Digital stabilization of strict feedback dynamics through immersion and invariance ⋆

M. Mattioni, S. Monaco * **and D. Normand-Cyrot** **

* *Dipartimento di Ingegneria Informatica, Automatica e Gestionale 'Antonio Ruberti', Università di Roma "La Sapienza", via Ariosto 25, 00185 Roma, Italy. ({mattioni, monaco}@dis.uniroma1.it)*
** *Laboratoire des Signaux et Systèmes, CNRS-Supelec, 3 Joliot Curie, 91190 Gif-sur-Yvette, France (cyrot@lss.supelec.fr)*

Abstract: This paper deals with the extension to sampled-data stabilization of strict feedback dynamics of the Immersion and Invariance procedure proposed in Astolfi and Ortega [2003]. A direct digital approach is developed in two steps: first the target dynamics and immersion mapping are defined for the equivalent discrete-time model; then the control law is built to drive the dynamics towards the invariant manifold. A simulated example illustrates the performances.

Keywords: Nonlinear systems, Stabilization, Sampled-data systems.

1. INTRODUCTION

Stabilization of continuous-time strict-feedback dynamics has been widely investigated in the last decades and several solutions have been proposed. Among them, backstepping is the most popular. It was firstly introduced in continuous time in Kokotović and Arcak [2001] and later on extended to the sampled-data context (Nešić and Teel [2006], Postoyan et al. [2009], Monaco et al. [2011]). Immersion and Invariance (I&I) proposed by Astolfi and Ortega [2003] represents an interesting design procedure as shown in Astolfi et al. [2008]; applications to different engineering domains have been also developed in (Mannarino and Mantegazza [2014], Hristea and Siguerdidjane [2011]). A first contribution in discrete time has been more recently proposed in Yalcin and Astolfi [2011].

I&I technique relies on the idea of defining a manifold where a lower dimensional dynamics is known to be stable and setting a control law making it attractive and invariant. In this way, the problem reduces to drive the dynamics to the invariant manifold with boundedness of the complete state trajectory.

In this paper, I&I is addressed in the sampled-data (SD) context starting from a continuous-time dynamics in strict feedback form

$$\begin{aligned}\dot{x}_1 &= f(x_1) + g(x_1)x_2 \\ \dot{x}_2 &= a(x_1,x_2) + b(x_1,x_2)u\end{aligned} \qquad (1)$$

which admits a backstepping stabilizer ensuring global asymptotic stability (GAS) of the equilibrium. A first result is proposed in Mattei et al. [2015] where a SD controller is designed to ensure partial input-to-state matching of the continuous off the manifold component. This assures attractivity under sampled-data control but not invariance, which holds in that case, because the external control is assumed equal to zero on the manifold.

⋆ This work is supported by a public grant overseen by the French National research Agency (ANR) as part of the Investissement d'Avenir program, through the "iCODE" project funded by the IDEX Paris-Saclay, ANR-11-IDEX-0003-02"

In the present work, a general solution is proposed for strict feedback dynamics. The design, worked out on the equivalent sampled-data model, provides a "direct digital" controller which is developed in two steps. First, it is shown that a target dynamics can be defined in the sampled-data domain while ensuring the invariance of the consequent manifold. Then, it is shown how to design a SD controller preserving manifold invariance and attractivity with trajectory boundedness. The computation of approximate solutions is also addressed. Computational issues for sampled-data design are in Monaco and Normand-Cyrot [2007].

The paper is organized as follows: in Section 2 some brief recalls on continuous-time backstepping and I&I design are provided. Section 3 deals with sampled-data I&I stabilization. An example is worked out in Section 4 with simulations illustrating the performances.

2. PRELIMINARIES

2.1 Some recalls

In the sequel, some recalls on continuous-time backstepping Khalil [1996] and I&I design Astolfi and Ortega [2003] are made for dynamics (1). The vector fields $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are assumed complete and without loss of generality the origin is an equilibrium.

Theorem 2.1. (Khalil [1996]) Consider (1). If there exist a smooth function $\gamma(x_1)$ with $\gamma(0) = 0$ and a positive-definite W: $\mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial W}{\partial x_1}(f(x_1) + g(x_1)\gamma(x_1)) < 0 \qquad \forall x_1 \in \mathbb{R}^n/\{0\}$$

then the state feedback control law

$$u = b^{-1}(x_1,x_2)[\dot{\gamma}(x_1) - \frac{\partial W}{\partial x_1}g(x_1) - a(x_1,x_2) - K(x_2 - \gamma(x_1))]$$

globally asymptotically stabilizes the origin with $K > 0$. ◁◁

Set, for simplicity

$$\bar{f}(x) = \begin{pmatrix} f(x_1) + x_2 g(x_1) \\ a(x_1, x_2) \end{pmatrix}; \; \bar{g}(x) = \begin{pmatrix} 0 \\ b(x_1, x_2) \end{pmatrix}.$$

Theorem 2.1 can be reformulated in the I&I framework by setting $\xi \in \mathbb{R}^n$ and

$$x = \pi(\xi) = \begin{pmatrix} \xi \\ \gamma(\xi) \end{pmatrix}; \qquad c(\xi) = b^{-1}(\xi)[\dot{\gamma}(\xi) - a(\xi, \gamma(\xi))];$$
$$z = \phi(x_1, x_2) = x_2 - \gamma(x_1). \tag{2}$$

Following Astolfi and Ortega [2003], the Corollary below sets the problem in the I&I context.

Corollary 2.1. Consider the system in (1) under the hypotheses of Theorem 2.1. Then, it is I&I stabilizable with target dynamics $\dot{\xi} = f(\xi) + g(\xi)\gamma(\xi)$ by means of the feedback

$$\psi(x, z) = b^{-1}(x_1, x_2)[\dot{\gamma}(x_1) - a(x_1, x_2) - K(x_1, x_2)z] \tag{3}$$

with $K(x_1, x_2) \geq K > 0$; i.e. there exist smooth functions

$$\alpha : \mathbb{R}^n \to \mathbb{R}^n, \quad \pi : \mathbb{R}^n \to \mathbb{R}^{n+1}, \qquad c : \mathbb{R}^n \to \mathbb{R}$$
$$\phi : \mathbb{R}^{n+1} \to \mathbb{R}, \; \psi : \mathbb{R}^{(n+1)\times 1} \times \mathbb{R} \to \mathbb{R}$$

such that the following hold

H1 (Target Dynamics) the system

$$\dot{\xi} = f(\xi) + g(\xi)\gamma(\xi) = \alpha(\xi) \tag{4}$$

with $\xi \in \mathbb{R}^n$ has a globally asymptotically stable equilibrium at ξ_e and $x_e = \pi(\xi_e)$.

H2 (Immersion Condition) For all $\xi \in \mathbb{R}^n$, $\exists\, \pi(\xi) = (\xi \;\; \gamma(\xi))^T$ and $c(\xi)$ such that

$$\bar{f}(\pi(\xi)) + \bar{g}(\pi(\xi))c(\xi) = \frac{\partial \pi}{\partial \xi}[f(\xi) + g(\xi)\gamma(\xi) \tag{5}$$

H3 (Implicit manifold) The following identity holds

$$\{x \in \mathbb{R}^n | \phi(x) = 0\} = \{x \in \mathbb{R}^n | x = \pi(\xi) \; for \; some \; \xi \in \mathbb{R}^n\}$$

with $\phi(x) = x_2 - \gamma(x_1)$ and $z_0 = \phi(x_0)$.

H4 (Manifold attractivity and trajectory boundedness) All trajectories of the system

$$\begin{aligned} \dot{z} &= \frac{\partial \phi}{\partial x}[\bar{f}(x) + \bar{g}(x)\psi(x, z)] \\ \dot{x} &= \bar{f}(x) + \bar{g}(x)\psi(x, z) \end{aligned} \tag{6}$$

are bounded with

$$\lim_{t \to \infty} z(t) = 0, \;\; \psi(\pi(\xi), \mathbf{0}) = c(\xi). \tag{7}$$

◁

Proof. The target dynamics is given in (4); accordingly, **H2** and **H3** are satisfied by the choice in (2) with z, the off-the-manifold variable. The feedback

$$\psi(x, z) = b^{-1}(x_1, x_2)[\dot{\gamma}(x_1) - a(x_1, x_2) - K(x_1, x_2)z]$$

with $K(x_1, x_2) \geq k > 0$ brings to the closed-loop dynamics

$$\begin{aligned} \dot{x}_1 &= f(x_1) + g(x_1)x_2 \\ \dot{x}_2 &= \dot{\gamma}(x_1) - K(x_1, x_2)z \\ \dot{z} &= -K(x_1, x_2)z \end{aligned}$$

with z converging to zero as the time increases. To prove boundedness of the trajectories, it is sufficient to show boundedness of x_1 and z. Hence, set any $M > 0$ such that W, as defined in (2.1), verifies $(L_f + \gamma L_g)W(x_1) < 0$, $\forall \|x_1\| > M$. Consider now the Lyapunov function $V(x_1, z) = W(x_1) + \frac{1}{2}z^2$ and its derivative along the trajectories of the system. Hence, for any smooth function $\rho(x_1) > 0$, one has

$$\dot{V}(x_1, z) \leq (L_f + \gamma L_g)W + \frac{\|L_g W\|^2}{\rho(x_1)} + \rho(x_1)z^2 - \tilde{K}(x_1, z)z^2 \tag{8}$$

with $\tilde{K}(x_1, z) = K(x_1, z + \gamma(x_1))$. Finally, (8) is made negative by defining a and $\tilde{K}$ such that

$$\begin{aligned} &(L_f + \gamma L_g)W + \frac{\|L_g W\|^2}{\rho(x_1)} < 0 \;\; \forall \|x_1\| > M \\ &\tilde{K}(x_1, z) > \rho(x_1). \end{aligned} \tag{9}$$

Hence the thesis. ◁

2.2 *Problem settlement and the class of system under study*

It is assumed in the sequel that the control input u is piecewise constant over intervals of fixed length δ, the sampling period. One looks for a controller u, possibly δ-dependent, that makes the equilibrium of the closed-loop system globally asymptotically stable at the sampling instants. With this in mind, the following definition is set.

Definition 2.1. A system described by equation

$$\dot{x} = f(x) + g(x)u$$

is said to be sampled-data I&I stabilizable if there exist $T > 0$ and for each $\delta \in]0, T^*[$, a piecewise constant control u constant over time intervals of length δ; i.e. $u(t) = u_k$ for $t \in [k\delta, (k+1)\delta[$ such that the equivalent sampled-data dynamics satisfy **H1d**,**H2d**, **H3d** and **H4d** provided in the sequel.

Hereafter, for simplicity of notations, we consider the simplest strict-feedback form

$$\begin{aligned} \dot{x}_1 &= f(x_1) + g(x_1)x_2 \\ \dot{x}_2 &= u \end{aligned} \tag{10}$$

under the assumption of Theorem 2.1.

Indeed, the theory developed in the sequel can be extended to any system in the general form (1) by first carrying out the design for the integrator dynamics $\dot{x}_2 = v_k$ and then finding u_k such that

$$u_k = (\int_{k\delta}^{(k+1)\delta} b(x(\tau))d\tau)^{-1} \int_{k\delta}^{(k+1)\delta} [v_k - a(x(\tau))]d\tau$$

providing an implicit definition of u_k.

Under sampling, dynamics (10) takes the form Monaco and Normand-Cyrot [2007]

$$\begin{aligned} x_{1k+1} &= F_1^\delta(x_{1k}, x_{2k}) + \frac{\delta^2}{2!}u_k G^\delta(x_{1k}, x_{2k}, u_k) \\ x_{2k+1} &= x_{2k} + \delta u_k \end{aligned} \tag{11}$$

with

$$F_1^\delta(x_{1k}, x_{2k}) = e^{\delta(f + x_{2k}g)}x_1\big|_{x_1 = x_{1k}}$$
$$G^\delta(x_{1k}, x_{2k}, u_k) = g(x_{1k}) + \frac{\delta}{3}(2L_f L_g + L_g L_f + 3x_{2k}L_g^2)x_1\big|_{x_1 = x_{1k}} + O(\delta^2).$$

The strict-feedback structure is lost under sampling since the dynamics of x_1 is directly influenced by the control which acts with terms in $O(\delta^2)$ at least.

3. SAMPLED-DATA I&I STABILIZATION

In this section, the design of the I&I controller is directly performed on the SD dynamics (11): namely, according to the sampled-data nature of the closed-loop system, the conditions **H1**, **H2**, **H3** and **H4** of Corollary 2.1 will be reformulated and shown to hold.

3.1 On the choice of the digital target dynamics

Given (10) and its equivalent sampled-data dynamics in (11), let us define the SD immersion mapping $\pi^\delta : \mathbb{R}^n \to \mathbb{R}^{n+1}$ as

$$x = \pi^\delta(\xi) = \left(\xi' \;\; \gamma^\delta(\xi)'\right)' \tag{12}$$

with $\xi \in R^n$ and the SD target dynamics as

$$\xi_{k+1} = F_1^\delta(\xi_k, \gamma^\delta(\xi_k)) + \frac{\delta^2}{2!}c^\delta(\xi_k)G^\delta(\xi_k, \gamma^\delta(\xi_k), c^\delta(\xi_k)) \tag{13}$$

where $\gamma^\delta(\xi)$ and $c^\delta(\xi)$ must be computed to satisfy

$$W(\xi_{k+1}) = W(\xi_k) + \int_{k\delta}^{(k+1)\delta} L_{(f+g\gamma)}W(\xi(\tau))d\tau \tag{14}$$

$$\gamma^\delta(\xi_{k+1}) = \gamma^\delta(\xi_k) + \delta c^\delta(\xi_k). \tag{15}$$

The equality (14) ensures Lyapunov Matching of $W(\cdot)$ under $\gamma^\delta(\cdot)$ while (15) satisfies the immersion and invariance condition (sampled-data version of the **H2**). Accordingly, a different invariant manifold is now obtained by setting

$$z^\delta = \phi^\delta(x) = x_2 - \gamma^\delta(x_1).$$

It is important to point out that $c^\delta(\cdot)$ is implicitly defined in terms of $\gamma^\delta(\cdot)$ in (15). Its computation can be nevertheless worked out by substituting $\gamma^\delta(\xi_{k+1})$ into (15) with its Taylor expansion in a neighbourhood of ξ_k

$$\gamma^\delta(\xi_{k+1}) = \gamma^\delta(\xi_k) + \sum_{i\geq 1}\frac{1}{i!}\frac{\partial^i\gamma^\delta}{\partial\xi^i}\Big|_{\xi_k}(\xi_{k+1}-\xi_k)^i \tag{16}$$

with ξ_{k+1} defined as in (13). Setting now the following structure for $c^\delta(\cdot)$ and $\gamma^\delta(\cdot)$ (Monaco and Normand-Cyrot [2007])

$$c^\delta(\xi) = c_0(\xi) + \sum_{i\geq 1}\frac{\delta^i}{(i+1)!}c_i(\xi) \tag{17}$$

$$\gamma^\delta(\xi) = \gamma_0(\xi) + \sum_{i\geq 1}\frac{\delta^i}{(i+1)!}\gamma_i(\xi) \tag{18}$$

easy but even tedious computations enable us to show that the first terms are solutions of the following equalities

$$c_0(\xi_k) = (L_f + \gamma_0 L_g)\gamma_0\big|_{\xi_k}$$
$$c_1(\xi_k) = [\gamma_1 L_g\gamma_0 + c_0 L_g\gamma_0 + (L_f+\gamma_0 L_g)\gamma_1 + (L_f+\gamma_0 L_g)^2\gamma_0]_{\xi_k}$$
$$\cdots.$$

Substituting now $c^\delta(\cdot)$ with such expressions into (14), one reformulates (14) as an equality in $\gamma^\delta(\cdot)$ which can be iteratively solved so getting for the first terms

$$\gamma_0(\xi_k) = \gamma(\xi)\big|_{\xi_k}$$
$$\gamma_1(\xi_k) = 0$$
$$\gamma_2(\xi_k) = [\ddot{\gamma}(\xi) - \frac{1}{2}c_1(\xi)]_{\xi_k}$$
$$\cdots.$$

and thus according to (17)

$$c_0(\xi_k) = \dot{\gamma}(\xi)\big|_{\xi_k}$$
$$c_1(\xi_k) = [c_0 L_g\gamma_0 + \ddot{\gamma}]_{\xi_k}.$$

The following result can now be stated.

Proposition 3.1. Consider (10) under the hypotheses of Theorem 2.1 with $L_gW \neq 0$. Then, there exist $T^* > 0$ and, for each $\delta \in]0, T^*[$, smooth functions

$$\alpha^\delta : \mathbb{R}^n \to \mathbb{R}^n;\; \pi^\delta : \mathbb{R}^n \to \mathbb{R}^{n+1}$$
$$c^\delta : \mathbb{R}^n \to \mathbb{R}; \quad \phi^\delta : \mathbb{R} \to \mathbb{R}$$

such that the following conditions hold true:

H1d (Target Dynamics) The equation (13) rewritten as

$$\xi_{k+1} = \alpha^\delta(\xi)$$

with $\xi \in \mathbb{R}^n$ has a globally asymptotically stable equilibrium at ξ_e and $\pi^\delta(\xi_e) = x_e$;

H2d (Immersion Condition) For all $\xi \in \mathbb{R}^n$, define $\pi^\delta(\xi) = \left(\pi_1^\delta(\xi)\;\; \pi_2^\delta(\xi)\right)^T$ and $c^\delta(\xi)$ such that

$$\pi_1^\delta \circ \alpha^\delta(\xi) = F_1^\delta(\pi_1^\delta(\xi), \pi_2^\delta(\xi)) + \frac{\delta^2}{2!}c^\delta(\xi)G^\delta(\pi_1^\delta(\xi), \pi_1^\delta(\xi), c^\delta(\xi))$$
$$\pi_2^\delta \circ \alpha^\delta(\xi) = \pi_2^\delta(\xi) + \delta c^\delta(\xi);$$

H3d (Implicit manifold) The following identity holds

$$\{x\in\mathbb{R}^n | \phi^\delta(x) = 0\} = \{x\in\mathbb{R}^n | x = \pi^\delta(\xi)\; for\; some\; \xi\in\mathbb{R}^n\}$$

with $\phi^\delta(x) = x_2 - \gamma^\delta(x_1)$ and $z_0 = \phi^\delta(x_0)$.

◁

Proof. First, one rewrites (14) as a formal series equality

$$\delta Q(\xi,\delta,\gamma^\delta) = W(\xi_{k+1}) - W(\xi_k) - e^{\delta(L_f+\gamma L_g)}W\big|_{\xi_k} = 0$$

with ξ_{k+1} as in (13) and one looks for $\gamma^\delta(\cdot)$ satisfying

$$Q(\xi,\delta,\gamma^\delta(\xi)) = 0 \;\; \forall \xi \in \mathbb{R}^n \tag{19}$$

in which, by definition, $Q(\xi,\delta,\dot{}) := Q_0(\xi,.) + \sum_{i\geq 1}\delta^i Q_i(\xi,.)$. It is immediately verified that $\gamma_0(\xi) = \gamma(\xi)$ satisfies (19) for $\delta = 0$; i.e.

$$Q_0(\xi,\gamma_0) = (L_f+\gamma_0 L_g)W\big|_\xi - (L_f+\gamma L_g)W\big|_\xi = 0.$$

Furthermore, provided the rank condition

$$\frac{\partial Q(\xi,\delta,\gamma^\delta)}{\partial\gamma^\delta}\Big|_{\delta=0,\gamma^\delta=\gamma_0} = L_gW(\xi) \neq 0 \tag{20}$$

holds, one concludes from the Implicit Function Theorem, the existence of a T^* small enough such that for any $\delta \in]0, T^*[$, (19) admits a solution in the form of asymptotic expansion (18) around $\gamma(\xi)$. GAS of the equilibrium of the digital target dynamics follows from the property of the continuous-time dynamics: i.e., by construction of α^δ, the Lyapunov matching condition with respect to W holds

$$W(\xi_{k+1}) - W(\xi_k) = \int_{k\delta}^{(k+1)\delta}(L_f+\gamma L_g)W(\xi(\tau))d\tau < 0$$

provided that $(L_f+\gamma L_g)W(\xi) < 0$. As far as invariance is concerned, one easily shows by comparing the terms of the same power in δ in the expansion of the respective sides of the equality (15) to be satisfied, the existence of $c^\delta(\cdot)$ in the form of (17) such that (15) holds. We note that $c^\delta(\cdot)$ appears in the construction of $\gamma^\delta(\cdot)$ and vice versa. Nevertheless, no algebraic loop is introduced in the computation because each term γ_i depends on the previous terms γ_j and c_j for $0 \leq j \leq i-1$) and each c_j depends itself on γ_p for $p \leq j$. ◁

Remark Proposition 3.1 states the existence of a digital target dynamics, defined according to $c^\delta(\cdot)$ and $\gamma^\delta(\cdot)$, which results, in general, to be different from the continuous-time one as well as the immersion mapping π^δ. Indeed, as could be expected, in the sampled-data context the whole design procedure is parametrized by δ; so that by changing the sampling period the manifold and the mappings take different forms. Such a re-shaping takes into account the possible mismatch with respect to the continuous manifold occurring under sampled-data control. ◁

3.2 On the digital controller

In the previous section the SD dynamics was found and the global asymptotic stability of its equilibrium proved by means of a γ^δ which was computed by solving the one-step input Lyapunov matching problem (ILM-P) with respect to W (as in (14)).

At this point one has that the immersion is defined as in (12) and that the complete dynamics is defined by

$$\begin{aligned} x_{1k+1} &= \bar{F}_1^\delta(x_{1k}) + z_k P^\delta(x_{1k}, z_k) + \\ &\quad \frac{\delta^2}{2!}\psi^\delta(x_{1k}, z_k)\bar{G}^\delta(x_{1k}, z_k, \psi^\delta(x_{1k}, z_k)) \\ x_{2k+1} &= x_{2k} + \delta\psi^\delta(x_{1k}, x_{2k}, z_k) \\ z_{k+1} &= z_k + \delta\psi^\delta(x_{1k}, x_{2k}, z_k) - \gamma^\delta(x_{1k+1}) + \gamma^\delta(x_{1k}) \end{aligned} \quad (21)$$

with

$$\begin{aligned} z_k &= \phi^\delta(x_{1k}, x_{2k}) = x_{2k} - \gamma^\delta(x_{1k}) \\ \bar{F}_1^\delta(x_{1k}) &= F_1^\delta(x_{1k}, \gamma^\delta(x_{1k})) \\ P^\delta(x_{1k}, z_k) &= \sum_{i\geq 1} \frac{1}{i!}\frac{\partial^i F_1^\delta}{\partial {x_2}^i}\Big|_{\gamma^\delta(x_{1k})} z_k^{i-1} \\ \bar{G}^\delta(x_{1k}, z_k, \psi^\delta) &= G^\delta(x_{1k}, z_k + \gamma^\delta(x_{1k}), \psi^\delta). \end{aligned}$$

In particular, the digital control law is computed by solving an input-output problem on z_k. Manifold attractivity and trajectories boundedness are proved by Theorem 2.1.

Theorem 3.1. Under the hypotheses of Proposition 3.1, there exist $T^* > 0$ and for each $\delta \in]0, T^*[$ a sampled-data feedback $\psi^\delta : \mathbb{R}^{n+1} \to \mathbb{R}$ of the form

$$\psi^\delta(x_1, x_2, z) = \psi_0(x_1, x_2, z) + \sum_{i\geq 1} \frac{\delta^i}{(i+1)!}\psi_i(x_1, x_2, z) \quad (22)$$

which ensures digital I&I stabilization of the equilibrium of the dynamics in (10); namely, one has

H4d (Manifold Invariance and trajectory boundedness) all trajectories of the system

$$\begin{aligned} x_{1k+1} &= \bar{F}_1^\delta(x_1, x_2) + z_k P^\delta(x_1, x_2, z) + \\ &\quad \frac{\delta^2}{2!}\psi^\delta(x_1, x_2, z)\bar{G}^\delta(x_1, x_2, z, \psi^\delta(x_1, x_2, z)) \\ x_{2k+1} &= x_2 + \delta\psi^\delta(x_1, x_2, z) \\ z_{k+1} &= z + \delta\psi^\delta(x_1, x_2, z) - \gamma^\delta(x_{1k+1}) + \gamma^\delta(x_1) \end{aligned}$$

are bounded and satisfy

$$\lim_{k\to\infty} z_k = 0 \quad \psi^\delta(\pi^\delta(\xi), \mathbf{0}) = c^\delta(\xi). \quad (23)$$

◁

Proof. As already stated in Proposition 3.1, **H1d**, **H2d** and **H3d** hold, and the dynamics on the manifold defined by $\{x \in \mathbb{R}^{n+1} : x = \pi^\delta(\xi)\}$ has a globally asymptotically stable equilibrium. It remains to show that it is possible to design a control law $u_k = \psi^\delta(x_k, z_k)$ such that the trajectories of the closed-loop system are bounded and satisfy (23). To this end, consider the continuous-time I&I controller, namely $\psi(x, z)$. As in Fossard and Normand-Cyrot [1996], one can get sampeld-data matching of the evolution of the closed-loop continuous-time z-dynamics when the feedback control law u_c is applied to (10); i.e.,

$$\phi^\delta(x_{k+1}) = e^{\delta(L_{\bar{f}} + u_c L_{\bar{g}})}\phi\big|_{t=k\delta}. \quad (24)$$

By rewriting (24) as a formal series

$$\delta S(x_k, z_k, \delta, \psi^\delta) = \phi^\delta(x_{k+1}) - e^{\delta(L_{\bar{f}} + u_c L_{\bar{g}})}\phi\big|_{t=k\delta}$$

, one concludes from the Implicit Function Theorem pointing out that since

$$\frac{\partial S(x_k, z_k, \delta, u)}{\partial u}\Big|_{\delta=0, u=\psi} = L_{\bar{g}}\phi = 1 \neq 0 \quad (25)$$

a solution in the form (22) exists for $\delta \in]0, T^*[$ in a neighborhood of the continuous-time solution.

It results that input-partial state matching is satisfied in the absence of finite escape time, which is ensured by the hypothesis of Theorem 2.1. As a direct consequence one obtains that z_k converges to zero as k increases with boundedness of the whole state trajectories and $\psi^\delta(\pi^\delta(\xi_k), 0) = c^\delta(\xi_k)$. ◁

Each term of the digital controller can be derived by equating the terms in (24) with the same power of δ. In this way one gets that each term is computed by solving a linear equation in the previous components ψ_j. For the first terms, one has

$$\begin{aligned} \psi_0(x_k, z_k) &= \psi(x, z)\big|_{x_k, z_k}, \quad \psi_1(x_k, z_k) = \dot{\psi}(x, z)\big|_{x_k, z_k} + \frac{1}{3}\gamma_2(x_k) \\ \psi_2(x_k) &= \ddot{\psi}(x, z)\big|_{x_k, z_k} + \frac{1}{4}\gamma_3(x_k) + [(L_f + x_2 L_g)\gamma_2 + \\ &\quad \frac{1}{2}(\dot{\psi} + \gamma_2)L_g\gamma_0]_{x_k, z_k}. \end{aligned}$$

Remark Setting $\delta = 0$, the sampled-data control law $\psi^\delta(\cdot)$ and immersion mapping $\pi^\delta(\cdot)$ (and, consequently, $c^\delta(\cdot)$ and $\gamma^\delta(\cdot)$) reduce to their continuous-time analogs. ◁

The so far introduced controller is characterized through its asymptotic expansion, but only approximate solutions can be computed. A *(p,q)-th order approximate solution* takes the form

$$\psi^{\delta,[p,q]}(x_k, z_k) = \psi_0^{[p]} + \sum_{i=1}^{q} \frac{\delta^i}{(i+1)!}\psi_i^{[p]}$$

where p is the order of approximation of $\gamma^\delta(\cdot)$. One has to set δ small enough so that the hypotheses in Theorem 3.1 hold even when an approximated controller is applied.

4. AN ACADEMIC EXAMPLE

Consider the system

$$\dot{x}_1 = x_1^2 + x_2 \;\; \dot{x}_2 = u \quad (26)$$

whose equilibrium $x_e = (0,\ 0)^T$ has to be stabilized. It verifies the hypotheses of Theorem 2.1: i.e. there exist $W = \frac{1}{2}x_1^2$ and $\gamma(x_1) = -x_1 - x_1^2$ such that $(L_f + \gamma L_g)W < 0$ for $x_1 \in \mathbb{R} - \{0\}$. Hence, one can set $z = \phi(x_1) = x_2 + x_1 + x_1^2$.

4.1 Continuous-time design

In the continuous time case, the I&I control law which makes the origin globally asymptotically stable is

$$u_c(x) = -K(x_2 + x_1 + x_1^2) - (1 + 2x_1)(x_1^2 + x_2) \;\; K > a > 1$$

with $K = 2$. The immersion mapping and invariant manifold are defined as in Theorem 2.1. The target dynamics is $\dot{\xi} = -\xi$.

4.2 Digital design

First, introduce the sampled-data equivalent model associated to (26) in $O(\delta^3)$, which is provided by

$$x_{1k+1} = x_{1k} + \delta(x_{1k}^2 + x_{2k}) + \delta^2 x_{1k}(x_{1k}^2 + x_{2k}) + \frac{\delta^2}{2!}u_k + O(\delta^3)$$
$$x_{2k+1} = x_{2k} + \delta u_k.$$

We use the second order approximated sampled-data equivalent model since we shall compute the second order approximate controller $\psi^{\delta[p,q]}$. In this case, the target dynamics is defined as

$$\xi_{k+1} = \xi_k + \delta(\xi_k^2 + \gamma_0(\xi_k)) + \frac{\delta^2}{2!}\gamma_1(\xi_k) + \frac{\delta^2}{2!}[2\xi_k(\xi_k^2 + \gamma_0(\xi_k))] + \frac{\delta^2}{2!}c_0(\xi_k) + O(\delta^2)$$

where γ_0, γ_1, and c_0 are the terms defining $\gamma^{\delta,[2]}$ and $c^{\delta,[1]}$. According to Theorem 3.1 they are computed as

$$\gamma_0(\xi_k) = -\xi_k - \xi_k^2 \quad \gamma_2(\xi_k) = 2\xi_k^3$$
$$c_0(\xi_k) = (\xi_k + 2\xi_k^2) \;\; c_1(\xi_k) = -2\xi_k - 8\xi_k^2 - 4\xi_k^3.$$

The second-order approximated SD I&I control law is defined according to Theorem 3.1

$$\begin{aligned}
u_0(x_k) =& -2(x_{2k} + x_{1k} + x_{1k}^2) - (1 + 2x_{1k})(x_{1k}^2 + x_{2k}) \\
u_1(x_k) =& -2x_{1k}^4 + 6x_{1k}^3 - 4x_{1k}^2 x_{2k} + 11x_{1k}^2 + \\
& 6x_{1k}x_{2k} + 6x_{1k} - 2x_{2k}^2 + 7x_{2k} + \frac{1}{3}\gamma_2 \\
u_2(x_k) =& \; 18x_{1k}^4 - 2x_{1k}^3 + 36x_{1k}^2 x_{2k} - 27x_{1k}^2 \\
& -2x_{1k}x_{2k} - 14x_{1k} + 18x_{2k}^2 - 15x_{2k} + 6x_{1k}^2(x_{1k}^2 + x_{2k}) + \\
& (\frac{3}{2}u_2 - \dot{u}_c)(-1 - 2x_{1k}).
\end{aligned} \tag{27}$$

It is nasty but easy to verify that by setting $x_{2k} = \gamma^\delta(x_{1k})$ in (27) one gets exactly the expression of $c^\delta(x_{1k})$.

4.3 Simulations

The designed controller is compared to the continuous-time one and its discrete emulated version and the results are illustrated in the sequel. The simulations are performed with different sampling periods δ (0.1, 0.4 and 0.7 seconds) and initial condition $x = (0.5, 0.5)^T$. First of all, it can be pointed out that the attractive manifold defined by the sampled-data approach is not the same as the continuous-time one (it is itself a function of δ); this appears evident from Figs for increasing values of δ. The performances of the proposed controller are depicted in Figs 1-2 and 3. It results that while no sensible differences can be detected for small sampling periods, when δ increases the proposed controller better fits the requirements and is still capable to assure stability when the emulated controller fails ($\delta = 0.7s$).

Concerning the control law, similar results can be observed; the amplitudes of the efforts are comparable for small sampling periods; while significant improvements are obtained when it increases. In particular, the effort under the approximate second order II controller is lower even with respect to the continuous solution; moreover it ensures smoother trajectories and faster transient than the emulated controller.

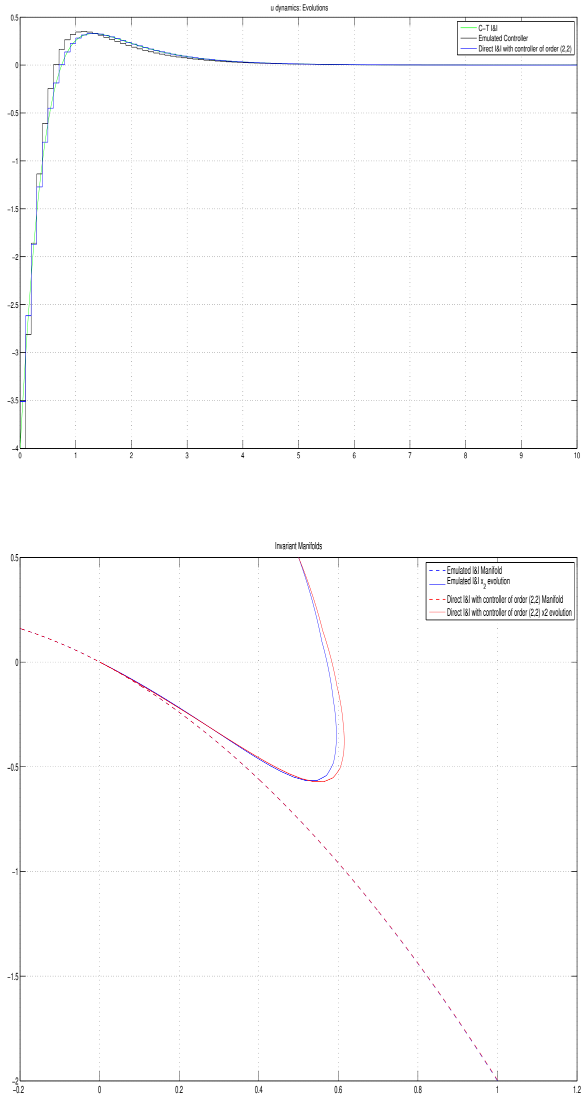

Fig. 1. Simulations with $\delta = 0.1$ s and $x_0 = (0.5, 0.5)^T$

5. CONCLUSIONS

Assuming the existence of a continuous controller asymptotically stabilizing a strict-feedback dynamics, a sampled-data I&I control law has been proposed in the present work. The major point stands in re-defining the immersion mapping and the attractive manifold, which characterize the I&I design procedure, according to the the sampled-equivalent model of the system. As pointed out it enables to compensate the mismatch of the state evolutions over the invariant manifold under sampled-data controller, so improving the over all performances. Some simple simulations confirm the interest of the proposed solution. Future works include the extension of this procedure to more general strict-feedback structures and investigations on the maximum allowed sampling period (MASP).

REFERENCES

Astolfi, A., Karagiannis, D., and Ortega, R. (2008). *Nonlinear and adaptive control with applications*. Springer Publishing Company.

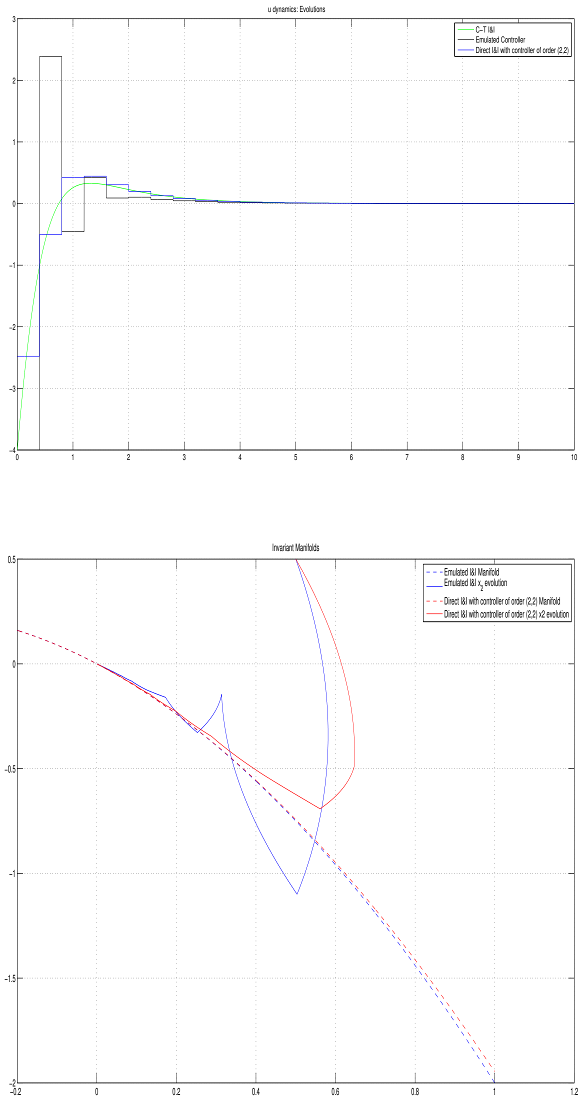

Fig. 2. Simulations with $\delta = 0.4$ s and $x_0 = (0.5, 0.5)^T$

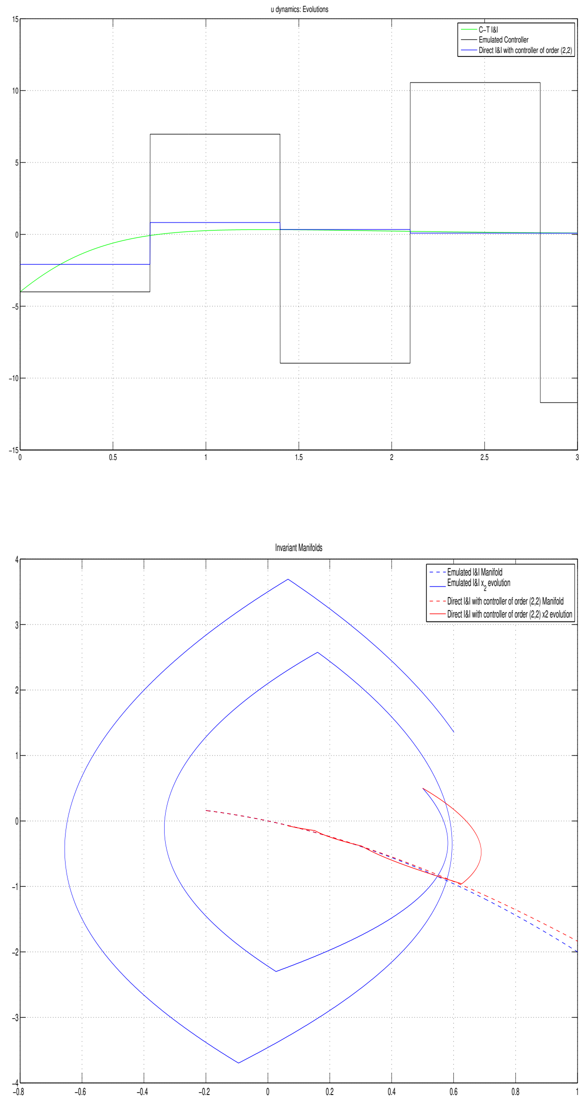

Fig. 3. Simulations with $\delta = 0.7$ s and $x_0 = (0.5, 0.5)^T$

Astolfi, A. and Ortega, R. (2003). Immersion and invariance: a new tool for stabilization and adaptive control of nonlinear systems. *IEEE Transactions on Automatic Control*, 48(4), 590–606.

Fossard, A. and Normand-Cyrot, D. (1996). *Nonlinear systems*, volume 3. Springer Science Business Media.

Hristea, E. and Siguerdidjane, H. (2011). Stabilization of a magnetic suspension by immersion and invariance and experimental robustness study. In *Proc. of 18th IFAC World Congress, Milano*, volume 18.

Khalil, H. (1996). *Nonlinear systems*. Prentice-Hall, Englewood Cliffs, NJ.

Kokotović, P. and Arcak, M. (2001). Constructive nonlinear control: a historical perspective. *Automatica*, 37(5), 637–662.

Mannarino, A. and Mantegazza, P. (2014). Multifidelity control of aeroelastic systems: an immersion and invariance approach. *Journal of Guidance*, 37(5), 1–15.

Mattei, G., Monaco, S., and Normand-Cyrot, D. (2015). Multirate sampled-data immersion and invariance stabilization of a class of nonlinear systems. In *Proc. ECC'15, Linz, Austria*.

Monaco, S. and Normand-Cyrot, D. (2007). Advanced tools for nonlinear sampled-data systems, analysis and control, minitutorial. *Special Issue "Fundamental Issues in Control", European Journal of Control*, 13(2,3), 221–241.

Monaco, S., Normand-Cyrot, D., and Tiefensee, F. (2011). Sampled-data stabilization; a pbc approach. *IEEE Trans. on Automatic Control*, 56, 907–912.

Nešić, D. and Teel, A. (2006). Stabilization of sampled-data nonlinear systems via backstepping on their euler approximate model. *Automatica*, 42, 1801 – 1808.

Postoyan, R., Ahmed-Ali, T., and Lamnabhi-Lagarrigue, F. (2009). Robust backstepping for the euler approximate model of sampled-data strict-feedback systems. *Automatica*, 45(9), 2164–2168.

Yalcin, Y. and Astolfi, A. (2011). Discrete time immersion and invariance adaptive control for systems in strict feedback form. In *50 th IEEE-CDC and EUCA-ECC*, 343–347.