General Information Conditioned by a Variable Event

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ABSTRACT

The aim of this paper is to present, by axiomatic way, an idea about the general conditional information of a single, fixed fuzzy set when the conditioning fuzzy event is variable. The properties of this conditional information are translated in a system of functional equations. Some classes of solutions of this functional system have been found.

1. INTRODUCTION

In these last years, on crisp setting, *measures of information J*, have been studied by many authors [1] [2-4]. Later particular researches have been done to these information measures defined without probability. The last information measures are called *general* because they are defined without any probability [5-8]. Analogous studies were presented in [9] on fuzzy setting.

In order to study the integration in information theory without probability, in [10-12] the families $\mathcal{N}, \mathcal{F}, \mathcal{I}_\infty, \mathcal{I}_0$ have been introduced. The mentioned classes of crisp sets have replaced the families of null sets in the classical theory of integration (*i.e.* with respect to an additive measure or a probability). A detailed overview can be found in [13].

Later, since 2005, on fuzzy setting, measures of conditional information, when the conditional event is fixed, have been considered and studied (see [14-16]).

Indeed, in this paper, a definition of the general conditional information of a fixed fuzzy set A, when the **conditioning fuzzy set** H is **variable** will be presented and it will be indicated by $J(A|H)$. For this reason, the families mentioned above will be adapted to the fuzzy setting and the corresponding class $\mathcal{H}_A$ will be introduced.

Example: let A be the fuzzy set of old men, H_1, H_2, H_3 the fuzzy set of those old men, who seem ill, who are ill, who are seriously ill, respectively. These different conditions H_1, H_2, H_3 are the conditioning variable events. The conditional information: $J(A|H_1), J(A|H_2), J(A|H_3)$ measure the influence of the grade of the illness in the old men.

The paper is organized in the following way: in Sect. 2 some preliminaries are recalled; in Sect. 3 the

definition of general information conditioned by a variable event is given. The statement of the problem is presented in Sect. 4 and in Sect. 5 the properties of the form of conditional information are translated in a system of functional equations [17], for which some classes of solutions are shown. Sect. 6 is devoted to the conclusions.

2. PRELIMINAIRES

In this paragraph, the definition of general information for fuzzy sets is recalled [9]. The concept of fuzzy set was introduced by Zadeh in [18], for all knowledge see [19, 20].

Let X be an abstract space and $\mathcal{A}$ a-algebra of all fuzzy sets of X, such that $(X, \mathcal{A})$ is a measurable space.

Definition 2.1 *In the fuzzy setting* measure of the general information *is a map*
$J(\cdot): \mathcal{A} \rightarrow [0, +\infty]$
such that $\forall A_1, A_2 \in \mathcal{A}$:

(i) $A_1 \supset A_2 \Rightarrow J(A_1) \leq J(A_2)$,

(ii) $J(\varnothing) = +\infty, J(X) = 0$.

Following the idea presented in [10, 12], assigned an information measure J, the following families are introduced:

$$\mathcal{I}_0 = \{N \in \mathcal{A} / J(N) = 0\}; \tag{1}$$

$$\mathcal{I}_{+\infty} = \{F \in \mathcal{A} / J(F) = +\infty\}. \tag{2}$$

The family (1) is not empty because it contains the whole set X and all supersets N' of $N \in \mathcal{I}_0$:

$$N \in \mathcal{I}_0, J(N) = 0, \forall N' \in \mathcal{A}, N \subset N', J(N') \leq J(N) = 0 \Rightarrow N' \in \mathcal{I}_0.$$

$\mathcal{I}_0$ is not an ideal [21, 22] because it is not stable with respect to the union between fuzzy sets.
The family (2) is not empty because it contains the empty set $\varnothing$ and all subsets F' of $F \in \mathcal{I}_{+\infty}$:

$$F \in \mathcal{I}_{+\infty}, J(F) = +\infty, \forall F' \in \mathcal{A}, F' \subset F, J(F) \geq J(F') = +\infty \Rightarrow F' \in \mathcal{I}_{+\infty}.$$

$\mathcal{I}_{+\infty}$ is not an filter [21, 22] because it is not stable with respect to the intersection between fuzzy sets.

3. MEASURE OF GENERAL CONDITIONAL INFORMATION BY A VARIABLE EVENT

From now on, the family $\mathcal{H} = \mathcal{A} - \mathcal{I}_{+\infty}$ shall be considered and measure of general conditional information of a fixed fuzzy set $A \in \mathcal{A}$, $J(A|\cdot)$ defined on the family $\mathcal{H}$ will be introduced.

Definition 3.1 Measure of general information of a fixed $A \in \mathcal{A}$, conditioned by a variable event $H \in \mathcal{H}$ is a map

$$J(A|\cdot): H \rightarrow [0, +\infty]$$

such that

(j) $H \supset H' \Rightarrow J(A|H) \leq J(A|H'), \ \forall H, H' \in \mathcal{H}$,

(jj) $J(A|N) = J(A)$ if $N \in \mathcal{I}_0$.

From the previous axioms, it follows that $J(A|X) = J(A)$.

The condition (j) is the monotonicity, the (jj) means that all null sets $N \in \mathcal{N}_0$ don't condition any fuzzy set $A \in \mathcal{A}$.

4. STATEMENT OF THE PROBLEM

Taking into account the previous axiomatic statement, fixed an information measure J on $\mathcal{A}$ and any fuzzy set $A \in \mathcal{A}$, some classes of measures $J(A|H)$, will be sought by supposing that $J(A|H)$ de-

pends only on $J(A), J(H)$ and $J(A \cap H)$. Now it is necessary to specify the family where H belongs.

Fixed A, our definition is restricted to the following family

$$\mathcal{H}_A = \{H \in \mathcal{H} \, / \, J(A \cap H) \neq +\infty\}$$

at least $\mathcal{H}_A$ contains the whole space X, so this family is not empty. The condition $J(A \cap H) \neq +\infty$ ensures that $J(H) \neq +\infty$: in fact if $J(H) = +\infty$ from the monotonicity of J, it is $J(A \cap H) = +\infty$.

So, the information $J(A|\cdot)$ is the function

$$J(A|\cdot) : \mathcal{H}_A \to [0, +\infty]$$

expressed by a function $\Phi : V \to (0, +\infty)$, such that

$$J(A|H) = \Phi(J(A), J(H), J(A \cap H))$$

with

$$V = \{(x, y, z) / x = J(A), y = J(H), z = J(A \cap H), x, y, z \in [0, +\infty], x \leq z, y \leq z\}.$$

This justifies the domain of the function Φ.

From (3), (j), (jj), $\forall H_1, H_2 \in \mathcal{H}$, the function Φ shall satisfy the following properties:

(I) $H_1 \supset H_2 \Rightarrow \Phi(J(A), J(H_1), J(A \cap H_1)) \leq \Phi(J(A), J(H_2), J(A \cap H_2))$,

(II) $\Phi(J(A), J(N), J(A \cap N)) = J(A), \quad \forall N \in \mathcal{I}_0$.

Setting $J(A) = x$, $J(H_1) = y_1$, $J(H_2) = y_2$, $J(A \cap H_1) = z_1$, $J(A \cap H_2) = z_2$, $J(A \cap N) = t$ with $x, y_1, y_2, z_1, z_2 \in [0, +\infty)$ and $x \leq z_1$, $y_1 \leq z_1$, $x \leq z_2$, $y_1 \leq z_2$, $x \leq t$, the following system of functional equations is obtained:

$$\begin{cases} (1) \ \Phi(x, y_1, z_1) \leq \Phi(x, y_2, z_2), & x \leq z_1, y_1 \leq z_1, x \leq z_2, y_2 \leq z_2 \\ (2) \ \Phi(x, 0, t) = x, & x \leq t. \end{cases}$$

5. SOLUTION OF THE PROBLEM

A function Φ continuous defined in the following set:

$$D = \{(x, y, z) : x, y, z \in [0, +\infty), x \leq z, y \leq z\} \tag{3}$$

will be sought as an universal law in the sense that the equation and the inequality about the function of the system must be satisfied for all values and variables in their proper space, which satisfies the system [(1)-(2)].

Now, the following results are proved:

Proposition 4.1 *A class of solutions of the system* [(1)-(2)] *is:*

$$\Phi_h(x, y, z) = h^{-1}([h(x) + h(y)] \wedge h(z)), \tag{4}$$

where $h : [0, +\infty) \to [0, +\infty)$ is any continuous function, strictly increasing with $h(0) = 0$.

Proof. The condition (1) follows by the monotonicity of the function h. The second one (2) results from the value $h(0) = 0$. and the property of the function h.

Proposition 4.2 *A class of solutions of the system* [(1)-(2)] *is:*

$$\Phi_h(x, y, z) = h^{-1}([h(x) \vee h(y)] \wedge h(z)) = h^{-1}(h(x) \vee h(y)), \tag{5}$$

where $h : [0, +\infty) \to [0, +\infty)$ is any continuous function, strictly increasing with $h(0) = 0$.

Proof. The proof is immediate. $\square$

Proposition 4.3 *A class of solutions of the system* [(1)-(2)] *is:*

$$\Phi_h(x, y, z) = h^{-1}([h(x) \wedge h(y)] \vee h(z)) = z, \tag{6}$$

where $h : [0, +\infty) \to [0, +\infty)$ is any continuous function, strictly increasing with $h(0) = 0$.

Proof. The proof is immediate. $\qquad\square$

From (4) (5) and (6), the following expressions of conditional information have been obtained, respectively:

$$J_h(A \mid H) = h^{-1}\left(\left[h(J(A)) + h(J(H))\right] \wedge h(J(A \cap H))\right); \tag{7}$$

$$J_h(A \mid H) = h^{-1}\left(h(J(A))\right) \vee h(J(H)), \tag{8}$$

where $h : [0, +\infty) \to [0, +\infty)$ is any continuous function, strictly increasing with $h(0) = 0$, and

$$J(A \mid H) = J(A \cap H). \tag{9}$$

6. CONCLUSIONS

In this paper, for the first time, we present an axiomatic definition of the information $J(A|H)$, when the conditioning event is variable.

We think that this axiomatic approach could be useful for future applications.

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