Bethe Equations for a $\mathfrak{g}_2$ Model

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Abstract

We prove, using the coordinate Bethe ansatz, the exact solvability of a model of three particles whose point-like interactions are determined by the root system of $\mathfrak{g}_2$. The statistics of the wavefunction are left unspecified. Using the properties of the Weyl group, we are also able to find Bethe equations. It is notable that the method relies on a certain generalized version of the well-known Yang-Baxter equation. A particular class of non-trivial solutions to this equation emerges naturally.

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Using the coordinate Bethe ansatz [1], C.N.Yang [2] solved a model of n particles, with unspecified statistics, interacting via contact interactions. This procedure led him to discover the celebrated Yang-Baxter equation, found also, in the context of statistical physics, by R.J.Baxter [3]. The potential used in the approach of C.N.Yang is intimately linked to the simple root system of $\mathfrak{sl}_n$. The generalization to the other root systems has also been intensively studied (see for example [4, 5]). In the case of the $\mathfrak{so}_n$ and $\mathfrak{sp}_n$ root systems, a new type of equation, the so-called *reflection equation* [6, 7], is obtained. It plays a fundamental role in the study of integrable system with boundaries.

In this letter, we demonstrate, using the procedure of [2], the exact solvability of a model based on the $\mathfrak{g}_2$ root system. This gives rise to a generalized version of the Yang-Baxter equation [4, 8, 6] which is peculiar to $\mathfrak{g}_2$ amongst the simple Lie algebras and is distinct from the usual Yang-Baxter and reflection equations. We then find the Bethe equations for the model, which involves some interesting subtleties.

The Model

Let us consider a system of three particles with positions x_1, x_2 and x_3, whose interactions are specified by the Hamiltonian

$$H = -\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + 2g_S \sum_{\substack{i,j=1 \\ i\neq j}}^{3} \delta(x_i - x_j) + 2g_L \sum_{\substack{i,j,k=1 \\ i\neq j\neq k\neq i}}^{3} \delta(x_k - \frac{1}{2}(x_i + x_j)). \tag{1}$$

Here g_S, g_L are real parameters characterizing the strength of two types of interactions. Physically, the g_S term is the usual contact term – specifying how particles interact when they collide – while the g_L term may be thought of as describing a contact interaction between each particle and the centre of mass of the remaining pair. We wish to solve the spectral problem

$$H\phi(x_1, x_2, x_3) = E\phi(x_1, x_2, x_3). \tag{2}$$

The motivation for the Hamiltonian (1) is that it is related to the root system of the exceptional Lie algebra $\mathfrak{g}_2$ [5], just as the model of n particles with purely contact interactions is related to that of $\mathfrak{s}l_n$: the root system of $\mathfrak{g}_2$ is

$$\Delta = \{\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j - 2\epsilon_k | 1 \leqslant i \neq j \neq k \neq i \leqslant 3\}, \tag{3}$$

where $\{\epsilon_i\}$ an orthonormal basis of $\mathbb{R}^3$. In terms of these roots, the Hamiltonian reads as

$$H = -\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + \sum_{\alpha\in\Delta} g_{\langle\alpha,\alpha\rangle}\ \delta(\langle\alpha, x\rangle) \tag{4}$$

where $x = \sum_i x_i\epsilon_i$, $g_2 = g_S$, $g_6 = g_L$ and $\langle\cdot,\cdot\rangle$ is the usual scalar product.

Coordinate Bethe Ansatz

To translate the problem into one which may be solved by the coordinate Bethe ansatz, it is useful to consider the Weyl group associated to the Lie algebra $\mathfrak{g}_2$ [4]. To each root $\alpha \in \Delta$ is associated a reflection in the hyperplane $\mathcal{H}_\alpha$ perpendicular to α:

$$s_\alpha(\beta) = \beta - 2\,\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\,\alpha\ . \tag{5}$$

The set of these reflections, together with the identity I, form the Weyl group of $\mathfrak{g}_2$, which maps the set of roots to itself. The group is generated by the reflections in the two simple roots

$$\alpha_1 = \epsilon_1 - \epsilon_2 \qquad \text{and} \quad \alpha_2 = \epsilon_2 + \epsilon_3 - 2\epsilon_1, \tag{6}$$

which we denote by, respectively, T and R. The relations obeyed by these generators

$$(T)^2 = I \qquad , \quad (R)^2 = I \qquad \text{and} \quad (TR)^6 = I\ , \tag{7}$$

completely specify the group, which is isomorphic to the dihedral group D_6.

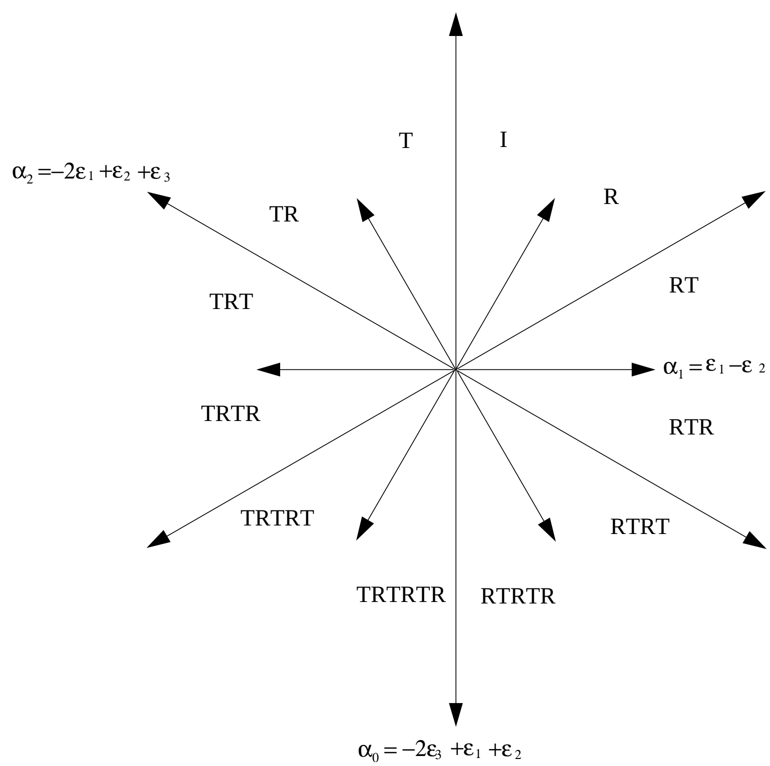

Figure 1: Root system of $\mathfrak{g}_2$, showing the labelling of Weyl chambers.

The twelve hyperplanes $\mathcal{H}_\alpha$ define twelve domains, called Weyl chambers, in $\mathbb{R}^3$. These chambers are characterized by

$$\mathcal{W}_{s_\alpha} :\ 0 < x_{s_\alpha 1} - x_{s_\alpha 2} < x_{s_\alpha 3} - x_{s_\alpha 1}\ , \tag{8}$$

where $x_{s_\alpha i}$ is the i^{th} component of the vector $\big(s_\alpha\big)^{-1}(x)$. Let us remark that conditions (8) are equivalent to

$$\mathcal{W}_{s_\alpha}: \; x_{s_\alpha 2} < x_{s_\alpha 1} \qquad \text{and} \quad x_{s_\alpha 1} < \frac{x_{s_\alpha 2} + x_{s_\alpha 3}}{2}. \tag{9}$$

In each Weyl chamber, the three particles are in a given order, as are the centre of mass of the two extreme particles and the middle particle. The importance of the definition of the Weyl group in this context is two-fold [4]: the hyperplanes, $\mathcal{H}_\alpha$, are the domains of the configuration space where the interactions take place; and the action successively of R and T allows us to describe, starting from one region $\mathcal{W}_{s_\alpha}$, the eleven other regions. (See figure 1.) Using these two properties, the spectral problem (2) may be written equivalently as a free Hamiltonian, for $(x_1, x_2, x_3) \notin \{\mathcal{H}_\alpha \mid \alpha \in \Delta\}$,

$$-\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \phi(x_1, x_2, x_3) = E\phi(x_1, x_2, x_3) \tag{10}$$

and boundary conditions on hyperplanes $\mathcal{H}_\alpha$ $(\alpha \in \Delta)$

$$\phi|_{\langle\alpha,x\rangle=0^+} = \phi|_{\langle\alpha,x\rangle=0^-} \quad \text{and} \quad \langle\alpha,\nabla\rangle\phi|_{\langle\alpha,x\rangle=0^+} = \big(\langle\alpha,\nabla\rangle + 2g_{\langle\alpha,\alpha\rangle}\big)\,\phi|_{\langle\alpha,x\rangle=0^-} \tag{11}$$

where $\nabla = \sum_i \epsilon_i \frac{\partial}{\partial x_i}$. The equivalence between the δ-potentials in the Hamiltonian (4) and these boundary conditions (11) is well-known (see for example [4]).

We now determine the eigenfunctions of the Hamiltonian by solving equation (10) with boundary conditions (11). We make the following ansatz for ϕ: in the region $\mathcal{W}_Q$ (with $Q \in D_6$), the eigenfunction of the Hamiltonian is written as follows

$$\phi_Q(x_1, x_2, x_3) = \sum_{P \in D_6} \exp(i\langle k_P, x_Q\rangle) A_P(Q). \tag{12}$$

This ansatz is similar to the one suggested by H.Bethe [1]: the only difference is that here the sum is over a dihedral, rather than permutation, group. These eigenfunctions obviously satisfy relation (10) with

$$E = \sum_{i=1}^{3} k_i^2 \; . \tag{13}$$

We need to determine the parameters $A_P(Q)$ present in the ansatz so that boundary conditions (11) are satisfied. Because of definition (8) of the region $\mathcal{W}_Q$ $(Q \in D_6)$, the eigenfunction ϕ_Q adjoins two boundaries: $x_{Q1} = \frac{x_{Q2}+x_{Q3}}{2}$ and $x_{Q1} = x_{Q2}$. The boundary conditions imply the following constraints between the vectors A_P (whose components are $A_P(Q)$, $P, Q \in D_6$):

$$A_{PR} = B(\langle k_P, \alpha_2\rangle) A_P \qquad \text{and} \quad A_{PT} = Y(\langle k_P, \alpha_1\rangle) A_P \tag{14}$$

where

$$B(k) = \frac{k\widehat{R} + ig_L}{k - ig_L} \qquad \text{and} \quad Y(k) = \frac{k\widehat{T} + ig_S}{k - ig_S} \tag{15}$$

and the operators $\widehat{R}$ and $\widehat{T}$, which provide a realization of the group D_6, are defined by

$$\widehat{R}A_P(Q) = A_P(QR) \qquad \text{and} \quad \widehat{T}A_P(Q) = A_P(QT) \ . \tag{16}$$

One can now use relations (14) to calculate recursively all the A_P starting from, for example, A_I. However, due to the relations (7) satisfied by the generators of D_6, the following consistency relations appear between the operators (15)

$$B(k) = (B(-k))^{-1} \qquad , \quad Y(k) = (Y(-k))^{-1} \tag{17}$$

and

$$\begin{aligned} &Y(k_1-k_2)B(2k_1-k_2-k_3)Y(k_1-k_3)B(k_1+k_2-2k_3)Y(k_2-k_3)B(2k_2-k_1-k_3) \\ &\quad = B(2k_2-k_1-k_3)Y(k_2-k_3)B(k_1+k_2-2k_3)Y(k_1-k_3)B(2k_1-k_2-k_3)Y(k_1-k_2) \end{aligned} \tag{18}$$

Using solely that $\widehat{R}$ and $\widehat{T}$ satisfy relations (7), one can verify by direct computation that these equations hold, finishing our argument about the exact solvability of the $\mathfrak{g}_2$ model. The relations (17) are the usual unitarity relations whereas (18) is a generalization of the Yang-Baxter equation [4, 8, 6]. We discuss the implications of its appearance briefly below.

Bethe Equations

In order to find the Bethe equations we suppose that the three particles live on a circle of finite circumference $2L$. One must specify carefully how the particles interact in this case. As on the infinite line, there are interactions due to direct collisions of particles. But, since on a circle there is no preferred notion of which particle lies on the left, on the right, or in the middle, it is now most natural to assume that *each* of the three particles interacts with the midpoint of the remaining pair. What is more, for each pair there are really two "midpoints", and we assume that the third particle interacts with both. (The midpoint of two particles would otherwise jump discontinuously as they pass opposite points, which seems physically unappealing.)

To make contact with the preceding section we need a prescription to identify the set of possible configurations of the particles on a circle with the group D_6. We make the following choice. We "cut" the circle at the point opposite the mid-point of the closest pair of particles (call them A and B – of course, the particles cannot be equally spaced for then each would lie on the mid-point of the other two). We then unwrap the circle to get an interval of length $2L$, and the positions of the particles on this interval correspond to a unique element of $Q \in D_6$, just as in the previous section. (Figure 2 illustrates an example in which $Q = I$.)

By construction, neither A nor B can lie at an endpoint of this interval (in fact, neither can be closer than $2L/3$ to an endpoint) so we need only specify the boundary conditions for the remaining particle, C.[3] On examining the definitions of the various regions, one sees that C can

[3] Note also that the prescription ensures that – to take the example in figure 2 – 1 reaches the "internal" midpoint (23) *before* 2 has a chance to reach the "external" midpoint (13). So the boundary condition on particle C (3, in this case) is genuinely the only new condition, not present in the model on the line.

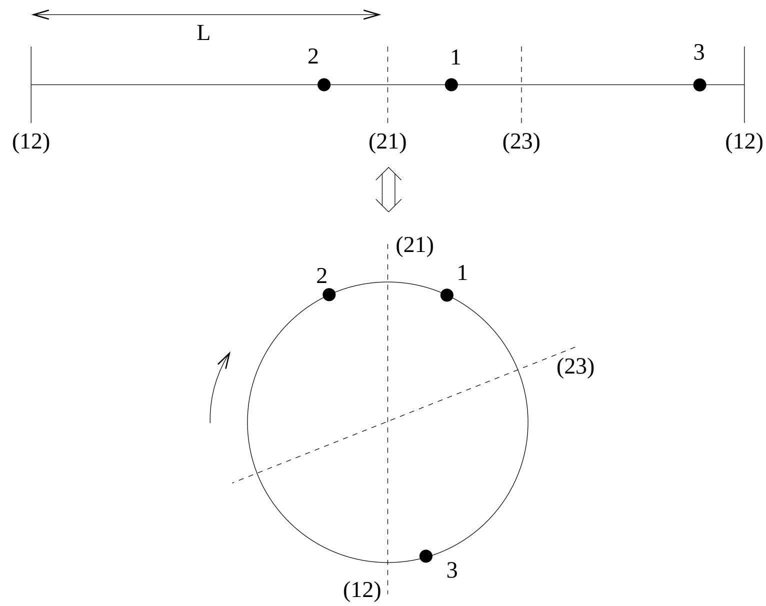

Figure 2: A configuration in region $\mathcal{W}_I$. (The midpoint line (13) is not shown.)

reach only the right endpoint for all configurations corresponding to

$$Q \in \mathcal{B}^+ := \{I,\ T,\ RTR,\ TRTR,\ RTRT,\ TRTRT\} \tag{19}$$

and that it can reach only the left endpoint for all configurations corresponding to

$$Q \in \mathcal{B}^+ W = \left\{RTRTR,\ (RT)^3,\ TR,\ R,\ TRT,\ RT\right\} \tag{20}$$

where we define $W = RTRTR$. More specifically, when one starts in a region $\mathcal{W}_Q$, with $Q \in \mathcal{B}^+$, and moves particle C through the right boundary, one reaches the region $\mathcal{W}_{QW}$. As it crosses the boundary, C interacts with the "opposite" (on the circle) midpoint of A and B, so the conditions are continuity with a jump in the first derivative, just in (11) above. Explicitly, one finds

$$\phi|_{\langle\alpha_0, x_Q\rangle = -2L^-} = \phi|_{\langle\alpha_0, x_Q\rangle = +2L^-} \tag{21}$$

and

$$\langle\alpha_0, \nabla_Q\rangle\phi|_{\langle\alpha_0, x_Q\rangle = -2L^-} = \left(\langle\alpha_0, \nabla_Q\rangle + 2g_L\right)\phi|_{\langle\alpha_0, x_Q\rangle = +2L^-}, \tag{22}$$

where $\alpha_0 = \epsilon_1 + \epsilon_2 - 2\epsilon_3$ is the lowest root. Note that $W = RTRTR$ is the reflection in the plane orthogonal to α_0, so that

$$\langle\alpha_0, x_Q\rangle = -\langle\alpha_0, x_{QW}\rangle \tag{23}$$

and hence one can replace Q with QW in these boundary conditions without altering their content – as expected, since this is supposed to describe the boundary between $\mathcal{W}_Q$ and $\mathcal{W}_{QW}$. In fact the boundary conditions may be re-written most symmetrically as

$$\phi|_{\langle\alpha_0, x_Q\rangle = -2L^-} = \phi|_{\langle\alpha_0, x_{QW}\rangle = -2L^-} \tag{24}$$

and

$$\langle\alpha_0,\nabla_Q\rangle\phi|_{\langle\alpha_0,x_Q\rangle=-2L^-} = \left(\langle\alpha_0,\nabla_Q\rangle + 2g_L\right)\phi|_{\langle\alpha_0,x_{QW}\rangle=-2L^-}. \tag{25}$$

Substituting the Bethe ansatz for ϕ (in region $\mathcal{W}_Q$ on the left and region $\mathcal{W}_{QW}$ on the right), one has, from the first boundary condition,

$$\sum_{P\in D_6} A_P(Q)e^{i\langle k_P,x_Q\rangle}|_{\langle\alpha_0,x_Q\rangle=-2L^-} = \sum_{P\in D_6} A_P(QW)e^{i\langle k_P,x_{QW}\rangle}|_{\langle\alpha_0,x_{QW}\rangle=-2L^-}. \tag{26}$$

Now the vectors $n=\frac{1}{\sqrt{3}}(1,1,1)$, α_1 and α_0 are orthogonal, and hence

$$\langle k_P,x_Q\rangle = \langle k_P,n\rangle\langle n,x_Q\rangle + \frac{1}{2}\langle k_P,\alpha_1\rangle\langle\alpha_1,x_Q\rangle + \frac{1}{6}\langle k_P,\alpha_0\rangle\langle\alpha_0,x_Q\rangle; \tag{27}$$

furthermore W fixes n and α_1 and inverts α_0. Thus

$$\begin{aligned} &\sum_{P\in\mathcal{B}^+} e^{i\langle k_P,n\rangle\langle n,x_Q\rangle+\frac{i}{2}\langle k_P,\alpha_1\rangle\langle\alpha_1,x_Q\rangle}\left(A_P(Q)e^{-\frac{iL}{3}\langle k_P,\alpha_0\rangle} + A_{PW}(Q)e^{+\frac{iL}{3}\langle k_P,\alpha_0\rangle}\right) \\ =\ &\sum_{P\in\mathcal{B}^+} e^{i\langle k_P,n\rangle\langle n,x_Q\rangle+\frac{i}{2}\langle k_P,\alpha_1\rangle\langle\alpha_1,x_Q\rangle}\left(A_P(QW)e^{-\frac{iL}{3}\langle k_P,\alpha_0\rangle} + A_{PW}(QW)e^{+\frac{iL}{3}\langle k_P,\alpha_0\rangle}\right), \end{aligned} \tag{28}$$

which can hold for general x only if, for all $P\in\mathcal{B}^+$,

$$A_P(Q)e^{-\frac{iL}{3}\langle k_P,\alpha_0\rangle} + A_{PW}(Q)e^{+\frac{iL}{3}\langle k_P,\alpha_0\rangle} = A_P(QW)e^{-\frac{iL}{3}\langle k_P,\alpha_0\rangle} + A_{PW}(QW)e^{+\frac{iL}{3}\langle k_P,\alpha_0\rangle}. \tag{29}$$

Meanwhile, from the boundary condition on the first derivative of ϕ one finds, by very similar reasoning, that

$$\begin{aligned} &i\langle k_P,\alpha_0\rangle A_P(Q)e^{-\frac{iL}{3}\langle k_P,\alpha_0\rangle} - i\langle k_P,\alpha_0\rangle A_{PW}(Q)e^{+\frac{iL}{3}\langle k_P,\alpha_0\rangle} \\ =\ &\left(-i\langle k_P,\alpha_0\rangle + 2g_L\right)A_P(QW)e^{-\frac{iL}{3}\langle k_P,\alpha_0\rangle} + \left(i\langle k_P,\alpha_0\rangle + 2g_L\right)A_{PW}(QW)e^{+\frac{iL}{3}\langle k_P,\alpha_0\rangle} \end{aligned} \tag{30}$$

(here we used $\langle k_P,x_{QW}\rangle = \langle k_{PW},x_Q\rangle$ for the first term on the right).

After eliminating $A_{PW}(QW)$ in the second equation using the first, one arrives at

$$e^{\frac{2iL}{3}\langle k_P,\alpha_0\rangle}A_{PW} = \frac{\langle k_P,\alpha_0\rangle\hat{W} + ig_L}{\langle k_P,\alpha_0\rangle - ig_L}A_P \tag{31}$$

where $\hat{W}A_P(Q) = A_P(QW)$. This is true for all $P\in\mathcal{B}^+$,[4] but there is some redundancy, for if (31) holds for P it also holds for PT:

$$\begin{aligned} e^{\frac{2iL}{3}\langle k_{PT},\alpha_0\rangle}A_{PTW} &= e^{\frac{2iL}{3}\langle k_{PT},\alpha_0\rangle}A_{PWT} \\ &= e^{\frac{2iL}{3}\langle k_P,\alpha_0\rangle}Y(\langle k_{PW},\alpha_1\rangle)A_{PW} \\ &= Y(\langle k_{PW},\alpha_1\rangle)\frac{\langle k_P,\alpha_0\rangle\hat{W} + ig_L}{\langle k_P,\alpha_0\rangle - ig_L}A_P \\ &= \frac{\langle k_P,\alpha_0\rangle\hat{W} + ig_L}{\langle k_P,\alpha_0\rangle - ig_L}Y(\langle k_{PW},\alpha_1\rangle)A_P \\ &= \frac{\langle k_{PT},\alpha_0\rangle\hat{W} + ig_L}{\langle k_{PT},\alpha_0\rangle - ig_L}A_{PT} \end{aligned} \tag{32}$$

[4]In fact it is, as an immediate consequence, true for all $P\in D_6$: on setting $P=P'W$ one finds the same equation for P'.

(the essential point is that α_0 and α_1 are orthogonal, so $W = s_{\alpha_0}$ fixes α_1, $T = s_{\alpha_1}$ fixes α_0, and $[W,T] = 0$). It therefore suffices to consider

$$P \in \{I, \quad RTR, \quad TRTR\}\,. \tag{33}$$

Using now the relations (14), which come from the "interior" boundary conditions, (31) yields three equations for $A_I(Q)$. For example, in the case $P = I$,

$$A_W = A_{RTRTR} = B(\langle k_{RTRT}, \alpha_2\rangle)Y(\langle k_{RTR}, \alpha_1\rangle)B(\langle k_{RT}, \alpha_2\rangle)Y(\langle k_R, \alpha_1\rangle)B(\langle k, \alpha_2\rangle)A_I. \tag{34}$$

To make the content of the resulting equations clearer, it is helpful to define some new operators. First, for every root α, let $\hat{s}_\alpha A_P(Q) = A_P(Qs_\alpha)$ and define

$$Z_\alpha(k) = \frac{\langle k, \alpha\rangle + ig_{\langle\alpha,\alpha\rangle}\hat{s}_\alpha}{\langle k, \alpha\rangle - ig_{\langle\alpha,\alpha\rangle}}, \tag{35}$$

which has the property that $Z_\alpha(k)^{-1} = Z_\alpha(-k)$. This notation is compact but rather opaque, so it is useful to define also

$$S_{12} = Z_{\epsilon_1-\epsilon_2}, \quad S_{23} = Z_{\epsilon_2-\epsilon_3}, \quad S_{31} = Z_{\epsilon_3-\epsilon_1}, \tag{36}$$

(whose inverses we write as S_{21}, S_{32}, and S_{13}) and similarly for the long roots

$$K^1_{23} = Z_{\epsilon_2+\epsilon_3-2\epsilon_1}, \quad K^2_{31} = Z_{\epsilon_3+\epsilon_1-2\epsilon_2}, \quad K^3_{12} = Z_{\epsilon_1+\epsilon_2-2\epsilon_3}. \tag{37}$$

In terms of these operators we find, for $P = RTR$, $P = TRTR$ and $P = I$ respectively,

$$e^{\frac{2iL}{3}\langle k,\epsilon_2+\epsilon_3-2\epsilon_1\rangle}A_I = \mathscr{R}_1 A_I := S_{21}(k)K^2_{13}(k)^{-1}K^1_{23}(k)K^3_{12}(k)^{-1}S_{31}(k)K^1_{23}(k)A_I, \tag{38}$$

$$e^{\frac{2iL}{3}\langle k,\epsilon_3+\epsilon_1-2\epsilon_2\rangle}A_I = \mathscr{R}_2 A_I := K^1_{23}(k)^{-1}K^2_{13}(k)K^3_{12}(k)^{-1}S_{32}(k)K^2_{13}(k)S_{12}(k)A_I, \tag{39}$$

$$e^{\frac{2iL}{3}\langle k,\epsilon_1+\epsilon_2-2\epsilon_3\rangle}A_I = \mathscr{R}_3 A_I := K^1_{23}(k)^{-1}S_{13}(k)K^3_{12}(k)S_{23}(k)K^2_{13}(k)^{-1}K^3_{12}(k)A_I. \tag{40}$$

These are the Bethe equations for the problem, and the task is to show that the $\mathscr{R}_i$ commute. But, before this, it is important to observe that there is further redundancy. The exponentials on the left hand sides contain the three long roots of $\mathfrak{g}_2$, which are of course co-planar and indeed sum to zero. Thus there are really only *two* independent equations, and, multiplying the equations above together, we find that we must have

$$\mathscr{R}_1\mathscr{R}_2\mathscr{R}_3 A_I = A_I. \tag{41}$$

To see that this is in fact true – which is a good consistency check – and to verify the commutation relations $[\mathscr{R}_i, \mathscr{R}_j] = 0$, one use the following properties of S and K:

$$S_{12}K^3_{12} = K^3_{12}S_{12} \tag{42}$$

$$K_{12}^{3}\left(K_{13}^{2}\right)^{-1}K_{23}^{1}=K_{23}^{1}\left(K_{13}^{2}\right)^{-1}K_{12}^{3} \tag{43}$$

$$S_{12}K_{23}^{1}S_{13}\left(K_{12}^{3}\right)^{-1}S_{23}K_{13}^{2}=K_{13}^{2}S_{23}\left(K_{12}^{3}\right)^{-1}S_{13}K_{23}^{1}S_{12}. \tag{44}$$

These may be verified directly. (See also [9, 10, 11].)

The reason there are only two equations – even though there are three momenta k_i – is that we chose to apply periodic boundary conditions in a fashion which made no reference to any fixed point on the circle. (It is more usual [12, 2, 13] to take, for example $\phi|_{x_1=0}=\phi|_{x_1=2L}$, but in the present case the nature of the interactions make this technically inconvenient.) Thus the symmetry of the problem under rotations was kept manifest throughout and the corresponding conserved quantity, the total (angular) momentum

$$P=k_1+k_2+k_3=\sqrt{3}\langle k,n\rangle, \tag{45}$$

dropped out of the calculation. But of course, as in any quantum-mechanical system of particles on a circle, P is the generator of rigid rotations, and the invariance of the problem under a rotation through one complete turn produces the quantization condition

$$1=e^{2iLP}=e^{2iL\langle k,\epsilon_1+\epsilon_2+\epsilon_3\rangle}. \tag{46}$$

This, together with any two of (38-40), gives the complete set of quantization conditions on the momenta k_i.

It is interesting to note that, at least in the centre of momentum frame $k_1+k_2+k_3=0$, the equations (38-40) have the intuitive interpretation one would expect: for example, (39) becomes

$$e^{2iLk_2}A_I=K_{23}^{1}(k)^{-1}K_{13}^{2}(k)K_{12}^{3}(k)^{-1}S_{32}(k)K_{13}^{2}(k)S_{12}(k)A_I \tag{47}$$

and describes the process of moving particle 2 clockwise through one complete revolution while the other particles remain fixed. Thus, the first event is the scattering of 1 and 2 (hence S_{12}) followed by particle 2 interacting with the midpoint of 1 and 3 (giving K_{13}^{2}), scattering with 3 (S_{32}), and so on.

Conclusion

To conclude, let us comment briefly on the generalized Yang-Baxter equation (44) we obtained. Like the Yang-Baxter and reflection equations, (44) may be represented diagrammatically. This is shown in Figure 3, where, to simplify the picture, we restrict ourselves to the case where $x_1+x_2+x_3=0$. The three arrows represent the "trajectories" of the three particles and the double-line that of their centre of mass. The intersection between the two arrows a and b corresponds to the scattering of S_{ab} of particles a and b. The intersection between the arrow a with the double-line corresponds to the scattering K_{bc}^{a} $(b,c\neq a)$ between a and the centre of mass. Obviously, we recover the usual representation of the Yang-Baxter equation by removing the double-line. This occurs in the limit $g_L=0$, for then $K_{bc}^{a}=1$.

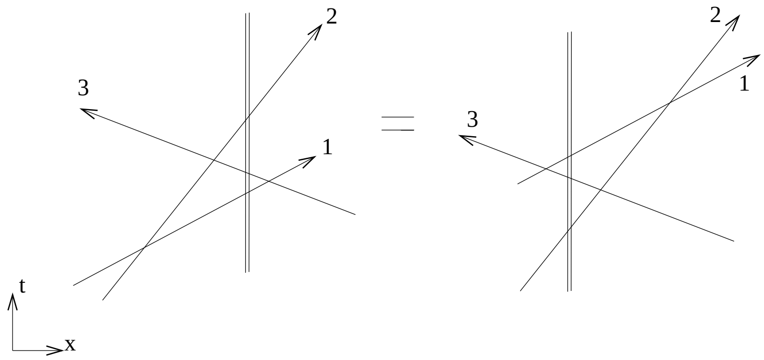

Figure 3: Pictorial representation of the generalized Yang-Baxter equation (44)

Since the Yang-Baxter and reflection equations play a fundamental role in the development of integrable models and quantum groups, it is natural to speculate that the generalized Yang-Baxter equation (44) might also have interesting applications. In particular, we hope that they will allow one to study integrable models where the interactions between three particles are not factorisable.

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References

[1] H. Bethe, *Zur theorie der metalle. Eigenwerte und eingenfunktionen atomkete,* Zeitschrift für Physik **71** (1931) 205.

[2] C. N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction,* Phys. Rev. Lett **19** (1967) 1312.

[3] R. J. Baxter, *Partition function of the eight-vertex lattice model,* Ann. Phys. **70** (1972) 193 et J. Stat. Phys. **8** (1973) 25; *Exactly solved models in statistical mechanics,* (Academic Press, 1982).

[4] M. Gaudin, *Boundary Energy of a Bose Gas in One Dimension,* Phys. Rev. **A4** (1971) 386.

[5] M.A. Olshanetsky and A.M. Perelomov, *Quantum integrable systems related to the Lie algebras,* Phys. Reports **94** (1983) 313.

[6] I. V. Cherednik, *Factorizing particles on a half line and root systems,* Theor. Math. Phys. **61** (1984) 977.

[7] E. K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. **A21** (1988) 2375.

[8] B. Sutherland, *Nondiffractive Scattering: Scattering From Kaleidoscopes*, J. Math. Phys. **21** (1980) 1770.

[9] I. Cherednik, *Difference elliptic operators and root systems*, arXiv:hep-th/9410188.

[10] Chapter 16, V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press (1994)

[11] V. G. Drinfeld, *Quasi-Hopf Algebras*, Lengingrad Math. J. **1** (1990) 1419 [Alg. Anal. **1N6** (1989) 114].

[12] E. H. Lieb and W. Liniger, *Exact Analysis Of An Interacting Bose Gas. 1. The General Solution And The Ground State*, Phys. Rev. **130** (1963) 1605.

[13] Z. Q. Ma, *Yang-Baxter equation and quantum enveloping algebras*, World Scientific (1993)