J. Math. Anal. Appl. 457 (2018) 1492–1501

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the comparison principle for unbounded solutions of elliptic equations with first order terms

Tommaso Leonori [a], Alessio Porretta [b,*]

[a] Departamento de Análisis Matematico, Universidad de Granada, Granada, Spain

[b] Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica 1, 00133 Roma, Italy

A R T I C L E I N F O

Article history:
Received 20 January 2017
Available online 13 April 2017
Submitted by H. Frankowska

Keywords:
Nonlinear elliptic equations with lower order terms
Comparison principle for unbounded solutions

A B S T R A C T

We prove a comparison principle for unbounded weak sub/super solutions of the equation

$$\lambda u - \operatorname{div}(A(x)Du) = H(x, Du) \quad \text{in } \Omega$$

where $A(x)$ is a bounded coercive matrix with measurable ingredients, $\lambda \geq 0$ and $\xi \mapsto H(x,\xi)$ has a super linear growth and is convex at infinity. We improve earlier results where the convexity of $H(x,\cdot)$ was required to hold globally.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain and let $A(x) = (a_{i,j}(x))$ be a coercive matrix of $L^\infty(\Omega)$ functions. This note is concerned with the uniqueness of *unbounded* solutions to the elliptic problem

$$\begin{cases} \lambda u - \operatorname{div}(A(x)Du) = H(x, Du) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $H(x, Du)$ is measurable with respect to x, locally Lipschitz with respect to ξ and has a super linear growth with respect to the gradient, namely

$$|H(x,\xi)| \leq \gamma|\xi|^q + f(x) \qquad \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^N, \tag{1.2}$$

for some $q > 1$ and $f(x)$ belonging to some Lebesgue space $L^m(\Omega)$, which will be detailed later.

* Corresponding author.
E-mail addresses: leonori@ugr.es (T. Leonori), porretta@mat.uniroma2.it (A. Porretta).

http://dx.doi.org/10.1016/j.jmaa.2017.04.018

It is well known that, if $\xi \mapsto H(x,\xi)$ is locally Lipschitz and has at most linear growth, then problem (1.1) admits a unique weak solution in the Sobolev space $H_0^1(\Omega)$. This is no longer true in case of super linear growth of the first order terms, and uniqueness may fail. For example, the function

$$u(x) = c_{q,N}\left(|x|^{-\frac{2-q}{q-1}} - 1\right) \tag{1.3}$$

is a nontrivial solution of the problem

$$\begin{cases} -\Delta u = |Du|^q & \text{in } B_1(0), \\ u = 0 & \text{on } \partial B_1(0), \end{cases} \tag{1.4}$$

in the distributional sense, if $N/(N-1) < q < 2$ and for a suitable choice of the constant $c_{q,N} > 0$. In particular, this is also a nontrivial $H_0^1(\Omega)$ solution if $1 + \frac{2}{N} < q < 2$.

This shows that the comparison principle does not hold straightforwardly for elliptic equations with super linear first order terms in the class of unbounded solutions, so this issue should be handled with care.

Let us give a brief summary of what is known in the literature. First of all, we recall that the case of super quadratic growth ($q > 2$ in (1.2)) has different features; in this case uniqueness fails even for continuous $H_0^1(\Omega)$ solutions, and one needs to use viscosity solutions, as in first order problems, in order to have comparison principles (see [5,8]). However this approach is restricted to continuous coefficients, and falls outside the spirit of this note, where we deal, in particular, with just measurable x-dependence and possibly unbounded data.

The case $q = 2$ is also a bit special; comparison principles were proved in [6] (for bounded weak solutions) and in [4] for solutions having a certain exponential integrability, namely such that $e^{\gamma u} \in H^1(\Omega)$ for some $\gamma > 0$ depending on the growth of H (1.2). As is well known, this exponential integrability is related to the Hopf–Cole change of unknown which transforms a problem with purely quadratic Hamiltonian into a linear one, where uniqueness is depending on the Fredholm alternative. This is very peculiar and restricted to the case $q = 2$, which therefore appears as a special threshold for this kind of problems. We also refer to [1] for a discussion of the quadratic case and a classification of the possible multiplicity of unbounded solutions.

For $\frac{N}{N-1} < q < 2$, it was shown in [7] that uniqueness holds for solutions of (1.1) such that

$$(1 + |u|)^{\sigma-1} u \in H_0^1(\Omega)\,, \qquad \text{where } \sigma = \frac{(N-2)(q-1)}{2(2-q)}. \tag{1.5}$$

Notice that the exponent σ is such that $\sigma \to \infty$ as $q \to 2$, which is somehow consistent with the quadratic growth case (as mentioned above, in that case the uniqueness result needs some exponential of u in $H_0^1(\Omega)$). Moreover, the uniqueness class (1.5) is shown to be optimal by example (1.3)–(1.4), which exhibits a radial nontrivial solution satisfying $|u|^{\rho-1}u \in H_0^1(\Omega)$ for all $\rho < \sigma$.

Two different methods were used in [7] to prove the uniqueness of solutions in the class (1.5). One approach stands on a linearization argument; roughly speaking, if u_1 and u_2 are two solutions, then one estimates

$$|H(x, Du_1) - H(x, Du_2)| \lesssim c\,[b(x) + |Du_1|^{q-1} + |Du_2|^{q-1}]|D(u_1 - u_2)|$$

for some function $b(x)$ in a consistent Lebesgue class. Therefore the difference $u_1 - u_2$ nearly satisfies a linear equation

$$\begin{cases} \lambda w - \operatorname{div}(A(x)Dw) \lesssim B(x)|Dw| & \text{in } \Omega \\ w \le 0 & \text{on } \partial\Omega\,. \end{cases}$$

It is well known that one needs here $B(x) \in L^N(\Omega)$ in order to conclude that $w \leq 0$ in Ω. In our concrete case, this requires to know that $Du_i \in L^{N(q-1)}(\Omega)$, $i = 1, 2$. But unfortunately, this kind of regularity of the gradient can not always be proved to hold true. In [7], this approach is used, and employed under sufficiently general assumptions, whenever $q \leq 1 + \frac{2}{N}$. In particular, if $\frac{N}{N-1} < q \leq 1 + \frac{2}{N}$, any solution in the class (1.5) satisfies the $L^{N(q-1)}$-integrability of the gradient which is needed for this argument to apply. If $q < \frac{N}{N-1}$, this even works for $W^{1,q}$ solutions (see also [2]), since in this case $N(q-1) < q$ and this regularity is actually already included in the requirement that the equation holds in $L^1(\Omega)$ (this is why the counterexample (1.4) only works for $q > \frac{N}{N-1}$). Generalizations of this approach even apply to more general nonlinear settings including the p-Laplace operator, see e.g. [9,15].

On the other hand, if $1+\frac{2}{N} < q$, the $L^{N(q-1)}$-integrability of the gradient required by the above linearization argument can not be justified unless one has suitable higher estimates, equivalent, roughly speaking, to the $W^{2,p}$ Calderon–Zygmund regularity. But this is not allowed if we only deal with just measurable coefficients. Therefore, for $1 + \frac{2}{N} < q < 2$ a different approach was used in [7] relying on the convexity of $H(x, \xi)$ with respect to ξ. In this case one tries to obtain an estimate like $H(x, Du_1) - H(x, Du_2) \lesssim CH(x, D(u_1 - u_2))$, which can be made rigorous joining the convexity of H with a perturbation argument. Unfortunately this idea, which is not unusual in Hamilton–Jacobi and viscosity solutions theory, requires stronger conditions for the uniqueness results in the range $1 + \frac{2}{N} < q < 2$ rather than those used in the range $q \leq 1 + \frac{2}{N}$.

The purpose of this note is, precisely, to improve the results obtained in [7] for the range $1 + \frac{2}{N} < q < 2$. By refining the idea mentioned above, we will obtain a uniqueness result which only requires the Hamiltonian $H(x, \xi)$ to be convex at infinity (i.e. for $|\xi|$ large). In particular, we show that the comparison, and uniqueness, hold for solutions of (1.1) which belong to the class (1.5), whenever $\xi \mapsto H(x, \xi)$ is a smooth function which is convex for $|\xi|$ sufficiently large and satisfies the growth (1.2). By relaxing the global convexity required in [7], this extension is more satisfactory as far as the generality of the nonlinearity H is concerned.

Let us now be more detailed by stating the main result below. First of all, we recall that a weak sub-solution (or super-solution) of the equation

$$\lambda u - \operatorname{div}(A(x)Du) = H(x, Du) \text{ in } \Omega \tag{1.6}$$

is a function $u \in H^1(\Omega)$ such that

$$\lambda \int_\Omega u\,\varphi\, dx + \int_\Omega A(x)DuD\varphi\, dx \leq (\geq) \int_\Omega H(x, Du)\varphi\, dx \qquad \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)\,.$$

We will say that u is a *strict* sub solution if it satisfies, in the above weak sense,

$$\lambda u - \operatorname{div}(A(x)Du) \leq H(x, Du) - \delta$$

for some $\delta > 0$. Similarly, v is a strict super solution if

$$\lambda v - \operatorname{div}(A(x)Dv) \geq H(x, Dv) + \delta$$

for some $\delta > 0$.

As usual, for two functions $u, v \in H^1(\Omega)$, we say that $u \leq v$ on $\partial\Omega$ whenever $(u - v)_+ \in H_0^1(\Omega)$.

We suppose that $H(x, \xi)$ is a Carathéodory function (measurable with respect to x and continuous with respect to ξ) such that $\xi \mapsto H(x, \xi)$ is locally semi-convex, namely

$$\forall\ K > 0 \quad \exists\, c_K : \xi \mapsto H(x, \xi) + c_K|\xi|^2 \quad \text{is convex in } B_K(0) := \{\xi \in \mathbb{R}^N : |\xi| \leq K\} \tag{1.7}$$

and, in addition, $H(x,\xi)$ is convex at infinity, namely

$$\exists\, R>0: \quad \xi \mapsto H(x,\xi) \qquad \text{is convex in } B_R(0)^c := \{\xi \in \mathbb{R}^N : |\xi| > R\}. \tag{1.8}$$

Let us stress that the above conditions are satisfied by any C^2 function $h(\xi)$ which is convex as $|\xi|$ is sufficiently large. This is clearly a simple though general case for the following statement to be used.

Theorem 1.1. *Let $A(x) = (a_{i,j}(x))$ be a matrix such that*

$$a_{i,j}(x) \in L^\infty(\Omega) \qquad \textit{and} \qquad A(x) \geq \alpha\, I\,, \qquad \textit{for some } \ \alpha > 0. \tag{1.9}$$

Assume that $\xi \mapsto H(x,\xi)$ satisfies (1.7)–(1.8) and the growth condition (1.2), with $1+\frac{2}{N} \leq q < 2$ and $f \in L^{\frac{N}{q'}}(\Omega)$, where $N > 2$ and $q' = \frac{q}{q-1}$.

Let u and v be a (weak) subsolution and a supersolution, respectively, of (1.6) such that $(1+|u|)^{\sigma-1}u, (1+|v|)^{\sigma-1}v \in H^1(\Omega)$ with $\sigma = \frac{(N-2)(q-1)}{2(2-q)}$. Assume that either $\lambda > 0$ or $\lambda = 0$ and one between u and v is strict. If $u \leq v$ at $\partial\Omega$, then $u \leq v$ in Ω.

In particular, under the above assumptions, problem (1.1) has a unique solution in the class (1.5).

Several comments and remarks follow below as a complement of the above statement.

(1) As mentioned above, see also the example (1.3)–(1.4), the above regularity class of sub and super solutions is optimal for the comparison principle to hold. We notice that $\sigma \geq 1$ for $1+\frac{2}{N} \leq q$, so this class is a strict subset of the space H^1. For $\lambda > 0$, Theorem 1.1 is an extension of [7, Theorem 2.1], where a global convexity condition was required upon H.

(2) The restriction $1+\frac{2}{N} \leq q < 2$ is meaningful only for $N > 2$. We observe that, for $1+\frac{2}{N} \leq q < 2$, we have $\frac{2N}{N+2} \leq \frac{N}{q'} < \frac{N}{2}$. This means that, by Sobolev embedding, we have $L^{\frac{N}{q'}}(\Omega) \subset H^1(\Omega)^*$, meaning that the data still belong to the dual space of H^1. This is consistent with considering solutions in H^1. On the other hand, we notice that whenever the data $f(x)$ belong to $L^m(\Omega)$ with $m > \frac{N}{2}$, then solutions are expected to be bounded, and the uniqueness would follow from earlier results. Therefore, leaving aside the case of $f \in L^{\frac{N}{2}}(\Omega)$, which is borderline (related to the case $q = 2$ and to the exponential integrability of solutions, as mentioned before, see [4]), the interesting class of data which lead to unbounded solutions precisely occurs for $f \in L^{\frac{N}{q'}}(\Omega)$ with $q < 2$.

(3) Let us stress that a similar result would also hold for $\frac{N}{N-1} < q < 1+\frac{2}{N}$, however in this range $L^{\frac{N}{q'}}(\Omega)$ is not included in the dual space $H^1(\Omega)^*$, so one would not be allowed to use standard H^1 weak solutions. For a suitable comparison principle, in that case one needs to use the more general framework of renormalized solutions. On one hand, up to changing weak with renormalized formulations, the same result would be true, and would result as an extension of [7, Theorem 3.1]. On another hand, if $1 < q < \frac{N}{N-1}$, the class (1.5) does not bring any significant extra information and it is no more required for comparison principles, which can be directly proved for renormalized solutions.
However, we stress once more that in the range $q \leq 1+\frac{2}{N}$ the approach by linearization already provides different, and general, results, see [7,10,11].

(4) As is well known, the comparison principle is more delicate when $\lambda = 0$. This is why we asked the extra condition that one between u or v is a strict (sub solution or super solution respectively). When $\xi \mapsto H(x,\xi)$ is convex, this extra condition can be replaced by the assumption that (1.6) admits a strict sub solution (or a strict super solution). In that case, if for example φ is a strict sub solution of (1.6), the convexity of H would imply that $(1-\varepsilon)u + \varepsilon\varphi$ is itself a strict sub solution. One could therefore compare $(1-\varepsilon)u + \varepsilon\varphi$ with v and conclude by letting $\varepsilon \to 0$.

Actually, we conjecture that whenever (1.6) admits a sub-solution (in the right class, i.e. (1.5)), then it also admits a strict sub solution. This is true for bounded sub solutions, at least if $H(x,\cdot)$ is convex: in that case, the existence of a bounded sub solution of (1.6) implies the existence of a solution to (1.1) and this implies that there exists some $c > 0$ and some $\varphi \in H^1 \cap L^\infty(\Omega)$ such that $-\mathrm{div}(A(x)D\varphi) + c = H(x, D\varphi)$. This fact follows from [16, Proposition 1.1] (at least for $H(x,\xi) = |\xi|^q + f(x)$) and is related to the characterization of the solvability of (1.1) in terms of the strict sign of the so-called ergodic constant of the state constraint problem. It would be interesting to extend this result to the unbounded case; if the same conclusion remains true, then the comparison principle for the case $\lambda = 0$ would not need extra conditions.

The proof of Theorem 1.1 will be given in the next section. Actually, we will derive this result as a corollary of a slightly more general one, in which we assume that the function H can be split as, roughly speaking, the sum of a convex and a Lipschitz function. We will show later that this is always possible when H satisfies the structure conditions of Theorem 1.1.

2. Proof of the result

Here we state the comparison principle in a slightly more general version. Let us consider the equation

$$\lambda u - \mathrm{div}(A(x)Du) + H(x, Du) = 0 \qquad \text{in } \Omega \tag{2.1}$$

where $A(x) \in L^\infty(\Omega)^{N\times N}$ satisfies (1.9) and $H(x,\xi)$ is a Carathéodory function satisfying the growth condition (1.2). We assume additionally that H can be decomposed as

$$H(x,\xi) = H_1(x,\xi) + H_2(x,\xi) \tag{2.2}$$

where $H_i(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $i = 1, 2$ are Carathéodory functions which satisfy (1.2) and for a.e. $x \in \Omega$ and for any $\xi, \eta \in \mathbb{R}^N$:

$$\xi \mapsto H_1(x,\xi) \text{ is convex} \tag{2.3}$$

and

$$\begin{aligned} &\exists\, L \geq 0 : \quad |H_2(x,\xi) - H_2(x,\eta)| \leq L\,|\xi - \eta| \\ &H_2(x,\xi) - (1-\varepsilon) H_2(x, \frac{\xi}{1-\varepsilon}) \leq 0 \end{aligned} \tag{2.4}$$

for ε sufficiently small.

Hence we have the following comparison result.

Theorem 2.1. *Assume that $A(x)$ satisfies (1.9) and that $H(x,\xi)$ satisfies (1.2) (with $1 + \frac{2}{N} \leq q < 2$ and $f \in L^{\frac{N}{q'}}(\Omega)$) and (2.2)–(2.4). Let u and v be a (weak) subsolution and a supersolution, respectively, of (2.1) such that $(1+|u|)^{\sigma-1}u, (1+|v|)^{\sigma-1}v \in H^1(\Omega)$ with $\sigma = \frac{(N-2)(q-1)}{2(2-q)}$.*

Assume that either $\lambda > 0$ or one between u, v is strict. If $u \leq v$ at $\partial\Omega$, then $u \leq v$ in Ω.

Proof. Let us set $\mathcal{A}(z) := -\mathrm{div}(A(x)Dz)$ and, for any function z, denote by $T_n(z) := \min(n, \max(z, -n))$ the standard truncation function. By [7, Lemma 2.1], $u_n := T_n(u)$ satisfies

$$\lambda u_n + \mathcal{A}(u_n) + H(x, Du_n) \leq I_n^u \qquad \text{in } \Omega,$$

for some $I_n^u \in L^1(\Omega)$ such that

$$\lim_{n\to+\infty} n^{2\sigma-1} \|I_n^u\|_{L^1(\Omega)} = 0\,. \tag{2.5}$$

Similarly, $v_n := T_n(v)$ satisfies

$$\lambda v_n + A(v_n) + H(x, Dv_n) \geq I_n^v$$

where I_n^v also satisfies (2.5). Now we define $u_{n,\varepsilon} = (1-\varepsilon)u_n$. Hence we obtain, subtracting the two equations and using (2.2):

$$\begin{aligned}\lambda(u_{n,\varepsilon} - v_n) + A(u_{n,\varepsilon} - v_n) &\leq H_1(x, Dv_n) - (1-\varepsilon)H_1\Big(x, \frac{Du_{n,\varepsilon}}{1-\varepsilon}\Big) \\ &+ H_2(x, Dv_n) - (1-\varepsilon)H_2(x, \frac{Du_{n,\varepsilon}}{1-\varepsilon}) + (1-\varepsilon)I_n^u - I_n^v - \delta(1-\varepsilon)\,,\end{aligned} \tag{2.6}$$

where $\delta = 0$ unless one between u and v is a strict sub or super solution.

Using the convexity of H_1, we have

$$H_1(x, Dv_n) - (1-\varepsilon)H_1\Big(x, \frac{Du_{n,\varepsilon}}{1-\varepsilon}\Big) \leq \varepsilon H_1\Big(x, \frac{Dv_n - Du_{n,\varepsilon}}{\varepsilon}\Big)$$

while thanks to (2.4) we have that

$$H_2(x, Dv_n) - (1-\varepsilon)H_2(x, \frac{Du_{n,\varepsilon}}{1-\varepsilon}) \leq H_2(x, Dv_n) - H_2(x, Du_{n,\varepsilon}) \leq L\,|Dv_n - Du_{n,\varepsilon}|\,.$$

Plugging the above inequalities into (2.6) we get

$$\begin{aligned}\lambda(u_{n,\varepsilon} - v_n) + A(u_{n,\varepsilon} - v_n) &\leq \varepsilon H_1\Big(x, \frac{Dv_n - Du_{n,\varepsilon}}{\varepsilon}\Big) \\ &\quad + L\,|D(u_{n,\varepsilon} - v_n)| + (1-\varepsilon)I_n^u - I_n^v - \delta(1-\varepsilon) \\ \leq \varepsilon\,\gamma\,\left|\frac{Dv_n - Du_{n,\varepsilon}}{\varepsilon}\right|^q &+ \varepsilon f(x) + L\,|D(u_{n,\varepsilon} - v_n)| + (1-\varepsilon)I_n^u - I_n^v - \delta(1-\varepsilon) \quad \text{in} \quad \Omega\,,\end{aligned}$$

where we used, in the last step, that $H_1 = H - H_2$ together with the growth conditions (1.2) and (2.4).

Define now $w_{n,\varepsilon} := \frac{u_{n,\varepsilon} - v_n}{\varepsilon}$; dividing by ε the above inequality we get

$$\lambda w_{n,\varepsilon} + A(w_{n,\varepsilon}) \leq \gamma\,|Dw_{n,\varepsilon}|^q + L\,|Dw_{n,\varepsilon}| + f(x) + \frac{1}{\varepsilon}[(1-\varepsilon)I_n^u - I_n^v] - \frac{\delta}{\varepsilon}\,(1-\varepsilon)\,. \tag{2.7}$$

It is known (see e.g. [13,14] and [3,12] specifically for $\lambda = 0$) that an a priori estimate in a suitable Lebesgue space holds for sub solutions of

$$\lambda z + A(z) = C\,|Dz|^q + F(x)$$

whenever $q \in (1,2)$ and $F \in L^{\frac{N}{q'}}(\Omega)$ if either $\lambda > 0$ or the norm of F in $L^{\frac{N}{q'}}(\Omega)$ is sufficiently small. Following the above references, we get here similar estimates, the only difference being the error term depending on n. For the reader's convenience, we detail below all the steps leading to those estimates.

We multiply the inequality (2.7) with the test function $(w_{n,\varepsilon} - k)_+^{2\sigma-1}$, $k > 0$. Notice that this test function vanishes on $\partial\Omega$ (in weak sense), since $u \leq v$, in particular we have $(w_{n,\varepsilon} - k)_+^{2\sigma-1} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and is an admissible test function.

In addition, since $(w_{n,\varepsilon}-k)_+^{2\sigma-1}\le c\,n^{2\sigma-1}$, we obtain

$$\lambda\int_\Omega w_{n,\varepsilon}(w_{n,\varepsilon}-k)_+^{2\sigma-1}+\int_\Omega A(x)Dw_{n,\varepsilon}\nabla[(w_{n,\varepsilon}-k)_+^{2\sigma-1}]\le\gamma\int_\Omega |Dw_{n,\varepsilon}|^q(w_{n,\varepsilon}-k)_+^{2\sigma-1}$$
$$+L\int_\Omega |Dw_{n,\varepsilon}|(w_{n,\varepsilon}-k)_+^{2\sigma-1}+\int_\Omega\Big[f(x)-\frac{\delta}{\varepsilon}(1-\varepsilon)\Big](w_{n,\varepsilon}-k)_+^{2\sigma-1}$$
$$+\frac{c}{\varepsilon}n^{2\sigma-1}(\|I_n^u\|_{L^1(\Omega)}+\|I_n^v\|_{L^1(\Omega)})\,.$$

Using $|\xi|\le 1+|\xi|^q$, we can absorb the linear growth term of the right-hand side obtaining

$$\lambda\int_\Omega w_{n,\varepsilon}(w_{n,\varepsilon}-k)_+^{2\sigma-1}+\int_\Omega A(x)Dw_{n,\varepsilon}\nabla[(w_{n,\varepsilon}-k)_+^{2\sigma-1}]\le\tilde\gamma\int_\Omega |Dw_{n,\varepsilon}|^q(w_{n,\varepsilon}-k)_+^{2\sigma-1}$$
$$+\int_\Omega\Big[\tilde f(x)-\frac{\delta}{\varepsilon}(1-\varepsilon)\Big](w_{n,\varepsilon}-k)_+^{2\sigma-1}+\frac{c}{\varepsilon}n^{2\sigma-1}(\|I_n^u\|_{L^1(\Omega)}+\|I_n^v\|_{L^1(\Omega)})\,,\tag{2.8}$$

for, say, $\tilde\gamma=\gamma+L$ and $\tilde f=f+L$.

Let us first deal with the case that $\lambda>0$ (δ can be taken to be zero in this case). Then we estimate

$$\int_\Omega \tilde f(x)(w_{n,\varepsilon}-k)_+^{2\sigma-1}\le\lambda\,k\int_\Omega (w_{n,\varepsilon}-k)_+^{2\sigma-1}+\int_{\{\tilde f>\lambda\,k\}}\tilde f(x)(w_{n,\varepsilon}-k)_+^{2\sigma-1}$$
$$\le\lambda\int_\Omega w_{n,\varepsilon}(w_{n,\varepsilon}-k)_+^{2\sigma-1}+\int_{\{\tilde f>\lambda\,k\}}\tilde f(x)(w_{n,\varepsilon}-k)_+^{2\sigma-1}\,.$$

Hence from (2.8) we deduce

$$\int_\Omega A(x)Dw_{n,\varepsilon}\nabla[(w_{n,\varepsilon}-k)_+^{2\sigma-1}]\le\tilde\gamma\int_\Omega |Dw_{n,\varepsilon}|^q(w_{n,\varepsilon}-k)_+^{2\sigma-1}$$
$$+\int_{\{\tilde f>\lambda\,k\}}\tilde f(x)(w_{n,\varepsilon}-k)_+^{2\sigma-1}+\frac{c}{\varepsilon}n^{2\sigma-1}(\|I_n^u\|_{L^1(\Omega)}+\|I_n^v\|_{L^1(\Omega)})\,.$$

Using the coercivity of $A(x)$ and since $Dw_{n,\varepsilon}\nabla[(w_{n,\varepsilon}-k)_+^{2\sigma-1}]=\frac{2\sigma-1}{\sigma^2}|D(w_{n,\varepsilon}-k)_+^\sigma|^2$ we get, for some (possibly different) constants c only depending on q,N,

$$\int_\Omega |D(w_{n,\varepsilon}-k)_+^\sigma|^2\le c\int_\Omega |Dw_{n,\varepsilon}|^q(w_{n,\varepsilon}-k)_+^{2\sigma-1}$$
$$+c\int_{\{\tilde f>\lambda\,k\}}\tilde f(x)(w_{n,\varepsilon}-k)_+^{2\sigma-1}+\frac{c}{\varepsilon}n^{2\sigma-1}(\|I_n^u\|_{L^1(\Omega)}+\|I_n^v\|_{L^1(\Omega)})\,,$$

which implies

$$\int_\Omega |D(w_{n,\varepsilon}-k)_+^\sigma|^2 \le c\int_\Omega |D(w_{n,\varepsilon}-k)_+^\sigma|^q (w_{n,\varepsilon}-k)_+^{2\sigma-1-q(\sigma-1)} + c\int_{\{\tilde f>\lambda\, k\}} \tilde f(x)(w_{n,\varepsilon}-k)_+^{2\sigma-1} + \frac{c}{\varepsilon}\, n^{2\sigma-1}(\|I_n^u\|_{L^1(\Omega)} + \|I_n^v\|_{L^1(\Omega)})\,.$$

We use Hölder's inequality in the two integrals of the right-hand side, obtaining

$$\begin{aligned}\int_\Omega |D(w_{n,\varepsilon}-k)_+^\sigma|^2 \le c\left(\int_\Omega |D(w_{n,\varepsilon}-k)_+^\sigma|^2\right)^{\frac q2}\left(\int_\Omega (w_{n,\varepsilon}-k)_+^{(2\sigma-1-q(\sigma-1))\frac{2}{2-q}}\right)^{1-\frac q2}\\ + c\,\|\tilde f(x)\,\chi_{\{\tilde f>\lambda\, k\}}\|_{L^{\frac{N}{q'}}(\Omega)}\|(w_{n,\varepsilon}-k)_+^{2\sigma-1}\|_{L^{\frac{N}{N-q'}}(\Omega)}\\ + \frac{c}{\varepsilon}\, n^{2\sigma-1}(\|I_n^u\|_{L^1(\Omega)} + \|I_n^v\|_{L^1(\Omega)})\,,\end{aligned} \tag{2.9}$$

where, for any set $E\subset\Omega$, we denote by χ_E the indicatrix function of the set E.

The precise value of $\sigma = \frac{(q-1)(N-2)}{2(2-q)}$ gives

$$(2\sigma-1-q(\sigma-1))\frac{2}{2-q} = 2^*\sigma \qquad \text{and} \qquad (2\sigma-1)\frac{N}{N-q'} = 2^*\sigma$$

where $2^* = \frac{2N}{N-2}$ is the Sobolev exponent. Therefore, we deduce from (2.9) that

$$\begin{aligned}\int_\Omega |D(w_{n,\varepsilon}-k)_+^\sigma|^2 \le c\left(\int_\Omega |D(w_{n,\varepsilon}-k)_+^\sigma|^2\right)^{\frac q2}\left(\int_\Omega (w_{n,\varepsilon}-k)_+^{2^*\sigma}\right)^{1-\frac q2}\\ + c\,\|\tilde f(x)\,\chi_{\{\tilde f>\lambda\, k\}}\|_{L^{\frac{N}{q'}}(\Omega)}\|(w_{n,\varepsilon}-k)_+\|_{L^{2^*\sigma}(\Omega)}^{2\sigma-1}\\ + \frac{c}{\varepsilon}\, n^{2\sigma-1}(\|I_n^u\|_{L^1(\Omega)} + \|I_n^v\|_{L^1(\Omega)})\,.\end{aligned}$$

Young's inequality leads us to

$$\begin{aligned}\int_\Omega |D(w_{n,\varepsilon}-k)_+^\sigma|^2 \le c\int_\Omega (w_{n,\varepsilon}-k)_+^{2^*\sigma}\\ + c\,\|\tilde f(x)\,\chi_{\{\tilde f>\lambda\, k\}}\|_{L^{\frac{N}{q'}}(\Omega)}\|(w_{n,\varepsilon}-k)_+\|_{L^{2^*\sigma}(\Omega)}^{2\sigma-1}\\ + \frac{c}{\varepsilon}\, n^{2\sigma-1}(\|I_n^u\|_{L^1(\Omega)} + \|I_n^v\|_{L^1(\Omega)})\,.\end{aligned}$$

Then, we use Sobolev inequality and we get

$$\begin{aligned}\left(\int_\Omega |(w_{n,\varepsilon}-k)_+|^{2^*\sigma}\right)^{\frac{2}{2^*}} \le c\int_\Omega (w_{n,\varepsilon}-k)_+^{2^*\sigma}\\ + c\,\|\tilde f(x)\,\chi_{\{\tilde f>\lambda\, k\}}\|_{L^{\frac{N}{q'}}(\Omega)}\|(w_{n,\varepsilon}-k)_+\|_{L^{2^*\sigma}(\Omega)}^{2\sigma-1}\\ + \frac{c}{\varepsilon}\, n^{2\sigma-1}(\|I_n^u\|_{L^1(\Omega)} + \|I_n^v\|_{L^1(\Omega)})\,.\end{aligned} \tag{2.10}$$

We can now let n go to infinity. Indeed, by assumption we know that u, v satisfy (1.5), which implies that $u, v \in L^{2^*\sigma}(\Omega)$. Therefore $w_{n,\varepsilon}$ converges to $w_\varepsilon := \frac{(1-\varepsilon)u-v}{\varepsilon}$ in $L^{2^*\sigma}(\Omega)$ as $n \to \infty$. Moreover, due to (2.5), the last term in (2.10) vanishes as $n \to \infty$. Finally we obtain

$$\left(\int_\Omega |(w_\varepsilon - k)_+|^{2^*\sigma}\right)^{\frac{2}{2^*}} \le c\int_\Omega (w_\varepsilon - k)_+^{2^*\sigma} + c\,\|\tilde f(x)\,\chi_{\{\tilde f>\lambda\, k\}}\|_{L^{\frac{N}{q'}}(\Omega)}\|(w_\varepsilon - k)_+\|_{L^{2^*\sigma}(\Omega)}^{2\sigma-1}$$

which implies

$$\|(w_\varepsilon - k)_+\|_{L^{2^*\sigma}(\Omega)} \le c\,\|(w_\varepsilon - k)_+\|_{L^{2^*\sigma}(\Omega)}^{\frac{q}{2-q}} + c\,\|\tilde f(x)\,\chi_{\{\tilde f>\lambda\, k\}}\|_{L^{\frac{N}{q'}}(\Omega)}, \tag{2.11}$$

where we used that $2^*\sigma - (2\sigma - 1) = \frac{q}{2-q}$. Notice that $\frac{q}{2-q} > 1$ since $q > 1$. The above inequality reads as

$$Y_k \le c\,Y_k^\beta + \ell_k$$

for some $\beta > 1$, where $Y_k = \|(w_\varepsilon - k)_+\|_{L^{2^*\sigma}(\Omega)}$ and $\ell_k = c\,\|\tilde f(x)\,\chi_{\{\tilde f>\lambda\, k\}}\|_{L^{\frac{N}{q'}}(\Omega)}$.

The conclusion of the estimate is exactly as in [13]: since $\tilde f \in L^{\frac{N}{q'}}(\Omega)$, it is possible to choose k_0 such that $\ell_k < \max\limits_{Y\in(0,\infty)} [Y - cY^\beta]$ for every $k \ge k_0$. Then, a continuity argument (based on the fact that Y_k is continuous with respect to k and vanishes for $k \to \infty$) allows us to conclude that $Y_k \le C$ for every $k \ge k_0$, where C is a constant only depending on q, N. Recalling the definition of Y_k, this readily implies a global estimate of w_ε in $L^{\sigma\, 2^*}(\Omega)$, namely

$$\|(w_\varepsilon)_+\|_{L^{2^*\sigma}(\Omega)} \le \|(w_\varepsilon - k_0)_+\|_{L^{2^*\sigma}(\Omega)} + k_0 \le C + k_0\,.$$

Recalling the definition of w_ε, we finally deduce

$$\|(1-\varepsilon)u - v)_+\|_{L^{\sigma\, 2^*}(\Omega)} \le \varepsilon\,(C + k_0)$$

and letting $\varepsilon \to 0$, we conclude that $u \le v$ in Ω.

In the case that $\lambda = 0$ and one of the two (sub solution or super solution) is strict, we have $\delta > 0$ in (2.8). In this case we estimate

$$\int_\Omega \Big[\tilde f(x) - \frac{\delta}{\varepsilon}\,(1-\varepsilon)\Big](w_{n,\varepsilon} - k)_+^{2\sigma-1} \le \int_{\{\tilde f>\frac{\delta}{\varepsilon}\,(1-\varepsilon)\}} \tilde f(x)(w_{n,\varepsilon} - k)_+^{2\sigma-1}$$

and we proceed as before. Finally we obtain the same as (2.11), which would read as

$$\|(w_\varepsilon - k)_+\|_{L^{2^*\sigma}(\Omega)} \le c\,\|(w_\varepsilon - k)_+\|_{L^{2^*\sigma}(\Omega)}^{\frac{q}{2-q}} + c\,\|\tilde f(x)\,\chi_{\{\tilde f>\frac{\delta}{\varepsilon}\,(1-\varepsilon)\}}\|_{L^{\frac{N}{q'}}(\Omega)}.$$

Note that last term can be made arbitrarily small provided ε is small enough. In particular we can assume that ε is sufficiently small so that

$$c\,\|\tilde f(x)\,\chi_{\{\tilde f>\frac{\delta}{\varepsilon}\,(1-\varepsilon)\}}\|_{L^{\frac{N}{q'}}(\Omega)} < \max_{Y\in(0,\infty)} [Y - cY^\beta]$$

where $\beta = \frac{q}{2-q}$. Therefore, the same continuity argument as before, made in terms of the parameter k for the whole range $k \in (0,\infty)$, allows one to conclude that $\|(w_\varepsilon - k)_+\|_{L^{2^*\sigma}(\Omega)} \le C$ for every $k > 0$, where

C is a constant only depending on q, N. This means that, as $k \to 0$, we have

$$\|(w_\varepsilon)_+\|_{L^{2^*\sigma}(\Omega)} \le C$$

and we conclude as before letting $\varepsilon \to 0$. $\square$

The proof of Theorem 1.1 now follows as corollary of the above result.

Proof of Theorem 1.1. It is sufficient to observe that, on account of (1.7)–(1.8), the function

$$H_1(x,\xi) = H(x,\xi) + \alpha\sqrt{1+|\xi|^2} \qquad \alpha \in \mathbb{R}^+$$

is globally convex with respect to ξ, provided α is chosen sufficiently large. On the other hand, the function $H_2(x,\xi) = -\alpha\sqrt{1+|\xi|^2}$ satisfies (2.4). Hence we can apply Theorem 2.1 to conclude. $\square$

References

[1] B. Abdellaoui, A. Dall'Aglio, I. Peral, Some remarks on elliptic problems with critical growth in the gradient, J. Differential Equations 222 (2006) 21–62.
[2] N. Alaa, M. Pierre, Weak solutions of some quasilinear elliptic equations with data measures, SIAM J. Math. Anal. 24 (1) (1993) 23–35.
[3] A. Alvino, V. Ferone, A. Mercaldo, Sharp a priori estimates for a class of nonlinear elliptic equations with lower order terms, Ann. Mat. Pura Appl. 194 (2015) 1169–1201.
[4] G. Barles, A. Blanc, C. Georgelin, M. Kobylanski, Remarks on the maximum principle for nonlinear elliptic PDEs with quadratic growth conditions, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 28 (1999) 381–404.
[5] G. Barles, F. Da Lio, On the generalized Dirichlet problem for viscous Hamilton–Jacobi equations, J. Math. Pures Appl. (9) 83 (1) (2004) 53–75.
[6] G. Barles, F. Murat, Uniqueness and the maximum principle for quasilinear elliptic equations with quadratic growth conditions, Arch. Ration. Mech. Anal. 133 (1995) 77–101.
[7] G. Barles, A. Porretta, Uniqueness for unbounded solutions to stationary viscous Hamilton–Jacobi equations, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 5 (2006) 107–136.
[8] G. Barles, E. Rouy, P.E. Souganidis, Remarks on the Dirichlet problem for quasilinear elliptic and parabolic equations, in: W.M. McEneaney, G.G. Yin, Q. Zhang (Eds.), Stochastic Analysis, Control, Optimization and Applications. A Volume in Honor of W.H. Fleming, Birkhäuser, Boston, 1999, pp. 209–222.
[9] F. Betta, R. Di Nardo, A. Mercaldo, A. Perrotta, Gradient estimates and comparison principle for some nonlinear elliptic equations, Commun. Pure Appl. Anal. 14 (2015) 897–922.
[10] F. Betta, A. Mercaldo, F. Murat, M. Porzio, Uniqueness of renormalized solutions to nonlinear elliptic equations with a lower order term and right-hand side in $L^1(\Omega)$. A tribute to J.L. Lions, ESAIM Control Optim. Calc. Var. 8 (2002) 239–272.
[11] F. Betta, A. Mercaldo, F. Murat, M. Porzio, Uniqueness results for nonlinear elliptic equations with a lower order term, Nonlinear Anal. 63 (2005) 153–170.
[12] V. Ferone, B. Messano, Comparison and existence results for classes of nonlinear elliptic equations with general growth in the gradient, Adv. Nonlinear Stud. 7 (2007) 31–46.
[13] N. Grenon, F. Murat, A. Porretta, Existence and a priori estimate for elliptic problems with subquadratic gradient dependent terms, C. R. Acad. Sci. Paris, Ser. I 342 (2006) 23–28.
[14] N. Grenon, F. Murat, A. Porretta, A priori estimates and existence for elliptic equations with gradient dependent terms, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) XIII (2014) 137–205.
[15] A. Porretta, On the comparison principle for p-Laplace type operators with first order terms, in: On the Notions of Solution to Nonlinear Elliptic Problems: Results and Developments, in: Quad. Mat., vol. 23, Dept. Math., Seconda Univ. Napoli, Caserta, 2008, pp. 459–497.
[16] A. Porretta, The "ergodic limit" for a viscous Hamilton–Jacobi equation with Dirichlet conditions, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 21 (2010) 59–78.