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Sutherland Models for Complex Reflection Groups

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ABSTRACT

There are known to be integrable Sutherland models associated to every real root system – or, which is almost equivalent, to every real reflection group. Real reflection groups are special cases of *complex* reflection groups. In this paper we associate certain integrable Sutherland models to the classical family of complex reflection groups. Internal degrees of freedom are introduced, defining dynamical spin chains, and the freezing limit taken to obtain static chains of Haldane-Shastry type. By considering the relation of these models to the usual BC_N case, we are led to systems with both real and complex reflection groups as symmetries. We demonstrate their integrability by means of new Dunkl operators, associated to wreath products of dihedral groups.

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1 Introduction

The Sutherland model [1] is an important and much-studied integrable quantum-mechanical system. It describes N particles moving on a circle, whose pairwise interactions are determined by a potential proportional to the inverse square of the chord-length separating the particles (as in figure 1). The Hamiltonian, in the simplest case of identical spinless bosons, is

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \lambda\sum_{i\neq j}\frac{1}{\sin^2\left(\frac{1}{2}\left(x_i - x_j\right)\right)}. \tag{1.1}$$

The model was first introduced in [2]. It, and the wider family of Calogero-Sutherland-Moser models [3] to which it belongs, have since appeared in areas physics apparently far removed from the original condensed-matter context: see for example [4]. Operator methods were used to solve the system in [5, 6]; the Yangian symmetry of the model was derived in [7, 8]. For recent reviews and references to the extensive literature see [9, 10, 11].

The Hamiltonian (1.1) is closely related to the Coxeter group A_{N-1}, because the potential can be written as $\sum_{\alpha\in\Delta}\sin^{-2}(\frac{1}{2}(\mathbf{x}\cdot\boldsymbol{\alpha}))$, where $\Delta = \{\boldsymbol{\epsilon}_i - \boldsymbol{\epsilon}_j : i\neq j\}$ is the root system of A_{N-1}. Similar integrable models exist also for all other finite Coxeter groups [12, 13, 14, 15, 16, 17]. The classical families BC_N and D_N describe systems with boundaries, via a kind of method of images: in the case of D_N, whose roots are $\pm\boldsymbol{\epsilon}_i \pm \boldsymbol{\epsilon}_j$, one has

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \lambda\sum_{i\neq j}\left(\frac{1}{\sin^2\left(\frac{1}{2}\left(x_i - x_j\right)\right)} + \frac{1}{\sin^2\left(\frac{1}{2}\left(x_i + x_j\right)\right)}\right), \tag{1.2}$$

and the second term describes the interaction of particle i with the image of particle j in the boundary (see figure 1). In the B and C cases, the extra roots $\propto \pm\boldsymbol{\epsilon}_i$ give an interaction between the particles and the boundary.

These families, $ABCD$, of course exhaust the classical irreducible finite Coxeter groups. Finite Coxeter groups are finite real reflection groups [18]: that is, subgroups of the orthogonal group generated by a finite number of elements $s\in O(N)$ such that

$$s^2 = 1, \qquad s \text{ has eigenvalue } +1 \text{ with multiplicity } N-1. \tag{1.3}$$

But it is possible to weaken the first of these requirements and consider subgroups of $U(N)$ generated by finitely many $s\in U(N)$ obeying

$$s^{n(s)} = 1, \qquad s \text{ has eigenvalue } +1 \text{ with multiplicity } N-1 \tag{1.4}$$

for some $n(s)\in\mathbb{N}$. By doing so one obtains *complex reflection groups*. The irreducible finite complex reflection groups were classified in [19]. There is one 'classical' three-parameter family

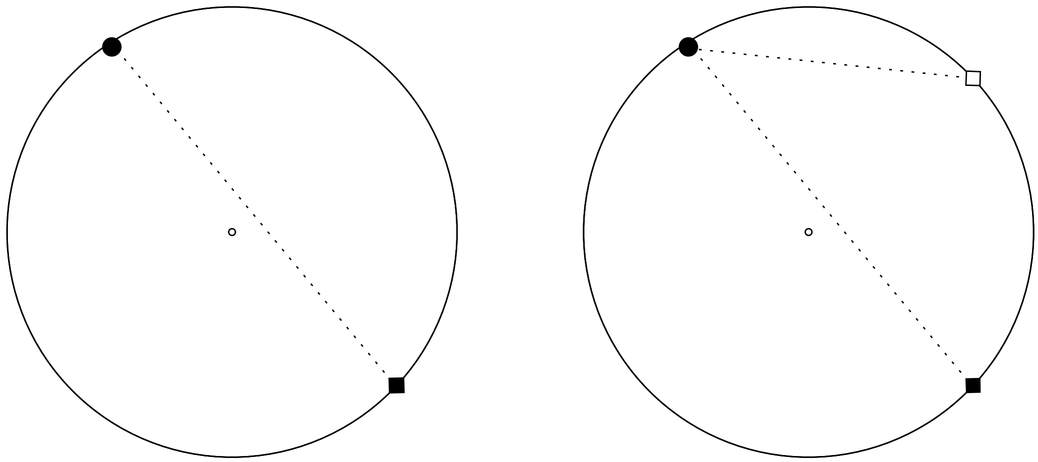

Figure 1: Interactions between two particles ($\bullet$ and $\blacklozenge$) in the Sutherland models associated to the A (left) and D (right) series of real reflection groups. The image of $\blacklozenge$ under reflection is drawn as $\lozenge$.

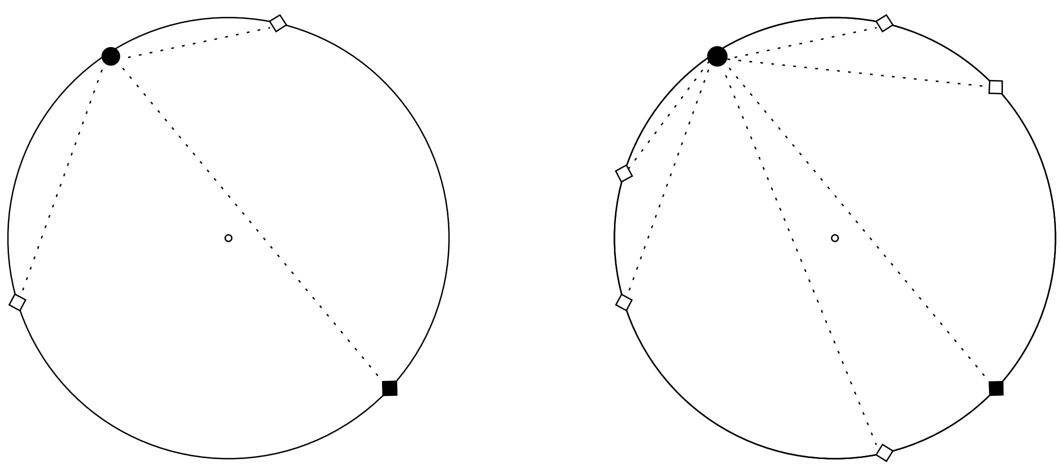

Figure 2: Interactions between particles $\bullet$ and $\blacklozenge$ for the Sutherland models associated to the classical complex reflection group $G(3,1,N)$ (left) and a wreath product of a dihedral group (right). The images of $\blacklozenge$ are drawn as $\lozenge$.

$G(pr, p, N)$, $p, r, N \in \mathbb{N}$, which includes the four classical families of real reflection groups as special cases, and then 34 exceptional cases.

In this paper our main goal is to construct Sutherland models for classical complex reflection groups. At first sight it is perhaps not clear that one should expect this to work, because complex reflection groups lack a great deal of the usual structure that comes with real reflection groups. The notions of root system, length function and Coxeter graph are either absent or, at best, less natural in the complex case [20] – and the definition of the BCD Sutherland models sketched above appears to rely explicitly on the root system data. Nevertheless, it turns out that there do exist integrable models of Sutherland type associated to complex reflection groups in a very natural fashion. The basic idea is sketched in figure 2 – each particle has a number of images, but now these images are generated by rotations.

The plan of this paper is as follows. In section 2 we begin with some algebraic preliminaries on complex reflection groups, and then introduce a key tool, Dunkl operators [21], from which we construct integrable Hamiltonians of Sutherland type. These models turn out to be members of a class of Calogero-Sutherland models first introduced and solved in [22, 23]. In section 4 we introduce models with internal "spin" degrees of freedom, and in section 5 static or "frozen" chains in which the spins are in fact the only degrees of freedom.

In section 6 we turn to models in which the set of images of each particle is generated by a dihedral group, as illustrated in figure 2. As we shall see, these models possess both a complex reflection group and a real reflection group as symmetries, embedded within a larger group which will turn out to be a wreath product of a dihedral group. An important part of our construction will be the introduction of new Dunkl operators, associated to such wreath products. We then go on to introduce spin degrees of freedom and static chains with dihedral symmetry.

We conclude by noting some open questions – primarily of solution and Hamiltonian reduction – concerning the new models of this paper, and some broader reasons for investigating integrable systems with complex reflection groups as symmetries.

2 Complex Reflection Algebras

Classical complex reflection groups

The complex reflection group $G(m, 1, N)$ is generated by $\{\, a,\, e_1,\, e_2, \ldots,\, e_{N-1}\}$, subject to the relations

$$\begin{array}{lll} e_i^2 = 1 & e_i\, e_{i+1}\, e_i = e_{i+1}\, e_i\, e_{i+1} & e_i\, e_j = e_j\, e_i \quad (|i-j| > 2) \\ a^m = 1 & a\, e_1\, a\, e_1 = e_1\, a\, e_1\, a & a\, e_j = e_j\, a \quad (j > 1) \end{array} \tag{2.1}$$

The e_i generate a copy of the permutation group S_N on N objects, a generates a copy of $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$, and the full group is a semidirect product $G(m,1,N) = (\mathbb{Z}_m)^N \rtimes S_N$. This structure is sometimes referred to as the *wreath product* of $\mathbb{Z}_m$ with S_N, denoted $\mathbb{Z}_m \wr S_N$. It will be convenient to write

$$\mathscr{P}_{ij} = \mathscr{P}_{ji} = e_i\, e_{i+1} \ldots e_{j-1} \ldots e_{i+1}\, e_i \quad (i < j) \tag{2.2}$$

for the transposition $i \leftrightarrow j$ (in particular $e_i = \mathscr{P}_{ii+1}$) and

$$\mathscr{Q}_1 = a \tag{2.3}$$

$$\mathscr{Q}_i = \mathscr{P}_{i1}\mathscr{Q}_1\mathscr{P}_{i1} \quad (i > 1). \tag{2.4}$$

In terms of these elements the defining relations imply, and can be recovered from,

$$\begin{array}{lll} \mathscr{P}_{ij}^2 = 1 & \mathscr{P}_{ij}\mathscr{P}_{jk} = \mathscr{P}_{ik}\mathscr{P}_{ij} = \mathscr{P}_{jk}\mathscr{P}_{ik} & \mathscr{P}_{ij}\mathscr{P}_{kl} = \mathscr{P}_{kl}\mathscr{P}_{ij} \\ & \mathscr{P}_{ij}\mathscr{Q}_i = \mathscr{Q}_j\mathscr{P}_{ij} & \mathscr{P}_{ij}\mathscr{Q}_k = \mathscr{Q}_k\mathscr{P}_{ij} \\ \mathscr{Q}_i^m = 1 & \mathscr{Q}_i\mathscr{Q}_j = \mathscr{Q}_j\mathscr{Q}_i & \qquad (i,j,k,l \text{ all distinct}). \end{array} \tag{2.5}$$

For any divisor p of m, the complex reflection group $G(m,p,N)$ is the subgroup of $G(m,1,N)$ generated by

$$\{\, a^p,\, a^{-1}\, e_1\, a,\, e_1,\, e_2, \ldots,\, e_{N-1}\}. \tag{2.6}$$

The classical *real* reflection groups occur as the special cases $A_{N-1} = S_N = G(1,1,N)$, $BC_N = G(2,1,N)$ and $D_N = G(2,2,N)$.[3]

Extended degenerate affine Hecke algebras

We will overload notation slightly by using $G(m,1,N)$ also to refer to the group algebra of $G(m,1,N)$ over $\mathbb{C}$. Let us define $H_\lambda(m,1,N)$, $\lambda \in \mathbb{C}$, to be the algebra generated by

$$\{\, a,\, d,\, e_1,\, e_2, \ldots,\, e_{N-1}\}, \tag{2.7}$$

obeying (2.1) and the further relations

$$a\, d = d\, a \qquad d\, e_1\, a\, e_1 = e_1\, a\, e_1\, d \qquad e_j\, d = d\, e_j \quad (j > 1) \tag{2.8}$$

$$d\, e_1\, d\, e_1 + \lambda\, d \sum_{s \in \mathbb{Z}_m} a^s\, e_1\, a^{-s} = e_1\, d\, e_1\, d + \lambda \sum_{s \in \mathbb{Z}_m} a^s\, e_1\, a^{-s}\, d\,. \tag{2.9}$$

[3]But note that, in what follows, the models we construct do not reduce, in the BC and D cases, to the standard Sutherland Hamiltonians, à la (1.2), for these groups. We return to this point in section 6.

In addition to the $\mathscr{Q}_i$ and $\mathscr{P}_{ij}$ of (2.2–2.4), it is also useful to introduce $d_1, \ldots, d_N$, defined recursively by

$$d_1 = d \tag{2.10}$$

$$d_{i+1} = \mathscr{P}_{ii+1} d_i \mathscr{P}_{ii+1} + \lambda \sum_{s\in\mathbb{Z}_m} \mathscr{Q}_i^s \mathscr{P}_{ii+1} \mathscr{Q}_i^{-s} \quad (i = 1, \ldots, N-1). \tag{2.11}$$

It follows from (2.8–2.9) that

$$[\, d_i, d_j] = 0, \qquad [\, d_i, \mathscr{Q}_j] = 0 \tag{2.12}$$

$H_\lambda(m, 1, N)$ can be regarded as an affinization of $G(m, 1, N)$, with d in a very loose sense a "lowest root". Indeed, when $m = 1$ the sums above collapse and one recovers the relations $\mathscr{P}_{ii+1} d_i = d_{i+1} P_{ii+1} + \lambda$ of the degenerate affine Hecke algebra, first introduced in [25].

We may define also $H_\lambda(pr, p, N)$, an affinization of $G(pr, p, N)$, to be the subalgebra of $H_\lambda(pr, 1, N)$ generated by $\{\, a^p,\, d,\, a^{-1}\, e_1\, a,\, e_1,\, e_2, \ldots,\, e_{N-1}\}$. Note that the relation (2.9) does not conflict with closure, because $G(pr, p, N)$ does contain all the elements

$$a^{-s}\, e_1\, a^s = \, e_1 \left(e_1\, a^{-1}\, e_1\, a \right)^s . \tag{2.13}$$

Extended degenerate affine Hecke algebra associated with the BC_N reflection groups appeared previously in [33, 39]. They differ from definition of $H_\lambda(2, 1, N)$ here though: in particular, the relation $[d_i, \mathscr{Q}_j] = 0$ does not hold there.

3 Dunkl operators and Hamiltonians

Realization of $H_\lambda(m, 1, N)$

The next stage is to realize these abstract algebraic relations in a concrete physical model. Consider a quantum-mechanical system of N particles on the unit circle. Let $q_i = \exp(ix_i)$ be the position operator of the i^{th} particle and write the position-space wavefunction as

$$\psi(q_1, q_2, \ldots, q_N). \tag{3.1}$$

Let $\mathscr{P}_{ij}$ be the operator which transposes the positions of particles i and j,

$$\mathscr{P}_{ij} \psi(\ldots, q_i, \ldots, q_j, \ldots) = \psi(\ldots, q_j, \ldots, q_i, \ldots)\ , \tag{3.2}$$

and $\mathscr{Q}_i$ the operator which rotates particle i through $(\frac{1}{m})^{th}$ of a revolution,

$$\mathscr{Q}_i \psi(\ldots, q_i, \ldots) = \psi(\ldots, \tau q_i \ldots), \quad \text{where} \quad \tau = \exp\left(\frac{2\pi i}{m}\right) \tag{3.3}$$

It is easy to see that these $\mathscr{P}_{ij}$ and $\mathscr{Q}_i$ satisfy the defining relations (2.5) of $G(m,1,N)$. We also have that

$$\mathscr{Q}_i\; q_i = \tau\; q_i\; \mathscr{Q}_i \qquad \mathscr{Q}_i\; \frac{\partial}{\partial q_i} = \tau^{-1}\; \frac{\partial}{\partial q_i}\; \mathscr{Q}_i \tag{3.4}$$

$$\mathscr{P}_{ij} q_i = q_j \mathscr{P}_{ij} \qquad \mathscr{P}_{ij} \frac{\partial}{\partial q_i} = \frac{\partial}{\partial q_j} \mathscr{P}_{ij}. \tag{3.5}$$

The crucial step is the introduction of differential operators, called Dunkl operators [26, 27, 5], that realize the algebraic relations of the d_i. The problem of finding such operators for complex reflection groups was solved in [21]. Following that paper, with minor modifications that will allow us to obtain a slightly more elegant Hamiltonian, we define

$$d_i = q_i \frac{\partial}{\partial q_i} + \lambda \sum_{j\neq i} \sum_{s\in\mathbb{Z}_m} \frac{q_i}{q_i - \tau^s q_j} \mathscr{Q}_i^{-s} \mathscr{P}_{ij} \mathscr{Q}_i^{s} - \lambda \sum_{j>i} \sum_{s\in\mathbb{Z}_m} \mathscr{Q}_i^{-s} \mathscr{P}_{ij} \mathscr{Q}_i^{s} \tag{3.6}$$

$$= q_i \frac{\partial}{\partial q_i} + \lambda \sum_{j<i} \sum_{s\in\mathbb{Z}_m} \frac{q_i}{q_i - \tau^s q_j} \mathscr{Q}_i^{-s} \mathscr{P}_{ij} \mathscr{Q}_i^{s} + \lambda \sum_{j>i} \sum_{s\in\mathbb{Z}_m} \frac{\tau^s q_j}{q_i - \tau^s q_j} \mathscr{Q}_i^{-s} \mathscr{P}_{ij} \mathscr{Q}_i^{s}. \tag{3.7}$$

Theorem 3.1 *These provide a realisation of $H_\lambda(m,1,N)$.*

This is essentially theorem (3.8) of [21], and the same strategy of proof works here. But when we come to introduce new Dunkl operators for wreath products of dihedral groups, in section 6, it will be useful to have noted the following alternative

Proof. It is easy to verify that (2.11) holds: were it not for the final term on the right of (3.6), the d_i would obey $\mathscr{P}_{ij}\, d_i \mathscr{P}_{ij} = d_j$. The final term involves an ordering of the particles and is responsible for the extra piece in (2.11).

It remains to show that $d_1 = d$ obeys (2.8) and (2.9). The first of these is straightforward to check by direct computation[4]. The second is nothing but the statement that the Dunkl operators commute:

$$[\, d_1,\, d_2] = 0, \tag{3.8}$$

which is really the key property. To prove it, first recall the Dunkl operators of the A_{N-1} case:

$$Z_i = q_i \frac{\partial}{\partial q_i} + m\lambda \sum_{j\neq i} \frac{q_i}{q_i - q_j} \mathscr{P}_{ij} - m\lambda \sum_{j>i} \mathscr{P}_{ij} \tag{3.9}$$

where we have chosen the coupling to be $m\lambda$. Observe that then

$$d_i = \frac{1}{m} \sum_{s\in\mathbb{Z}_m} \mathscr{Q}_i^{-s} Z_i \mathscr{Q}_i^{s}. \tag{3.10}$$

[4]We sketch the arguments, for the more involved case of dihedral groups to be considered in section 6, in an appendix.

This motivates the definition of a family of projectors: we write

$$\mathrm{Ad}(\mathscr{Q})(X) = \mathscr{Q}^{-1} X \mathscr{Q} \tag{3.11}$$

and define

$$\Pi_i^r = \frac{1}{m} \sum_{s\in\mathbb{Z}_m} \tau^{sr} \mathrm{Ad}(\mathscr{Q}_i^s). \tag{3.12}$$

These obey

$$id = \sum_{r\in\mathbb{Z}_m} \Pi^r, \qquad\qquad \Pi^r\Pi^t = \delta^{r,t}\Pi^r. \tag{3.13}$$

Also, for any A and B,

$$\mathscr{Q}\left(\Pi^r A\right)\left(\Pi^t B\right) = \tau^r \left(\Pi^r A\right) \mathscr{Q} \left(\Pi^t B\right) = \tau^{r+t} \left(\Pi^r A\right)\left(\Pi^t B\right)\mathscr{Q}, \tag{3.14}$$

so $\Pi^{r+t}\left(\Pi^r A\right)\left(\Pi^t B\right) = \left(\Pi^r A\right)\left(\Pi^t B\right)$. It follows that

$$\Pi^0 AB = \sum_{r,t\in\mathbb{Z}_m} \Pi^0 \left(\Pi^r A\right)\left(\Pi^t B\right) = \sum_{r\in\mathbb{Z}_m} \left(\Pi^r A\right)\left(\Pi^{-r} B\right). \tag{3.15}$$

Armed with these facts we argue as follows. Given the result [26, 27, 5] that

$$[Z_i, Z_j] = 0 \tag{3.16}$$

we have in particular that

$$0 = \Pi_i^0\Pi_j^0 [Z_i, Z_j] = \sum_{r,t\in\mathbb{Z}_m} \left[\Pi_i^r\Pi_j^t Z_i, \Pi_j^{-t}\Pi_i^{-r} Z_j\right]. \tag{3.17}$$

But one may compute, for all $i \neq j$,

$$\begin{aligned}\Pi_i^r\Pi_j^t Z_i \;\; = \;\; & \delta^{r,0}\delta^{t,0} q_i \frac{\partial}{\partial q_i} + \delta^{t,0}\lambda \sum_{s\in\mathbb{Z}_m} \tau^{rs} \left(\sum_{h\notin\{i,j\}} \frac{q_i}{q_i - \tau^s q_h} \mathscr{Q}_i^{-s}\mathscr{P}_{ih}\mathscr{Q}_i^s - \sum_{h>i, h\neq j} \mathscr{Q}_i^{-s}\mathscr{P}_{ih}\mathscr{Q}_i^s \right) \\ & + \delta^{r+t,0}\lambda \sum_{s\in\mathbb{Z}_m} \tau^{rs} \left(\frac{q_i}{q_i - \tau^s q_j} \mathscr{Q}_i^{-s}\mathscr{P}_{ij}\mathscr{Q}_i^s - \theta^{j>i} \mathscr{Q}_i^{-s}\mathscr{P}_{ij}\mathscr{Q}_i^s \right).\end{aligned} \tag{3.18}$$

The only terms in (3.17) which can survive in view of the δ's here are

$$0 \;\; = \;\; [\Pi_i^0\Pi_j^0 Z_i, \Pi_i^0\Pi_j^0 Z_j] + \sum_{t\in\mathbb{Z}_m, t\neq 0} \left[\Pi_i^t\Pi_j^{-t} Z_i, \Pi_j^t\Pi_i^{-t} Z_j\right]. \tag{3.19}$$

The second term (with $i > j$, without loss of generality) is

$$\begin{aligned} & \sum_{t\in\mathbb{Z}_m, t\neq 0} \left[\sum_{s\in\mathbb{Z}_m} \tau^{ts} \frac{q_i}{q_i - \tau^s q_j} \mathscr{Q}_i^{-s}\mathscr{P}_{ij}\mathscr{Q}_i^s, \sum_{s'\in\mathbb{Z}_m} \tau^{ts'} \left(\frac{q_j}{q_j - \tau^{s'} q_i} - 1 \right) \mathscr{Q}_j^{-s'}\mathscr{P}_{ji}\mathscr{Q}_j^{s'} \right] \\ = \; & - \sum_{t\in\mathbb{Z}_m, t\neq 0} \left[\sum_{s\in\mathbb{Z}_m} \tau^{ts} \frac{q_i}{q_i - \tau^s q_j} \mathscr{Q}_i^{-s}\mathscr{P}_{ij}\mathscr{Q}_i^s, \sum_{s'\in\mathbb{Z}_m} \tau^{-ts'} \frac{q_i}{q_i - \tau^{s'} q_j} \mathscr{Q}_i^{-s'}\mathscr{P}_{ij}\mathscr{Q}_i^{s'} \right],\end{aligned} \tag{3.20}$$

and this vanishes. (In the sum over t, the summands cancel in conjugate pairs, t against $-t$, except for the possible $\tau^t = -1$ term which is zero by itself.) But now since $\Pi_i^0 \Pi_j^0 Z_i = d_i$ by (3.18), we have that indeed

$$0 = [d_i, d_j]\,, \tag{3.21}$$

completing the proof. ■

Hamiltonians

It follows from the discussion above that the quantities

$$I^{(k)} = \sum_{i=1}^{N} d_i^k \tag{3.22}$$

form a commuting set. The $I^{(k)}$ are algebraically independent for $k = 1, 2, \ldots N$, and these give N commuting conserved quantities of the model with Hamiltonian

$$\begin{aligned} H = I^{(2)} &= \sum_{i=1}^{N} \left(q_i \frac{\partial}{\partial q_i} \right)^2 - 2\lambda \sum_{i<j} \sum_{s \in \mathbb{Z}_m} \frac{\tau^s q_i q_j}{(q_i - \tau^s q_j)^2} (\lambda + \mathscr{Q}_i^{-s} \mathscr{P}_{ij} \mathscr{Q}_i^s) & (3.23) \\ &= \sum_{i=1}^{N} \left(q_i \frac{\partial}{\partial q_i} \right)^2 - \lambda \sum_{i \neq j} \sum_{s \in \mathbb{Z}_m} \frac{\tau^s q_i q_j}{(q_i - \tau^s q_j)^2} (\lambda + \mathscr{Q}_i^{-s} \mathscr{P}_{ij} \mathscr{Q}_i^s) & (3.24) \end{aligned}$$

which is therefore, by construction, integrable. After the change of coordinates $q_j = \exp(ix_j)$ the Hamiltonian takes the form

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{\lambda}{4} \sum_{i \neq j} \sum_{s \in \mathbb{Z}_m} \frac{1}{\sin^2\left(\frac{1}{2}\left(x_i - x_j + \frac{2\pi s}{m}\right)\right)} \left(\lambda + \mathscr{Q}_i^{-s} \mathscr{P}_{ij} \mathscr{Q}_i^s\right) \tag{3.25}$$

One sees that each particle x_i interacts with every other particle x_j both directly and via its images under rotations. A sketch of the case $G(3, 1, 2)$ is shown in figure 3. Note that the Hamiltonian is not local, because of the final term which exchanges (and moves) particles. To find local Hamiltonians it is useful to introduce spins, as follows.

4 Particles with spin

We now generalize the models above to particles with internal 'spin' degrees of freedom, $\vec{s} \in \mathbb{C}^n$. Let us introduce a map $Q \in U(n)$ of order m ($Q^m = 1$) and write Q_i for Q acting on the spin $\vec{s}_i$ of the i^{th} particle. Let also P_{ij} be the operator which exchanges the spins of the the i^{th} and j^{th} particles. P_{ij} and Q_i then obey the same defining relations of $G(m, 1, N)$ as $\mathscr{P}_{ij}, \mathscr{Q}_i$ in (2.5). The two copies of $G(m, 1, N)$ commute.

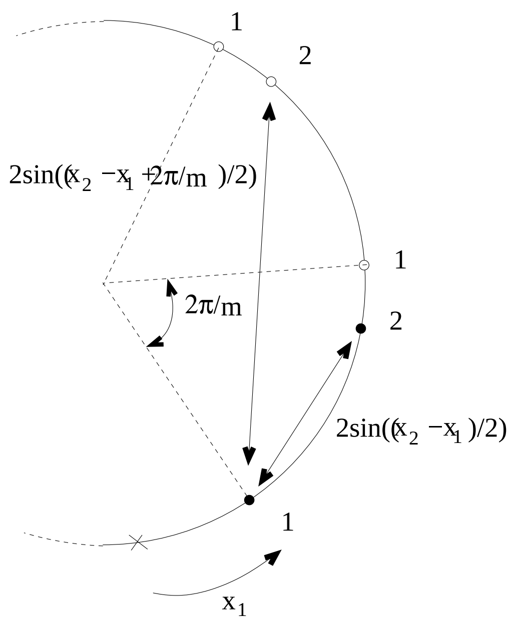

Figure 3: Partial picture of the model for $N = 2$. The particles are represented by full circles whereas their images by rotations of $2\pi/m$ are represented by empty circles.

The introduction of spins typically makes the restriction to the case of identical particles much richer and more interesting. For bosons (fermions) the wavefunction should now be (anti)symmetric under exchange of positions *and* spins – that is, under

$$\mathscr{P}_{ij} P_{ij}. \tag{4.1}$$

These generate the group S_N of exchange symmetries. In the original A_N-series Sutherland models, on wavefunctions with definite exchange statistics it is possible [28, 8] systematically to eliminate $\mathscr{P}_{ij}$ in favour of P_{ij} in the Hamiltonian and higher conserved quantities, and so obtain a purely local model with spin-spin interactions.

We would like to do something similar in the present case. Here the exchange-symmetry group is contained in several larger groups of discrete symmetries (involving the Q_i and $\mathscr{Q}_i$). It is natural to pick one of these as a group of "generalized exchange symmetries" and demand definite behaviour of the wavefunction under it. There are a number of possibilities: one could for example demand that a full copy of $G(m, 1, N)$, generated by e.g. $\mathscr{P}_{ij} P_{ij}$ and $\mathscr{Q}_i Q_i$, be promoted in this sense. But to do so would be overly restrictive on physical wavefunctions; instead, it will suffice to demand invariance under

$$\mathscr{Q}_i^{-s} \mathscr{P}_{ij} \mathscr{Q}_i^{s} \, Q_i^{-s} P_{ij} Q_i^{s} \tag{4.2}$$

for all $i \neq j$ and for all $s = 0, 1, \ldots, m-1$. These generate a copy of $G(m, m, N)$.

(We focus for definiteness on +1-eigenstates of (4.2), but the more general case with arbitrary eigenvalues $p_s \in \{\pm 1\}$ for each value of s – in particular, $p_0 = -1$, giving fermions – can be treated very similarly.)

Let Λ be the projector onto such states. To write Λ explicitly, let $g \mapsto P_g$ and $g \mapsto \mathscr{P}_g$ be the maps representing abstract elements $g \in G(m,m,N)$ on, respectively, spins and positions. Then

$$\Lambda = \frac{1}{N!\, m^{N-1}} \sum_{g \in G(m,m,N)} P_g \mathscr{P}_g \tag{4.3}$$

and we consider wavefunctions such that

$$\psi = \Lambda \psi. \tag{4.4}$$

Define $I^{(k)}_{\text{spin}}$ to be the operator obtained by first moving all the $\mathscr{P}_{ij}$, $\mathscr{Q}_i$ in $I^{(n)}$, as defined in (3.22), to the right of all positions x_i and derivatives $\frac{\partial}{\partial x_i}$, and then replacing them $\mathscr{P}_{ij} \mapsto P_{ij}$, $\mathscr{Q}_i \mapsto Q_i$.[5] It follows from the property

$$\left(\mathscr{Q}_i^{-s} \mathscr{P}_{ij} \mathscr{Q}_i^{s}\right) \Lambda = \left(Q_i^{-s} P_{ij} Q_i^{s}\right) \Lambda, \tag{4.6}$$

of Λ that

$$I^{(k)}_{\text{spin}} \Lambda = I^{(k)} \Lambda \tag{4.7}$$

so that these operators agree on wavefunctions obeying (4.4).

Note next the following properties of the $I^{(k)}$:

$$\left[I^{(k)}, \mathscr{P}_{ij}\right] = 0, \quad \left[I^{(k)}, \mathscr{Q}_i\right] = 0 \tag{4.8}$$

which follow from the definition $I^{(k)} = \sum_{i=1}^N d_i^k$ and the algebra (2.11–2.12) of the Dunkl operators. It is also trivially the case that

$$\left[I^{(k)}, P_{ij}\right] = 0, \quad \left[I^{(k)}, Q_i\right] = 0. \tag{4.9}$$

Consequently, for any monomial $M(\{I^{(k)}\})$ in the $I^{(k)}$, $M(\{I^{(k)}\})\Lambda$ obeys the same relations (4.6) as Λ itself,

$$\left(\mathscr{Q}_i^{-s} \mathscr{P}_{ij} \mathscr{Q}_i^{s}\right) M(\{I^{(k)}\})\Lambda = \left(Q_i^{-s} P_{ij} Q_i^{s}\right) M(\{I^{(k)}\})\Lambda, \tag{4.10}$$

and thus

$$I^{(k)}_{\text{spin}} M(\{I^{(p)}\})\Lambda = I^{(k)} M(\{I^{(p)}\})\Lambda. \tag{4.11}$$

Given now any string of $I^{(k)}_{\text{spin}}$'s, not *a priori* assumed to commute, repeated use of this fact allows one to replace each $I^{(k)}_{\text{spin}}$ by $I^{(k)}$, working from the inside out:

$$\left(\dots I^{(k)}_{\text{spin}} I^{(\ell)}_{\text{spin}}\right)\Lambda = \left(\dots I^{(k)}_{\text{spin}} I^{(\ell)}\right)\Lambda = \left(\dots I^{(k)} I^{(\ell)}\right)\Lambda = \dots. \tag{4.12}$$

[5] The replacement map could also be defined, following [8], as the projection

$$\pi : G(m,1,N)_{\langle \mathscr{P}P, \mathscr{Q}Q^{-1}\rangle} \ltimes G(m,1,N)_{\langle P,Q\rangle} \to G(m,1,N)_{\langle P,Q\rangle}\ ;\ AB \mapsto B. \tag{4.5}$$

Then indeed $\pi \mathscr{P}_{ij} = \pi(\mathscr{P}_{ij} P_{ij}) P_{ij} = P_{ij}$ and $\pi \mathscr{Q}_i = \pi(\mathscr{Q}_i Q_i^{-1}) Q_i = Q_i$.

Having done so, the result

$$\left[I^{(k)}, I^{(\ell)}\right] = 0, \tag{4.13}$$

may be used to reorder the $I^{(k)}$ at will, and the above procedure then reversed to return $I^{(k)} \to I^{(k)}_{\rm spin}$. Thus, in particular, we have that the N independent evolution operators

$$U_k(t) = e^{itI^{(k)}_{\rm spin}}, \quad k = 1, \ldots, N \tag{4.14}$$

commute amongst themselves when acting on physical wavefunctions:

$$U_k(t)U_\ell(t')\Lambda = U_\ell(t')U_k(t)\Lambda. \tag{4.15}$$

Therefore the model described by the Hamiltonian

$$H_{spin} = I^{(2)}_{\rm spin} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{\lambda}{4}\sum_{i\neq j}\sum_{s\in\mathbb{Z}_m} \frac{1}{\sin^2\left(\frac{1}{2}\left(x_i - x_j + \frac{2\pi s}{m}\right)\right)}\left(\lambda + Q_i^{-s}P_{ij}Q_i^s\right) \tag{4.16}$$

is integrable.

5 Static spin chain

In this section, we find static integrable spin models from the new integrable models introduced above. Indeed, it is well-known that from the A_N Sutherland model it is possible to find static spin chains [29, 30], called usually Haldane–Shastry models [31, 32]. Using similar methods, we will prove that the following Hamiltonian

$$\overline{H} = \sum_{i\neq j}\sum_{s\in\mathbb{Z}_m} \frac{\tau^s\ q_iq_j}{(q_i - \tau^s q_j)^2}\ \mathscr{Q}_i^{-s}\mathscr{P}_{ij}\mathscr{Q}_i^s \tag{5.1}$$

is integrable for some particular values of the positions q_i.

First, we introduce the following operators

$$\overline{d}_i \;=\; \sum_{j<i}\sum_{s\in\mathbb{Z}_m} \frac{q_i}{q_i - \tau^s q_j}\ \mathscr{Q}_i^{-s}\mathscr{P}_{ij}\mathscr{Q}_i^s + \sum_{j>i}\sum_{s\in\mathbb{Z}_m} \frac{\tau^s q_j}{q_i - \tau^s q_j}\ \mathscr{Q}_i^{-s}\mathscr{P}_{ij}\mathscr{Q}_i^s \tag{5.2}$$

Note that $d_i = q_i\frac{\partial}{\partial q_i} + \lambda\overline{d}_i$. Since the relation $[d_j, d_k] = 0$ is valid for any λ, we deduce that

$$[\overline{d}_j, \overline{d}_k] = 0\ . \tag{5.3}$$

Similarly, from the relation $[H, d_i] = 0$, it follows that, for any i,

$$[\overline{H}, \overline{d}_i] = \left[q_ip_i\ ,\ \sum_{j\neq\ell}\sum_{s\in\mathbb{Z}_m} \frac{\tau^s\ q_jq_\ell}{(q_l - \tau^s q_j)^2}\right] \tag{5.4}$$

The commutators $[\overline{H}, \overline{d}_i]$ therefore vanish if and only if

$$\sum_{j\neq i}\sum_{s\in\mathbb{Z}_m} \tau^s \frac{q_i q_j (q_i + \tau^s q_j)}{(q_i - \tau^s q_j)^3} = 0 \quad , \forall i = 1, 2, \ldots, N \tag{5.5}$$

and these conditions are fulfilled if

$$q_k = \exp\left(\frac{2ik}{mN}\right) \ . \tag{5.6}$$

Then, the Hamiltonian (5.1) with the particular values of the positions given by (5.6) is integrable and may be written as follows

$$\overline{H} = -\frac{1}{4}\sum_{k\neq\ell}\sum_{s\in\mathbb{Z}_m} \frac{1}{\sin^2\left(\frac{\pi}{mN}(k-\ell-Ns)\right)} \ \mathscr{Q}_k^{-s}\mathscr{P}_{k\ell}\mathscr{Q}_k^s \ . \tag{5.7}$$

By the same procedure as in the section 4, we can obtain an integrable Hamiltonian acting on spins

$$\overline{H}_{spin} = -\frac{1}{4}\sum_{k\neq\ell}\sum_{s\in\mathbb{Z}_m} \frac{1}{\sin^2\left(\frac{\pi}{mN}(k-\ell-Ns)\right)} \ Q_k^{-s}P_{k\ell}Q_k^s \tag{5.8}$$

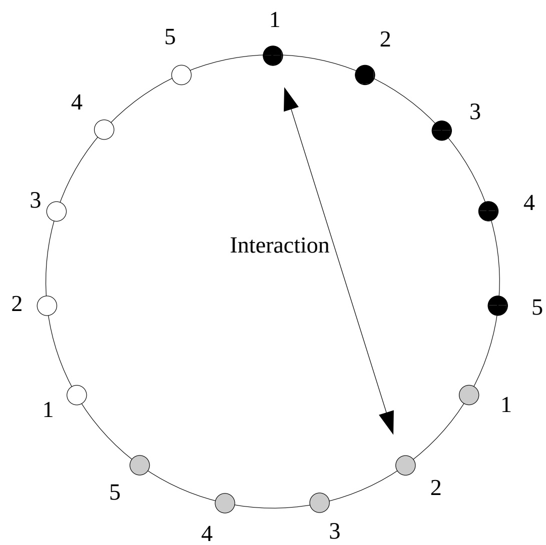

Figure 4: Position of the spins on a circle of diameter 1 for $N = 5$ and $m = 3$.

6 Models with dihedral symmetry

In the previous sections we introduced Sutherland models based on the complex reflection group $G(m, 1, N)$. They reduce to the original Sutherland models, as in (1.1), in the special case $G(1, 1, N) = A_{N-1}$, but for other values of m they are new. In particular, although there is an isomorphism of groups

$$G(2, 1, N) \cong BC_N \cong (\mathbb{Z}_2)^N \rtimes S_N, \tag{6.1}$$

as we noted above, our models certainly do not coincide with the usual BC_N Sutherland models in this case, because the $\mathbb{Z}_2$ generator is realized in different ways. In the $G(m,1,N)$ models, recall,

$$\mathscr{Q}_i\psi(\ldots,q_i,\ldots) = \psi(\ldots,\tau q_i,\ldots), \tag{6.2}$$

where $\tau = e^{2\pi i/m}$, whereas in the BCD_N case the action of the $\mathbb{Z}_2$ generator is [30],

$$\mathscr{K}_i\psi(\ldots,q_i,\ldots) = \psi(\ldots,q_i^{-1},\ldots)\ . \tag{6.3}$$

In this section we show that it is possible to include both types of symmetry, rotation and reflection.

Dunkl operators for wreath products of dihedral groups

To take into account the new operators $\mathscr{K}_i$, we must find the group $W(m,N)$ generated by the $\mathscr{Q}_i$, $\mathscr{K}_i$ and $\mathscr{P}_{ij}$, which will contain as subgroups both D_N and $G(m,1,N)$. First note that $\mathscr{Q}_i$ and $\mathscr{K}_i$ satisfy, for each i,

$$\mathscr{Q}_i^m = 1 \quad , \quad \mathscr{K}_i^2 = 1 \quad \text{and} \quad \mathscr{K}_i\mathscr{Q}_i = (\mathscr{Q}_i)^{-1}\mathscr{K}_i\ , \tag{6.4}$$

which are the defining relations of the dihedral group of order m, denoted Dih_m. We deduce that the group $W(m,N)$ must be the wreath product

$$W(m,N) = \mathrm{Dih}_m \wr S_N = (\mathrm{Dih}_m)^N \rtimes S_N. \tag{6.5}$$

A minimal set of generators for $W(m,N)$ is

$$\{\, a,\, b,\, e_1,\, e_2, \ldots,\, e_{N-1}\}, \tag{6.6}$$

obeying the relations

$$\begin{array}{lllll} & e_i^2 = 1 & e_i\, e_{i+1}\, e_i = \, e_{i+1}\, e_i\, e_{i+1} & e_i\, e_j = \, e_j\, e_i \quad (|i-j|>2) \\ & a^m = 1 & a\, e_1\, a\, e_1 = \, e_1\, a\, e_1\, a & a\, e_j = \, e_j\, a \quad (j>1) \\ b\, a = \, a^{-1}\, b & b^2 = 1 & b\, e_1\, b\, e_1 = \, e_1\, b\, e_1\, b & b\, e_j = \, e_j\, b \quad (j>1) \end{array} \tag{6.7}$$

and in terms of these $\mathscr{P}_{ij}$ and $\mathscr{Q}_i$ are again defined as in (2.2) and (2.4) while

$$\mathscr{K}_1 \;=\; b \tag{6.8}$$

$$\mathscr{K}_i \;=\; \mathscr{P}_{i1}\mathscr{K}_1\mathscr{P}_{i1} \quad (i>1). \tag{6.9}$$

To construct integrable models, we must extend this algebra as explained in the previous sections. Define $H_{\lambda,\mu}(W(m,N))$, $\lambda,\mu \in \mathbb{C}$, to be the algebra generated by $\{\, D,\, b,\, a,\, e_1,\, e_2, \ldots,\, e_{N-1}\}$, obeying (6.7) and the further relations

$$b\, D = -\, D\, b + \mu \sum_{s\in\mathbb{Z}_m} a^{2s} \qquad , \qquad a\, D = \, D\, a \tag{6.10}$$

$$\left(D+\lambda\sum_{s\in\mathbb{Z}_m} a^{-s}\,e_1\,a^s\right)e_1\,b\,e_1 = e_1\,b\,e_1\left(D+\lambda\sum_{s\in\mathbb{Z}_m} a^{-s}\,e_1\,a^s\right) \quad , \qquad D\,e_1\,a\,e_1 = e_1\,a\,e_1\,D \tag{6.11}$$

$$D\left(e_1\,D\,e_1+\lambda\sum_{s\in\mathbb{Z}_m} a^{s}\,e_1\,a^{-s}\right) = \left(e_1\,D\,e_1+\lambda\sum_{s\in\mathbb{Z}_m} a^{s}\,e_1\,a^{-s}\right)D \quad , \qquad e_j\,D = D\,e_j \quad (j>1). \tag{6.12}$$

Defining as before

$$\begin{aligned} D_1 &= D \\ D_{i+1} &= \mathscr{P}_{ii+1}\,D_i\,\mathscr{P}_{ii+1}+\lambda\sum_{s\in\mathbb{Z}_m}\mathscr{Q}_i^s\,\mathscr{P}_{ii+1}\,\mathscr{Q}_i^{-s} \quad (i=1,\ldots,N-1) \end{aligned} \tag{6.13}$$

it follows from (6.10-6.12) that

$$[\,D_i,\,D_j]=0, \qquad [\,D_i,\mathscr{Q}_j]=0. \tag{6.14}$$

The hard step, just in the case of $G(m,1,N)$, is to find a concrete realization of the D_i satisfying these abstract relations.

Theorem 6.1 *For any $\rho\in\mathbb{C}$, the differential operators*

$$\begin{aligned} D_i &= q_i\frac{\partial}{\partial q_i}+\lambda\sum_{j\neq i}\sum_{s\in\mathbb{Z}_m}\left(\frac{q_i}{q_i-\tau^s q_j}\mathscr{Q}_i^{-s}\mathscr{P}_{ij}\mathscr{Q}_i^{s}+\frac{q_i}{q_i-\tau^{-s}q_j^{-1}}\mathscr{K}_i\mathscr{Q}_i^{-s}\mathscr{P}_{ij}\mathscr{Q}_i^{s}\mathscr{K}_i\right) \\ &\quad+\sum_{s\in\mathbb{Z}_m}\frac{\mu\tau^s q_i-\rho}{\tau^s q_i-\tau^{-s}q_i^{-1}}\mathscr{Q}_i^{2s}\,\mathscr{K}_i \\ &\quad-\lambda\sum_{j>i}\sum_{s\in\mathbb{Z}_m}\mathscr{Q}_i^{-s}\mathscr{P}_{ij}\mathscr{Q}_i^{s} \end{aligned} \tag{6.15}$$

obey (6.10-6.12) and (6.13).

We have verified this by direct computation, which is conceptually straightforward though somewhat laborious. It is also possible to adapt the proof we used in the $G(m,1,N)$ case above, as follows.

Proof. Some details of the verification of (6.10) are given in an appendix, but as in the $G(m,1,N)$ case the important and difficult step is to show that

$$[\,D_i,\,D_j]=0. \tag{6.16}$$

The Dunkl operators of the BC_N case [33] may be written

$$\begin{aligned} Y_i &= q_i\frac{\partial}{\partial q_i}+m\lambda\sum_{j\neq i}\left(\frac{q_i}{q_i-q_j}\mathscr{P}_{ij}+\frac{q_i}{q_i-q_j^{-1}}\mathscr{K}_i\mathscr{P}_{ij}\mathscr{K}_i\right) \\ &\quad+\frac{\mu q_i-\rho}{q_i-q_i^{-1}}\mathscr{K}_i-m\lambda\sum_{j>i}\mathscr{P}_{ij}. \end{aligned} \tag{6.17}$$

It follows from the result

$$[Y_i, Y_j] = 0 \tag{6.18}$$

that, with the projectors Π_i^r as defined in (3.12),

$$0 = \Pi_i^0 \Pi_j^0 [Y_i, Y_j] = \sum_{r,t \in \mathbb{Z}_m} \left[\Pi_i^r \Pi_j^t Y_i, \Pi_i^{-r} \Pi_j^{-t} Y_j \right]. \tag{6.19}$$

Now one can compute, for $i \neq j$,

$$\begin{aligned} \Pi_i^r \Pi_j^t Y_i \;=\; & \delta^{r,0}\delta^{t,0} q_i \frac{\partial}{\partial q_i} + \delta^{t,0}\lambda \sum_{s\in\mathbb{Z}_m} \tau^{rs} \left(\sum_{h \notin \{i,j\}} \frac{q_i}{q_i - \tau^s q_h} \mathscr{Q}_i^{-s} \mathscr{P}_{ih} \mathscr{Q}_i^s - \sum_{h>i, h\neq j} \mathscr{Q}_i^{-s} \mathscr{P}_{ih} \mathscr{Q}_i^s \right) \\ & + \delta^{t,0}\lambda \sum_{s\in\mathbb{Z}_m} \tau^{-rs} \left(\sum_{h\notin\{i,j\}} \frac{q_i}{q_i - \tau^{-s} q_h^{-1}} \mathscr{K}_i \mathscr{Q}_i^{-s} \mathscr{P}_{ih} \mathscr{Q}_i^s \mathscr{K}_i + \frac{\mu\tau^s q_i - \rho}{\tau^s q_i - \tau^{-s} q_i^{-1}} \mathscr{Q}^{2s} \mathscr{K}_i \right) \\ & + \delta^{r+t,0}\lambda \sum_{s\in\mathbb{Z}_m} \tau^{rs} \left(\frac{q_i}{q_i - \tau^s q_j} \mathscr{Q}_i^{-s} \mathscr{P}_{ij} \mathscr{Q}_i^s + \frac{q_i}{q_i - \tau^{-s} q_j^{-1}} \mathscr{K}_i \mathscr{Q}_i^{-s} \mathscr{P}_{ij} \mathscr{Q}_i^s \mathscr{K}_i - \theta^{j>i} \mathscr{Q}_i^{-s} \mathscr{P}_{ij} \mathscr{Q}_i^s \right). \end{aligned} \tag{6.20}$$

The δ's mean that the only terms that can possibly survive in (6.19) are

$$0 \;=\; \left[\Pi_i^0 \Pi_j^0 Y_i, \Pi_i^0 \Pi_j^0 Y_j \right] + \sum_{t \in \mathbb{Z}_m, t\neq 0} \left[\Pi_i^t \Pi_j^{-t} Y_i, \Pi_i^{-t} \Pi_j^t Y_j \right]. \tag{6.21}$$

As before, the second term vanishes on closer inspection, and since $\Pi_i^0 \Pi_j^0 Y_i = D_i$ by (6.20), we have established (6.16) as required. ■

Integrable models

Setting $\beta = (\mu+\rho)/2$ and $\gamma = (\mu-\rho)/2$, we may rewrite the Dunkl operators (6.15) as

$$D_\ell = d_\ell + \lambda \sum_{k\neq\ell} \sum_{s=0}^{m-1} \frac{\tau^s q_\ell \; q_k}{\tau^s q_\ell \; q_k - 1} \mathscr{K}_\ell \mathscr{Q}_\ell^{-s} \mathscr{P}_{\ell k} \mathscr{Q}_\ell^s \mathscr{K}_\ell + \sum_{s=0}^{m-1} \left(\frac{\beta \tau^s q_\ell}{\tau^s q_\ell + 1} + \frac{\gamma \tau^s q_\ell}{\tau^s q_\ell - 1} \right) \mathscr{Q}_\ell^{2s} \; \mathscr{K}_\ell \tag{6.22}$$

This equivalent form is useful for computing Hamiltonians. As explained in the previous sections, we know that the model described by the Hamiltonian $\mathcal{H} = \sum_{\ell=1}^N D_\ell^2$ is integrable because it is one of a set of N independent mutually-commuting conserved quantities, namely

$$J^{(k)} = \sum_{\ell=1}^N \; D_\ell^k \; . \tag{6.23}$$

The explicit form of the Hamiltonian depends on the parity of m: for m odd,

$$\begin{aligned} \mathcal{H}^{odd} \;=\; & H - \lambda \sum_{k\neq\ell} \sum_{s\in\mathbb{Z}_m} \frac{\tau^s q_\ell q_k}{(\tau^s q_\ell q_k - 1)^2} (\lambda + \mathscr{K}_\ell \mathscr{Q}_\ell^{-s} \mathscr{P}_{\ell k} \mathscr{Q}_\ell^s \mathscr{K}_\ell) \\ & + \sum_\ell \sum_{s\in\mathbb{Z}_m} \left(\frac{\beta\tau^s q_\ell}{(1+\tau^s q_\ell)^2} (\beta + \mathscr{Q}_\ell^{2s} \mathscr{K}_\ell) - \frac{\gamma\tau^s q_\ell}{(1-\tau^s q_\ell)^2} (\gamma + \mathscr{Q}_\ell^{2s} \mathscr{K}_\ell) \right) \end{aligned} \tag{6.24}$$

– where H is given by (3.23) – while for m even

$$\begin{aligned}\mathcal{H}^{even} &= H - \lambda \sum_{k\neq\ell}\sum_{s\in\mathbb{Z}_m} \frac{\tau^s q_\ell q_k}{(\tau^s q_\ell q_k - 1)^2}(\lambda + \mathscr{K}_\ell \mathscr{Q}_\ell^{-s}\mathscr{P}_{\ell k}\mathscr{Q}_\ell^s \mathscr{K}_\ell) \\ &\quad -\mu \sum_\ell \sum_{s\in\mathbb{Z}_m} \frac{\tau^s q_\ell}{(1-\tau^s q_\ell)^2}(\mu + \mathscr{Q}_\ell^{2s}\mathscr{K}_\ell). \end{aligned} \tag{6.25}$$

In the case $\beta = \gamma$ (i.e. $\rho = 0$), the boundary term in the Hamiltonian, for m odd, can be simplified and becomes

$$-\sum_\ell \sum_{s\in\mathbb{Z}_{2m}} \frac{\beta\sqrt{\tau^s} q_\ell}{(1-\sqrt{\tau^s}q_\ell)^2}(\beta + \mathscr{Q}_\ell^s \mathscr{K}_\ell) \tag{6.26}$$

After the change of coordinates $q_\ell = \exp(ix_\ell)$, the Hamiltonians are

$$\mathcal{H}^{odd} = H + \frac{\lambda}{4}\sum_{k\neq\ell}\sum_{s\in\mathbb{Z}_m} \frac{1}{\sin^2\left(\frac{1}{2}\left(x_\ell + x_k + \frac{2\pi s}{m}\right)\right)}(\lambda + \mathscr{K}_\ell \mathscr{Q}_\ell^{-s}\mathscr{P}_{\ell k}\mathscr{Q}_\ell^s \mathscr{K}_\ell) \tag{6.27}$$

$$+ \sum_\ell \sum_{s\in\mathbb{Z}_m}\left(\frac{\beta/4}{\cos^2\left(\frac{1}{2}\left(x_\ell + \frac{2\pi s}{m}\right)\right)}(\beta + \mathscr{Q}_\ell^{2s}\mathscr{K}_\ell) + \frac{\gamma/4}{\sin^2\left(\frac{1}{2}\left(x_\ell + \frac{2\pi s}{m}\right)\right)}(\gamma + \mathscr{Q}_\ell^{2s}\mathscr{K}_\ell)\right)$$

$$\begin{aligned}\mathcal{H}^{even} &= H + \frac{\lambda}{4}\sum_{k\neq\ell}\sum_{s\in\mathbb{Z}_m} \frac{1}{\sin^2\left(\frac{1}{2}\left(x_\ell + x_k + \frac{2\pi s}{m}\right)\right)}(\lambda + \mathscr{K}_\ell \mathscr{Q}_\ell^{-s}\mathscr{P}_{\ell k}\mathscr{Q}_\ell^s \mathscr{K}_\ell) \\ &\quad + \frac{\mu}{4}\sum_\ell \sum_{s\in\mathbb{Z}_m} \frac{1}{\sin^2\left(\frac{1}{2}\left(x_\ell + \frac{2\pi s}{m}\right)\right)}(\mu + \mathscr{Q}_\ell^{2s}\mathscr{K}_\ell) \end{aligned} \tag{6.28}$$

where now H is given by (3.25).

Models on spins

As explained in the section 4, it is possible to construct models acting on spins using the suitable projectors on the wavefunctions. In addition to the map Q introduced at the beginning of the section 4, we introduce now $K \in U(n)$ such that

$$K^2 = 1 \qquad\qquad KQ = Q^{-1}K\ . \tag{6.29}$$

Such matrices certainly exist: for example

$$Q = \mathrm{diag}(\tau^{a_1}, \ldots, \tau^{a_n}) \quad \text{with} \quad a_i = -a_{n+1-i} \quad \text{and} \quad K = \mathrm{antidiag}(1, \ldots, 1)\ . \tag{6.30}$$

In addition to the condtions (4.2) on the wavefunctions, we demand that the physical wave functions be invariant under

$$\mathscr{Q}_i^{2s}\mathscr{K}_i\ Q_i^{2s}K_i \qquad\qquad \forall i, s \tag{6.31}$$

The explicit form of the projector is the product $\Lambda\Lambda_b$ where Λ is defined by (4.3) and

$$\Lambda_b = \frac{1}{(2s)^N}\prod_j \left(\sum_{s\in\mathbb{Z}_m} \mathscr{Q}_j^{2s} Q_j^{2s}\right)(1 + \mathscr{K}_j K_j)\ . \tag{6.32}$$

At this point, a supplementary difficulty appears in comparison to the previous case (the same problem appears in the usual BC_N case in comparison to the A_N case) because we get $[J^{(k)},\mathscr{Q}_i]=0$ and $[J^{(k)},\mathscr{P}_{ij}]=0$ but

$$[J^{(k)},\mathscr{K}_i]\neq 0. \tag{6.33}$$

Fortunately, we can show that this commutator vanishes when k is even and, in particular, for $k=2$ which corresponds to the Hamiltonian. Up to this restriction, we can use the same procedure to the section 4 with the projector $\Lambda\Lambda_b$ and deduce that the dynamical spin model described by the Hamiltonian, for m even,

$$\begin{aligned}\mathcal{H}^{even}_{spin} &= H_{spin}+\frac{\lambda}{4}\sum_{k\neq\ell}\sum_{s\in\mathbb{Z}_m}\frac{1}{\sin^2\left(\frac{1}{2}\left(x_\ell+x_k+\frac{2\pi s}{m}\right)\right)}(\lambda+K_\ell Q_\ell^{-s}P_{\ell k}Q_\ell^s K_\ell)\\ &\quad+\frac{\mu}{4}\sum_{\ell}\sum_{s\in\mathbb{Z}_m}\frac{1}{\sin^2\left(\frac{1}{2}\left(x_\ell+\frac{2\pi s}{m}\right)\right)}(\mu+Q_\ell^{2s}K_\ell)\end{aligned} \tag{6.34}$$

or, for m odd,

$$\begin{aligned}\mathcal{H}^{odd}_{spin} &= H_{spin}+\frac{\lambda}{4}\sum_{k\neq\ell}\sum_{s\in\mathbb{Z}_m}\frac{1}{\sin^2\left(\frac{1}{2}\left(x_\ell+x_k+\frac{2\pi s}{m}\right)\right)}(\lambda+K_\ell Q_\ell^{-s}P_{\ell k}Q_\ell^s K_\ell) \qquad (6.35)\\ &\quad+\sum_{\ell}\sum_{s\in\mathbb{Z}_m}\left(\frac{\beta/4}{\cos^2\left(\frac{1}{2}\left(x_\ell+\frac{2\pi s}{m}\right)\right)}(\beta+Q_\ell^{2s}K_\ell)+\frac{\gamma/4}{\sin^2\left(\frac{1}{2}\left(x_\ell+\frac{2\pi s}{m}\right)\right)}(\gamma+Q_\ell^{2s}K_\ell)\right)\end{aligned}$$

is integrable.

Spin chain

As explained in the section 5, it is possible find an integrable static spin chain from a dynamical one. Using this procedure[6], we can prove that the following Hamiltonian, for m odd,

$$\begin{aligned}\overline{\mathcal{H}}^{odd} &= \sum_{k\neq\ell}\sum_{s\in\mathbb{Z}_m}\left(\frac{\tau^s q_\ell q_k}{(q_k-\tau^s q_\ell)^2}\mathscr{Q}_\ell^{-s}\mathscr{P}_{\ell k}\mathscr{Q}_\ell^s+\frac{\tau^s q_\ell q_k}{(\tau^s q_\ell q_k-1)^2}\mathscr{K}_\ell\mathscr{Q}_\ell^{-s}\mathscr{P}_{\ell k}\mathscr{Q}_\ell^s\mathscr{K}_\ell\right)\\ &\quad+\sum_{\ell}\sum_{s\in\mathbb{Z}_m}\left(\frac{\gamma\tau^s q_\ell}{(1-\tau^s q_\ell)^2}-\frac{\beta\tau^s q_\ell}{(1+\tau^s q_\ell)^2}\right)\mathscr{Q}_\ell^{2s}\mathscr{K}_\ell\end{aligned} \tag{6.36}$$

is integrable if, for $\ell=1,\dots,N$,

$$\sum_{s\in\mathbb{Z}_m}\tau^s\left(2\sum_{j\neq\ell}\left(\frac{q_j(q_\ell+\tau^s q_j)}{(q_\ell-\tau^s q_j)^3}+\frac{q_j(\tau^s q_\ell q_j+1)}{(\tau^s q_\ell q_j-1)^3}\right)+\beta^2\frac{1-\tau^s q_\ell}{(1+\tau^s q_\ell)^3}-\gamma^2\frac{1+\tau^s q_\ell}{(1-\tau^s q_\ell)^3}\right)=0 \tag{6.37}$$

Similarly, for m even, we prove that the Hamiltonian

$$\begin{aligned}\overline{\mathcal{H}}^{even} &= \sum_{k\neq\ell}\sum_{s\in\mathbb{Z}_m}\left(\frac{\tau^s q_\ell q_k}{(q_k-\tau^s q_\ell)^2}\mathscr{Q}_\ell^{-s}\mathscr{P}_{\ell k}\mathscr{Q}_\ell^s+\frac{\tau^s q_\ell q_k}{(\tau^s q_\ell q_k-1)^2}\mathscr{K}_\ell\mathscr{Q}_\ell^{-s}\mathscr{P}_{\ell k}\mathscr{Q}_\ell^s\mathscr{K}_\ell\right)\\ &\quad+\sum_{\ell}\sum_{s\in\mathbb{Z}_m}\frac{\mu\tau^s q_\ell}{(1-\tau^s q_\ell)^2}\mathscr{Q}_\ell^{2s}\mathscr{K}_\ell\end{aligned} \tag{6.38}$$

[6]It is convenient to rescale the coupling constants of the boundary: $\beta\to\lambda\beta$ and $\gamma\to\lambda\gamma$

is integrable if, for $\ell = 1, \dots, N$,

$$\sum_{s\in\mathbb{Z}_m} \tau^s \left(2 \sum_{j\neq\ell} \left(\frac{q_j(q_\ell + \tau^s q_j)}{(q_\ell - \tau^s q_j)^3} + \frac{q_j(\tau^s q_\ell q_j + 1)}{(\tau^s q_\ell q_j - 1)^3} \right) - \mu^2 \frac{1+\tau^s q_\ell}{(1-\tau^s q_\ell)^3} \right) = 0 \tag{6.39}$$

As discussed above, we can now replaced in the Hamiltonians $\overline{\mathcal{H}}^{odd}$ and $\overline{\mathcal{H}}^{even}$ the operators acting on the positions by the operators acting on spins, while preserved integrability. We finish this section by discussing the different solutions of relations (6.37) and (6.39) in which the positions are equidistant. The solutions depend on the value of the coupling constant β and γ (or μ and ρ). Let us define L to be the number of sites – which may differ from N, the number of spins – and let $\omega_L = e^{2i\pi/L}$. Different possible distributions of the coordinates q_i, for m odd, are given in the following table:

L	β^2	γ^2	q_k	Figure
$2Nm$	$\frac{1}{4}$	$\frac{1}{4}$	$\omega_L^{k-\frac{1}{2}}$	5
$2Nm+m$	$\frac{9}{4}$	$\frac{1}{4}$	$\omega_L^{k-\frac{1}{2}}$	6
	$\frac{1}{4}$	$\frac{9}{4}$	ω_L^{k}	7
$2(N+1)m$	$\frac{9}{4}$	$\frac{9}{4}$	ω_L^{k}	8

The number in the column *Figure* corresponds to the labels of the figures below where the particular case $m = 3$ is taken. In these figures, the black circles represent the positions of the original spins whereas the white circles represent the images of these spins. The grey circles are empty sites. Of course, we can recover the usual BC_N cases studied in [30] when we put $m = 1$ in the previous table. The case when m is even seems more complicated: we found no solution for relation (6.39).

7 Conclusions and outlook

In this paper we considered two families of Sutherland models, in which each particle possesses a set of images determined by a cyclic or dihedral group. The former we were led to by the desire to find models in which complex reflection groups act as symmetries; in the latter, the role of the reflection group is played by a wreath product of a dihedral group. The Dunkl operators were the key ingredient in demonstrating integrability. In the cyclic case these had been found in [21], but in the dihedral case they have not appeared previously, to the authors' knowledge.

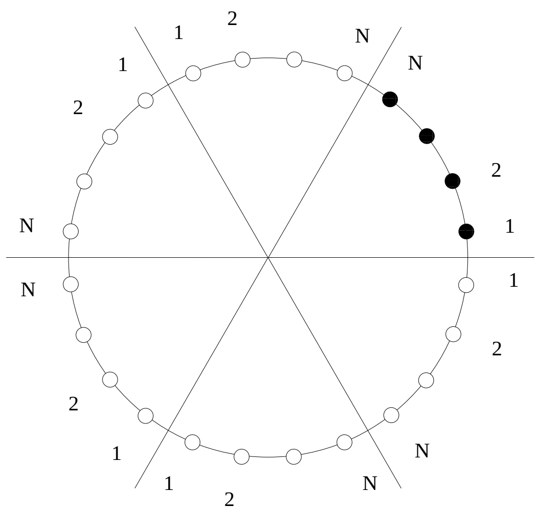

Figure 5: Position of the sites for $m = 3$ and $L = 2Nm$

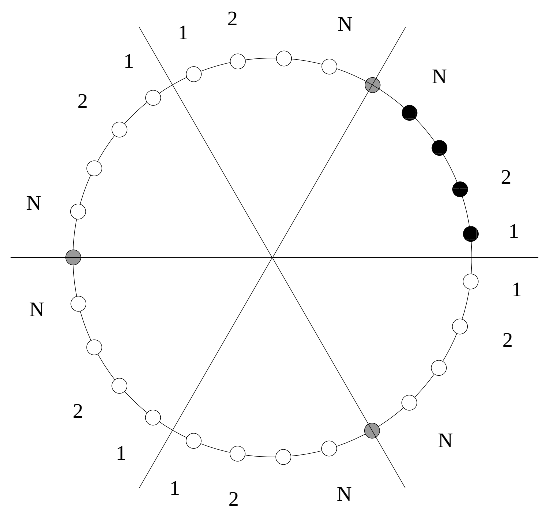

Figure 6: Position of the sites for $m = 3$, $L = 2Nm + m$ and $q_k = \omega_L^{k-\frac{1}{2}}$

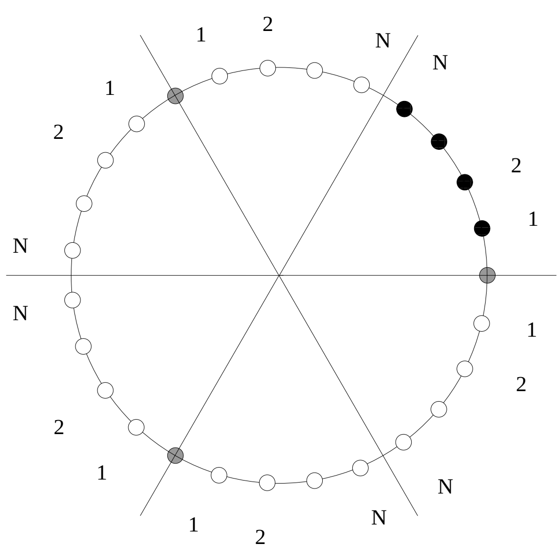

Figure 7: Position of the sites for $m = 3$, $L = 2Nm + m$ and $q_k = \omega_L^k$

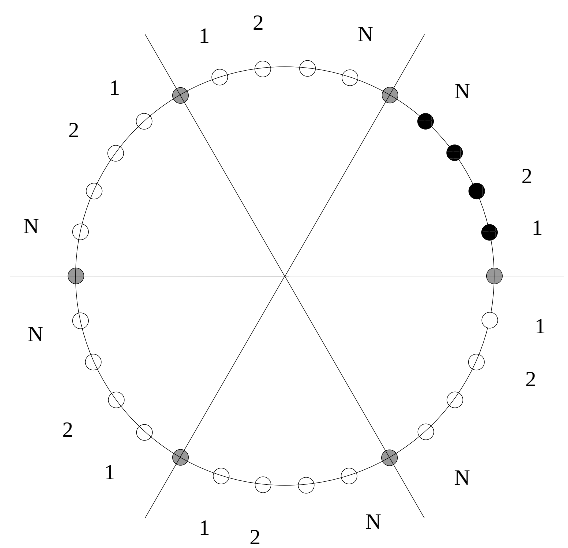

Figure 8: Position of the sites for $m = 3$ and $L = 2(N+1)m$.

We sought to emphasise the link between the models and complex reflection groups. In the cyclic cases the models themselves are a special case of systems previously obtained by appropriate reduction of a matrix model [22] and of rational spin-Calogero models [23]; in the latter case the equivalence may be seen by re-writing the $\sin^{-2}$ potential as an infinite sum of inverse squares. (Further generalizations of these models involving a “twisted” symmetry element were found in [24].) In principle our models with dihedral symmetry could also be obtained by reductions of rational models involving parity in addition to translation symmetry, though this has not been done explicitly.

There are a number of interesting open questions concerning these models. First, one should be able to solve for the energy eigenstates exactly. This could be achieved by simultaneously diagonalizing the Dunkl operators by means of (suitably generalized) Jack polynomials – see e.g. [6]. One also strongly expects, looking at figure 2, that it should be possible to obtain all the models here from the standard A-series Sutherland model via a suitable reduction procedure [9, 11, 14], just as is true of the BC_N case. This is usually related to folding of Dynkin diagrams (see e.g. [16]); here we expect broader notion of folding will come into play, and the projectors used in our proof of commutation the Dunkl operators (theorem 3.1) seem rather suggestive.

We stress however that these models are of interest in their own right, regardless of their origin via reduction. In particular they should possess some extended symmetry algebra, analogous to the Yangian and reflection-algebra symmetries of, respectively, the A and BC Sutherland models, but respecting the underlying complex reflection group. It is worth remarking here on intriguing hints in the mathematics literature that, the lack of root systems and so on notwithstanding, certain complex reflection groups are actually closely analogous to real crystallographic ones (i.e. Weyl groups) with – very loosely speaking – the role of Lie algebras being played by objects called “spetses” [35]. These remain somewhat mysterious, and one can speculate that the machinery of integrable models (Hopf algebras, R-matrices and so on) might provide a helpful new perspective, as it has in the past for Lie algebras and their representation theory.

Finally let us note a few more open questions. Do there exist Sutherland models for the exceptional complex reflection groups, perhaps via reduction as in [36]? The Sutherland model is the trigonometric member of the Calogero-Moser family: can one generalize to elliptic potentials? What models, presumably conformal field theories [37], are obtained in the limit of large N?

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A Properties of Dunkl operators

Here we give some details of the argument that $D_1 = D$ obeys (6.10-6.12). Consider the relation in (6.10) involving $b = \mathscr{K}_1$. For the terms at order λ, one finds first that

$$\begin{aligned}
&\mathscr{K}_1 \sum_{j\neq 1} \sum_{s\in\mathbb{Z}_m} \left(\frac{q_1}{q_1 - \tau^s q_j} \mathscr{Q}_1^{-s} \mathscr{P}_{1j} \mathscr{Q}_1^s + \frac{q_1}{q_1 - \tau^{-s} q_j^{-1}} \mathscr{K}_1 \mathscr{Q}_1^{-s} \mathscr{P}_{1j} \mathscr{Q}_1^s \mathscr{K}_1 \right) \qquad \text{(A.1)}\\
&+ \sum_{j\neq 1} \sum_{s\in\mathbb{Z}_m} \left(\frac{q_1}{q_1 - \tau^s q_j} \mathscr{Q}_1^{-s} \mathscr{P}_{1j} \mathscr{Q}_1^s + \frac{q_1}{q_1 - \tau^{-s} q_j^{-1}} \mathscr{K}_1 \mathscr{Q}_1^{-s} \mathscr{P}_{1j} \mathscr{Q}_1^s \mathscr{K}_1 \right) \mathscr{K}_1\\
= &\sum_{j\neq 1} \sum_{s\in\mathbb{Z}_m} \left(\frac{q_1^{-1}}{q_1^{-1} - \tau^s q_j} + \frac{q_1}{q_1 - \tau^{-s} q_j^{-1}} \right) \mathscr{K}_1 \mathscr{Q}_1^{-s} \mathscr{P}_{1j} \mathscr{Q}_1^s + \left(\frac{q_1^{-1}}{q_1^{-1} - \tau^{-s} q_j^{-1}} + \frac{q_1}{q_1 - \tau^s q_j} \right) \mathscr{Q}_1^{-s} \mathscr{P}_{1j} \mathscr{Q}_1^s \mathscr{K}_1\\
= &\sum_{j\neq 1} \sum_{s\in\mathbb{Z}_m} \mathscr{K}_1 \mathscr{Q}_1^{-s} \mathscr{P}_{1j} \mathscr{Q}_1^s + \mathscr{Q}_1^{-s} \mathscr{P}_{1j} \mathscr{Q}_1^s \mathscr{K}_1
\end{aligned}$$

which then is precisely cancelled by the contribution from the other order-λ piece in (6.15). Note of course that the algebra of $\mathscr{K}_i$ with q_i, implicit the action (6.3) of $\mathscr{K}_i$ on wavefunctions, is $\mathscr{K}_i \; q_i = q_i^{-1} \; \mathscr{K}_i$ and $\mathscr{K}_i \frac{\partial}{\partial q_i} = \frac{\partial}{\partial q_i^{-1}} \; \mathscr{K}_i$. Since also

$$\mathscr{K}_1 q_1 \frac{\partial}{\partial q_1} = -q_1 \frac{\partial}{\partial q_1} \mathscr{K}_i \qquad \text{(A.2)}$$

(because $q_i^{-1} = q_i^{-1}$) we have that $\mathscr{K}_1 \, D_1 = - D_1 \mathscr{K}_1$ up to the terms involving μ and ρ. It is straightforward to verify that these give

$$\mathscr{K}_1 \, D_1 = - D_1 \mathscr{K}_1 + \mu \sum_{s\in\mathbb{Z}_m} \mathscr{Q}_1^{2s} \qquad \text{(A.3)}$$

as claimed. The other relation in (6.10), $\mathscr{Q}_1 \, D_1 = D_1 \mathscr{Q}_1$ is almost immediate.

Next consider (6.11): $\mathscr{K}_2 = e_1 \, b \, e_1$ commutes term by term with the right hand side of

$$\begin{aligned}
D_1 + \lambda \sum_{s\in\mathbb{Z}_m} a^{-s} \, e_1 \, a^s \quad = \quad & q_1 \frac{\partial}{\partial q_1} + \lambda \sum_{j\neq 1} \sum_{s\in\mathbb{Z}_m} \left(\frac{q_1}{q_1 - \tau^s q_j} \mathscr{Q}_1^{-s} \mathscr{P}_{1j} \mathscr{Q}_1^s + \frac{q_1}{q_1 - \tau^{-s} q_j^{-1}} \mathscr{K}_1 \mathscr{Q}_1^{-s} \mathscr{P}_{1j} \mathscr{Q}_1^s \mathscr{K}_1 \right)\\
& + \sum_{s\in\mathbb{Z}_m} \frac{\mu \tau^s q_i - \rho}{\tau^s q_i - \tau^{-s} q_i^{-1}} \mathscr{Q}_1^{2s} \; \mathscr{K}_1\\
& + \lambda \sum_{j>2} \sum_{s\in\mathbb{Z}_m} \mathscr{Q}_i^{-s} \mathscr{P}_{ij} \mathscr{Q}_i^s \qquad \text{(A.4)}
\end{aligned}$$

except when $j = 2$ in the sum: but there

$$\begin{aligned}
&\mathscr{K}_2 \sum_{s\in\mathbb{Z}_m} \left(\frac{q_1}{q_1 - \tau^s q_2} \mathscr{Q}_1^{-s} \mathscr{P}_{12} \mathscr{Q}_1^s + \frac{q_1}{q_1 - \tau^{-s} q_2^{-1}} \mathscr{K}_1 \mathscr{Q}_1^{-s} \mathscr{P}_{12} \mathscr{Q}_1^s \mathscr{K}_1 \right)\\
= &\sum_{s\in\mathbb{Z}_m} \left(\frac{q_1}{q_1 - \tau^s q_2^{-1}} \mathscr{K}_2 \mathscr{Q}_1^{-s} \mathscr{P}_{12} \mathscr{Q}_1^s \mathscr{K}_2 + \frac{q_1}{q_1 - \tau^{-s} q_2} \mathscr{K}_2 \mathscr{K}_1 \mathscr{Q}_1^{-s} \mathscr{P}_{12} \mathscr{Q}_1^s \mathscr{K}_1 \mathscr{K}_2 \right) \mathscr{K}_2\\
= &\sum_{s\in\mathbb{Z}_m} \left(\frac{q_1}{q_1 - \tau^s q_2^{-1}} \mathscr{K}_1 \mathscr{Q}_1^{s} \mathscr{P}_{12} \mathscr{Q}_1^{-s} \mathscr{K}_1 + \frac{q_1}{q_1 - \tau^{-s} q_2} \mathscr{Q}_1^{s} \mathscr{P}_{12} \mathscr{Q}_1^{-s} \right) \mathscr{K}_2
\end{aligned}$$

and, on renaming $s \to -s$, one sees that the two terms are merely exchanged. Thus indeed $\mathscr{K}_2$ commutes with $D + \lambda \sum_{s\in\mathbb{Z}_m} a^{-s}\, e_1\, a^s$. To show that $\mathscr{Q}_2\, D_1 = D_1 \mathscr{Q}_2$ is straightforward.

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