On the W-action on B-sheets in positive characteristic

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Abstract. Let G be a connected reductive group defined over an algebraically closed base field of characteristic $p \geq 0$, let $B \subseteq G$ be a Borel subgroup, and let X be a G-variety. We denote the (finite) set of closed B-invariant irreducible subvarieties of X that are of maximal complexity by $\mathfrak{B}_0(X)$. The first named author has shown that for $p = 0$ there is a natural action of the Weyl group W on $\mathfrak{B}_0(X)$ and conjectured that the same construction yields a W-action whenever $p \neq 2$. In the present paper, we prove this conjecture.

1. Introduction

Let G be a connected reductive group defined over an algebraically closed base field $\Bbbk$ with Borel subgroup $B \subseteq G$. For any G-variety X let $\mathfrak{B}(X)$ be the set of all B-stable irreducible closed subvarieties of X. The *complexity* $c(Y)$ of $Y \in \mathfrak{B}(X)$ is the codimension of a generic B-orbit in Y or, equivalently, the transcendence degree of $\Bbbk(Y)^B$. It is a result of Vinberg [Vi86] that the complexity takes its maximal value for $Y = X$. Of particular interest is therefore the subset

$$\mathfrak{B}_0(X) := \{Y \in \mathfrak{B}(X) \mid c(Y) = c(X)\}.$$

This set contains X and is finite since it consists of the closures of all B-sheets with a maximal number of parameters (see [Kn95, Proposition 4.1]). The most important case is that of a spherical variety (i.e. $c(X) = 0$) when $\mathfrak{B}_0(X) = \mathfrak{B}(X)$ is just the set of all B-orbit closures.

Let W be the Weyl group of G. In [Kn95], an action of W on $\mathfrak{B}_0(X)$ was constructed whenever the base field has characteristic zero. On the other side, in the same paper an example was given showing that the construction does not work in characteristic 2. In any other characteristic, the situation was unclear so far. The purpose of this paper is to close this gap by showing that the method of [Kn95] does indeed define a W-action on $\mathfrak{B}_0(X)$ in every characteristic $\neq 2$.

It was already indicated [Kn95] that the problem can be reduced to the following special case: the characteristic p of $\Bbbk$ is > 2, the group G is semisimple of rank 2, and the variety X is of the form $X = G/H$ where H is a connected non-spherical subgroup of G. Moreover, we may replace $\mathfrak{B}_0(X)$ by a certain subset $\mathfrak{B}_{00}(X)$ (see §**2** for its definition).

It is then enough to consider only those H for which $\mathfrak{B}_{00}(G/H)$ consists of more than one element. Remarkably, under such assumptions the proof can be completed by showing that there exist a spherical subgroup $K \subseteq G$ and a bijection $\mathfrak{B}_{00}(G/H) \to \mathfrak{B}_{00}(G/K)$ that is compatible with the operation of the simple reflections of W.

2010 *Mathematics Subject Classification.* 20G15, 14M17, 14L30, 20G05.

While several subgroups require case-by-case considerations, others can be treated with general arguments, e.g. solvable subgroups.

Notation. All varieties are defined over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. We denote by G a connected reductive group, we fix a Borel subgroup $B \subseteq G$, whose unipotent radical is denoted by U, and a maximal torus $T \subseteq B$. The opposite Borel subgroup with respect to T, and its unipotent radical, are denoted by B^- and U^-, respectively.

Denote by R the set of roots with respect to T, by R^+ the set of positive roots corresponding to B, and by $S \subseteq R$ the set of simple roots. If G is simple, then the simple roots $\alpha_1, \alpha_2, \ldots$ and the fundamental dominant weights $\omega_1, \omega_2, \ldots$ will be numbered as in [Bou, Planches I–IX]. The 1-dimensional unipotent subgroup of G associated to a root γ is denoted by U_γ, and we choose once and for all an isomorphism $u_\gamma : \mathbb{G}_a \to U_\gamma$. The Weyl group of G is denoted by W, its longest element is denoted by w_0, and the simple reflection associated to $\alpha \in S$ is denoted by s_α.

If ω is a dominant weight, then $V_G(\omega)$ denotes the irreducible G-module of highest weight ω. If no confusion arises, we simply write $V(\omega)$.

If α is a simple root of G then P_α denotes the minimal parabolic subgroup of G which is generated by B and $U_{-\alpha}$, and $\pi_\alpha \colon G/B \to G/P_\alpha$ the natural map $gB \mapsto gP_\alpha$. If $g \in G$ and $H \subseteq G$, we use the notation ${}^gH = gHg^{-1}$. For any algebraic group H, we denote by H' its commutator subgroup, by H^r (resp. H^u) its radical (resp. unipotent radical), and by $\mathcal{X}(H)$ its group of characters, i.e. the set of all algebraic group homomorphisms $H \to \mathbb{G}_m$.

2. The action of the Weyl group

We recall some definitions and facts from [Kn95]. Let X be an algebraic variety equipped with a G-action. To avoid confusion, a B-stable irreducible closed subvariety Z of X is denoted sometimes by (Z) if we are referring to it as an element of $\mathfrak{B}(X)$.

We define the *character group* $\mathcal{X}(Z)$ of Z as the group of B-eigenvalues of B-eigenvectors in $\Bbbk(Z)$. The *rank* $r(Z)$ of Z is the rank of the free abelian group $\mathcal{X}(Z)$.

We also define the following subset of $\mathfrak{B}_0(X)$:

$$\mathfrak{B}_{00}(X) \quad = \quad \{(Z) \in \mathfrak{B}(X) \mid c(Z) = c(X),\ r(Z) = r(X)\}\,.$$

If $X = G/H$ is homogeneous, then there is a canonical bijection between $\mathfrak{B}(X)$ and the set of H-stable H-irreducible closed subsets of G/B. Here "H-irreducible" means that H acts transitively on the set of irreducible components. Throughout the paper we will sometimes implicitly make use of this bijection.

Let $\widetilde{W}$ be the group defined by generators $s_\alpha, \alpha \in S$ and relations $s_\alpha^2 = e, \alpha \in S$. In [Kn95] an action of $\widetilde{W}$ of $\mathfrak{B}_0(X)$ has been defined as follows. Let α be a simple root, and recall that $P_\alpha \supset B$ is the minimal parabolic subgroup of G corresponding to α. Then $Z \mapsto P_\alpha Z$ is an idempotent selfmap of $\mathfrak{B}_0(X)$. Its image $\mathfrak{B}_0^\alpha(X)$ consists of those $Z \in \mathfrak{B}_0(X)$ which are P_α-stable. Thus, for $Y \in \mathfrak{B}_0^\alpha(X)$ the fibers

$$\mathfrak{B}_0(Y, P_\alpha) = \{Z \in \mathfrak{B}_0(X) \mid P_\alpha Z = Y\}$$

form partition of $\mathfrak{B}_0(X)$. Now, the element s_α acts on each block $\mathfrak{B}_0(Y, P_\alpha)$ as an involution according to the following table:

Type	$\mathfrak{B}_0(Y, P_\alpha)$		s_α–action
(G)	$\{Y\}$		$s_\alpha \cdot (Y) = (Y)$
(U)	$\{Y, Z\}$,	$r(Z) = r(Y)$	$s_\alpha \cdot (Y) = (Z),\ s_\alpha \cdot (Z) = (Y)$
(N)	$\{Y, Z\}$,	$r(Z) < r(Y)$	$s_\alpha \cdot (Y) = (Y),\ s_\alpha \cdot (Z) = (Z)$
(T)	$\{Y, Z_0, Z_\infty\}$,	$r(Z_0) = r(Z_\infty) < r(Y)$	$s_\alpha \cdot (Y) = (Y),\ s_\alpha \cdot (Z_0) = (Z_\infty)$
			$s_\alpha \cdot (Z_\infty) = (Z_0)$

No other cases can occur (see [Kn95, §4]). This definition is based on a construction of Lusztig and Vogan in the case of symmetric spaces (see [LV83, §3]). Under the hypothesis that B has a dense orbit on G/H (in which case G/H is called a *spherical* homogeneous space, and H a *spherical* subgroup of G), the link is explained in more details in [Kn95, §1 and §5].

Let us further analyze the four above cases in case that $X = G/H$ is a homogeneous variety.

Suppose that $Y \in \mathfrak{B}_0(G/H)$ is P_α-stable. Then, considered as an H-stable subset of G/B, it satisfies $\pi^{-1}(\pi(Y)) = Y$ where $\pi = \pi_\alpha \colon G/B \to G/P_\alpha$.

The fiber $\pi^{-1}(x)$ over any $x \in \pi(Y)$ is isomorphic to $\mathbb{P}^1$ and equipped with a natural transitive action of G_x; we fix the isomorphism $\pi^{-1}(x) \cong \mathbb{P}^1$ and the corresponding homomorphism $\Phi\colon G_x \twoheadrightarrow \mathrm{PGL}(2) = \mathrm{Aut}\,\mathbb{P}^1$. The fiber $\pi^{-1}(x)$ intersects each $Z \in \mathfrak{B}_0(Y, P_\alpha)$ in an H_x-stable subset. If x is general in $\pi(Y)$ then the four cases of the above table correspond resp. to the following:

- (G) $\Phi(H_x)$ is $\mathrm{PGL}(2)$ or finite,
- (U) the unipotent radical of $\Phi(H_x)$ is a maximal unipotent subgroup of $\mathrm{PGL}(2)$,
- (N) $\Phi(H_x)$ has trivial unipotent radical and $|\mathfrak{B}_0(Y, P_\alpha)| = 2$,
- (T) $\Phi(H_x)$ is a maximal torus of $\mathrm{PGL}(2)$.

In [Kn95] it is shown that the $\widetilde{W}$-action on $\mathfrak{B}_0(G/H)$ descends to an action of the Weyl group W of G under several different assumptions, see [Kn95, Theorem 4.2]. In particular, this is true if H is a spherical subgroup of G and $\mathrm{char}\ \Bbbk \neq 2$.

We come to our main result.

2.1. Theorem. *Let* $\mathrm{char}\ \Bbbk > 2$ *and let* G *be semisimple of rank* 2*. For any connected subgroup* $H \subseteq G$*, the* $\widetilde{W}$*-action defined above induces a* W*-action on* $\mathfrak{B}_{00}(G/H)$*.*

Sections **3**–**6** are devoted to the proof. Precisely, the theorem follows from Corollaries **3.3**, **4.4**, **5.2** and **6.6**.

Thanks to [Kn95, §7], the above theorem implies the following.

2.2. Corollary. *Let* $\mathrm{char}\ \Bbbk \neq 2$*. Then for any* G*-variety* X *the* $\widetilde{W}$*-action defined above induces a* W*-action on* $\mathfrak{B}_0(X)$*.*

Before going into the details of the proof of Theorem **2.1**, we report two general results on $\mathfrak{B}_{00}(G/H)$.

2.3. Proposition. *Let $X = G/H$ be a homogeneous G-variety and $Z \in \mathfrak{B}_{00}(X)$ with $Z \neq X$. Then there is $\alpha \in S$ such that $\dim s_\alpha \cdot (Z) = \dim Z + 1$. In particular:*

- *The $\widetilde{W}$-action on $\mathfrak{B}_{00}(X)$ is transitive.*
- *If $|\mathfrak{B}_{00}(X)| > 1$ then there is $Z \in \mathfrak{B}_{00}(X)$ with $\operatorname{codim}_X Z = 1$.*

Proof. Suppose $\dim s_\alpha \cdot (Z) \leq \dim Z$. Then the definition of the s_α-action and the fact that Z is of maximal rank implies $P_\alpha Z = Z$. Since $Z \neq X$ and since X is homogeneous there is $\alpha \in S$ with $P_\alpha Z \neq Z$. Then $\dim s_\alpha \cdot (Z) = \dim Z + 1$ since s_α increases dimension by at most 1.

This implies that the action is transitive since any Z can be moved in finitely many steps to (X). Finally, starting from any $Z \neq X$ the next to the last step will be of codimension one. □

2.4. Lemma. *Let $H \subseteq K \subseteq G$ be subgroups such that H is normal in K and the quotient K/H is diagonalizable. Let $\pi\colon G/H \to G/K$ be the standard projection. Then the map $Y \mapsto \pi^{-1}(Y)$ is a $\widetilde{W}$-equivariant bijection $\mathfrak{B}_{00}(G/K) \to \mathfrak{B}_{00}(G/H)$.*

Proof. We show by induction on $\operatorname{codim}_{G/K} Z$ that $\pi^{-1}(Z) \in \mathfrak{B}_{00}(G/H)$ for all $Z \in \mathfrak{B}_{00}(G/K)$. This is true if $Z = G/K$, hence we can suppose $Z \subsetneq G/K$.

Let $Y \in \mathfrak{B}_{00}(G/K)$ and $\alpha \in S$ be such that $\dim Y = \dim Z + 1$ and $s_\alpha \cdot (Z) = (Y)$; notice that such elements exist thanks to Proposition 2.3. In particular, Z is the unique element of $\mathfrak{B}_{00}(G/K)$ such that $P_\alpha Z = Y$.

By induction $Y' = \pi^{-1}(Y)$ is in $\mathfrak{B}_{00}(G/H)$. Then for a general $y \in Y$ the image of the stabilizer $(P_\alpha)_y$ into $\operatorname{Aut}(P_\alpha/B) = \operatorname{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2)$ is a proper subgroup that contains a maximal unipotent subgroup.

Let $y' = gH$ be a general point of Y': its stabilizer $(P_\alpha)_{y'}$ is equal to $P_\alpha \cap {}^gH$, and we may compare the latter with $(P_\alpha)_y = P_\alpha \cap {}^gK$ where $y = gK = \pi(y')$. We have that $P_\alpha \cap {}^gH$ is normal in $P_\alpha \cap {}^gK$, and the quotient is a diagonalizable group. Therefore the image of $(P_\alpha)_{y'}$ into $\mathrm{PGL}(2)$ is again proper and contains a maximal unipotent subgroup. Then the P_α-orbit of y' in this case is the union of two B-orbits, and there exists a unique $Z' \in \mathfrak{B}(G/H)$ such that $P_\alpha Z' = Y'$. It follows that $P_\alpha \pi^{-1}(Z) = Y'$, so $Z' = \pi^{-1}(Z)$, the latter lies in $\mathfrak{B}_{00}(G/H)$, and $s_\alpha \cdot (Y') = (Z')$.

Denote with $f\colon \mathfrak{B}_{00}(G/K) \to \mathfrak{B}_{00}(G/H)$ the map $Y \mapsto \pi^{-1}(Y)$. We have shown that f is injective, and that whenever s_α exchanges two elements $Y, Z \in \mathfrak{B}_{00}(G/K)$ then it exchanges $f(Y)$ and $f(Z)$.

To show that f is $\widetilde{W}$-equivariant, which also implies the surjectivity of f since both sets are $\widetilde{W}$-orbits, it remains only to consider $Y \in \mathfrak{B}_{00}(G/K)$ and $\alpha \in S$ such that Y is P_α-stable and $s_\alpha \cdot (Y) = (Y)$, and to check that $s_\alpha \cdot (f(Y)) = (f(Y))$. In this case, the image of $P_\alpha \cap {}^gK$ in $\mathrm{PGL}(2)$ is either the whole $\mathrm{PGL}(2)$, a maximal torus, the normalizer of a maximal torus, or a finite group. Since $P_\alpha \cap {}^gH$ is a normal subgroup of $P_\alpha \cap {}^gK$ such that the quotient is diagonalizable, we conclude that its image in $\mathrm{PGL}(2)$ also belongs to one of the above four possible types of subgroups. Hence $s_\alpha \cdot (f(Y)) = (f(Y))$. □

3. Solvable subgroups

We assume from now on that $p := \mathrm{char}\, \Bbbk > 2$.

3.1. Lemma. *Suppose that G is semisimple of rank 2, and let α, β be the two simple roots. Let N be the normal subgroup of U generated by U_α. Then $N = P^u_\beta$.*

Proof. Clearly, $N \subseteq P^u_\beta$. To show the reverse inclusion we recall that

$$u_\beta(y)^{-1} u_\alpha(x) u_\beta(y) = \prod_{\gamma \in R^+ \setminus \{\beta\}} u_\gamma(f_\gamma(x, y)) \in N \tag{1}$$

where each $f_\gamma(x, y)$ is a non-constant polynomial (see [SGA3, Exposé XXIII: Proposition 3.1.2(iii), Proposition 3.2.1(iii), Proposition 3.3.1(iii), Proposition 3.4.1(iii)] for groups of type $\mathsf{A}_1 \times \mathsf{A}_1$, A_2, B_2 and G_2, respectively). Since N is normalized by T we have $N = \prod_{\gamma \in R'} U_\gamma$ where $R' \subseteq R^+$ is a subset. Thus, all factors of the right hand side of (1) are also in N. This shows $R^+ \setminus \{\beta\} \subseteq R'$ and therefore $P^\beta_u \subseteq N$, as claimed. □

3.2. Proposition. *Let G be semisimple of rank 2 and $H \subseteq G$ be a connected solvable subgroup. Then $|\mathfrak{B}_{00}(G/H)| = 1$, or H contains the unipotent radical of a parabolic subgroup of G.*

Proof. Suppose $|\mathfrak{B}_{00}(G/H)| > 1$. Then there is a simple root α and $Z \in \mathfrak{B}_{00}(G/H)$ with $Z \neq G/H$ and $s_\alpha(Z) = (G/H)$ (Proposition **2.3**). Let $Z' \subset G/B$ correspond to Z. The definition of the s_α -action implies that $\pi_\alpha(Z') = G/P_\alpha$ and that the generic isotropy group of H in Z', hence in G/P_α, contains a subgroup isomorphic to $\mathbb{G}_a$. Without loss of generality we may assume that $H \subseteq B^-$. The U^--orbit of $e_G P_\alpha$ is dense in G/P_α and, for $u \in U^-$, the isotropy group of $x = uP_\alpha$ in B^- is $B^-_x = {}^u(TU_{-\alpha}) = {}^uT\,{}^uU_{-\alpha}$. Because of $H_x \subseteq B^-_x$ we conclude that ${}^uU_{-\alpha} \subseteq H$ for $u \in U^-$ general and therefore, by continuity, for all $u \in U^-$. Then Lemma **3.1** implies $H \supseteq P^u_{-\beta}$ where β is the simple root different from α. □

3.3. Corollary. *Theorem* **2.1** *holds for all connected solvable subgroups $H \subset G$.*

Proof. If $|\mathfrak{B}_{00}(G/H)| = 1$ there is nothing to prove. Otherwise, without loss of generality, the unipotent radical H^u is either U or P^u_α for some simple root α (Proposition **3.2**). The claim is also known to be true if H is spherical. This leaves to check only $H = \widetilde{T} P^u_\alpha$ where either $\widetilde{T} = \{e\}$ or $\widetilde{T} = (\ker \alpha)^\circ \subset T$. In either case Lemma **2.4** applies to H and $K = TP^u_\alpha$; since TP^u_α is spherical, the corollary follows. □

4. Reductive subgroups

In this section, we consider the case where H is a reductive subgroups of G. Clearly, by the preceding section we may assume that H is not a torus. So, its semisimple rank is one or two.

4.1. Proposition. *Let G be a semisimple group of rank 2 and $H \subseteq G$ a connected reductive subgroup of semisimple rank 2. Then H is spherical.*

Proof. The root system of H is given by a subroot system of rank 2 of the root system of G. These are easily determined: in case $G = \mathsf{A}_1 \times \mathsf{A}_1$ or $G = \mathsf{A}_2$ then only $H = G$ is possible. If $G = \mathsf{B}_2$ then there is additionally $H = \mathsf{A}_1 \times \mathsf{A}_1$. Finally, if $G = \mathsf{G}_2$ there is an extra-complication if $p = 3$. In general, $H = \mathsf{A}_1 \times \mathsf{A}_1$ and $H = \mathsf{A}_2$ is possible, the latter corresponding to the set of long roots. If $p = 3$ then there is another subgroup of type A_2 corresponding to the short roots. But that is mapped by a special isogeny (see [BT73, §3.3]) to the former. We conclude that H is a spherical subgroup in every case thanks to [Br98, Theorem 4.3]. □

We are left with the case where H has semisimple rank one.

4.2. Lemma. *Let G be a semisimple group of rank 2 and $H \subset G$ a connected reductive subgroup with $\operatorname{rk}_{\mathrm{ss}} H = 1$ and $|\mathfrak{B}_{00}(G/H)| > 1$. Let $U_H \cong \mathbb{G}_a$ be a maximal connected unipotent subgroup of H. Then there is a simple root $\alpha \in S$ such that the fixed point set $(G/P_\alpha)^{U_H}$ has a component of codimension at most one in G/P_α.*

Proof. Arguing as in the proof of Proposition **3.2**, there is a simple root α such that the isotropy group H_y is not reductive for general $y \in G/P_\alpha$. Then $\operatorname{rk}_{\mathrm{ss}} H = 1$ implies that H_y contains a conjugate of U_H for general $y \in G/P_\alpha$ or, in other words, a general H-orbit in G/P_α contains an U_H-fixed point. The normalizer of U_H in H is of codimension one. Thus, the U_H-fixed points are of codimension at most one in any general orbit Hy. We conclude that $(G/P_\alpha)^{U_H}$ has a component C which is of codimension at most one in G/P_α.

□

Now we analyze the situation of the preceding lemma further.

4.3. Lemma. *Let G, H, U_H be as in Lemma **4.2**. If G is of type B_2 or G_2 then H is conjugate to either L or L' where L is a Levi subgroup of a parabolic subgroup of G.*

Proof. By replacing U_H by a conjugate, we may assume that $U_H \subset B^-$. Let C be a component of $(G/P_\alpha)^{U_H}$ of codimension one. Then there are two possibilities: either C meets the open B^--orbit in G/P_α or not. We claim that in both cases U_H is conjugate in G to a root subgroup U_γ for some simple root γ. Let $y \in C$ be general.

In the first case, B_y^- is conjugate to $TU_{-\alpha}$. Because of $U_H \subset B_y^-$ we conclude that U_H is G-conjugate to U_α, which implies the claim with $\gamma = \alpha$.

In the second case, C equals the Bruhat cell of codimension one. In that case B_y^- is B^--conjugate to TU_0 with $U_0 = U_{-\beta}U_{-\gamma}$ where β is the simple root different from α and $\gamma = s_\beta(\alpha)$. But now U_H is contained in every B^--conjugate of TU_0, i.e., in the biggest connected subgroup U_1 of U_0 which is normal in B^-.

Since U_1 is normalized by T, it is either trivial, $U_{-\beta}$, $U_{-\gamma}$, or U_0. It can't be trivial since it contains U_H. Moreover, a short case-by-case consideration shows that if $G = \mathsf{G}_2$ then none of the other three groups are normal in B^-.

Then G is of type B_2. Another short case-by-case consideration shows that U_1 is either trivial (which is impossible since $U_H \subseteq U_1$) or $U_{-\gamma}$ (where in this case γ is long), proving the claim.

Now we may assume that $U_H = U_\gamma$. Since both a maximal torus S of H and the maximal torus T of G are contained in the normalizer of U_γ in G, we can replace H by a conjugate such that $B \cap H$ contains both S and U_γ. The intersection $B \cap H$ is then a Borel subgroup of H. Let L be the Levi subgroup corresponding to γ. Then $B \cap H \subseteq L \cap H$ is parabolic in H. Thus $H/L \cap H$ is a projective subvariety of the affine variety G/L. Thus $H \subseteq L$ proving the lemma. □

4.4. Corollary. *Let G be a semisimple group of rank 2 and $H \subset G$ a connected reductive subgroup of semisimple rank 1. Then H is spherical, or $|\mathfrak{B}_{00}(G/H)| = 1$, or $H = L$ or L' where L is a Levi subgroup of a proper parabolic subgroup P of G.*

Proof. If the rank of H is 2 then H is a Levi subgroup of a parabolic subgroup of G.

If the rank of H is 1 then H is of type A_1. If $G = \mathsf{A}_1 \times \mathsf{A}_1$ then H is either one of the factors, in which case $|\mathfrak{B}_{00}(G/H)| = 1$, or H is embedded diagonally, possibly by a power of the Frobenius morphism. Thus, there is an inseparable isogeny ϕ of G such that $\phi(H)$ is the diagonal subgroup of G. This shows that H is spherical.

If $G = \mathsf{A}_2$ then H is embedded into G via a 3-dimensional representation V. Because $p \neq 2$, there are no non-trivial extensions of a two dimensional representation $\Bbbk^2$ and the trivial representation $\Bbbk$ (see e.g. [St10, Proposition 2.6]). Thus, $V = \Bbbk^2 \oplus \Bbbk$, or $V = \Bbbk^3$ is irreducible. In the first case $H = \mathrm{SL}(2) \times 1 \subset G = \mathrm{SL}(3)$, in the second case $H = \mathrm{SO}(3) \subset G = \mathrm{SL}(3)$. In both cases H is spherical.

Ir remains to check the case $G = \mathsf{B}_2$ or $G = \mathsf{G}_2$ and H is of semisimple rank one. In this case the corollary follow from Lemma **4.3**. □

5. Levi subgroups

In this section we discuss subgroups H of G that are Levi subgroups of some parabolic, up to a $\Bbbk^*$-factor.

5.1. Proposition. *Let G be simple of rank 2, $p \geq 3$, let $P \supsetneq B$ be a proper parabolic subgroup of G, and let H be a Levi subgroup of P. If G has type $\mathsf{A}_1 \times \mathsf{A}_1$, A_2 or B_2 then H is spherical; if G has type G_2 then H is spherical or $|\mathfrak{B}_{00}(G/H)| = 1$.*

Proof. We may assume that G is simply connected. If it has type $\mathsf{A}_1 \times \mathsf{A}_1$ then H is spherical because a maximal torus of SL(2) is spherical in SL(2) (see [Br98, Theorem 4.3]).

If G has type A_2 or B_2 then H itself appears in [Br98, Table 1], hence we apply again [Br98, Theorem 4.3].

Suppose that G has type G_2, and consider the case $P = P_{\alpha_1}$. Assume also $p \geq 5$: indeed, if $p = 3$ then P_{α_1} is sent onto P_{α_2} by a special isogeny of G, so we refer to our later discussion of the case $P = P_{\alpha_2}$.

Thanks to the assumption $p \geq 5$ the module $V = V_G(\omega_1)$ is also the Weyl module of G associated to ω_1 (see e.g. [Pr88, Theorem 1]), and G/P_{α_1} is a subvariety of $\mathbb{P}(V)$. Let U_H be a maximal unipotent subgroup of H. The weights of the G-module V imply that the latter is the sum of three irreducible H-modules, therefore $(G/P_{\alpha_1})^{U_H}$ has components of dimension at most 2.

If $|\mathfrak{B}_{00}(G/H)| > 1$ then Lemma **4.2** applies: since $\dim G/B = 6$, then $(G/P_{\alpha_1})^{U_H}$ or $(G/P_{\alpha_2})^{U_H}$ has a component C of dimension 4. The first case is excluded, so consider the second case. The unipotent group U_H has at least one fixed point in $\pi_{\alpha_2}^{-1}(c)$ for all $c \in C$. It follows that $(G/P_{\alpha_1})^{U_H}$ has a component of dimension at least 3, which is also impossible. Hence$|\mathfrak{B}_{00}(G/H)| = 1$

It remains the case where $P = P_{\alpha_2}$ and $p \geq 3$. Let U_H be a maximal connected unipotent subgroup of H. We claim that the set of U_H-fixed points on G/P_{α_i} has components of dimension at most 3 for both $i = 1, 2$: from Lemma **4.2** we obtain $|\mathfrak{B}_{00}(G/H)| = 1$ as above.

To prove the claim, we use the commutation relations of [SGA3, Exposé XXIII: Proposition 3.4.1(iii)] as in the proof of Lemma **3.1**. To simplify notations, write $\alpha = \alpha_1$ and $\beta = \alpha_2$.

Denoting $Q = P_\alpha$, we have

$$G/Q = (Q^u w_0 Q/Q) \cup (Q^u s_\beta w_0 Q/Q) \cup (\text{subvarieties of dimension} \leq 3).$$

Let us compute the U_H-fixed points on $Q^u w_0 Q/Q$. We may assume that U_H is the set of elements of the form $u = u_\beta(x)$ for $x \in \Bbbk$.

If $v \in Q^u$, then $v w_0 Q \in Q^u w_0 Q/Q$ is fixed under the action of u if and only if $v^{-1} u v \in {}^{w_0}Q$. Now write v^{-1} as a product:

$$v^{-1} = u_{3\alpha+\beta}(y_1) u_{2\alpha+\beta}(y_2) u_{\alpha+\beta} y_3) u_{3\alpha+2\beta}(y_4) u_\beta(y_5).$$

Then:

$$v^{-1} u_\beta(x) v = u_{3\alpha+2\beta}(-x y_1) u_\beta(x).$$

This belongs to ${}^{w_0}Q$ only if $x = 0$, therefore there are no U_H-fixed points on $Q^u w_0 Q/Q$.

Let us do the same with $Q^u s_\beta w_0 Q/Q$, and call $w_1 = s_\beta w_0$. This set is an affine space of dimension 4, and we may take $y_1, \ldots, y_4$ as its coordinates.

A point $v w_1 Q$ is fixed by u if and only if $v^{-1} u v \in {}^{w_1}Q$. This time, $u_{3\alpha+2\beta}(-x y_1) u_\beta(x)$ lies in ${}^{w_1}Q$ for all x if and only if $y_1 = 0$. It follows that the set of U_H-fixed points on $Q^u s_\beta w_0 Q/Q$ is irreducible of dimension 3.

Finally, we discuss the other parabolic $P = P_\beta$ with the same procedure. Write

$$G/P = (P^u w_0 P/P) \cup (P^u s_\alpha w_0 P/P) \cup (\text{subvarieties of dimension} \leq 3),$$

and consider $v \in P^u$. The point $v w_0 P$ is fixed by $u \in U_H$ if and only if $v^{-1} u v \in {}^{w_0}P$. If:

$$v^{-1} = u_\alpha(y_1) u_{3\alpha+\beta}(y_2) u_{3\alpha+2\beta} y_3) u_{2\alpha+\beta}(y_4) u_{\alpha+\beta}(y_5)$$

and $u = u_\beta(x)$, then:

$$\begin{aligned} v^{-1} u v &= \big(u_\alpha(-y_1) u_{3\alpha+\beta}(-y_2)\big) u_\beta(x) \big(u_{3\alpha+\beta}(y_2) u_\alpha(y_1)\big) \\ &= u_\beta(x) u_{\alpha+\beta}(x y_1) u_{2\alpha+\beta}(x y_1^2) u_{3\alpha+\beta}(x y_1^3) u_{3\alpha+2\beta}(x^2 y_1^3 - x y_2). \end{aligned}$$

It belongs to ${}^{w_0}P$ for all x if and only if $y_1 = y_2 = 0$, hence the set of U_H-fixed points on $P^u w_0 P/P$ is 3-dimensional. On the other hand, for no $v \in P^u$ we have $v^{-1} u v \in {}^{s_\alpha w_0}P$ for all x, therefore U_H has no fixed points on $P^u s_\alpha w_0 P/P$, and the proof is complete. □

5.2. Corollary. *Let $H \subset G$ be a connected subgroup. Suppose that $L' \subseteq H \subseteq L$, where L is a Levi subgroup of some proper parabolic subgroup $P \supsetneq B$. Then Theorem* **2.1** *holds for H.*

Proof. Suppose that G has type $\mathsf{A}_1 \times \mathsf{A}_1$ and is simply connected, so $G = \mathrm{SL}(2) \times \mathrm{SL}(2)$. Without loss of generality $P = \mathrm{SL}(2) \times B_{\mathrm{SL}(2)}$, where $B_{\mathrm{SL}(2)}$ is a Borel subgroup of $\mathrm{SL}(2)$. It follows that $H = \mathrm{SL}(2) \times K$ where K is a subgroup of a maximal torus of $B_{\mathrm{SL}(2)}$. Therefore the reflection associated to one of the simple roots of G acts trivially on $\mathfrak{B}_{00}(G/H)$, and Theorem **2.1** follows.

If G is simple, apply Lemma **2.4** to $H \subseteq L$ and then Proposition **5.1** to L: the corollary follows. □

6. Other subgroups

In this section we finish the proof of Theorem **2.1**, discussing the remaining connected subgroups H of G.

The first result regards the representation theory of SL(2) and will be useful in subsequent proofs.

6.1. Lemma. *Let V be a finite dimensional $\mathrm{SL}(2)$-module, and $R \subseteq V$ an $\mathrm{SL}(2)$-stable additive subgroup. Suppose that one of the following two conditions is satisfied:*

(1) The module V is nontrivial and simple.

(2) The characteristic p of $\Bbbk$ is $\neq 2$, the subgroup R is closed and connected, and $V = V(0) \oplus V'$ where V' is nontrivial and simple.

Then R is a submodule of V.

Proof. We may suppose that $R \neq \{0\}$. Assume (1): then the union of the sets aR for all $a \in \Bbbk$ is a non-zero $\mathrm{SL}(2)$-submodule of V, therefore equal to V. It follows that R contains a highest weight vector v, therefore all its multiples and all linear combinations of elements of the form gv for $g \in \mathrm{SL}(2)$. Hence $R = V$.

Assume now (2), and let R' be the projection of R on V' along $V(0)$. Since it is an SL(2)-stable additive subgroup of V', as in the proof of the first part of the Lemma we conclude that R' is either $\{0\}$, or contains a highest weight vector $v \in V'$.

In the first case R is either $\{0\}$ or $V(0)$, since it is closed and connected. In the second case, let $r \in R$ project to v: since $p \neq 2$ it is elementary to show that R also contains both projections of r in $V(0)$ and V', and this completes the proof. □

6.2. Lemma. *Let G be semisimple of rank 2 and let $p \geq 3$. Then any connected subgroup H of G has a Levi subgroup. If H is contained in a parabolic subgroup P of G, then any Levi subgroup of H is contained in a Levi subgroup of P.*

Proof. The proposition is true if H is solvable or very reductive, i.e. not contained in any proper parabolic subgroup of G. Therefore we may assume that H is contained in a proper parabolic subgroup $P \supsetneq B$ but not in any Borel subgroup of G. In this case HP^u/P^u is not contained in any proper parabolic subgroup of P/P^u, which implies $H/H \cap P^u \cong HP^u/P^u$ reductive, and hence $H^u = H \cap P^u$.

Denote by $L \supset T$ the standard Levi subgroup of P; according to the decomposition $P = L \ltimes P^u$ we define two projections $\pi_\ell \colon P \to L$ and $\pi_u \colon P \to P^u$. Notice that L is the quotient of $\mathrm{SL}(2) \times \mathbb{G}_m$ by a finite central subgroup scheme.

If H contains a maximal torus of G then it contains T up to conjugation by an element of P; we may then suppose that $H \supset T$. It follows that H contains $\pi_\ell(H)$, because $\pi_\ell(h) \in \overline{\{tht^{-1} \mid t \in T\}}$ for any $h \in H$. This implies that $H = \pi_\ell(H) \ltimes (H \cap P^u)$, whence both statements of the proposition.

We are left with the case where H doesn't contain any maximal torus of G, hence $\pi_\ell(H) \cong H/H^u$ is semisimple of rank 1. With this assumption, we show that the two statements of the proposition follow from the vanishing of $H^1(L', R)$ and $H^2(L', R)$ for certain subquotients R of P^u.

Consider the lower central series $P^u = P^u_0 \supseteq P^u_1 \supseteq P^u_2 \supseteq \ldots$ of P^u, and the projection $\pi_1 \colon P \to P/P^u_1$. Then $\pi_1(H^u)$ is a $\pi_\ell(H)$-stable subgroup of P^u/P^u_1, which is a vector group where $\pi_\ell(H)$ acts linearly by conjugation (see [SGA3, Exposé XXVI: Proposition 2.1]). If $H^2(\pi_\ell(H), \pi_1(H^u)) = 0$ then $\pi_1(H)$ is isomorphic to the semidirect product of $\pi_\ell(H)$ and $\pi_1(H^u)$, thus it has a Levi subgroup $L_1 \subseteq P/P^u_1$. In particular $L_1 \cap \pi_1(H^u)$ is trivial.

Let now $H_1 = H \cap \pi_1^{-1}(L_1)$: this group has unipotent radical contained in P^u_1, and satisfies $\pi_\ell(H) = \pi_\ell(H_1)$. We may go on applying the same procedure to the group H_1 using the projection onto the quotient P/P^u_2, provided that the corresponding cohomology groups vanish. We obtain a sequence $H \supseteq H_1 \supseteq H_2 \supseteq \ldots$ of subgroups of H satisfying $\pi_\ell(H) = \pi_\ell(H_i)$ and $H^u_i \subseteq P^u_i$ for all i. If n is big enough so that P^u_n is trivial, the subgroup H_n of H is reductive and isomorphic to $\pi_\ell(H)$, hence it is a Levi subgroup of H.

Denote now by L_H a Levi subgroup of H, and consider the map $(\pi_1 \circ \pi_u)|_{L_H} \colon L_H \to P^u/P^u_1$. It is a 1-cocycle of L_H with values in the module P^u/P^u_1. If the group $H^1(L_H, P^u/P^u_1)$ vanishes, then $(\pi_1 \circ \pi_u)|_{L_H}$ is a coboundary, whence $\pi_1(L_H)$ is contained in $\pi_1(L)$ up to conjugation by an element of P^u/P^u_1.

Therefore we may assume that $L_H \subset LP^u_1$. Proceeding as above using the projections on P/P^u_i for $i \in \{2, 3, \ldots\}$, provided that the needed cohomology groups vanish, we obtain that L_H is contained in L up to conjugation by an element of P^u.

To finish the proof we must show the vanishing of the cohomology groups involved. We notice that for all of them we may replace the group with its image under π_ℓ, i.e. with L'. Then it remains to show that $H^n(L', R) = 0$, where $n \in \{1, 2\}$ and R is an L'-stable additive subgroup of P^u_j/P^u_{j+1}. Using the long exact sequence of group cohomology the problem is reduced to the case where R is an L'-stable subgroup of a simple subquotient Q of P^u_j/P^u_{j+1}.

If Q is the trivial simple L'-module then L' acts trivially on R; this implies $H^1(L', R) = 0$ because SL(2) has no non-trivial abelian quotient. In this case also $H^2(L', R) = 0$ follows, using the long exact sequence of group cohomology, the vanishing of $H^1(L', Q/R)$ by the above argument applied to Q/R instead of R, and the vanishing of $H^2(L', Q)$ (see [St10, Theorem 1]). We may now assume that Q is not the trivial simple L'-module, and that $R \neq \{0\}$.

From Lemma **6.1**, part (1) it follows that $R = Q$. Moreover, by inspection on all rank 2 root systems, the L'-module Q (viewed as an SL(2)-module) is isomorphic to $V(\omega_1)$, $V(2\omega_1)$ or $V(3\omega_1)$. Finally, the vanishing of $H^1(L', Q)$ follows from [St10, Proposition 2.6], and the vanishing of $H^2(L', Q)$ follows from [St10, Theorem 1]. Notice that here is the step where we use the assumption $p > 2$, since e.g. $H^1(\mathrm{SL}(2), V(\omega_1)) \neq 0$ in characteristic 2. □

6.3. Lemma. *If $H \subseteq G$ has an open orbit on G/P_α for each simple root α but not on G/B, then $|\mathfrak{B}_{00}(G/H)| = 1$.*

Proof. If $|\mathfrak{B}_{00}(G/H)| > 1$ then Proposition **2.3** implies the existence of an element $Y \in \mathfrak{B}_{00}(G/H)$ of codimension 1 in G/H satisfying $s_\alpha \cdot (G/H) = (Y)$ for some simple root α. Since H has no open orbit on G/B, we have $c(Y) = c(G/H) > 0$. Then $s_\alpha \cdot (G/H) = (Y)$ implies that the generic H-sheet of G/P_α has positive complexity, which contradicts our assumptions. □

6.4. Lemma. *Let P and P_- be two opposite parabolic subgroups of G, set $L = P \cap P_-$ and let I be either L or L'. Let also $H \subseteq P$ be a connected subgroup containing I. Then $H^u \subseteq P^u$; if P^u/H^u has an open I-orbit then H has an open orbit on G/P_-, and if P^u/H^u is spherical under the action of I then H is a spherical subgroup of G.*

Proof. The inclusion $H^u \subseteq P^u$ stems from the inclusion $H^u \subseteq P^r$, which holds because $HP^r/P^r = P/P^r$ is reductive, therefore H^u is in the kernel of the projection $P \to P/P^r$.

Consider a Borel subgroup $B_L \subseteq L$. Then $B_- = B_L P_-^u$ is a Borel subgroup of G, and its subgroup P_-^u has an open orbit on G/P. Then L has an open orbit on P/H if and only if P_- has an open orbit on G/H, and B_L has an open orbit on P/H if and only if B_- has an open orbit on G/H.

This completes the proof if $I = L$, because then $P/H = P^u/H^u$; in this case the two last statements of the lemma are even equivalences. If $I = L'$ but H anyway contains L then again $P/H = P^u/H^u$ and the two last statements of the lemma follow from the case $I = L$. Hence we can suppose $H \not\supseteq L$.

In this case $P/H = P^r/H^u$: if L' has an open orbit on P^u/H^u then L has an open orbit on P^r/H^u thus P_- has an open orbit on G/H, and if P^u/H^u is spherical under the action of L' then P^r/H^u is spherical under the action of L thus H is a spherical subgroup of G. □

6.5. Proposition. *Let G be semisimple of rank 2 and $p \geq 3$. Let $P \supsetneq B$ be a proper parabolic subgroup of G and H a connected non-reductive subgroup of P containing a Levi subgroup L of P. Then H is spherical or $|\mathfrak{B}_{00}(G/H)| = 1$.*

Proof. If G has type $\mathsf{A}_1 \times \mathsf{A}_1$, A_2 or B_2 then L is spherical (Proposition 5.1). This implies that H is also spherical.

So we suppose that G has type G_2, and also that $L \supset T$. Write for brevity $\alpha = \alpha_1$ and $\beta = \alpha_2$.

Since $H \supseteq L$, P is minimal among the parabolic subgroups containing H: it follows that $H^u \subseteq P^u$. The quotient $P^u/(P^u)'$ is a vector group where L acts linearly, and $H^u(P^u)'/(P^u)'$ is an additive subgroup. Since the center of L acts on $P^u/(P^u)'$ non-trivially by homotheties and H^u is L-stable, the group $H^u(P^u)'/(P^u)'$ is an L-submodule of $P^u/(P^u)'$.

Moreover, we may suppose that it is a proper submodule, otherwise H^u is the whole P^u, and hence $H = P$ is spherical. Notice that here $(P^u)'$ is abelian, so also a vector group where L acts linearly. Let us then recall the structure of the L-modules under consideration (viewed as $\mathrm{SL}(2) \times \mathbb{G}_m$-modules):

$$\begin{aligned} P_\alpha^u/(P_\alpha^u)' &\cong \mathrm{Sym}^3(V(\omega_1)) \otimes V(\beta|_{\mathbb{G}_m}), \\ (P_\alpha^u)' &\cong V(0) \otimes V(2\beta|_{\mathbb{G}_m}), \end{aligned}$$

and

$$\begin{aligned} P_\beta^u/(P_\beta^u)' &\cong V(\omega_1) \otimes V(\alpha|_{\mathbb{G}_m}), \\ (P_\beta^u)' &\cong \big(V(0) \otimes V(2\alpha|_{\mathbb{G}_m})\big) \oplus \big(V(\omega_1) \otimes V(3\alpha|_{\mathbb{G}_m})\big). \end{aligned}$$

If now $p > 3$ or if $P = P_\beta$ then $H^u(P^u)'/(P^u)'$ is trivial, because with either of these two assumptions $P^u/(P^u)'$ is a simple L-module. For $p = 3$ and $P = P_\alpha$, the L- module

$P^u/(P^u)'$ contains a unique nontrivial and proper L-submodule (of dimension 2). However, if $H^u(P^u)'/(P^u)'$ contains this submodule then P^u/H^u is spherical under the action of L, therefore H is spherical thanks to Lemma **6.4**.

As a consequence, we may suppose from now on that $H^u \subseteq (P^u)'$. Now $(P^u)'$ as an SL(2)-module it is either $V(0)$ or $V(0) \oplus V(\omega_1)$, and we observe that in both cases H^u is an L-submodule. This is obvious in the first case since H^u is connected, and in the second case it follows from Lemma **6.1**, part (2).

In addition, if $P = P_\beta$ and H^u contains the 2-dimensional L-submodule of $(P^u)'$, then again P^u/H^u is L-spherical, so H is spherical in G.

This leaves only one subgroup H for each of the two choices of P, namely the one where H^u is equal to the 1-dimensional summand of $(P^u)'$. We claim that in both cases H has an open orbit on G/P_α and G/P_β. This proves the proposition thanks to Lemma **6.3**, since H has no open orbit on G/B for dimension reasons.

Let us prove the claim for $P = P_\alpha$, and consider first G/P_α. Thanks to Lemma **6.4**, to prove that $H \subset P_\alpha$ has an open orbit on G/P_α it is enough to prove that L has an open orbit on P_α^u/H^u. Both L and $P_\alpha^u/H^u \cong \mathrm{Sym}^3(V(\omega_1)) \otimes V(\beta|_{\mathbb{G}_m})$ have dimension 4, so our claim follows from the fact that the point $e_1e_2(e_1+e_2)$ has finite stabilizer in L, where e_1, e_2 is the standard basis of $V(\omega_1) \cong \Bbbk^2$.

We show now that $H \subset P_\alpha$ has an open orbit on G/P_β. Notice that a Levi subgroup of P_β and a Levi subgroup of P_α are not conjugated in G, whence the group $({}^gP_\beta \cap H)^\circ$ is solvable for all $g \in G$. It is then contained in a Borel subgroup of H. Since the flag variety of H has dimension 1 and $B \cap H$ is a Borel of H, the inclusion

$$({}^gP_\beta \cap H)^\circ \subseteq B \cap H$$

holds for all g such that $gP_\beta \in D$, where D is a subvariety of G/P_β of codimension 1. If H has no open orbit on G/P_β then $({}^gP_\beta \cap H)$ has positive dimension for all $g \in G$, therefore for all g such that $gP_\beta \in D$ we have

$$\dim({}^gP_\beta \cap B \cap H) > 0. \tag{2}$$

We prove that this is impossible, by checking that that the intersection of the locus where (2) is satisfied with any Schubert cell of G/P_β has codimension at least 2 in G/P_β.

We first consider the open Schubert cell, i.e. g is of the form $g = uw_0$ with $u \in U$. Then

$$({}^gP_\beta \cap B) \cap H = ({}^{uw_0}P_\beta \cap B) \cap H = {}^u(TU_\beta) \cap H$$

We fix an isomorphism of the open Schubert cell with $\Bbbk^5$ in such a way that

$$u = u_{\alpha+\beta}(x_1)u_{2\alpha+\beta}(x_2)u_{3\alpha+\beta}(x_3)$$

where the parameters x_1, x_2 and x_3 are coordinate functions of $\Bbbk^5$. This is possible since $U_\alpha U_{3\alpha+2\beta} \subset H$. An elementary computation shows that ${}^u(TU_\beta) \cap H$ is infinite only if two of x_1, x_2, x_3 are zero.

Consider now the codimension 1 Schubert cell, i.e. g is of the form uw_0s_α. Then

$$({}^gP_\beta \cap B) \cap H = ({}^{uw_0s_\alpha}P_\beta \cap B) \cap H = {}^u(TU_\alpha U_{3\alpha+\beta}) \cap H$$

Here we can assume that

$$u = u_\beta(x_1)u_{\alpha+\beta}(x_2)u_{2\alpha+\beta}(x_3)$$

where x_1, x_2, x_3 are coordinate functions on the 4-dimensional Schubert cell. Consider also elements $t \in T$, $u_\alpha(y_1)$ and $u_{3\alpha+\beta}(y_2)$ with $y_1, y_2 \in \Bbbk$. Then ${}^u(tu_\alpha(y_1)u_{3\alpha+\beta}(y_2))$ is equal to

$$tu_\beta(A_\beta)u_{\alpha+\beta}(A_{\alpha+\beta})u_{2\alpha+\beta}(A_{2\alpha+\beta})u_{3\alpha+\beta}(A_{3\alpha+\beta})u_{3\alpha+2\beta}(A_{3\alpha+2\beta})u_\alpha(A_\alpha)$$

where

$$\begin{array}{lcl} A_\beta & = & (b-1)x_1 \\ A_{\alpha+\beta} & = & (ab-1)x_2 + x_1y_1 \\ A_{2\alpha+\beta} & = & (a^2b-1)x_3 + 2x_2y_1 - x_1y_1^2 \\ A_{3\alpha+\beta} & = & y_2 - 6x_2y_1^2 + x_1y_1^3 \end{array}$$

and $a = \alpha(t^{-1})$, $b = \beta(t^{-1})$. This element is in H only if $A_\beta = A_{\alpha+\beta} = A_{2\alpha+\beta} = A_{3\alpha+\beta} = 0$. For general x_1, x_2 and x_3 the resulting system has only finitely many solutions in a, b, y_1, y_2. This finishes the proof that (2) cannot be satisfied for gP_β lying on a subvariety of G/P_β of codimension 1.

Finally, we prove the claim for $P = P_\beta$ using the same method. Here $H \subset P_\beta$, and we consider first G/P_β. The quotient P_β^u/H^u is abelian and, as an $\mathrm{SL}(2) \times \mathbb{G}_m$-module, isomorphic to the sum

$$(V(\omega_1) \otimes V(\alpha|_{\mathbb{G}_m})) \oplus \big(V(\omega_1) \otimes V(3\alpha|_{\mathbb{G}_m})\big) \cong \Bbbk^2 \oplus \Bbbk^2.$$

The group L has the open orbit

$$\{(v, w) \in \Bbbk^2 \oplus \Bbbk^2 \mid v \neq 0 \neq w, \, \Bbbk v \neq \Bbbk w\},$$

hence H has an open orbit on G/P_β.

To show that $H \subset P_\beta$ has an open orbit on G/P_α we proceed as above, showing that the locus where gP_α satisfies

$$\dim({}^gP_\alpha \cap B \cap H) > 0$$

has codimension at least 2 in G/P_α.

Suppose first that gP_α is in the open Schubert cell, i.e. g is of the form uw_0 for $u \in U$. Then

$$({}^gP_\alpha \cap B) \cap H = ({}^{uw_0}P_\alpha \cap B) \cap H = {}^u(TU_\alpha) \cap H$$

We can write

$$u = u_{\alpha+\beta}(x_1)u_{3\alpha+\beta}(x_2)u_{3\alpha+2\beta}(x_3)$$

since here $U_\beta U_{2\alpha+\beta} \subset H$. Again, the intersection ${}^u(TU_\alpha) \cap H$ is infinite only if two of x_1, x_2, x_3 are zero.

Consider now the codimension 1 Schubert cell, i.e. g is of the form uw_0s_β. Then

$$({}^gP_\alpha \cap B) \cap H = ({}^{uw_0s_\beta}P_\beta \cap B) \cap H = {}^u(TU_\beta U_{\alpha+\beta}) \cap H$$

Here we can assume that

$$u = u_\alpha(x_1)u_{3\alpha+\beta}(x_2)u_{3\alpha+2\beta}(x_3)$$

where x_1, x_2, x_3 are coordinate functions on the 4-dimensional Schubert cell. Consider also elements $t \in T$, $u_\beta(y_1)$ and $u_{\alpha+\beta}(y_2)$ with $y_1, y_2 \in \Bbbk$. Then ${}^u(tu_\beta(y_1)u_{\alpha+\beta}(y_2))$ is equal to

$$tu_\alpha(A_\alpha)u_{3\alpha+\beta}(A_{3\alpha+\beta})u_{3\alpha+2\beta}(A_{3\alpha+2\beta})u_{2\alpha+\beta}(A_{2\alpha+\beta})u_\beta(A_\beta)u_{\alpha+\beta}(A_{\alpha+\beta})$$

where

$$\begin{array}{lcl} A_\alpha & = & (a-1)x_1 \\ A_{3\alpha+\beta} & = & (a^3b-1)x_2 + x_1^3y_1 - 3x_1^2y_2 \\ A_{3\alpha+2\beta} & = & (a^3b^2-1)x_3 - x_2y_1 - 9x_1^2y_1y_2 - x_1^3y_1^2 - 3x_1y_2^2 \\ A_{\alpha+\beta} & = & x_1y_1 + y_2 \end{array}$$

and $a = \alpha(t^{-1})$, $b = \beta(t^{-1})$. This element is in H only if $A_\alpha = A_{3\alpha+\beta} = A_{3\alpha+2\beta} = A_{\alpha+\beta} = 0$. For general x_1, x_2 and x_3 the resulting system has finitely many solutions in a, b, y_1, y_2. This finishes the proof. □

6.6. Corollary. *Theorem* **2.1** *holds for every connected subgroup H of G such that H is neither solvable nor very reductive, and does not satisfy $L' \subseteq H \subseteq L$ where $P \supsetneq B$ is a proper parabolic subgroup of G and L is a Levi of P.*

Proof. Since H is not solvable nor very reductive, it is contained in a proper parabolic subgroup P, which may be assumed to properly contain B. Denote by L a Levi subgroup of P. Thanks to Lemma **6.2**, H has a Levi subgroup inside L and containing L'. Moreover, the inclusion $H^u \subseteq P^u$ holds thanks to Lemma **6.4**, and H^u is not trivial thanks to our assumptions.

To prove the corollary it is enough to show that H is spherical or L^r normalizes H. Indeed, if L^r normalizes H, then by Lemma **2.4** the sets $\mathfrak{B}_{00}(G/H)$ and $\mathfrak{B}_{00}(G/L^rH)$ are $\widetilde{W}$-equivariantly isomorphic. In this case the corollary stems from Proposition **6.5**, since L^rH contains L.

Suppose first that P^u is abelian and a trivial or simple L'-module, which is the case if G has type $\mathsf{A}_1 \times \mathsf{A}_1$ or A_2, or G has type B_2 and $P = P_{\alpha_2}$. Then by Lemma **6.1**, part (1), we conclude that H^u is a L-stable submodule of P^u, which implies that H is normalized by L^r.

If G has type B_2 and $P = P_{\alpha_1}$, then $P^u/(P^u)'$ under the action of L' is the SL(2)-module $V(\omega_1)$ and the image K of H^u in the quotient $P^u/(P^u)'$ is an L'-stable additive subgroup. Lemma **6.1**, part (1) implies that K is either the whole $V(\omega_1)$ or trivial. In the first case then $H^u = P^u$. In the second case $H^u \subseteq (P^u)'$, and since the former is non-trivial while the latter is one-dimensional we have $H^u = (P^u)'$. In any case H is both spherical and normalized by L^r, and this concludes the proof in the case G is not of type G_2.

We may now suppose that G has type G_2. We start with the case $P = P_{\alpha_1}$.

If $p = 3$ then P^u is abelian, therefore SL(2)-isomorphic to $\mathrm{Sym}^3(V(\omega_1)) \oplus V(0)$. The first summand has a composition series $\mathrm{Sym}^3(V(\omega_1)) \supseteq V_1 \supseteq \{0\}$ of SL(2)-submodules with both V_1 and V/V_1 simple and 2-dimensional.

Consider the projection K of H^u in $\mathrm{Sym}^3(V(\omega_1))$. By Lemma **6.1**, part (1), the intersection $K \cap V_1$ and the projection of K on V/V_1 are SL(2)-submodules, therefore either 0- or 2-dimensional.

If both are 0-dimensional then H^u is nontrivial and contained in $V(0)$, therefore $H^u = V(0)$. This implies that H is normalized by L^r. Suppose that at least one is 2-dimensional, and consider the projection J of H^u on $V(0) \subset P^u$. Then either J is $V(0)$, which implies that P^u/H^u is SL(2)-spherical and thus H is a spherical subgroup of G, or J is trivial, which implies that L^r normalizes H.

If $p > 3$ then $V = P^u/(P^u)' \cong \mathrm{Sym}^3(V(\omega_1))$ is irreducible under the action of SL(2). It follows from Lemma **6.1**, part (1) that the projection of H^u on V is an SL(2)-submodule, so either trivial, and hence H is normalized by L^r, or the full V, which implies that $H^u = P^u$ and that H is spherical.

We deal now with the case $P = P_{\alpha_2}$. If $H^u \subseteq (P^u)'$, since the latter is abelian and the sum of a trivial and a 2-dimensional simple SL(2)-module, we conclude from Lemma **6.1**, part (2) that H^u is an SL(2)-submodule, thus normalized by L^r.

Otherwise H^u projects surjectively on $P^u/(P^u)'$, because the latter is a simple SL(2)-module. Then $H^u = P^u$ and H is a spherical subgroup of G. □

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