Energy exchange between nonlinear oscillators: an entropy foundation

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Abstract

In the field of vibrations of complex structures, energy methods like SEA and a series of mid-frequency methods, represent an important resource for computational analysis. All these methods are based in general on a linear formulation of the elastic problem. However, when nonlinearities are present, for example related to clearance or stiffening of joints, these methods, in principle, cannot be applied. This paper, on the basis of a theory presented recently by one of the authors, proposes a foundation of a new energy method able to deal with nonlinearities when studying the energy exchange between subsystems. The idea relies on the concept of a thermodynamic vibroacoustic temperature, that can be directly defined when introducing the entropy of a vibrating structure. The theory is introduced in general, and examples of calculation of the power flow between nonlinear resonators are presented introducing stiffening and clearences for systems with many degrees of freedom.

1   Introduction

This paper introduces the concept of Khinchin's entropy [1] aimed at introducing a thermodynamic temperature for Hamiltonian systems, a quantity that is able to control the energy flow between two coupled linear or nonlinear sub-systems.

This subject represents the backbone of many problems of actual and historical interest both in physics as well as in structural engineering. The common root of the problem has its origins in deriving macroscopic relationships from first principles to which the detailed microscopic structure of a mechanical system obeys.

Heat conduction Fourier law is well known to describe the macroscopic behavior of thermal processes and the temperature evolution in heated bodies. The analysis of the same phenomenon starting form a molecular description of solids, is indeed much less obvious, and it is still the subject of several theoretical as well as numerical investigations [2-5]. In fact, the investigation of thermal vibration and energy migration among sub-systems, is a phenomenon occurring at very high frequency and related to molecular and atomic motion, and in some cases involving even quantum-mechanical effects [6]. Equilibrium and non-equilibrium thermodynamics would be involved in this context, and the response of a test particle in a thermal bath also belongs to this class of problems. Prototypical approach of microscopic modeling of macroscopic thermal behavior, was the backbone of the Boltzmann's ideas on Thermodynamics and also considered, in the early years of 1900, in the context of molecular physics and Brownian movement [7]. There, for the first time and by the Langevin's equation, a macroscopic dissipation term is introduced to describe the energy released by the test particle to the thermal bath: the first is an elemental point-mass particle, the second is modeled by a plurality of randomly oscillating masses [8-12].

Another class of interesting problems in this frame is certainly related to the presence of continuum elastic systems equipped with a micro-structure, that exhibit analogies with the molecular structure of the matter at the origin of its macroscopic thermal behavior, as previously mentioned. In this field, recent analyses have been developed in the context of porous materials and micro-structured materials [13-15], where very

intriguing energy considerations arise also show phenomena of energy transfer from the macro- to the micro-structure of the system [16-18].

Also thermodynamically and energy transfer inspiration is used in the analysis of a large class of dynamic dampers, either mechanical or due to some form of electromechanical coupling. The goal of an effective dynamic damper is the fast energy transfer from a principal oscillator to a secondary oscillator, or set of oscillators. Power flow between the principal and the attached secondary resonators is, again, the goal of the analysis [19-23].

Remarkably, nowadays the distinction between thermal vibration, occurring at a molecular level, and macroscopic vibration of continuous elastic system, is in some context becoming less obvious than in the past. In fact, nanotechnology advances introduce very intriguing devices made by a relatively small and countable number of atoms, where thermal vibration energy becomes comparable with the vibrational energy response excited by the actuation forces, making impossible to separate elastic and thermal vibrations.

This paper considers the problem of energy flow transmission due to classical mechanical vibrations in coupled continuum elastic structures. The element of primary interest is the role of the collective energies of sub-systems made of a large number of oscillating masses. These phenomena occur in general at characteristic wavelengths that are much longer than the typical interatomic distance of the lattice, i.e. far away from the range of thermal vibration. In this context, the chance of usual simplifications used in statistical thermodynamics, relying on averaging over an extraordinary large number of particles, is missed. Vibrations of engineering structures, exhibit indeed a moderately large number of degrees of freedom: in some cases it is too large to allow for a detailed analysis of the system, and too small to rely on a complete thermodynamic energy average approach. This is a new kingdom, sometime called mesoscale thermodynamics.

The chance of a thermodynamic analogy could suggest that the energy flow would be proportional to the difference of the vibrational energy in each of the two coupled sub-systems. This hypothesis is referred in the following as Fourier-like energy transmission or vibrational conductivity. However, for mechanical vibrations, as it appears from some investigations presented in the last three decades, a Fourier-like energy transmission law does not strictly hold, except under special circumstances. However, the question has been debated in depth, and we can reasonably conclude that, in general, under certain specific assumptions, and for some type of systems, some form of vibrational thermal conductivity can be derived [24-28] even for structural vibrations.

Numerical methods were presented during the years that are based on the assumption of a vibrational conductivity law. As an engineering tool, this methodology, in some cases, can provide significant results and permits important computational simplifications and drastic reduction of computational costs. In fact, vibration of the elastic continuum is classically described by hyperbolic equations, while a thermal conductivity approach would lead to describe the vibration energy diffusion through parabolic equations, with the mentioned numerical advantages. In this context, the engineering community made an impressive effort in the last four decades for predicting energy flow among mechanical sub-systems: Statistical Energy Analysis – SEA, is indeed the engineering vibration theory more closely inspired to a thermal analogy approach. In the next sections, our findings will be compared with those of SEA to underline differences and analogies.

This scenario can explain the effort of some investigators to attack the problem from another front, recognizing entropy having a central role in thermodynamic foundation of energy methods in elastic vibrations [29-33].

In the present paper, reanalyzing some results contained in [34], we adopt this thermodynamic point of view. Entropy is the root to understand the concept of thermodynamic temperature, the quantity that controls the power flow among two Hamiltonian systems. Among the beneficial effects of this approach, the method is applicable to linear and to nonlinear systems as well.

Therefore, this paper can serve to corroborate the theoretical background of the vibrational energy methods, opening new perspectives for including nonlinear systems in this context.

2   Temperature, energy and entropy

A quantitative formulation of the second principle of thermodynamics is based on the Boltzmann's inequality:

$$d\,H\,/\,dt \geq 0 \tag{1}$$

where H is the entropy of an isolated system subjected to a thermodynamic transformation and t is the time. Equation (1) states that the entropy of an isolated system never decreases. This can be considered in some theories an axiom or an empirical property or, as in the original approach of Boltzmann, even a mathematical deduction form mechanical first principles through the famous H-theorem.

Let us show how equation (1) plays a crucial role in the problem of energy transmission between two sub-systems discussed in this paper.

Assume H for a mechanical N-dof resonator can be defined to satisfy equation (1) -we see later how this quantity can be introduced. Assume also H exhibits an additive property, i.e. the entropy of a composite system is the sum of the individual entropy of each component.

For example, if two conservative resonators $\mathbf{R_1}$ and $\mathbf{R_2}$ are considered, their total energy and entropy are E_1, H_1 and E_2, H_2, respectively. The energy flow between $\mathbf{R_1}$ and $\mathbf{R_2}$ can be approached by using the energy balance and two entropy based properties:

energy conservation: $\quad dE_1 \,/\, dt + dE_2 \,/\, dt = 0$ (2)

Boltzmann's inequality: $\quad d\,H\,/\,dt \geq 0$ (3)

additive entropy property: $\quad H = H_1 + H_2$ (4)

By combining the last two:

$$\frac{dH_1}{dt} + \frac{dH_2}{dt} \geq 0 \quad \Rightarrow \quad \frac{dH_1}{dE_1}\frac{dE_1}{dt} + \frac{dH_2}{dE_2}\frac{dE_2}{dt} \geq 0 \tag{5}$$

and using the energy balance:

$$\left[\frac{dH_1}{dE_1} - \frac{dH_2}{dE_2}\right]\dot{W} \geq 0 \quad , \quad \dot{W} = \frac{dE_1}{dt} = -\frac{dE_2}{dt} \tag{6}$$

where $\dot{W}$ is the energy flow between $\mathbf{R_1}$ and $\mathbf{R_2}$. Inequality (6) states that the power flow direction, i.e. the sign of $\dot{W}$, depends on the sign of the difference between the entropy rates of the two resonators.

This fact, suggests the quantity dH/dE is a measure of the capability of the system to absorb external energy, and its inverse is the system's capability to release its energy. Thus:

$$1/T = dH/dE \tag{7}$$

defines the thermodynamic temperature T of the system. On the basis of inequality (6), the fundamental statement follows: the energy flow migrates from the system at higher temperature to the system at lower temperature.

The previous statement and equation (7) are key points of the present paper. In equation (7), three central quantities are involved: energy, temperature and entropy. This point of view has a direct implication in the study of the power flow between the two resonators. Since, as stated by equation (6), T_1 - T_2 controls the power flow, we can conclude that some functional dependency between T_1 - T_2 and $\dot{W}$, must exist. Therefore, two basic steps are the fundaments of a system-theoretic approach to vibrational energy sharing:

(i) how a suitable entropy function can be defined, so that by equation (7) the temperature of the resonator can be determined, and (ii) how a relationship between $\dot{W}$ and T_1 - T_2 can be established.

The following sections are aimed at studying these two points. To make the paper self-contained, some parts of the theory presented in [34] are resumed.

3   Entropy fundamentals

The dynamical system **R** has state variables represented by $x(t)$, $\dot{x}(t)$. If their value is known at t=0, they are sufficient to predict the system time history for any t>0. R is assumed to be conservative and characterised by an invariant total energy E, *i.e.* $\dot{E}(x,\dot{x})=0$.

The micro-states of **R** are associated with all the possible values of the pair x, $\dot{x}$; the system macro-states are indeed associated with the value of a global function depending on x, $\dot{x}$, e.g., *E*. It is clear that infinite different states x, $\dot{x}$ can lead to the same value of *E*, i.e. infinite micro-states can be associated to a single macro-state.

The infinite micro-states associated with a given energy level E_0 (macro-state) are determined simply by imposing the constraint:

$$E(x,\dot{x}) = E_0 \tag{8}$$

Consider the phase space Γ defined by the variables x_i, $\dot{x}_i$ (components of x, $\dot{x}$ for i=1,N). The last equation defines an hyper-surface Σ belonging to Γ. For example, if the system is linear, then *E* is a quadratic form of x, $\dot{x}$ and Σ is an hyper-ellipsoid. The points belonging to Σ are system's micro-states, while the whole set of points on Σ is the macro-state associated to the given energy level E_0. When a certain energy level *E* characterizes the macro-state of **R** at *t=0*, then for any *t>0* the micro-states are constrained over the surface Σ of Γ and the point *P* of co-ordinates $x_1, x_2,...x_N, \dot{x}_1, \dot{x}_2,...\dot{x}_N$ describes a trajectory over the surface Σ.

In the present study we are interested in the energy exchange between two (or more) sub-systems. **R₁** and **R₂** are defined sub-systems of **R** if (i) the sum of their energies E_1 and E_2 equals the energy of **R** and (ii) if E_1 and E_2 depend on two sets of variables that have an empty intersection and whose union provides the set of variables of **R**. Introducing the state vectors **v**, **v₁** and **v₂** associated to **R**, **R₁** and **R₂**, respectively, the previous statement is formalized as:

$$E(v) = E_1(v_1) + E_2(v_2), \qquad v = v_1 + v_2$$
$$v = \{x, \dot{x}\}^T = \{x_1, x_2, \dot{x}_1, \dot{x}_2\}^T, \quad v_1 = \{x_1, 0, \dot{x}_1, 0\}^T \quad v_2 = \{0, x_2, 0, \dot{x}_2\}^T \tag{9}$$

This decomposability property is not trivial, even for simple linear systems. In fact, in general two coupled sub-systems, have total energy:

$$E(v) = E_1(v_1) + E_2(v_2) + E_{12}(v_1, v_2) \tag{10}$$

being the third term on the right hand side a mixed energy contribution. However, in some cases, the mixed energy term is negligible in comparison with the other energy contributions. This is in the following denoted as weak coupling condition, and permits to write:

$$E(v) \approx E_1(v_1) + E_2(v_2) \tag{11}$$

Let us now introduce the structure and the generating functions.

The equal energy surface Σ wraps the volume V_Σ. The micro-states with energy E belong to Σ and the set of micro-states $\mathbf{x}, \dot{\mathbf{x}}$ having energy $E' < E$ fall inside the volume wrapped by Σ. The measure of V_Σ is:

$$V(E) = \int_{V_\Sigma} dx_1 dx_2 ... dx_N \, d\dot{x}_1 d\dot{x}_2 ... d\dot{x}_N \tag{12}$$

that depends on E. Let:

$$\Omega(E) = \frac{dV}{dE} \tag{13}$$

named structure function of $\mathbf{R}$. It is shown in [1] that Ω is measure of the number of micro-states associated to a given energy level E since it provides a measure of the surface Σ.

If $\mathbf{R_1}$ and $\mathbf{R_2}$ are sub-systems of $\mathbf{R}$ with respective structure functions $\Omega_1(E_1)$ and $\Omega_2(E_2)$ it can be demonstrated [1] that $\Omega = \Omega_1 * \Omega_2$ or taking the Laplace transforms $\Phi = \Phi_1 \Phi_2$, where $\Phi(s)$ is named the generating function of $\mathbf{R}$. The Khinchin's entropy is introduced as [1]:

$$H(E, \sigma) = \log\left[e^{\sigma E} \Phi(\sigma)\right] + const \tag{14}$$

where H depends on the parameter σ and on the energy E, that are not independent, requiring the derivative of H with respect to σ, for any E, vanishes:

$$\frac{\partial H(E, \sigma)}{\partial \sigma} = 0 \qquad \Rightarrow \qquad \frac{\Phi'(\sigma)}{\Phi(\sigma)} = -E \tag{15}$$

that makes H a function of E only, through the previous introduced definitions.

Considering now the individual entropies of the two sub-systems $\mathbf{R_1}$ and $\mathbf{R_2}$, and that of the resonator $\mathbf{R}$, we can show [1] that when the two sub-systems reach the equilibrium, that is for $\sigma_1 = \sigma_2 = \sigma*$, their entropies sum equal the constant value H_R. Moreover, [34] the following inequality holds:

$$H_R = H_1(E_1^*) + H_2(E_2^*) \geq H_1(E_{10}) + H_2(E_{20}) \tag{16}$$

i.e. the sum of the entropy increments of the resonators is always greater-equal than zero, proceedings from an adiabatic towards the equilibrium condition. If the resonators are observed in conditions close to the thermodynamic equilibrium, then:

$$\frac{dH_1}{dt} + \frac{dH_2}{dt} \geq 0 \tag{17}$$

and the analysis of section 2 holds and it produces:

$$\sigma = \frac{1}{T} \tag{18}$$

This permits to conclude that a strict definition of temperature, as the quantity controlling the energy flow between two coupled resonators, is provided by:

$$\frac{\Phi'(1/T)}{\Phi(1/T)} = -E \tag{19}$$

4 Linear systems: temperature of a finite-dimension Hamiltonian system

In this section we intend to calculate the thermodynamic temperature of a linear dynamical system. We can assume it directly consists of N degrees of freedom, for example a set of N linear elastically interconnected lumped masses.

The kinetic and potential energies K and U can be expressed respectively:

$$K = \frac{1}{2}\dot{q}^T\dot{q}, \qquad U = \frac{1}{2}q^T \Lambda q \tag{20}$$

where q is the vector the collects the normal coordinates of the resonator and Λ the related diagonal eigenvalues matrix. The equal energy surface Σ, associated to the energy level E, is represented in the phase space Γ by the equation:

$$\frac{1}{2}\dot{q}^T\dot{q} + \frac{1}{2}q^T \Lambda q - E = 0 \tag{21}$$

This is an ellipsoid in Γ whose semi-axis lengths are easily determined. In fact, the intersections of this ellipsoid Σ with the q_i axes are given by:

$$\frac{1}{2}q_i^2 \lambda_i = E, \qquad q_i = \sqrt{2E/\lambda_i} \qquad i = 1,...N \tag{22}$$

where N is the number of degrees of freedom and also the number of modes of the resonator. Analogous consideration holds when the intersection between Σ and the $\dot{q}_i$ axis is determined:

$$\frac{1}{2}\dot{q}_i^2 = E, \qquad \dot{q}_i = \sqrt{2E} \qquad i = 1,...N \tag{23}$$

The volume V of the ellipsoid of given semi-axes $q_i = \sqrt{2E/\lambda_i}$ and $\dot{q}_i = \sqrt{2E}$ is simply proportional to the products of all the semi-axis lengths, therefore the following relationships follow:

$$V = 2^N A \frac{E^N}{\sqrt{\prod_{i=1}^{N}\lambda_i}}, \quad \Omega = 2^N A \frac{N E^{N-1}}{\sqrt{\prod_{i=1}^{N}\lambda_i}}, \quad \Phi(s) = \alpha(N)\frac{1}{s^N}, \quad \alpha(N) = 2^N A \frac{N!}{\sqrt{\prod_{i=1}^{N}\lambda_i}} \tag{24}$$

The temperature, accordingly to equation (19) is:

$$\sigma = \frac{N}{E} \qquad \rightarrow \qquad T = \frac{E}{N} \tag{25}$$

It is relevant to remark that this equation states the quantity actually controlling the power flow is the average modal energy E/N. This results is reminding a fundamental SEA statement.

5 Energy flow and temperatures

In section 3 a general method to determine the temperature of the resonator has been presented and it has been applied in section 4 to the case of N-dof linear oscillators.

Although we can state that $\dot{W}$ depends on T_2 and T_1, the form of this functional relationship is not explicitly provided by the previous approach, as it clear also form the theory outlined in section 2. More precisely,

since, as it is shown in the previous section, $T=E/N$ is a positive quantity, a simple manipulation of inequality (6) allows to conclude that $\dot{W}$ depends on $T_2 - T_1 = \Delta T$.

For example in [34] for a two degrees of freedom elastic system it is shown the relationship holds: $a\,\dot{W}^2 + \Delta T^2 = E_0^2$. It appears that for a simple two-dof system, the relationship $\dot{W} = f(\Delta T)$ is approximately represented by an elliptic orbit and it is not a single valued function (is a constant depending on some parameters of the system).

In the presence of a large number of degrees of freedom systems, the relationship between temperature difference and power flow assumes different forms. In particular it becomes a single-valued function.

For example in SEA, a simple linear relationship is obtained under certain assumptions, for steady-state vibration and random excitation. For transient excitation, and when an ensemble average over a stochastic population of structures is considered, it has been shown in [24] that indeed a nonlinear relationship can be written.

As a general conjecture, a form of the type $\dot{W} = \sum_{i=1}^{N} \eta_i \left(T_2 - T_1\right)^{\alpha_i}$ would describe the energy flow between systems equipped by a rather large number of dof's, where the exponents are not necessarily integers [24]. However, it appears that close to equilibrium conditions and for a steady-state process, an assumption of linearity $\dot{W} = \eta\left(T_2 - T_1\right)$ can be reasonable and represents the first order approximation of more complicated relationships.

6   Nonlinear potentials

Considering as the simplest case the single dof resonators, nonlinear effects make the equation representing the equal energy curve as:

$$E = \frac{1}{2} m\, \dot{x}^2 + \frac{1}{2} k_1\, x^2 + U_{nl}(\,x\,) \tag{26}$$

where U_{nl} is the anharmonic contribution to the potential energy because of nonlinear restoring forces. The determination of the structure function needs the measure of the volume lying inside the equal energy surface. Although when using numerical tools the problem can be always solved, in the present section, we are interested to derive some closed form formulas for the thermodynamic temperature of some nonlinear resonators. In some cases this is directly possible, as for the case of the dead-zone nonlinearity (frequently met in engineering due to clearances), in other cases some simplified hypotheses can be introduced to obtain closed formulas, such as the weak nature of the nonlinearities.

A resonator with a dead-zone g (gap), is characterized by the equation of motion:

$$
\begin{aligned}
& m\,\ddot{x} + f_{e+}(\,x\,) + f_{e-}(\,x\,) = 0 \\
& f_{e-}(\,x\,) = k\,W_-(\,x\,)(\,x + g\,) \\
& f_{e+}(\,x\,) = k\,W_+(\,x\,)(\,x - g\,) \\
& W_-(\,x\,) = \begin{cases} 1 \ \ for\ x < -g \\ 0 \ \ for\ x > -g \end{cases} \\
& W_+(\,x\,) = \begin{cases} 0 \ \ for\ x < g \\ 1 \ \ for\ x > g \end{cases}
\end{aligned}
\tag{27}
$$

where k is the elastic constant and g is the spring gap that characterizes the size of the dead-zone. The total elastic energy is:

$$E = \frac{1}{2} m \dot{x}^2 + U(x)$$

$$U(x) = \frac{1}{2} k W_+(x)(x-g)^2 + \frac{1}{2} k W_-(x)(x+g)^2 \tag{28}$$

Σ is represented by the previous equation over the phase plane $x, \dot{x}$. The area V_Σ is simply evaluated by the closed form expression:

$$V_\Sigma(E) = 4 \left(\frac{2\pi E}{\sqrt{mk}} + g \sqrt{\frac{2E}{m}} \right) \tag{29}$$

This implies that the structure and the generating functions, and thermodynamic temperature are, respectively:

$$\Omega(E) = 4 \left(\frac{2\pi}{\sqrt{mk}} + g \frac{1}{2} \sqrt{\frac{2}{m}} \frac{1}{\sqrt{E}} \right), \quad \Phi(s) = 4 \left(\frac{2\pi}{\sqrt{mk}} \frac{1}{s} + g \frac{1}{2} \sqrt{\frac{2\pi}{m}} \frac{1}{\sqrt{s}} \right)$$

$$\frac{\frac{8\pi}{\sqrt{mk}} T^2 + g \sqrt{\frac{2\pi}{m}} T^{\frac{3}{2}}}{\frac{8\pi}{\sqrt{mk}} T + 2g \sqrt{\frac{2\pi}{m}} T^{\frac{1}{2}}} = E \tag{30}$$

an exact result. An approximate solution $T(E)$, under the hypothesis of a small gap g is:

$$T = E + \frac{g}{8} \sqrt{\frac{2k}{\pi}} E^{\frac{1}{2}} \tag{31}$$

that is the desired temperature-energy relationship.

Let us consider polynomial nonlinearities:

$$m \ddot{x} + k_1 x + \varepsilon \sum_{i=2}^{P} k_i x^i = 0 \tag{32}$$

where the nonlinear restoring force is supposed to be controlled by the small ε parameter.
The elastic potential U is made by a linear and nonlinear contribution:

$$U = U_l(x) + U_{nl}(x) = \frac{1}{2} k_1 x^2 + \varepsilon \sum_{i=2}^{P} \frac{k_i x^{i+1}}{i+1} \tag{33}$$

Thus, the total energy of the oscillator, sum of the kinetic and the potential energies, is:

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k_1 x^2 + U_{nl}(x) \tag{34}$$

that represents the equal energy surface Σ, a closed curve belonging to the phase plane $x, \dot{x}$, whose implicit equation is given by the previous relationship. An explicit form can be determined by:

$$\dot{x}(x) = \sqrt{\frac{2}{m}} \left[E - \frac{1}{2} k_1 x^2 - U_{nl}(x) \right]^{\frac{1}{2}} \tag{35}$$

To simplify the calculations, limit our attention to a cubic hardening stiffness (the Duffing oscillator):

$$U_{nl}(x) = \varepsilon k_3 \frac{x^4}{4} \tag{36}$$

so that the curve represented by

$$\dot{x}(x,\varepsilon) = \sqrt{\frac{2}{m}} \left[E - \frac{1}{2}k_1 x^2 - \varepsilon\, k_3 \frac{x^4}{4} \right]^{\frac{1}{2}} \tag{37}$$

is symmetric with respect to both the $\dot{x}$ and x axes, and the area V_Σ wrapped by this curve can be calculated as:

$$V_\Sigma = 4 \int_0^{x_M(\varepsilon)} \dot{x}(x,\varepsilon)\,dx \tag{38}$$

x_M is the abscissa of intersection between Σ and the x axis. Note that both the integrand as well as x_M depend on ε. For a small ε, V_Σ can be written, after some perturbation analysis, in the simplified form:

$$V_\Sigma = \pi \sqrt{\frac{1}{mk_1}} \left(2E - \frac{3}{4}\frac{k_3}{(k_1)^{3/2}} E^2 \varepsilon \right) \tag{39}$$

Therefore, the generating and structure function and equation (19) take, respectively, the forms:

$$\Omega(E) = \pi\sqrt{\frac{1}{mk_1}} \left(2 - \frac{3}{2}\frac{k_3}{(k_1)^{3/2}} E\varepsilon \right), \quad \Phi(s) = \pi\sqrt{\frac{1}{mk_1}} \left(\frac{2}{s} - \frac{3}{2}\frac{k_3}{(k_1)^{3/2}} \frac{1}{s^2}\varepsilon \right), \quad \frac{T - \frac{3}{2}\frac{k_3}{(k_1)^{3/2}} T^2 \varepsilon}{1 - \frac{3}{4}\frac{k_3}{(k_1)^{3/2}} T \varepsilon} = E \tag{40}$$

Form the last expression, let us look for an approximate solution $T(E)$ for a small ε:

$$f(E) = \frac{3}{4}\frac{k_3}{(k_1)^{3/2}} E^2 \quad \rightarrow \quad T = E + \varepsilon \frac{3}{4}\frac{k_3}{(k_1)^{3/2}} E^2 \tag{41}$$

providing the temperature-energy relationship.

It can be shown, with the same order of linear ε approximation used in the previous calculation, that if the potential energy includes a complete polynomial of order $2P$, the volume V_Σ depends only on the terms related to the even powers of the potential energy, while those related to the odd ones contribute only at higher ε orders. Therefore, as a general result, the previous expression of the potential energy remains associated to a temperature-energy relationship of the form $T = \sum_{i=1}^{P} c_i\, E^i$, where the c's coefficients are easily calculated following a procedure identical to that previously described.

The analysis of multi-degrees-of-freedom systems does not differ conceptually from the study of the simpler nonlinear single-dof resonator illustrated in the previous section.

Let us consider the entire procedure as an example of application to a two-dof system. It consists of a simple spring-mass chain, where the two masses have the same value m and the three equal linear springs of constant k_1 connect the two masses between them and to a fixed frame. The nonlinear element is introduced between one of the masses and the fixed frame and has a cubic elastic characteristic, i.e. the restoring force is $\varepsilon k_3 x_1^3$. The eigenvectors of the linearized system are $\Psi^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ and $\Psi^{(2)} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$ while the associated eigenvalues are $\lambda_1 = \dfrac{m}{k_1}$ and $\lambda_2 = \dfrac{3m}{k_1}$ respectively. Obviously it is:

$$U_{nl}(x_1) = \varepsilon \int k_3 x_1^3 = \varepsilon \frac{k_3}{4} x_1^4 \tag{42}$$

that, by the modal co-ordinates transformation:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \tag{43}$$

produces the anharmonic potential component:

$$U_{nl}(q_1,q_2) = \varepsilon \frac{k_3}{4m}(q_1 - q_2)^4 \tag{44}$$

Therefore, the equation of the equal energy surface is:

$$\frac{1}{2}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2 + \frac{1}{2}\lambda_1 q_1^2 + \frac{1}{2}\lambda_2 q_2^2 + \varepsilon \frac{k_3}{4m}(q_1 - q_2)^4 - E = 0 \tag{45}$$

and the lengths of its four semi-axes are:

$$\dot{q}_{M1} = \sqrt{2E} \qquad \dot{q}_{M2} = \sqrt{2E}$$

$$q_{M1} = \left(\frac{2E}{\lambda_1}\right)^{\frac{1}{2}} - \varepsilon \frac{k_3}{4m\lambda_1}\left(\frac{2E}{\lambda_1}\right)^{\frac{3}{2}} \qquad q_{M2} = \left(\frac{2E}{\lambda_2}\right)^{\frac{1}{2}} - \varepsilon \frac{k_3}{4m\lambda_2}\left(\frac{2E}{\lambda_2}\right)^{\frac{3}{2}} \tag{46}$$

Therefore, the usual quantities are determined:

$$V_\Sigma \approx 2A\left[\frac{2E^2}{\sqrt{\lambda_1\lambda_2}} - \varepsilon \frac{k_3 E^3}{m\lambda_1\lambda_2}\left(\frac{\lambda_1}{\sqrt{\lambda_1\lambda_2^3}} + \frac{\lambda_2}{\sqrt{\lambda_2\lambda_1^3}}\right)\right]$$

$$\Omega(E) = A\frac{8}{\sqrt{\lambda_1\lambda_2}}\left[E - \varepsilon \frac{3k_3 E^2}{4m}\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right)\right], \quad \Phi(s) = A\frac{8}{\sqrt{\lambda_1\lambda_2}}\left[\frac{1}{s^2} + \varepsilon \frac{6k_3}{4ms^3}\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right)\right] \tag{47}$$

$$\frac{2T + \varepsilon \dfrac{9k_3}{2m}T^2\left(\dfrac{1}{\lambda_1^2} + \dfrac{1}{\lambda_2^2}\right)}{1 + \varepsilon \dfrac{3k_3}{2m}T\left(\dfrac{1}{\lambda_1^2} + \dfrac{1}{\lambda_2^2}\right)} = E$$

The thermodynamic temperature in terms of the energy E, is obtained by the inverse of the previous relationship. Looking for an approximate solution $T(E)$ for a small ε:

$$T = \frac{E}{2} - \varepsilon \frac{15}{2}\frac{k_3}{m}\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right)\left(\frac{E}{2}\right)^2 \tag{48}$$

7   Conclusions

The entropy thermodynamic approach here presented, is a powerful tool and permits to include nonlinearities in the context of an energy-vibration approach. The class of nonlinearities here examined are rather general, since polynomial elastic restoring forces can represents a wide category of systems. Moreover, the case of switch-on switch-off nonlinearity shows the analysis is not limited to the case of small nonlinearities but also strong nonlinear effects can be considered in principle without difficulties.

The possibility outlined in this paper of using entropy as the key to formulate a system-theoretic approach to energy flow among vibrating structures, seems to have the chance of adding theoretical robustness to energy flow methods in vibration and offers a potential important chance of generalization.

The authors would like to suggest a possible strategy for a practical use of the presented findings in the context of SEA. In fact, the large number of investigations made in the context of Statistical Energy Analysis of linear systems, suggest a simple proportionality between power flow and modal energy difference is determined, that is:

$$\dot{W} = \eta \left(\frac{E_2}{N_2} - \frac{E_1}{N_1} \right) \tag{49}$$

Note that the determination of the proportionality coefficient η (named coupling loss factor in SEA) is a very difficult problem, to which a large effort has been dedicated in the scientific literature, and a large amount of data are today available for a large class of structures and related couplings, and dedicated software solve in an excellent way this point. Note that, on the basis of equation (19), since the thermodynamic temperature for linear systems is demonstrated to be $T=E/N$, we can write:

$$\dot{W} = \eta (T_2 - T_1) \tag{50}$$

One can conjecture that this relationship is a linear approximation valid close to equilibrium conditions, and that it is still valid in the presence of nonlinearities, except that in this last case the temperatures and the energies are not anymore proportional. In fact, the present approach leads to the nonlinear dependencies $T_1(E_1)$ and $T_2(E_2)$, and the procedure to calculate them and examples are given. Therefore, a nonlinear Statistical Energy Analysis, can be based on the use of equations [34]:

$$\begin{cases} \dot{W} = \eta \left[T_2(E_2) - T_1(E_1) \right] \\ \\ \dfrac{\Phi_1'(1/T_1)}{\Phi_1(1/T_1)} = -E_1 \\ \\ \dfrac{\Phi_2'(1/T_2)}{\Phi_2(1/T_2)} = -E_2 \end{cases} \tag{51}$$

where the Φ's are the generating functions of the two systems determined as illustrated in the present paper, while the coupling loss factor η would remain as it is determined in the context of SEA literature for the two considered systems.

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