The case  $n=3$  differs slightly from the general pattern, because of some special behaviour of the low-dimensional representations involved; but this has already been treated above as one of the complex Grassmannians, since  $SO(6)/U(3) = SU(4)/S(U(3) \times U(1))$ ).

There is also a Pfaffian amongst the primitive  $d$  invariants on  $\mathfrak{so}(2n)$ , which gives rise to a  $\Lambda$  tensor. However, this tensor, and the corresponding current, are not independent of those already written above. This is because the  $\varepsilon$  tensor of  $\mathfrak{so}(2n)$  which appears in the Pfaffian, is a product  $\varepsilon_{i_1 \dots i_n} \varepsilon^{j_1 \dots j_n}$  in terms of  $H = U(n)$  invariants. But such a product can be re-written in terms of  $\delta_i^j$  tensors, and hence in terms of traces.

The family  $Sp(n)/U(n)$  is very similar. We can again identify  $A + iB \in \mathfrak{u}(n)$  with the matrix  $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$  and elements of  $\mathfrak{m}$  take the form  $i \begin{pmatrix} C & D \\ D & -C \end{pmatrix}$  where  $C$  and  $D$  are now  $n \times n$  real and symmetric and can be combined in a complex symmetric matrix  $C + iD$ . The invariants are the same as the trace-type invariants in the previous case, and the reasoning is very similar.

## 5 Overview of results obtained and those to follow

### 5.1 Summary of primitive currents

The work of sections 3 and 4 determines, by exhaustion, the primitive  $\Lambda$  and  $\Omega$  tensors for each classical symmetric space and we now summarize the results.

$G/H$	$s : \Lambda^{(d)}$ primitive, $d$ of degree $s+1$
$SO(p+q)/SO(p) \times SO(q) \quad p \leq q$	$1, 3, \dots, \begin{cases} p & p \text{ odd} \\ p-1 & p \text{ even} \end{cases}$ and $\frac{1}{2}(p+q)-1, p, q \text{ even}$
$SU(p+q)/S(U(p) \times U(q)) \quad p \leq q$	$1, 2, \dots, p$
$Sp(p+q)/Sp(p) \times Sp(q) \quad p \leq q$	$1, 3, \dots, 2p-1$
$SU(n)/SO(n) \quad n \geq 2$	$1, 3, \dots, \begin{cases} n-2 & n \text{ odd} \\ n-1 & n \text{ even} \end{cases}$
$SU(2n)/Sp(n) \quad n \geq 3$	$1, 3, \dots, 2n-1$
$SO(2n)/U(n) \quad n \geq 4$	$1, 3, \dots, \begin{cases} n & n \text{ odd} \\ n-1 & n \text{ even} \end{cases}$
$Sp(n)/U(n) \quad n \geq 2$	$1, 3, \dots, \begin{cases} n & n \text{ odd} \\ n-1 & n \text{ even} \end{cases}$

The first table provides a list of primitive  $\Lambda$  tensors and the integer  $s$  is the spin of the top component, bosonic conserved charge. The first member of each list has  $s = 1$ , corresponding to the super-energy-momentum tensor. The second table specifies the degrees of the primitive cohomology generators, or  $\Omega$  tensors (these results could also be found using techniques such as those in [25]-[28]).

$G/H$	degrees of $\Omega$ : cohomology generators
$SO(p+q)/SO(p) \times SO(q) \quad p \leq q$	$4, 8, \dots, \begin{cases} 2p-2 & p \text{ odd} \\ 2p-4, & \text{and } p, p \text{ even} \\ & \text{and } q, q \text{ even} \end{cases}$
$SU(p+q)/S(U(p) \times U(q)) \quad p \leq q$	$2, 4, \dots, 2p$
$Sp(p+q)/Sp(p) \times Sp(q) \quad p \leq q$	$4, 8, \dots, 4p$
$SU(n)/SO(n) \quad n \geq 2$	$5, 9, \dots, \begin{cases} 2n-1 & n \text{ odd} \\ 2n-3, & \text{and } n, n \text{ even} \end{cases}$
$SU(2n)/Sp(n) \quad n \geq 3$	$5, 9, \dots, 4n-3$
$SO(2n)/U(n) \quad n \geq 3$	$2, 6, \dots, \begin{cases} 2n & n \text{ odd} \\ 2n-2 & n \text{ even} \end{cases}$
$Sp(n)/U(n) \quad n \geq 2$	$2, 6, \dots, \begin{cases} 2n & n \text{ odd} \\ 2n-2 & n \text{ even} \end{cases}$

Note that some of the families above resemble Lie groups in that all (or almost all) their cohomology generators are of odd degree. The symmetric spaces with  $\text{rank}(\mathfrak{g}) = \text{rank}(\mathfrak{h})$ , on the other hand, have all their cohomology generators of even degree.

## 5.2 Poisson brackets

Having arrived at a coherent description of the primitive classical conserved currents in each supersymmetric  $G/H$  model, it is natural to ask about their Poisson bracket algebra. Even for the case of groups, however, calculations of the full classical current algebra are lengthy and involved [19]. We will therefore confine our attention to a particularly interesting aspect of this current algebra: the Poisson brackets of the top component charges arising from all the primitive currents. One reason these are of special significance is that it was shown in [19] that the primitive currents for the SPCMs could be chosen so that their bosonic, top component charges all Poisson-commute, and it is natural to ask whether this generalizes to symmetric spaces. A second reason is that it is the top component charges, and these alone, that are most likely to survive quantization. Although our results in this paper are purely classical, possible future implications for the quantized theories are also

a strong motivation for our work here. Let us recall why it is the top component charges which are significant for the quantum theory.

The superholomorphic form of the classical conservation laws (23) is intimately associated with the superconformal symmetry of the classical theory—which is why, of course, the conservation of the classical super-energy-momentum tensor itself takes this form,  $D_-(\mathcal{K}_+^\alpha \mathcal{D}_+ \mathcal{K}_+^\alpha) = 0$ . It is reasonable to anticipate that (at least some of) the local conservation laws we have found survive in the quantum theory [13, 20, 14]. But since superconformal invariance is broken quantum-mechanically, we should expect that any conservation equations which survive will appear in the more general, modified form:

$$D_- \mathcal{C} = D_+ \tilde{\mathcal{C}} , \quad (77)$$

where  $\tilde{\mathcal{C}}$  is some new superfield, and  $\mathcal{C}$  too may receive corrections. It is easy to show, simply by introducing component expansions for  $\mathcal{C}$  and  $\tilde{\mathcal{C}}$ , that the equation (77) contains a generalization of the top component conservation equation in (23), with charge  $S$ , but that there is, *no* generalization of the bottom component conservation law, in general (for more details, see [19, 23]). This is consistent with the fact that the original top component charge  $S$  and any quantum generalization of it are both invariant under supersymmetry.

We will denote by  $F^{(e)}$  or  $B^{(d)}$  the top component charges arising from a current  $\mathcal{C}_\Omega$  based on forms  $\Omega^{(e)}$  or  $\Omega^{(d)}$  respectively; these charges are indeed fermionic and bosonic, as the notation suggests. We will denote by  $P^{(d)}$  the top component charge arising from a current  $\mathcal{C}_\Lambda$  based on  $\Lambda^{(d)}$ ; these are always bosonic and the first member of the sequence is the momentum, arising from the choice  $\Lambda_{\alpha\beta} = \delta_{\alpha\beta}$ . Our aim in the remainder of this paper is to compute the Poisson bracket algebra of these quantities defined using particular primitive tensors  $d$  and  $e$ . We will carry out the calculations in section 7, after analyzing the canonical structure of the sigma models in section 6, but it is useful to state the results in advance as a guide to the calculations which follow.

We will prove that for the special family of  $d$  tensors obeying (59), all bosonic top component charges Poisson-commute:

$$\{B^{(d)}, B^{(\tilde{d})}\} = 0 \quad (78)$$

$$\{B^{(d)}, P^{(\tilde{d})}\} = 0 \quad (79)$$

$$\{P^{(d)}, P^{(\tilde{d})}\} = 0 \quad (80)$$

We will also find that (graded) Poisson brackets involving the fermionic charges close in

the following pattern:

$$\{F^{(e)}, B^{(d)}\} = 0 \quad (81)$$

$$\{F^{(e)}, P^{(d)}\} = \hat{F} \quad (82)$$

$$\{F^{(e)}, F^{(\bar{e})}\} = \hat{P} \quad (83)$$

Here  $\hat{F}$  denotes a top component fermionic charge arising from a current consisting of terms of the form  $\mathcal{C}_{\Omega'}\partial_+\mathcal{C}_{\Omega''}$ , while  $\hat{P}$  denotes a bosonic charge arising from a current  $\mathcal{C}_\Lambda$  but with  $\Lambda$  a compound tensor, in general.

It is certainly satisfactory that we can find classically-commuting sets of bosonic charges in all models, generalizing the results of [19]. The fermionic top component charges are a novel feature of the symmetric space sigma models, however, with no counterparts in the SPCMs. It is worth emphasizing that we should *not* expect to find sets of these fermionic charges with *vanishing* Poisson brackets. This is because the Poisson brackets are graded, so  $\{F, F\} = 0$  becomes the operator equation  $F^2 = 0$  in the quantum theory, but if  $F$  is hermitian this implies  $F = 0$ , and the charge is trivial (for a positive-definite Hilbert space). The algebra (83) can be regarded as some higher-spin generalization of supersymmetry. We should also mention that a highly non-trivial check of our results on primitive currents is that the non-zero expressions found on the right hand sides of (82) and (83) have integrands which are indeed differential polynomials in the primitive currents we have identified.

### 5.3 Hermitian symmetric spaces and $\mathcal{N} = 2$ supersymmetry

It is a famous result that a (two-dimensional) supersymmetric sigma model admits an additional supersymmetry iff its target space is a Kähler manifold [4]. The symmetric spaces  $G/H$  which are Kähler, otherwise known as *hermitian* symmetric spaces, are those for which  $H$  is the product of  $U(1)$  and some semi-simple factor [24]. We see from the second table in section 5.1 that in precisely these cases there is a cohomology generator of degree 2, which is just the Kähler form. The corresponding superfield current has spin-1, and so is not really ‘higher-spin’ at all, but rather a conventional, Noether current. Its top component is fermionic and is the spin-3/2 supercurrent for the second supersymmetry, while its bottom, bosonic, component is the current for a chiral  $\mathcal{R}$ -symmetry which rotates the supercharges in the  $\mathcal{N} = 2$  algebra into one another.

To elaborate on this, consider the component expansions of the Kähler holomorphic spin-1 current and its anti-holomorphic counterpart:

$$r_+ + \theta^+ s_+ \quad \text{and} \quad r_- + \theta^- s_- \quad (84)$$

The conserved charges  $R_{(\pm)}$  arising from  $r_{\pm}$  are Lorentz scalars; they rotate the original supersymmetry charges,  $Q_{\pm}$ , into the additional supercharges,  $S_{\pm}$ , arising from the spin-3/2 currents  $s_{\pm}$ , while leaving  $Q_{\mp}$  unchanged. It is for this reason that we refer to these transformations as *chiral R-symmetries*. These symmetries may be more familiar [2, 3] in the combinations  $R = R_{(+)} + R_{(-)}$  and  $\tilde{R} = R_{(+)} - R_{(-)}$  which correspond to the current conservation equations taken in the forms

$$\partial_- r_+ \pm \partial_+ r_- = 0 . \quad (85)$$

These currents are duals of one another (as vectors or one-forms in two-dimensional Minkowski space), the first being a vector, while the second is a pseudo-vector under Lorentz transformations together with reflections.

Note that the general form of the charge algebra given in (83) gives exactly what we expect when applied to the Kähler currents: the Poisson brackets of the new supercharges  $S_{\pm}$  ( $F$  type) with themselves yield energy-momentum ( $P$  type). It is also worth emphasizing that well-known properties of certain  $\mathcal{N} = 2$  models [2, 3] confirm our expectation that bottom component symmetries need not survive quantization. In these models, the  $\mathcal{N} = 2$  supersymmetries and the charge  $R$ , corresponding to the first combination in (85), all persist at the quantum level, but there there is no conserved charge  $\tilde{R}$ . The quantum violation of this symmetry is just the usual chiral anomaly, in two-dimensions.<sup>3</sup>

## 6 Component fields and canonical structure

### 6.1 Component lagrangian

To calculate Poisson brackets one needs the ordinary  $x$ -space form of the superspace Lagrangian (14). Let us expand the basic  $G$ -valued superfield in term of real component fields as follows

$$\mathcal{G}(x, \theta) = g(x) \exp (i\theta^+ \psi_+(x) + i\theta^- \psi_-(x) + i\theta^+ \theta^- \mu(x)) . \quad (86)$$

where  $g(x) \in G$  and  $\psi_{\pm}(x), \mu(x) \in \mathfrak{g}$ . The superfield currents defined in (6) are then

$$\begin{aligned} \mathcal{J}_+ &= i\psi_+ - i\theta^+ (j_+ + i\psi_+^2) + i\theta^- \left( \mu - \frac{i}{2} [\psi_+, \psi_-] \right) \\ &\quad + \theta^+ \theta^- \left( \partial_+ \psi_- + [j_+, \psi_-] + \frac{i}{2} [\psi_+^2, \psi_-] + [\mu, \psi_+] \right) , \end{aligned} \quad (87)$$

$$\begin{aligned} \mathcal{J}_- &= i\psi_- - i\theta^- (j_- + i\psi_-^2) + i\theta^+ \left( -\mu - \frac{i}{2} [\psi_-, \psi_+] \right) \\ &\quad + \theta^- \theta^+ \left( \partial_- \psi_+ + [j_-, \psi_+] + \frac{i}{2} [\psi_-^2, \psi_+] - [\mu, \psi_-] \right) , \end{aligned} \quad (88)$$

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<sup>3</sup>There is actually a discrete remnant of the  $\tilde{R}$  symmetry in the quantum theory and this is *spontaneously* broken, which is important in understanding the spectrum [2, 3].

where we have introduced bosonic currents

$$j_{\pm} = g^{-1} \partial_{\pm} g \quad \in \quad \mathfrak{g} . \quad (89)$$

The resulting component lagrangian for the  $G/H$  sigma model is

$$\begin{aligned} L = & \frac{1}{2} (j_+ + i\psi_+^2)^\alpha (j_- + i\psi_-^2)^\alpha - \frac{1}{2} (\mu - \frac{i}{2}[\psi_+, \psi_-])^\alpha (-\mu - \frac{i}{2}[\psi_-, \psi_+])^\alpha \\ & + \frac{i}{2} \psi_+^\alpha (\partial_- \psi_+ + [j_-, \psi_+] + \frac{i}{2}[\psi_-^2, \psi_+] - [\mu, \psi_-])^\alpha \\ & + \frac{i}{2} \psi_-^\alpha (\partial_+ \psi_- + [j_+, \psi_-] + \frac{i}{2}[\psi_+^2, \psi_-] + [\mu, \psi_+])^\alpha . \end{aligned} \quad (90)$$

This lagrangian, although written in terms of component fields, still possesses the full superspace  $H$ -gauge symmetry, under which  $\mathcal{G} \mapsto \mathcal{GH}$  and  $\mathcal{J}_{\pm} = \mathcal{K}_{\pm} + \mathcal{A}_{\pm}$  transforms as in (16) and (18). It is best to partially fix this so as to leave only conventional,  $x$ -space, gauge-transformations. To see how this can be done, consider a component expansion

$$\mathcal{H}(x, \theta) = h(x) \exp(i\theta^+ \eta_+(x) + i\theta^- \eta_-(x) + i\theta^+ \theta^- \nu(x)) \quad (91)$$

with  $h(x) \in H$  and  $\eta_{\pm}(x), \nu(x) \in \mathfrak{h}$ . It follows that under  $\mathcal{G} \mapsto \mathcal{GH}$  we have

$$\psi_{\pm}^{\alpha} \mapsto (h^{-1} \psi_{\pm} h)^{\alpha} , \quad \psi_{\pm}^{\hat{\alpha}} \mapsto (h^{-1} \psi_{\pm} h + \eta_{\pm})^{\hat{\alpha}} \quad (92)$$

and so one can always impose  $\psi_{\pm}^{\hat{\alpha}} = 0$  by a unique choice of  $\eta_{\pm}^{\hat{\alpha}}$ . Similarly, the gauge freedom inherent in  $\nu^{\hat{\alpha}}$  can be used up by setting  $\mu^{\hat{\alpha}} = 0$ , and in fact  $\mu^{\alpha} = 0$  then follows from the lagrangian as an equation of motion.

This achieves the aim of reducing the gauge redundancy to  $x$ -space gauge transformations (the restrictions we have imposed constitute the *Wess-Zumino gauge* for this superspace gauge theory, which can also be expressed by the condition  $\theta^+ \mathcal{A}_+ + \theta^- \mathcal{A}_- = 0$ ). Having done this, we have component expansions of the form

$$\mathcal{K}_{\pm}^{\alpha} = i\psi_{\pm}^{\alpha} - i\theta^{\pm} k_{\pm}^{\alpha} + O(\theta^{\mp}) \quad (93)$$

$$\mathcal{A}_{\pm}^{\hat{\alpha}} = -i\theta^{\pm} A_{\pm}^{\hat{\alpha}} + O(\theta^{\mp}) \quad (94)$$

(higher components are functions of those given explicitly) where

$$j_{\pm} = k_{\pm} + A_{\pm} \quad \text{with} \quad k_{\pm} \in \mathfrak{m} , \quad A_{\pm} \in \mathfrak{h} . \quad (95)$$

Finally, then, the component formulation of the theory involves fields  $g(x^{\mu})$ , and  $\psi_{\pm}^{\alpha}$ , with lagrangian

$$\begin{aligned} L = & \frac{1}{2} (k_0^{\alpha} k_0^{\alpha} - k_1^{\alpha} k_1^{\alpha}) + \frac{i}{2} \psi_+^{\alpha} \partial_- \psi_+^{\alpha} + \frac{i}{2} \psi_-^{\alpha} \partial_+ \psi_-^{\alpha} \\ & - iA_0^{\hat{\alpha}} (h_+^2 + h_-^2)^{\hat{\alpha}} + iA_1^{\hat{\alpha}} (h_+^2 - h_-^2)^{\hat{\alpha}} + h_+^{\hat{\alpha}} h_-^{\hat{\alpha}} , \end{aligned} \quad (96)$$

where the various combinations of fields which appear are defined by (89) and (95) together with

$$h_{\pm}^{\hat{\alpha}} = (\psi_{\pm}^2)^{\hat{\alpha}} = \frac{1}{2} f^{\hat{\alpha}\beta\gamma} \psi_{\pm}^{\beta} \psi_{\pm}^{\gamma} . \quad (97)$$



## 6.2 Poisson brackets

We can now carry out a standard canonical analysis of the lagrangian in (96). All Poisson brackets are graded and hold at equal times, where appropriate.

The classical brackets for the real (Majorana) fermions are simply

$$\left\{ \psi_{\pm}^{\alpha}(x), \psi_{\pm}^{\beta}(y) \right\} = -i\delta^{\alpha\beta}\delta(x-y) , \quad (98)$$

with others vanishing. It is useful to note that these imply

$$\left\{ h_{\pm}^{\hat{\alpha}}(x), \psi_{\pm}^{\beta}(y) \right\} = if^{\hat{\alpha}\beta\gamma}\psi_{\pm}^{\gamma}(x)\delta(x-y) \quad (99)$$

$$\left\{ h_{\pm}^{\hat{\alpha}}(x), h_{\pm}^{\hat{\beta}}(y) \right\} = if^{\hat{\alpha}\hat{\beta}\hat{\gamma}}h_{\pm}^{\hat{\gamma}}(x)\delta(x-y) \quad (100)$$

(with  $h^{\hat{\alpha}}$  defined in (97)).

To calculate the brackets of the bosonic currents  $j = k + A$  we introduce a set of coordinates  $\{\phi^i\}$  on the group manifold  $G$  and regard the field as depending on  $x$  through these:  $g(\phi^i(x))$ . Vielbeins for the group will be denoted

$$E_i^a(\phi) = (g^{-1}\partial_i g)^a \quad \Rightarrow \quad k_{\pm}^{\alpha} = E_i^{\alpha}\partial_{\pm}\phi^i , \quad A_{\pm}^{\hat{\alpha}} = E_i^{\hat{\alpha}}\partial_{\pm}\phi^i . \quad (101)$$

From (96), the momentum conjugate to  $\phi^i$  is

$$\pi_i = \frac{\partial L}{\partial(\partial_0\phi^i)} = E_i^{\alpha}E_j^{\alpha}\partial_0\phi^j - iE_i^{\hat{\alpha}}(h_+ + h_-)^{\hat{\alpha}} . \quad (102)$$

The spatial components of the current  $j$  can be expressed

$$j_1^a = E_i^a(\phi)\partial_1\phi^i \quad \Rightarrow \quad k_1^{\alpha} = E_i^{\alpha}(\phi)\partial_1\phi^i , \quad A_1^{\hat{\alpha}} = E_i^{\hat{\alpha}}(\phi)\partial_1\phi^i \quad (103)$$

and it is convenient to define a new current, related to momentum:

$$J^a = E^{ai}(\phi)\pi_i \quad \Rightarrow \quad J^{\alpha} = k_0^{\alpha} , \quad J^{\hat{\alpha}} = -i(h_+ + h_-)^{\hat{\alpha}} . \quad (104)$$

The Poisson brackets of all these quantities can now be calculated from

$$\left\{ \phi^i(x), \pi_j(y) \right\} = \delta^i_j \delta(x-y) \quad (105)$$

(similar calculations are described in appendices to [16, 19]). The results are:

$$\begin{aligned} \left\{ J^a(x), J^b(y) \right\} &= -f^{abc}J^c(x)\delta(x-y) \\ \left\{ J^a(x), j_1^b(y) \right\} &= -f^{abc}j_1^c(x)\delta(x-y) + \delta^{ab}\delta'(x-y) \\ \left\{ j_1^a(x), j_1^b(y) \right\} &= 0 . \end{aligned} \quad (106)$$

and the brackets of  $J^a$  and  $j_1^a$  with the fermions  $\psi_\pm^\alpha$  all vanish.

However, one of the formulas in (104) constitutes a constraint on the canonical variables

$$\Phi^{\hat{\alpha}} = J^{\hat{\alpha}} + i(h_+ + h_-)^{\hat{\alpha}} \approx 0 \quad (107)$$

The Hamiltonian density for the system is, therefore, only weakly determined

$$H \approx \frac{1}{2}k_0^\alpha k_0^\alpha + \frac{1}{2}k_1^\alpha k_1^\alpha - iA_1^{\hat{\alpha}}(h_+ - h_-)^{\hat{\alpha}} + \frac{1}{2}i\psi_+^\alpha \partial_1 \psi_+^\alpha - \frac{1}{2}i\psi_-^\alpha \partial_1 \psi_-^\alpha - h_+^{\hat{\alpha}} h_-^{\hat{\alpha}}. \quad (108)$$

From the brackets (106) and (98) it is easy to show that

$$\{H, \Phi^{\hat{\alpha}}\} \approx 0, \quad (109)$$

so that there are no secondary constraints, and also

$$\{\Phi^{\hat{\alpha}}(x), \Phi^{\hat{\beta}}(y)\} = -f^{\hat{\alpha}\hat{\beta}\hat{\gamma}}\Phi^{\hat{\gamma}}(x)\delta(x-y), \quad (110)$$

from (106) and (100), so that the constraints are first class. The canonical formalism has thus been consistently completed.

The general nature of these results is expected: the lagrangian (96) has an  $x$ -space  $H$ -gauge symmetry, which is generated by the first-class constraints  $\Phi^{\hat{\alpha}}$ . It is important to emphasize that we will *not* fix this gauge symmetry in any of the calculations which follow. To do so would require the introduction of Dirac brackets, which would certainly differ from the Poisson brackets above, in general. However, we will be interested, ultimately in the Poisson brackets of the gauge-invariant charges of type  $B$ ,  $F$  and  $P$ , and Poisson brackets and Dirac brackets coincide for any gauge-invariant quantities.

### 6.3 Component forms of the charges

From the component expansion of the current  $\mathcal{K}_+$  given in (93), it is easy to see that the top component charges associated with  $\Omega$  tensors are (up to irrelevant, overall factors)

$$B^{(d)} = \int dx \Omega_{\alpha_1 \dots \alpha_{2s} \alpha}^{(d)} \psi_+^{\alpha_1} \dots \psi_+^{\alpha_{2s}} k_+^\alpha \quad (111)$$

$$F^{(e)} = \int dx \Omega_{\beta_1 \dots \beta_{2s+2}}^{(e)} \psi_+^{\beta_1} \dots \psi_+^{\beta_{2s+1}} k_+^{\beta_{2s+2}} \quad (112)$$

and they have spins  $s$  and  $s + 1/2$  respectively, where  $s$  is an integer and  $s+1$  is the degree of the symmetric tensor  $d$  or  $e$ .

The top component charges associated with  $\Lambda$  tensors look rather more complicated. First note that

$$\mathcal{D}_+ \mathcal{K}_+^\alpha = -ik_+^\alpha + \theta^+(\partial_+ \psi_+ + [A_+, \psi_+])^\alpha + O(\theta^-), \quad (113)$$

and that to express this in terms of good canonical variables it is necessary to eliminate  $\partial_0\psi_+$  using the equation of motion

$$\partial_-\psi_+ = -[A_- + ih_-, \psi_+] \quad \Rightarrow \quad \partial_+\psi_+ = -[A_- + ih_-, \psi_+] + 2\psi_+^{\prime\alpha} \quad (114)$$

(this is most easily derived from  $\mathcal{D}_-\mathcal{K}_+ = 0$ , rather than using the component lagrangian). It follows that

$$\mathcal{D}_+\mathcal{K}_+^\alpha = -ik_+^\alpha + \theta^+ (2\psi_+^{\prime\alpha} + [2A_1 - ih_-, \psi_+]^\alpha) + O(\theta^-) \quad (115)$$

and hence

$$P^{(d)} = - \int dx \Lambda_{\alpha_1 \dots \alpha_{2s-1}; \alpha}^{(d)} \left[ (2s-1)k_+^\alpha k_+^{\alpha_{2s-1}} - i(2\psi_+^{\prime\alpha} - f^{\alpha\hat{\beta}\gamma} I^{\hat{\beta}} \psi_+^{\prime\gamma}) \psi_+^{\alpha_{2s-1}} \right] \psi_+^{\alpha_1} \dots \psi_+^{\alpha_{2s-2}} \quad (116)$$

is the charge of spin  $s$  resulting from a  $d$  tensor of degree  $s+1$ , where

$$I^{\hat{\alpha}} = i(h_+ + h_-)^{\hat{\alpha}} - 2A_1^{\hat{\alpha}}. \quad (117)$$

We have used invariance of  $\Lambda$  to include the  $h_+$  term in this definition. This proves convenient because  $I$  is then the quantity which appears in the bracket of  $k_+$  with itself:

$$\left\{ k_+^\alpha(x), k_+^\beta(y) \right\} = f^{\alpha\beta\hat{\gamma}} I^{\hat{\gamma}}(x) \delta(x-y) + 2\delta^{\alpha\beta} \delta'(x-y), \quad (118)$$

which follows from (106).

We now have all the information we need and we proceed to compute Poisson brackets and establish the results announced in section 5.

## 7 Calculations of Poisson Brackets of charges

### 7.1 Brackets amongst $B$ and $F$ type charges

Consider the bracket of two charges of type (111) or (112) based on tensors  $\Omega_{\alpha_1\alpha_2\dots\alpha_p}$  and  $\tilde{\Omega}_{\beta_1\beta_2\dots\beta_q}$  where, for the moment,  $p$  and  $q$  can be even or odd integers, so that the charges can be type  $B$  or  $F$ . From (118) and the fact that  $k_+^\alpha$  and  $\psi_+^\beta$  Poisson-commute we find the result:

$$(q-1) \int dx \Omega_{\alpha_1 \dots \alpha_{p-2} \alpha_{p-1} \gamma} \tilde{\Omega}_{\beta_1 \beta_2 \dots \beta_{q-1} \gamma} \psi_+^{\alpha_1} \dots \psi_+^{\alpha_{p-2}} \psi_+^{\beta_2} \dots \psi_+^{\beta_{q-1}} \\ \times \left[ -i(p-1)k_+^{\alpha_{p-1}} k_+^{\beta_1} + \psi_+^{\alpha_{p-1}} (2\psi_+^{\prime\beta_1} - f^{\beta_1\hat{\gamma}\beta} I^{\hat{\gamma}} \psi_+^{\beta}) \right], \quad (119)$$

which is obtained after integration by parts and using the invariance of  $\tilde{\Omega}$ . Taking into account various factors arising from antisymmetrization of tensor indices, this integrand is proportional to

$$\Omega_{[\alpha_1 \dots \alpha_{p-1}} \gamma \tilde{\Omega}_{\beta_1 \dots \beta_{q-2}] \beta \gamma} \psi_+^{\alpha_1} \dots \psi_+^{\alpha_{p-1}} \psi_+^{\beta_1} \dots \psi_+^{\beta_{q-3}} \times \left[ (p+q-3) k_+^\beta k_+^{\beta_{q-2}} - i \psi_+^{\beta_{q-2}} (2 \psi_+^{\beta'} - f^{\beta \gamma \delta} I^{\hat{\gamma}} \psi_+^\delta) \right]. \quad (120)$$

Because we have taken the bracket of two conserved charges, the result must be another conserved quantity. We should therefore be able to express the integrand above in terms of our known, primitive conserved currents. It clearly has the general form required for a charge of type  $P$  given by (116), if the  $\Lambda$  tensor is taken as

$$\Omega_{[\alpha_1 \dots \alpha_{p-1}} \gamma \tilde{\Omega}_{\beta_1 \dots \beta_{q-2}] \beta \gamma}. \quad (121)$$

But we must check that such an expression really can arise from our construction of  $\Lambda$  tensors given in sections 3 and 4 (if not, we would have found a new conserved quantity, not expressible in terms of the currents we claim are primitive).

In fact, if either one of the original tensors, say  $\tilde{\Omega}$ , is of odd degree, then (120) actually *vanishes*. The crucial point is that for

$$\tilde{\Omega}_{\alpha_1 \dots \alpha_{2s+1}} = \frac{1}{2^s} f_{\alpha_1 [\alpha_2}^{\hat{\rho}_1} \dots f_{\alpha_{2s-1} \alpha_{2s}] }^{\hat{\rho}_s} d_{\alpha_{2s+1} \hat{\rho}_1 \dots \hat{\rho}_s},$$

(referring to (41) and setting  $q = 2s + 1$ ) the explicit antisymmetrization is only necessary over  $2s-1$  of the indices. When this is substituted into (120), all the required antisymmetrization is enforced by the presence of the fermions  $\psi_+$ . The whole integrand then vanishes by invariance of the tensor  $\Omega$  (the term with  $\psi_+^{\beta'}$  immediately, and the  $k_+ k_+$  term because invariance of  $\Omega$  produces an expression with both bosons  $k_+$  contracted with the antisymmetric  $\tilde{\Omega}$ ). This establishes the relations  $\{B^{(d)}, B^{(\tilde{d})}\} = \{B^{(d)}, F^{(e)}\} = 0$ .

More interesting is the result for the remaining kind of bracket,  $\{F^{(e)}, F^{(\tilde{e})}\}$ , which requires both  $\Omega$  and  $\tilde{\Omega}$  to be of even degree. We will show that when  $e$  and  $\tilde{e}$  are single trace invariants or Pfaffians, the tensor (121) is indeed a known  $\Lambda$  tensor (possibly composite, or even zero in some cases). The generalization to products of traces and Pfaffians presents no difficulties of principle, the arguments would just be more cumbersome to write down.

Let us deal with the trace-type  $e$  invariants first, which occur for the Grassmannians and for the families  $SO(2n)/U(n)$  and  $Sp(n)/U(n)$ . These symmetric spaces are defined by an automorphism on  $\mathfrak{g}$  of the form  $X \mapsto NXN$  for some matrix  $N$ . Referring to the block forms introduced in section 4, we have  $N = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  for the Grassmannians and  $N = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$  for the remaining two families. By definition,  $N$  commutes with generators  $t_{\hat{\alpha}}$  belonging to  $\mathfrak{h}$  but anticommutes with generators  $t_{\alpha}$  belonging to  $\mathfrak{m}$ .

Recall that the tensors  $e$  are *not* the restrictions of invariant tensors on  $\mathfrak{g}$ . Nevertheless, those involving a single trace can be expressed

$$e_{\hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_k} = \text{Tr} \left( N t_{(\hat{\alpha}_1} t_{\hat{\alpha}_2} \dots t_{\hat{\alpha}_k)} \right) \quad (122)$$

(the additional factor of  $N$  in the trace prevents this from being the restriction to  $\mathfrak{h}$  of a quantity like (38)). The corresponding antisymmetric tensor can be written

$$\Omega_{\alpha_1 \dots \alpha_{2k-1} \alpha_{2k}} = \text{Tr}(N t_{[\alpha_1} \dots t_{\alpha_{2k-1}} t_{\alpha_{2k}]}) = \text{Tr}(N t_{[\alpha_1} \dots t_{\alpha_{2k-1}}] t_{\alpha_{2k}}) \quad (123)$$

and the fact that the last expression is automatically totally antisymmetric will be important below. In addition, whenever this  $\Omega$  tensor is non-zero, we have

$$N t_{[\alpha_1} t_{\alpha_2} \dots t_{\alpha_{2k-1}]} = -t_\gamma \text{Tr} \left( N t_{[\alpha_1} t_{\alpha_2} \dots t_{\alpha_{2k-1}]} t_\gamma \right) = -t_\gamma \Omega_{\alpha_1 \alpha_2 \dots \alpha_{2k-1} \gamma} \cdot \quad (124)$$

This is a consequence of the fact that the matrix on the far left actually belongs to  $\mathfrak{m}$ , or in some cases the complexification of  $\mathfrak{m}$ , and so can be expanded as a (real or complex) linear combination of generators. It is straightforward to check this claim e.g. using the block forms given in section 4.

We now return to (121) for the case of two even-degree tensors based on single traces. Setting  $p = 2k$  and  $q = 2\ell$ , we have

$$\begin{aligned} & \Omega_{\alpha_1 \dots \alpha_{2k-1} \gamma} \tilde{\Omega}_{\beta_1 \dots \beta_{2\ell-2} \beta \gamma} \\ &= \text{Tr} \left( N t_{[\alpha_1} \dots t_{\alpha_{2k-1}]} t_\gamma \right) \text{Tr} \left( t_\gamma N t_{[\beta_1} \dots t_{\beta_{2\ell-2}}] t_\beta \right) \\ &= -\text{Tr} \left( N t_{[\alpha_1} \dots t_{\alpha_{2k-1}]} N t_{[\beta_1} \dots t_{\beta_{2\ell-2}}] t_\beta \right) \\ &= \text{Tr} \left( t_{[\alpha_1} \dots t_{\alpha_{2k-1}]} t_{[\beta_1} \dots t_{\beta_{2\ell-2}}] t_\beta \right) \end{aligned}$$

and on imposing the correct antisymmetrization,

$$\begin{aligned} & \Omega_{[\alpha_1 \dots \alpha_{2k-1}} \gamma \tilde{\Omega}_{\beta_1 \dots \beta_{2\ell-2}] \beta \gamma} \\ &= \text{Tr} \left( t_{[\alpha_1} \dots t_{\alpha_{2k-1}}] t_{\beta_1} \dots t_{\beta_{2\ell-2}}] t_\beta \right) \\ &\propto f^{\hat{\gamma}_1}_{[\alpha_1 \alpha_2} \dots f^{\hat{\gamma}_k}_{\alpha_{2k-1} \beta_1} \dots f^{\hat{\gamma}_{\ell+k-1}}_{\beta_{2\ell-4} \beta_{2\ell-3}} \text{Tr} \left( t_{\beta_{2\ell-2}}] t_\beta t_{\hat{\gamma}_1} \dots t_{\hat{\gamma}_{\ell+k-1}} \right) \\ &\propto f^{\hat{\gamma}_1}_{[\alpha_1 \alpha_2} \dots f^{\hat{\gamma}_k}_{\alpha_{2k-1} \beta_1} \dots f^{\hat{\gamma}_{\ell+k-1}}_{\beta_{2\ell-4} \beta_{2\ell-3}} d_{\beta_{2\ell-2}] \beta \hat{\gamma}_1 \dots \hat{\gamma}_{\ell+k-1}} \\ &\propto \Lambda_{\alpha_1 \alpha_2 \dots \alpha_{2k-1} \beta_1 \beta_2 \dots \beta_{2\ell-2}; \beta}^{(d)} \quad (125) \end{aligned}$$

where  $d$  is the symmetric  $G$ -invariant on  $\mathfrak{g}$  associated with the invariant polynomial  $\text{Tr} X^{k+\ell+1}$ . This is indeed one of our standard tensors, although neither  $d$  nor  $\Lambda$  need be primitive. As mentioned above, this analysis can be extended to products of traces.

The even-degree  $\Omega$  tensors we have not yet considered are constructed from Pfaffians. They occur only for the real Grassmannians and for the spaces  $SU(2n)/SO(2n)$ . The

special nature of the Pfaffian invariants means that it is convenient to adopt a notation similar to that used in (67)-(71) and (74). We shall not give a detailed account of the calculations involving these charges, but we will show how the final result is once again based on a  $\Lambda$  tensor (possibly composite) of known form.

Considering first the Grassmannians  $SO(p+q)/SO(p) \times SO(q)$ , we can regard  $(\psi_+)_{i\ell}$  and  $(k_+)_{i\ell}$  as  $p \times q$  matrices transforming under  $SO(p)$  from the left and  $SO(q)$  from the right, as in (63). Suppose  $p$  is even, so we have a Pfaffian current (67) in addition to trace-type currents (66). Setting  $p = 2n$ , it is sufficient for our purposes to investigate the bracket of the *bottom* component of the Pfaffian current, proportional to

$$\varepsilon_{i_1 j_1 \dots i_n j_n} (X)_{i_1 j_1} \dots (X)_{i_n j_n} \quad \text{where} \quad X_{ij} = (\psi_+)_{i\ell} (\psi_+)_{j\ell} \quad (126)$$

with the top component of the trace current, proportional to

$$\text{Tr}(Y X^{r-1}) \quad \text{where} \quad Y_{ij} = (\psi_+)_{i\ell} (k_+)_{j\ell} \quad (127)$$

and to identify the result as the *bottom* component of a superfield current based on some  $\Lambda$  tensor. The relationship between the top component charges then follows by supersymmetry (under which bottom component charges transform into top component charges, while top component charges are unchanged).

A short calculation reveals that the Poisson bracket of the quantities (126) and (127) produces an expression proportional to

$$\varepsilon_{i_1 j_1 i_2 j_2 \dots i_n j_n} (Y X^{r-1})_{i_1 j_1} X_{i_2 j_2} \dots X_{i_n j_n} \quad (128)$$

The crucial point now is that

$$\varepsilon_{i_1 j_1 i_2 j_2 \dots i_n j_n} X_{k j_1} X_{i_2 j_2} \dots X_{i_n j_n} = 2^{n-1} (n-1)! \text{Pf}(X) \delta_{i_1 k} \quad (129)$$

for *any* antisymmetric matrix  $X$ . Hence the expression above factorizes into a Pfaffian and a trace and we recognize this as the bottom component of a superfield current which is a product of terms (66) and (70).

If both  $p$  and  $q$  are even, then there are Pfaffian currents (67) and (68) corresponding to both  $SO(p)$  and  $SO(q)$ . Proceeding similarly, it is not difficult to see that the Poisson bracket of the top component  $F$  charges gives a  $P$  charge based on a  $\Lambda$  tensor built from the Pfaffian of  $SO(p+q)$ . Finally, we can consider the bracket of a Pfaffian  $F$  charge with itself, either in one of the real Grassmannians or in  $SU(2n)/SO(2n)$  (in the latter case this is the unique  $F$  charge). Given the identity

$$\varepsilon_{i_1 i_2 \dots i_{2n-1} i_{2n}} \varepsilon^{j_1 j_2 \dots j_{2n-1} j_{2n}} = (2n)! \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} \dots \delta_{i_{2n-1}}^{j_{2n-1}} \delta_{i_{2n}}^{j_{2n}}, \quad (130)$$

it is straightforward to verify that the result is a sum of terms, each a product of factors of type (66) together with a single factor of type (70).

This completes the argument that the brackets of  $F$  charges close onto  $P$  charges.

## 7.2 Brackets amongst $P$ type charges

We turn now to computing the bracket of two charges of type  $P^{(d)}$ , based on tensors of type  $\Lambda^{(d)}$ . We will show that, for special choices of the tensors  $d$ , these charges actually Poisson-commute. The calculations are rather involved, but we can follow the same strategy as in [19]. Setting  $s = m + 1$  in (116), it is useful to write

$$P^{(d)} = - \int dx d_{\alpha\beta\hat{\gamma}_1\dots\hat{\gamma}_m} \left[ (m+1)k_+^\alpha k_+^\beta - i(2\psi_+^{\prime\alpha} - f^{\alpha\hat{\beta}\gamma} I^{\hat{\beta}} \psi_+^\gamma) \psi_+^\beta \right] h_+^{\hat{\gamma}_1} \dots h_+^{\hat{\gamma}_m}, \quad (131)$$

which follows on using the  $G$ -invariance of  $d$ . Hence

$$P^{(d)} = -U - V - W, \quad (132)$$

where

$$U = 2i \int dx d_{\alpha\beta\hat{\gamma}_1\dots\hat{\gamma}_m} \psi_+^\alpha \psi_+^{\prime\beta} h_+^{\hat{\gamma}_1} \dots h_+^{\hat{\gamma}_m}, \quad (133)$$

$$V = (m+1) \int dx d_{\alpha\beta\hat{\gamma}_1\dots\hat{\gamma}_m} k_+^\alpha k_+^\beta h_+^{\hat{\gamma}_1} \dots h_+^{\hat{\gamma}_m}, \quad (134)$$

$$\begin{aligned} W &= i \int dx d_{\alpha\beta\hat{\gamma}_1\dots\hat{\gamma}_m} f^{\alpha\hat{\delta}\epsilon} I^{\hat{\delta}} \psi_+^\epsilon \psi_+^\beta h_+^{\hat{\gamma}_1} \dots h_+^{\hat{\gamma}_m}, \\ &= 2i \int dx d_{\hat{\delta}\hat{\gamma}_1\dots\hat{\gamma}_{m+1}} I^{\hat{\delta}} h_+^{\hat{\gamma}_1} \dots h_+^{\hat{\gamma}_{m+1}}, \end{aligned} \quad (135)$$

(the last equality again relies on invariance of  $d$ ). We will compute the Poisson bracket of  $P^{(d)}$  with a second charge,  $P^{(\tilde{d})}$ , expressed similarly as

$$P^{(\tilde{d})} = -\tilde{U} - \tilde{V} - \tilde{W} \quad (136)$$

with  $s = l + 1$ . This will involve consideration of various groups of terms arising from the brackets of  $U, V$  and  $W$  with  $\tilde{U}, \tilde{V}$  and  $\tilde{W}$ .

First,  $\{W, \tilde{W}\}$  can be shown to vanish by repeated use of invariance of  $d$  and  $\tilde{d}$ . Turning next to  $\{U, \tilde{W}\} + \{W, \tilde{U}\}$ , we find that

$$\{U, h_+^{\hat{\gamma}}(x)\} = -4d_{\hat{\gamma}\hat{\alpha}_1\dots\hat{\alpha}_{m+1}} \partial_x (h_+^{\hat{\alpha}_1} \dots h_+^{\hat{\alpha}_{m+1}}), \quad (137)$$

after extensive use of invariance of  $d$ , and hence

$$\{U, \tilde{W}\} + \{W, \tilde{U}\} = -8i \int dx d_{\hat{\gamma}\hat{\alpha}_1\dots\hat{\alpha}_m} \tilde{d}_{\hat{\gamma}\hat{\beta}_1\dots\hat{\beta}_l} h_+^{\hat{\alpha}_1} \dots h_+^{\hat{\alpha}_m} h_+^{\hat{\beta}_1} \dots h_+^{\hat{\beta}_l} \left( h_+^{\prime\hat{\alpha}} I^{\hat{\beta}} - h_+^{\prime\hat{\beta}} I^{\hat{\alpha}} \right) \quad (138)$$

This can be shown to vanish as follows. The antisymmetry in  $\hat{\alpha} \leftrightarrow \hat{\beta}$  imposed by the last factor allows us to symmetrize over the indices  $(\hat{\alpha}, \hat{\alpha}_1, \dots, \hat{\alpha}_m, \hat{\beta}_1, \dots, \hat{\beta}_l)$  on the  $d$  tensors. But  $d_{\gamma\hat{\alpha}\hat{\alpha}_1\dots\hat{\alpha}_m} = 0$ , from (46), so that the repeated index  $\hat{\gamma}$  in (138) can be extended to a repeated index  $c$ , running over the whole Lie algebra  $\mathfrak{g}$ . The tensor structure in (138) is therefore

$$d_{c(\hat{\alpha}\hat{\alpha}_1\dots\hat{\alpha}_m\tilde{d}_{\hat{\beta}_1\dots\hat{\beta}_l}^c} = d_{c(\hat{\alpha}\hat{\alpha}_1\dots\hat{\alpha}_m\tilde{d}_{\hat{\beta}_1\dots\hat{\beta}_l}^c} , \quad (139)$$

provided the tensors  $d$  and  $\tilde{d}$  have the special property (59). For these special choices, then, the symmetrization is actually over *all* the indices  $(\hat{\alpha}, \hat{\beta}, \hat{\alpha}_1, \dots, \hat{\alpha}_m, \hat{\beta}_1, \dots, \hat{\beta}_l)$ , but this implies that the contraction with the rest of the integrand in (138) gives zero, because the last factor is antisymmetric in  $(\hat{\alpha}, \hat{\beta})$ .

Now consider the terms

$$\begin{aligned} \{V, \tilde{V}\} &= (m+1)(l+1) \int dx d_{\alpha\beta\hat{\gamma}_1\dots\hat{\gamma}_m} \tilde{d}_{\delta\epsilon\hat{\rho}_1\dots\hat{\rho}_l} \left( 4f_{\alpha\delta\hat{\kappa}} I^{\hat{\kappa}} k_+^\beta k_+^\epsilon h_+^{\hat{\gamma}_1} \dots h_+^{\hat{\gamma}_m} h_+^{\hat{\rho}_1} \dots h_+^{\hat{\rho}_l} \right. \\ &\quad \left. + iml k_+^\alpha k_+^\beta k_+^\delta k_+^\epsilon f_{\hat{\gamma}_1\hat{\rho}_1\hat{\kappa}} h_+^{\hat{\kappa}} h_+^{\hat{\gamma}_2} \dots h_+^{\hat{\gamma}_m} h_+^{\hat{\rho}_2} \dots h_+^{\hat{\rho}_l} \right) + \dots \end{aligned} \quad (140)$$

$$\begin{aligned} \{V, \tilde{W}\} &= 2i(m+1) \int dx d_{\alpha\beta\hat{\gamma}_1\dots\hat{\gamma}_m} \tilde{d}_{\hat{\delta}\hat{\rho}_1\dots\hat{\rho}_{l+1}} \left( + 4f_{\alpha\hat{\delta}\hat{\kappa}} k_+^\kappa k_+^\beta h_+^{\hat{\gamma}_1} \dots h_+^{\hat{\gamma}_m} h_+^{\hat{\rho}_1} \dots h_+^{\hat{\rho}_{l+1}} \right. \\ &\quad \left. + im(l+1) f_{\hat{\gamma}_1\hat{\rho}_1\hat{\kappa}} h_+^{\hat{\kappa}} k_+^\alpha k_+^\beta I^{\hat{\delta}} h_+^{\hat{\gamma}_2} \dots h_+^{\hat{\gamma}_m} h_+^{\hat{\rho}_2} \dots h_+^{\hat{\rho}_{l+1}} \right) \end{aligned} \quad (141)$$

$$\begin{aligned} \{W, \tilde{V}\} &= -2i(l+1) \int dx d_{\hat{\delta}\hat{\gamma}_1\dots\hat{\gamma}_{m+1}} \tilde{d}_{\alpha\beta\hat{\rho}_1\dots\hat{\rho}_l} \left( + 4f_{\alpha\hat{\delta}\hat{\kappa}} k_+^\kappa k_+^\beta h_+^{\hat{\gamma}_1} \dots h_+^{\hat{\gamma}_{m+1}} h_+^{\hat{\rho}_1} \dots h_+^{\hat{\rho}_l} \right. \\ &\quad \left. + il(m+1) f_{\hat{\rho}_1\hat{\gamma}_1\hat{\kappa}} h_+^{\hat{\kappa}} k_+^\alpha k_+^\beta I^{\hat{\delta}} h_+^{\hat{\gamma}_2} \dots h_+^{\hat{\gamma}_{m+1}} h_+^{\hat{\rho}_2} \dots h_+^{\hat{\rho}_l} \right) \end{aligned} \quad (142)$$

where we have neglected the non-ultra-local contributions (involving  $\delta'$ ) to  $\{V, \tilde{V}\}$  for the moment. The  $k_+ k_+ k_+$  term may be shown to vanish by repeated use of invariance of  $d$  and  $\tilde{d}$ . Now observe that, again by invariance of  $d$  and  $\tilde{d}$ ,

$$0 = 4f_{a_1 b_1 \hat{\kappa}} I^{\hat{\kappa}} d_{a_1 \dots a_{m+2}} \tilde{d}_{b_1 \dots b_{l+2}} \eta_{a_2} \dots \eta_{a_{m+2}} \eta_{b_2} \dots \eta_{b_{l+2}} \Big|_{\lambda^2}, \quad (143)$$

where

$$\eta = h_+ + \lambda k_+, \quad (144)$$

and in fact when this expression is expanded out in terms of  $h_+$ 's and  $k_+$ 's, it produces precisely the  $k_+ k_+$  terms in  $\{V, \tilde{V}\} + \{W, \tilde{V}\} + \{V, \tilde{W}\}$ . The remaining terms in  $\{W, \tilde{V}\} + \{V, \tilde{W}\}$  may be written, up to a factor, as

$$d_{ac_1 \dots c_{m+1}} \tilde{d}_{bd_1 \dots d_{l+1}} f_{ab\hat{\kappa}} k_+^{\hat{\kappa}} \eta^{c_1} \dots \eta^{c_{m+1}} \eta^{d_1} \dots \eta^{d_{l+1}} \Big|_{\lambda} \quad (145)$$

which also vanishes.



Thus, all the ultra-local terms in  $\{V, \tilde{V}\} + \{W, \tilde{V}\} + \{V, \tilde{W}\}$  vanish. The only non-ultra-local contribution comes from  $\{V, \tilde{V}\}$  and is

$$8(m+1)(l+1) \int dx d_{\alpha\beta\hat{\gamma}_1\dots\hat{\gamma}_m} \tilde{d}_{\alpha\epsilon\hat{\rho}_1\dots\hat{\rho}_l} k_+^\beta h_+^{\hat{\gamma}_1} \dots h_+^{\hat{\gamma}_m} \partial_x (k_+^\epsilon h_+^{\hat{\rho}_1} \dots h_+^{\hat{\rho}_l}) . \quad (146)$$

Meanwhile, using the expression obtained previously for  $\{U, h_+\}$ , we have

$$\{U, \tilde{V}\} = 4l(l+1) \int dx d_{\hat{\delta}\hat{\alpha}_1\dots\hat{\alpha}_{m+1}} \tilde{d}_{\hat{\delta}\hat{\alpha}\hat{\rho}_1\dots\hat{\rho}_{l-1}} h_+^{\hat{\alpha}_1} \dots h_+^{\hat{\alpha}_{m+1}} \partial_x (k_+^\alpha k_+^\beta h_+^{\hat{\rho}_1} \dots h_+^{\hat{\rho}_{l-1}})$$

and

$$\{V, \tilde{U}\} = 4m(m+1) \int dx d_{\hat{\delta}\alpha\hat{\rho}_1\dots\hat{\rho}_{m-1}} \tilde{d}_{\hat{\delta}\hat{\alpha}_1\dots\hat{\alpha}_{l+1}} k_+^\alpha k_+^\beta h_+^{\hat{\rho}_1} \dots h_+^{\hat{\rho}_{m-1}} \partial_x (h_+^{\hat{\alpha}_1} \dots h_+^{\hat{\alpha}_{l+1}}) .$$

The sum of these last three terms is

$$8 \int dx d_{ab_1\dots b_{m+1}} \tilde{d}_{ac_1\dots c_{l+1}} \eta^{b_1} \dots \eta^{b_{m+1}} \partial_x (\eta^{c_1} \dots \eta^{c_{l+1}}) \Big|_{\lambda^2} \quad (147)$$

$$\propto \int dx d_{a(b_1\dots b_{m+1}} \tilde{d}_{c_1\dots c_{l+1})c_{l+1}} \eta^{b_1} \dots \eta^{b_{m+1}} \eta^{c_1} \dots \eta^{c_l} \partial_x \eta^{c_{l+1}} \Big|_{\lambda^2} . \quad (148)$$

Now we can invoke the property (59) and extend the symmetrization over all the indices  $(b_1, \dots, c_{l+1})$ . The integrand is then a total derivative, and so the whole expression vanishes.

There is only one remaining piece to consider in the bracket of  $P^{(d)}$  with  $P^{(\tilde{d})}$ , namely  $\{U, \tilde{U}\}$ . This can be shown to be proportional to

$$d_{\alpha\beta\hat{\gamma}_1\dots\hat{\gamma}_m} \tilde{d}_{\alpha\epsilon\hat{\rho}_1\dots\hat{\rho}_l} h_+^{\hat{\gamma}_1} \dots h_+^{\hat{\gamma}_m} h_+^{\hat{\rho}_1} \dots h_+^{\hat{\rho}_l} \psi_+^{\prime\beta} \psi_+^{\prime\epsilon} , \quad (149)$$

which again vanishes given the property (59) for  $d$  and  $\tilde{d}$ . This completes the argument that

$$\{P^{(d)}, P^{(\tilde{d})}\} = 0 \quad (150)$$

for the special set of primitive invariants  $d$  defined in (59).

### 7.3 Brackets of $B$ or $F$ charges with $P$ charges

We now consider the Poisson brackets between top component charges based on  $\Omega$  tensors (odd or even degree) and those based on  $\Lambda$  tensors.

First, the bracket

$$\{B^{(\tilde{d})}, P^{(d)}\} = - \{B^{(\tilde{d})}, U + V + W\} . \quad (151)$$

can be found by calculations similar to those in the last section. Given (137), we find

$$\{U, B^{(\tilde{d})}\} = -4(m+1)(l+1) \int dx d_{\hat{\delta}\hat{\gamma}_1\dots\hat{\gamma}_{m+1}} \tilde{d}_{\hat{\beta}\hat{\rho}_1\dots\hat{\rho}_l\hat{\delta}} k_+^\beta h_+^{\hat{\rho}_1} \dots h_+^{\hat{\rho}_l} h_+^{\hat{\gamma}_1} \dots h_+^{\hat{\gamma}_m} h_+^{\hat{\gamma}_{m+1}} . \quad (152)$$

Then

$$\begin{aligned} \{V, B^{(\tilde{d})}\} &= 4(m+1)(l+1) \int dx d_{\alpha\delta\hat{\gamma}_1\dots\hat{\gamma}_m} \tilde{d}_{\delta\hat{\rho}_1\dots\hat{\rho}_{l+1}} k_+^\alpha h_+^{\hat{\gamma}_1} \dots h_+^{\hat{\gamma}_m} h_+^{\hat{\rho}_1} \dots h_+^{\hat{\rho}_{l+1}} \\ &\quad + 2(m+1) \int dx d_{\alpha\beta\hat{\gamma}_1\dots\hat{\gamma}_m} \tilde{d}_{\delta\hat{\rho}_1\dots\hat{\rho}_{l+1}} k_+^\alpha f^{\beta\delta\hat{\kappa}} I^{\hat{\kappa}} h_+^{\hat{\gamma}_1} \dots h_+^{\hat{\gamma}_m} h_+^{\hat{\rho}_1} \dots h_+^{\hat{\rho}_{l+1}} \end{aligned} \quad (153)$$

(the non-zero terms are those from brackets of  $k_+$ 's with  $k_+$ 's). The first line can be combined with the bracket  $\{U, B^{(\tilde{d})}\}$  to produce the expression

$$d_{ad\hat{\gamma}_1\dots\hat{\gamma}_m} \tilde{d}_{bd\hat{\rho}_1\dots\hat{\rho}_l} h_+^{\hat{\gamma}_1} \dots h_+^{\hat{\gamma}_m} h_+^{\hat{\rho}_1} \dots h_+^{\hat{\rho}_l} (k_+^a h_+^b - k_+^b h_+^a), \quad (154)$$

and this vanishes by the same argument used for (138). But in addition we have

$$\begin{aligned} \{W, B^{(\tilde{d})}\} &= 2i \int dx d_{\delta\hat{\gamma}_1\dots\hat{\gamma}_{m+1}} \tilde{d}_{\alpha\hat{\beta}_1\dots\hat{\beta}_{l+1}} \left( 2f_{\delta\alpha\hat{\kappa}} k_+^{\hat{\kappa}} h_+^{\hat{\gamma}_1} \dots h_+^{\hat{\gamma}_{m+1}} h_+^{\hat{\beta}_1} \dots h_+^{\hat{\beta}_{l+1}} \right. \\ &\quad \left. + i(m+1)(l+1) I^{\hat{\delta}} k_+^\alpha f_{\hat{\gamma}_1\hat{\beta}_1\hat{\kappa}} h_+^{\hat{\kappa}} h_+^{\hat{\gamma}_2} \dots h_+^{\hat{\gamma}_{m+1}} h_+^{\hat{\beta}_2} \dots h_+^{\hat{\beta}_{l+1}} \right). \end{aligned} \quad (155)$$

The first of these terms is zero by invariance of  $d$  and  $\tilde{d}$ , and the second cancels the last remaining term in  $\{V, B^{(\tilde{d})}\}$ . Hence

$$\{P^{(d)}, B^{(\tilde{d})}\} = 0. \quad (156)$$

The final bracket we must consider is

$$\{F^{(e)}, P^{(d)}\} = -\{F^{(e)}, U + V + W\}. \quad (157)$$

It is actually significantly easier to calculate brackets involving the bosonic, bottom component charge

$$A^{(e)} = \int dx \Omega^{(e)}(\psi_+, \dots, \psi_+) = \int dx e_{\hat{\alpha}_1\dots\hat{\alpha}_p} h_+^{\hat{\alpha}_1} \dots h_+^{\hat{\alpha}_p}. \quad (158)$$

rather than the top component charge  $F^{(e)}$ , and the results are easily related by supersymmetry, as explained in section 7.1.

We find that  $\{A^{(e)}, V\} = \{A^{(e)}, W\} = 0$  and so, using (137),

$$\begin{aligned} \{A^{(e)}, P^{(d)}\} &= \{A^{(e)}, U\} \\ &= 4p \int dx e_{\hat{\beta}\hat{\alpha}_1\dots\hat{\alpha}_{p-1}} d_{\hat{\beta}\hat{\gamma}_1\dots\hat{\gamma}_{m+1}} \left( \partial_x h_+^{\hat{\alpha}_1} \dots h_+^{\hat{\alpha}_{p-1}} \right) h_+^{\hat{\gamma}_1} \dots h_+^{\hat{\gamma}_{m+1}}. \end{aligned} \quad (159)$$

There are various ways in which this can be evaluated, but it is convenient to note that the structure is identical to results obtained when investigating the current algebra of the bosonic PCM based on  $H$  [16] (see section 4.1 of that paper; this deals with  $H$  simple, but the conclusions are easily extended to direct products of groups). We find, by comparison, that the bracket vanishes in many cases, for instance if  $d$  and  $e$  are both single trace invariants, but for *any* choice of  $d$  and  $e$ , the results of [16] ensure that the integrand in (159) can be written as a polynomial in currents of type  $\Omega^{(e)}$  and their  $\partial_+$  derivatives. Thus  $\{F^{(e)}, P^{(d)}\}$  does indeed have the form described in section 5.

## 8 Comments

We have presented a thorough account of two classes of local conservation laws in supersymmetric sigma models based on classical symmetric spaces  $G/H$ . One variety of conserved quantity, based on  $\Omega$  forms, corresponds to the cohomology of the target manifold. The other variety, based on  $\Lambda$  tensors, are higher-spin generalizations of energy-momentum. In each case we have found primitive sets of currents, from which all others of  $\Omega$  or  $\Lambda$  type can be constructed as differential polynomials. The mathematics underlying the  $\Omega$  type currents is, of course, very well-understood; nevertheless, it is not entirely straightforward to extract the relevant sorts of results on primitive generators from the standard sources such as [25]-[28], and so the concrete presentation of the cohomology generators which we have derived may, perhaps, be useful elsewhere.

We have not proved that the  $\Omega$  and  $\Lambda$  type currents (and differential polynomials in them) give the *only* local conserved quantities in these sigma models, but all our results are consistent with this suggestion. In particular, the Poisson bracket calculations of the top component charges which we carried out in the last section (and whose results were announced in section 5) always generate answers which correspond to currents of recognizable form.

Our Poisson bracket results generalize those obtained for SPCMs [19] and, as in this earlier work, we were able to prove the existence of commuting families of bosonic (top component) charges. A new feature of the symmetric space models, however, is the appearance of fermionic top component charges, which close under Poisson brackets onto the bosonic charges. The presence of commuting charges for the bosonic symmetric space model has been linked to the Drinfeld-Sokolov/mKdV hierarchies [18] and it would be interesting to understand how these super-generalizations fit into this framework.

The most obvious problem for future work is the study of these conserved quantities at the quantum level (some recent progress for the analogous bosonic models has recently been made in [14]). Given the way in which commuting sets of bosonic charges are known to constrain the S-matrix, it would be very interesting to understand what additional implications are imposed by the fermionic charges. This might help in the construction or confirmation of S-matrices, about which we still know comparatively little for these models (the exceptions are the  $S^n$  model [3], the  $CP^n$  model [42], the  $SU(n)$  super PCM [43], and some Grassmannian theories [44]).

Finally, there has been intense recent investigation of various integrable structures in string theory on AdS backgrounds, which are very closely related to symmetric spaces [37]-[41]. The models we have studied here correspond to world-sheet, rather than target

space supersymmetry in string theory. Nevertheless, it would clearly be worthwhile to investigate and clarify how the structures of conserved quantities might coincide, or differ, in these two classes of supersymmetric theories.

**Acknowledgements:** We are very grateful to Ivan Smith for helpful conversations concerning the cohomology of symmetric spaces. The research of CASY was supported by a PPARC studentship; the work of JME is supported in part by Gonville and Caius College, Cambridge.

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