Sampled-data Stabilization of Nonlinear Dynamics with Input Delays through Immersion and Invariance

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***Abstract*—In this paper, we show that Immersion and Invariance is a natural framework for the design of sampled-data stabilizing controllers for input-delayed systems. Assuming the existence of a continuous-time feedback in the delay free case, Immersion and Invariance stabilizability of the equivalent sampled-data dynamics is proven. The proof is constructive for the stabilizing controller. Two simulated examples illustrate the performances.**

***Index Terms*—Nonlinear Systems, Systems with Delays, Sampled-Data Control, Nonlinear Predictive Control**

I. Introduction

Compensation of delays is widely discussed in the literature through different approaches ([1], [2], [3] and the references therein). Recent works extend the reduction method or predictor-based methodologies to the nonlinear context [4], [5], [6]. Nowadays, a growing interest is addressed to the sampling problem in presence of input delays ([7], [8], [9], [10], [11], [12], [13], [14], [15]). This paper focuses on both the problems of sampling and compensation of constant input delays for nonlinear systems. We show how Immersion and Invariance (I&I) ([16], [17], [18]) provides a natural and fruitful framework for the design of sampled-data state-feedback for dynamics with input delays. Furthermore, we underline that such a strategy notably simplifies the design with respect to other predictor-based ones.

Nonlinear stabilization is addressed with reference to a single-input-affine continuous-time system with non-negative input-delay $\tau \in \mathbb{R}$. The problem is set in the sampled-data context, i.e. the state measures are available at the sampling instants $t = k\delta, k \geq 0$, and the control variable is constant over each sampling period. The existence of a continuous-time feedback ensuring Global Asymptotic Stability (GAS) of the continuous-time delay free dynamics (i.e., when $\tau = 0$) is assumed. Furthermore, as usual in the concerned literature [19], the sampling period δ is set as multiple of the delay length (i.e., $\tau = N\delta$, $N \in \mathbb{N}^+$). It is well known that, under such assumptions, the input-delayed dynamics (intrinsically infinite-dimensional) admits a finite dimensional equivalent sampled-data model. This has been recently exploited in the nonlinear context ([19], [10], [9], [11]) as a simplifying approach to handle the presence of a delay on the input. However, the computation of nonlinear sampled-data predictor-based controllers remains a difficult problem ([11], [12], [13], [14]).

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In this paper the problem is set and solved in the I&I context: a sampled-data stabilizing feedback is firstly designed on the delay-free dynamics following the lines of [20], [21]; then, the delay-free controlled dynamics identifies the target system evoling over a stable manifold; finally, I&I stabilization in discrete-time is achieved by driving the off-the manifold components to zero. The manifold invariance guarantees that the on-the-manifold closed-loop dynamics recover the sampled-data stable target ones.

In comparison with sampled-data predictor-based techniques (see [19], [10], [9], [11]), the proposed I&I feedback achieves asymptotic convergence of the controlled trajectories onto the manifold where they recover the predictor-based ones. With this in mind, such a strategy prevents from big control effort with improved robustness in particular when sampled-data predictor-based controllers cannot be easily computed [13]. To conclude the paper, two simulated examples illustrate the performances even in the case of uncertain delay length. In addition to the well known van der Pol oscillator, admitting an exactly computable predictor [19], an academic example is developed to illustrate the robusteness of the proposed approach when only approximate predictors can be computed. A preliminary version of this work is in [22] where the linear case illustrates the strategy.

The paper is organized as follows: instrumental tools and results are given in Section II; the main result is proposed and discussed in Section III; finally, simulated examples are in Sections IV and V.

Notations: M_U (M_U^I) denotes the space of measurable and locally bounded functions $u: \mathbb{R}^+ \to U$ ($u: I \to U$, $I \subset \mathbb{R}^+$) with $U \subseteq \mathbb{R}$. A system $\dot{x} = f(x) + g(x)u$ (with $x \in \mathbb{R}^n$, $u \in U$) is said forward complete if for every $x_0 \in \mathbb{R}^n$ and $u \in M_U$ its solution $x(t)$ with $x(0) = x_0 \in \mathbb{R}^n$ exists for all $t \geq 0$. Given a vector field f, L_f denotes the Lie derivative operator, $L_f = \sum_{i=1}^n f_i(\cdot)\frac{\partial}{\partial x_i}$. e^{L_f}(or e^f, when no confusion arises) denotes the associated Lie series operator, $e^f := 1 + \sum_{i\geq 1}\frac{L_f^i}{i!}$. Given vector fields f, g, their Lie bracket is defined as $ad_f g := [f,g] := [L_f, L_g] := L_f \circ L_g - L_g \circ L_f$ and, iteratively, $ad_f^i g := [f, ad_f^{i-1} g]$, with $ad_f^0 g := g$. We denote by the same $\circ$ the composition of functions and operators. Given a n-dimensional real vector v, v' denotes its transposed.

II. Problem settlement and preliminaries

A. Problem settlement

Consider the retarded single input-affine dynamics

$$\dot{x}(t) = f(x) + u(t-\tau)g(x) \tag{1}$$

where f and g are smooth (i.e. C^∞) vector fields on $\mathbb{R}^n$; x_* denotes the equilibrium $f(x_*) = 0$; the delay τ is known. We shall refer to the dynamics (1) as *delay free dynamics* (or *delay free system*) when no delay is acting on the control input (i.e. $\tau = 0$); i.e.

$$\dot{x}(t) = f(x) + u(t)g(x). \tag{2}$$

The following standing assumptions are set:

- measures are available only at the sampling instants $t = k\delta$ ($k \geq 0$) and the control is constant over time intervals of length $\delta \in]0, T^*[$, where δ is the sampling period and T^* is the *maximum allowable sampling period*;
- maps and vector fields are smooth (i.e. infinitely differentiable - C^∞) and the system (2) is forward complete;
- δ is chosen so that $\tau = N\delta$ for a suitable integer N;
- Assumption **A** - The delay free system (2) is smoothly stabilizable; i. e. there exists a feedback $u(t) = \gamma(x)$ with $\gamma(x_*) = 0$

and a proper [1] Lyapunov function $V:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ with $V(x_*)=0$ such that $\dot{V}=(\mathrm{L}_f+\gamma\mathrm{L}_g)V<0$.

Accordingly, the objective of this work is to design a sampled-data control law which stabilizes the closed-loop equilibrium of (1).

Remark 2.1 ([19]): Assuming that the delay-free system (2) is forward complete ensures that the delayed one (1) is complete too: for any x_0 and $u\in M_U^{[-\tau,\infty)}$ the solution $x(t)$ of (1) with initial condition $x(0)=x_0\in\mathbb{R}^n$ corresponding to $u\in M_U^{[-\tau,\infty)}$ exists for all $t\geq 0$.

Remark 2.2: Assuming $\tau=N\delta$ (i.e., N steps delay) is classical in the case of nonlinear purely discrete-time and sampled-data retarded systems [9], [19]. We argue that is not too restrictive under sampling since actuators are assumed to provide piecewise constant control with periodic sampling. Finally, a mismatch between the lengths of the sampling period and the delay is allowed so providing a further degree of freedom for the sampled-data design.

B. Sampled-data stabilization in the delay free case - recalls

To define the discrete-time target dynamics let us first recall that assumption **A** is sufficient to prove the existence of a piecewise constant control preserving GAS of the equilibrium at the sampling instants. Consider (2) and assume the input constant over successive intervals of length $\delta>0$; i.e. $u(t)=u_k$ for $t\in[k\delta,(k+1)\delta[$

$$\dot{x}(t)=f(x)+u_kg(x). \tag{3}$$

Through integration over the same time-interval with initial condition $x_k=x|_{t=k\delta}$, one describes the *equivalent sampled-data dynamics* in the form of a map as

$$x_{k+1}=F_1^{\delta}(x_k,u_k)=e^{\delta(f+u_kg)}x\Big|_{x_k}. \tag{4}$$

Following [20], from Assumption **A** one directly infers the existence of a sampled-data feedback $u_k=\gamma^{\delta}(x_k)$ which stabilizes (4) by guaranteeing, at the sampling instants, the same performances ensured by the continuous-time controller.

Theorem 2.1 ([21]): Given the dynamics (2) and the smooth feedback $u=\gamma(x)$ satisfying **A** with Lyapunov function V, if $\mathrm{L}_gV(x)\neq x_*$ for any $x\neq 0$, then there exists a state-feedback $u_k=\gamma^{\delta}(x_k)$ which achieves *input-Lyapunov matching* for the closed loop dynamics; i.e. it is is the unique solution of the equality

$$V(F^{\delta}(x_k,u_k))-V(x_k)=\int_{k\delta}^{(k+1)\delta}(\mathrm{L}_fV+\gamma\mathrm{L}_gV)(x(s))\mathrm{d}s \tag{5}$$

in the unknown u_k. Such a controller, which admits the series expansion

$$\gamma^{\delta}(x)=\gamma_0(x)+\sum_{i\geq 1}\frac{\delta^i}{(i+1)!}\gamma_i(x) \tag{6}$$

with $\gamma^{\delta}(x_*)=0$, ensures global asymptotic stabilization of the equilibrium of (4) with the same control Lyapunov function $V(\cdot)$.

The proof of the above result is constructive and works out by equating the terms of the same power in δ in both sides of the equality (5). For the first terms one computes

$$\gamma_0(x_k)=\gamma(x(t))|_{t=k\delta} \tag{7}$$

$$\gamma_1(x_k)=\dot{\gamma}(x(t))|_{t=k\delta} \tag{8}$$

$$\gamma_2(x_k)=\ddot{\gamma}(x(t))|_{t=k\delta}+\frac{\gamma_1(x_k)}{2L_gV(x(t))|_{t=k\delta}}ad_{[f,g]}V(x(t))|_{t=k\delta}. \tag{9}$$

recovering respectively the continuous-time solution (7) and its time derivative (8) computed at $t=k\delta$. The higher order terms $\gamma_q(\cdot)$, $q>2$, can be computed from the previous ones through an executable algorithm.

Remark 2.3: The hypothesis $\mathrm{L}_gV(x)\neq 0$ for any $x\neq x_*$ is here assumed to guarantee the existence and uniqueness of the solution of the formal power series associated to (5). Less rescrictive conditions could be identified by renouncing to exploit the function $V(\cdot)$ to get the sampled-data feedback; this would correspond to address the problem of extending the concept of control Lyapunov function under sampling ([23], [24]) .

Remark 2.4: $\gamma_0(x_k)$ restitutes the emulated solution which satisfies (5) with an error in $O(\delta)$. Denoting by $\gamma^{[q]}(x_k)$ with $q\geq 0$, the q^{th}-order approximate controller (truncation in $O(\delta^{q+1})$ of the exact solution (6)), one has by construction that $\gamma^{[1]}(x_k):=\gamma_0(x_k)+\frac{\delta}{2}\gamma_1(x_k)$ satisfies (5) with an error in $O(\delta^2)$ and $\gamma^{[q]}(x_k)$ with an error in $O(\delta^{q+1})$.

Remark 2.5: In [21], it was shown that Lyapunov-based approximate controllers yield practical global asymptotic stability of the closed-loop equilibrium.

C. I&I stabilization in discrete time

I&I was firstly introduced in [16] and applied to several domains. I&I stabilizability in discrete time was discussed in [18] and is reformulated below.

Theorem 2.2: Consider a nonlinear discrete-time dynamics in the form of a map

$$x_{k+1}=F(x_k,u_k) \tag{10}$$

with state $x\in\mathbb{R}^n$ and control $u\in U$ and equilibrium state x_* to be stabilized. Let $p<n$ and assume that we can find mappings

$$\alpha(\cdot):\mathbb{R}^p\to\mathbb{R}^p;\qquad \pi(\cdot):\mathbb{R}^p\to\mathbb{R}^n;\qquad c(\cdot):\mathbb{R}^p\to\mathbb{R}$$
$$\phi(\cdot):\mathbb{R}^n\to\mathbb{R}^{n-p};\quad \psi(\cdot,\cdot):\mathbb{R}^{n\times(n-p)}\to\mathbb{R}$$

such that the following four conditions hold:

H1) (*Target dynamics*) - The dynamics with state $\xi\in\mathbb{R}^p$

$$\xi_{k+1}=\alpha(\xi_k) \tag{11}$$

has a GAS equilibrium at $\xi_*\in\mathbb{R}^p$ and $x_*=\pi(\xi_*)$.

H2) (*Immersion and invariance condition*) - For all $\xi\in\mathbb{R}^p$, there exists $c(\cdot):\mathbb{R}^p\to R$ such that

$$F(\pi(\xi_k),c(\xi_k))=\pi(\alpha(\xi_k)). \tag{12}$$

H3) (*Implicit manifold*) - The following identity between sets holds

$$\{x\in\mathbb{R}^n|\phi(x)=0\}=\{x\in\mathbb{R}^n|x=\pi(\xi)\text{ for }\xi\in\mathbb{R}^p\}. \tag{13}$$

H4) (*Manifold attractivity and trajectory boundedness*) - All the trajectories of the system

$$z_{k+1}=\phi(F(x_k,\psi(x_k,z_k))) \tag{14a}$$
$$x_{k+1}=F(x_k,\psi(x_k,z_k)) \tag{14b}$$

with $z=\phi(x)$ and $z_0=\phi(x_0)$ are bounded for all $k\geq 0$ and satisfy

$$\lim_{k\to\infty}z_k=0\quad\text{ and }\quad\psi(\pi(\xi),0)=c(\xi). \tag{15}$$

Then x_* is a globally asymptotically stable equilibrium of the closed loop dynamics

$$x_{k+1}=F(x_k,\psi(x_k,\phi(x_k))). \tag{16}$$

Definition 2.1: Any discrete-time dynamics of the form (10) satisfying the hypotheses H1) to H4) of Theorem 2.2 is said to be *I&I stabilizable* with target dynamics $\xi_{k+1}=\alpha(\xi_k)$.

[1] $V:\mathbb{R}^n\to\mathbb{R}$ is *proper* if $\forall\ r>0,\ V^{-1}([0,r])=\{x\in\mathbb{R}^n\ \ V(x)\geq r\}$ is compact.

Remark 2.6: Rewriting (10) as $F(x,u) = F(x) + G(x,u)$ with $G(x,0) = 0$, condition H4) can be relaxed. To prove asymptotic convergence of x_k to x_*, it is sufficient to require

$$\lim_{k\to\infty}(G(x_k, \psi(x_k, z_k)) - G(x_k, \psi(x_k, 0))) = 0. \tag{17}$$

D. Hybrid representation and target dynamics

Setting $\tau = N\delta$ and assuming $u(t) = u_k$ for $t \in [k\delta, (k+1)\delta[$, it is immediate to represent the continuous-time retarded system (1) as an *hybrid dynamics* over $\mathbb{R}^{n+N}$; one sets

$$\begin{aligned} \dot{x}(t) =& f(x) + v^1_k g(x); \quad \forall t \in [k\delta, (k+1)\delta[\\ v^1_{k+1} =& v^2_k; \quad \dots \quad ; v^N_{k+1} = u_k. \end{aligned} \tag{18}$$

On these bases, the stabilizing problem can be set on an extended but finite dimensional dynamics strictly related to the delay length. Accordingly, the discrete-time target dynamics is defined as (4), the sampled-data equivalent of the continuous-time delay free system (3) when $u_k = \gamma^\delta(x_k)$; i.e. for $t \in [k\delta, (k+1)\delta[$ the target gets the form

$$x_{k+1} = F^\delta_1(x_k, \gamma^\delta(x_k)) = e^{\delta(f+\gamma^\delta(x_k)g)}x_k \tag{19}$$

with GAS equilibrium x_* by construction of $\gamma^\delta(\cdot)$.

III. The main result

Consider the discrete-time equivalent of the hybrid system (18)

$$x_{k+1} = F^\delta_1(x_k, v^1_k) = e^{\delta(f+v^1_k g)}x_k;\ v^1_{k+1} = v^2_k;\ \dots;\ v^N_{k+1} = u_k \tag{20}$$

rewritten in compact form (10) as

$$x^e_{k+1} = F^\delta(x^e_k, u_k) \tag{21}$$

with $x^e = (x', \bar{v}') \in \mathbb{R}^{n+N}$, $\bar{v} = (v^1, ..., v^N)' \in \mathbb{R}^N$ and equilibrium $x^e_* = (x'_*, 0)'$. Following Theorem 2.2, one introduces

$$\bar{z} = \phi^\delta(x^e) = (\phi^\delta_1(x^e), ..., \phi^\delta_N(x^e))'$$

with

$$z^i_k = \phi^\delta_i(x_k, \bar{v}_k) = v^i_k - \gamma^\delta(x_{k+i-1}), \quad i = 1, \dots, N \tag{22}$$

and $\pi^\delta(\cdot) = (\cdot, \gamma^\delta(\cdot), \dots, \gamma^\delta((\alpha^\delta)^{N-1}(\cdot)))'$ with $\alpha^\delta(\cdot) = F^\delta(\cdot, \gamma^\delta(\cdot))$. The above mappings refer to the ones in Theorem 2.2 where the superscript $^\delta$ is added so to underline their parametrization by the sampling period.

In (22), x_{k+i} stands for the i-times composition of the function F^δ_1

$$x_{k+i} = F^\delta_1(\cdot, v^i) \circ \dots \circ F^\delta_1(x_k, v^1) = e^{\delta(f+v^1_k g)} \circ \dots \circ e^{\delta(f+v^i_k g)} x\big|_{x_k}.$$

The following definition is instrumental.

Definition 3.1: The continuous-time system (1) is said to be *sampled-data I&I stabilizable* if its sampled-data equivalent model (21) is I&I stabilizable in the discrete-time sense of Definition 2.1.

The main result can now be set.

Theorem 3.1: The input-affine continuous-time dynamics (1) under Assumption **A** with $\mathrm{L}_g V(x) \neq x_*$ for any $x \neq 0$ and $\tau = N\delta$ is sampled-data I&I stabilizable. Equivalently, (21) is I&I stabilizable with target dynamics over $\mathbb{R}^n$

$$\xi_{k+1} = F^\delta_1(\xi_k, \gamma^\delta(\xi_k)) := \alpha^\delta(\xi_k) \tag{23}$$

where $\gamma^\delta(\cdot) : \mathbb{R}^n \to \mathbb{R}$ denotes the controller of the delay free system defined in Theorem 2.1.

Proof. For, one has to show that the four conditions in Theorem 2.2 are satisfied. Given $\gamma^\delta(\cdot) : \mathbb{R}^n \to \mathbb{R}$ defined according to (5) with control Lyapunov function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$, let $\xi \in \mathbb{R}^n$ and define $\pi^\delta(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n+N}$ as $\pi^\delta(\xi_k) = (\xi_k, \gamma^\delta(\xi_k), ..., \gamma^\delta(\xi_{k+N-1}))'$. It is a matter of computations to very that the conditions H1), H2) and H3) of Theorem 2.2 are satisfied by construction with $\phi^\delta(\cdot) : \mathbb{R}^{n+N} \to \mathbb{R}$ described in (22) and

$$c^\delta(\xi_k) = \gamma^\delta(\xi_{k+N}) = \gamma^\delta \circ \alpha^\delta \circ \dots \circ \alpha^\delta(\xi_k), \quad N \text{ times.} \tag{24}$$

On these bases, it follows directly from Theorem 2.2 that any feedback $u = \psi^\delta(x, \bar{v}, \bar{z})$ such that $\psi^\delta(\pi^\delta(\xi), 0) = c^\delta(\xi)$ designed to drive $\bar{z}$ to zero while ensuring boundedness of the state trajectories of the extended dynamics

$$z^i_{k+1} = \phi^\delta_i(F^\delta(x^e_k, \psi^\delta(x^e_k, \bar{z}_k))) \tag{25a}$$

$$x^e_{k+1} = F^\delta(x^e_k, \psi^\delta(x^e_k, \bar{z}_k)) \tag{25b}$$

for $i = 1, \dots, N$ achieves global asymptotic stability of the equilibrium of the closed loop dynamics $x^e_{k+1} = F^\delta(x^e_k, \psi^\delta(x^e_k, \phi^\delta(x^e_k)))$. ◁

Remark 3.1: Theorem 3.1 shows that the problem admits a solution in the discrete-time I&I context when: there exists a sampled-data solution to the continuous time delay free stabilization problem (Theorem 2.1) and the hybrid retarded dynamics (18) is finite dimensional.

Remark 3.2: The feedback $c^\delta(\cdot)$ in (24) corresponds to the N-steps ahead predictor-based feedback $\gamma^\delta(\cdot)$ with target dynamics (23) as predictor. Hence, when the stable manifold is reached ($\bar{z} = 0$), one recovers the predictor-based solution $\gamma^\delta \circ (\alpha^\delta)^N(\cdot)$ while in the delay free case $c^\delta(\cdot) = \gamma^\delta(\cdot)$.

A. On the I&I stabilizing feedback design

According to the specific structure of the extended dynamics (20) and the definition of $\phi^\delta(x^e)$ in (22), it is a matter of computations to verify that the off-the manifold dynamics (25a) simplifies so that the system (25) rewrites as

$$z^i_{k+1} = z^{i+1}_k; \qquad z^N_{k+1} = u_k - \gamma^\delta(x_{k+N}) \tag{26a}$$

$$x^e_{k+1} = F^\delta(x^e, 0) + Bu_k \tag{26b}$$

for $i = 1, \dots, N-1$ with $x_{k+N} = e^{\delta(f+v^1_k g)} \circ \dots \circ e^{\delta(f+v^N_k g)} x\big|_{x_k}$ and

$$F^\delta(x^e, 0) = (F^\delta_1(x, v^1)',\ v^2,\ \dots\ v^N,\ 0)'; \quad B = (\mathbf{0}', \quad 1)'.$$

As a consequence of the structure of the x^e-dynamics, condition H4) of Theorem 2.2 can be relaxed to (17) so getting the following result about the construction of the control law.

Proposition 3.1: The extended dynamics (21) with target dynamics (23) is I&I stabilized by the feedback

$$u_k = \psi^\delta_k(x^e_k, \bar{z}_k) = l\bar{z}_k + \gamma^\delta(x_{k+N}) \tag{27}$$

with $x_{k+N} = F^\delta_1(., v^N_k) \circ \dots \circ F^\delta_1(x_k, v^1_k)$ and $l = (l_1 \dots l_N)$ chosen such that the matrix

$$L = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & 0 & \dots & 1 \\ l_1 & l_2 & l_3 & \dots & l_N \end{pmatrix}$$

is Schur[2]. Equivalently, the piecewise constant control (27) globally asymptotically stabilizes the equilibrium of (1) under Assumption **A** with $L_g V(x) \neq 0$ for any $x \neq x_*$ and $\tau = N\delta$. Setting $l = \mathbf{0}$ one has convergence to the manifold in exactly N steps (*deadbeat*).

Proof. According to Remark 2.6, it can be easily deduced that $\lim_{k\to\infty} l\,\bar{z}_k = 0$ is sufficient to prove global asymptotical convergence to x^e_* since the dynamics $\bar{z}_{k+1} = L\bar{z}_k$ is globally asymptotically stable by construction. Suppose $l = \mathbf{0}$ to bring $\bar{z}$ to zero in exactly N steps, then $x_{k+N} = e^{\delta(f+v^1_k g)} \circ \dots \circ e^{\delta(f+v^N_k g)} x\big|_{x_k}$ coincides with the stable delay-free target (19). Hence, (17) still holds. ◁

[2]A matrix $N \in \mathbb{R}^{n\times n}$ is said to be *Schur* if all its eigenvalues are within' the unit circle

Remark 3.3: Proposition 3.1 shows that (27) linearizes the off-the-manifold $\bar{z}$-dynamics. It results that the closed-loop system is minimum-phase with respect to the output $\bar{z}$ with zero-dynamics coinciding with the target one.

B. Approximate solutions: the case $\tau = \delta$

The solution (27) with constant gain l is referred to the case in which the sampled-data equivalent model (20) can be finitely computed. This is seldom the case so that, in general, only approximate expressions of $\gamma^\delta(x_{k+N})$ in (27) can be worked out. For this reason, a constant gain l as in (27) may not be robust with respect to unmodeled higher order dynamics in (20). Hereinafter, we propose an alternative approximate solution in which the closed-loop $\bar{z}$-dynamics becomes nonlinear in x when choosing a dynamic gain $l^\delta(x)$.
Constructive aspects are sketched to characterize the I&I solution around the predictor-based feedback when $N = 1$. In such a case, dynamics (26a) and (26b) rewrite as

$$z_{k+1} = u_k - \gamma^\delta(x_{k+1}) \tag{28a}$$
$$x_{k+1} = F_1^\delta(x_k, \gamma^\delta(x_k) + z_k) \tag{28b}$$
$$v_{k+1} = u_k. \tag{28c}$$

The following proposition states the result in the case in which a finite sampled-data equivalent model cannot be computed.

Proposition 3.2: The input-affine continuous-time dynamics (1) under Assumption **A** with $\tau = \delta$ is sampled-data locally I&I stabilized by the piecewise constant feedback

$$u_k = \gamma^\delta(\alpha^\delta(x_k)) + l^\delta(x_k) z_k \tag{29}$$

with dynamic gain $l^\delta(x)$ such that for all $x_k \in \mathbb{R}^n$

$$|l^\delta(x_k) - \frac{\partial \gamma^\delta(x)}{\partial x}\Big|_{\alpha^\delta(x_k)} G^\delta(\alpha^\delta(x_k), \gamma^\delta(x_k))| < 1 \tag{30}$$

and $G^\delta(x,v) = \int_0^\delta e^{-s\mathrm{ad}_{f+vg}} g(x)\mathrm{d}s$. Equivalently, the control (29) locally asymptotically stabilizes the equilibrium of (28b)-(28c).
Proof. (28b) can be expanded around the target dynamics $\alpha^\delta(\cdot) = F_1^\delta(\cdot, \gamma^\delta(\cdot))$ as

$$x_{k+1} = \alpha^\delta(x_k) + \frac{\partial F_1^\delta(x_k, v)}{\partial v}\Big|_{\gamma^\delta(x_k)} z_k + O(z^2).$$

Similarly, in (28a), one has

$$\gamma^\delta(x_{k+1}) = \gamma^\delta(\alpha^\delta(x)) + \frac{\partial \gamma^\delta(x)}{\partial x}\Big|_{\alpha^\delta(x)} \frac{\partial F_1^\delta(x, v)}{\partial v}\Big|_{\gamma^\delta(x)} z + O(z^2)$$

with by definition (see [20] for more details)

$$G^\delta(\alpha^\delta(x_k), \gamma^\delta(x_k)) = \frac{\partial F_1^\delta(x_k, v)}{\partial v}\Big|_{\gamma^\delta(x_k)}. \tag{31}$$

Choosing the feedback $\psi^\delta(x, v, z)$ as in (29) with dynamic gain $l^\delta(x)$ such that (30) holds, one gets in $O(z^2)$

$$z_{k+1} = \Big(l^\delta(x) - \frac{\partial \gamma^\delta(x)}{\partial x}\Big|_{\alpha^\delta(x)} G^\delta(\alpha^\delta(x_k), \gamma^\delta(x_k))\Big) z_k$$
$$x_{k+1} = \alpha^\delta(x_k) + G^\delta(\alpha^\delta(x_k), \gamma^\delta(x_k)) z_k$$
$$v_{k+1} = \gamma^\delta(\alpha^\delta(x_k)) + l^\delta(x_k) z_k = c^\delta(x_k) + l^\delta(x_k) z_k$$

where $x_{k+1} = \alpha^\delta(x_k)$ is the target dynamics and $\gamma^\delta(\alpha^\delta(x_k)) = c^\delta(x_k)$ the predictor-based feedback solution.

Under the feedback (29), condition (17) is verified in $O(z^2)$; i.e. local sampled-data asymptotic stabilization of the input-delay dynamics (1) is achieved. ◁

Further approximations in δ give the following condition which specifies the constraint in terms of the continuous-time dynamics and the sampling period only. By construction, $G^\delta(\cdot, \gamma^\delta) = \delta g + O(\delta^2)$, so that the gain condition (30) gives in $O(\delta^2)$

$$|l^\delta(x_k) - \delta \mathrm{L}_g \gamma^\delta(x_k)| < 1. \tag{32}$$

Remark 3.4: The structure of the feedback (29) underlines that the I&I solution recovers the predictor-based controller when $z = 0$. The additional term $l^\delta(x)z$ is referred to the prediction error dynamics. This guarantees robustness improvements with respect to discarded higher order approximations of the sampled-data equivalent model.

Remark 3.5: Easy manipulations show that in the general case $\tau = N\delta$, $\gamma^\delta(x_{k+N})$ rewrites as

$$\gamma^\delta(x_{k+N}) = e^{\delta(\tilde{f} + z_k^1 g)} \circ \cdots \circ e^{\delta(\tilde{f} + z_k^N g)} \gamma^\delta(x_k)$$

which can be expanded around the predictor-based solution $e^{N\delta\tilde{f}}\gamma^\delta(x_k) = c^\delta(x_k)$. Hence, the gain function $l^\delta(x)z$ in (29) generalizes as

$$\bar{l}^\delta(x)\bar{z} = \delta \sum_{i=1}^N z^i \mathrm{L}_g(e^{N\delta\tilde{f}}\gamma^\delta(x)) + \delta\Omega(x, \bar{z}, \delta)$$

where $\tilde{f} = f + \gamma^\delta g$ and $\Omega(x, \bar{z}, \delta)$ contains the remaining higher order terms in δ with $\Omega(x, 0, \delta) = 0$ and $\Omega(x, \bar{z}, 0) = 0$.

IV. The van der Pol example

Let us consider the system in strict-feedforward form studied in [19], [10] which represents the van der Pol oscillator

$$\dot{x}_1(t) = x_2(t) - x_2^2(t)u(t-\tau), \quad \dot{x}_2(t) = u(t-\tau).$$

When $\tau = 0$, the stabilizing continuous-time controller is provided as $\gamma(x) = -x_1 - 2x_2 - \frac{x_2^3}{3}$ with control Lyapunov function $V(x) = \frac{1}{2}(x_1 + \frac{x_2^3}{3})^2 + \frac{1}{2}(x_2 + 2x_1 + \frac{2}{3}x_2^3)$. Assuming, $u(t) = u_k$ for $t \in [k\delta, (k+1)\delta)$, one gets the exact sampled-data equivalent dynamics

$$x_{1k+1} = x_{1k} + \delta(x_{2k} - x_{2k}^2 u_k) + \frac{\delta^2}{2!}(u_k - 2x_{2k}u_k^2) - \frac{\delta^3}{3}u_k^3 \tag{33}$$
$$x_{2k+1} = x_{2k} + \delta u_k. \tag{34}$$

The equivalent sampled-data delay-free stabilizing controller $\gamma^\delta(x)$ is computed according to Section II to satisfy the equality (5). It is approximated as

$$\gamma^\delta(x) = \gamma(x) + \frac{\delta}{2}\dot{\gamma}(x) + O(\delta^2) \tag{35}$$

with $\dot{\gamma}(x) = \frac{2}{3}x_2^3 + 3x_2 + 2x_1$.

A. Predictor-based controller

Assuming the delay equal to δ ($N = 1$) and setting $x_k^p = x_{k+1}$, the predicted state, the sampled-data predictor-based feedback is $u_k = \gamma^\delta(x_k^p)$ with

$$x_{1k}^p = x_{1k} + \delta(x_{2k} - x_{2k}^2 u_{k-1}) + \frac{\delta^2}{2!}(u_{k-1} - 2x_{2k}u_{k-1}^2) - \frac{\delta^3}{3}u_{k-1}^3$$
$$x_{2k}^p = x_{2k} + \delta u_{k-1}.$$

Hence, as well known, prediction is performed in open-loop.

B. I&I sampled data controller

According to Section III, one sets the extended dynamics as

$$x_{1k+1} = x_{1k} + \delta(x_{2k} - x_{2k}^2 v_k) + \frac{\delta^2}{2!}(v_k - 2x_{2k}v_k^2) - \frac{\delta^3}{3}v_k^3$$
$$x_{2k+1} = x_{2k} + \delta v_k, \qquad v_{k+1} = u_k$$

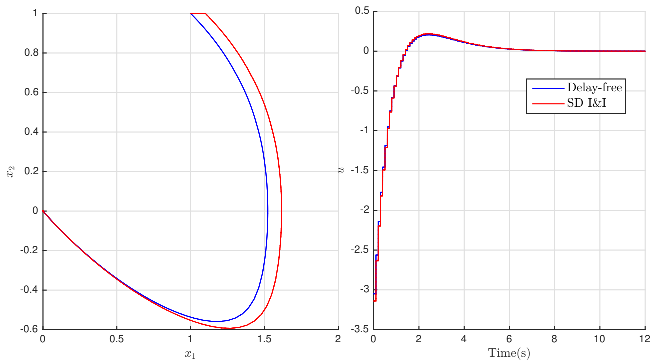

Fig. 1: The Van der Pol oscillator: $\tau = \delta = 0.1$ s.

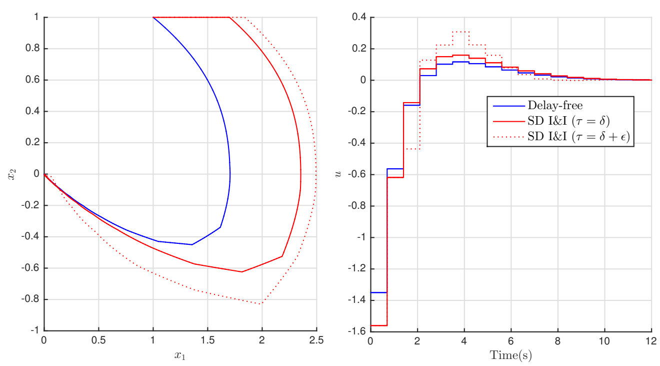

Fig. 2: The Van der Pol oscillator: $\delta = 0.7$ s, $\varepsilon = 0.2\delta$.

and the target dynamics as (dropping the k-subscript in the r.h.s.)

$$\xi_{1k+1} = \xi_1 + \delta(\xi_2 - \xi_2^2 \gamma^\delta(\xi)) + \frac{\delta^2}{2!}(\gamma^\delta(\xi) - 2\xi_2\gamma^\delta(\xi)^2) - \frac{\delta^3}{3}\gamma^\delta(\xi)^3$$
$$\xi_{2k+1} = \xi_2 + \delta\gamma^\delta(\xi)$$

with $\gamma^\delta(\cdot)$ as in (35). Finally, the manifold is implicitly defined as $\phi^\delta(x) = v - \gamma^\delta(x) = 0$ and the off-the-manifold component as $z = \phi^\delta(x)$, with $z_0 = \phi^\delta(x_0)$. Setting $\psi^\delta(x_k, v_k, z_k) = l z_k + \gamma^\delta(x_{k+1})$, one ensures that Proposition 3.1 holds so concluding GAS of the equilibrium of the closed loop sampled-data dynamics

$$x_{1k+1} = x_{1k} + \delta(x_{2k} - x_{2k}^2 v_k) + \frac{\delta^2}{2!}(v_k - 2x_{2k}v_k^2) - \frac{\delta^3}{3}v_k^3$$
$$x_{2k+1} = x_{2k} + \delta v_k; \qquad v_{k+1} = l(v_k - \gamma^\delta(x_k)) + \gamma^\delta(x_{k+1}).$$

When $z = 0$ (and, hence, $v = \gamma^\delta(x)$), the manifold is reached and one recovers $\psi^\delta(\pi^\delta(\xi), 0) = \gamma^\delta(\alpha^\delta(\xi))$, the closed loop stable dynamics on the manifold. Since the van der Pol dynamics admits a finite sampled-data equivalent model, the term $\gamma^\delta(x_{k+1})$ can be exactly computed.

C. Simulations

Simulations are carried out on the van der Pol Example. The I&I strategy (SD I&I) is implemented for the case $\tau = \delta$ with $l = 0$. The delay-free closed-loop trajectories under the feedback (35) are depicted as well. The initial conditions are set at $x_0 = (1\ 1)'$ and $u(t) = 0$ for $t \in [-\tau, 0)$.

Figure 1 depicts the results in the nominal case $\tau = N\delta$ with $N = 1$. Figure 2 puts in light two aspects: good performances even when singnificantly increasing the sampling period (from 0.1 to 0.3 seconds); robustness with respect to unmodeled uncertainties over δ (i.e., on τ when $\tau = \delta + \varepsilon$, with small $\varepsilon \geq 0$). Further simulations (for $\varepsilon = 0$) were carried out for comparing the proposed solution with the result in [19] showing that both approaches yield good and satisfactory closed-loop performances with limited control effort.

V. An academic example

An interesting comparison with the approach proposed in [9] is provided by the following simple academic example over $\mathbb{R}^2$

$$\dot{x}_1 = x_1^2 + x_2; \qquad \dot{x}_2 = u(t - \tau). \tag{36}$$

Setting $\phi(x_1) = -x_1 - x_1^2$, the control law $u_c = \gamma(x) = \dot{\phi}(x_1) - x_1 - K(x_2 - \phi(x_1))$, $K > 0$, makes the origin of (36) GAS when $\tau = 0$, with $V(x_1, x_2) = \frac{1}{2}(x_1^2 + (x_2 - \phi(x_1))^2)$. According to Theorem 2.1 one computes $u_D = \gamma^\delta(x) = \gamma(x) + \frac{\delta}{2}\dot{\gamma}(x) + O(\delta^2)$, with

$$\dot{\gamma}(x) = (x_1 + (2x_1 + 1)(x_1^2 + x_2) + K(x_1^2 + x_1 + x_2))(K + 2x_1 + 1)$$
$$- (x_1^2 + x_2)(2x_2 + K(2x_1 + 1) + 2x_1(2x_1 + 1) + 2x_1^2 + 1)$$

so ensuring GAS of the equilibrium of (36) under sampling when $\tau = 0$. Since in this case an exact predictor cannot be computed as (36) does not admit a finite sampled-data equivalent model, we will show robustness with respect to the discarded higher order dynamics in δ^p (p approximate order) of the I&I predictor based-controller. Setting $\tau = \delta$, the extended hybrid dynamics (20) is detailed as

$$\dot{x}_1 = x_1^2 + x_2; \qquad \dot{x}_2 = v_k; \qquad v_{k+1} = u_k$$

with 3^{rd}-order approximate sampled-data model described as

$$x_{1k+1} = x_1 + \delta(x_1^2 + x_2) + \frac{\delta^2}{2}(v + 2x_1(x_1^2 + x_2))$$
$$+ \frac{\delta^3}{3}(3x_1^4 + 4x_1^2 x_2 + x_1 v + x_2^2) + O(\delta^4) \tag{37}$$
$$x_{2k+1} = x_2 + \delta v; \qquad v_{k+1} = u.$$

Accordingly, one sets $z = v - \gamma^\delta(x)$ and gets

$$z_{k+1} = u - \gamma^\delta(x_{k+1})$$
$$x_{1k+1} = x_1 + \delta(x_1^2 + x_2) + \frac{\delta^2}{2}(\gamma^\delta(x) + 2x_1(x_1^2 + x_2))$$
$$+ \frac{\delta^3}{3}(3x_1^4 + 4x_1^2 x_2 + x_1\gamma^\delta(x) + x_2^2) + \frac{\delta^2}{2}z + \frac{\delta^3}{3}x_1 z + O(\delta^4)$$
$$x_{2k+1} = x_2 + \delta\gamma^\delta(x) + \delta z; \qquad v_{k+1} = u.$$

The feedback

$$u = lz + \gamma^\delta(x_{k+1}) \tag{38}$$

ensures I&I stabilization of the dynamics (36) by ensuring boundedness of the trajectories of the (z, x, v)-extended dynamics. As pointed out, an exact predictor for (36) does not exist. Similarly, only approximations of the control (38) can be implemented. According to Section III-B, we define the approximate control

$$u^{app} = l^\delta(x, z)z + \gamma^\delta(\alpha^{\delta[3]}(\xi)) \tag{39}$$

with $\gamma^\delta(\cdot)$ computed on the approximate target dynamics $\xi_{k+1} = \alpha^{\delta[3]}(\xi)$ provided as

$$\xi_{1k+1} = \xi_1 + \delta(\xi_1^2 + \xi_2) + \frac{\delta^2}{2}(\gamma^\delta(\xi) + 2\xi_1(\xi_1^2 + \xi_2)) +$$
$$\frac{\delta^3}{3}(3\xi_1^4 + 4\xi_1^2\xi_2 + \xi_1\gamma^\delta(\xi) + \xi_2^2)$$
$$\xi_{2k+2} = \xi_2 + \delta\gamma^\delta(\xi).$$

Furthermore, the term $l^\delta(x, z)z$ with $l^\delta(x, z) = l + \delta(1 + K + 2x_1) + O(\delta^2)$ represents a feedback on the prediction error which makes the closed-loop system under approximate control more robust with respect to discarded higher order predicted-state dynamics in $O(\delta^4)$.

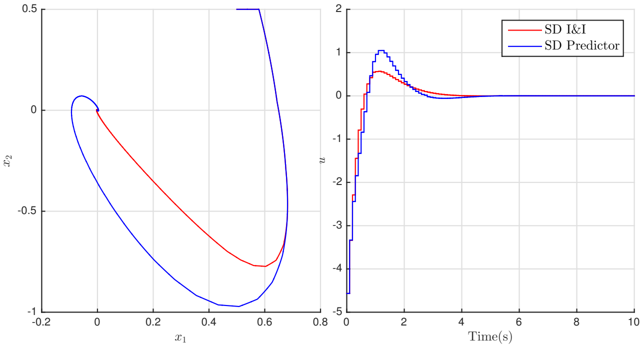

Fig. 3: $\delta = 0.1$ s and $\tau = \delta$.

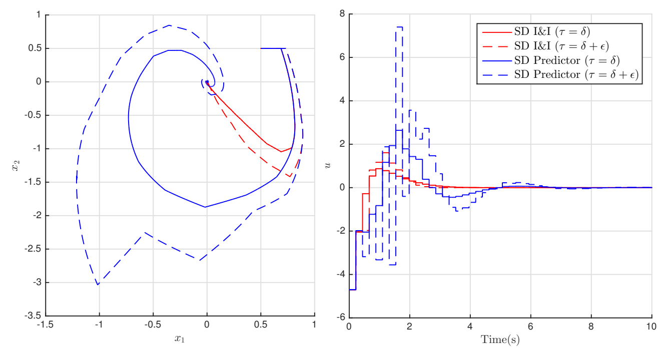

Fig. 4: Academic example: $\delta = 0.2$ s and $\varepsilon = 0.2\delta$.

A. Simulations

We simulate the closed-loop performances of the dynamics (36) under the proposed approximate sampled-data I&I control (39). We set $x_0 = (0.5, 0.5)^T$ and $u(t) = 0$ for $t \in [-\tau, 0)$. Comparisons are carried out with respect to the mere approximate sampled-data predicted-based control [19] $u_k = \gamma^{\delta[1]}(x_{k+1})$ that is implemented with x_{k+1} as in (37). In particular, we consider two scenarios: Figure 3 depicts the result with rather fast sampling and when the delay is assumed known with N = 1 and $\tau = \delta = 0.1$ s; in Figure 4 we report simulations for the case of uncerataínty on the delay length (i.e., we simulate $\tau = \delta + \varepsilon$ with small $\varepsilon > 0$) and rather large value of δ. In all the simulations, we apply the same order of approximations in both the I&I and predictor based control schemes. As the sampling period increases, the predictor-based control yields degrading performances (both in the state trajectories and control effort) even when the delay is assumed exactly known. As a matter of fact, the effect of the uncertainties in the delay length is better handled by the proposed I&I strategy so yielding robustness even with respect to both unmodeled higher order dynamics and approximation of the control solutions.

VI. Conclusions

The I&I stabilizing approach has been used to investigate sampled-data feedback stabilization of input-affine-delayed continuous-time dynamics. The sampled-data predictor-based controller is recovered as a particular case. The method applies to multi-input dynamics along the same lines. Extensions will concern other types of delays as in the state, distributed delays or when the delay cannot be assumed an entire of the sampling period. In this last case, a multirate sampled-data strategy should be applied along the lines in [7].

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