Information-Based Jumps, Asymmetry and Dependence in Financial Modelling

by

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Abstract

In mathematical finance, economies are often presented with the specification of a probability space equipped with a filtration that encodes information flow. The information-based framework of Brody, Hughston and Macrina (BHM) emphasises the role of market information in deriving asset price dynamics, instead of assuming price behaviour from the start. We extend the BHM framework by (i) modelling the nature of access to information through information blockages and activations of new information sources, and (ii) introducing a new class of multivariate Markov processes that we call Generalised Liouville Processes (GLPs) which can model the flow of information about vectors of assets. The analysis of access to information allows us to derive price dynamics with jumps. It additionally enables us to develop an information-switching framework, and price derivatives under regime-switching economies. We also indicate some geometrical aspects of appearances of new information sources. We represent information jumps on the unit sphere in the Hilbert space of square-integrable functions, and on hyperbolic spaces. We use differential geometry, information theory and what we call n-order piecewise enlargements of filtrations to dynamically quantify the impact of sudden changes in the sources of information. This helps us to model the stochastic evolution of what may be viewed as information asymmetry. In related work, we construct GLPs on finite time horizons by splitting so-called Lévy random bridges into non-overlapping subprocesses. The terminal values of GLPs have generalised multivariate Liouville distributions, and GLPs can model a wide spectrum of information-driven dependence structures between assets. The law of an n-dimensional GLP under an equivalent measure is that of an n-vector of independent Lévy processes. We focus on a special type of GLPs that we call Archimedean Survival Processes (ASPs). The terminal value of an ASP has an ℓ_1-norm symmetric distribution, and hence, an Archimedean survival copula.

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The work presented in this thesis is my own.

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Chapter 1

Introduction and Summary

Mathematicians and economists face a non-trivial problem when developing a realistic asset-pricing model, since one can write a long list of interacting features that play a part in the formation of prices. A desirable framework would be flexible enough to represent a wide range of financial behaviour, and would also be able to deliver meaningful and interpretable results in developing our understanding of finance. Satisfying these requirements is already an ambitious challenge. Hence, it is not suprising that the analysis usually starts in a relatively simpler framework, which is gradually elaborated.

Considering information as a mathematical concept advanced many important physical applications in various scientific areas including electronic engineering, computer science and quantum mechanics. In probability theory, a rigorous analysis of a stochastic model relies heavily on the treatment of information. Thus, it is perhaps not surprising to see why information plays such a significant role in mathematical finance, where financial markets are often presented with the specification of a probability space equipped with a filtration that encodes the revelation of information.

In the asset-pricing literature, many stochastic models have been proposed for price processes, and these prices are usually adapted to some filtration. As a standard example, in the Black-Scholes-Merton theory, a great deal of analytic tractability is attained by choosing the underlying asset price to follow a geometric Brownian motion adapted to a Brownian filtration. However, when one assumes price behaviour from the start, one may lose the interpretation of how market information affects price dynamics. In particular, new information that the market has about an asset causes asset prices to change. Therefore, reversing this approach by first specifying the market information, and modelling the flow of information as a driver of price movements presents itself as potentially fruitful in the quest to understand asset price behaviour.

There may also exist small traders who are relatively more informed than the market, and who may exploit their additional information for profit. This scenerio presents the question as to how to model and quantify information asymmetry.

The objective of this thesis is to provide an information-driven framework which (i) admits the derivation of a rich class of asset price dynamics, (ii) allows dynamic representations and quantifications of information asymmetries, and (iii) enables the modelling of a broad range of dependence structures between assets. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ equipped with a filtration $\{\mathcal{F}_t\}_{0\leq t\leq\infty}$, where $\mathcal{F}_\infty = \mathcal{F}$ and $\mathbb{Q}$ is the pricing measure. We assume the existence of a pricing kernel and the absence of arbitrage to ensure the existence of a pricing measure (see, for example, Cochrane, 2005). We consider an asset that pays a random cash flow X_T at a predetermined time $T < \infty$. The cash flow X_T can be expressed as a function of a collection of independent market factors, say M_T^α for $\alpha = 1, 2, \ldots$. For instance, $X_T = g(M_T^1, M_T^2, \ldots, M_T^m)$, where $g : \mathbb{R}^m \to \mathbb{R}$ is a suitably chosen function. The pricing models we discuss in this work can easily be extended to the case where there are multiple cash flows at different times. An example is provided in Chapter 4. We assume the existence of an information process $\{\xi_t\}$ that provides noisy information about the value of X_T and that generates the market filtration $\{\mathcal{F}_t^\xi\}$, where $\mathcal{F}_t^\xi \subset \mathcal{F}_t$. We shall model this information process explicitly. Then, we define the asset price as the expectation of the discounted cash flow, conditional on the market filtration. More specifically, denoting P_{tT} as the deterministic discount factor, the price at time $t < T$ is given by $X_t = P_{tT}\mathbb{E}^{\mathbb{Q}}[X_T|\mathcal{F}_t^\xi]$. This work is organised as follows:

Chapter 2 gives a brief introduction to the information-based asset pricing framework of Brody, Hughston and Macrina (BHM), where an asset is defined by its cash-flow structure. First, we discuss the so-called Brownian information process, which consists of a signal component plus an independent Brownian bridge noise component. The signal component is the cash flow X_T, and the Brownian bridge spans the time interval $[0, T]$. Such an additive construction of the information process is natural from the standpoint of filtering theory, and the bridge property of the noise process ensures the revelation of the value of the cash flow X_T at time T. The cash flow can be represented as a function of various independent market factors, each associated with a Brownian information process that generates the market filtration. We provide the stochastic differential equations of asset price processes, and also provide the value of a European option, which is of the Black-Scholes-Merton type. In the context of aggregate claims (which may arise in insurance problems), where the cash flow is determined by the terminal value of a cumulative process, we briefly discuss what one may call a gamma information process. Such a process consists of a signal component (i.e. the cash flow) multiplied by an independent gamma bridge noise component that spans $[0, T]$. We leave many relevant results on gamma information processes (or what we also call gamma random bridges) to Chapter 7, when constructing Archimedean survival processes.

In Chapter 3, we model new information sources appearing in the market by the activation of additional information processes that generate the market filtration. The market filtration is generated by Brownian information processes that carry partial information about the cash flow X_T, and we represent the availability of new sources of information at independent stopping times. More specifically, we partition σ-algebras into subalgebras with respect to their time dimension, and initiate the subalgebras at stopping times independent of the information processes. The market filtration is then defined in terms of a σ-algebra that contains all the collections of these subalgebras. In this respect, the stopping times may be viewed as what one may call 'measurable start-up times'. We prove a strong Markov property of Brownian information processes and analyze the impact of availability of new information sources on conditional expectations of X_T. We show that the appearance of new sources of information induces jumps in the conditional probability density process (given that X_T is a continuous random variable with a density), and thus the price process. We provide the stochastic differential equations of the conditional probability density process and the price process. The conditional probability density process (and the price process) is càdlàg, and hence its paths are elements of a Skorokhod space. There exists a random jump measure naturally associated with the conditional probability density process. We show that the price process has jump-diffusion dynamics, and the jump sizes are determined by the difference of two dependent exponential Brownian motions with stochastic volatilities. It is a direct outcome of our framework that the price process has stochastic volatility with jumps, since the volatility is a function of the number of information processes provided to the market. In the Hilbert space of square-integrable functions, denoted by $\mathcal{L}^2$, we project the square-root of conditional probability densities onto orthogonal subspaces. The impact of appearances of new information sources can then be measured geometrically on the positive orthant of the unit sphere $\mathcal{S}^+ \subset \mathcal{L}^2$. More precisely, information jumps can be characterised by the spherical distance (or the Bhattacharyya angle) between the Fourier coefficients of the square-root of the conditional probability density. We provide a generalisation of the setting for the case when X_T can be expressed as a function of independent market factors. We associate different sequences of stopping times to different market factors. Hence, the number of sources of information about each market factor may be different at any time. This results in an elaborate expression for the price processes represented in terms of Kronecker products, Hadamard products and matrix norms. Finally, we provide an alternative way of modelling the availability of new sources of information at stopping times. In doing so, we start with a larger filtration and project it to a smaller one that we assume to be the market filtration. The market filtration is generated by information processes that become *alive* starting from the stopping times.

In Chapter 4, we develop an information-based regime-switching framework. Our primary interpretation is that regime switches coincide with price jumps caused by entries of new information sources to the market. One can then argue that, in between jumps, each volatility process (of the price process) belongs to a different regime. This is a common view in the regime-switching literature. We value European options while admitting activations of new sources of information. The option value is the weighted sum of different option values induced by different number of information sources, where the nth weight equals the probability of n information processes driving the market at maturity. We use a sequence of measure changes to value European options. In a special case, we can obtain an option value very similar to that of Merton (1976). In addition, letting $X_T \in \{0, 1\}$, we value credit-risky bonds and credit default swaps. The values of European options and credit-based products may be interpreted as the values under regime-switching economies. Since it is still a rather restrictive viewpoint to expect a jump in the price process at every regime switch, we develop a more elaborate framework, where we view regime switches as changes in the sources of market information. By changes, we do not neccesarily mean appearances of new information sources, it may as well be that the information provided to the market stops flowing. Thus, we also model a scenerio when information ceases to flow, by stopping information processes at stopping times. By starting and stopping information processes, we construct deactivation-reactivation dynamics for price processes. This leads to scenerios where conditional expectations of cash flows may stick to a value for a random period of time, which may arise in illiquid markets. We generalise the setting to the multiple market factor case, where the source of information associated with a market factor may be switched on or switched off at a given time. This allows the possibility to have random numbers and allocations of active and inactive information processes in the market, where each stopping time does not neccesarily induce a jump in the price dynamics. Since the Brownian information process is strong Markov with respect to the given filtration, the price process is determined by the last observations of the switched off information processes, and the new information coming from the switched on information processes. If the total number of information sources is k, then there are 2^k possible economic states at a given time. Finally, as a special example, we construct a σ-algebra where each stopping time induces a switch from one source of information to another. That is, while each stopping time stops an information flow, it simultaneously acts as a start-up time of another information source. It follows that the price process jumps at each information switch. This example provides an alternative view on regime switches that coincide with price jumps.

Chapter 5 focuses on addressing the following question: How can one dynamically quantify the impact of changes in the source of information about a cash flow X_T? The motivation arises from the wish to measure the informational advantage of a small trader who is more informed than the market. A similar approach is presented in Brody *et al.* (2009), where there is an informed trader who has access to extra information from time $t = 0$. The value of the excess information is measured in terms of the difference between the mutual information of the market and the informed trader. Chapter 5 may be seen as a generalisation of this scenerio with an alternative information-theoretic perspective. We construct what we call information asymmetry processes on $[0, T]$ by using information-theoretic measures and enlargements of filtrations. We assume that the filtration of an informed trader is what we call an n-order piecewise enlargement of the market filtration. We focus on a specific case where a small trader may receive additional sources of information at stopping times, and where the market filtration is generated by a single Brownian information process. Using f-divergences and piecewise enlargements of filtrations, we generate what we call the Kullback-Leibler (KL) and the Squared-Hellinger (SH) asymmetry processes. The KL and the SH asymmetry processes are jump-diffusion processes taking the value zero at time T, and the jumps occur when the informed trader receives a new source of information. Thus, each jump quantifies the impact of a change in information sources. We also build a competitive setting involving two informed traders who can not see each others' actions, and whose filtrations are different piecewise enlargements of the same market filtration. We focus on a scenerio where the informed traders receive additional information at different stopping times. This way, we are able to quantify the competitive advantage of an informed trader with respect to another, using the language of information theory. We model financial mispricing as a class of information asymmetry, and construct what we call mispricing processes. We initially let an economy receive incorrect information about a future cash flow as opposed to correct fundamental information. Therefore, the mispricing process represents the dynamic evolution of the informational asymmetry between the market and the fundamentals. The mispricing process jumps to zero if the market instantaneously receives the fundamental information flow, which represents a sudden market correction. This chapter also provides the stochastic differential equation of a Shannon entropy process defined in terms of an n-order piecewise enlargement of a filtration. In this particular example, we show that the Shannon entropy process is a supermartingale.

In Chapter 6, we address the same question to that of Chapter 5, but with a slight modification: Can one dynamically quantify the impact of changes in the source of information using geometry? The motivation partly arises from the fact that the SH asymmetry process can be defined in terms of Bhattacharyya angles on the unit sphere $\mathcal{S}^+ \subset \mathcal{L}^2$ between the square-roots of conditional probability densities. Thus, in a way, this angle provides a geometric perspective on information asymmetry. It follows that the Bhattacharyya angle process is the inverse cosine of a jump-diffusion process and it takes the value zero at time T. To take matters further, we assume that X_T is a Gaussian random variable. We parameterise the conditional probability distributions to form a parametric class of Gaussian distributions, in which the parameters (the mean and the variance) are functions of Brownian information processes. This induces a natural geometry on a Riemannian manifold of which the points are determined by Gaussian distributions. More specifically, the manifold is a hyperbolic space that we denote by $\mathcal{P}$, which is endowed with the Fisher metric tensor. It follows that for each fixed time $t < T$, a Brownian information process determines a point on $\mathcal{P}$. We include the boundary of $\mathcal{P}$, which we denote by $\partial\mathcal{P}$, using Dirac measures as limits of Gaussian distributions, and define a manifold with boundary: $\mathcal{M} = \mathcal{P} \bigcup \partial\mathcal{P}$. Then, we are able to construct what we call the Fisher-Rao (FR) asymmetry process on $[0, T]$, using points on $\mathcal{M}$ that are determined by different numbers of information sources. The FR asymmetry between points on the boundary $\partial\mathcal{P}$ takes the value zero at time T, and the FR asymmetry process for $t < T$ jumps when a new information source appears. In addition, at points on $\mathcal{P}$, infinitesimally close to each other, both the KL and the SH asymmetries coincide with the FR asymmetry. The jumps of the SH and the FR asymmetry processes induce spherical triangles and hyperbolic triangles on $\mathcal{S}^+$ and $\mathcal{P}$, respectively. The triangular surfaces allow us to measure the jump sizes of conditional probability densities using angles between geodesics and the curvatures of the underlying manifolds. These surfaces provide alternative ways of quantifying the impact of the activation of new information sources. Also in a way, these surfaces enable us to view information asymmetry as a geometric shape instead of just a quantity. We introduce a mathematical analogy between the SH asymmetry and an isometric invariant of the Poincaré disc under the action of the general Möbius group. The analogy encourages us to propose the use of the isometric invariant as an alternative way of measuring information asymmetry in the Gaussian setting. The isometric invariant is zero if there is no information asymmetry, and is strictly positive otherwise. In addition, similar to Chapter 5, we create a competitive environment between two informed traders and quantify the competitive advantage with respect to each other geometrically. We also model financial mispricing as a type of information asymmetry, and since the SH and the FR asymmetry processes provide geodesic distances on $\mathcal{S}^+$ and $\mathcal{P}$, respectively, they offer a geometric perspective on quantifying sudden market corrections.

In Chapter 7, we introduce a class of multivariate processes that we name Archimedean Survival Processes (ASPs) and we present some of their properties. An ASP is defined over a finite time horizon and its terminal value has an ℓ_1-norm symmetric distribution and an Archimedean survival copula. Indeed, there is a bijection between the class of Archimedean copulas and the class of ASPs. We construct ASPs by splitting so-called gamma random bridges (a gamma random bridge is the product of a gamma bridge with an independent positive random variable) into non-overlapping pieces. The one-dimensional marginal processes of an ASP are gamma random bridges. These marginal processes are increasing and, in general, not independent, but they are identical in law. ASPs are Markov processes and their increments have multivariate Liouville distributions. The ℓ_1-norm of an ASP is a one-dimensional gamma random bridge. We also provide the first and second order moments of ASPs. The law of an n-dimensional ASP is equivalent to that of a vector of n independent gamma processes, and we provide details of the associated change of measure. The law of an n-dimensional ASP is identical to the law of a positive random variable multiplied by the Hadamard product of an n-dimensional Dirichlet random variable and an n-vector of independent gamma bridges. An ASP may be viewed as a multivariate information process, where each marginal process carries partial information about an aggregate claim determined by the terminal value of a cumulative gains process. Therefore, ASPs can model a rich class of dependence structures between cash flows by the use of Archimedean copulas. One of the attractive features of copulas is that they allow the fitting of one-dimensional marginal distributions to be performed separately from the fitting of cross-sectional dependence. Archimedean copulas have received particular attention in the literature for both their analytical tractability and practical convenience. ASPs present an avenue to extend the theory and application of Archimedean copulas in multi-period and continuous-time frameworks. The results presented in this chapter can also be found in Hoyle and Mengütürk (2012). The material in this chapter and in Hoyle and Mengütürk (2012) are based on the collaborative work with Ed Hoyle. The two authors contributed equally in this effort.

Chapter 8 introduces a family of multivariate Markov processes that we call Generalised Liouville Processes (GLPs). GLPs are defined over a finite time horizon, and their terminal values and increments have generalised multivariate Liouville distributions. We construct GLPs by splitting so-called Lévy random bridges into non-overlapping pieces and by employing deterministic time changes. Lévy random bridges are introduced in Hoyle *et al.* (2011) to model the flow of information as an extension to the BHM framework. A Lévy random bridge (or a Lévy information process) is identical in law to a Lévy process conditioned to have a fixed marginal law (say, the a priori law of the future cash flow) at a finite future time. The one-dimensional marginal processes of GLPs are Lévy random bridges. Hence, GLPs may be viewed as multivariate information processes, where each marginal is a Lévy information for a cash flow. The sum of marginals of GLPs are one-dimensional Lévy random bridges, and the law of an n-dimensional GLP under an equivalent measure is that of a vector of n independent Lévy processes. GLPs generalise ASPs and allow us to model a wide spectrum of dependence structures between cash flows that have a generalised multivariate Liouville distribution. From an information-based viewpoint, the law of a GLP determines the distribution of the prices of a vector of assets.

Chapter 2

Information-Based Framework

We provide a general account of the information-based framework of Brody, Hughston and Macrina (BHM), developed in Macrina (2006) and Brody *et al.* (2008a). The BHM framework is applied to credit risk modelling in Brody *et al.* (2007), Rutkowski and Yu (2007) and Brody *et al.* (2010), to interest rate theory in Hughston and Macrina (2008), to insurance problems in Brody *et al.* (2008b), and to insider trading in Brody *et al.* (2009). The framework is extended in Hoyle *et al.* (2011) with the introducion of a larger class of information processes called Lévy random bridges. The BHM framework is closely related to other partial information approaches in the literature such as Giesecke (1994), Duffie and Lando (2001), and Jarrow and Protter (2004).

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{0\leq t\leq\infty}$, where $\mathcal{F}_\infty = \mathcal{F}$. The probability measure $\mathbb{Q}$ is the pricing measure. We assume that all filtrations under consideration are right-continuous and complete, and we fix a finite time horizon $[0, T]$. We let $X_T \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{Q})$ be an $\mathcal{F}_0$-measurable square-integrable continuous random variable with state-space $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, and continuous density $q(x) > 0$ for $x \in \mathbb{X}$. Here, $\mathcal{L}^2$ is the Hilbert space of square-integrable functions and $\mathcal{B}(\mathbb{X})$ is the Borel σ-field (it is straightforward to rewrite the following results if X_T is a discrete random variable). We shall be using X_T to model a cash flow at time T, and we assume $\mathbb{X} \subset \mathbb{R}$. One may generalise the topological conditions on $\mathbb{X}$ such that it is a complete separable metric space. Since we are working in a financial context, $\mathbb{X} \subset \mathbb{R}$ is a natural choice. We postulate the existence of an $\mathcal{F}_t$-adapted càdlàg process $\{\xi_t\}_{t\in[0,T]}$, which generates the filtration $\{\mathcal{F}^\xi_t\}$, i.e.,

$$\mathcal{F}^\xi_t = \sigma(\{\xi_s\}_{0\leq s\leq t}), \tag{2.1}$$

for $0 \leq t \leq T$. We assume $\{\mathcal{F}^\xi_t\}$ is the market filtration where the process $\{\xi_t\}$ provides noisy information about the cash flow X_T. In other words, the σ-algebra $\mathcal{F}^\xi_t \subset \mathcal{F}_t$ is all the information that the market has about X_T at time t.

Brody *et al.* (2008a) model $\{\xi_t\}$ through an additive construction, in particular, with a signal component (i.e., X_T) plus an independent Brownian bridge noise component. Such an additive construction of the information process is natural from the standpoint of filtering theory (see, for example, Davis and Marcus, 1981, and Krishnan, 2005). More specifically, the market information process $\{\xi_t\}$ is

$$\xi_t = \kappa X_T t + B_{tT}, \tag{2.2}$$

where $\{B_{tT}\}_{t\in[0,T]}$ is a Brownian bridge independent of X_T to the value zero, and is not $\mathcal{F}_t^\xi$-adapted. Also, the random variable X_T is $\mathcal{F}_T^\xi$-measurable, but is not $\mathcal{F}_t^\xi$-measurable for $t < T$. Note that the Brownian bridge $\{B_{tT}\}$ can be represented as

$$B_{tT} = B_t - \frac{t}{T} B_T, \tag{2.3}$$

where $\{B_t\}$ is a $\mathbb{Q}$-Brownian motion. The value of X_T is obscured by the noise $\{B_{tT}\}$ for $0 < t < T$, and it is finally revealed without noise at time T. We call $\{\xi_t\}$, defined as in (2.2), a Brownian information process. Setting $\xi_T = \kappa X_T T$ ensures that the marginal law of the Brownian information process at T is the a priori law of $\kappa X_T T$. We assume $\kappa > 0$ is finite and call it the speed coefficient of $\{\xi_t\}$, since it controls the speed at which the true value of X_T is revealed to the market. Brody *et al.* (2007) proves that $\{\xi_t\}$ is a Markov process with respect to $\{\mathcal{F}_t^\xi\}$. In Chapter 3, we shall prove a strong Markov property of $\{\xi_t\}$.

Following BHM, we let the risk-free system of interest rates be deterministic. We denote the corresponding system of discount functions by $\{P_{0t}\}_{0\le t<\infty}$ and assume that P_{0t} is differentiable, strictly decreasing and satisfies $0 < P_{0t} \le 1$ and $\lim_{t\to\infty} P_{0t} = 0$. The no-arbitrage condition implies that $P_{tT} = P_{0T}/P_{0t}$ for $t \le T$. If $\{r_t\}$ is the risk-free rate process such that $r_t > 0$ and $\int_t^\infty r_s \,\mathrm{d}s = \infty$, the discount function P_{tT} is the no-arbitrage price of a zero-coupon risk-free bond (paying unity) with maturity T:

$$P_{tT} = \exp\left(- \int_t^T r_s \,\mathrm{d}s \right). \tag{2.4}$$

Then the price of a cash flow X_T at time t for $0 \le t < T$, which we denote by X_t, is given by the $\mathcal{F}_t^\xi$-conditional expectation of X_T discounted by P_{tT}:

$$X_t = P_{tT} \mathbb{E}^{\mathbb{Q}} \left[X_T \,\middle|\, \mathcal{F}_t^\xi \right], \tag{2.5}$$

where $\mathbb{E}^{\mathbb{Q}}[.]$ denotes the expectation under $\mathbb{Q}$. Note that since $\{\xi_t\}$ generates the information provided to the market, the dynamics of the price process are dependent on the law of $\{\xi_t\}$.

For $x \in \mathbb{X}$, we denote the conditional probability density of X_T at time t by $\psi_t(x)$, i.e.,

$$\psi_t(x) = q(x|\mathcal{F}_t^\xi) = q(x|\xi_t), \tag{2.6}$$

for $0 \leq t < T$. If $B_b(\mathbb{X})$ is the space of bounded $\mathcal{B}(\mathbb{X})$-measurable functions, the following can be written for any $g \in B_b(\mathbb{X})$:

$$\mathbb{E}^{\mathbb{Q}}\left[g(X_T)\,\middle|\,\mathcal{F}_t^{\xi}\right] = \int_{\mathbb{X}} g(x)\psi_t(x)\,\mathrm{d}x. \tag{2.7}$$

Therefore, when we express X_T as a function of $m \in \mathbb{N}_+$ independent market factors, we shall choose a function from $B_b(\mathbb{X}^m)$. Note that the right-hand side of (2.7) is defined for $t \in [0, T)$, excluding time T, whereas the left-hand side is defined including time T. In the following chapters, we shall make use of distributions for calculating expectations including time T. In fact, the reason why we introduce only the densities in this chapter is to demonstrate their use in deriving the stochastic differential equation (SDE) of the price process.

Note that the Brownian bridge $\{B_{tT}\}$ is a Gaussian process with mean zero, and the covariance between B_{uT} and B_{tT} is $u(T-t)/T$ for $u \leq t$. It follows from the Markovian property of $\{\xi_t\}$ and the Bayes formula that

$$\psi_t(x) = \frac{\exp\left[\frac{T}{(T-t)}(\kappa x\xi_t - \frac{1}{2}(\kappa x)^2 t)\right] q(x)}{\int_{\mathbb{X}} \exp\left[\frac{T}{(T-t)}(\kappa x\xi_t - \frac{1}{2}(\kappa x)^2 t)\right] q(x)\,\mathrm{d}x}, \tag{2.8}$$

for $0 \leq t < T$.

The SDE of $\{\psi_t\}_{t\in[0,T)}$ can be calculated by the use of Ito's lemma applied to (2.8). More specifically, it can be shown that the process $\{\psi_t\}$ is governed by

$$\mathrm{d}\psi_t(x) = \sigma_t(x)\psi_t(x)\,\mathrm{d}W_t, \tag{2.9}$$

for $0 \leq t < T$, where the coefficient $\{\sigma_t\}_{t\in[0,T)}$ is defined by

$$\sigma_t(x) = \frac{T\kappa(x - \mathbb{E}^{\mathbb{Q}}[X_T \,|\, \xi_t])}{(T-t)}, \tag{2.10}$$

and where $\{W_t\}_{t\in[0,T)}$ is a $\mathbb{Q}$-Brownian motion given by

$$W_t = \xi_t + \int_0^t \frac{1}{T-s}\xi_s\,\mathrm{d}s - T\kappa\int_0^t \frac{1}{T-s}\mathbb{E}^{\mathbb{Q}}[X_T \,|\, \xi_s]\,\mathrm{d}s. \tag{2.11}$$

It follows from (2.5) and (2.8) that the price X_t, for $0 \leq t < T$, can be expressed as

$$X_t = P_{tT}\frac{\int_{\mathbb{X}} x\exp\left[\frac{T}{(T-t)}(\kappa x\xi_t - \frac{1}{2}(\kappa x)^2 t)\right] q(x)\,\mathrm{d}x}{\int_{\mathbb{X}} \exp\left[\frac{T}{(T-t)}(\kappa x\xi_t - \frac{1}{2}(\kappa x)^2 t)\right] q(x)\,\mathrm{d}x}. \tag{2.12}$$

From (2.2) and (2.4), $P_{TT}\mathbb{E}^{\mathbb{Q}}\left[X_T\,\middle|\,\mathcal{F}_T^{\xi}\right] = X_T$. The price of the asset is X_T at time T.

Making use of (2.9), it can be shown that the dynamics of the price process $\{X_t\}$ are governed by the following SDE:

$$\mathrm{d}X_t = r_t X_t \,\mathrm{d}t + P_{tT} \frac{T\kappa}{T-t} \mathrm{Var}^{\mathbb{Q}} \left[X_T \,|\, \xi_t \right] \mathrm{d}W_t, \tag{2.13}$$

for $0 \leq t < T$, where $\mathrm{Var}^{\mathbb{Q}}[X_T \,|\, \xi_t] = \mathbb{E}^{\mathbb{Q}} \left[X_T^2 \,|\, \xi_t \right] - \mathbb{E}^{\mathbb{Q}} \left[X_T \,|\, \xi_t \right]^2$ is the conditional variance of X_T under $\mathbb{Q}$. We provide a proof of a generalised version of (2.13) in Chapter 3. Note that we do not specify a stochastic model for the price process from the start. The dynamics of the price process are derived by specifying an information process that generates the market filtration.

At time 0, the value of a European call option with strike K, exercisable at time t, is

$$C_0 = P_{0t} \mathbb{E}^{\mathbb{Q}} \left[(X_t - K)^+ \right], \tag{2.14}$$

for $0 \leq t < T$, where $(y)^+ = \max(y, 0)$. Hence, the option is exercisable on the time-t price of an asset that has the cash flow X_T at time T. Brody *et al.* (2008a) show that the value of this option is given by

$$\begin{aligned} C_0 = {} & P_{0t} \int_{\mathbb{X}} x q(x) \mathcal{N} \left(-z_t + \kappa x \sqrt{\frac{tT}{T-t}} \right) \mathrm{d}x \\ & - P_{0t} K \int_{\mathbb{X}} q(x) \mathcal{N} \left(-z_t + \kappa x \sqrt{\frac{tT}{T-t}} \right) \mathrm{d}x, \end{aligned} \tag{2.15}$$

where $\mathcal{N}(.)$ is the standard normal distribution function, and $z_t = \xi^* \sqrt{\frac{T}{t(T-t)}}$, where ξ^* solves the following:

$$\int_{\mathbb{X}} (P_{tT} x - K) \exp \left[\frac{T}{(T-t)} \left(\kappa x \xi^* - \frac{1}{2} (\kappa x)^2 t \right) \right] q(x) \,\mathrm{d}x = 0. \tag{2.16}$$

Note that the information-based setting leads to a Black-Scholes-Merton type European call option price. We provide a proof of a generalised version of (2.15) in Chapter 4.

In the BHM framework, the cash flow X_T is represented as a function of a set of independent random variables, say M_T^α, for $\alpha = 1, \ldots, m$, with continuous densities $q^\alpha(x) > 0$. The random variable M_T^α is called a market factor, and each market factor determines the value of the cash flow X_T. Without loss of generality, we assume that the state-space of each M_T^α is $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. We choose a function $g \in B_b(\mathbb{X}^m)$ such that $g : \mathbb{X}^m \to \mathbb{X}$, and express X_T as follows:

$$X_T = g(M_T^1, \ldots, M_T^m). \tag{2.17}$$

We associate a Brownian information process, that we denote by $\{\xi^{\alpha}_t\}_{t\in[0,T]}$, with each market factor M^{α}_T such that

$$\xi^{\alpha}_t = \kappa^{\alpha} M^{\alpha}_T t + B^{\alpha}_{tT}, \tag{2.18}$$

where $\{B^{\alpha}_{tT}\}_{t\in[0,T]}$, $\alpha = 1,\dots,m$ are mutually independent Brownian bridges to the value zero and which are independent of each M^{α}_T. The market filtration $\{\mathcal{F}^{\xi}_t\}$ is then given by

$$\mathcal{F}^{\xi}_t = \sigma(\{\xi^{\alpha}_s\}_{0\le s\le t} : \alpha = 1,\dots,m), \tag{2.19}$$

for $0 \le t \le T$. For each α, M^{α}_T is $\mathcal{F}^{\xi}_T$-measurable, but not $\mathcal{F}^{\xi}_t$-measurable for $t < T$.

For $x \in \mathbb{X}$, we denote the conditional density of M^{α}_T at time t by $\psi^{\alpha}_t(x)$:

$$\begin{aligned}\psi^{\alpha}_t(x) &= q^{\alpha}(x|\mathcal{F}^{\xi}_t) = q^{\alpha}(x|\xi^{\alpha}_t)\\ &= \frac{\exp\left[\frac{T}{(T-t)}(\kappa^{\alpha}x\xi^{\alpha}_t - \frac{1}{2}(\kappa^{\alpha}x)^2 t)\right] q^{\alpha}(x)}{\int_{\mathbb{X}} \exp\left[\frac{T}{(T-t)}(\kappa^{\alpha}x\xi^{\alpha}_t - \frac{1}{2}(\kappa^{\alpha}x)^2 t)\right] q^{\alpha}(x)\,\mathrm{d}x},\end{aligned} \tag{2.20}$$

for $0 \le t < T$, since $\{\xi^{\alpha}_t\}$, $\alpha = 1,\dots,m$ are Markovian and mutually independent. From (2.7) it follows that the time-t price of X_T is

$$\begin{aligned}X_t &= P_{tT}\mathbb{E}^{\mathbb{Q}}\left[g(M^1_T,\dots,M^m_T)\,\middle|\,\xi^1_t,\dots,\xi^m_t\right]\\ &= P_{tT}\int_{\mathbb{X}^m} g(x^1,\dots,x^m)\psi^1_t(x_1)\cdots\psi^m_t(x_m)\,\mathrm{d}x_1\cdots\mathrm{d}x_m.\end{aligned} \tag{2.21}$$

Then the dynamics of the price process $\{X_t\}$ are governed by the following SDE:

$$\mathrm{d}X_t = r_t X_t\,\mathrm{d}t + P_{tT}\sum_{\alpha=1}^{m}\frac{T\kappa^{\alpha}}{T-t}\mathrm{Cov}^{\mathbb{Q}}\left[X_T, M^{\alpha}_T\,\middle|\,\xi^1_t,\dots,\xi^m_t\right]\,\mathrm{d}W^{\alpha}_t, \tag{2.22}$$

for $0 \le t < T$, where $\{W^{\alpha}_t\}_{t\in[0,T)}$ is a $\mathbb{Q}$-Brownian motion satisfying

$$W^{\alpha}_t = \xi^{\alpha}_t + \int_0^t \frac{1}{T-s}\xi^{\alpha}_s\,\mathrm{d}s - T\kappa^{\alpha}\int_0^t \frac{1}{T-s}\mathbb{E}^{\mathbb{Q}}\left[M^{\alpha}_T\,|\,\xi^{\alpha}_s\right]\,\mathrm{d}s, \tag{2.23}$$

for $\alpha = 1,\dots,m$. $\mathrm{Cov}^{\mathbb{Q}}[X_T, M^{\alpha}_T\,|\,\mathcal{F}^{\xi}_t]$ is the conditional covariance of X_T and M^{α}_T under $\mathbb{Q}$.

In Chapters 3-6, we shall work with Brownian information processes having the functional form as shown in (2.2). In Chapter 7, when we introduce Archimedean survival processes, we make use of what one may call gamma information processes (or what we also call gamma random bridges). Gamma information processes are used within the BHM framework (see Brody *et al.*, 2008b) in the modelling of aggregate claims. Although we discuss these processes in detail in Chapter 7, we shall give a brief overview here of how gamma information processes are used in the BHM framework.

Let $\{\gamma_t\}$ be a gamma process, which is an increasing Lévy process with gamma distributed increments (see for example, Bertoin 1996, and Sato, 1999). If $X_T > 0$ is a cash flow, then

$$\{\xi^*_t\}_{t\in[0,T]} = \{X_T \frac{\gamma_t}{\gamma_T}\}_{t\in[0,T]}, \tag{2.24}$$

is a gamma information process, where $\{\gamma_t/\gamma_T\}$ is a gamma bridge to the value 1 and independent of X_T. We can view X_T as a signal and the gamma bridge as independent multiplicative noise. Brody *et al.* (2008b) argue that such a representation is natural from the standpoint of filtering theory, since many additive properties of the Brownian bridge are analogues to multiplicative properties of the gamma bridge. We refer the reader to Emery and Yor (2004) and Yor (2007) for some of these properties.

Note that if the market filtration is generated by a process of the form (2.24), the value of X_T is revealed without noise at time T. Let $\{\mathcal{F}^{\xi^*}_t\}$ be the market filtration given by

$$\mathcal{F}^{\xi^*}_t = \sigma(\{\xi^*_s\}_{0\le s\le t}), \tag{2.25}$$

for $0 \le t \le T$. Brody *et al.* (2008b) show that $\{\xi^*_t\}$ is a Markov process with respect to $\{\mathcal{F}^{\xi^*}_t\}$. It follows that

$$X_t = P_{tT}\mathbb{E}^{\mathbb{Q}}[X_T \,|\, \mathcal{F}^{\xi^*}_t] = P_{tT}\frac{\int_{\xi^*_t}^{\infty} x^{2-mT}(x-\xi^*_t)^{m(T-t)-1}q(x)\,\mathrm{d}x}{\int_{\xi^*_t}^{\infty} x^{1-mT}(x-\xi^*_t)^{m(T-t)-1}q(x)\,\mathrm{d}x}, \tag{2.26}$$

is the price of an asset with cash flow X_T at time T, provided that $m > 0$ is a parameter of the gamma process $\{\gamma_t\}$.

Hoyle *et al.* (2011) introduce the so-called Lévy random bridges (LRBs) to model the flow of market information using a broader family of stochastic processes. LRBs are Markov processes, and both Brownian information processes and gamma information processes are examples of LRBs. We provide a formal definition of LRBs in Chapter 8 when we introduce Generalised Liouville Processes. Briefly, an LRB is identical in law to a Lévy process conditioned to have a fixed marginal law (say, the a priori law of the future cash flow) at a fixed future time. The time-t price of an asset that pays X_T at time T is calculated by the discounted conditional expectation of X_T with respect to the market filtration generated by an LRB.

Chapter 3

Brownian Information Processes and Jump-Diffusion Dynamics

In this chapter, we model the appearance of new sources of information by the activation of additional information processes that generate the market filtration. In particular, we analyze the access of the market to new sources of information at stopping times. This allows us to investigate how the flow of information may lead to jumps in prices.

Information about an asset influences the behaviour of the price of that asset. In other words, new information that the market has about an asset causes its price to change. Emprically speaking, changes in asset price dynamics occasionally exhibit large jumps, usually as a response to an announcement or a relevant newscast made to the market. Merton (1976) decomposes price changes into two parts: marginal and non-marginal changes. If information about an asset arrives gradually and continuously in time, then over short time intervals, the impact of information on price dynamics is marginal. On the other hand, important news about an asset may arrive infrequently, and new information sources may appear at discrete points in time. Then it is reasonable to expect that the impact of a new broadcast or the revelation of a new source of information is not marginal.

In the BHM framework, one specifies the law of information processes generating some market filtration, and price dynamics are derived. Since the price of an asset is the expectation of its discounted cash flow conditional on the market filtration, one may expect that the price dynamics are continuous if the information processes are continuous, and price dynamics have jumps if the information processes have jumps. Although these statements are true in general, we introduce a way to derive price dynamics with jumps even though the information processes are continuous. We do this by modelling the appearance of new sources of information at stopping times, where we assume that the market filtration is generated by Brownian information processes. Our framework is analytically tractable and provides an alternative perspective on information-based price jumps in an economy.

In order to model the appearance of new sources of information, we first partition σ-algebras into subalgebras (by a subalgebra, we mean a σ-algebra that is contained in another σ-algebra) with respect to time, and admit arbitrary starting times. We postulate the existence of Brownian information processes $\{\xi^i_t\}_{t\in[0,T]}$, $i = 1, \ldots, k$, and admit these information processes to enter the market filtration at stopping times. First, we define

$$\mathcal{F}^{\xi^i}_{u,t} = \begin{cases} \sigma(\{\xi^i_s\}_{u\le s\le t}) & u \le t, \\ \{\Omega, \varnothing\} & u > t, \end{cases} \tag{3.1}$$

for a fixed time u and for $0 \le t \le T$. The set $\{\Omega, \varnothing\}$ is the trivial σ-algebra, and $\mathcal{F}^{\xi^i}_{u,t}$ is a subalgebra of $\mathcal{F}^{\xi^i}_t = \sigma(\{\xi^i_s\})_{0\le s\le t}$, since $\mathcal{F}^{\xi^i}_{u,t} \subseteq \mathcal{F}^{\xi^i}_t$ for all $u \ge 0$. We then employ stopping times to initiate these subalgebras (instead of a fixed deterministic time u), which allows us to represent random appearances of new sources of information in the market.

We shall first define a stopping time. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{Q})$ be a filtered probability space. Then, a random time $\tau : \Omega \to \mathbb{R}_+$ is an $\mathcal{F}_t$-stopping time, if $\{\tau \le t\} \in \mathcal{F}_t$. There are various ways one can model stopping times, and our framework offers the flexibility to consider a broad range of such models. One common example of a stopping time is the first hitting time of a continuous process. For example, let $\{L_t\}_{t\ge0}$ be a continuous process adapted to a filtration $\{\mathcal{F}^L_t\}$ (where $\mathcal{F}^L_t \subset \mathcal{F}_t$), then $\tau_B : \Omega \to \mathbb{R}_+$ defined by

$$\tau_B(\omega) = \inf\{s \ge 0 : L_s(\omega) \in B\}, \tag{3.2}$$

is an $\mathcal{F}^L_t$-hitting time for $\omega \in \Omega$ and $B \in \mathcal{B}$, and hence, it is an $\mathcal{F}^L_t$-stopping time. The random variable τ_B is the first time the process $\{L_t\}$ enters B.

Throughout this chapter, we assume that the stopping times are independent of the information processes. We define the market filtration as the union of the filtrations of subalgebras that are initiated at stopping times. Without loss of generality, we choose to model stopping times as the jump times of Heaviside processes. We would like to note that Heaviside processes are not what we call information processes (especially since they are independent of the cash flows), and they simply serve as processes that indicate when the new sources of information appear in the market. As noted above, one may model the stopping times as the first hitting times of càdlàg processes instead of Heaviside processes. If the càdlàg processes are continuous, then we can define a market filtration that is generated only by processes that are continuous, and still be able to derive price dynamics with jumps. We shall see that this follows since, instead of having jumps in the evolution of information processes, we define the market filtration in such way that the filtration itself 'jumps' at a stopping time, due to a sudden expansion of the σ-algebra.

We shall explicitly show how the appearance of a new source of information induces a jump in the price process. In fact, we shall show that the price process follows jump-diffusion dynamics as a natural result of sudden appearances of Brownian information processes. This is consistent with the jump-diffusion model of Merton (1976, pp. 127) who quotes:

> "*By its very nature, important information arrives only at discrete points in time. This component is modeled by a jump process reflecting the non-marginal impact of the information.*"

This chapter draws particular attention to the time dimension of filtrations. Although we present our results within the BHM framework, the models can be generalised within stochastic filtering theory. Our work may also be regarded as an extension to a stream of literature that assigns emphasis on the time dimension of filtrations (see for example, Jacod and Skorohod, 1994, Jeanblanc and Valchev, 2005, and Guo *et al.*, 2009). Jeanblanc and Valchev (2005) use discrete-time filtrations and model default hazard processes, and Guo *et al.* (2009) introduce delayed filtrations and model credit risk using time changes.

This chapter is organised as follows: Section 1 is the mathematical setting. Section 2 explains our model for the appearances of new information sources at stopping times. Section 3 generalises the setting to the multiple market factor representation of the future cash flow. Section 4 provides an alternative way of modelling the availability of new sources of information at stopping times. Section 5 is price simulations.

3.1 Mathematical Setting

3.1.1 Hilbert Space Setting

Hilbert spaces allow us to measure the non-marginal impact of appearances of new information sources using functional analysis and geometry. The reader may refer to Rudin (1987), and Riesz and Nagy (1990) for more details on Hilbert spaces. We shall provide some notations:

We denote a Hilbert space by $\mathcal{H}$, a normed vector space associated with a metric and endowed with an inner product $\langle\, . \,\rangle$, such that every Cauchy sequence in $\mathcal{H}$ converges in $\mathcal{H}$. If $\mathcal{V}$ is a vector space and F is a field, then the inner product $\langle\, . \,\rangle$ is a mapping $\langle\, . \,\rangle : \mathcal{V} \times \mathcal{V} \to F$ which satisfies linearity, conjugate symmetry, and positive definiteness. We denote the norm of $g \in \mathcal{H}$ by $||g||$.

The elements $g, h \in \mathcal{H}$ are orthogonal if $\langle g, h \rangle = 0$, and if g and h are orthogonal we write $g \perp h$. Also, if M, N are two subspaces of $\mathcal{H}$, and if all elements of M are orthogonal to all elements of N, then M is orthogonal to N, denoted by $M \perp N$. Since $\langle g, h \rangle = 0$ implies $\langle h, g \rangle = 0$, the relation $\perp$ is symmetric. If M is the orthogonal complement of N, then any $g \in \mathcal{H}$ can uniquely be represented as $g = h^{(1)} + h^{(2)}$, where $h^{(1)} \in M$ and $h^{(2)} \in N$.

Similarly, if $\{M_1, M_2, \ldots, M_n\}$ is a collection of mutually orthogonal closed subspaces of $\mathcal{H}$, spanning $\mathcal{H}$, then any $g \in \mathcal{H}$ can uniquely be represented as

$$g = h^{(1)} + \cdots + h^{(n)}, \tag{3.3}$$

where $h^{(i)} \in M_i$. If every element in the vector sum of the mutually orthogonal sets M_i admits a unique representation of the form $h^{(1)} + \cdots + h^{(n)}$, the direct sum, which we denote by $\oplus$, of the M_i's is $\mathcal{H}$, i.e.,

$$\mathcal{H} = \bigoplus_{i=1}^{n} M_i = M_1 \oplus \cdots \oplus M_n. \tag{3.4}$$

The elements $e_m \in \mathcal{H}$ for $m = 1, 2, \ldots$ are orthonormal, if in addition to being orthogonal each satisfies $||e_m|| = 1$. An orthonormal sequence $e_m \in \mathcal{H}$ for $m = 1, 2, \ldots$ is complete if the only member of $\mathcal{H}$ which is orthogonal to all e_m, for $m = 1, 2, \ldots$, is the zero vector.

We state a theorem and a definition from functional analysis that we shall later refer to:

Theorem 3.1.1. *Let e_m for $m = 1, 2, \ldots$ be a complete orthonormal sequence in $\mathcal{H}$. Then for every $g \in \mathcal{H}$,*

$$g = \sum_{m=1}^{\infty} \langle g, e_m \rangle e_m. \tag{3.5}$$

Definition 3.1.2. *The coefficients $\langle g, e_m \rangle$ which appear in (3.5) are called the* Fourier coefficients *of $g \in \mathcal{H}$.*

One of the most important examples of a Hilbert space is the space of square-integrable functions defined on some measurable set. A square-integrable function g on B satisfies:

$$\int_B |g(t)|^2 \mu(\mathrm{d}t) < \infty, \tag{3.6}$$

where μ is the Lebesgue measure. We denote this space by $\mathcal{L}^2(B)$. The space $\mathcal{L}^2(B)$ is the collection of Borel-measurable, square-integrable functions g on B, with the inner product

$$\langle g, h \rangle = \int_B g(t)h(t)\mu(\mathrm{d}t), \tag{3.7}$$

for some $h \in \mathcal{L}^2(B)$, and the norm

$$\|g - h\| = \left(\int_B |g(t) - h(t)|^2 \mu(\mathrm{d}t) \right)^{\frac{1}{2}}. \tag{3.8}$$

The integrals shown in (3.6)-(3.8) are Lebesgue integrals.

3.1.2 Information-Based Setting

We let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{0\leq t\leq\infty}$, where $\mathcal{F}_\infty = \mathcal{F}$. We assume that all filtrations under consideration are right-continuous and complete, and we fix a time horizon $[0, T]$, where $T < \infty$. We set $\mathbb{Q}$ to be the pricing measure. We let $X_T \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{Q})$ be an $\mathcal{F}_0$-measurable square-integrable cash flow at time T. That is, X_T is a continuous random variable with state-space $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, $\mathbb{X} \subset \mathbb{R}$, and with continuous density $q(x) > 0$ for $x \in \mathbb{X}$.

For $k \in \mathbb{N}_+$, we postulate the existence of k $\{\mathcal{F}_t\}$-adapted càdlàg processes that we denote by $\{\xi^i_t\}_{t\in[0,T]}$, $i = 1, \ldots, k$. In addition, we let $\{\mathcal{F}^{\xi^i}_t\}$ be the filtration of a subalgebra $\mathcal{F}^{\xi^i}_t \subset \mathcal{F}_t$ for $i = 1, \ldots, k$, i.e.,

$$\mathcal{F}^{\xi^i}_t = \sigma(\{\xi^i_s\}_{0\leq s\leq t}), \tag{3.9}$$

for $0 \leq t \leq T$. Throughout this chapter, $\{\xi^i_t\}$ is a Brownian information process for $i = 1, \ldots, k$:

$$\xi^i_t = \kappa^i X_T t + B^i_{tT}, \tag{3.10}$$

where $\{B^i_{tT}\}_{t\in[0,T]}$ is a Brownian bridge independent of X_T that takes the value 0 at time T, and is not $\mathcal{F}^{\xi^i}_t$-adapted. The $\{B^i_{tT}\}$'s may be correlated and we assume that the speed coefficients κ^i's are positive and finite.

We denote by $\mathcal{Q}(\mathbb{X})$ the space of probability measures over $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Since $\mathbb{X} \subset \mathbb{R}$ is a complete seperable metric space, using Theorem 2.1 in Bain and Crisan (2009), we can proceed by defining a $\mathcal{Q}(\mathbb{X})$-valued, $\mathcal{F}^{\xi^i}_t$-adapted stochastic process $\{\pi^i_t\}_{t\in[0,T]}$ as follows:

$$\pi^i_t(\omega)(A) = \mathbb{Q}(X_T \in A | \mathcal{F}^{\xi^i}_t)(\omega), \tag{3.11}$$

for $\omega \in \Omega$, $i = 1, \ldots, k$ and $A \in \mathcal{B}(\mathbb{X})$. For the rest of this work, we fix some $\omega \in \Omega$ outside a $\mathbb{Q}$-null set, and drop it from the expressions. Also, we fix $A \in \mathcal{B}(\mathbb{X})$. π^i_t is a conditional distribution (or a random probability measure) of X_T with respect to $\mathcal{F}^{\xi^i}_t$. For pairwise disjoint sets $A_m \in \mathcal{B}(\mathbb{X})$ for $m \geq 1$, π^i_t satisfies the σ-additivity condition:

$$\pi^i_t\left(\bigcup_m A_m\right) = \sum_m \pi^i_t(A_m). \tag{3.12}$$

For any $g \in B_b(\mathbb{X})$, we can write

$$\mathbb{E}^{\mathbb{Q}}\left[g(X_T) \,\middle|\, \mathcal{F}^{\xi^i}_t\right] = \int_{\mathbb{X}} g(x)\pi^i_t(\mathrm{d}x). \tag{3.13}$$

Until Section 3.3, we consider $g(x) = x$. Then, we express X_T as a function of $m \in \mathbb{N}_+$ independent market factors, and generalise to other $g \in B_b(\mathbb{X}^m)$.

For $x \in \mathbb{X}$, we denote

$$\psi_t^i(x)\,\mathrm{d}x = q(x|\mathcal{F}_t^{\xi^i})\,\mathrm{d}x = \pi_t^i(\mathrm{d}x), \tag{3.14}$$

for $0 \leq t < T$. Hence, ψ_t^i is a conditional density of X_T. Note that we can write

$$\pi_t^i(\mathrm{d}x) = \mathbb{Q}(X_T \in \,\mathrm{d}x|\xi_t^i), \tag{3.15}$$

since $\{\xi_t^i\}$ is a Markov process with respect to $\{\mathcal{F}_t^{\xi^i}\}$, and X_T is a function of ξ_T^i. Then,

$$\psi_t^i(x) = \frac{\exp\left[\frac{T}{(T-t)}(\kappa^i x\xi_t^i - \frac{1}{2}(\kappa^i x)^2 t)\right] q(x)}{\int_{\mathbb{X}} \exp\left[\frac{T}{(T-t)}(\kappa^i x\xi_t^i - \frac{1}{2}(\kappa^i x)^2 t)\right] q(x)\,\mathrm{d}x}, \tag{3.16}$$

for $0 \leq t < T$ and $i = 1, \ldots, k$. Since $\{\psi_t^i\}$ is càdlàg and $\mathcal{F}_t^{\xi^i}$-adapted, $\{\psi_t^i\}$ is progressively measurable (see, for example, Karatzas and Shreve, 1991) such that $\psi^i : ([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{F}_t^{\xi^i}) \to (\mathcal{Q}(\mathbb{X}), \mathcal{B}(\mathcal{Q}(\mathbb{X})))$, $t \mapsto \psi_t^i$ is measurable for any $T > 0$, where $\otimes$ is the tensor product.

The stochastic differential equation (SDE) of $\{\psi_t^i\}$ can be derived by the use of Ito's lemma. The process $\{\psi_t^i\}$ is governed by the following SDE:

$$\mathrm{d}\psi_t^i(x) = \sigma_t^i(x)\psi_t^i(x)\,\mathrm{d}W_t^i, \tag{3.17}$$

for $0 \leq t < T$, where the coefficient $\{\sigma_t^i\}_{t\in[0,T)}$ is defined by

$$\sigma_t^i(x) = \frac{T\kappa^i(x - X_t^i)}{(T-t)}, \tag{3.18}$$

where $X_t^i = \mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \xi_t^i\right]$, and $\{W_t^i\}_{t\in[0,T)}$ is a $\mathbb{Q}$-Brownian motion, satisfying

$$W_t^i = \xi_t^i + \int_0^t \frac{1}{T-s}\xi_s^i\,\mathrm{d}s - T\kappa^i \int_0^t \frac{1}{T-s}X_s^i\,\mathrm{d}s. \tag{3.19}$$

Note that we have not yet defined the market filtration. We simply introduced a cash flow X_T, the Brownian information processes $\{\xi_t^i\}$, $i = 1, \ldots, k$, the process $\{\pi_t^i\}$, and the conditional density process $\{\psi_t^i\}$.

3.2 Appearance of New Sources of Information

This section provides a model for the entrance of new information sources to the market. We first introduce an $\mathcal{F}_t$-stopping time $\tau : \Omega \to \mathbb{R}_+$. We assume that for some finite $M \in \mathbb{R}_+$, $0 < \tau \leq M < \infty$.

We define the Heaviside function at τ by

$$H_\tau(t) = \begin{cases} 1 & \text{if } \tau \leq t, \\ 0 & \text{otherwise.} \end{cases} \tag{3.20}$$

Note that $\{H_\tau(t)\}$ is a càdlàg process, and $H_\tau(t)$ can equivalently be viewed as a Dirac measure. The following can be written:

$$H_\tau(t) = \int_0^t \mathrm{d}H_\tau(u) = \int_0^t \delta_\tau(\mathrm{d}u), \tag{3.21}$$

where $\delta_\tau(.)$ is the Dirac measure centered at the stopping time τ. The first integral is a Riemann-Stieltjes integral, and the second integral is a Lebesgue integral.

We shall first prove a strong Markov property of the Brownian information process. Before we state our proposition, we would like to review that, from Hoyle *et al.* (2011),

$$\mathbb{Q}(\xi_{t_1} \in \mathrm{d}y_1, \ldots, \xi_{t_m} \in \mathrm{d}y_m | \xi_T = x) = \mathbb{Q}(B_{t_1} \in \mathrm{d}y_1, \ldots, B_{t_m} \in \mathrm{d}y_m | B_T = x), \tag{3.22}$$

where $\{B_t\}$ is a $\mathbb{Q}$-Brownian motion as in (2.3). In fact, one can interpret $\{\xi_t\}$ as a Brownian motion conditioned to have the marginal density q at time T.

Given that $\{B_t\}$ is a $\mathbb{Q}$-Brownian motion, we denote $\mathcal{F}_t^B = \sigma(\{B_s\}_{0 \leq s \leq t}) \subset \mathcal{F}_t$. Then $\{B_t\}$ is a strong Markov process with respect to $\{\mathcal{F}_t^B\}$. We denote the probability density of B_t by f_t; thus $\mathbb{Q}(B_t \in \mathrm{d}x) = f_t(x)\,\mathrm{d}x$. Also, we define

$$\theta_0(\mathrm{d}x; y) = q(x)\,\mathrm{d}x, \qquad \text{and} \qquad \theta_t(\mathrm{d}x; y) = \frac{f_{T-t}(x-y)}{f_T(x)} q(x)\,\mathrm{d}x, \tag{3.23}$$

for $t \in (0, T)$. We are now in the position to state our proposition:

Proposition 3.2.1. *Assume that $\tau : \Omega \to \mathbb{R}_+$ and $\tau^* : \Omega \to \mathbb{R}_+$ are random times.*

1. *Let τ be an $\mathcal{F}_t^\xi$-stopping time and τ^* be an $\mathcal{F}_t^B$-stopping time, such that*

$$\begin{aligned} &\mathbb{Q}(\xi_{t_1} \in \mathrm{d}y_1, \ldots, \xi_{t_m} \in \mathrm{d}y_m | \xi_T = x, \tau = t_m)\mathbb{Q}(\tau \in \mathrm{d}t_m | \xi_T = x) = \\ &\quad = \mathbb{Q}(B_{t_1} \in \mathrm{d}y_1, \ldots, B_{t_m} \in \mathrm{d}y_m | B_T = x, \tau^* = t_m)\mathbb{Q}(\tau^* \in \mathrm{d}t_m | B_T = x). \end{aligned} \tag{3.24}$$

Then $\{\xi_t\}$ is a strong Markov process with respect to $\{\mathcal{F}_t^\xi\}$, with transition law:

$$\mathbb{Q}(\xi_T \in \mathrm{d}x | \xi_\tau = y) = \frac{\theta_{\tau^*}(\mathrm{d}x; y)}{\int_{\mathbb{X}} \theta_{\tau^*}(\mathrm{d}x; y)}, \tag{3.25}$$

$$\mathbb{Q}(\xi_t \in \mathrm{d}z | \xi_\tau = y) = \frac{\int_{\mathbb{X}} \theta_t(\mathrm{d}x; z)}{\int_{\mathbb{X}} \theta_{\tau^*}(\mathrm{d}x; y)} f_{t-\tau^*}(z-y)\,\mathrm{d}z. \tag{3.26}$$

2. *Let $\mathcal{Z}_t = \mathcal{F}_t^\xi \bigvee \sigma(\tau)$, where τ is a $\mathcal{Z}_t$-stopping time independent of $\{\xi_t\}$. Then $\{\xi_t\}$ is a strong Markov process with respect to $\{\mathcal{Z}_t\}$, with transition law:*

$$\mathbb{Q}(\xi_T \in \mathrm{d}x|\xi_\tau = y) = \int_{\mathbb{R}_+} \frac{\theta_s(\mathrm{d}x; y)}{\int_{\mathbb{X}} \theta_s(\mathrm{d}x; y)} \delta_\tau(\mathrm{d}s), \tag{3.27}$$

$$\mathbb{Q}(\xi_t \in \mathrm{d}z|\xi_\tau = y) = \int_{\mathbb{R}_+} \frac{\int_{\mathbb{X}} \theta_t(\mathrm{d}x; z)}{\int_{\mathbb{X}} \theta_s(\mathrm{d}x; y)} f_{t-s}(z-y)\,\mathrm{d}z\delta_\tau(\mathrm{d}s). \tag{3.28}$$

Proof. For the first part of the statement, let τ be an $\mathcal{F}_t^\xi$-stopping time and τ^* be an $\mathcal{F}_t^B$-stopping time. Assume (3.24) holds. Then, first we need to show that

$$\mathbb{Q}(\xi_t \le y \,|\, \xi_{t_1}, \ldots, \xi_{t_n}, \xi_\tau) = \mathbb{Q}(\xi_t \le y \,|\, \xi_\tau), \tag{3.29}$$

for $0 < t_1 < \ldots < t_n < \tau < t \le T$. Without loss of generality, we assume $\kappa = 1$ and $T = 1$. Hence, $\xi_T = X_T$. Then from (3.22),

$$\mathbb{Q}(\xi_{t_1} \in \mathrm{d}y_1, \ldots, \xi_{t_n} \in \mathrm{d}y_n, \xi_T \in \mathrm{d}x) = \prod_{i=1}^{n} (f_{t_i - t_{i-1}}(y_i - y_{i-1})\,\mathrm{d}y_i)\theta_{t_n}(\mathrm{d}x; y_n). \tag{3.30}$$

Since Brownian motion $\{B_t\}$ is a strong Markov process with respect to $\{\mathcal{F}_t^B\}$, it follows that $B_T - B_{\tau^*}$ is independent of $\mathcal{F}_{\tau^*}^B$, where the σ-algebra $\mathcal{F}_{\tau^*}^B = \{A \in \mathcal{F}^B : A \bigcap \{\tau^* \le s\} \in \mathcal{F}_s^B\}$ for every $s \ge 0$. Then, using the law of total probability, equation (3.24) and the strong Markov property of $\{B_t\}$, we have

$$\begin{aligned}
\mathbb{Q}(\xi_T \in \mathrm{d}x|\xi_\tau = y_m) &= \frac{\int_{\mathbb{R}_+} \mathbb{Q}(\xi_{t_m} \in \mathrm{d}y_m|\xi_T = x, \tau = t_m)\mathbb{Q}(\tau \in \mathrm{d}t_m|\xi_T = x)q(x)\,\mathrm{d}x}{\int_{\mathbb{X}}\int_{\mathbb{R}_+} \mathbb{Q}(\xi_{t_m} \in \mathrm{d}y_m|\xi_T = x, \tau = t_m)\mathbb{Q}(\tau \in \mathrm{d}t_m|\xi_T = x)q(x)\,\mathrm{d}x} \\
&= \frac{\int_{\mathbb{R}_+} \mathbb{Q}(B_{t_m} \in \mathrm{d}y_m|B_T = x, \tau^* = t_m)\mathbb{Q}(\tau^* \in \mathrm{d}t_m|B_T = x)q(x)\,\mathrm{d}x}{\int_{\mathbb{X}}\int_{\mathbb{R}_+} \mathbb{Q}(B_{t_m} \in \mathrm{d}y_m|B_T = x, \tau^* = t_m)\mathbb{Q}(\tau^* \in \mathrm{d}t_m|B_T = x)q(x)\,\mathrm{d}x} \\
&= \frac{\mathbb{Q}(B_{\tau^*} \in \mathrm{d}y_m|B_T = x)q(x)\,\mathrm{d}x}{\int_{\mathbb{X}} \mathbb{Q}(B_{\tau^*} \in \mathrm{d}y_m|B_T = x)q(x)\,\mathrm{d}x} \\
&= \frac{\theta_{\tau^*}(\mathrm{d}x; y_m)}{\int_{\mathbb{X}} \theta_{\tau^*}(\mathrm{d}x; y_m)}.
\end{aligned} \tag{3.31}$$

Similarly, we can write the following:

$$\begin{aligned}
&\mathbb{Q}(\xi_T \in \mathrm{d}x|\xi_{t_1} = y_1, \ldots, \xi_{t_n} = y_n, \xi_\tau = y_m) = \\
&= \frac{\int_{\mathbb{R}_+} \mathbb{Q}(\xi_{t_1} \in \mathrm{d}y_1, \ldots, \xi_{t_n} \in \mathrm{d}y_n, \xi_{t_m} \in \mathrm{d}y_m|\xi_T = x, \tau = t_m)\mathbb{Q}(\tau \in \mathrm{d}t_m|\xi_T = x)q(x)\,\mathrm{d}x}{\int_{\mathbb{X}}\int_{\mathbb{R}_+} \mathbb{Q}(\xi_{t_1} \in \mathrm{d}y_1, \ldots, \xi_{t_n} \in \mathrm{d}y_n, \xi_{t_m} \in \mathrm{d}y_m|\xi_T = x, \tau = t_m)\mathbb{Q}(\tau \in \mathrm{d}t_m|\xi_T = x)q(x)\,\mathrm{d}x} \\
&= \frac{\mathbb{Q}(B_{t_1} \in \mathrm{d}y_1, \ldots, B_{t_n} \in \mathrm{d}y_n, B_{\tau^*} \in \mathrm{d}y_m|B_T = x)q(x)\,\mathrm{d}x}{\int_{\mathbb{X}} \mathbb{Q}(B_{t_1} \in \mathrm{d}y_1, \ldots, B_{t_n} \in \mathrm{d}y_n, B_{\tau^*} \in \mathrm{d}y_m|B_T = x)q(x)\,\mathrm{d}x}.
\end{aligned} \tag{3.32}$$

Then, from the strong Markov property of $\{B_t\}$ and (3.32), it follows that

$$\mathbb{Q}(\xi_T \in dx|\xi_{t_1} = y_1, \ldots, \xi_{t_n} = y_n, \xi_\tau = y_m) = \frac{\theta_{\tau^*}(\mathrm{d}x; y_m)}{\int_{\mathbb{X}} \theta_{\tau^*}(\mathrm{d}x; y_m)}. \tag{3.33}$$

Hence, (3.29) holds for $t = T$. For $t < T$, using (3.33), we can write

$$\begin{aligned}\mathbb{Q}(\xi_t \leq y|\xi_{t_1}, \ldots, \xi_{t_n}, \xi_\tau) &= \int_{\mathbb{X}} \mathbb{Q}(\xi_t \leq y|\xi_{t_1}, \ldots, \xi_{t_n}, \xi_\tau, \xi_T = x)\mathbb{Q}(\xi_T \in \mathrm{d}x|\xi_\tau) \\ &= \int_{\mathbb{X}} \mathbb{Q}(\xi_t \leq y|xt_1 + B_{t_1T}, \ldots, x\tau + B_{\tau T}, \xi_T = x)\mathbb{Q}(\xi_T \in \mathrm{d}x|\xi_\tau). \end{aligned} \tag{3.34}$$

The process $\{xt + B_{tT}\}_{0\leq t\leq T}$ is a Brownian bridge to the value x at time $T = 1$. From Fitzsimmons *et al.* (1993), Brownian bridges are strong Markov processes (also, see Howard and Zumbrun, 1998). Thus,

$$\begin{aligned}\mathbb{Q}(\xi_t \leq y|\xi_{t_1}, \ldots, \xi_{t_n}, \xi_\tau) &= \int_{\mathbb{X}} \mathbb{Q}(xt + B_{tT} \leq y|x\tau + B_{\tau T}, \xi_T = x)\mathbb{Q}(\xi_T \in \mathrm{d}x|\xi_\tau) \\ &= \mathbb{Q}(\xi_t \leq y|\xi_\tau). \end{aligned} \tag{3.35}$$

From (3.33) and (3.35), $\{\xi_t\}$ is a strong Markov process with respect to $\{\mathcal{F}_t^\xi\}$. Since $0 < \kappa < \infty$ and time $T < \infty$ can be chosen arbitrarily, it follows that $\{\xi_t\}$ is a strong Markov process with respect to $\{\mathcal{F}_t^\xi\}$. The transition law $\mathbb{Q}(\xi_t \in \mathrm{d}z|\xi_\tau = y)$ for $t < T$ is

$$\begin{aligned}\mathbb{Q}(\xi_t \in \mathrm{d}z|\xi_\tau = y) &= \int_{\mathbb{X}} \mathbb{Q}(\xi_t \in \mathrm{d}z|\xi_\tau = y, \xi_T = x)\mathbb{Q}(\xi_T \in \mathrm{d}x|\xi_\tau = y) \\ &= \int_{\mathbb{X}} \frac{\mathbb{Q}(B_t \in \mathrm{d}z, B_{\tau^*} \in \mathrm{d}y|B_T = x)}{\mathbb{Q}(B_{\tau^*} \in \mathrm{d}y|B_T = x)} \frac{\theta_{\tau^*}(\mathrm{d}x; y)}{\int_{\mathbb{X}} \theta_{\tau^*}(\mathrm{d}x; y)} \\ &= \frac{\int_{\mathbb{X}} \theta_t(\mathrm{d}x; z)}{\int_{\mathbb{X}} \theta_{\tau^*}(\mathrm{d}x; y)} f_{t-\tau^*}(z - y)\,\mathrm{d}z, \end{aligned} \tag{3.36}$$

which completes the proof of the first part of the statement.

For the second part of the statement, let $\mathcal{Z}_t = \mathcal{F}_t^\xi \bigvee \sigma(\tau)$. Assume τ is independent of $\{\xi_t\}$. Then, $\{\xi_t\}$ is a strong Markov process with respect to $\{\mathcal{Z}_t\}$, since

$$\begin{aligned}\mathbb{Q}(\xi_t \leq y|\mathcal{Z}_\tau) &= \int_{\mathbb{R}_+} \mathbb{Q}(\xi_t \leq y|\mathcal{F}_s^\xi \bigvee \sigma(\tau), \tau = s)\mathbb{Q}(\tau \in \mathrm{d}s|\mathcal{Z}_\tau) \\ &= \int_{\mathbb{R}_+} \mathbb{Q}(\xi_t \leq y|\xi_s)\delta_\tau(\mathrm{d}s) = \mathbb{Q}(\xi_t \leq y|\xi_\tau), \end{aligned} \tag{3.37}$$

for $0 < \tau < t \leq T$. Equation (3.37) follows since $\{\xi_t\}$ is a Markov process with respect to $\{\mathcal{F}_t^\xi\}$ and $\mathbb{Q}(\tau \in \mathrm{d}s|\mathcal{Z}_\tau)$ is the Dirac measure centered at τ. The transition laws $\mathbb{Q}(\xi_T \in \mathrm{d}x|\xi_s = y)$ and $\mathbb{Q}(\xi_t \in \mathrm{d}z|\xi_s = y)$ follow from (3.30). □

From this point on, to focus attention on modelling the appearance of new sources of information, we consider only the parsimonious case where the stopping times are independent of the information processes $\{\xi_t^i\}$, $i = 1, \dots, k$.

3.2.1 A New Information Source

It is stated in Chung (1982) that stopping times are the most effective tools to "tame the continuum of time". We shall use stopping times as what one may call 'measurable start-up times', so that the market receives a new information source starting from a stopping time.

First, for a fixed time u and for $0 \le t \le T$, we denote by $\{\mathcal{F}_{u,t}^{\xi^i}\}$ the filtration given by

$$\mathcal{F}_{u,t}^{\xi^i} = \begin{cases} \sigma(\{\xi_s^i\}_{u \le s \le t}) & u \le t, \\ \{\Omega, \varnothing\} & u > t, \end{cases} \tag{3.38}$$

where $\{\Omega, \varnothing\}$ is the trivial σ-algebra. Note that $\mathcal{F}_{u,t}^{\xi^i} \subseteq \mathcal{F}_t^{\xi^i}$ for all $u \ge 0$, and $\mathcal{F}_{t,t}^{\xi^i} = \sigma(\xi_t^i)$.

We are now in the position to define our market filtration. We start with the case where the market is already provided with an information process $\{\xi_t^1\}$, and the market receives an additional source of information $\{\xi_t^2\}$ at time τ. To use τ as a start-up time, we define $\{\mathcal{V}_t^{\xi^2}\}$ as the filtration of the subalgebra $\mathcal{V}_t^{\xi^2} \subset \mathcal{F}_t$ given by

$$\mathcal{V}_t^{\xi^2} = \begin{cases} \sigma(\{H_\tau(s)\}_{0 \le s \le t}) & \tau > t, \\ \sigma(\{H_\tau(s)\}_{0 \le s \le t}, \{\xi_s^2\}_{\tau \le s \le t}) & \tau \le t, \end{cases} \tag{3.39}$$

for $0 \le t \le T$, where τ is a $\mathcal{V}_t^{\xi^2}$-stopping time independent of $\{\xi_t^1\}$ and $\{\xi_t^2\}$. Note that the σ-algebra $\mathcal{V}_t^{\xi^2}$ is generated by the Heaviside process independent of the information processes, and if $\tau \le t$, it additionally encodes the information provided by $\{\xi_s^2\}_{s \ge \tau}$ for $s \le t$.

Define the filtration $\{\mathcal{G}_t\}$ by

$$\mathcal{G}_t = \mathcal{F}_t^{\xi^1} \bigvee \mathcal{V}_t^{\xi^2} = \begin{cases} \sigma(\{H_\tau(s)\}_{0 \le s \le t}, \{\xi_s^1\}_{0 \le s \le t}) & \tau > t, \\ \sigma(\{H_\tau(s)\}_{0 \le s \le t}, \{\xi_s^1\}_{0 \le s \le t}, \{\xi_s^2\}_{\tau \le s \le t}) & \tau \le t. \end{cases} \tag{3.40}$$

We assume $\{\mathcal{G}_t\}$ is the market filtration. The σ-algebra $\mathcal{G}_t \subset \mathcal{F}_t$ encodes all the information that market has about the cash flow X_T. For example, $\{\xi_t^2\}$ may provide idiosyncratic information about X_T that has leaked to the market at time τ, or τ may represent the time of an announcement regarding X_T.

One may replace the Heaviside process in (3.39) and (3.40) with a continuous process (independent of $\{\xi_t^1\}$ and $\{\xi_t^2\}$), and model τ as the first hitting time of this process. As an example, if $\{L_t\}_{t \ge 0}$ is this process (which may represent an economic variable), we can define $\tau = \inf\{s \geq 0 : L_s \in B\}$ for some $B \in \mathcal{B}(\mathbb{X})$. Then τ is the first time $\{L_t\}$ enters B, which indicates the moment when the new information process $\{\xi_t^2\}$ appears in the market.

We denote the t-price of an asset with cash flow X_T by $\overline{X}_t$, and define $\overline{X}_t$ as follows:

$$\overline{X}_t = P_{tT}\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \mathcal{G}_t\right], \qquad 0 \leq t < T. \tag{3.41}$$

Brody *et. al* (2009) detail an orthogonalization procedure to compactify the information $\{\xi_t^1, \xi_t^2\}$ into the information, say $\{\widehat{\xi}_t^{(2)}\}$. In particular, given that $|\rho| < 1$ is the correlation between $\{B_{tT}^1\}$ and $\{B_{tT}^2\}$, Brody *et. al* (2009) show that

$$\mathbb{Q}(X_T \in \,\mathrm{d}x|\xi_t^1, \xi_t^2) = \mathbb{Q}(X_T \in \,\mathrm{d}x|\widehat{\xi}_t^{(2)}), \tag{3.42}$$

where $\{\widehat{\xi}_t^{(2)}\}_{0\leq t\leq T}$ is the effective Brownian information process given by

$$\widehat{\xi}_t^{(2)} = \widehat{\kappa}^{(2)} X_T t + \widehat{B}_{tT}^{(2)}, \tag{3.43}$$

provided that

$$\widehat{\kappa}^{(2)} = \sqrt{\frac{(\kappa^1)^2 - 2\rho\kappa^1\kappa^2 + (\kappa^2)^2}{(1-\rho^2)}}, \tag{3.44}$$

$$\widehat{B}_{tT}^{(2)} = \frac{1}{\kappa^{(2)}}\left[\frac{\kappa^1 - \rho\kappa^2}{1-\rho^2} B_{tT}^1 + \frac{\kappa^2 - \rho\kappa^1}{1-\rho^2} B_{tT}^2\right]. \tag{3.45}$$

Note that $\{\widehat{B}_{tT}^{(2)}\}_{t\in[0,T]}$ is a Brownian bridge and the speed coefficient $\widehat{\kappa}^{(2)}$ is a function of κ^1 and κ^2. Equation (3.42) simplifies calculations considerably.

Define a $\mathcal{Q}(\mathbb{X})$-valued stochastic process $\{\overline{\pi}_t\}_{t\in[0,T]}$ by

$$\overline{\pi}_t(A) = \mathbb{Q}(X_T \in A|\mathcal{G}_t), \tag{3.46}$$

for $A \in \mathcal{B}(\mathbb{X})$. $\overline{\pi}_t$ is a random probability measure. We further define

$$\psi_t^{(1)}(x)\,\mathrm{d}x = \pi_t^{(1)}(\mathrm{d}x) = \mathbb{Q}(X_T \in \,\mathrm{d}x|\xi_t^1), \tag{3.47}$$

$$\psi_t^{(2)}(x)\,\mathrm{d}x = \pi_t^{(2)}(\mathrm{d}x) = \mathbb{Q}(X_T \in \,\mathrm{d}x|\widehat{\xi}_t^{(2)}), \tag{3.48}$$

for $0 \leq t < T$. We shall be absolutely clear with our notation: $\{\xi_t^i\}$ is the ith information process as defined in (3.10), and $\{\widehat{\xi}_t^{(i)}\}$ is the *effective* information process constructed using the first i information processes $\{\xi_t^1\}, \ldots, \{\xi_t^i\}$. Hence, $\xi_t^1 = \widehat{\xi}_t^{(1)}$, but $\xi_t^2 \neq \widehat{\xi}_t^{(2)}$ for $t \in (0, T]$.

Proposition 3.2.2. *The random probability measure $\overline{\pi}_t$ can be represented as*

$$\overline{\pi}_t(A) = \pi_t^{(1)}(A)\,(1 - H_\tau(t)) + \pi_t^{(2)}(A) H_\tau(t), \tag{3.49}$$

and the asset price $\overline{X}_t$ is given by

$$\overline{X}_t = P_{tT}\left(\frac{\int_{\mathbb{X}} x \exp\left[\frac{T}{(T-t)}(\kappa^1 x\xi^1_t - \frac{1}{2}(\kappa^1 x)^2 t)\right] q(x)\,\mathrm{d}x}{\int_{\mathbb{X}} \exp\left[\frac{T}{(T-t)}(\kappa^1 x\xi_t - \frac{1}{2}(\kappa^1 x)^2 t)\right] q(x)\,\mathrm{d}x}\right)(1 - H_\tau(t))$$
$$+ P_{tT}\left(\frac{\int_{\mathbb{X}} x \exp\left[\frac{T}{(T-t)}(\widehat{\kappa}^{(2)} x\widehat{\xi}^{(2)}_t - \frac{1}{2}(\widehat{\kappa}^{(2)} x)^2 t)\right] q(x)\,\mathrm{d}x}{\int_{\mathbb{X}} \exp\left[\frac{T}{(T-t)}(\widehat{\kappa}^{(2)} x\widehat{\xi}^{(2)}_t - \frac{1}{2}(\widehat{\kappa}^{(2)} x)^2 t)\right] q(x)\,\mathrm{d}x}\right) H_\tau(t). \tag{3.50}$$

Proof. The law of total probability can be used to project $\overline{\pi}_t(A)$ onto the two orthogonal subspaces $\{t < \tau\}$ and $\{\tau \leq t\}$. If we denote $\mathbb{Q}(\tau \in \mathrm{d}u) = v_\tau(\mathrm{d}u)$, then since $0 < \tau \leq M$ for some finite M:

$$\begin{aligned}\mathbb{Q}(X_T \in A|\mathcal{G}_t) &= \int_t^M \mathbb{Q}(X_T \in A|\mathcal{F}^{\xi^1}_t \bigvee \sigma(\{H_\tau(s)\}_{0\leq s\leq t}), \tau = u)\nu_\tau(\mathrm{d}u|\mathcal{V}^{\xi^2}_t) \\ &+ \int_0^t \mathbb{Q}(X_T \in A|\mathcal{F}^{\xi^1}_t \bigvee \mathcal{F}^{\xi^2}_{u,t} \bigvee \sigma(\{H_\tau(s)\}_{0\leq s\leq t}), \tau = u)\nu_\tau(\mathrm{d}u|\mathcal{V}^{\xi^2}_t) \\ &= \int_t^M \mathbb{Q}(X_T \in A|\xi^1_t)\nu_\tau(\mathrm{d}u|\mathcal{V}^{\xi^2}_t) + \int_0^t \mathbb{Q}(X_T \in A|\xi^1_t, \xi^2_t)\nu_\tau(\mathrm{d}u|\mathcal{V}^{\xi^2}_t),\end{aligned} \tag{3.51}$$

since $\{\xi^1_t\}$ and $\{\xi^2_t\}$ are Markovian, and independent of τ. From (3.42), it follows that

$$\begin{aligned}\overline{\pi}_t(A) &= \mathbb{Q}(X_T \in A|\xi^1_t)\left(\int_t^M \nu_\tau(\mathrm{d}u|\mathcal{V}^{\xi^2}_t)\right) + \mathbb{Q}(X_T \in A|\widehat{\xi}^{(2)}_t)\left(\int_0^t \nu_\tau(\mathrm{d}u|\mathcal{V}^{\xi^2}_t)\right) \\ &= \pi^{(1)}_t(A)\left(\int_t^M \nu_\tau(\mathrm{d}u|\mathcal{V}^{\xi^2}_t)\right) + \pi^{(2)}_t(A)\left(\int_0^t \nu_\tau(\mathrm{d}u|\mathcal{V}^{\xi^2}_t)\right).\end{aligned} \tag{3.52}$$

Since τ is a $\mathcal{V}^{\xi^2}_t$-stopping time, we can write

$$\int_0^t \nu_\tau(\mathrm{d}u|\mathcal{V}^{\xi^2}_t) = \int_0^t \delta_\tau(\mathrm{d}u) = \int_0^t \mathrm{d}H_\tau(u). \tag{3.53}$$

Thus, the first integral in (3.52) is equal to $1 - H_\tau(t)$, and the second integral is $H_\tau(t)$. From the law of total probability, the independence of τ and the strong Markov property of $\{\xi^i_t\}$,

$$\mathbb{Q}(X_T \in A|\mathcal{G}_\tau) = \mathbb{Q}(X_T \in A|\xi^1_\tau, \xi^2_\tau) = \mathbb{Q}(X_T \in A|\widehat{\xi}^{(2)}_\tau), \tag{3.54}$$

for $\tau < T$. Hence, (3.52) and (3.54) are consistent. Equation (3.49) follows. The expression for $\psi^{(1)}_t(x)$ is already given in (3.16). From (3.42), the expression for $\psi^{(2)}_t(x)$ is

$$\psi^{(2)}_t(x) = \frac{\exp\left[\frac{T}{(T-t)}(\widehat{\kappa}^{(2)} x\widehat{\xi}^{(2)}_t - \frac{1}{2}(\widehat{\kappa}^{(2)} x)^2 t)\right] q(x)}{\int_{\mathbb{X}} \exp\left[\frac{T}{(T-t)}(\widehat{\kappa}^{(2)} x\widehat{\xi}^{(2)}_t - \frac{1}{2}(\widehat{\kappa}^{(2)} x)^2 t)\right] q(x)\,\mathrm{d}x}, \tag{3.55}$$

for $0 \le t < T$. From (3.41), the price $\overline{X}_t$ is

$$\overline{X}_t = P_{tT} \int_{\mathbb{X}} x \overline{\pi}_t(\mathrm{d}x), \tag{3.56}$$

for $0 \le t < T$ and the expression in (3.50) follows since τ is independent. □

The price $\overline{X}_t$ is expressed in terms of one information process if $\tau > t$, and two processes if $\tau \le t$. The market adjusts the price after the appearance of a new information source.

3.2.2 Multiple Information Sources

We extend the model by introducing $n \in \mathbb{N}_+$ stopping times. We define an n-sequence of $\mathcal{F}_t$-stopping times $\{\tau_i\}_{i=1}^n$ such that for some finite $M \in \mathbb{R}_+$, $0 < \tau_1 < \tau_2 < ... < \tau_n \le M < \infty$. We assume the existence of $n+1$ information processes mutually independent of each τ_i. We denote the associated Heaviside functions centered at τ_i by H_{τ_i}, and define $\mathcal{V}_t^{\xi^i} \subset \mathcal{F}_t$ by

$$\mathcal{V}_t^{\xi^{i+1}} = \begin{cases} \sigma(\{H_{\tau_i}(s)\}_{0\le s\le t}) & \tau_i > t, \\ \sigma(\{H_{\tau_i}(s)\}_{0\le s\le t}, \{\xi_s^{i+1}\}_{\tau_i \le s \le t}) & \tau_i \le t, \end{cases} \tag{3.57}$$

for $i = 1, \dots, n$ and $0 \le t \le T$. We define the filtration $\{\mathcal{G}_t\}$ by

$$\mathcal{G}_t = \mathcal{F}_t^{\xi^1} \bigvee_{i=1}^{n} \mathcal{V}_t^{\xi^{i+1}}, \tag{3.58}$$

and assume $\{\mathcal{G}_t\}$ is the market filtration. Note that the market is provided with ξ^1 from time $t = 0$, and it receives additional information sources at stopping times.

In order to derive price dynamics in this market, we first define a $\mathcal{Q}(\mathbb{X})$-valued process $\{\overline{\pi}_t\}_{t\in[0,T]}$ by

$$\overline{\pi}_t(A) = \mathbb{Q}(X_T \in A | \mathcal{G}_t), \tag{3.59}$$

for $A \in \mathcal{B}(\mathbb{X})$, with density

$$\overline{\psi}_t(x)\,\mathrm{d}x = \overline{\pi}_t(\mathrm{d}x), \tag{3.60}$$

for $0 \le t < T$. We also define the processes $\{\pi_t^{(i)}\}_{t\in[0,T]}$, $i = 1, ..., n+1$, by

$$\pi_t^{(i)}(A) = \mathbb{Q}(X_T \in A | \xi_t^1, ..., \xi_t^i), \tag{3.61}$$

and their conditional densities are

$$\psi_t^{(i)}(x)\,\mathrm{d}x = \pi_t^{(i)}(\mathrm{d}x), \tag{3.62}$$

for $0 \leq t < T$. From (3.43)-(3.45), iterating the orthogonalization procedure detailed in Brody *et. al* (2009) using pairs of information processes, we can write

$$\pi_t^{(i)}(\mathrm{d}x) = \mathbb{Q}(X_T \in \mathrm{d}x|\xi_t^1, ..., \xi_t^i) = \mathbb{Q}(X_T \in \mathrm{d}x|\widehat{\xi}_t^{(i)}), \tag{3.63}$$

where the effective Brownian information process $\{\widehat{\xi}_t^{(i)}\}_{t\in[0.T]}$ is defined by

$$\widehat{\xi}_t^{(i)} = \widehat{\kappa}^{(i)} X_T t + \widehat{B}_{tT}^{(i)}, \tag{3.64}$$

for $i = 1, \ldots, n+1$, given that

$$\widehat{\kappa}^{(i)} = \sqrt{\frac{(\widehat{\kappa}^{(i-1)})^2 - 2\widehat{\rho}^{(i)}\widehat{\kappa}^{(i-1)}\kappa^i + (\kappa^i)^2}{(1-(\widehat{\rho}^{(i)})^2)}}, \tag{3.65}$$

$$\widehat{B}_{tT}^{(i)} = \frac{1}{\widehat{\kappa}^{(i)}}\left[\frac{\widehat{\kappa}^{(i-1)} - \widehat{\rho}^{(i)}\kappa^i}{(1-(\widehat{\rho}^{(i)})^2)}\widehat{B}_{tT}^{(i-1)} + \frac{\kappa^i - \widehat{\rho}^{(i)}\widehat{\kappa}^{(i-1)}}{(1-(\widehat{\rho}^{(i)})^2)}B_{tT}^i\right], \tag{3.66}$$

where $\widehat{\kappa}^{(0)} = 0$, $\widehat{B}_{tT}^{(0)} = 0$, and $\widehat{\rho}^{(1)} = 0$. Note that $\{\widehat{B}_{tT}^{(i)}\}_{t\in[0,T]}$ is a Brownian bridge for $i = 1, \ldots, n+1$, where $|\widehat{\rho}^{(i)}| < 1$ is the correlation between $\{\widehat{B}_{tT}^{(i-1)}\}$ and $\{B_{tT}^i\}$ for $i = 2, \ldots, n+1$. Also, $\widehat{\xi}_t^{(1)} = \xi_t^1$, $\widehat{\kappa}^{(1)} = \kappa^1$, $\widehat{B}_{tT}^{(1)} = B_{tT}^1$, but such equalities do not hold for $i = 2, \ldots, n+1$. Finally, we define the following vectors:

$$\mathbf{P}_t(A) = \begin{bmatrix} \pi_t^{(1)}(A) \\ \vdots \\ \pi_t^{(i)}(A) \\ \vdots \\ \pi_t^{(n+1)}(A) \end{bmatrix} \quad \text{and} \quad \mathbf{I}_t = \begin{bmatrix} 1 - H_{\tau_1}(t) \\ \vdots \\ H_{\tau_{i-1}}(t)(1-H_{\tau_i}(t)) \\ \vdots \\ H_{\tau_n}(t) \end{bmatrix}. \tag{3.67}$$

Proposition 3.2.3. *The random probability measure* $\overline{\pi}_t$ *can be represented as*

$$\overline{\pi}_t(A) = \mathbf{P}_t^\top(A)\mathbf{I}_t, \tag{3.68}$$

where the conditional density $\psi_t^{(i)}$ *is given by*

$$\psi_t^{(i)}(x) = \frac{\exp\left[\frac{T}{(T-t)}\left(\widehat{\kappa}^{(i)} x \widehat{\xi}_t^{(i)} - \frac{1}{2}(\widehat{\kappa}^{(i)} x)^2 t\right)\right] q(x)}{\int_{\mathbb{X}} \exp\left[\frac{T}{(T-t)}\left(\widehat{\kappa}^{(i)} x \widehat{\xi}_t^{(i)} - \frac{1}{2}(\widehat{\kappa}^{(i)} x)^2 t\right)\right] q(x)\,\mathrm{d}x}. \tag{3.69}$$

Proof. We can project $\overline{\pi}_t$ onto $n+1$ orthogonal subspaces with respect to time so that

$$\overline{\pi}_t(A) = \sum_{i=0}^{n} \mathbb{Q}(X_T \in A|\mathcal{G}_t, \tau_i \leq t < \tau_{i+1})\mathbb{Q}(\tau_i \leq t < \tau_{i+1}|\mathcal{G}_t), \tag{3.70}$$

where we set $\tau_0 = 0$ and $t < \tau_{n+1}$. Each information process ξ^i is Markov and is mutually independent of the τ_i's. Since each τ_i is a $\mathcal{G}_t$-stopping time, $\mathbb{Q}(\tau_i \leq t < \tau_{i+1}|\mathcal{G}_t)$ is a Dirac measure. Following similar steps as done in the proof of Proposition 3.2.2, we have

$$\begin{aligned}
\overline{\pi}_t(A) &= \mathbb{Q}(X_T \in A|\xi_t^1) \int_t^M \mathbb{Q}(\tau_1 \in \mathrm{d}u_1|\mathcal{G}_t) + \mathbb{Q}(X_T \in A|\xi_t^1, \xi_t^2) \int_0^t \mathbb{Q}(\tau_1 \in \mathrm{d}u_1, t < \tau_2|\mathcal{G}_t) \\
&\quad + \cdots + \mathbb{Q}(X_T \in A|\xi_t^1, \ldots, \xi_t^{n+1}) \int_{[0,t]^n} \mathbb{Q}(\tau_1 \in \mathrm{d}u_1, \ldots, \tau_n \in \mathrm{d}u_n|\mathcal{G}_t) \\
&= \pi_t^{(1)}(A)(1 - H_{\tau_1}(t)) + \pi_t^{(2)}(A)H_{\tau_1}(t)(1 - H_{\tau_2}(t)) \\
&\quad + \cdots + \pi_t^{(n+1)}(A)H_{\tau_n}(t).
\end{aligned} \tag{3.71}$$

From the law of total probability, independence of τ_i and the strong Markov property of $\{\xi_t^i\}$,

$$\mathbb{Q}(X_T \in A|\mathcal{G}_{\tau_i}) = \mathbb{Q}(X_T \in A|\xi_{\tau_i}^1, \ldots, \xi_{\tau_i}^{i+1}) = \mathbb{Q}(X_T \in A|\widehat{\xi}_{\tau_i}^{(i+1)}), \tag{3.72}$$

for $\tau_i < T$. Hence, (3.71) and (3.72) are consistent. Equation (3.68) follows. Equation (3.69) is from (3.63)-(3.66) and the Bayes formula. □

Lemma 3.2.4. *Let $\mathbf{I}_t(i)$ be the ith element of $\mathbf{I}_t$ for $i = 1, \ldots, n+1$. Then,*

$$d\mathbf{I}_t(i+1) = \delta_{\tau_i}(\mathrm{d}t) - \delta_{\tau_{i+1}}(\mathrm{d}t), \tag{3.73}$$

for $i = 0, \ldots, n$, provided that $\tau_0 < t < \tau_{n+1}$, and $\delta_{\tau_{n+1}}(\mathrm{d}t) = \delta_{\tau_0}(\mathrm{d}t) = 0$.

Proof. $\mathbf{I}_t$ is defined in (3.67) and $\mathbf{I}_t(i)$ is the ith element of $\mathbf{I}_t$ for $i = 1, \ldots, n+1$. Then,

$$d\mathbf{I}_t(i+1) = \delta_{\tau_i}(\mathrm{d}t)(1 - H_{\tau_{i+1}}(t)) + H_{\tau_i}(t)(-1)\delta_{\tau_{i+1}}(\mathrm{d}t), \tag{3.74}$$

where $\tau_0 < t < \tau_{n+1}$. If $\tau_i = t$, the condition $t < \tau_{i+1}$ is immediately satisfied, and hence, $\delta_{\tau_i}(\mathrm{d}t)(1-H_{\tau_{i+1}}(t)) = \delta_{\tau_i}(\mathrm{d}t)$. If $\tau_i \neq t$, $\delta_{\tau_i}(\mathrm{d}t) = 0$ and so $\delta_{\tau_i}(\mathrm{d}t)(1-H_{\tau_{i+1}}(t)) = 0$. If $\tau_{i+1} = t$, the condition $\tau_i \leq t$ is immediately satisfied, and hence, $H_{\tau_i}(t)(-1)\delta_{\tau_{i+1}}(\mathrm{d}t) = -\delta_{\tau_{i+1}}(\mathrm{d}t)$. If $\tau_{i+1} \neq t$, $\delta_{\tau_{i+1}}(\mathrm{d}t) = 0$ and so $H_{\tau_i}(t)(-1)\delta_{\tau_{i+1}}(\mathrm{d}t) = 0$. □

We are now in the position to provide the SDE of the conditional density process $\{\overline{\psi}_t\}_{t\in[0,T)}$. First, we define the process $\{\sigma_t^{(i)}\}_{t\in[0,T)}$ as follows:

$$\sigma_t^{(i)}(x) = \frac{T\widehat{\kappa}^{(i)}}{(T-t)} \left(x - \mathbb{E}^{\mathbb{Q}} \left[X_T \,\middle|\, \widehat{\xi}_t^{(i)} \right] \right), \tag{3.75}$$

for $i = 1, \ldots, n+1$ and $0 \leq t < T$. Note that from (3.18), $\sigma_t^{(1)} = \sigma_t^1$, but $\sigma_t^{(i)} \neq \sigma_t^i$ for $i = 2, \ldots, n+1$. That is, the random variable σ_t^i is defined in terms of the ith information process, whereas $\sigma_t^{(i)}$ is defined in terms of i sources of information processes.

We also note that $\{W_t^{(i)}\}_{t\in[0,T)}$ satisfying

$$W_t^{(i)} = \widehat{\xi}_t^{(i)} + \int_0^t \frac{1}{T-s}\widehat{\xi}_s^{(i)}\,\mathrm{d}s - T\widehat{\kappa}^{(i)}\int_0^t \frac{1}{T-s}\mathbb{E}^{\mathbb{Q}}\left[X_T \,\middle|\, \widehat{\xi}_s^{(i)}\right]\,\mathrm{d}s, \tag{3.76}$$

is a $\mathbb{Q}$-Brownian motion for $i = 1,\dots,n+1$, by Lévy's characterisation.

Proposition 3.2.5. *The dynamics of $\{\overline{\psi}_t\}$ are governed by the following SDE:*

$$\mathrm{d}\overline{\psi}_t(x) = \sum_{i=1}^{n+1} \sigma_t^{(i)}(x)\psi_t^{(i)}(x)\,\mathrm{d}W_t^{(i)}\mathbf{I}_t(i) + \sum_{i=2}^{n+1}\left(\psi_t^{(i)}(x) - \psi_t^{(i-1)}(x)\right)\delta_{\tau_{i-1}}(\mathrm{d}t). \tag{3.77}$$

Proof. The expression for $\psi_t^{(i)}(x)$ is given in (3.69). Then applying Ito's lemma,

$$\mathrm{d}\psi_t^{(i)}(x) = \frac{T\widehat{\kappa}^{(i)}}{(T-t)}\left(x - \mathbb{E}^{\mathbb{Q}}\left[X_T \,\middle|\, \widehat{\xi}_t^{(i)}\right]\right)\psi_t^{(i)}(x)\,\mathrm{d}W_t^{(i)}, \tag{3.78}$$

for $0 \le t < T$. Then (3.77) follows from Proposition 3.2.3 and Lemma 3.2.4. □

The process $\{\overline{\psi}_t\}$ has jump-diffusion dynamics. Each $\{\psi_t^{(i)}\}$ is an exponential Brownian motion with a different stochastic diffusion coefficient. Then every time a new information source appears in the market, diffusion coefficient of $\{\overline{\psi}_t\}$ jumps.

For a fixed $x \in \mathbb{X}$ and $\tau_i = t$, a jump of size $(\psi_t^{(i+1)}(x) - \psi_t^{(i)}(x))$ occurs in $\overline{\psi}_t(x)$. Thus, the law of the jump size of $\overline{\psi}_t(x)$ at $\tau_i = t$ is characterised by the joint law of $\psi_t^{(i)}(x)$ and $\psi_t^{(i+1)}(x)$. For the fixed $\tau_i = t$ and $x \in \mathbb{X}$, setting $Y_t^i(x) = \psi_t^{(i+1)}(x) - \psi_t^{(i)}(x)$ and $R_t^i(x) = \psi_t^{(i+1)}(x)$, the Jacobian $\mathrm{Jac}(Y_t^i(x), R_t^i(x)) = 1$. From multivariate transformation theorem, if $h_t(p_i, p_{i+1})$ is the joint density of $\psi_t^{(i)}(x)$ and $\psi_t^{(i+1)}(x)$, $\mathbb{Q}(Y_t^i(x) \in \mathrm{d}y)/\,\mathrm{d}y$ is

$$\int_{\mathbb{R}_+} h_t(r-y, r)|\mathrm{Jac}(Y_t^i, R_t^i)|\,\mathrm{d}r = \int_{\mathbb{R}_+} h_t(r-y, r)\,\mathrm{d}r. \tag{3.79}$$

The conditional density process $\{\overline{\psi}_t\}$ is a càdlàg process. It is possible to define a topology along with the concept of convergence on the space of càdlàg functions. With this topology and the Borel σ-algebra, the paths of $\{\overline{\psi}_t\}$ are elements of a Skorokhod space. In addition, for every càdlàg process with jumps, taking values in $\mathbb{M} \subseteq \mathbb{R}^d$, one can naturally associate it with a random measure on $[0,T] \times \mathbb{M}$, which can be called the random jump measure. For a fixed $x \in \mathbb{X}$, each jump size $Y_{\tau_i}^i(x)$ is $\mathcal{G}_{\tau_i}$-measurable for $\tau_i < T$, and the process $\{\tau_i, Y_{\tau_i}^i(x)\}$ contains all the information about the jump times and the jump sizes of $\{\overline{\psi}_t(x)\}$. Then we can construct a random jump measure for $\{\overline{\psi}_t\}$, denoted by $J_{\overline{\psi}}$ as follows:

$$J_{\overline{\psi}(x)}(\omega, .) = \sum_{i=1}^{n} \delta_{(\tau_i(\omega), Y_{\tau_i}^i(x)(\omega))}, \tag{3.80}$$

for $\omega \in \Omega$, where we can view $J_{\overline{\psi}(x)}([0,T] \times A)$, $A \subset \mathbb{R}$, as the number of jumps of $\{\overline{\psi}_t(x)\}$ on $[0,T]$, whose size belongs to A. The random jump measure $J_{\overline{\psi}}$ contains all the information about the jumps of $\{\overline{\psi}_t\}$, but does not contain information about the continuous part.

We shall now provide the price dynamics. Using Proposition 3.2.3, the price $\overline{X}_t$ is given by

$$\overline{X}_t = P_{tT}\mathbb{E}^{\mathbb{Q}}[X_T|\mathcal{G}_t] = P_{tT}\int_{\mathbb{X}} x\mathbf{P}'_t(\mathrm{d}x)\mathbf{I}_t, \qquad 0 \le t < T. \tag{3.81}$$

Proposition 3.2.6. *The dynamics of the price $\{\overline{X}_t\}$ are governed by the following SDE:*

$$\begin{aligned}\mathrm{d}\overline{X}_t &= r_t\overline{X}_t\,\mathrm{d}t + P_{tT}\sum_{i=1}^{n+1}\frac{T\widehat{\kappa}^{(i)}}{(T-t)}\left(Var^{\mathbb{Q}}\left[X_T\,\middle|\,\widehat{\xi}_t^{(i)}\right]\right)\mathrm{d}W_t^{(i)}\mathbf{I}_t(i)\\ &\quad + P_{tT}\sum_{i=2}^{n+1}\left(\mathbb{E}^{\mathbb{Q}}\left[X_T\,\middle|\,\widehat{\xi}_t^{(i)}\right] - \mathbb{E}^{\mathbb{Q}}\left[X_T\,\middle|\,\widehat{\xi}_t^{(i-1)}\right]\right)\delta_{\tau_{i-1}}(\mathrm{d}t), \end{aligned} \tag{3.82}$$

for $0 \le t < T$, where $Var^{\mathbb{Q}}[X_T\,|\,\widehat{\xi}_t^{(i)}]$ is a $\mathbb{Q}$-supermartingale.

Proof. Using the Lebesgue Dominated Convergence, Proposition 3.2.5 and (3.81), we have

$$\begin{aligned}\mathrm{d}\overline{X}_t &= r_t\overline{X}_t\,\mathrm{d}t + P_{tT}\sum_{i=1}^{n+1}\left(\int_{\mathbb{X}} x\sigma_t^{(i)}(x)\psi_t^{(i)}(x)\,\mathrm{d}x\right)\mathrm{d}W_t^{(i)}\mathbf{I}_t(i)\\ &+ P_{tT}\sum_{i=2}^{n+1}\left(\int_{\mathbb{X}} x\left(\psi_t^{(i)}(x) - \psi_t^{(i-1)}(x)\right)\mathrm{d}x\right)\delta_{\tau_{i-1}}(\mathrm{d}t)\\ &= r_t\overline{X}_t\,\mathrm{d}t + P_{tT}\sum_{i=1}^{n+1}\left(\int_{\mathbb{X}} x\frac{T\widehat{\kappa}^{(i)}}{(T-t)}\left(x - \mathbb{E}^{\mathbb{Q}}\left[X_T\,\middle|\,\widehat{\xi}_t^{(i)}\right]\right)\psi_t^{(i)}(x)\,\mathrm{d}x\right)\mathrm{d}W_t^{(i)}\mathbf{I}_t(i)\\ &+ P_{tT}\sum_{i=2}^{n+1}\left(\int_{\mathbb{X}} x\psi_t^{(i)}(x)\,\mathrm{d}x - \int_{\mathbb{X}} x\psi_t^{(i-1)}(x)\,\mathrm{d}x\right)\delta_{\tau_{i-1}}(\mathrm{d}t)\\ &= r_t\overline{X}_t\,\mathrm{d}t + P_{tT}\sum_{i=1}^{n+1}\frac{T\widehat{\kappa}^{(i)}}{(T-t)}\left(\mathbb{E}^{\mathbb{Q}}\left[(X_T)^2\,\middle|\,\widehat{\xi}_t^{(i)}\right] - \mathbb{E}^{\mathbb{Q}}\left[X_T\,\middle|\,\widehat{\xi}_t^{(i)}\right]^2\right)\mathrm{d}W_t^{(i)}\mathbf{I}_t(i)\\ &+ P_{tT}\sum_{i=2}^{n+1}\left(\mathbb{E}^{\mathbb{Q}}\left[X_T\,\middle|\,\widehat{\xi}_t^{(i)}\right] - \mathbb{E}^{\mathbb{Q}}\left[X_T\,\middle|\,\widehat{\xi}_t^{(i-1)}\right]\right)\delta_{\tau_{i-1}}(\mathrm{d}t), \end{aligned} \tag{3.83}$$

for $0 \le t < T$. $\mathbb{E}^{\mathbb{Q}}[(X_T)^2\,|\,\widehat{\xi}_t^{(i)}]$ is a $\mathbb{Q}$-martingale and $\mathbb{E}^{\mathbb{Q}}[X_T\,|\,\widehat{\xi}_t^{(i)}]^2$ is a $\mathbb{Q}$-submartingale. Hence, $\mathrm{Var}^{\mathbb{Q}}[X_T\,|\,\widehat{\xi}_t^{(i)}] = \mathbb{E}^{\mathbb{Q}}[(X_T)^2\,|\,\widehat{\xi}_t^{(i)}] - \mathbb{E}^{\mathbb{Q}}[X_T\,|\,\widehat{\xi}_t^{(i)}]^2$ is a $\mathbb{Q}$-supermartingale. □

By allowing new Brownian information sources to appear randomly in the market, we see that it is a natural outcome of this framework that the price process $\{\overline{X}_t\}$ follows jump-diffusion dynamics, and that it has stochastic volatility with jumps.

3.2.3 A Hilbert Space Perspective on New Information Sources

We can view Proposition 3.2.3 from a Hilbert space perspective. We shall briefly discuss the insight that the Hilbert space setting brings to the representation of $\{\overline{\psi}_t\}$. This insight allows us to measure the impact of new information sources geometrically.

We let $G \subset \mathbb{R}^2$ be a measurable set and assume that the following orthogonal decomposition of $\mathcal{L}^2(G)$ holds:

$$\mathcal{L}^2(G) = \bigoplus_{i=1}^{n+1} \mathcal{L}_i^2(G), \tag{3.84}$$

where $\mathcal{L}_i^2(G)$ and $\mathcal{L}_j^2(G)$ are mutually orthogonal closed subspaces of $\mathcal{L}^2(G)$ for $i \neq j$, such that any function in $\mathcal{L}^2(G)$ can uniquely be represented by the sum of its projections onto the subspaces $\mathcal{L}_i^2(G)$ for $i = 1, \ldots, n+1$ that span $\mathcal{L}^2(G)$.

Since $\overline{\psi}_t$ is a probability density for $0 \leq t < T$, it satisfies: $\int_{\mathbb{X}} \overline{\psi}_t(x)\,\mathrm{d}x = 1$. We define

$$\overline{\rho}_t(x) = \sqrt{q(x|\mathcal{G}_t)}. \tag{3.85}$$

Note that $\overline{\rho}$ is a square-integrable function such that $\overline{\rho} \in \mathcal{L}^2(\mathbb{X} \times [0,T))$. Further we define

$$\rho_t^{(i)}(x) = \sqrt{q(x|\xi_t^1, ..., \xi_t^i)}. \tag{3.86}$$

The function $\rho^{(i)}$ is square-integrable for $i = 1, \ldots, n+1$, where $\rho^{(i)} \in \mathcal{L}^2(\mathbb{X} \times [0,T))$. Let G be the domain of the measurable functions $\overline{\rho}$ and $\rho^{(i)}$ so that $G = \mathbb{X} \times \mathbb{T}$, where $\mathbb{T} = \{t : 0 \leq t < T\}$. That is, setting $G = \mathbb{X} \times \mathbb{T}$, we consider the case where the orthogonal decomposition in (3.84) can be written as

$$\mathcal{L}^2(\mathbb{X} \times \mathbb{T}) = \bigoplus_{i=1}^{n+1} \mathcal{L}_i^2(\mathbb{X} \times \mathbb{T}). \tag{3.87}$$

Let the disjoint sets $\mathbb{W}_i$, for $i = 1, ..., n+1$ be such that $\mathbb{W}_1 = \{t \in \mathbb{T} : t < \tau_1\}$, $\mathbb{W}_i = \{t \in \mathbb{T} : \tau_{i-1} \leq t < \tau_i\}$ for $i = 2, ..., n$, and $\mathbb{W}_{n+1} = \{t \in \mathbb{T} : \tau_n \leq t\}$. Note that $\mathbb{T} = \bigcup_{i=1}^{n+1} \mathbb{W}_i$. Now, we define the measurable function $\widehat{\pi}^{(i)}$, for $i = 1, ..., n+1$ by

$$\widehat{\pi}_t^{(i)}(x) = \begin{cases} \sqrt{q(x|\mathcal{G}_t)} & \text{if } t \in \mathbb{W}_i, \\ 0 & \text{otherwise.} \end{cases} \tag{3.88}$$

From the strong Markovian property of $\{\xi_t^i\}$ and the independence of τ_i's, it follows that

$$\widehat{\pi}_t^{(i)}(x) = \begin{cases} \sqrt{q(x|\xi_t^1, ..., \xi_t^i)} & \text{if } t \in \mathbb{W}_i, \\ 0 & \text{otherwise.} \end{cases} \tag{3.89}$$

Note that $\widehat{\pi}^{(i)} \perp \widehat{\pi}^{(j)}$ for $i \neq j$ on G. That is, $\langle \widehat{\pi}^{(i)}, \widehat{\pi}^{(j)} \rangle = 0$ for $i \neq j$ on $\mathbb{X} \times \mathbb{T}$. We write $\widehat{\pi}^{(i)} \in \mathcal{L}_i^2(\mathbb{X} \times \mathbb{T})$ and $\widehat{\pi}^{(j)} \in \mathcal{L}_j^2(\mathbb{X} \times \mathbb{T})$, and we have the following representation:

$$\overline{\rho} = \widehat{\pi}^{(1)} + \cdots + \widehat{\pi}^{(n+1)}, \tag{3.90}$$

in $\mathcal{L}^2(\mathbb{X} \times \mathbb{T})$. The Heaviside function $H_{\tau_i}(t)$ for $i = 1, ..., n$ is measurable at each $t \leq T$, and is an element of the Hilbert space of square-integrable functions on $\mathbb{T}$. Recall,

$$\{\mathbf{I}_t\}_{0 \leq t \leq T} = \left\{ \left[1 - H_{\tau_1}(t), \ldots, H_{\tau_{i-1}}(t)(1 - H_{\tau_i}(t)), \ldots, H_{\tau_n}(t)\right]^\top \right\}_{0 \leq t \leq T}. \tag{3.91}$$

Then $\sigma(\{\mathbf{I}_s\})_{0 \leq s \leq t} \subset \mathcal{G}_t$. Using (3.90) and denoting $\mathbf{I}_t(i)$ as the ith element of $\mathbf{I}_t$, the function $\overline{\rho}$ can be represented as

$$\begin{aligned} \overline{\rho}_t(x) &= \widehat{\pi}_t^{(1)}(x) + \cdots + \widehat{\pi}_t^{(n+1)}(x) \\ &= \rho_t^{(1)}(x)\mathbf{I}_t(1) + \cdots + \rho_t^{(n+1)}(x)\mathbf{I}_t(n+1), \end{aligned} \tag{3.92}$$

in $\mathbb{R}$ for some $x \in \mathbb{X}$. Note that by squaring $\overline{\rho}$ in (3.92), equation (3.68) is recovered:

$$\begin{aligned} \overline{\psi}_t(x) &= \left(\rho_t^{(1)}(x)\mathbf{I}_t(1) + \cdots + \rho_t^{(n+1)}(x)\mathbf{I}_t(n+1)\right)^2 \\ &= \psi_t^{(1)}(x)\mathbf{I}_t(1) + \cdots + \psi_t^{(n+1)}(x)\mathbf{I}_t(n+1), \end{aligned} \tag{3.93}$$

since $\mathbf{I}_t(i)\mathbf{I}_t(j) = 0$ for $i \neq j$ and $\mathbf{I}_t(i)\mathbf{I}_t(i) = \mathbf{I}_t(i)$.

Since each $\rho_t^{(i)}(x)\mathbf{I}_t(i)$ takes values in $\mathbb{R}$ for $i = 1, \ldots, n+1$ and $x \in \mathbb{X}$, we can as well work with any $\mathcal{H}$ isomorphic to $\mathbb{R}^{n+1}$ by using (3.92). That is, we can canonically represent $\overline{\rho}$ as an $(n+1)$-tuple

$$\overline{\rho} = \left[\rho^{(1)}, \ldots, \rho^{(i)}, \ldots, \rho^{(n+1)}\right]^\top, \tag{3.94}$$

in $\mathbb{R}^{n+1}$, and represent each $\mathbf{I}(i)$ for $i = 1, \ldots, n+1$ as

$$\mathbf{I}(i) = e_i = [0, \ldots, 1, \ldots, 0]^\top, \tag{3.95}$$

where the ith element is 1 and the remaining n elements are 0. For $\mathcal{H} \cong \mathbb{R}^{n+1}$, e_i's form a complete orthonormal sequence $e_i \in \mathcal{H}$ for $i = 1, \ldots, n+1$. Then, we have

$$\overline{\rho} = \sum_{i=1}^{n+1} \langle \overline{\rho}, e_i \rangle e_i = \sum_{i=1}^{n+1} \rho^{(i)} e_i, \tag{3.96}$$

in $\mathcal{H}$ by Theorem 3.1.1. The representation (3.96) is equivalent to (3.92). From Definition 3.1.2, we shall refer to $\rho^{(i)}$'s as the Fourier coefficients for $i = 1, \ldots, n+1$.

The insight gained from the Hilbert space brings forth a geometrical interpretation. The function $\overline{\rho}$ is a non-negative function, and for a fixed time t, the integral of the square of $\overline{\rho}_t$ on $\mathbb{X}$ is unity. Thus, from the transformation $\overline{\psi}_t \mapsto \overline{\rho}_t$ for fixed t, $\overline{\rho}_t$ determines a point on the positive orthant of the unit sphere $\mathcal{S} \subset \mathcal{L}^2$. Therefore, the process $\{\overline{\rho}_t\}$ determines a stochastic trajectory on $\mathcal{S}^+$, where $\mathcal{S}^+$ is the positive orthant of $\mathcal{S}$. Also, each Fourier coefficient determines a stochastic evolution on $\mathcal{S}^+$, since each is a non-negative function and the integral of their square on $\mathbb{X}$ is unity for a fixed time t.

The unit sphere $\mathcal{S}$ is a differentiable manifold, and, if equipped with a Riemannian metric, it is a Riemannian manifold (see, for example Do Carmo, 1992). We provide a formal account of Riemannian manifolds in Chapter 6 when quantifying information asymmetry. We shall give a brief overview here of how geometry interacts with information:

The distance between the points determined by the Fourier coefficients $\rho^{(i)}$ and $\rho^{(j)}$, which are defined by different numbers of information sources, has a natural geometry on $\mathcal{S}^+$. Any two points on $\mathcal{S}$ lie on a circle with center coinciding with the center of $\mathcal{S}$. The circle and its segments are geodesics and the spherical distance between the points determined by $\rho_t^{(i)}$ and $\rho_t^{(j)}$ is the length of the geodesic connecting these two points on $\mathcal{S}^+$. Since each $\{\rho_t^{(i)}\}$ determines a trajectory on $\mathcal{S}^+$, the spherical distance between the points determined by $\rho_t^{(i)}$ and $\rho_t^{(j)}$ can vary in the interval $[0, \pi/2]$. Since $\rho^{(i)}$ and $\rho^{(i+1)}$ are defined in terms of i and $i+1$ information processes respectively, the spherical distance between the two points measures the effect of having the additional information source.

The Fourier coefficients $\rho^{(i)}$ and $\rho^{(j)}$ can also be used to define an angle process $\Theta^{ij} = \{\Theta_t^{ij}\}_{0 \le t < T}$, for $i, j = 1, \ldots, n+1$ by the $\mathcal{L}^2$-inner product,

$$\cos \Theta_t^{ij} = \int_{\mathbb{X}} \rho_t^{(i)}(x) \rho_t^{(j)}(x) \, \mathrm{d}x = 1 - \frac{1}{2} \int_{\mathbb{X}} \left(\rho_t^{(i)}(x) - \rho_t^{(j)}(x) \right)^2 \mathrm{d}x, \tag{3.97}$$

where $\Theta_t^{ij} = \Theta_t^{ji}$ is called the Bhattacharyya angle (see Bhattacharyya, 1946) between $\rho_t^{(i)}$ and $\rho_t^{(j)}$. This is the angle from the center of $\mathcal{S}$ subtended to the endpoints on $\mathcal{S}^+$. Note that Θ^{ij} is stochastic, and the maximum angle between the Fourier coefficients is $\pi/2$ radians. Given that $i \neq j$, the angle Θ_t^{ij} can be used as a geometric quantity (equivalent to the spherical distance) that measures the influence of additional information sources.

Note that the trajectory determined by $\{\overline{\rho}_t\}$ on $\mathcal{S}^+$ has jumps. Each jump size on $\mathcal{S}^+$ can be measured geometrically at stopping times.

Remark 3.2.7. *The non-marginal impact of a new information source can be measured by the spherical distance (or the Bhattacharyya angle) between the Fourier coefficients $\rho^{(i)}$ and $\rho^{(i+1)}$ on $\mathcal{S}^+$ at stopping times.*

Remark 3.2.7 is what partly motivates Chapter 6, where we dynamically quantify the impact of changes in information sources geometrically. We shall provide a more detailed account of it in Chapter 6.

3.3 Multiple Market Factor Generalisation

We generalise our framework to the case where X_T can be represented as a function of a set of independent random variables. We address the question as to how to represent a market, in which, new sources of information about different market factors may appear at different stopping times. This is a valid problem since the market may receive a broadcast about a particular market factor, but may not receive any for another market factor at that time.

We assume that the cash flow X_T can be expressed as a function of a set of independent random variables, say M_T^α, for $\alpha = 1, \ldots, m$, with state-space $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ and with continuous densities $q^\alpha(x) > 0$. The random variables M_T^α are the market factors and they govern the value of the cash flow X_T. Choosing a function $g \in B_b(\mathbb{X}^m)$ such that $g : \mathbb{X}^m \to \mathbb{X}$, we represent X_T as follows:

$$X_T = g(M_T^1, M_T^2, \ldots, M_T^m). \tag{3.98}$$

We associate a sequence of $\mathcal{F}_t$-stopping times to each market factor M_T^α denoted by $\{\tau_i^\alpha\}_{i=1}^n$ for $\alpha = 1, \ldots, m$. For fixed α, we let $\tau_1^\alpha < \tau_2^\alpha < \ldots < \tau_n^\alpha$. However, for each i and j, $\mathbb{Q}(\tau_i^\alpha < \tau_j^{\alpha+1}) \neq 1$. Hence, the stopping times do not necessarily occur in a sequential order across α. We denote the associated Heaviside functions at τ_i^α by $H_{\tau_i}^\alpha$:

$$H_{\tau_i}^\alpha(t) = \begin{cases} 1 & \text{if } \tau_i^\alpha \leq t, \\ 0 & \text{otherwise.} \end{cases} \tag{3.99}$$

We use each stopping time to model the appearance of a new source of information in the market. Accordingly, we associate multiple Brownian information processes $\{\xi_t^{\alpha,i}\}_{t\in[0,T]}$, $i = 1, \ldots, n+1$, with each M_T^α, such that

$$\xi_t^{\alpha,i} = \kappa^{\alpha,i} M_T^\alpha t + B_{tT}^{\alpha,i}, \tag{3.100}$$

where $\{B_{tT}^{\alpha,i}\}_{t\in[0,T]}$ is a Brownian bridge to the value zero. We assume that $\{B_{tT}^{\alpha,i}\}$'s are mutually independent across α (i.e., $\{B_{tT}^{\alpha,i}\}$ and $\{B_{tT}^{\beta,j}\}$ are independent) and independent from each M_T^α. Hence, the information processes are mutually independent across α. However, for a fixed α, $\{B_{tT}^{\alpha,i}\}$ and $\{B_{tT}^{\alpha,j}\}$ can be correlated. We further assume that each sequence of stopping times is mutually independent from each other and mutually independent of each information process.

For fixed α, we define $\{\mathcal{V}_t^{\xi^{\alpha,i}}\}$ as the filtration of the subalgebra $\mathcal{V}_t^{\xi^{\alpha,i}} \subset \mathcal{F}_t$ such that

$$\mathcal{V}_t^{\xi^{\alpha,i+1}} = \begin{cases} \sigma(\{H_{\tau_i}^\alpha(s)\}_{0\leq s\leq t}) & \tau_i^\alpha > t, \\ \sigma(\{H_{\tau_i}^\alpha(s)\}_{0\leq s\leq t}, \{\xi_s^{\alpha,i+1}\}_{\tau_i^\alpha \leq s\leq t}) & \tau_i^\alpha \leq t, \end{cases} \tag{3.101}$$

for $i = 1\ldots,n$ and $0 \le t \le T$. Let $\mathcal{F}_t^{\xi^{\alpha,1}} = \sigma(\{\xi_s^{\alpha,i}\}_{0\le s\le t})$ for $0 \le t \le T$. Then we define

$$\mathcal{Z}_t = \bigvee_{\alpha=1}^{m} \mathcal{F}_t^{\xi^{\alpha,1}} \bigvee_{i=1}^{n} \mathcal{V}_t^{\xi^{\alpha,i+1}}. \tag{3.102}$$

We assume that $\{\mathcal{Z}_t\}$ is the market filtration. Hence, $\mathcal{Z}_t \subset \mathcal{F}_t$ is all the information that the market receives about the cash flow X_T. We define a $\mathcal{Q}(\mathbb{X})$-valued process $\{\overline{\pi}_t\}_{t\in[0,T]}$ by

$$\overline{\pi}_t(\mathbf{A}) = \mathbb{Q}([M_T^1, M_T^2, \ldots, M_T^m] \in \mathbf{A}|\mathcal{Z}_t), \tag{3.103}$$

for a fixed $\mathbf{A} \in \mathcal{B}(\mathbb{X}^m) = \otimes_{i=1}^m \mathcal{B}(\mathbb{X})$. $\overline{\pi}_t$ is a joint conditional distribution of the vector of market factors. Note that due to the independence properties we imposed above, the following can be written:

$$\begin{aligned}\overline{\pi}_t(\mathbf{A}) = \mathbb{Q}(M_T^1 \in A_1|\mathcal{F}_t^{\xi^{1,1}} \bigvee_{i=1}^{n} \mathcal{V}_t^{\xi^{1,i+1}}) \times \mathbb{Q}(M_T^2 \in A_2|\mathcal{F}_t^{\xi^{2,1}} \bigvee_{i=1}^{n} \mathcal{V}_t^{\xi^{2,i+1}}) \\ \times \cdots \times \mathbb{Q}(M_T^m \in A_m|\mathcal{F}_t^{\xi^{m,1}} \bigvee_{i=1}^{n} \mathcal{V}_t^{\xi^{m,i+1}}),\end{aligned} \tag{3.104}$$

for $\mathbf{A} = [A_1, A_2, \ldots, A_m] \in \mathcal{B}(\mathbb{X}^m)$. We denote the conditional density by

$$\overline{\psi}_t(\mathbf{x})\,\mathrm{d}x_1 \cdots \mathrm{d}x_m = \overline{\pi}_t(\mathrm{d}\mathbf{x}), \tag{3.105}$$

for $0 \le t < T$ and $\mathbf{x} = [x_1, \ldots, x_m] \in \mathbb{X}^m$. We also define the process $\{\pi_t^{(i)}\}_{t\in[0,T]}$ by

$$\pi_t^{\alpha,(i)}(A_\alpha) = \mathbb{Q}(M_T^\alpha \in A_\alpha|\xi_t^{\alpha,1}, ..., \xi_t^{\alpha,i}), \tag{3.106}$$

for $i = 1, ..., n+1$ and $\alpha = 1, \ldots, m$, and the conditonal density by

$$\psi_t^{\alpha,(i)}(x_\alpha)\,\mathrm{d}x_\alpha = \pi_t^{\alpha,(i)}(\mathrm{d}x_\alpha), \tag{3.107}$$

for $0 \le t < T$ and $x_\alpha \in \mathbb{X}$.

In order to derive asset price dynamics, we additionally define:

$$\mathbf{P}_t^1 = \begin{bmatrix} \pi_t^{1,(1)} \\ \vdots \\ \pi_t^{1,(i)} \\ \vdots \\ \pi_t^{1,(n+1)} \end{bmatrix}, \quad \mathbf{P}_t^2 = \begin{bmatrix} \pi_t^{2,(1)} \\ \vdots \\ \pi_t^{2,(i)} \\ \vdots \\ \pi_t^{2,(n+1)} \end{bmatrix}^\top, \quad \ldots, \quad \mathbf{P}_t^m = \begin{bmatrix} \pi_t^{m,(1)} \\ \vdots \\ \pi_t^{m,(i)} \\ \vdots \\ \pi_t^{m,(n+1)} \end{bmatrix}^\top. \tag{3.108}$$

Note that each $\mathbf{P}_t^\alpha$ is a vector of conditional distributions associated with each market factor

M_T^α for $\alpha = 1, \dots, m$, where each vector element is determined by different number of information processes. We also define the following vectors of Heaviside processes:

$$\mathbf{I}_t^1 = \begin{bmatrix} 1 - H_{\tau_1}^1(t) \\ \vdots \\ H_{\tau_{i-1}}^1(t)(1 - H_{\tau_i}^1(t)) \\ \vdots \\ H_{\tau_n}^1(t) \end{bmatrix} \quad \text{and} \quad \{\mathbf{I}_t^\alpha\}_{\alpha=2}^m = \left\{ \begin{bmatrix} 1 - H_{\tau_1}^\alpha(t) \\ \vdots \\ H_{\tau_{i-1}}^\alpha(t)(1 - H_{\tau_i}^\alpha(t)) \\ \vdots \\ H_{\tau_n}^\alpha(t) \end{bmatrix}^\top \right\}_{\alpha=2}^m .$$

Note that each $\mathbf{I}_t^\alpha$ is a vector of Heaviside processes associated with each market factor M_T^α for $\alpha = 1, \dots, m$.

The following proposition makes use of the Kronecker product, Hadamard product and the entry-wise norm of a matrix. We shall provide a brief account of these operations:

If $\mathbf{X} \in \mathbb{R}^{m \times n}$ is a matrix and $\mathbf{Y} \in \mathbb{R}^{k \times l}$, we denote the Kronecker product of $\mathbf{X}$ and $\mathbf{Y}$ by $(\mathbf{X} \otimes \mathbf{Y}) \in \mathbb{R}^{mk \times nl}$, such that

$$\mathbf{X} \otimes \mathbf{Y} = \begin{bmatrix} \mathbf{X}_{11}\mathbf{Y} & \mathbf{X}_{12}\mathbf{Y} & \dots & \mathbf{X}_{1n}\mathbf{Y} \\ \mathbf{X}_{21}\mathbf{Y} & \mathbf{X}_{22}\mathbf{Y} & \dots & \mathbf{X}_{2n}\mathbf{Y} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{X}_{m1}\mathbf{Y} & \mathbf{X}_{m2}\mathbf{Y} & \dots & \mathbf{X}_{mn}\mathbf{Y} \end{bmatrix}, \tag{3.109}$$

where $\mathbf{X}_{ij}$ is the ijth element of the matrix $\mathbf{X}$. The Kronecker product is a type of the tensor product (this is why we choose to denote the Kronecker product by $\otimes$ as well), hence, it is associative: $(\mathbf{X} \otimes \mathbf{Y}) \otimes \mathbf{Z} = \mathbf{X} \otimes (\mathbf{Y} \otimes \mathbf{Z})$. For matrices $\mathbf{X}^i$ for $i = 1 \dots, m$, we write

$$\bigotimes_{i=1}^m \mathbf{X}^i = \mathbf{X}^1 \otimes \dots \otimes \mathbf{X}^i \otimes \dots \otimes \mathbf{X}^m. \tag{3.110}$$

For matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$, we denote their Hadamard product by $(\mathbf{A} \circ \mathbf{B}) \in \mathbb{R}^{m \times n}$, such that

$$(\mathbf{A} \circ \mathbf{B})_{ij} = \mathbf{A}_{ij}\mathbf{B}_{ij}. \tag{3.111}$$

The Hadamard product is commutative: $(\mathbf{A} \circ \mathbf{B}) = (\mathbf{B} \circ \mathbf{A})$. For a matix $\mathbf{C} \in \mathbb{R}^{m \times n}$, we denote the entry-wise p-norm of $\mathbf{C}$ by $||\mathbf{C}||_p$, given by

$$||\mathbf{C}||_p = \left(\sum_{i=1}^m \sum_{j=1}^n |c_{ij}|^p \right)^{1/p}. \tag{3.112}$$

We shall make use of the entry-wise 1-norm. To simplify the notation, we denote the entry-wise 1-norm by $||\mathbf{C}|| = ||\mathbf{C}||_1$.

We define an information process associated with M_T^α by $\{\widehat{\xi}_t^{\alpha,(i)}\}_{t\in[0,T)}$ where

$$\widehat{\xi}_t^{\alpha,(i)} = \widehat{\kappa}^{\alpha,(i)} M_T^\alpha t + \widehat{B}_{tT}^{\alpha,(i)}, \tag{3.113}$$

for $i = 1, \ldots, n+1$, such that

$$\widehat{\kappa}^{\alpha,(i)} = \sqrt{\frac{(\widehat{\kappa}^{\alpha,(i-1)})^2 - 2\widehat{\rho}^{\alpha,(i)}\widehat{\kappa}^{\alpha,(i-1)}\kappa^{\alpha,i} + (\kappa^{\alpha,i})^2}{(1-(\widehat{\rho}^{\alpha,(i)})^2)}}, \tag{3.114}$$

$$\widehat{B}_{tT}^{\alpha,(i)} = \frac{1}{\widehat{\kappa}^{\alpha,(i)}}\left[\frac{\widehat{\kappa}^{\alpha,(i-1)} - \widehat{\rho}^{\alpha,(i)}\kappa^{\alpha,i}}{(1-(\widehat{\rho}^{\alpha,(i)})^2)}\widehat{B}_{tT}^{\alpha,(i-1)} + \frac{\kappa^{\alpha,i} - \widehat{\rho}^{\alpha,(i)}\widehat{\kappa}^{\alpha,(i-1)}}{(1-(\widehat{\rho}^{\alpha,(i)})^2)}B_{tT}^{\alpha,i}\right], \tag{3.115}$$

$\widehat{\kappa}^{\alpha,(0)} = 0$, $\widehat{B}_{tT}^{\alpha,(0)} = 0$, and $\widehat{\rho}^{\alpha,(1)} = 0$. $\{\widehat{B}_{tT}^{\alpha,(i)}\}_{t\in[0,T]}$ is a Brownian bridge for $i = 1, \ldots, n+1$, and $|\widehat{\rho}^{\alpha,(i)}| < 1$ for a fixed α is the correlation between $\{\widehat{B}_{tT}^{\alpha,(i-1)}\}$ and $\{B_{tT}^{\alpha,i}\}$ for $i = 2, \ldots, n+1$. Also, $\widehat{\xi}_t^{\alpha,(1)} = \xi_t^{\alpha,1}$, $\widehat{\kappa}^{\alpha,(1)} = \kappa^{\alpha,1}$, $\widehat{B}_{tT}^{\alpha,(1)} = B_{tT}^{\alpha,1}$, but such equalities do not hold for $i = 2, \ldots, n+1$.

Proposition 3.3.1. *The random probability measure $\overline{\pi}_t$ can be represented as*

$$\overline{\pi}_t(\mathbf{A}) = \left\| \left(\bigotimes_{i=1}^m \mathbf{P}_t^i(A_i)\right) \circ \left(\bigotimes_{i=1}^m \mathbf{I}_t^i\right) \right\|, \tag{3.116}$$

where the conditional density $\psi_t^{\alpha,(i)}$ is given by

$$\psi_t^{\alpha,(i)}(x) = \frac{\exp\left[\frac{T}{(T-t)}\left(\widehat{\kappa}^{\alpha,(i)}x\widehat{\xi}_t^{\alpha,(i)} - \frac{1}{2}(\widehat{\kappa}^{\alpha,(i)}x)^2 t\right)\right] q^\alpha(x)}{\int_{\mathbb{X}} \exp\left[\frac{T}{(T-t)}\left(\widehat{\kappa}^{\alpha,(i)}x\widehat{\xi}_t^{\alpha,(i)} - \frac{1}{2}(\widehat{\kappa}^{\alpha,(i)}x)^2 t\right)\right] q^\alpha(x)\,\mathrm{d}x}. \tag{3.117}$$

Proof. The proof is similar to that of Proposition 3.2.3. Each stopping time τ_i^α is a $\mathcal{Z}_t$-stopping times. Hence, $\mathbb{Q}(\tau_i^\alpha \le t < \tau_{i+1}^\alpha | \mathcal{Z}_t)$'s are Dirac measures for $i = 1, \ldots, n$ and $\alpha = 1, \ldots, m$. $\{\xi_t^{\alpha,i}\}$ is strong Markov and the stopping times are independent from each M_T^α and each $\{\xi_t^{\alpha,i}\}$. Also, each sequence of stopping times are independent from each other. Then, the $(n+1 \times (n+1)^{m-1})$-dimensional matrix $(\mathbf{P}_t^1 \otimes \mathbf{P}_t^2 \otimes \cdots \otimes \mathbf{P}_t^m)$ encodes all possible combinations of the conditional distributions by the use of law of total probability, which follows from the fact that the information processes $\{\xi_t^{\alpha,i}\}$ are all independent across α. The $(n+1 \times (n+1)^{m-1})$-dimensional matrix $(\mathbf{I}_t^1 \otimes \mathbf{I}_t^2 \otimes \cdots \otimes \mathbf{I}_t^m)$ encodes all possible combinations of the number of information processes provided to the market on each market factor. The Hadamard product associates each element of $(\mathbf{P}_t^1 \otimes \mathbf{P}_t^2 \otimes \cdots \otimes \mathbf{P}_t^m)$ with the correct element of $(\mathbf{I}_t^1 \otimes \mathbf{I}_t^2 \otimes \cdots \otimes \mathbf{I}_t^m)$. The entry-wise norm of the resulting matrix is due to the law of total probability. The expression (3.117) follows from (3.113)-(3.115) and the Bayes formula. □

The $\mathcal{Q}(\mathbb{X})$-valued process $\{\overline{\pi}_t\}$ represents an economy in which the market is provided with different numbers of information sources about different market factors. We are now in the position to provide a representation of the price of an asset with the cash flow $X_T = g(M_T^1, M_T^2, \ldots, M_T^m)$. We denote the price by $\overline{X}_t$, which is

$$\overline{X}_t = P_{tT}\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \mathcal{Z}_t\right], \qquad 0 \le t < T. \tag{3.118}$$

Proposition 3.3.2. *The price $\overline{X}_t$ can be written as*

$$\overline{X}_t = P_{tT}\int_{\mathbb{X}^m} g(x^1, \ldots, x^m) \left\| \left(\bigotimes_{i=1}^{m} \mathbf{P}_t^i(\mathrm{d}x_i)\right) \circ \left(\bigotimes_{i=1}^{m} \mathbf{I}_t^i\right) \right\|. \tag{3.119}$$

Proof. The statement follows from (3.13), (3.116) and (3.118). □

The processes $\{\overline{\pi}_t\}$ and $\{\overline{\psi}_t\}$ jump at stopping times. Thus, the price process $\{\overline{X}_t\}$ jumps at every appearance of a new source of information about any of the market factors. Since there are $(n + 1 \times (n+1)^{m-1})$ possible states at a given time, $\{\overline{X}_t\}$ may jump a maximum of $(n + 1 \times (n+1)^{m-1}) - 1$ times during the time interval $[0, T]$.

Note that all elements of the matrix $(\mathbf{I}_t^1 \otimes \mathbf{I}_t^2 \otimes \cdots \otimes \mathbf{I}_t^m)$ are pairwise orthogonal functions in $\mathcal{L}^2([0,T])$. Hence, the square-root of the elements of the matrix $(\mathbf{P}_t^1 \otimes \cdots \otimes \mathbf{P}_t^m) \circ (\mathbf{I}_t^1 \otimes \mathbf{I}_t^2 \otimes \cdots \otimes \mathbf{I}_t^m)$, written in terms of densities, are pairwise orthogonal functions in $\mathcal{L}^2(\mathbb{X}^m \times [0,T))$. Following the arguments presented in Section 3.2, one can see that the square-root of each density of $(\mathbf{P}_t^1 \otimes \cdots \otimes \mathbf{P}_t^m)$ is a Fourier coefficient of the function $\sqrt{\psi}$, which determines a stochastic trajectory on $\mathcal{S}^+$. Then again, the non-marginal impact of new information sources can be measured by the spherical distance between the points on $\mathcal{S}^+$, determined by the Fourier coefficients of $\sqrt{\psi}$ at stopping times τ_i^α, $i = 1, \ldots, n$ and $\alpha = 1, \ldots, m$.

3.3.1 A Simplification: One Sequence of Stopping Times

Let's assume there is only one sequence $\{\tau_i^\alpha\}_{i=1}^n = \{\tau_i\}_{i=1}^n$ for $\alpha = 1, \ldots, m$. Hence, $\{H_{\tau_i}^\alpha\}_{\alpha=1}^m = H_{\tau_i}$ for $i = 1, \ldots, n$. The market filtration is defined in (3.102), the conditional distribution $\overline{\pi}_t$ and density $\psi_t^{\alpha,(i)}$ are given in (3.103) and (3.107), respectively. Also, we define

$$\mathbf{P}_t(\mathbf{A}) = \begin{bmatrix} \psi_t^{1,(1)}(A_1) \times \psi_t^{2,(1)}(A_2) \times \cdots \times \psi_t^{m,(1)}(A_m) \\ \vdots \\ \psi_t^{1,(i)}(A_1) \times \psi_t^{2,(i)}(A_2) \times \cdots \times \psi_t^{m,(i)}(A_m) \\ \vdots \\ \psi_t^{1,(n+1)}(A_1) \times \psi_t^{2,(n+1)}(A_2) \times \cdots \times \psi_t^{m,(n+1)}(A_m) \end{bmatrix}, \tag{3.120}$$

for $\mathbf{A} = [A_1, \ldots, A_m] \in \mathcal{B}(\mathbb{X}^m)$. The way $\mathbf{P}_t(\mathbf{A})$ is defined makes sense, since the market factors and information processes in each row are all independent from each other.

Proposition 3.3.3. *The random probability measure $\overline{\pi}_t$ can be represented as*

$$\overline{\pi}_t(\mathbf{A}) = \mathbf{P}_t^{\top}(\mathbf{A})\mathbf{I}_t. \tag{3.121}$$

Proof. Proposition 3.3.3 is a special case of Proposition 3.3.1. By the law of total probability, there are $n+1$ orthogonal states at each time t, represented by each row of $\mathbf{P}_t(\mathbf{A})$. The rest of the proof is very similar to that of Proposition 3.3.1. □

In this setting, the SDE of the price process has an elegant representation. First, we note that $\{W_t^{\alpha,(i)}\}_{t\in[0,T)}$ satisfying

$$W_t^{\alpha,(i)} = \widehat{\xi}_t^{\alpha,(i)} + \int_0^t \frac{1}{T-s}\widehat{\xi}_s^{\alpha,(i)}\,\mathrm{d}s - T\widehat{\kappa}^{\alpha,(i)}\int_0^t \frac{1}{T-s}\mathbb{E}^{\mathbb{Q}}\left[M_T^{\alpha}\,\middle|\,\widehat{\xi}_s^{\alpha,(i)}\right]\mathrm{d}s, \tag{3.122}$$

is a $\mathbb{Q}$-Brownian motion by Lévy's characterisation.

Proposition 3.3.4. *The dynamics of the price $\{\overline{X}_t\}$ are governed by the following SDE:*

$$\begin{aligned}\mathrm{d}\overline{X}_t &= r_t\overline{X}_t\,\mathrm{d}t + P_{tT}\sum_{\alpha=1}^{m}\sum_{i=1}^{n+1}\frac{T\widehat{\kappa}^{\alpha,(i)}}{(T-t)}\left(Cov^{\mathbb{Q}}\left[X_T, M_T^{\alpha}\,\middle|\,\widehat{\xi}_t^{1,(i)},\widehat{\xi}_t^{2(i)}\ldots,\widehat{\xi}_t^{m,(i)}\right]\right)\mathrm{d}W_t^{\alpha,(i)}\mathbf{I}_t(i)\\ &\quad + P_{tT}\sum_{i=2}^{n+1}\left(\mathbb{E}^{\mathbb{Q}}\left[X_T\,\middle|\,\widehat{\xi}_t^{1,(i)},\ldots,\widehat{\xi}_t^{m,(i)}\right] - \mathbb{E}^{\mathbb{Q}}\left[X_T\,\middle|\,\widehat{\xi}_t^{1,(i-1)},\ldots,\widehat{\xi}_t^{m,(i-1)}\right]\right)\delta_{\tau_{i-1}}(\mathrm{d}t),\end{aligned}$$

for $0 \le t < T$.

Proof. The statement follows from Proposition 3.3.3, (2.22), (3.118) and Lemma 3.2.4. □

3.4 An Alternative Model for New Information Sources

We briefly present an alternative way of modelling the availability of new information sources at stopping times. The idea is to start with a larger filtration, generated by Brownian information processes, and project it to a smaller one.

Let $\{\overline{\mathcal{Y}}_t\}_{0\le t\le T}$ be the filtration of the subalgebra $\overline{\mathcal{Y}}_t \subset \mathcal{F}_t$ such that

$$\overline{\mathcal{Y}}_t = \sigma(\{\xi_s^i\}_{0\le s\le t}, \{H_{\tau_i}(s)\}_{0\le s\le t} : i = 1,\ldots,n+1), \tag{3.123}$$

where each τ_i is a $\overline{\mathcal{Y}}_t$-stopping time, independent of the Brownian information processes $\{\xi_t^i\}$, $i = 1,\ldots,n+1$. In order to project the σ-algebra $\overline{\mathcal{Y}}_t$ to a smaller σ-algebra, we first define the following information process:

$$\overline{\xi}_t^{i+1} = \xi_t^{i+1}H_{\tau_i}(t), \tag{3.124}$$

for $i = 1, \ldots, n$. Note that the process $\{\overline{\xi}_t^{i+1}\}_{0 \leq t \leq T}$ is zero for $t < \tau_i$ and $\{\xi_t^{i+1}\}$ for $\tau_i \leq t$. In addition, $\{\overline{\xi}_t^{i+1}\}$ is a càdlàg process. We define

$$\mathcal{Y}_t = \mathcal{F}_t^{\xi^1} \bigvee_{i=1}^{n} \sigma(\{\overline{\xi}_s^{i+1}\}_{0 \leq s \leq t}, \{H_{\tau_i}(s)\}_{0 \leq s \leq t}), \tag{3.125}$$

for $0 \leq t \leq T$, so that $\mathcal{Y}_t \subset \overline{\mathcal{Y}}_t$. We assume $\{\mathcal{Y}_t\}$ is the market filtration for X_T. Note that the market filtration is generated by information processes that become alive starting from stopping times. Following similar steps as done in the proof of Proposition 3.2.3,

$$\begin{aligned}\mathbb{Q}(X_T \in A|\mathcal{Y}_t) &= \sum_{i=0}^{n} \mathbb{Q}(X_T \in A|\mathcal{Y}_t, \tau_i \leq t < \tau_{i+1})\mathbb{Q}(\tau_i \leq t < \tau_{i+1}|\mathcal{Y}_t) \\ &= \mathbb{Q}(X_T \in A|\xi_t^1)(1 - H_{\tau_1}(t)) + \mathbb{Q}(X_T \in A|\xi_t^1, \xi_t^2)H_{\tau_1}(t)(1 - H_{\tau_2}(t)) \\ &+ \cdots + \mathbb{Q}(X_T \in A|\xi_t^1, \xi_t^2, \ldots, \xi_t^{n+1})H_{\tau_n}(t),\end{aligned} \tag{3.126}$$

since $\{\xi_t^i\}$ is a strong Markov process, independent from each $\mathcal{Y}_t$-stopping time τ_i. Then

$$\mathbb{Q}(X_T \in A|\mathcal{Y}_t) = \mathbb{Q}(X_T \in A|\mathcal{G}_t), \tag{3.127}$$

where $\mathcal{G}_t$ is defined in (3.58). Therefore, the results in this chapter involving $\{\mathcal{G}_t\}$ follow equivalently if $\{\mathcal{Y}_t\}$ is the market filtration. However, there is a subtle difference between the insights gained from $\{\mathcal{G}_t\}$ and $\{\mathcal{Y}_t\}$. The way $\mathcal{G}_t$ is defined suggests that the filtration $\{\mathcal{G}_t\}$ 'jumps' at stopping times by expanding with new sources of information. The way $\mathcal{Y}_t$ is defined suggests that the information processes that generate $\{\mathcal{Y}_t\}$ jump at stopping times.

The way $\{\mathcal{G}_t\}$ is defined offers flexibility in modelling the stopping times. Following similar steps as shown in the proofs of this chapter, one can verify that by replacing the $\{H_{\tau_i}(t)\}$'s in $\mathcal{G}_t$ with continuous processes independent of the information processes, and defining the stopping times as the first hitting times of these processes, one can still derive dynamics with jumps for conditional densities. This would also enable us to introduce previsible jump times in this framework, which we leave for future research.

3.5 Simulations

We shall provide some simulations of price processes. In the figures below, different colours represent different numbers of information sources available to the market. Hence, each colour is associated with a different volatility process and a Brownian motion governing the price process. One may view each colour as a different economic regime, suggesting that each jump is a regime switch. We shall develop a more general regime-switching framework in Chapter 4.

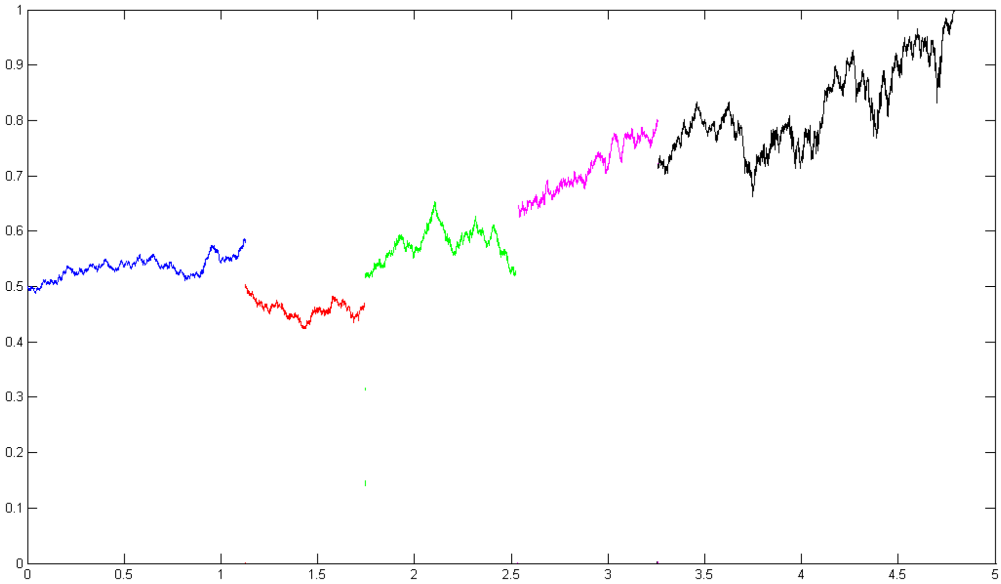

Figure 3.1: A price process with four jumps. Different colours represent different economic regimes: Blue regime, red regime, green regime and etc. The price process is governed by a different Brownian motion and a stochastic volatility process during each regime. Cash flow: $X_T = 1$. Parameters: $T = 5$, $r_t = 0$, $\kappa^i = 1/T$ and $\rho^i = 0.5$. Stopping times are uniformly distributed on $[0, T]$.

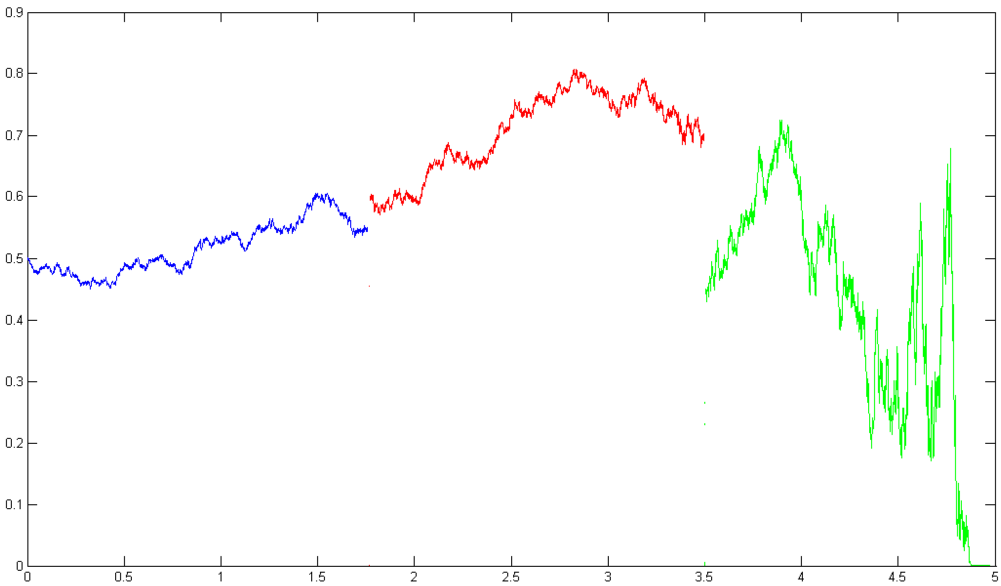

Figure 3.2: A price process with two jumps. There are three different regimes. Cash flow: $X_T = 0$. Parameters: $T = 5$, $r_t = 0$, $\kappa^i = 1/T$ and $\rho^i = 0.5$. Stopping times are uniformly distributed on $[0, T]$.

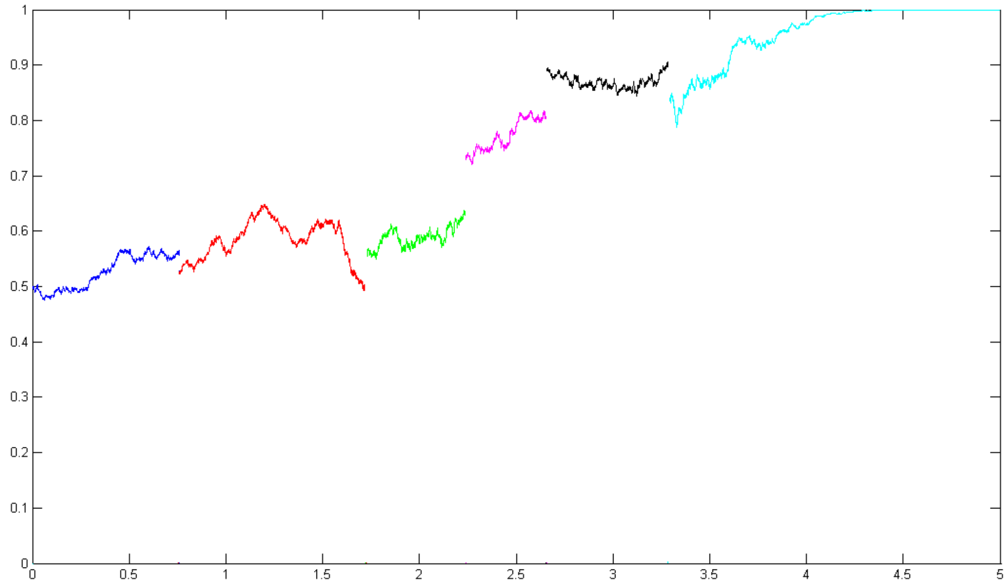

Figure 3.3: A price process with five jumps. There are six different regimes. Cash flow: $X_T = 1$. Parameters: $T = 5$, $r_t = 0$, $\kappa^i = 1/T$ and $\rho^i = 0.5$. Stopping times are uniformly distributed on $[0, T]$.

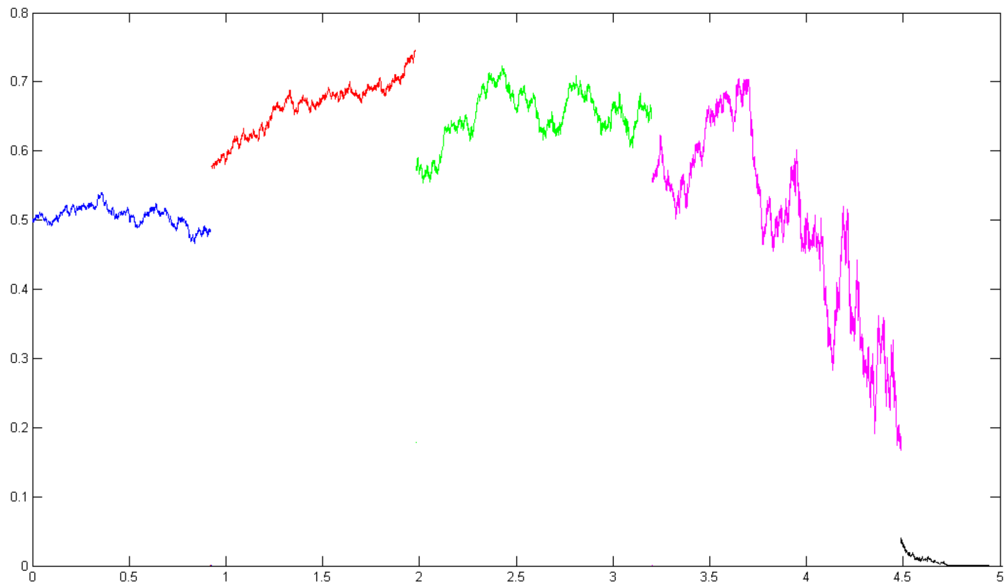

Figure 3.4: A price process with four jumps. There are five different regimes. Cash flow: $X_T = 0$. Parameters: $T = 5$, $r_t = 0$, $\kappa^i = 1/T$ and $\rho^i = 0.5$. Stopping times are uniformly distributed on $[0, T]$.

Chapter 4

Random Deactivation-Reactivation of Information and Regime Switches

The main aim of this chapter is to develop an information-based framework to model regime switches in a given economy. In a way, in Chapter 3, we have already presented an approach for modelling regime switches. More precisely, one may argue that there is a bicausal relationship between appearances of new information sources (or public announcements) and passing from one economic regime to another. From the results presented in Chapter 3, this suggests that every regime switch coincides with a jump in the price process. However, we believe that it is still a rather restrictive viewpoint to expect a price jump at every regime switch. Therefore, we would like to adapt a more elaborate information-based standpoint in our approach. In general terms, we prefer to view regime switches as events that coincide with changes in the sources of information in the market. By changes of information sources, we do not neccesarily mean appearances of new information sources. It may as well be that a source of market information stops flowing for a random period of time before it is active again.

There is a vast stream of mathematical literature on regime switches. For example, the continuous-time version of the stochastic regime-switching model of Hamilton (1989) (also see Hamilton, 1996) implies that asset prices switch between two states where the switches are governed by a Markov point process, and prices are continuous during each state. In a given economic regime, the continuous changes of a price process are governed by a diffusion process with its own volatility. Diffusion processes together with Markov point processes can be analysed under Hidden Markov models, which have a wide spectrum of applications in mathematics (see, for example, Elliott, Aggoun, and Moore, 1997). Cecchetti, Lang and Mark (1990), and Driffill and Sola (1998) model dividends using two-state Markov-switching models to represent the US stock market. Kim, Piger and Startz (2005) discuss the estimation of Markov regime switch models where the switches are endogenous. Driffill, Kenc and Sola (2002) price perpetual American call options when the underlying prices are modelled as regime-switching processes which have stochastic dividends that switch between two economic states characterised by different volatilities. Naik (1993), Bollen, Gray and Whalley (2000) and Chourdakis and Tzavalis (2000) are few other examples of option pricing under regime-switching economies.

In the literature, it is common to start with a model of a price process that has the characteristics to represent regime switches in an economy. This motivates us to ask whether it is possible to reverse this approach. More precisely, we start by specifying the flow of information first, and derive price processes under regime-switching economies, where regime switches are events that coincide with changes in the sources of information in the market. In addition, we would still like our price processes to exhibit similar behaviour as assumed in the current literature. For example, price processes are usually assumed to have different volatilities during different economic regimes. In this respect, the material presented in Chapter 3 can be interpreted with a regime switch perspective, since we have seen that price processes are governed by different diffusion and volatility processes in between stopping times. Then, each time interval between the stopping times (say, between important newscasts) can be interpreted as a different economic state. Our aim is to further develop an information-based framework that allows us to derive a rich class of price dynamics under regime-switching economies, and which potentially sheds light on our understanding of how regime switches may arise in a given economy.

This chapter is organised as follows: Section 2 provides a brief mathematical setting. Section 3 is the pricing of financial derivatives when new sources of information appear at stopping times. This section includes European options and few examples of credit-based products. In Section 4, we stop the flow of information. Section 5 presents the random deactivation-reactivation of information sources. We generalise the deactivation-reactivation setting to the multiple market factor scenerio. In addition, as a special example, we introduce a market filtration where each stopping time induces a switch from one information source to another.

4.1 Mathematical Setting

The mathematical setting in this chapter is almost exactly the same as the one in Chapter 3. To save space, we do not restate everything that we have already stated, and refer the reader to Chapter 3.2.

We let $(\Omega, \mathcal{F}, \mathbb{Q})$ be the probability space equipped with $\{\mathcal{F}_t\}_{0\leq t\leq\infty}$, where $\mathbb{Q}$ is the pricing measure. We assume that all filtrations are right-continuous and complete, and we fix a finite time horizon $[0, T]$. The minor difference with respect to the previous chapter arises in our view of the cash flow X_T. In this chapter, X_T is not neccessarily continuous.

If X_T is discrete, we denote its probability mass function by $p(x_j) > 0$ (i.e., $\mathbb{Q}(X_T = x_j)$) for some index j where $x_j \in \mathbb{X}$, and its conditional mass function at time t by $\phi^i_t(x_j)$:

$$\phi^i_t(x_j) = p(x_j|\mathcal{F}^{\xi^i}_t) = p(x_j|\xi^i_t), \tag{4.1}$$

given that $\{\xi^i_t\}$ is a Brownian information process for $i = 1, 2, \ldots$. There may be countably infinite information processes. We have

$$\phi^i_t(x_j) = \frac{\exp\left[\frac{T}{(T-t)}(\kappa^i x_j \xi^i_t - \frac{1}{2}(\kappa^i x_j)^2 t)\right] p(x_j)}{\sum_{\mathbb{X}} \exp\left[\frac{T}{(T-t)}(\kappa^i x_j \xi^i_t - \frac{1}{2}(\kappa^i x_j)^2 t)\right] p(x_j)}, \tag{4.2}$$

for $i = 1, 2, \ldots$, and $0 \leq t < T$. By the use of Ito's lemma, we can write

$$\mathrm{d}\phi^i_t(x_j) = \sigma^i_t(x_j)\phi^i_t(x_j)\,\mathrm{d}W^i_t, \tag{4.3}$$

for $0 \leq t < T$. The coefficient $\{\sigma^i_t\}_{t\in[0,T)}$ is defined by

$$\sigma^i_t(x_j) = \frac{T\kappa^i(x_j - X^i_t)}{(T-t)}, \tag{4.4}$$

where $X^i_t = \mathbb{E}^{\mathbb{Q}}[X_T \,|\, \xi^i_t]$, and $\{W^i_t\}_{t\in[0,T)}$ which is given by

$$W^i_t = \xi^i_t + \int_0^t \frac{1}{T-s}\xi^i_s\,\mathrm{d}s - T\kappa^i \int_0^t \frac{1}{T-s}X^i_s\,\mathrm{d}s, \tag{4.5}$$

is a $\mathbb{Q}$-Brownian motion.

Note that nothing much changes when X_T is a discrete random variable. In fact, our primary motivation to introduce the discrete scenerio is to be able to let $X_T \in \{0, 1\}$, and price risky-bonds and credit default swaps under regime-switching economies.

4.2 Pricing Derivatives Under Regime Switches

For this section, we assume there is a bicausal relationship between appearances of new information sources and regime switches. There are infinite $\mathcal{F}_t$-stopping times τ_i such that $\tau_i < \tau_{i+1}$. The market receives additional sources of information at these stopping times, where $\{\mathcal{G}_t\}$ as defined in (3.58) is the market filtration (though, as an ∞ union). If X_T is discrete, $\mathcal{G}_t$ is the market information about the discrete cash flow X_T. Recall that at each activation of a new information source, the asset price jumps and it is governed by a different diffusion and volatility process. We assume that the time intervals between the price jumps represent different economic states. This is a common viewpoint in the current literature.

4.2.1 Pricing European Options

We let $\mathbb{X} \subset \mathbb{R}$ such that X_T is continuous and bounded. We are interested in pricing a European call option where the underlying asset has the time-t price $\overline{X}_t$ given by

$$\overline{X}_t = P_{tT}\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \mathcal{G}_t\right] = P_{tT}\int_{\mathbb{X}} x\overline{\pi}_t(\mathrm{d}x), \tag{4.6}$$

for $0 \leq t < T$. The conditional density is $\overline{\psi}_t(x) = q(x|\mathcal{G}_t)$, and the stopping times are independent of the information processes. We want to price a European call option with strike K that is exercisable at a fixed time t:

$$C_0 = P_{0t}\mathbb{E}^{\mathbb{Q}}\left[(\overline{X}_t - K)^+\right], \tag{4.7}$$

for $0 \leq t < T$. Let's now define

$$X_t^{(i)} = P_{tT}\mathbb{E}^{\mathbb{Q}}\left[X_T \,\middle|\, \xi_t^1, ..., \xi_t^i\right] = P_{tT}\int_{\mathbb{X}} x\pi_t^{(i)}(\mathrm{d}x), \tag{4.8}$$

for $0 \leq t < T$, where the conditional density is $\psi_t^{(i)}(x) = q(x|\xi_t^1, ..., \xi_t^i)$ for $i = 1, 2, ...$ We also define a measurable process $Y = \{Y_t\}_{t\in[0,T]} = \sup\{y : \tau_y \leq t\}$, independent of the information processes, with state-space $(\mathbb{Y} = \{0, 1, 2, ...\}, \mathcal{B}(\mathbb{Y}))$. The process Y counts the number of stopping times. For the remaining part of this section, we set $\tau_0 = 0$.

Lemma 4.2.1. *The value of C_0 is*

$$C_0 = \sum_{i=1}^{\infty} \mathbb{Q}(Y_t = i-1)C_0^{(i)}, \tag{4.9}$$

where $C_0^{(i)}$ is given by

$$C_0^{(i)} = P_{0t}\mathbb{E}^{\mathbb{Q}}\left[(X_t^{(i)} - K)^+\right]. \tag{4.10}$$

Proof. The random variable Y_t is the number of jumps until t. Using law of total expectation,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[(\overline{X}_t - K)^+\right] &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[(\overline{X}_t - K)^+ \,|\, Y_t\right]\right] \\
&= \sum_{i=1}^{\infty} \mathbb{E}^{\mathbb{Q}}\left[(\overline{X}_t - K)^+ \,|\, Y_t = i-1\right]\mathbb{Q}(Y_t = i-1) \\
&= \sum_{i=1}^{\infty} \mathbb{E}^{\mathbb{Q}}\left[(X_t^{(i)} - K)^+\right]\mathbb{Q}(Y_t = i-1).
\end{aligned} \tag{4.11}$$

The last equality follows since $\{\xi_t^i\}$ is Markovian and independent of the process Y. □

Lemma 4.2.1 states that the option price is the weighted sum of different option prices induced by different number of information processes, where the nth weight equals the probability of n information processes driving the market at maturity.

Due to the appearance of new information sources, the underlying process $\{\overline{X}_t\}$ has jump-diffusion dynamics:

$$
\begin{aligned}
\overline{X}_t = P_{0T}\mathbb{E}^{\mathbb{Q}}\left[X_T\right] + \int_0^t r_s \overline{X}_s \,\mathrm{d}s + \sum_{i=1}^{\infty}\int_0^t P_{sT}\frac{T\widehat{\kappa}^{(i)}}{(T-s)}\left(\mathrm{Var}^{\mathbb{Q}}\left[X_T \,\middle|\, \widehat{\xi}_s^{(i)}\right]\right)\mathrm{d}W_s^{(i)}\mathbf{I}_s(i) \\
+ \sum_{i=2}^{\infty}\int_0^t P_{sT}\left(\mathbb{E}^{\mathbb{Q}}\left[X_T \,\middle|\, \widehat{\xi}_s^{(i)}\right] - \mathbb{E}^{\mathbb{Q}}\left[X_T \,\middle|\, \widehat{\xi}_s^{(i-1)}\right]\right)\delta_{\tau_{i-1}}(\mathrm{d}s), \qquad (4.12)
\end{aligned}
$$

where $\mathbf{I}_s$ has infinite rows. Note that if we fix $r_s = 0$ for every $s \in [0, T]$, $P_{0T} = P_{sT} = 1$. Then, one may regard (4.12) as a martingale representation of $\{\overline{X}_t\}$.

There are various ways to find option prices when the underlying asset price has jump-diffusion dynamics. For example, Cont and Tankov (2004) discuss how an option value can be calculated by solving a partial integro-differential equation (PIDE), when the underlying price has jumps. However, generally speaking, it is difficult to solve PIDEs, and one may need to use viscosity solutions introduced by Crandall and Lions (1983).

We provide an explicit price for C_0 by using Lemma 4.2.1 and by introducing a sequence of measure changes. First, we let $\widehat{z}_t^{(i)} = \widehat{\varsigma}^{(i)}\sqrt{T/t(T-t)}$, where $\widehat{\varsigma}^{(i)}$ solves

$$
\int_{\mathbb{X}}(P_{tT}x - K)\exp\left[\frac{T}{(T-t)}\left(\widehat{\kappa}^{(i)}x\widehat{\varsigma}^{(i)} - \frac{1}{2}(\widehat{\kappa}^{(i)}x)^2 t\right)\right]q(x)\,\mathrm{d}x = 0. \qquad (4.13)
$$

We are now in the position to provide the price of C_0:

Proposition 4.2.2. *The price of the European call option C_0 is*

$$
\begin{aligned}
C_0 = P_{0t}\sum_{i=1}^{\infty}\mathbb{Q}(Y_t = i-1)\int_{\mathbb{X}} xq(x)\mathcal{N}\left(-\widehat{z}_t^{(i)} + \widehat{\kappa}^{(i)}x\sqrt{\frac{tT}{T-t}}\right)\mathrm{d}x \\
- P_{0t}\sum_{i=1}^{\infty}\mathbb{Q}(Y_t = i-1)K\int_{\mathbb{X}} q(x)\mathcal{N}\left(-\widehat{z}_t^{(i)} + \widehat{\kappa}^{(i)}x\sqrt{\frac{tT}{T-t}}\right)\mathrm{d}x, \qquad (4.14)
\end{aligned}
$$

where $\mathcal{N}(.)$ is the standard normal distribution function.

Proof. The functional form for the call price $C_0^{(1)}$ is given in (2.15), where $\xi_t^1 = \widehat{\xi}_t^{(1)}$. The call prices $C_0^{(i)}$ for $i = 2, ...$ have the same functional form, only with modified parameters. More specifically, from (3.69):

$$
\psi_t^{(i)}(x) = \frac{\exp\left[\frac{T}{(T-t)}\left(\widehat{\kappa}^{(i)}x\widehat{\xi}_t^{(i)} - \frac{1}{2}(\widehat{\kappa}^{(i)}x)^2 t\right)\right]q(x)}{\int_{\mathbb{X}}\exp\left[\frac{T}{(T-t)}\left(\widehat{\kappa}^{(i)}x\widehat{\xi}_t^{(i)} - \frac{1}{2}(\widehat{\kappa}^{(i)}x)^2 t\right)\right]q(x)\,\mathrm{d}x}, \qquad (4.15)
$$

for $0 \leq t < T$. Following similar steps as done in Brody *et al.* (2008a), we define

$$\chi_t^{(i)}(x) = \exp\left[\frac{T}{(T-t)}\left(\widehat{\kappa}^{(i)} x \widehat{\xi}_t^{(i)} - \frac{1}{2}(\widehat{\kappa}^{(i)} x)^2 t\right)\right], \tag{4.16}$$

and write (4.15) as follows:

$$\psi_t^{(i)}(x) = \frac{\chi_t^{(i)}(x) q(x)}{\int_{\mathbb{X}} \chi_t^{(i)}(x) q(x)\, \mathrm{d}x}, \tag{4.17}$$

for $0 \leq t < T$. From (4.10), the value of the option induced by i information processes is

$$C_0^{(i)} = P_{0t}\mathbb{E}^{\mathbb{Q}}\left[\left(P_{tT}\int_{\mathbb{X}} x\psi_t^{(i)}(x)\, \mathrm{d}x - K\right)^+\right]. \tag{4.18}$$

Substituting (4.17) into (4.18), the value of the option is

$$C_0^{(i)} = P_{0t}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\Phi_t^{(i)}}\left(\int_{\mathbb{X}}(P_{tT}x - K)\chi_t^{(i)}(x) q(x)\, \mathrm{d}x\right)^+\right], \tag{4.19}$$

where

$$\Phi_t^{(i)} = \int_{\mathbb{X}} \chi_t^{(i)}(x) q(x)\, \mathrm{d}x. \tag{4.20}$$

Brody *et al.* (2008a) prove that $1/\Phi_t^{(1)}$ for $0 \leq t < T$ can be used as a Radon-Nikodym derivative to introduce a measure $\mathbb{B}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$. Similarly, we can define an infinite sequence $\{1/\Phi_t^{(i)}\}_{i=1}^{\infty}$ and introduce the measure $\mathbb{B}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ as

$$\left\{\left.\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{Q}}\right|_{\sigma(\widehat{\xi}_t^{(i)})}\right\}_{i=1}^{\infty} = \left\{\frac{1}{\Phi_t^{(i)}}\right\}_{i=1}^{\infty}, \tag{4.21}$$

which is a sequence of Radon-Nikodym derivatives. This follows since $\{1/\Phi_t^{(i)}\}$ is a $\mathbb{Q}$-martingale: $\mathbb{E}^{\mathbb{Q}}\left[1/\Phi_t^{(i)} | \widehat{\xi}_s^{(i)}\right] = 1/\Phi_s^{(i)}$ for $s < t$, and also $\Phi_0^{(i)} = 1$ and $\Phi_t^{(i)} > 0$. In particular,

$$(\Phi_t^{(i)})^{-1} = \exp\left(-\int_0^t \frac{T\widehat{\kappa}^{(i)}}{T-s}\mathbb{E}^{\mathbb{Q}}\left[X_T|\widehat{\xi}_s^{(i)}\right]\, \mathrm{d}W_s^{(i)} - \frac{1}{2}\int_0^t \frac{(T\widehat{\kappa}^{(i)})^2}{(T-s)^2}\mathbb{E}^{\mathbb{Q}}\left[X_T|\widehat{\xi}_s^{(i)}\right]\, \mathrm{d}s\right), \tag{4.22}$$

and the Novikov's condition

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\frac{1}{2}\int_0^t \frac{(T\widehat{\kappa}^{(i)})^2}{(T-s)^2}\mathbb{E}^{\mathbb{Q}}\left[X_T|\widehat{\xi}_s^{(i)}\right]\, \mathrm{d}s\right)\right] < \infty, \tag{4.23}$$

is satisfied. The martingale property follows.

Under the measure $\mathbb{B}$, the random variable $\widehat{\xi}_t^{(i)}$ is Gaussian with mean 0 and variance $t(T-t)/T$ for $0 \leq t < T$. This follows directly from Brody *et al.* (2008a) and (3.64)-(3.66). Then, we can define an infinite sequence of call option prices:

$$\begin{aligned}
\{C_0^{(i)}\}_{i=1}^{\infty} &= \left\{ P_{0t}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\Phi_t^{(i)}} \left(\int_{\mathbb{X}} (P_{tT}x - K)\chi_t^{(i)}(x)q(x)\,\mathrm{d}x \right)^{+} \right] \right\}_{i=1}^{\infty} \\
&= \left\{ P_{0t}\mathbb{E}^{\mathbb{B}}\left[\left(\int_{\mathbb{X}} (P_{tT}x - K)\chi_t^{(i)}(x)q(x)\,\mathrm{d}x \right)^{+} \right] \right\}_{i=1}^{\infty}. \qquad (4.24)
\end{aligned}$$

Computing the constant critical value which we denote by $\widehat{\varsigma}^{(i)}$ that solves (4.13), the expectation of each term in the sequence (4.24) is

$$\begin{aligned}
C_0^{(i)} &= P_{0t}\int_{\mathbb{X}} xq(x)\mathcal{N}\left(-\widehat{z}_t^{(i)} + \widehat{\kappa}^{(i)}x\sqrt{\frac{tT}{T-t}} \right)\mathrm{d}x \\
&\quad - P_{0t}K\int_{\mathbb{X}} q(x)\mathcal{N}\left(-\widehat{z}_t^{(i)} + \widehat{\kappa}^{(i)}x\sqrt{\frac{tT}{T-t}} \right)\mathrm{d}x, \qquad (4.25)
\end{aligned}$$

where $\mathcal{N}(.)$ is the standard normal distribution function. From Lemma 4.2.1, the price of the European call option is

$$C_0 = P_{0t}\sum_{i=1}^{\infty}\mathbb{Q}(Y_t = i-1)\mathbb{E}^{\mathbb{B}}\left[\left(\int_{\mathbb{X}} (P_{tT}x - K)\chi_t^{(i)}(x)q(x)\,\mathrm{d}x \right)^{+} \right], \qquad (4.26)$$

which completes the proof. □

Proposition 4.2.2 shows that the option price can be represented as the weighted sum of the Black-Scholes-Merton prices induced by different number of information processes. We have not yet specified any distribution for the stopping times, $\mathbb{Q}(Y_t = i-1)$ is arbitrary at this point. Any reasonable distribution can be used to generate a large class of call prices.

Corollary 4.2.3. *Let τ_i be a jump time of an independent Poisson process with intensity λ. Then,*

$$\begin{aligned}
C_0 &= \sum_{i=1}^{\infty}\frac{e^{-\lambda t}(\lambda t)^{i-1}}{(i-1)!}C_0^{(i)} \\
&= P_{0t}\sum_{i=1}^{\infty}\frac{e^{-\lambda t}(\lambda t)^{i-1}}{(i-1)!}\int_{\mathbb{X}} xq(x)\mathcal{N}\left(-\widehat{z}^{(i)} + \widehat{\kappa}^{(i)}x\sqrt{\frac{tT}{T-t}} \right)\mathrm{d}x \\
&\quad - P_{0t}\sum_{i=1}^{\infty}\frac{e^{-\lambda t}(\lambda t)^{i-1}}{(i-1)!}K\int_{\mathbb{X}} q(x)\mathcal{N}\left(-\widehat{z}^{(i)} + \widehat{\kappa}^{(i)}x\sqrt{\frac{tT}{T-t}} \right)\mathrm{d}x. \qquad (4.27)
\end{aligned}$$

Proof. The statement follows from Proposition 4.2.2. □

The option price in (4.27) is very similar to that of Merton (1976). We note that Merton (1976) assumes the price process to have jump-diffusion dynamics, where the jumps are that of a Poisson process. In our framework, we derive the price dynamics and do not need to specify the distribution of the jumps from the start.

Remark 4.2.4. *The sequence of measure changes as shown in (4.21) may be viewed as a sequence of different regimes represented in terms of the Radon-Nikodym derivatives.*

4.2.2 Pricing Credit-Risky Bonds and Credit Default Swaps

We first price a credit-risky bond without coupons. Let $X_T \in \{0,1\}$ be the payoff of a risky bond with maturity T. More precisely, let $X_T = 1 - H_{\tau^*}(T)$ where τ^* is the possible default time of the bond. Hence, $X_T = 1$ if $T < \tau^*$ and $X_T = 0$ if $\tau^* \leq T$. Define

$$\overline{\phi}_t(x_j) = p(x_j|\mathcal{G}_t), \tag{4.28}$$

for $0 \leq t < T$ and $x_j \in \{0,1\}$. Also let $\mathbf{R}_t$ be the vector of probability mass functions $\phi_t^{(i)}$, for $i = 1, 2, \ldots$, such that

$$\mathbf{R}_t(x_j) = [\phi_t^{(1)}(x_j), \ldots, \phi_t^{(i)}(x_j), \ldots]^\top, \tag{4.29}$$

where, from (4.2), we can write

$$\phi_t^{(i)}(1) = \frac{p(1)\exp\left[\frac{T}{(T-t)}\left(\widehat{\kappa}^{(i)}\widehat{\xi}_t^{(i)} - \frac{1}{2}(\widehat{\kappa}^{(i)})^2 t\right)\right]}{p(0) + p(1)\exp\left[\frac{T}{(T-t)}\left(\widehat{\kappa}^{(i)}\widehat{\xi}_t^{(i)} - \frac{1}{2}(\widehat{\kappa}^{(i)})^2 t\right)\right]}, \tag{4.30}$$

$$\phi_t^{(i)}(0) = p(0)\left(p(0) + p(1)\exp\left[\frac{T}{(T-t)}\left(\widehat{\kappa}^{(i)}\widehat{\xi}_t^{(i)} - \frac{1}{2}(\widehat{\kappa}^{(i)})^2 t\right)\right]\right)^{-1}. \tag{4.31}$$

Proposition 4.2.5. *The price of the bond is*

$$\overline{X}_t = P_{tT}\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \mathcal{G}_t\right] = P_{tT}\mathbf{R}_t^\top(1)\mathbf{I}_t, \qquad 0 \leq t < T. \tag{4.32}$$

Proof. Similar to the proof of Proposition 3.2.3, one can show that $\overline{\phi}_t = \mathbf{R}_t^\top \mathbf{I}_t$. Equation (4.32) follows since $X_T \in \{0,1\}$. □

From (4.3) and (4.32), one can verify that the bond price $\{\overline{X}_t\}$ follows jump-diffusion dynamics. Note that $\mathbb{Q}(T < \tau^*|\mathcal{G}_t) = 1 - \mathbb{E}^{\mathbb{Q}}\left[H_{\tau^*}(T)\,|\,\mathcal{G}_t\right] = \mathbf{R}_t^\top(1)\mathbf{I}_t$ is the conditional survival probability of the bond. Hence, at each regime switch, the market assigns a new survival probability to the risky bond in a discontinuous way.

We now assume that the market receives partial information about future coupons and the principal. For illustration purposes, we consider the case of a risky bond that has two payments. We represent the coupon payment as c, and the principal as p. We let R_1 and R_2 denote the effective recovery rates on the first and second payments, respectively. Following Macrina (2006), we denote the payments by

$$C_{T_1} = cX_{T_1} + R_1(c+p)(1 - X_{T_1}), \tag{4.33}$$

$$C_{T_2} = (c+p)X_{T_1}X_{T_2} + R_2(c+p)X_{T_1}(1 - X_{T_2}), \tag{4.34}$$

at times T_1 and T_2, respectively, where we set $T = T_2$, and $T_1 < T_2$. We assume that $X_{T_1} \in \{0,1\}$ and $X_{T_2} \in \{0,1\}$ are independent random variables, and let

$$\xi_t^{1,i} = \kappa^{1,i} X_{T_1} t + B_{tT_1}^{1,i} \tag{4.35}$$

$$\xi_t^{2,i} = \kappa^{2,i} X_{T_2} t + B_{tT_2}^{2,i}, \tag{4.36}$$

where the Brownian bridges $\{B_{tT_1}^{1,i}\}$ and $\{B_{tT_2}^{2,i}\}$ are independent of each other and of X_{T_1} and X_{T_2}. In Macrina (2006), the market filtration is generated by $\{\xi_t^{1,1}\}$ and $\{\xi_t^{2,1}\}$. Then, the time-t price of a risky bond that pays C_{T_1} and C_{T_2}, which we denote by V_t, is given by

$$\begin{aligned} V_t = P_{tT_2} &\left((c+p)\mathbb{E}^{\mathbb{Q}}[X_{T_1} \,|\, \xi_t^{1,1}]\mathbb{E}^{\mathbb{Q}}[X_{T_2} \,|\, \xi_t^{2,1}] + R_2(c+p)\mathbb{E}^{\mathbb{Q}}[X_{T_1} \,|\, \xi_t^{1,1}]\mathbb{E}^{\mathbb{Q}}[(1 - X_{T_2}) \,|\, \xi_t^{2,1}]\right) \\ &+ P_{tT_1}\left(c\mathbb{E}^{\mathbb{Q}}[X_{T_1} \,|\, \xi_t^{1,1}] + R_1(c+p)\mathbb{E}^{\mathbb{Q}}[(1 - X_{T_1}) \,|\, \xi_t^{1,1}]\right), \end{aligned} \tag{4.37}$$

for $t < T_1$. We generalise this statement to regime-switching economies. We define $\{\mathcal{V}_t^{\xi^{\alpha,i}}\}$ as the filtration of the subalgebra $\mathcal{V}_t^{\xi^{\alpha,i}} \subset \mathcal{F}_t$ such that

$$\mathcal{V}_t^{\xi^{\alpha,i+1}} = \begin{cases} \sigma(\{H_{\tau_i^\alpha}(s)\}_{0 \le s \le t}) & \tau_i^\alpha > t, \\ \sigma(\{H_{\tau_i^\alpha}(s)\}_{0 \le s \le t}, \{\xi_s^{\alpha,i+1}\}_{\tau_i^\alpha \le s \le t}) & \tau_i^\alpha \le t, \end{cases} \tag{4.38}$$

for $0 \le t \le T_\alpha$, $i = 1, 2, \ldots$, and $\alpha = 1, 2$. We assume that $\{\tau_i^1\}$ and $\{\tau_i^2\}$ are independent of each other, and independent of $\{\xi_t^{1,i}\}$ and $\{\xi_t^{2,i}\}$. We also let

$$\mathbf{I}_t^1 = \begin{bmatrix} 1 - H_{\tau_1}^1(t) \\ H_{\tau_1}^1(t)(1 - H_{\tau_2}^1(t)) \\ \vdots \\ H_{\tau_n}^1(t)(1 - H_{\tau_{n+1}}^1(t)) \\ \vdots \end{bmatrix} \quad \text{and} \quad \mathbf{I}_t^2 = \begin{bmatrix} 1 - H_{\tau_1}^2(t) \\ H_{\tau_1}^2(t)(1 - H_{\tau_2}^2(t)) \\ \vdots \\ H_{\tau_n}^2(t)(1 - H_{\tau_{n+1}}^2(t)) \\ \vdots \end{bmatrix}. \tag{4.39}$$

We further define the filtration $\{\mathcal{Y}_t\}_{0 \le t \le T}$ by

$$\mathcal{Y}_t = \bigvee_{\alpha=1}^{2} \mathcal{F}_t^{\xi^{\alpha,1}} \bigvee_{i=1}^{\infty} \mathcal{V}_t^{\xi^{\alpha,i+1}}. \tag{4.40}$$

We assume $\{\mathcal{Y}_t\}$ is the market filtration. We define the time-t price of a risky bond that pays C_{T_1} and C_{T_2} by

$$\overline{V}_t = P_{tT_1}\mathbb{E}^{\mathbb{Q}}[C_{T_1}|\mathcal{Y}_t] + P_{tT_2}\mathbb{E}^{\mathbb{Q}}[C_{T_2}|\mathcal{Y}_t], \tag{4.41}$$

for $t < T_1$. At each regime switch, the price of the risky bond jumps:

Proposition 4.2.6. *The price $\overline{V}_t$ is*

$$\begin{aligned}\overline{V}_t = P_{tT_1}\sum_{i=1}^{\infty}&\left(c\mathbb{E}^{\mathbb{Q}}\left[X_{T_1}|\widehat{\xi}_t^{1,(i)}\right] + R_1(c+p)\mathbb{E}^{\mathbb{Q}}\left[(1-X_{T_1})|\widehat{\xi}_t^{1,(i)}\right]\right)\mathbf{I}_t^1(i)\\ + P_{tT_2}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}&\left(R_2(c+p)\mathbb{E}^{\mathbb{Q}}\left[X_{T_1}|\widehat{\xi}_t^{1,(i)}\right]\mathbb{E}^{\mathbb{Q}}\left[(1-X_{T_2})|\widehat{\xi}_t^{2,(j)}\right]\right)\mathbf{I}_t^1(i)\mathbf{I}_t^2(j)\\ + P_{tT_2}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}&\left((c+p)\mathbb{E}^{\mathbb{Q}}\left[X_{T_1}|\widehat{\xi}_t^{1,(i)}\right]\mathbb{E}^{\mathbb{Q}}\left[X_{T_2}|\widehat{\xi}_t^{2,(j)}\right]\right)\mathbf{I}_t^1(i)\mathbf{I}_t^2(j).\end{aligned} \tag{4.42}$$

Proof. Using the independence properties imposed above, Proposition 4.2.6 follows from Proposition 3.3.1, (4.37) and (4.41). □

Information-based approach provides a tractable framework in pricing swap-like instruments. As done in Macrina (2006), we consider a simple credit default swap (CDS) written on the risky bond we discussed above. There is a series of premiums denoted by v, payed by the protection buyer to the protection seller. For simplification, we assume that premiums are payed at coupon dates. The buyer continues paying unless the reference bond defaults, at which the protection seller makes a payment of h.

Given that the market filtration is generated by $\{\xi_t^{1,1}\}$ and $\{\xi_t^{2,1}\}$, Macrina (2006) shows that the time-t price of this CDS, which we denote by CDS_t, can be written as

$$\begin{aligned}\text{CDS}_t = {}&[(v+h)P_{tT_1} - hP_{tT_2}]\mathbb{E}^{\mathbb{Q}}[X_{T_1}\,|\,\xi_t^{1,1}] - hP_{tT_1}\\ &+ (v+h)P_{tT_2}\mathbb{E}^{\mathbb{Q}}[X_{T_1}\,|\,\xi_t^{1,1}]\mathbb{E}^{\mathbb{Q}}[X_{T_2}\,|\,\xi_t^{2,1}],\end{aligned} \tag{4.43}$$

for $t < T_1$. We aim to price this CDS under a regime-switching economy. From (4.43), we define the CDS price as follows:

$$\begin{aligned}\overline{\text{CDS}}_t = {}&[(v+h)P_{tT_1} - hP_{tT_2}]\mathbb{E}^{\mathbb{Q}}[X_{T_1}|\mathcal{Y}_t] - hP_{tT_1}\\ &+ (v+h)P_{tT_2}\mathbb{E}^{\mathbb{Q}}[X_{T_1}|\mathcal{Y}_t]\mathbb{E}^{\mathbb{Q}}[X_{T_2}|\mathcal{Y}_t],\end{aligned} \tag{4.44}$$

for $t < T_1$.

Similar to Proposition 4.2.6, the next proposition shows that the CDS price jumps at each regime switch:

Proposition 4.2.7. *The price* $\overline{CDS}_t$ *is*

$$\overline{CDS}_t = [(v+h)P_{tT_1} - hP_{tT_2}] \sum_{i=1}^{\infty} \mathbb{E}^{\mathbb{Q}} \left[X_{T_1} | \widehat{\xi}_t^{1,(i)} \right] \mathbf{I}_t^1(i) - hP_{tT_1}$$
$$+ (v+h)P_{tT_2} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \mathbb{E}^{\mathbb{Q}} \left[X_{T_1} | \widehat{\xi}_t^{1,(i)} \right] \mathbb{E}^{\mathbb{Q}} \left[X_{T_2} | \widehat{\xi}_t^{2,(j)} \right] \mathbf{I}_t^1(i) \mathbf{I}_t^2(j). \tag{4.45}$$

Proof. Using the independence properties imposed above, Proposition 4.2.7 follows from Proposition 3.3.1, (4.43) and (4.44). □

4.3 Randomly Stopping the Information Flow

This section considers the possibility when market information suddenly ceases to flow. For demonstration purposes, we consider a single information source. This particular approach is later used to model deactivation-reactivation of information sources, which allows us to generalise our view towards regime switches. We only discuss the case when X_T is continuous. It is straightforward to adapt the discrete scenerio to all the results that follow.

It is rather optimistic to assume that the market has non-interrupted access to every source of information. A particular information source may suddenly stop flowing, not being able to provide updates about X_T for a period of time. This may be understood as a possible information blockage in the market. In order to represent this scenerio, we wish the information process to stop at some measurable random instance. More formally, we model a stopped filtration. For this section, the Brownian information process $\{\xi_t^1\}$ is denoted by $\{\xi_t\}$, τ_1 is τ, and ψ_t^1 is ψ_t.

We define $\{\mathcal{F}_t^\eta\}$ as the filtration of the subalgebra $\mathcal{F}_t^\eta \subset \mathcal{F}_t$ such that

$$\mathcal{F}_t^\eta = \sigma(\{H_\tau(s)\}_{0 \le s \le t}, \{\eta_s\}_{0 \le s \le t}), \tag{4.46}$$

for $0 \le t \le T$, where the information process $\{\eta_t\}_{0 \le t \le T}$ is defined by

$$\eta_t = \xi_{t \wedge \tau}, \tag{4.47}$$

given that τ is a $\mathcal{F}_t^\eta$-stopping time independent of $\{\xi_t\}$, and where $t \wedge \tau = \min(t, \tau)$.

We define a $\mathcal{Q}(\mathbb{X})$-valued process $\{\Upsilon_t\}_{t \in [0,T]}$ by

$$\Upsilon_t(A) = \mathbb{Q}(X_T \in A | \mathcal{F}_t^\eta), \tag{4.48}$$

for $A \in \mathcal{B}(\mathbb{X})$. The process $\{\Upsilon_t\}$ may stop. This means that the market has the possibility of not being able to update the price of X_T.

Proposition 4.3.1. *The random probability measure* Υ_t *can be represented as*

$$\Upsilon_t(A) = \pi_t(A)\,(1 - H_\tau(t)) + \int_0^t \pi_u(A)\,\mathrm{d}H_\tau(u) = \pi_{t\wedge\tau}, \tag{4.49}$$

and the asset price $\overline{X}_t = P_{tT}\mathbb{E}^{\mathbb{Q}}[X_T \,|\, \mathcal{F}_t^\eta]$ *is*

$$\begin{aligned}\overline{X}_t = {} & P_{tT}\left(\frac{\int_{\mathbb{X}} x\exp\left[\frac{T}{(T-t)}(\kappa x\xi_t - \frac{1}{2}(\kappa x)^2 t)\right] q(x)\,\mathrm{d}x}{\int_{\mathbb{X}}\exp\left[\frac{T}{(T-t)}(\kappa x\xi_t - \frac{1}{2}(\kappa x)^2 t)\right] q(x)\,\mathrm{d}x}\right)(1 - H_\tau(t)) \\ & + P_{tT}\int_0^t \left(\frac{\int_{\mathbb{X}} x\exp\left[\frac{T}{(T-u)}(\kappa x\xi_u - \frac{1}{2}(\kappa x)^2 u)\right] q(x)\,\mathrm{d}x}{\int_{\mathbb{X}}\exp\left[\frac{T}{(T-u)}(\kappa x\xi_u - \frac{1}{2}(\kappa x)^2 u)\right] q(x)\,\mathrm{d}x}\right)\mathrm{d}H_\tau(u).\end{aligned} \tag{4.50}$$

Proof. We can project $\Upsilon_t(A)$ onto the two orthogonal subspaces $\{t < \tau\}$ and $\{\tau \leq t\}$. Then, denoting $\nu_\tau(.|\mathcal{F}_t^\eta)$ as the conditional distribution of τ with respect to $\mathcal{F}_t^\eta$,

$$\begin{aligned}\Upsilon_t(A) = {} & \int_t^M \mathbb{Q}(X_T \in A|\sigma(\{\xi_s\}_{0\leq s\leq t})\bigvee\sigma(\{H_\tau(s)\}_{0\leq s\leq t}, \tau = u)\nu_\tau(\mathrm{d}u|\mathcal{F}_t^\eta) \\ & + \int_0^t \mathbb{Q}(X_T \in A|\sigma(\{\xi_s\}_{0\leq s\leq u})\bigvee\sigma(\{H_\tau(s)\}_{0\leq s\leq t}, \tau = u)\nu_\tau(\mathrm{d}u|\mathcal{F}_t^\eta).\end{aligned} \tag{4.51}$$

It follows from the Markovian property of $\{\xi_t\}$ and the independence of τ that

$$\begin{aligned}\Upsilon_t(A) &= \mathbb{Q}(X_T \in A|\xi_t)\left(\int_t^M \nu_\tau(\mathrm{d}u|\mathcal{F}_t^\eta)\right) + \int_0^t \mathbb{Q}(X_T \in A|\xi_u)\nu_\tau(\mathrm{d}u|\mathcal{F}_t^\eta) \\ &= \pi_t(A)\left(\int_t^M \nu_\tau(\mathrm{d}u|\mathcal{F}_t^\eta)\right) + \int_0^t \pi_u(A)\nu_\tau(\mathrm{d}u|\mathcal{F}_t^\eta).\end{aligned} \tag{4.52}$$

Since τ is an $\mathcal{F}_t^\eta$-stopping time, we have

$$\int_0^t \pi_u(A)\nu_\tau(\mathrm{d}u|\mathcal{F}_t^\eta) = \int_0^t \pi_u(A)\delta_\tau(\mathrm{d}u) = \int_0^t \pi_u(A)\,\mathrm{d}H_\tau(u). \tag{4.53}$$

The first integral in (4.52) equals $1 - H_\tau(t)$. Since $\{\xi_t\}$ is strong Markov, for $\tau < T$,

$$\mathbb{Q}(X_T \in A|\mathcal{F}_\tau^\eta) = \mathbb{Q}(X_T \in A|\xi_\tau) = \pi_\tau(A). \tag{4.54}$$

Hence, (4.54) agrees with (4.52). Equation (4.49) follows. Having $\overline{X}_t = P_{tT}\mathbb{E}^{\mathbb{Q}}[X_T \,|\, \mathcal{F}_t^\eta]$,

$$\overline{X}_t = P_{tT}\int_{\mathbb{X}} x\Upsilon_t(\mathrm{d}x), \tag{4.55}$$

and (4.50) follows from the Bayes formula, Fubini's theorem and the independence of τ. □

Note that the integral that appears in (4.53) can also be written as

$$\int_0^t \pi_u(A)\,\mathrm{d}H_\tau(u) = \mathbb{Q}(X_T \in A|\xi_\tau)H_\tau(t) = \pi_\tau(A)H_\tau(t). \tag{4.56}$$

Proposition 4.3.2. *The dynamics of* $\{\psi_{t\wedge\tau}\}$ *are governed by the following SDE:*

$$\mathrm{d}\psi_{t\wedge\tau}(x) = (1 - H_\tau(t))\,\sigma_t(x)\psi_t(x)\,\mathrm{d}W_t. \tag{4.57}$$

Proof. The statement follows from (4.49). □

Proposition 4.3.3. *The dynamics of* $\{\eta_t\}$ *are governed by the following SDE:*

$$\mathrm{d}\eta_t = (1 - H_\tau(t))\left(\left(\frac{T\kappa X_t - \xi_t}{T - t}\right)\mathrm{d}t + \mathrm{d}W_t\right), \tag{4.58}$$

for $0 \le t < T$*, where* $X_t = \mathbb{E}^{\mathbb{Q}}[X_T \,|\, \xi_t]$.

Proof. Since τ is an $\mathcal{F}_t^\eta$-stopping time, the following representation of η_t can be written:

$$\eta_t = \xi_t\,(1 - H_\tau(t)) + \xi_\tau H_\tau(t), \tag{4.59}$$

for $0 \le t \le T$. Also, since $\mathrm{d}\xi_\tau = 0$, it follows that

$$\mathrm{d}\eta_t = \mathrm{d}\xi_t\,(1 - H_\tau(t)) - \xi_t\delta_\tau(\mathrm{d}t) + \xi_\tau\delta_\tau(\mathrm{d}t). \tag{4.60}$$

Note that the term $(\xi_\tau - \xi_t)\delta_\tau(\mathrm{d}t) = 0$. Then the statement follows from (3.19). □

4.4 Deactivation-Reactivation of Information Sources

We combine the models for appearances of new information sources and information blockages. This allows us to view regime switches not only as events coinciding with activation of new information sources, but also as events coinciding with stopped information. In other words, the sources of information may be switched on or switched off.

4.4.1 One Source of Information

We start with the case where there is a single information process. The source of information may be deactivated for a random period of time, and may suddenly reactivate at another random time. That is, the information flow may dry up for a period of time, and then may start providing updates again. For parsimony, we fix $n \in \mathbb{N}_+$ and consider an n-sequence of $\mathcal{F}_t$-stopping times $\{\tau_i\}_{i=1}^n$.

We shall denote the set of odd integers by O and define the filtration $\{\mathcal{R}_t\}$ by

$$\mathcal{R}_t = \mathcal{F}^{\eta}_t \bigvee_{i=1, i\in O}^{n} \begin{cases} \sigma(\{H_{\tau_{i+1}}(s)\}_{0\le s\le t}, \{H_{\tau_{i+2}}(s)\}_{0\le s\le t}) & \tau_{i+1} > t, \\ \sigma(\{H_{\tau_{i+1}}(s)\}_{0\le s\le t}, \{H_{\tau_{i+2}}(s)\}_{0\le s\le t}, \{\xi_{s\wedge\tau_{i+2}}\}_{\tau_{i+1}\le s\le t}) & \tau_{i+1} \le t, \end{cases} \tag{4.61}$$

for $0 \le t \le T$, where $t < \tau_{i+1}$ if $n < i+1$. We assume that $\{\mathcal{R}_t\}$ is the market filtration.

Note that if i is an odd integer, then τ_i stops the information process, and if i is an even integer, then τ_i acts as a start-up time. Keeping notations the same, we define a $\mathcal{Q}(\mathbb{X})$-valued process $\{\overline{\pi}_t\}_{t\in[0,T]}$ by

$$\overline{\pi}_t(A) = \mathbb{Q}(X_T \in A | \mathcal{R}_t), \tag{4.62}$$

for $A \in \mathcal{B}(\mathbb{X})$. We denote the associated conditional density by

$$\overline{\psi}_t(x)\,\mathrm{d}x = \overline{\pi}_t(\mathrm{d}x), \tag{4.63}$$

for $0 \le t < T$. Finally, we let E be the set of even integers.

Proposition 4.4.1. *The random probability measure $\overline{\pi}_t$ can be represented as*

$$\overline{\pi}_t(A) = \sum_{i=1, i\in O}^{n+1} \pi_t(A)\mathbf{I}_t(i) + \sum_{i=2, i\in E}^{n+1} \pi_{\tau_{i-1}}(A)\mathbf{I}_t(i). \tag{4.64}$$

The dynamics of $\{\overline{\psi}_t\}$ are governed by the following SDE:

$$\mathrm{d}\overline{\psi}_t(x) = \sum_{i=1, i\in O}^{n+1} \sigma_t(x)\psi_t(x)\,\mathrm{d}W_t\mathbf{I}_t(i) + \sum_{i=2, i\in E}^{n} \big(\psi_t(x) - \psi_{\tau_{i-1}}(x)\big)\,\delta_{\tau_i}(\mathrm{d}t). \tag{4.65}$$

Proof. The proof of the first part is similar to that of Proposition 3.2.3 and Proposition 4.3.1. In particular, using the law of total probability,

$$\begin{aligned} \mathbb{Q}(X_T \in A|\mathcal{R}_t) = & \sum_{i=1, i\in O}^{n+1} \mathbb{Q}(X_T \in A|\mathcal{R}_t, \tau_{i-1} \le t < \tau_i)\mathbb{Q}(\tau_{i-1} \le t < \tau_i|\mathcal{R}_t) \\ & \sum_{i=2, i\in E}^{n+1} \mathbb{Q}(X_T \in A|\mathcal{R}_t, \tau_{i-1} \le t < \tau_i)\mathbb{Q}(\tau_{i-1} \le t < \tau_i|\mathcal{R}_t), \end{aligned} \tag{4.66}$$

where we set $\tau_0 = 0$ and $t < \tau_{n+1}$. Equation (4.64) follows from the strong Markov property of $\{\xi_t\}$, the indepedence of τ and since $\mathbb{Q}(\tau_{i-1} \le t < \tau_i|\mathcal{R}_t)$ is a Dirac measure. The SDE of $\{\overline{\psi}_t\}$ follows from (4.64) and Lemma 3.2.4. □

The conditional density process $\{\overline{\psi}_t\}$ stops for random periods of time and jumps when the source of information is reactivated. We define the price by

$$\overline{X}_t = P_{tT}\mathbb{E}^{\mathbb{Q}}[X_T \,|\, \mathcal{R}_t], \qquad 0 \leq t < T. \tag{4.67}$$

Proposition 4.4.2. *The price $\{\overline{X}_t\}$ is governed by the following SDE:*

$$\begin{aligned} \mathrm{d}\overline{X}_t &= r_t\overline{X}_t\,\mathrm{d}t + P_{tT}\sum_{i=1,i\in O}^{n+1}\frac{T\kappa}{(T-t)}\left(Var^{\mathbb{Q}}\left[X_T \,|\, \xi_t\right]\right)\mathrm{d}W_t\mathbf{I}_t(i) \\ &\quad + P_{tT}\sum_{i=2,i\in E}^{n}\left(\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \xi_t\right] - \mathbb{E}^{\mathbb{Q}}\left[X_T \,\middle|\, \xi_{\tau_{i-1}}\right]\right)\delta_{\tau_i}(\mathrm{d}t), \end{aligned} \tag{4.68}$$

for $0 \leq t < T$, where $Var^{\mathbb{Q}}[X_T \,|\, \xi_t]$ is a $\mathbb{Q}$-supermartingale.

Proof. The proof is very similar to that of Proposition 3.2.6, and follows from (4.65). □

If i is an odd integer, then from τ_i until τ_{i+1}, the price change is zero. The source of information is active again at τ_{i+1} (until τ_{i+2}), and the price is governed by a Brownian motion. Such price behaviour may arise in illiquid markets. From (4.68) we can see that the conditional expectation of the cash flow sticks to a value when the information source is deactive, and it jumps when the information source is activated.

This scenerio inlcudes the possibility that the market never realizes the true value of X_T at time T, if it had deactivated at some time before T. To overcome this, we may first assume that $\tau_i \in (0, T)$ for $i = 1, \ldots, n$, so that all stopping times are realized during the lifespan of the asset price. Secondly, if we choose n to be an even number, then the market realizes the true value of the cash flow X_T at time T, since $t < \tau_{i+1}$ if $n < i + 1$.

At the end of this chapter, we provide two simulations of such price processes. Figure 4.1 is a simulation when the information source is deactivated-reactivated two times. We set $X_T = 1$ and $T = 5$, $r_t = 0$, $\kappa^i = 1/T$ and $\rho^i = 0.5$. Figure 4.2 is a simulation when the source of information is deactivated-reactivated three times. We set $X_T = 0$ and $T = 5$, $r_t = 0$, $\kappa^i = 1/T$ and $\rho^i = 0.5$. Stopping times are uniformly distributed on $[0, T]$.

4.4.2 Multiple Market Factor Generalisation

We generalise the setting to the case where X_T is represented as a function of independent market factors. We assume that there is a single information process for each market factor, which can switch on or off. As before, we represent the cash flow X_T as a function of a set of independent market factors M_T^{α}, $\alpha = 1, \ldots, m$, with state-space $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ and with continuous densities $q^{\alpha}(x) > 0$. Choosing a function $g \in B_b(\mathbb{X}^m)$ such that $g : \mathbb{X}^m \to \mathbb{X}$,

$$X_T = g(M_T^1, M_T^2, \ldots, M_T^m). \tag{4.69}$$

We associate a sequence of $\mathcal{F}_t$-stopping times to each M_T^α denoted by $\{\tau_i^\alpha\}_{i=1}^n$ for $\alpha = 1,\ldots,m$. For fixed α, we let $\tau_1^\alpha < \tau_2^\alpha < ... < \tau_n^\alpha$. For each i and j, $\mathbb{Q}(\tau_i^\alpha < \tau_j^{\alpha+1}) \neq 1$.

We associate a Brownian information process $\{\xi_t^\alpha\}_{t\in[0,T]}$ with each M_T^α:

$$\xi_t^\alpha = \kappa^\alpha M_T^\alpha t + B_{tT}^\alpha. \tag{4.70}$$

We assume that $\{B_{tT}^\alpha\}$'s are mutually independent from each other across α (i.e., $\{B_{tT}^\alpha\}$ and $\{B_{tT}^\beta\}$ are independent) and independent of each M_T^α. We further assume that each sequence of stopping times is mutually independent from each other and mutually independent of each information process.

We define the following σ-algebra:

$$\mathcal{F}_t^{\eta^\alpha} = \sigma(\{H_{\tau_1}^\alpha(s)\}_{0\le s\le t}, \{\xi_{s\wedge\tau_1^\alpha}^\alpha\}_{0\le s\le t}), \quad \text{for } \ 0 \le t \le T. \tag{4.71}$$

We introduce a sequence of σ-algebras $\{\mathcal{R}_t^\alpha\}_{\alpha=1}^m$ for $0 \le t \le T$, where

$$\mathcal{R}_t^\alpha = \mathcal{F}_t^{\eta^\alpha} \bigvee_{i=1, i\in O}^{n} \begin{cases} \sigma(\{H_{\tau_{i+1}}^\alpha(s)\}_{0\le s\le t}, \{H_{\tau_{i+2}}^\alpha(s)\}_{0\le s\le t}) & \tau_{i+1}^\alpha > t, \\ \sigma(\{H_{\tau_{i+1}}^\alpha(s)\}_{0\le s\le t}, \{H_{\tau_{i+2}}^\alpha(s)\}_{0\le s\le t}, \{\xi_{s\wedge\tau_{i+2}^\alpha}^\alpha\}_{\tau_{i+1}^\alpha\le s\le t}) & \tau_{i+1}^\alpha \le t, \end{cases} \tag{4.72}$$

Note that each sequence $\{\tau_i^\alpha\}_{i=1}^n$ is a sequence of $\mathcal{R}_t^\alpha$-stopping times. We assume $t < \tau_{i+1}^\alpha$ if $n < i+1$. Also, we define the filtration $\{\mathcal{R}_t\}$ by

$$\mathcal{R}_t = \bigvee_{\alpha=1}^{m} \mathcal{R}_t^\alpha, \tag{4.73}$$

and assume that $\{\mathcal{R}_t\}$ is the market filtration. The σ-algebra (4.73) is all the information that the market receives about X_T, where an information source may be active or inactive.

If we associate 1 to active information and 0 to inactive information, then we have 2^m different m-vectors of information processes, each representing a different economic state. For example, if $m = 5$, and [1 0 0 1 0] represents a state in which only $\{\xi_t^1\}$ and $\{\xi_t^4\}$ are active, then there are 31 additional vectors such as [1 0 1 0 1], [0 1 0 0 1], and etc., associated with different numbers and allocations of active and inactive information processes.

We define a $\mathcal{Q}(\mathbb{X})$-valued process $\{\overline{\pi}_t\}_{t\in[0,T]}$ by

$$\overline{\pi}_t(\mathbf{A}) = \mathbb{Q}([M_T^1, M_T^2, \ldots, M_T^m] \in \mathbf{A}|\mathcal{R}_t), \tag{4.74}$$

for fixed $\mathbf{A} \in \mathcal{B}(\mathbb{X}^m)$.

Note that due to the independence properties we imposed above, we have

$$\overline{\pi}_t(\mathbf{A}) = \mathbb{Q}(M_T^1 \in A_1|\mathcal{R}_t^1) \times \mathbb{Q}(M_T^2 \in A_2|\mathcal{R}_t^2) \times \cdots \times \mathbb{Q}(M_T^m \in A_m|\mathcal{R}_t^m), \tag{4.75}$$

for $\mathbf{A} = [A_1, A_2, \ldots, A_m] \in \mathcal{B}(\mathbb{X}^m)$. We denote the conditional density by

$$\overline{\psi}_t(\mathbf{x})\,\mathrm{d}x_1 \cdots \,\mathrm{d}x_m = \overline{\pi}_t(\mathrm{d}\mathbf{x}), \tag{4.76}$$

for $0 \le t < T$, and $\mathbf{x} = [x_1, \ldots, x_m] \in \mathbb{X}^m$. We also define the process $\{\pi_t^\alpha\}_{t\in[0,T]}$ by

$$\pi_t^\alpha(A_\alpha) = \mathbb{Q}(M_T^\alpha \in A_\alpha | \xi_t^\alpha), \tag{4.77}$$

for $\alpha = 1, \ldots, m$. We denote the associated conditonal density by

$$\psi_t^\alpha(x_\alpha)\,\mathrm{d}x_\alpha = \pi_t^\alpha(\mathrm{d}x_\alpha), \tag{4.78}$$

for $0 \le t < T$ and $\alpha = 1, \ldots, m$ and $x_\alpha \in \mathbb{X}$, where $\psi_t^\alpha(x_\alpha)$ is as given in (2.20).

We assume that $n \in E$ is an even number (it is straightforward to modify the following results if $n \in O$ is an odd number). Then, we define the following vectors:

$$\mathbf{Q}_t^1 = \begin{bmatrix} \pi_t^1 \\ \pi_{\tau_1^1}^1 \\ \pi_t^1 \\ \vdots \\ \pi_{\tau_{n-1}^1}^1 \\ \pi_t^1 \end{bmatrix}, \quad \mathbf{Q}_t^2 = \begin{bmatrix} \pi_t^2 \\ \pi_{\tau_1^2}^2 \\ \pi_t^2 \\ \vdots \\ \pi_{\tau_{n-1}^2}^2 \\ \pi_t^2 \end{bmatrix}^\top, \quad \ldots, \quad \mathbf{Q}_t^m = \begin{bmatrix} \pi_t^m \\ \pi_{\tau_1^m}^m \\ \pi_t^m \\ \vdots \\ \pi_{\tau_{n-1}^m}^m \\ \pi_t^m \end{bmatrix}^\top, \tag{4.79}$$

for $n \ge 2$. Note that $\mathbf{Q}_t^\alpha$ is the vector of conditional distributions associated with M_T^α. We also define the following vectors of Heaviside processes:

$$\mathbf{I}_t^1 = \begin{bmatrix} 1 - H_{\tau_1}^1(t) \\ H_{\tau_1}^1(t)(1 - H_{\tau_2}^1(t)) \\ \vdots \\ H_{\tau_{n-1}}^1(t)(1 - H_{\tau_n}^1(t)) \\ H_{\tau_n}^1(t) \end{bmatrix}, \quad \text{and} \quad \{\mathbf{I}_t^\alpha\} = \left\{ \begin{bmatrix} 1 - H_{\tau_1}^\alpha(t) \\ H_{\tau_1}^\alpha(t)(1 - H_{\tau_2}^\alpha(t)) \\ \vdots \\ H_{\tau_{n-1}}^\alpha(t)(1 - H_{\tau_n}^\alpha(t)) \\ H_{\tau_n}^\alpha(t) \end{bmatrix}^\top \right\}_{\alpha=2}^m.$$

Note that each $\mathbf{I}_t^\alpha$ is a vector associated with M_T^α for $\alpha = 1, \ldots, m$.

Proposition 4.4.3. *The random probability measure $\overline{\pi}_t$ can be represented as*

$$\overline{\pi}_t(\mathbf{A}) = \left\| \left(\bigotimes_{i=1}^m \mathbf{Q}_t^i(A_i) \right) \circ \left(\bigotimes_{i=1}^m \mathbf{I}_t^i \right) \right\|. \tag{4.80}$$

Proof. The proof is very similar to that of Proposition 3.3.1. All the stopping times τ_i^α's are $\mathcal{R}_t$-stopping times. Hence, $\mathbb{Q}(\tau_i^\alpha \le t < \tau_{i+1}^\alpha | \mathcal{R}_t)$'s are Dirac measures for $i = 1, \ldots, n$ and

$\alpha = 1, \ldots, m$. Note that $\{\xi_t^\alpha\}$'s are independent from each other across α and are strong Markov. The stopping times are independent from each M_T^α and each information process $\{\xi_t^\alpha\}$. Also, each sequence of stopping times is independent from each other as well. The statement follows from the law of total probability. □

We are now in the position to provide a representation of the price of the asset with cash flow $X_T = g(M_T^1, M_T^2, \ldots, M_T^m)$. The time-$t$ price, which we denote by $\overline{X}_t$, is

$$\overline{X}_t = P_{tT}\mathbb{E}^{\mathbb{Q}}[X_T \,|\, \mathcal{R}_t], \qquad 0 \leq t < T. \tag{4.81}$$

Proposition 4.4.4. *The price $\overline{X}_t$ is*

$$\overline{X}_t = P_{tT}\int_{\mathbb{X}^m} g(x^1, \ldots, x^m) \left\| \left(\bigotimes_{i=1}^m \mathbf{Q}_t^i(\mathrm{d}x_i) \right) \circ \left(\bigotimes_{i=1}^m \mathbf{I}_t^i \right) \right\|. \tag{4.82}$$

Proof. The statement follows from (3.13), (4.80) and (4.81). □

If inactive information is activated, then there is a jump in the price dynamics. If active information is deactivated, then there is no jump. If all sources of information are deactive, then the conditional expectation of X_T sticks to a value. Similar to Chapter 3, one can also employ $\sqrt{\psi}$ and each $\sqrt{\psi^\alpha}$ to bring forth a geometrical perspective.

As a simplification, similar to what is done in Chapter 3.3.1, let's assume there is only one sequence of stopping times associated to every market factor. Then we have the following representation for the SDE of $\{\overline{X}_t\}$:

Proposition 4.4.5. *Let $\{\tau_i^\alpha\}_{i=1}^n = \{\tau_i\}_{i=1}^n$ for $\alpha = 1, \ldots, m$, so that $\{H_{\tau_i}^\alpha\}_{\alpha=1}^m = H_{\tau_i}$ for $i = 1, \ldots, n$. Then,*

$$\begin{aligned}\mathrm{d}\overline{X}_t &= r_t\overline{X}_t\,\mathrm{d}t + P_{tT}\sum_{\alpha=1}^m \sum_{i=1, i\in O}^{n+1} \frac{T\kappa^\alpha}{T-t} Cov^{\mathbb{Q}}\left[X_T, M_T^\alpha \,|\, \xi_t^1, \ldots, \xi_t^m\right] \mathrm{d}W_t^\alpha \mathbf{I}_t(i) \\ &\quad + P_{tT}\sum_{i=2, i\in E}^{n} \left(\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \xi_t^1, \ldots, \xi_t^m\right] - \mathbb{E}^{\mathbb{Q}}\left[X_T \,\middle|\, \xi_{\tau_{i-1}}^1, \ldots, \xi_{\tau_{i-1}}^m\right]\right) \delta_{\tau_i}(\mathrm{d}t). \end{aligned} \tag{4.83}$$

for $0 \leq t < T$.

Proof. The statement follows from (2.22), (4.81) and Proposition 4.4.4. □

This framework is an alternative way of viewing regime switches. One may interpret that any given switch in an information source coincides with a switch from one regime to another. Each regime switch does not neccessarily coincide with a price jump, but rather with a change in the information source provided to the market.

4.4.3 Switching From One Source of Information to Another

We make a slight modification, and introduce a setting where each stopping time induces a switch from one source of information to another. More precisely, we develop this framework by initiating and stopping σ-algebras at stopping times such that each stopping time τ_i stops an information flow and simultaneously acts as a start-up time of another source of information.

Suppose there are $n \in \mathbb{N}_+$ $\mathcal{F}_t$-stopping times $\{\tau_i\}_{i=1}^n$ independent of $m \in \mathbb{N}_+$ information processes, where $m \leq n+1$. We denote the modular of two integers i and m by $\text{mod}(i,m)$ (i.e., if $m = 5$, $\text{mod}(1,5) = 1$, $\text{mod}(5,5) = 0$, $\text{mod}(7,5) = 2$, etc.).

We denote the $(\text{mod}(i,m)+1)$th Brownian information process for X_T by $\{\xi_t^{\text{mod}(i,m)+1}\}_{t\in[0,T]}$, where

$$\xi_t^{\text{mod}(i,m)+1} = \kappa^{\text{mod}(i,m)+1} X_T t + B_{tT}^{\text{mod}(i,m)+1}. \tag{4.84}$$

That is, if $m = 5$, then for example $\{\xi_t^{\text{mod}(1,5)+1}\} = \{\xi_t^2\}$ as defined in (3.10) and so on. We define the filtration $\{\mathcal{Z}_t\}_{0\leq t\leq T}$ by

$$\mathcal{Z}_t = \mathcal{F}_t^{\eta^1} \bigvee_{i=1}^{n} \begin{cases} \sigma(\{H_{\tau_i}(s)\}_{0\leq s\leq t}, \{H_{\tau_{i+1}}(s)\}_{0\leq s\leq t}) & \tau_i > t, \\ \sigma(\{H_{\tau_i}(s)\}_{0\leq s\leq t}, \{H_{\tau_{i+1}}(s)\}_{0\leq s\leq t}, \{\xi_{s\wedge\tau_{i+1}}^{\text{mod}(i,m)+1}\}_{\tau_i\leq s\leq t}) & \tau_i \leq t, \end{cases} \tag{4.85}$$

where $t < \tau_{n+1}$. We assume that $\{\mathcal{Z}_t\}$ is the market filtration. From (4.85) we can see that every time an information source stops flowing, another source of information becomes active. This scenerio can as well be interpreted as a sudden switch from one regime to another, while different sources of information are active during different economic states.

Keeping notations the same to that of previous sections, we define a $\mathcal{Q}(\mathbb{X})$-valued process $\{\overline{\pi}_t\}_{t\in[0,T]}$ by

$$\overline{\pi}_t(A) = \mathbb{Q}(X_T \in A|\mathcal{Z}_t), \tag{4.86}$$

for $A \in \mathcal{B}(\mathbb{X})$. We also define the following matrix:

$$\mathbf{N}_t = \begin{bmatrix} 1 - H_{\tau_1}(t) & 0 & \dots & 0 \\ 0 & H_{\tau_1}(t)(1 - H_{\tau_2}(t)) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & H_{\tau_{m-1}}(t)(1 - H_{\tau_m}(t)) \\ H_{\tau_m}(t)(1 - H_{\tau_{m+1}}(t)) & 0 & \dots & 0 \\ 0 & H_{\tau_{m+1}}(t)(1 - H_{\tau_{m+2}}(t)) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ H_{\tau_n}(t) & H_{\tau_n}(t) & H_{\tau_n}(t) & H_{\tau_n}(t) \end{bmatrix}.$$

We let $\mathbf{N}(i,j)$ denote the ith row and jth column element of the matrix $\mathbf{N}$. In addition, we let $\tau_k = 0$ for $k \leq 0$, which is a slight abuse of notation, but simplifies the following definitions:

$$\pi_t^{(i,1)}(A) = \mathbb{Q}(X_T \in A | \xi_t^1, \xi^2_{\tau_{i-(m-1)}}, \xi^3_{\tau_{i-(m-2)}}, \dots, \xi^m_{\tau_{i-1}}), \tag{4.87}$$

for $i = 1, 1+m, 1+2m, .. \leq n+1$. Also,

$$\pi_t^{(i,j)}(A) = \mathbb{Q}(X_T \in A | \xi^1_{\tau_{i-(j-1)}}, \dots, \xi^{j-1}_{\tau_{i-1}}, \xi^j_t, \xi^{j+1}_{\tau_{i-(m+j-(j+1))}}, \dots, \xi^m_{\tau_{i-j}}), \tag{4.88}$$

for $i = j, j+m, j+2m, ... \leq n+1$ and $1 < j < m$. Finally,

$$\pi_t^{(i,m)}(A) = \mathbb{Q}(X_T \in A | \xi^1_{\tau_{i-(m-1)}}, \xi^2_{\tau_{i-(m-2)}}, \dots, \xi^{m-1}_{\tau_{i-1}}, \xi^m_t), \tag{4.89}$$

for $i = m, 2m, 3m, ... \leq n+1$. Note that (4.87)-(4.89) make sense due to the strong Markovian property of the information processes. We denote the conditional density of $\overline{\pi}_t$ by

$$\overline{\psi}_t(x)\,\mathrm{d}x = \overline{\pi}_t(\mathrm{d}x), \tag{4.90}$$

for $0 \leq t < T$. In addition, for $i = 1, \dots, n+1$, and $1 \leq j \leq m \leq n+1$, we let

$$\psi_t^{(i,j)}(x)\,\mathrm{d}x = \pi_t^{(i,j)}(\mathrm{d}x), \tag{4.91}$$

for $0 \leq t < T$, and for a fixed m and n.

Denoting $\chi_{\{.\}}$ as the Kronecker delta, for $k \in \mathbb{N}_+ \cup \{0\}$, we also define

$$\mathbf{M}_t(A) = \begin{bmatrix} \pi_t^{(1,1)}(A) & 0 & \dots & 0 \\ 0 & \pi_t^{(2,2)}(A) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \pi_t^{(m,m)}(A) \\ \pi_t^{(m+1,1)}(A) & 0 & \dots & 0 \\ 0 & \pi_t^{(m+2,2)}(A) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \pi_t^{(n+1,1)}(A)\chi_{\{km+1=n+1\}} & \dots & 0 & \pi_t^{(n+1,m)}(A)\chi_{\{(k+1)m=n+1\}} \end{bmatrix}. \tag{4.92}$$

Proposition 4.4.6. *The random probability measure $\overline{\pi}_t$ can be represented as*

$$\overline{\pi}_t(A) = ||\mathbf{M}_t^\top(A)\mathbf{N}_t||, \qquad \textit{for} \quad k \in \mathbb{N}_+ \cup \{0\}. \tag{4.93}$$

Proof. The proof is similar to that of Proposition 3.2.3. Each stopping time τ_i is a $\mathcal{Z}_t$-

stopping time. By the use of law of total probability, we can write

$$\overline{\pi}_t(A) = \sum_{i=0}^{n} \mathbb{Q}(X_T \in A | \mathcal{Z}_t, \tau_i \leq t < \tau_{i+1}) \mathbb{Q}(\tau_i \leq t < \tau_{i+1} | \mathcal{Z}_t), \tag{4.94}$$

where we set $\tau_0 = 0$ and $t < \tau_{n+1}$. Each $\mathbb{Q}(\tau_i \leq t < \tau_{i+1}|\mathcal{Z}_t)$ is a Dirac measure. Then, from the strong Markov property of $\{\xi^i_t\}$, and the independence of the stopping times,

$$\begin{aligned}\overline{\pi}_t(A) = {} & \pi^{(1,1)}_t(A)\mathbf{N}_t(1,1) + \pi^{(2,2)}_t(A)\mathbf{N}_t(2,2) + \cdots + \pi^{(m,m)}_t(A)\mathbf{N}_t(m,m) \\ & + \pi^{(m+1,1)}_t(A)\mathbf{N}_t(m+1,1) + \cdots + \pi^{(n+1,1)}_t(A)\mathbf{N}_t(n+1,1)\chi_{\{km+1=n+1\}} + \cdots \\ & \cdots + \pi^{(n+1,m)}_t(A)\mathbf{N}_t(n+1,m)\chi_{\{(k+1)m=n+1\}}. \end{aligned} \tag{4.95}$$

The Kronecker delta, where $k \in \mathbb{N}_+ \cup \{0\}$ must be satisfied, ensures that the correct element of the last row of $\mathbf{M}_t$ is non-zero when $\tau_n \leq t$ for the fixed m and n. Since both $\mathbf{M}_t$ and $\mathbf{N}_t$ are $(n+1) \times m$ matrices, (4.93) follows by taking the transpose of $\mathbf{M}_t$. □

We call the $m \times m$ matrix $\mathbf{M}^\top_t \mathbf{N}_t$ the information-switching matrix. At every information switch, a jump in $\{\overline{\psi}_t\}$ occurs. From then on, since the information processes are strong Markov, $\{\overline{\psi}_t\}$ is driven by the last observations of the switched off information processes and the new observations of the single switched on information process. Also, since the price is

$$\overline{X}_t = P_{tT}\mathbb{E}^{\mathbb{Q}}[X_T \,|\, \mathcal{Z}_t] = P_{tT} \int_{\mathbb{X}} x \overline{\pi}_t(\mathrm{d}x), \qquad 0 \leq t < T, \tag{4.96}$$

every time a switch between two different information sources occurs, the process $\{\overline{X}_t\}$ jumps. This example provides an alternative way of viewing regime switches as events that coincide with price jumps.

For a demonstration, we shall give a simple example of the information switching detailed above:

Example 4.4.7. *Let $n = 2$, and $m = 2$. Hence, there are two switches between two information processes. Then, the conditional distribution is given by:*

$$\begin{aligned}\overline{\pi}_t(\mathrm{d}x) &= \pi^{(1,1)}_t(\mathrm{d}x)(1 - H_{\tau_1}(t)) + \pi^{(2,2)}_t(\mathrm{d}x)H_{\tau_1}(t)(1 - H_{\tau_2}(t)) + \pi^{(3,1)}_t(\mathrm{d}x)H_{\tau_2}(t) \\ &= \mathbb{Q}(X_T \in \mathrm{d}x|\xi^1_t)(1 - H_{\tau_1}(t)) + \mathbb{Q}(X_T \in \mathrm{d}x|\xi^1_{\tau_1}, \xi^2_t)H_{\tau_1}(t)(1 - H_{\tau_2}(t)) \\ &\quad + \mathbb{Q}(X_T \in \mathrm{d}x|\xi^1_t, \xi^2_{\tau_2})H_{\tau_2}(t). \end{aligned} \tag{4.97}$$

From (4.96) and (4.97), we can see that the price process $\{X_t\}$ is governed by $\{\xi^1_t\}$ for $t < \tau_1$, is governed by $\xi^1_{\tau_1}$ and $\{\xi^2_t\}$ for $\tau_1 \leq t < \tau_2$ and is governed by $\{\xi^1_t\}$ and $\xi^2_{\tau_2}$ for $\tau_2 \leq t$. That is, at each regime switch, one of the information sources is switched off and the other is switched on.

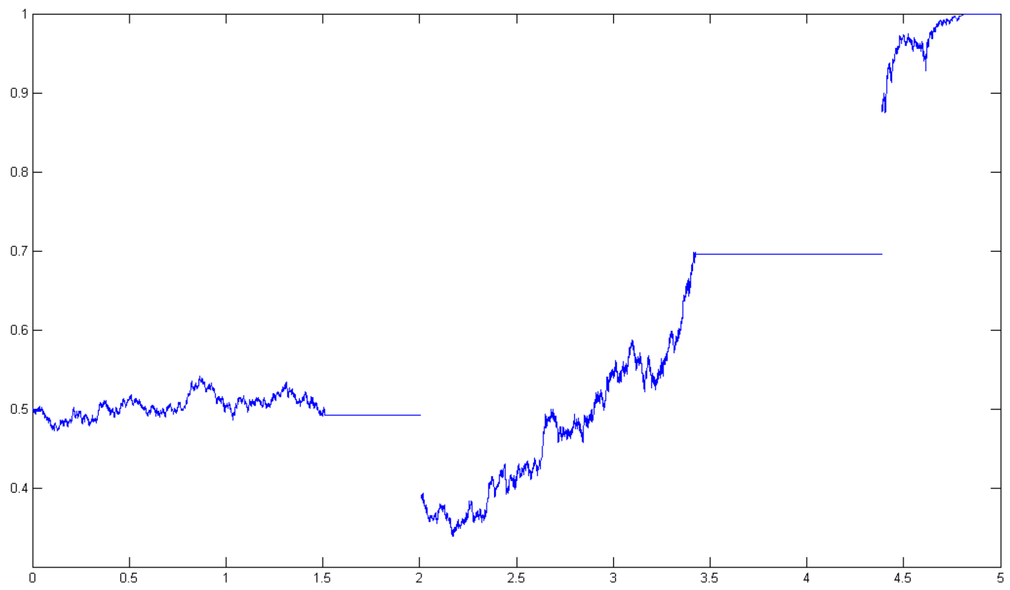

Figure 4.1: A price process. The single source of information is deactivated-reactivated two times. There are two regimes when no new information enters the market and when the price "sticks" to a value with zero interest rates. Cash flow: $X_T = 1$. Parameters: $T = 5$, $r_t = 0$, $\kappa^i = 1/T$ and $\rho^i = 0.5$. Stopping times are uniformly distributed on $[0, T]$.

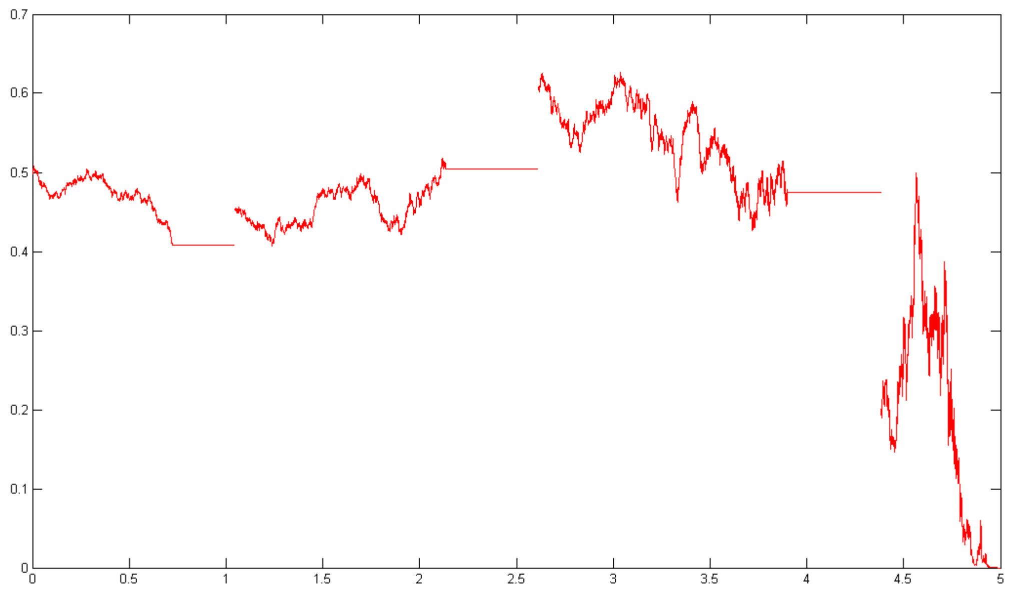

Figure 4.2: A price process. The single source of information is deactivated-reactivated three times. There are three regimes when no new information enters the market. Cash flow: $X_T = 0$. Parameters: $T = 5$, $r_t = 0$, $\kappa^i = 1/T$ and $\rho^i = 0.5$. Stopping times are uniformly distributed on $[0, T]$.

Chapter 5

Information-Theoretic Dynamics of Information Asymmetry

We construct what we call information asymmetry processes with jumps by using information-theoretic measures and enlargements of filtrations.

One main aim of constructing the so-called asymmetry processes is to address the question: How can one dynamically quantify the impact of changes in the source of information about a cash flow X_T? Our motivation stems from the aim of measuring the informational advantage of a small trader who is more informed than the market. A similar approach is considered in Brody *et al.* (2009), where there is an informed trader who is more susceptible to information than the market, and who is provided with an extra source of information from time $t = 0$. Brody *et al.* (2009) provide examples of how informed traders may be able to exploit statistical arbitrage opportunities by using their additional information, and demonstrate how this extra information transforms into profit. The value of excess information is measured by the difference of the mutual information between the market and the trader, which is shown to be nonnegative.

Information asymmetry in financial markets has attracted considerable attention in recent years, and the literature can be traced back to Kyle (1985), Duffie and Huang (1986), and Back (1992). Models generally consist of two agents making decisions based on different information. One of the agents behaves purely based on the knowledge of the evolution of the market, whereas the other agent (insider) has additional information. The insiders are usually assumed to be small, and cannot affect market price dynamics. One stream of models relies heavily on the works of Jeulin (1980), Jacod (1980) and Yor (1980) on enlargements of filtrations. These works laid the mathematical foundations later to be used in modelling information asymmetry between agents. Amongst many examples, Imkeller (1996), Amendinger *et al.* (1998), Grorud and Pontier (1998), and Biagini and Oksendal (2005) are few of the important papers to mention. In most of these works, the expected increase in the utility gained from the insider's additional information is analyzed.

Assume all filtrations under consideration are right-continuous and complete. Formally, an enlargement of $\{\mathcal{Y}_t\}$ in $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{Q})$ is a filtration $\{\mathcal{M}_t\}$, which satisfies: (i) $\mathcal{Y}_t \subset \mathcal{M}_t$ for all $t \in \mathbb{R}_+$, and (ii) the $\mathcal{F}_t$-stopping time $\tau < \infty$ is an $\mathcal{M}_t$-stopping time. In the information asymmetry literature, a considerable attention is directed towards what are called initial enlargements of filtrations and progressive enlargements of filtrations. An initial enlargement of a filtration $\{\mathcal{Y}_t\}$ is a filtration $\{\mathcal{J}_t\}$ given by $\mathcal{J}_t = \sigma(\tau) \bigvee \mathcal{Y}_t$. A progressive enlargement of $\{\mathcal{Y}_t\}$ is the minimal (smallest) filtration $\{\mathcal{J}_t^*\}$, which satisfies: (i) $\mathcal{Y}_t \subset \mathcal{J}_t^*$ for all $t \in \mathbb{R}_+$, and (ii) $\tau < \infty$ is a $\mathcal{J}_t^*$-stopping time. More explicitly, $\mathcal{J}_t^* = \sigma(\tau \wedge t) \bigvee \mathcal{Y}_t$. In the literature, it is usually the case that the filtrations of informed traders are assumed to be either initial or progressive enlargements of the market filtration.

We shall construct information asymmetry processes using enlargements of filtrations. Note that $\{\mathcal{G}_t\}$ as shown in (3.58) is an enlargement of $\{\mathcal{F}_t^{\xi^1}\}$. However, $\{\mathcal{G}_t\}$ is neither an initial nor a progressive enlargement of $\{\mathcal{F}_t^{\xi^1}\}$. We want the flexibility of being able to handle an informed trader who may have access to more information additional to the stopping time τ, and may start receiving extra information about new economic variables. For example, if τ is the default time of a bond (which is common in the current literature), an informed trader may start observing previously non-observed data starting from τ, especially if this default represents a serious economic shock or possibly an early warning signal for a financial turbulance. To formalize this mathematically, we introduce a new type of an enlargement of filtrations that we call an *n-order piecewise enlargement*. We choose this name due to the nature of the enlarged filtrations that expand at $n \in \mathbb{N}_+$ stopping times:

Definition 5.0.8. *Let $\{\tau_i\}_{i=1}^n$ for $n \in \mathbb{N}_+$ be an increasing sequence of $\mathcal{F}_t$-stopping times in $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{Q})$ such that $\tau_n < \infty$, and let $\{X_t^i\}_{t \in \mathbb{R}_+}$ be an $\mathcal{F}_t$-adapted càdlàg process for $i = 1, \ldots, n$. Then an* n-order piecewise enlargement *of a filtration $\{\mathcal{Y}_t\}$ in $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{Q})$ is a filtration $\{\mathcal{G}_t\}$, which satisfies: (i) $\mathcal{Y}_t \subset \mathcal{G}_t$ for all $t \in \mathbb{R}_+$, (ii) $\{\tau_i\}_{i=1}^n$ is an increasing sequence of $\mathcal{G}_t$-stopping times, and (iii) $\sigma(\{X_u^i\}_{\tau_i \le u \le t}) \subset \mathcal{G}_t$ if $\tau_i \le t$ for $i = 1, \ldots, n$.*

One can then consider what one may call an initial n-order piecewise enlargement $\{\mathcal{G}_t^I\}$ of $\{\mathcal{Y}_t\}$ given by $\mathcal{G}_t^I = \bigvee_{i=1}^n \sigma(\tau_i) \bigvee \mathcal{Y}_t \bigvee_{i=1}^n \sigma(\{X_u^i\}_{\tau_i \le u \le t})$ if $\tau_i \le t$ for all $i = 1, \ldots, n$, or a progressive n-order piecewise enlargement $\{\mathcal{G}_t^P\}$ of $\{\mathcal{Y}_t\}$ given by $\mathcal{G}_t^P = \bigvee_{i=1}^n \sigma(\tau_i \wedge t) \bigvee \mathcal{Y}_t \bigvee_{i=1}^n \sigma(\{X_u^i\}_{\tau_i \le u \le t})$ if $\tau_i \le t$ for all $i = 1, \ldots, n$.

In our framework, we assume the existence of a restricted number of small traders whose filtrations are n-order piecewise enlargements of the market filtration. Introducing n-order piecewise enlargements allows us to represent informed traders who may have access to more information additional to the stopping time. Note that $\{\mathcal{G}_t\}$ as shown in (3.58) is a (progressive) n-order piecewise enlargement of $\{\mathcal{F}_t^{\xi^1}\}$. Since we have this explicit example, we focus on a scenerio where informed trader's filtration is given by $\{\mathcal{G}_t\}$ and the market filtration is $\{\mathcal{F}_t^{\xi^1}\}$. We assume that the actions of informed traders do not affect price dynamics, and X_T is a continuous random variable (until we discuss the Shannon entropy).

Our work may be viewed as a generalisation of the framework presented in Brody *et al.* (2009), with the introduction of the n-order piecewise enlargements of filtrations. Also, instead of using mutual information, we refer to a broad class of information-theoretic measures, namely f-divergences, to quantify the impact of changes in the source of information. In particular, using f-divergences and piecewise enlargements of filtrations, we generate what we call the Kullback-Leibler (KL) and the Squared-Hellinger (SH) asymmetry processes. The KL divergence is commonly used to measure the information gain from passing from a prior distribution to a posterior distribution. The SH divergence measures the distance between two distributions, and it brings a geometrical perspective that motivates our next chapter.

We also build a competitive setting involving two informed traders whose filtrations are different piecewise enlargements of the same market filtration. We focus on a scenerio where the informed traders receive additional information at different stopping times. This allows us to dynamically quantify the competitive edge between two informed traders who have different accesibility to additional information. The informed traders can not see each others' actions, and at a given time, the trader who has access to more sources of information has an informational advantage over the other. Another motivation in constructing the asymmetry processes is to model financial mispricing as a type of information asymmetry. We assume that the market receives incorrect information about a future cash flow as opposed to correct information. The mispricing process represents the dynamic evolution of the information asymmetry between the market and the fundamentals. The mispricing process jumps to zero if the market receives the correct information flow, which represents a sudden market correction.

This chapter is organised as follows: Section 1 is a brief preliminary on f-divergences. Section 2 introduces the asymmetry processes. Section 3 is the competition between two informed traders. Section 4 models mispricing. Section 5 quantifies the level of uncertainty of an informed trader using the Shannon entropy. Section 6 is the Appendix.

5.1 Preliminaries

5.1.1 Information-Theoretic f-Divergences

For our purposes, we shall use the class of so-called f-divergences, introduced by Ali and Silvey (1966), Csiszár (1967). We let $\Delta_f[\,.\hookrightarrow .\,]$ denote an f-divergence. In a measure-theoretic sense, the f-divergence between equivalent probability measures $\mathbb{Q}$ and $\mathbb{P}$ is defined as follows:

$$\Delta_f[\mathbb{Q} \hookrightarrow \mathbb{P}] = \int_\Omega f\left(\frac{\mathrm{d}\mathbb{P}(\omega)}{\mathrm{d}\mathbb{Q}(\omega)}\right) \mathrm{d}\mathbb{Q}(\omega), \tag{5.1}$$

for $\omega \in \Omega$, where f is a convex function which satisfies $f(1) = 0$, and $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}$ is the Radon-Nikodym derivative of $\mathbb{P}$ over $\mathbb{Q}$. Also, $\Delta_f[\mathbb{P} \hookrightarrow \mathbb{Q}]$ is defined similarly only with $\mathbb{P}$ and $\mathbb{Q}$ interchanged in (5.1). Alternatively, an f-divergence can be defined in terms of probability densities (given that they exist):

$$\Delta_f[q \hookrightarrow p] = \int_{\mathbb{X}} f\left(\frac{p(x)}{q(x)}\right) q(x)\,\mathrm{d}x, \tag{5.2}$$

where $q(x) > 0$ and $p(x) > 0$ for $x \in \mathbb{X}$. An f-divergence $\Delta_f[\mathbb{Q} \hookrightarrow \mathbb{P}]$ (or $\Delta_f[q \hookrightarrow p]$) measures the discrepancy from $\mathbb{Q}$ to $\mathbb{P}$ (or q to p), which is not exactly a distance, since it may not satisfy properties such as symmetry and triangle inequality. In fact, we use the symbol $\hookrightarrow$ to emphasize the direction from $\mathbb{Q}$ to $\mathbb{P}$, since many examples exist such that $\Delta_f[\mathbb{Q} \hookrightarrow \mathbb{P}] \neq \Delta_f[\mathbb{P} \hookrightarrow \mathbb{Q}]$. We denote an f-divergence that satisfies the symmetry property $\Delta_f[\mathbb{Q} \hookrightarrow \mathbb{P}] = \Delta_f[\mathbb{P} \hookrightarrow \mathbb{Q}]$ (or $\Delta_f[q \hookrightarrow p] = \Delta_f[p \hookrightarrow q]$) by $\Delta_f[\mathbb{Q}||\mathbb{P}] = \Delta_f[\mathbb{P}||\mathbb{Q}]$ (or $\Delta_f[q||p] = \Delta_f[p||q]$). We refer to Csiszár (1967), Chentsov (1972), and Amari and Cichocki (2010) for some interesting properties of f-divergences.

The Kullback-Leibler (KL) divergence forms an important subclass of f-divergences, and is widely used in applied mathematics and engineering to measure the information gain from passing from a prior distribution to a posterior distribution. The KL divergence is

$$\Delta_{\mathrm{KL}}(\mathbb{Q} \hookrightarrow \mathbb{P}) = -\int_{\Omega} \log\left(\frac{\mathrm{d}\mathbb{P}(\omega)}{\mathrm{d}\mathbb{Q}(\omega)}\right) \mathrm{d}\mathbb{Q}(\omega) = \int_{\Omega} \log\left(\frac{\mathrm{d}\mathbb{Q}(\omega)}{\mathrm{d}\mathbb{P}(\omega)}\right) \mathrm{d}\mathbb{Q}(\omega). \tag{5.3}$$

The KL divergence is not a distance metric defined on the space of probability distributions, since $\Delta_{\mathrm{KL}}(\mathbb{P} \hookrightarrow \mathbb{Q}) \neq \Delta_{\mathrm{KL}}(\mathbb{Q} \hookrightarrow \mathbb{P})$, and it does not satisfy the triangle inequality.

The Squared-Hellinger (SH) divergence forms another important subclass of f-divergences, which is used in problems that involve measuring the distance between two different distributions. Unlike the KL divergence, the SH divergence is symmetric. Thus, we write $\Delta_{\mathrm{SH}}(\mathbb{P}||\mathbb{Q}) = \Delta_{\mathrm{SH}}(\mathbb{Q}||\mathbb{P})$. The SH divergence can be defined as

$$\Delta_{\mathrm{SH}}(\mathbb{Q}||\mathbb{P}) = \frac{1}{2}\int_{\Omega} \left(\sqrt{\frac{\mathrm{d}\mathbb{P}(\omega)}{\mathrm{d}\mathbb{Q}(\omega)}} - 1\right)^2 \mathrm{d}\mathbb{Q}(\omega) = \frac{1}{2}\int_{\Omega} \left(\sqrt{\frac{\mathrm{d}\mathbb{Q}(\omega)}{\mathrm{d}\mathbb{P}(\omega)}} - 1\right)^2 \mathrm{d}\mathbb{P}(\omega). \tag{5.4}$$

Also, if $\mathbb{L}$ denotes the Lebesgue measure, and $\mathbb{Q}$ and $\mathbb{P}$ are equivalent to $\mathbb{L}$, then we can write the following:

$$\Delta_{\mathrm{SH}}(\mathbb{Q}||\mathbb{P}) = \frac{1}{2}\int_{\Omega} \left(\sqrt{\frac{\mathrm{d}\mathbb{Q}(\omega)}{\mathrm{d}\mathbb{L}(\omega)}} - \sqrt{\frac{\mathrm{d}\mathbb{P}(\omega)}{\mathrm{d}\mathbb{L}(\omega)}}\right)^2 \mathrm{d}\mathbb{L}(\omega). \tag{5.5}$$

As we shall see later in Chapter 6, by the use of (5.5), the SH divergence brings forth a geometrical perspective.

5.2 Information Asymmetry, Piecewise Enlargements of Filtrations, f-Divergences

We would like the f-divergence to be symmetric in order to eliminate any bias towards a probability measure. In order to ensure this, we can do the following:

$$\Delta_f(\mathbb{Q}||\mathbb{P}) = \frac{1}{2}\left[\Delta_f(\mathbb{Q} \hookrightarrow \mathbb{P}) + \Delta_f(\mathbb{P} \hookrightarrow \mathbb{Q})\right] = \Delta_f(\mathbb{P}||\mathbb{Q}). \tag{5.6}$$

We shall define a Radon-Nikodym derivative to introduce what we call an f*-asymmetry* with respect to an n-order piecewise enlargement. For fixed $A \in \mathcal{B}(\mathbb{X})$, let $\mathcal{Y} \subset \mathcal{G}$ be two σ-algebras in $(\Omega, \mathcal{F}, \mathbb{Q})$, where $\{\mathcal{G}_t\}$ is an n-order piecewise enlargement of $\{\mathcal{Y}_t\}$. Define

$$\mathbb{Y}_t(X_T \in A) = \mathbb{Q}(X_T \in A|\mathcal{Y}_t) \qquad \text{and} \qquad \mathbb{G}_t(X_T \in A) = \mathbb{Q}(X_T \in A|\mathcal{G}_t), \tag{5.7}$$

such that $\mathbb{E}^{\mathbb{Y}_t}[X_T] = \mathbb{E}^{\mathbb{Q}}[X_T|\mathcal{Y}_t]$ and $\mathbb{E}^{\mathbb{G}_t}[X_T] = \mathbb{E}^{\mathbb{Q}}[X_T|\mathcal{G}_t]$. We shall denote the conditional measures in (5.7) as $\mathbb{Y}_t$ and $\mathbb{G}_t$, respectively. Let $\mathbb{Y}_t$ be equivalent to $\mathbb{G}_t$ and

$$Z_t = \frac{\mathrm{d}\mathbb{Y}_t}{\mathrm{d}\mathbb{G}_t}, \tag{5.8}$$

be a Radon-Nikodym derivative such that

$$\mathbb{E}^{\mathbb{G}_t}[X_T Z_t] = \int_\Omega X_T(\omega)\frac{\mathrm{d}\mathbb{Y}_t(\omega)}{\mathrm{d}\mathbb{G}_t(\omega)}\,\mathrm{d}\mathbb{G}_t(\omega) = \int_\Omega X_T(\omega)\,\mathrm{d}\mathbb{Y}_t(\omega) = \mathbb{E}^{\mathbb{Y}_t}[X_T]. \tag{5.9}$$

We define the time-t f-asymmetry $\Delta_f(.||.)$ between the probability measures $\mathbb{Y}_t$ and $\mathbb{G}_t$ by

$$\begin{aligned}\Delta_f(\mathbb{Y}_t||\mathbb{G}_t) &= \frac{1}{2}\left[\Delta_f(\mathbb{Y}_t \hookrightarrow \mathbb{G}_t) + \Delta_f(\mathbb{G}_t \hookrightarrow \mathbb{Y}_t)\right]\\ &= \frac{1}{2}\left[\int_\Omega f\left(\frac{1}{Z_t(\omega)}\right)\mathrm{d}\mathbb{Y}_t(\omega) + \int_\Omega f\left(Z_t(\omega)\right)\mathrm{d}\mathbb{G}_t(\omega)\right].\end{aligned} \tag{5.10}$$

Equation (5.10) makes sense since conditional probability distributions are probability distributions. Similarly, we shall use conditional probability densities (since they are probability densities) to derive the dynamics of the f-asymmetry process $\{\Delta_f(q(x|\mathcal{Y}_t)||q(x|\mathcal{G}_t))\}$, where

$$\begin{aligned}\Delta_f(q(x|\mathcal{Y}_t)||q(x|\mathcal{G}_t)) &= \frac{1}{2}\left[\Delta_f\left(q(x|\mathcal{Y}_t) \hookrightarrow q(x|\mathcal{G}_t)\right) + \Delta_f\left(q(x|\mathcal{G}_t) \hookrightarrow q(x|\mathcal{Y}_t)\right)\right]\\ &= \frac{1}{2}\left[\int_\mathbb{X}\left(f\left(\frac{q(x|\mathcal{G}_t)}{q(x|\mathcal{Y}_t)}\right)q(x|\mathcal{Y}_t) + f\left(\frac{q(x|\mathcal{Y}_t)}{q(x|\mathcal{G}_t)}\right)q(x|\mathcal{G}_t)\right)\mathrm{d}x\right].\end{aligned} \tag{5.11}$$

We shall focus on the case where the market filtration is $\{\mathcal{Y}_t\} = \{\mathcal{F}_t^\xi\}$ (we write $\xi = \xi^1$), and the filtration of the informed trader is $\{\mathcal{G}_t\}$ as shown in (3.58).

5.2.1 Kullback-Leibler Asymmetry

We define the time-t *KL asymmetry*, which we denote by $\mathrm{KL}(.||.)$, as

$$\mathrm{KL}_t(\mathbb{Y}_t||\mathbb{G}_t) = \frac{1}{2}\left[\int_\Omega \log\left(\frac{\mathrm{d}\mathbb{Y}_t(\omega)}{\mathrm{d}\mathbb{G}_t(\omega)}\right)\mathrm{d}\mathbb{Y}_t(\omega) + \int_\Omega \log\left(\frac{\mathrm{d}\mathbb{G}_t(\omega)}{\mathrm{d}\mathbb{Y}_t(\omega)}\right)\mathrm{d}\mathbb{G}_t(\omega)\right], \tag{5.12}$$

between $\mathbb{Y}_t$ and $\mathbb{G}_t$, which are the conditional measures given $\mathcal{F}^\xi_t$ and $\mathcal{G}_t$, respectively. We can now quantify the impact of activation of new information sources, and derive the dynamics of the information asymmetry process between the market and the informed trader. In order to do so, we define $\{\mathrm{KL}_t(\psi_t||\overline{\psi}_t)\}_{t\in[0,T]}$ as follows:

$$\mathrm{KL}_t(\psi_t||\overline{\psi}_t) = \begin{cases} \frac{1}{2}\int_{\mathbb{X}}\left(\psi_t(x)\log\left(\frac{\psi_t(x)}{\overline{\psi}_t(x)}\right) + \overline{\psi}_t(x)\log\left(\frac{\overline{\psi}_t(x)}{\psi_t(x)}\right)\right)\mathrm{d}x & \text{if } t<T, \\ 0 & \text{if } t=T, \end{cases} \tag{5.13}$$

where ψ_t and $\overline{\psi}_t$ are as shown in (3.47) and (3.60), respectively.

Lemma 5.2.1. *Let*

$$A^{(i)}_t = \frac{1}{2}\int_{\mathbb{X}}\psi_t(x)\log\left(\frac{\psi_t(x)}{\psi^{(i)}_t(x)}\right)\mathrm{d}x, \quad \textit{and} \quad B^{(i)}_t = \frac{1}{2}\int_{\mathbb{X}}\psi^{(i)}_t(x)\log\left(\frac{\psi^{(i)}_t(x)}{\psi_t(x)}\right)\mathrm{d}x, \tag{5.14}$$

for $0\le t<T$, where $A^{(i)}_T = B^{(i)}_T = 0$. Then,

$$KL_t(\psi_t||\overline{\psi}_t) = \sum_{i=1}^{n+1}(A^{(i)}_t + B^{(i)}_t)\mathbf{I}_t(i). \tag{5.15}$$

Proof. It's trivial when $t=T$. For some $t<T$, note that we can write

$$\begin{aligned} \int_{\mathbb{X}}\psi_t(x)\log\left(\frac{\psi_t(x)}{\overline{\psi}_t(x)}\right)\mathrm{d}x &= \int_{\mathbb{X}}\psi_t(x)\log\left(\frac{\psi_t(x)}{\sum_i\psi^{(i)}_t(x)\mathbf{I}_t(i)}\right)\mathrm{d}x = \\ = \int_{\mathbb{X}}\psi_t(x)\sum_i\log\left(\frac{\psi_t(x)}{\psi^{(i)}_t(x)}\right)\mathbf{I}_t(i)\,\mathrm{d}x &= \sum_i\left(\int_{\mathbb{X}}\psi_t(x)\log\left(\frac{\psi_t(x)}{\psi^{(i)}_t(x)}\right)\mathrm{d}x\right)\mathbf{I}_t(i), \end{aligned} \tag{5.16}$$

and similarly, we can write the following:

$$\begin{aligned} \int_{\mathbb{X}}\overline{\psi}_t(x)\log\left(\frac{\overline{\psi}_t(x)}{\psi_t(x)}\right)\mathrm{d}x &= \int_{\mathbb{X}}\left(\sum_i\psi^{(i)}_t(x)\mathbf{I}_t(i)\right)\sum_j\log\left(\frac{\psi^{(j)}_t(x)}{\psi_t(x)}\right)\mathbf{I}_t(j)\,\mathrm{d}x \\ &= \sum_i\left(\int_{\mathbb{X}}\psi^{(i)}_t(x)\log\left(\frac{\psi^{(i)}_t(x)}{\psi_t(x)}\right)\mathrm{d}x\right)\mathbf{I}_t(i). \end{aligned} \tag{5.17}$$

The statement (5.15) follows directly from (5.16) and (5.17). □

Recall that $\{W_t\}$ and $\{W_t^{(i)}\}$ are defined in (2.11) and (3.76), respectively. We also define

$$(\mu_t^{(i)})^{\hookrightarrow} = \frac{1}{4}\int_{\mathbb{X}} \psi_t(x)\left((\sigma_t^{(i)}(x))^2 - 2\sigma_t(x)\sigma_t^{(i)}(x)\rho^{(i)} + \sigma_t^2(x)\right)\,\mathrm{d}x, \tag{5.18}$$

$$(\mu_t^{(i)})^{\hookleftarrow} = \frac{1}{4}\int_{\mathbb{X}} \psi_t^{(i)}(x)\left((\sigma_t^{(i)}(x))^2 - 2\sigma_t(x)\sigma_t^{(i)}(x)\rho^{(i)} + \sigma_t^2(x)\right)\,\mathrm{d}x, \tag{5.19}$$

where $\rho^{(i)}$ is the correlation between $\{W_t\}$ and $\{W_t^{(i)}\}$. In addition,

$$(\sigma_t^{(i)})^{\hookrightarrow} = \frac{1}{2}\int_{\mathbb{X}} \sigma_t(x)\psi_t(x)\log\left(\frac{\psi_t(x)}{\psi_t^{(i)}(x)}\right)\,\mathrm{d}x \quad \text{and} \quad (\sigma_t^{(i)})^{\hookleftarrow} = \frac{1}{2}\int_{\mathbb{X}} \psi_t^{(i)}(x)\sigma_t(x)\,\mathrm{d}x, \tag{5.20}$$

and also

$$(\theta_t^{(i)})^{\hookrightarrow} = \frac{1}{2}\int_{\mathbb{X}} \psi_t(x)\sigma_t^{(i)}(x)\,\mathrm{d}x \ \text{ and } \ (\theta_t^{(i)})^{\hookleftarrow} = \frac{1}{2}\int_{\mathbb{X}} \sigma_t^{(i)}(x)\psi_t^{(i)}(x)\log\left(\frac{\psi_t^{(i)}(x)}{\psi_t(x)}\right)\,\mathrm{d}x. \tag{5.21}$$

Proposition 5.2.2. *Let $KL_t(\mathbb{Y}_t||\mathbb{G}_t)$ be the time-t KL asymmetry. Then,*

$$\begin{aligned}\mathrm{d}KL_t(\psi_t||\overline{\psi}_t) = &\sum_{i=1}^{n+1}((\mu_t^{(i)})^{\hookrightarrow} + (\mu_t^{(i)})^{\hookleftarrow})\mathbf{I}_t(i)\,\mathrm{d}t + \sum_{i=2}^{n+1}\left(A_t^{(i)} - A_t^{(i-1)} + B_t^{(i)} - B_t^{(i-1)}\right)\delta_{\tau_{i-1}}(\mathrm{d}t)\\ &+ \sum_{i=1}^{n+1}((\sigma_t^{(i)})^{\hookrightarrow} - (\sigma_t^{(i)})^{\hookleftarrow})\mathbf{I}_t(i)\,\mathrm{d}W_t + \sum_{i=1}^{n+1}\left((\theta_t^{(i)})^{\hookleftarrow} - (\theta_t^{(i)})^{\hookrightarrow}\right)\mathbf{I}_t(i)\,\mathrm{d}W_t^{(i)}.\end{aligned} \tag{5.22}$$

Proof. See Appendix 5.6.1. □

The KL asymmetry process between the market and the informed trader has jump-diffusion dynamics. For $t < \tau_1$, the process is zero. The drift and the diffusion coefficients of the asymmetry process jump, which quantify the impact of new information sources.

By definition, the KL asymmetry process takes the value zero at $t = T$. This is not simply an ad hoc condition we impose. Note that $\{\mathrm{KL}_t(\psi_t||\overline{\psi}_t)\}$ gets arbitrarily close to zero, as $t \to T$. This is due to $\lim_{t\to T}\pi_t(\mathrm{d}x) = \lim_{t\to T}\overline{\pi}_t(\mathrm{d}x) = \delta_{X_T}(\mathrm{d}x)$.

Remark 5.2.3. *The terms involving $\psi_t/\psi_t^{(i)}$ and $\psi_t^{(i)}/\psi_t$ can alternatively be written as*

$$\frac{\psi_t(x)}{\psi_t^{(i)}(x)} = C_t^{(i)}\zeta_t^{(i)}(x) \quad \textit{and} \quad \frac{\psi_t^{(i)}(x)}{\psi_t(x)} = \left(C_t^{(i)}\zeta_t^{(i)}(x)\right)^{-1}, \tag{5.23}$$

provided that

$$C_t^{(i)} = \frac{\int_{\mathbb{X}} \exp\left[\frac{T}{(T-t)}\left(\widehat{\kappa}^{(i)}x\widehat{\xi}_t^{(i)} - \frac{1}{2}(\widehat{\kappa}^{(i)}x)^2 t\right)\right]q(x)\,\mathrm{d}x}{\int_{\mathbb{X}} \exp\left[\frac{T}{(T-t)}\left(\kappa x\xi_t - \frac{1}{2}(\kappa x)^2 t\right)\right]q(x)\,\mathrm{d}x}, \tag{5.24}$$

and

$$\zeta_t^{(i)}(x) = \exp\left[\frac{T}{(T-t)}\left(\left(\kappa\xi_t - \widehat{\kappa}^{(i)}\widehat{\xi}_t^{(i)}\right)x - \frac{1}{2}\left((\kappa x)^2 - (\widehat{\kappa}^{(i)}x)^2\right)t\right)\right]. \tag{5.25}$$

Recall that $\widehat{\xi}_t^{(1)} = \xi_t$ *and* $\widehat{\kappa}^{(1)} = \kappa$*. Hence,* $C_t^{(1)} = 1$ *and* $\zeta_t^{(1)}(x) = 1$*. Also note that*

$$(\sigma_t^{(i)})^{\hookrightarrow} = \frac{T\kappa}{2(T-t)}\left(\int_{\mathbb{X}} x\psi_t(x)\log\left(C_t^{(i)}\zeta_t^{(i)}(x)\right)\mathrm{d}x - \mathbb{E}^{\mathbb{Q}}[X_T|\xi_t^1]A_t^{(i)}\right), \tag{5.26}$$

$$(\theta_t^{(i)})^{\hookleftarrow} = \frac{T\widehat{\kappa}^{(i)}}{2(T-t)}\left(\int_{\mathbb{X}} x\psi_t^{(i)}(x)\log\left(\left(C_t^{(i)}\zeta_t^{(i)}(x)\right)^{-1}\right)\mathrm{d}x - \mathbb{E}^{\mathbb{Q}}[X_T|\widehat{\xi}_t^{(i)}]B_t^{(i)}\right). \tag{5.27}$$

Figure 5.1 at the end of this chapter is a simulation of the KL asymmetry process. The process is zero until the informed trader receives an additional information source. Different colours represent different number of sources that the informed trader has. The parameters are $T = 5$, $\kappa^i = 1/T$ and $\rho^i = 0.25$. Stopping times are uniformly distributed on $[0, T]$.

5.2.2 Squared-Hellinger Asymmetry

We define the time-t *SH asymmetry*, which we denote by $\mathrm{SH}^2(.||.)$, as

$$\mathrm{SH}_t(\mathbb{Y}_t||\mathbb{G}_t) = \frac{1}{2}\int_{\Omega}\left(\sqrt{\frac{\mathrm{d}\mathbb{G}_t(\omega)}{\mathrm{d}\mathbb{Y}_t(\omega)}} - 1\right)^2 \mathrm{d}\mathbb{Y}_t(\omega) = \frac{1}{2}\int_{\Omega}\left(\sqrt{\frac{\mathrm{d}\mathbb{Y}_t(\omega)}{\mathrm{d}\mathbb{G}_t(\omega)}} - 1\right)^2 \mathrm{d}\mathbb{G}_t(\omega), \tag{5.28}$$

between $\mathbb{Y}_t$ and $\mathbb{G}_t$, which are the conditional measures given $\mathcal{F}_t^{\xi}$ and $\mathcal{G}_t$, respectively. Also, if $\mathbb{L}$ denotes the Lebesgue measure, and $\mathbb{Y}_t$ and $\mathbb{G}_t$ are equivalent to $\mathbb{L}$, then

$$\mathrm{SH}_t(\mathbb{Y}_t||\mathbb{G}_t) = \frac{1}{2}\int_{\Omega}\left(\sqrt{\frac{\mathrm{d}\mathbb{Y}_t(\omega)}{\mathrm{d}\mathbb{L}(\omega)}} - \sqrt{\frac{\mathrm{d}\mathbb{G}_t(\omega)}{\mathrm{d}\mathbb{L}(\omega)}}\right)^2 \mathrm{d}\mathbb{L}(\omega). \tag{5.29}$$

Using (5.29), we define $\{\mathrm{SH}_t(\psi_t||\overline{\psi}_t)\}_{t\in[0,T]}$ by

$$\mathrm{SH}_t(\psi_t||\overline{\psi}_t) = \begin{cases} \frac{1}{2}\int_{\mathbb{X}}\left(\sqrt{\psi_t} - \sqrt{\overline{\psi}_t}\right)^2 \mathrm{d}x & \text{if } t < T, \\ 0 & \text{if } t = T. \end{cases} \tag{5.30}$$

Following similar steps as done in the proof of Lemma 5.2.1, we have

$$\begin{aligned}\mathrm{SH}_t(\psi_t||\overline{\psi}_t) &= 1 - \int_{\mathbb{X}}\left(\sqrt{\psi_t(x)}\sqrt{\overline{\psi}_t(x)}\right)\mathrm{d}x \\ &= 1 - \sum_{i=1}^{n+1}\left(\int_{\mathbb{X}}\left(\sqrt{\psi_t(x)}\sqrt{\psi_t^{(i)}}\right)\mathrm{d}x\right)\mathbf{I}_t(i) = 1 - \sum_{i=1}^{n+1} M_t^{(i)}\mathbf{I}_t(i),\end{aligned} \tag{5.31}$$

for $0 \le t < T$. When we do calculations, we use the representation shown in (5.31).

We make the following definitions:

$$\mu_t^{(i)} = \frac{1}{8}\left(\int_{\mathbb{X}} \frac{\sqrt{\psi_t^{(i)}(x)}}{\sqrt{\psi_t^3(x)}}\sigma_t^2(x)\psi_t^2(x) + \frac{\sqrt{\psi_t(x)}}{\sqrt{(\psi_t^{(i)}(x))^3}}(\sigma_t^{(i)}(x))^2(\psi_t^{(i)}(x))^2\right)\mathrm{d}x$$
$$- \frac{1}{4}\int_{\mathbb{X}} \frac{\sigma_t^{(i)}(x)\psi_t^{(i)}(x)\sigma_t(x)\psi_t(x)\rho^{(i)}}{\sqrt{\psi_t(x)\psi_t^{(i)}(x)}}\,\mathrm{d}x, \tag{5.32}$$

and also

$$\upsilon_t^{(i)} = -\frac{1}{2}\int_{\mathbb{X}} \frac{\sqrt{\psi_t^{(i)}(x)}}{\sqrt{\psi_t(x)}}\sigma_t(x)\psi_t(x)\,\mathrm{d}x \quad\text{and}\quad \theta_t^{(i)} = -\frac{1}{2}\int_{\mathbb{X}} \frac{\sqrt{\psi_t(x)}}{\sqrt{\psi_t^{(i)}(x)}}\sigma_t^{(i)}(x)\psi_t^{(i)}(x)\,\mathrm{d}x. \tag{5.33}$$

Proposition 5.2.4. *Let $SH_t(\mathbb{Y}_t||\mathbb{G}_t)$ be the time-t SH asymmetry. Then,*

$$\mathrm{d}SH_t(\psi_t||\overline{\psi}_t) = \sum_{i=1}^{n+1}\mu_t^{(i)}\mathbf{I}_t(i)\,\mathrm{d}t + \sum_{i=1}^{n+1}\upsilon_t^{(i)}\mathbf{I}_t(i)\,\mathrm{d}W_t$$
$$+ \sum_{i=1}^{n+1}\theta_t^{(i)}\mathbf{I}_t(i)\,\mathrm{d}W_t^{(i)} - \sum_{i=2}^{n+1}(M_t^{(i)} - M_t^{(i-1)})\delta_{\tau_{i-1}}(\mathrm{d}t). \tag{5.34}$$

Proof. Using the Lebesgue Dominated Convergence,

$$-\mathrm{d}M_t^{(i)} = -\int_{\mathbb{X}}\left(\mathrm{d}\sqrt{\psi_t(x)}\sqrt{\psi_t^{(i)}(x)}\right)\mathrm{d}x - \int_{\mathbb{X}}\left(\sqrt{\psi_t(x)}\,\mathrm{d}\sqrt{\psi_t^{(i)}(x)}\right)\mathrm{d}x$$
$$- \int_{\mathbb{X}}\left(\mathrm{d}\sqrt{\psi_t(x)}\,\mathrm{d}\sqrt{\psi_t^{(i)}(x)}\right)\mathrm{d}x, \tag{5.35}$$

for $0 \le t < T$. We write $-\mathrm{d}M_t^{(i)} = J_1^* + J_2^* + J_3^*$, and define $g_t^{(i)}(x) = \sqrt{\psi_t^{(i)}(x)}$. Then,

$$\mathrm{d}g_t^{(i)} = \frac{1}{2\sqrt{\psi_t^{(i)}}}\sigma_t^{(i)}\psi_t^{(i)}\,\mathrm{d}W_t^{(i)} - \frac{1}{8(\sqrt{\psi_t^{(i)}})^3}(\sigma_t^{(i)})^2(\psi_t^{(i)})^2\,\mathrm{d}t. \tag{5.36}$$

It follows that

$$J_1^* = -\int_{\mathbb{X}}\left[\frac{1}{2}\frac{\sqrt{\psi_t^{(i)}(x)}}{\sqrt{\psi_t(x)}}\,\mathrm{d}\psi_t(x) - \frac{1}{8}\frac{\sqrt{\psi_t^{(i)}(x)}}{\sqrt{\psi_t^3(x)}}\sigma_t^2(x)\psi_t^2(x)\,\mathrm{d}t\right]\mathrm{d}x, \tag{5.37}$$

$$J_2^* = -\int_{\mathbb{X}}\left[\frac{1}{2}\frac{\sqrt{\psi_t(x)}}{\sqrt{\psi_t^{(i)}(x)}}\,\mathrm{d}\psi_t^{(i)}(x) - \frac{1}{8}\frac{\sqrt{\psi_t(x)}}{\sqrt{(\psi_t^{(i)}(x))^3}}(\sigma_t^{(i)}(x))^2(\psi_t^{(i)}(x))^2\,\mathrm{d}t\right]\mathrm{d}x, \tag{5.38}$$

and finally, we have

$$J_3^* = -\frac{1}{4}\int_{\mathbb{X}} \left[\frac{\sigma_t^{(i)}(x)\psi_t^{(i)}(x)\sigma_t(x)\psi_t(x)\rho^{(i)}\,\mathrm{d}t}{\sqrt{\psi_t(x)\psi_t^{(i)}(x)}} \right] \mathrm{d}x, \tag{5.39}$$

where $\rho^{(i)}$ is the correlation between $\{W_t\}$ and $\{W_t^{(i)}\}$. This completes the proof. □

The process $\{\mathrm{SH}_t(\psi_t||\overline{\psi}_t\}$ is a jump-diffusion process. It takes the value zero at $t = T$. This is not an ad hoc condition, since $\{\mathrm{SH}_t(\psi_t||\overline{\psi}_t)\}$ gets arbitrarily close to zero as $t \to T$.

Figure 5.2 at the end of this chapter is a simulation of the SH asymmetry process, where $T = 5$, $\kappa^i = 1/T$ and $\rho^i = 0.25$. Stopping times are uniformly distributed on $[0, T]$.

5.3 Competitive Edge in Information

We consider a financial setting where there are two informed traders who are unaware of each others' actions. We assume that the filtrations of the informed traders are different n-order piecewise enlargements of the same market filtration. This is a valid assumption, since not every informed trader has the same accessibility to extra information. Some informed traders may have better facilities to extract additional information compared to other informed traders. As an example, we focus on a scenerio where the informed traders are provided with extra sources of information (the same sources of information) at different stopping times. Then at a given time, an informed trader has a competitive edge with respect to the other if she has access to more information sources, which can be used to seek statistical arbitrage opportunities. We aim to quantify this competitive edge in a dynamic framework. We call these traders Agent 1 and Agent 2.

First, we define two independent sequences of stopping times $\{\tau_i\}_{i=1}^n$ and $\{\tau_i^*\}_{i=1}^n$ such that $\tau_1 < \tau_2 < \ldots < \tau_n$ and $\tau_1^* < \tau_2^* < \ldots < \tau_n^*$. We note that $\mathbb{Q}(\tau_i < \tau_j^*) \neq 1$ and $\mathbb{Q}(\tau_i^* < \tau_j) \neq 1$ for $i, j = 1, \ldots, n$. Also, $\tau_i \neq \tau_j^*$ for any i, j. We let $\{\mathcal{G}_t\}$ as shown in (3.58) be the filtration of Agent 1. We define $\{\mathcal{G}_t^*\}$ as the filtration of Agent 2, such that

$$\mathcal{G}_t^* = \mathcal{F}_t^{\xi^1} \bigvee_{i=1}^{n} \begin{cases} \sigma(\{H_{\tau_i^*}(s)\}_{0\le s\le t}) & \tau_i^* > t, \\ \sigma(\{H_{\tau_i^*}(s)\}_{0\le s\le t}, \{\xi_s^{i+1}\}_{\tau_i^*\le s\le t}) & \tau_i^* \le t, \end{cases} \tag{5.40}$$

for $0 \le t \le T$. Then τ_i^*'s are $\mathcal{G}_t^*$-stopping times. Also, we let

$$\mathbf{I}_t^* = \left[1 - H_{\tau_1^*}(t), \ldots, H_{\tau_{i-1}^*}(t)(1 - H_{\tau_i^*}(t)), \ldots H_{\tau_n^*}(t)\right]^\top. \tag{5.41}$$

For demonstration purposes, we shall quantify the competitive edge using only the KL asymmetry process. The SH asymmetry process can also be used in a similar sense.

We also define the following probability measure:

$$\mathbb{G}_t^*(X_T \in A) = \mathbb{Q}(X_T \in A|\mathcal{G}_t^*), \tag{5.42}$$

and denote it by $\mathbb{G}_t^*$. In addition, denoting $\psi_t^*(x) = q(x|\mathcal{G}_t^*)$, we write

$$\mathrm{KL}_t(\overline{\psi}_t||\psi_t^*) = \begin{cases} \frac{1}{2}\int_{\mathbb{X}} \left(\psi_t^*(x)\log\left(\frac{\psi_t^*(x)}{\overline{\psi}_t(x)}\right) + \overline{\psi}_t(x)\log\left(\frac{\overline{\psi}_t(x)}{\psi_t^*(x)}\right)\right)\,\mathrm{d}x & \text{if } t < T, \\ 0 & \text{if } t = T. \end{cases} \tag{5.43}$$

Lemma 5.3.1. *Let*

$$A_t^{(i,j)} = \frac{1}{2}\int_{\mathbb{X}} \psi_t^{(j)}(x)\log\left(\frac{\psi_t^{(j)}(x)}{\psi_t^{(i)}(x)}\right)\,\mathrm{d}x, \quad \textit{and} \quad B_t^{(i,j)} = \frac{1}{2}\int_{\mathbb{X}} \psi_t^{(i)}(x)\log\left(\frac{\psi_t^{(i)}(x)}{\psi_t^{(j)}(x)}\right)\,\mathrm{d}x, \tag{5.44}$$

for $0 \le t < T$*, where* $A_T^{(i,j)} = B_T^{(i,j)} = 0$*. Then,*

$$KL_t(\overline{\psi}_t||\psi_t^*) = \sum_{i=1}^{n+1}\sum_{j=1}^{n+1}(A_t^{(i,j)} + B_t^{(i,j)})\mathbf{I}_t^*(j)\mathbf{I}_t(i). \tag{5.45}$$

Proof. It's trivial when $t = T$. For some $t < T$, using Lemma 5.2.1, we have

$$\begin{aligned}
\int_{\mathbb{X}} \psi_t^*(x)\log\left(\frac{\psi_t^*(x)}{\overline{\psi}_t(x)}\right)\,\mathrm{d}x &= \sum_i \left(\int_{\mathbb{X}} \psi_t^*(x)\log\left(\frac{\psi_t^*(x)}{\psi_t^{(i)}(x)}\right)\,\mathrm{d}x\right)\mathbf{I}_t(i) \\
&= \sum_i \left(\int_{\mathbb{X}} \left(\sum_j \psi_t^{(j)}(x)\mathbf{I}_t^*(j)\sum_k \log\left(\frac{\psi_t^{(k)}(x)}{\psi_t^{(i)}(x)}\right)\mathbf{I}_t^*(k)\right)\,\mathrm{d}x\right)\mathbf{I}_t(i) \\
&= \sum_i\sum_j \left(\int_{\mathbb{X}} \psi_t^{(j)}(x)\log\left(\frac{\psi_t^{(j)}(x)}{\psi_t^{(i)}(x)}\right)\,\mathrm{d}x\right)\mathbf{I}_t^*(j)\mathbf{I}_t(i), \qquad (5.46)
\end{aligned}$$

and similarly, the following can be written:

$$\begin{aligned}
\int_{\mathbb{X}} \overline{\psi}_t(x)\log\left(\frac{\overline{\psi}_t(x)}{\psi_t^*(x)}\right)\,\mathrm{d}x &= \sum_i \left(\int_{\mathbb{X}} \psi_t^{(i)}(x)\log\left(\frac{\psi_t^{(i)}(x)}{\psi_t^*(x)}\right)\,\mathrm{d}x\right)\mathbf{I}_t(i) \\
&= \sum_i \left(\int_{\mathbb{X}} \psi_t^{(i)}(x)\sum_j \log\left(\frac{\psi_t^{(i)}(x)}{\psi_t^{(j)}(x)}\right)\mathbf{I}_t^*(j)\,\mathrm{d}x\right)\mathbf{I}_t(i) \\
&= \sum_i\sum_j \left(\int_{\mathbb{X}} \psi_t^{(i)}(x)\log\left(\frac{\psi_t^{(i)}(x)}{\psi_t^{(j)}(x)}\right)\,\mathrm{d}x\right)\mathbf{I}_t^*(j)\mathbf{I}_t(i), \qquad (5.47)
\end{aligned}$$

and the result follows. □

Lemma 5.3.1 implies the following: At some time t, where $\mathbf{I}_t^*(j)\mathbf{I}_t(i) = 1$ for the chosen

i and j, if $i = j$, there is no competitive edge between the informed traders, since $A_t^{(i,j)} = B_t^{(i,j)} = 0$. The information asymmetry between them is zero. On the other hand, if $i > j$, then $A_t^{(i,j)} \neq 0$ and $B_t^{(i,j)} \neq 0$ in favor of Agent 1. That is, Agent 1 has informational advantage over Agent 2. If $j > i$, then $A_t^{(i,j)} \neq 0$ and $B_t^{(i,j)} \neq 0$ in favor of Agent 2. We let

$$(\mu_t^{(i,j)})^{\hookrightarrow} = \frac{1}{4}\int_{\mathbb{X}} \psi_t^{(j)}(x)\left((\sigma_t^{(i)}(x))^2 - 2\sigma_t^{(j)}(x)\sigma_t^{(i)}(x)\rho^{(i,j)} + (\sigma_t^{(j)})^2(x)\right) \mathrm{d}x, \tag{5.48}$$

$$(\mu_t^{(i,j)})^{\hookleftarrow} = \frac{1}{4}\int_{\mathbb{X}} \psi_t^{(i)}(x)\left((\sigma_t^{(i)}(x))^2 - 2\sigma_t^{(j)}(x)\sigma_t^{(i)}(x)\rho^{(i,j)} + (\sigma_t^{(j)})^2(x)\right) \mathrm{d}x, \tag{5.49}$$

where $\rho^{(i,j)}$ is the correlation between $\{W_t^{(i)}\}$ and $\{W_t^{(j)}\}$, and we let

$$(\sigma_t^{(i,j)})^{\hookrightarrow} = \frac{T\widehat{\kappa}^{(j)}}{2(T-t)}\left(\int_{\mathbb{X}} x\psi_t^{(j)}(x)\log\left(\frac{\psi_t^{(j)}(x)}{\psi_t^{(i)}(x)}\right) \mathrm{d}x - \mathbb{E}^{\mathbb{Q}}[X_T|\widehat{\xi}_t^{(j)}]A_t^{(i,j)}\right), \tag{5.50}$$

$$(\sigma_t^{(i,j)})^{\hookleftarrow} = \frac{1}{2}\int_{\mathbb{X}} \psi_t^{(i)}(x)\sigma_t^{(j)}(x)\,\mathrm{d}x, \tag{5.51}$$

and also,

$$(\theta_t^{(i,j)})^{\hookrightarrow} = \frac{1}{2}\int_{\mathbb{X}} \psi_t^{(j)}(x)\sigma_t^{(i)}(x)\,\mathrm{d}x, \tag{5.52}$$

$$(\theta_t^{(i,j)})^{\hookleftarrow} = \frac{T\widehat{\kappa}^{(i)}}{2(T-t)}\left(\int_{\mathbb{X}} x\psi_t^{(i)}(x)\log\left(\frac{\psi_t^{(i)}(x)}{\psi_t^{(j)}(x)}\right) \mathrm{d}x - \mathbb{E}^{\mathbb{Q}}[X_T|\widehat{\xi}_t^{(i)}]B_t^{(i,j)}\right). \tag{5.53}$$

Proposition 5.3.2. *Let $KL_t(\mathbb{G}_t||\mathbb{G}_t^*)$ be the time-t KL asymmetry. Then, the competition between Agent 1 and Agent 2 has the following dynamics:*

$$\begin{aligned}
\mathrm{d}KL_t(\overline{\psi}_t||\psi_t^*) &= \sum_{j=1}^{n+1}\sum_{i=1}^{n+1}((\mu_t^{(i,j)})^{\hookrightarrow} + (\mu_t^{(i,j)})^{\hookleftarrow})\mathbf{I}_t(i)\mathbf{I}_t^*(j)\,\mathrm{d}t \\
&+ \sum_{j=1}^{n+1}\sum_{i=1}^{n+1}((\sigma_t^{(i,j)})^{\hookrightarrow} - (\sigma_t^{(i,j)})^{\hookleftarrow})\mathbf{I}_t(i)\mathbf{I}_t^*(j)\,\mathrm{d}W_t^{(j)} \\
&+ \sum_{j=1}^{n+1}\sum_{i=1}^{n+1}\left((\theta_t^{(i,j)})^{\hookleftarrow} - (\theta_t^{(i,j)})^{\hookrightarrow}\right)\mathbf{I}_t(i)\mathbf{I}_t^*(j)\,\mathrm{d}W_t^{(i)} \\
&+ \sum_{j=1}^{n+1}\sum_{i=2}^{n+1}\left(A_t^{(i,j)} - A_t^{(i-1,j)} + B_t^{(i,j)} - B_t^{(i-1,j)}\right)\delta_{\tau_{i-1}}(\mathrm{d}t)\mathbf{I}_t^*(j) \\
&+ \sum_{i=1}^{n+1}\sum_{j=2}^{n+1}\left(A_t^{(i,j)} - A_t^{(i,j-1)} + B_t^{(i,j)} - B_t^{(i,j-1)}\right)\delta_{\tau_{j-1}^*}(\mathrm{d}t)\mathbf{I}_t(i).
\end{aligned} \tag{5.54}$$

Proof. Using Lemma 5.3.1, the proof is almost the same as shown in Appendix 5.6.1. Note that since $\tau_i \neq \tau_j^*$ for any i, j, we have $\delta_{\tau_{j-1}^*}(\mathrm{d}t)\delta_{\tau_{i-1}}(\mathrm{d}t) = 0$. □

Figures 5.3 and 5.4 at the end of this chapter are simulations of the KL asymmetry process between two informed traders. The process is zero when both agents have the same number of information sources. If the colour is red, Agent 1 has an informational advantage over Agent 2. If the colour is blue, Agent 2 has an advantage over Agent 1. The parameters are $T = 5$, $\kappa^i = 1/T$ and $\rho^i = 0.25$. Stopping times are uniformly distributed on $[0, T]$.

5.4 Financial Mispricing and Information Asymmetry

We view financial mispricing as a special type of information asymmetry. The market receives incorrect information about X_T that will not be paid at $t = T$. We postulate the existence of a fundamental information flow, which carries information about the correct cash flow $X_T - c$, for some constant $c \in \mathbb{R}$. We call c the mispricing component. The market receives the fundamental information flow at some stopping time $\tau \in (0, T)$. Let

$$\mathcal{F}_t^{\xi^b} = \sigma(\{\xi_s^b\}_{0\leq s\leq t}) \qquad \text{where} \qquad \xi_t^b = \kappa^b X_T t + B_{tT}^b, \tag{5.55}$$

for $0 \leq t \leq T$ and $0 < \kappa^b < \infty$. We assume that $\{\xi_t^b\}_{t\in[0,T]}$ carries partial information about the wrong cash flow X_T. We call $\{\xi_t^b\}$ a mispriced information process. We let

$$\xi_t^c = \kappa^b X_T t - \kappa c t + B_{tT}^c, \tag{5.56}$$

be the fundamental information process, where $\{B_{tT}^c - \kappa c t\}$ is a Brownian bridge to the value $-\kappa c T$ at $t = T$ (see also, Andruszkiewicz and Brody, 2011, who include a drift in the noise term to model anomalous price dynamics). To simplify calculations, we set

$$\kappa = \sqrt{\frac{2(\kappa^b)^2(1-\rho)}{1-\rho^2}}, \tag{5.57}$$

where $|\rho| < 1$ is the correlation between $\{B_{tT}^b\}$ and $\{B_{tT}^c\}$. Note that $0 < \kappa < \infty$. We define

$$\mathcal{J}_t = \sigma(\{H_\tau(s)\}_{0\leq s\leq t}, \{\xi_s^b\}_{0\leq s\leq t}, \{\xi_s^c\}_{0\leq s\leq t}), \tag{5.58}$$

for $0 \leq t \leq T$, where τ is the independent $\mathcal{J}_t$-stopping time. We define

$$\xi_t^* = \kappa(X_T - c)t + B_{tT}^*, \tag{5.59}$$

where the Brownian bridge $\{B_{tT}^*\}_{t\in[0,T]}$ is

$$B_{tT}^* = \frac{1}{\kappa}\left[\frac{\kappa^b(1-\rho)}{(1-\rho^2)}(B_{tT}^b + B_{tT}^c)\right], \tag{5.60}$$

and where κ is as shown in (5.57). Note that (5.60) follows similarly to (3.66). Since a filtration generated by both $\{\xi_t^b\}$ and $\{\xi_t^c\}$ is equivalent to a filtration generated by $\{\xi_t^*\}$, we can write

$$\psi_t^*(x) = q(x|\mathcal{J}_t) = q(x|\xi_t^*) \quad \text{for } 0 \le t \le T. \tag{5.61}$$

Lemma 5.4.1. *The dynamics of* $\{\psi_t^*\}_{t\in[0,T)}$ *are governed by the following SDE:*

$$\mathrm{d}\psi_t^*(x) = \sigma_t^*(x)\psi_t^*(x)\,\mathrm{d}W_t^* + \frac{T\kappa c}{(T-t)}\sigma_t^*(x)\psi_t^*(x)\,\mathrm{d}t, \tag{5.62}$$

where $\{W_t^*\}_{t\in[0,T)}$ *is a* $\mathbb{Q}$*-Brownian motion with negative drift if* $c > 0$*, or with positive drift if* $c < 0$*, satisfying*

$$W_t^* = \xi_t^* + \int_0^t \frac{1}{T-s}\xi_s^*\,\mathrm{d}s - T\kappa\int_0^t \frac{1}{T-s}\mathbb{E}^{\mathbb{Q}}\left[X_T\,|\,\xi_s^*\right]\,\mathrm{d}s, \tag{5.63}$$

and where

$$\sigma_t^*(x) = \frac{T\kappa\left(x - \mathbb{E}^{\mathbb{Q}}\left[X_T\,|\,\xi_t^*\right]\right)}{(T-t)}. \tag{5.64}$$

Proof. See Appendix 5.6.2. □

Alternative to Appendix 5.6.2, to see that $\{W_t^*\}$ is a $\mathbb{Q}$-Brownian motion with drift, we let $Y_T = X_T - c$. Then,

$$\mathrm{d}\psi_t^*(y) = \sigma_t^*(y)\psi_t^*(y)\,\mathrm{d}Z_t, \tag{5.65}$$

with $y = x - c$, where $\{Z_t\}$ is a $\mathbb{Q}$-Brownian and

$$\sigma_t^*(y) = \frac{T\kappa(y - \mathbb{E}^{\mathbb{Q}}\left[Y_T\,|\,\xi_t^*\right])}{(T-t)} = \frac{T\kappa(x - \mathbb{E}^{\mathbb{Q}}\left[X_T\,|\,\xi_t^*\right])}{(T-t)} = \sigma_t^*(x). \tag{5.66}$$

More specifically, $\{Z_t\}$ is a $\mathbb{Q}$-Brownian motion such that

$$\mathrm{d}Z_t = \mathrm{d}W_t^* + \frac{T\kappa c}{(T-t)}\,\mathrm{d}t, \tag{5.67}$$

or in the integral form

$$Z_t = \xi_t^* + \int_0^t \frac{1}{T-s}\xi_s^*\,\mathrm{d}s - T\kappa\int_0^t \frac{1}{T-s}\mathbb{E}^{\mathbb{Q}}\left[X_T\,|\,\xi_s^*\right]\,\mathrm{d}s + T\kappa c\int_0^t \frac{1}{T-s}\,\mathrm{d}s. \tag{5.68}$$

Using (5.62) and (5.63) together with (5.67) and (5.68) is another way of seeing that $\{W_t^*\}$ is a $\mathbb{Q}$-Brownian motion with negative drift if $c > 0$, or with positive drift if $c < 0$. Then, from (5.67), equation (5.62) can be written as

$$\mathrm{d}\psi_t^*(x) = \sigma_t^*(x)\psi_t^*(x)\,\mathrm{d}Z_t, \tag{5.69}$$

which we shall make use of in the following statements. We define the filtration $\{\mathcal{Z}_t\}$ by

$$\mathcal{Z}_t = \mathcal{F}_t^{\xi^b} \bigvee \begin{cases} \sigma(\{H_\tau(s)\}_{0\leq s\leq t}) & \tau > t, \\ \sigma(\{H_\tau(s)\}_{0\leq s\leq t}, \{\xi_s^c\}_{\tau\leq s\leq t}) & \tau \leq t. \end{cases} \tag{5.70}$$

for $0 \leq t \leq T$, and assume that $\{\mathcal{Z}_t\}$ is the market filtration. Note that the market is initially provided with the incorrect information until the correct information appears at τ.

We define $\sigma_t^b(x) = T\kappa^b \left(x - \mathbb{E}^{\mathbb{Q}}\left[X_T \,\middle|\, \xi_t^b\right]\right)/(T-t)$. Also, $\{W_t^b\}$ is a $\mathbb{Q}$-Brownian motion given by $W_t^b = \xi_t^b + \int_0^t \frac{1}{T-s}\xi_s^b \,\mathrm{d}s - T\kappa^b \int_0^t \frac{1}{T-s}\mathbb{E}^{\mathbb{Q}}\left[X_T \,\middle|\, \xi_t^b\right] \,\mathrm{d}s$. We also denote $\psi_t^a(x) = q(x|\mathcal{Z}_t)$ and $\psi_t^b(x) = q(x|\mathcal{F}_t^{\xi^b})$.

Proposition 5.4.2. *The dynamics of $\{\psi_t^a\}_{t\in[0,T)}$ are governed by the following SDE:*

$$\mathrm{d}\psi_t^a(x) = \sigma_t^b(x)\psi_t^b(x)\,\mathrm{d}W_t^b\mathbf{I}_t(1) + \sigma_t^*(x)\psi_t^*(x)\,\mathrm{d}Z_t\mathbf{I}_t(2) + (\psi_t^*(x) - \psi_t^b(x))\delta_\tau(\mathrm{d}t). \tag{5.71}$$

Proof. The dynamics for $\{\psi_t^b\}$ follow directly from (3.17)-(3.18). The SDE of $\{\psi_t^a\}$ is derived by using the law of total probability and by following the steps as done in Chapter 3. □

5.4.1 Mispricing Processes

We shall only provide the dynamics of what we call the SH mispricing process. The SH mispricing between ψ_t^a and ψ_t^* is the SH asymmetry between ψ_t^a and ψ_t^*. The KL mispricing process can be introduced in a similar sense. We define the following probability measures:

$$\mathbb{Z}_t(X_T \in A) = \mathbb{Q}(X_T \in A|\mathcal{Z}_t) \qquad \text{and} \qquad \mathbb{J}_t(X_T \in A) = \mathbb{Q}(X_T \in A|\mathcal{J}_t), \tag{5.72}$$

and denote them as $\mathbb{Z}_t$ and $\mathbb{J}_t$, respectively. Note that

$$\mathrm{SH}_t(\psi_t^a||\psi_t^*) = \begin{cases} \left(1 - \left(\int_{\mathbb{X}}\left(\sqrt{\psi_t^b(x)}\sqrt{\psi_t^*}\right) dx\right)\right)\mathbf{I}_t(1) & \text{if } t < T, \\ 0 & \text{if } t = T, \end{cases}$$

and hence, we can write the following:

$$\mathrm{SH}_t(\psi_t^a||\psi_t^*) = \mathrm{SH}_t(\psi_t^b||\psi_t^*)\mathbf{I}_t(1). \tag{5.73}$$

We define

$$\begin{aligned} \mu_t = {} & \frac{1}{8}\left(\int_{\mathbb{X}} \frac{\sqrt{\psi_t^b(x)}}{\sqrt{(\psi_t^*)^3(x)}}(\sigma_t^*)^2(x)(\psi_t^*)^2(x) + \frac{\sqrt{\psi_t^*(x)}}{\sqrt{(\psi_t^b)^3(x)}}(\sigma_t^b)^2(x)(\psi_t^b)^2(x)\right)\mathrm{d}x \\ & - \frac{1}{4}\int_{\mathbb{X}} \frac{\sigma_t^b(x)\psi_t^b(x)\sigma_t^*(x)\psi_t^*(x)\rho^{b,*}}{\sqrt{\psi_t^*(x)\psi_t^b(x)}}\,\mathrm{d}x, \end{aligned} \tag{5.74}$$

where $\rho^{b,*}$ is the correlation between $\{W_t^b\}$ and $\{Z_t\}$, and

$$\upsilon_t = -\frac{1}{2}\int_{\mathbb{X}} \frac{\sqrt{\psi_t^b(x)}}{\sqrt{\psi_t^*(x)}}\sigma_t^*(x)\psi_t^*(x)\,\mathrm{d}x \qquad \text{and} \qquad \theta_t = -\frac{1}{2}\int_{\mathbb{X}} \frac{\sqrt{\psi_t^*(x)}}{\sqrt{\psi_t^b(x)}}\sigma_t^b(x)\psi_t^b(x)\,\mathrm{d}x. \tag{5.75}$$

Proposition 5.4.3. *Let $SH_t(\mathbb{J}_t||\mathbb{Z}_t)$ be the time-t SH mispricing. Then,*

$$\mathrm{d}SH_t(\psi_t^a||\psi_t^*) = \mu_t\mathbf{I}_t(1)\,\mathrm{d}t + \theta_t\mathbf{I}_t(1)\,\mathrm{d}W_t^b + \upsilon_t\mathbf{I}_t(1)\,\mathrm{d}Z_t - SH_t(\psi_t^b||\psi_t^*)\delta_\tau(\mathrm{d}t). \tag{5.76}$$

Proof. Using Proposition 5.4.2, the proof is almost exactly the same to that of Proposition 5.2.4. □

The SH mispricing process is a diffusion process with drift for $t < \tau$. At τ, the process jumps to zero and remains zero. In other words, when the correct information flow appears in the market at τ, the information asymmetry between the market and the fundamentals jumps to zero. This jump represents a sudden market correction.

One can interpret that the time-t price of the asset is not the correct price of the asset prior to the appearance of the fundamental information flow. Prior to this appearance, the market has incorrect expectations about the future cash flow, since the asset will actually pay $X_T - c$ instead of X_T at time T. Then, with the emergence of the correct information, the market abruptly changes the price, which represents the sudden market correction on the price of the asset.

5.5 Shannon Entropy

The Shannon entropy quantifies the level of uncertainty or the lack of information in a given system. The higher entropy is, the lower the information content is (see, for example, Jaynes, 1982, Cover and Thomas, 1991). We shall provide the dynamics of a Shannon entropy process with respect to an n-order piecewise enlargement to quantify the level of uncertainty of an informed trader.

For this section, we assume X_T is a discrete cash flow (see Chapter 4.1 for neccessary notations of the discrete setting). The Shannon entropy, which we denote by S, is

$$S = -\sum_{\mathbb{X}} p(x_j)\log p(x_j). \tag{5.77}$$

Equation (5.77) is the standard way to define the Shannon entropy. As a continuous extension, one may also define entropy using probability densities, which is often called the differential entropy. However, unlike the Shannon entropy, the differential entropy is usually not a good measure of uncertainty. For example, differential entropy can be negative, whereas $S \in \mathbb{R}_+$. It is possible to define a Shannon entropy process by introducing a time dimension into the setting (see, for instance, Brody and Hughston, 2002). We shall consider the case when new information sources appear at stopping times. Again, we assume that the filtration of the informed trader is given by (3.58). We define a Shannon entropy process

$$\overline{S}_t = -\sum_{\mathbb{X}} \overline{\phi}_t(x_j) \log \overline{\phi}_t(x_j). \tag{5.78}$$

We also define

$$\mu_t^{(i)} = -\sum_{\mathbb{X}} \frac{(\sigma_t^{(i)}(x_j))^2}{2} \phi_t^{(i)}(x_j), \tag{5.79}$$

$$\zeta_t^{(i)} = \frac{T\widehat{\kappa}^{(i)}}{(T-t)} \left(X_t^{(i)} S_t^{(i)} - \sum_{\mathbb{X}} x_j \phi_t^{(i)}(x_j) \log \phi_t^{(i)}(x_j) \right), \tag{5.80}$$

where $S_t^{(i)} = -\sum_{\mathbb{X}} \phi_t^{(i)}(x_j) \log \phi_t^{(i)}(x_j)$ and $X_t^{(i)} = \mathbb{E}^{\mathbb{Q}}[X_T \,|\, \widehat{\xi}_t^{(i)}]$.

Proposition 5.5.1. *The entropy process $\{\overline{S}_t\}$ is governed by the following SDE:*

$$\mathrm{d}\overline{S}_t = \sum_{i=1}^{n+1} \mu_t^{(i)} \mathbf{I}_t(i)\,\mathrm{d}t + \sum_{i=1}^{n+1} \zeta_t^{(i)} \mathbf{I}_t(i)\,\mathrm{d}W_t^{(i)} + \sum_{i=2}^{n+1} \left(S_t^{(i)} - S_t^{(i-1)} \right) \delta_{\tau_{i-1}}(\mathrm{d}t). \tag{5.81}$$

Proof. At a fixed time t, for $0 \le t < T$, the Shannon entropy $\overline{S}_t$ can be rewritten as

$$\begin{aligned}
\overline{S}_t &= -\sum_{\mathbb{X}} \mathbf{R}_t(x_j)\mathbf{I}_t \log\left(\mathbf{R}_t(x_j)\mathbf{I}_t\right) = -\sum_{\mathbb{X}} \left[\sum_{i=1}^{n+1} \phi_t^{(i)}(x_j)\mathbf{I}_t(i) \log\left(\sum_{j=1}^{n+1} \phi_t^{(j)}(x_j)\mathbf{I}_t(j) \right) \right] \\
&= -\sum_{\mathbb{X}} \left[\sum_{i=1}^{n+1} \phi_t^{(i)}(x_j)\mathbf{I}_t(i) \sum_{j=1}^{n+1} \log\left(\phi_t^{(j)}(x_j)\right) \mathbf{I}_t(j) \right] \\
&= -\sum_{i=1}^{n+1} \left(\sum_{\mathbb{X}} \phi_t^{(i)}(x_j) \log \phi_t^{(i)}(x_j) \right) \mathbf{I}_t(i).
\end{aligned} \tag{5.82}$$

Then, we can write

$$\begin{aligned}
\mathrm{d}S_t^{(i)} = &-\sum_{\mathbb{X}} \mathrm{d}\phi_t^{(i)}(x_j) \log \phi_t^{(i)}(x_j) - \sum_{\mathbb{X}} \phi_t^{(i)}(x_j)\,\mathrm{d}\log \phi_t^{(i)}(x_j) \\
&- \sum_{\mathbb{X}} \mathrm{d}\phi_t^{(i)}(x_j)\,\mathrm{d}\log \phi_t^{(i)}(x_j).
\end{aligned} \tag{5.83}$$

We define the function: $g_t^{(i)} = \log \phi_t^{(i)}$. Then,

$$\mathrm{d}g_t^{(i)} = \frac{1}{\phi_t^{(i)}}\,\mathrm{d}\phi_t^{(i)} - \frac{1}{2(\phi_t^{(i)})^2}(\mathrm{d}\phi_t^{(i)})^2 = \sigma_t^{(i)}\,\mathrm{d}W_t^{(i)} - \frac{1}{2}(\sigma_t^{(i)})^2\,\mathrm{d}t. \tag{5.84}$$

It follows that

$$-\sum_{\mathbb{X}} \mathrm{d}\phi_t^{(i)}(x_j)\,\mathrm{d}\log\phi_t^{(i)}(x_j) = -\sum_{\mathbb{X}} (\sigma_t^{(i)}(x_j))^2\phi_t^{(i)}(x_j)\,\mathrm{d}t. \tag{5.85}$$

In addition,

$$-\sum_{\mathbb{X}} \mathrm{d}\phi_t^{(i)}(x_j)\log\phi_t^{(i)}(x_j) = -\sum_{\mathbb{X}} \sigma_t^{(i)}(x_j)\phi_t^{(i)}(x_j)\,\mathrm{d}W_t\log\phi_t^{(i)}(x_j), \tag{5.86}$$

and the second term is

$$\begin{aligned}
-\sum_{\mathbb{X}} \phi_t^{(i)}(x_j)\,\mathrm{d}\log\phi_t^{(i)}(x_j) &= -\sum_{\mathbb{X}} \sigma_t^{(i)}(x_j)\phi_t^{(i)}(x_j)\,\mathrm{d}W_t + \sum_{\mathbb{X}} \frac{1}{2}(\sigma_t^{(i)}(x_j))^2\phi_t^{(i)}(x_j)\,\mathrm{d}t \\
&= \sum_{\mathbb{X}} \frac{1}{2}(\sigma_t^{(i)}(x_j))^2\phi_t^{(i)}(x_j)\,\mathrm{d}t - \sum_{\mathbb{X}} \mathrm{d}\phi_t^{(i)}(x_j) \\
&= \sum_{\mathbb{X}} \frac{1}{2}(\sigma_t^{(i)}(x_j))^2\phi_t^{(i)}(x_j)\,\mathrm{d}t.
\end{aligned} \tag{5.87}$$

The statement follows from Lemma 3.2.4. □

Note that the Shannon entropy process follows jump-diffusion dynamics. At each entry of a new information source, the level of uncertainty jumps to a new one. Also, it follows that

$$\mathbb{E}^{\mathbb{Q}}[S_t^{(i)}] \leq \mathbb{E}^{\mathbb{Q}}[S_t^{(i-1)}], \tag{5.88}$$

for $0 \leq t < T$, since $S_t^{(i)}$ is defined in terms of an additional information source about X_T when compared to $S_t^{(i-1)}$. Hence, the expected values of jump sizes of $\{\overline{S}_t\}$ are nonpositive. In addition, note that

$$\mathbb{E}^{\mathbb{Q}}[\mu_t^{(i)}] \leq 0. \tag{5.89}$$

Then the following remark can be written:

Remark 5.5.2. *The Shannon entropy process $\{\overline{S}_t\}$ is a $\mathbb{Q}$-supermartingale.*

The uncertainty of the informed trader is decreasing on average. In other words, the informed trader gains information on average. The level of uncertainty exhibits discontinuities at every appearance of a new source of information. Proposition 5.5.1 is a way of quantifying this qualitatively intuitive result.

5.6 Appendix

5.6.1 Proof of Proposition 5.2.2

Proof. Using the Lebesgue Dominated Convergence,

$$\begin{aligned}\mathrm{dKL}_t(\psi_t \hookrightarrow \psi_t^{(i)}) &= \int_{\mathbb{X}} \mathrm{d}\psi_t(x) \log\left(\frac{\psi_t(x)}{\psi_t^{(i)}(x)}\right) \mathrm{d}x + \int_{\mathbb{X}} \psi_t(x)\, \mathrm{d}\log\left(\frac{\psi_t(x)}{\psi_t^{(i)}(x)}\right) \mathrm{d}x \\ &+ \int_{\mathbb{X}} \mathrm{d}\psi_t(x)\, \mathrm{d}\log\left(\frac{\psi_t(x)}{\psi_t^{(i)}(x)}\right) \mathrm{d}x, \end{aligned} \tag{5.90}$$

for $0 \leq t < T$. We write $\mathrm{dKL}_t(\psi_t \hookrightarrow \psi_t^{(i)}) = J_1^{>} + J_2^{>} + J_3^{>}$. Define $g_t = \log(\psi_t/\psi_t^{(i)}) = \log(u_t)$. Denoting the quadratic variation by $\langle \, , \rangle^v$ and using Ito quotient rule:

$$\begin{aligned}\mathrm{d}u_t &= \frac{\psi_t}{\psi_t^{(i)}}\left(\frac{\mathrm{d}\psi_t}{\psi_t} - \frac{\mathrm{d}\psi_t^{(i)}}{\psi_t^{(i)}} + \frac{\mathrm{d}\left\langle \psi_t^{(i)}, \psi_t^{(i)}\right\rangle^v}{(\psi_t^{(i)})^2} - \frac{\mathrm{d}\left\langle \psi_t, \psi_t^{(i)}\right\rangle^v}{\psi_t\psi_t^{(i)}}\right) \\ &= \frac{\psi_t}{\psi_t^{(i)}}\left(\sigma_t\, \mathrm{d}W_t - \sigma_t^{(i)}\, \mathrm{d}W_t^{(i)} + (\sigma_t^{(i)})^2\, \mathrm{d}t - \sigma_t\sigma_t^{(i)}\rho^{(i)}\, \mathrm{d}t\right), \end{aligned} \tag{5.91}$$

where $\rho^{(i)}$ is the correlation between $\{W_t\}$ and $\{W_t^{(i)}\}$. Then from (5.91),

$$(\mathrm{d}u_t)^2 = \frac{\psi_t^2}{(\psi_t^{(i)})^2}\left(\sigma_t^2\, \mathrm{d}t - 2\sigma_t\sigma_t^{(i)}\rho^{(i)}\, \mathrm{d}t + (\sigma_t^{(i)})^2\, \mathrm{d}t\right). \tag{5.92}$$

It follows that

$$\mathrm{d}\log(u_t) = \sigma_t\, \mathrm{d}W_t - \sigma_t^{(i)}\, \mathrm{d}W_t^{(i)} + \frac{1}{2}\left((\sigma_t^{(i)})^2 - \sigma_t^2\right)\, \mathrm{d}t. \tag{5.93}$$

Then, having the expression for $\mathrm{d}\log(\psi_t/\psi_t^{(i)})$ as given above, it follows that:

$$\begin{aligned} J_2^{>} &= \int_{\mathbb{X}} \psi_t(x)\left(\sigma_t(x)\, \mathrm{d}W_t - \sigma_t^{(i)}(x)\, \mathrm{d}W_t^{(i)} + \frac{1}{2}\left((\sigma_t^{(i)}(x))^2 - \sigma_t^2(x)\right)\, \mathrm{d}t\right) \mathrm{d}x \\ &= \frac{1}{2}\left(\int_{\mathbb{X}} \psi_t(x)\left((\sigma_t^{(i)}(x))^2 - \sigma_t^2(x)\right)\, \mathrm{d}x\right) \mathrm{d}t - \left(\int_{\mathbb{X}} \psi_t(x)\sigma_t^{(i)}(x)\, \mathrm{d}x\right) \mathrm{d}W_t^{(i)}. \end{aligned} \tag{5.94}$$

In addition, the terms J_1 and J_3 are

$$J_1^{>} = \left(\int_{\mathbb{X}} \sigma_t(x)\psi_t(x) \log\left(\frac{\psi_t(x)}{\psi_t^{(i)}(x)}\right) \mathrm{d}x\right) \mathrm{d}W_t, \tag{5.95}$$

$$J_3^{>} = \left(\int_{\mathbb{X}} \left(\sigma_t^2(x)\psi_t(x) - \sigma_t(x)\sigma_t^{(i)}(x)\psi_t(x)\rho^{(i)}\right) \mathrm{d}x\right) \mathrm{d}t. \tag{5.96}$$

Then, the SDE for $\{\mathrm{KL}_t(\psi_t \hookrightarrow \psi_t^{(i)}\}$ is

$$\mathrm{dKL}_t(\psi_t \hookrightarrow \psi_t^{(i)}) = \left(\int_{\mathbb{X}} \sigma_t(x)\psi_t(x)\log\left(\frac{\psi_t(x)}{\psi_t^{(i)}(x)}\right)\mathrm{d}x\right)\mathrm{d}W_t - \left(\int_{\mathbb{X}} \psi_t(x)\sigma_t^{(i)}(x)\,\mathrm{d}x\right)\mathrm{d}W_t^{(i)}$$
$$+ \frac{1}{2}\left(\int_{\mathbb{X}} \psi_t(x)\left((\sigma_t^{(i)}(x))^2 - 2\sigma_t(x)\sigma_t^{(i)}(x)\rho^{(i)} + \sigma_t^2(x)\right)\mathrm{d}x\right)\mathrm{d}t. \tag{5.97}$$

For the dynamics of $\{\mathrm{KL}_t(\psi_t^{(i)} \hookrightarrow \psi_t)\}$, let $\mathrm{dKL}_t(\psi_t^{(i)} \hookrightarrow \psi_t) = J_1^< + J_2^< + J_3^<$. Then,

$$\begin{aligned}
J_1^< &= \int_{\mathbb{X}} \mathrm{d}\psi_t^{(i)}(x)\log\left(\frac{\psi_t^{(i)}(x)}{\psi_t(x)}\right)\mathrm{d}x \\
&= \left(\int_{\mathbb{X}} \sigma_t^{(i)}(x)\psi_t^{(i)}(x)\log\left(\frac{\psi_t^{(i)}(x)}{\psi_t(x)}\right)\mathrm{d}x\right)\mathrm{d}W_t^{(i)}, && (5.98)\\
J_2^< &= \int_{\mathbb{X}} \psi_t^{(i)}(x)\,\mathrm{d}\log\left(\frac{\psi_t^{(i)}(x)}{\psi_t(x)}\right)\mathrm{d}x \\
&= \frac{1}{2}\left(\int_{\mathbb{X}} \psi_t^{(i)}(x)\left(\sigma_t^2(x) - (\sigma_t^{(i)}(x))^2\right)\mathrm{d}x\right)\mathrm{d}t - \left(\int_{\mathbb{X}} \psi_t^{(i)}(x)\sigma_t(x)\,\mathrm{d}x\right)\mathrm{d}W_t, && (5.99)\\
J_3^< &= \int_{\mathbb{X}} \mathrm{d}\psi_t^{(i)}(x)\,\mathrm{d}\log\left(\frac{\psi_t^{(i)}(x)}{\psi_t(x)}\right)\mathrm{d}x \\
&= \left(\int_{\mathbb{X}} (\sigma_t^{(i)})^2(x)\psi_t^{(i)}(x)\,\mathrm{d}x\right)\mathrm{d}t - \left(\int_{\mathbb{X}} \sigma_t(x)\sigma_t^{(i)}(x)\psi_t^{(i)}(x)\rho^{(i)}\,\mathrm{d}x\right)\mathrm{d}t. && (5.100)
\end{aligned}$$

Thus, the SDE for $\{\mathrm{KL}_t(\psi_t^{(i)} \hookrightarrow \psi_t\}$ is

$$\mathrm{dKL}_t(\psi_t^{(i)} \hookrightarrow \psi_t) = \left(\int_{\mathbb{X}} \sigma_t^{(i)}(x)\psi_t^{(i)}(x)\log\left(\frac{\psi_t^{(i)}(x)}{\psi_t(x)}\right)\mathrm{d}x\right)\mathrm{d}W_t^{(i)} - \left(\int_{\mathbb{X}} \psi_t^{(i)}(x)\sigma_t(x)\,\mathrm{d}x\right)\mathrm{d}W_t$$
$$+ \frac{1}{2}\left(\int_{\mathbb{X}} \psi_t^{(i)}(x)\left((\sigma_t^{(i)}(x))^2 - 2\sigma_t(x)\sigma_t^{(i)}(x)\rho^{(i)} + \sigma_t^2(x)\right)\mathrm{d}x\right)\mathrm{d}t. \tag{5.101}$$

The SDE for $\{\mathrm{KL}_t(\overline{\psi}_t||\psi_t)\}$ follows from (5.97), (5.101), Lemma 3.2.4 and Lemma 5.2.1. □

5.6.2 Proof of Lemma 5.4.1

Proof. Note that the following can be written:

$$\begin{aligned}
\mathrm{d}\psi_t^*(x) &= \mathrm{d}\left(\frac{\exp\left[\frac{T}{(T-t)}\left(\kappa(x-c)\xi_t^* - \frac{1}{2}(\kappa(x-c))^2 t\right)\right]q(x)}{\int_{\mathbb{X}}\exp\left[\frac{T}{(T-t)}\left(\kappa(x-c)\xi_t^* - \frac{1}{2}(\kappa(x-c))^2 t\right)\right]q(x)\,\mathrm{d}x}\right) \\
&= \mathrm{d}\left(\frac{V_t^*(x)}{Y_t^*}\right), && (5.102)
\end{aligned}$$

for $0 \leq t < T$. By the Ito quotient rule,

$$\mathrm{d}\left(\frac{V_t^*}{Y_t^*}\right) = \left[\frac{V_t^*}{Y_t^*}\left(\frac{\mathrm{d}V_t^*}{V_t^*} - \frac{\mathrm{d}Y_t^*}{Y_t^*} + \frac{\mathrm{d}\left\langle Y_t^*, Y_t^*\right\rangle^v}{(Y_t^*)^2} - \frac{\mathrm{d}\left\langle V_t^*, Y_t^*\right\rangle^v}{V_t^* Y_t^*}\right)\right]. \tag{5.103}$$

Let the numerator be denoted by the function $V_t^* = g(t, \xi_t^*)$. Then, by Ito's lemma,

$$\mathrm{d}g(t, \xi_t^*) = \left[\frac{\partial g}{\partial t}\,\mathrm{d}t + \frac{\partial g}{\partial \xi_t^*}\,\mathrm{d}\xi_t^* + \frac{1}{2}\frac{\partial^2 g}{\partial (\xi_t^*)^2}(\mathrm{d}\xi_t^*)^2\right], \tag{5.104}$$

where the following can be written:

$$\begin{aligned}\frac{\partial g}{\partial t}\,\mathrm{d}t &= \left(\frac{T}{(T-t)^2}\left(\kappa(x-c)\xi_t^* - \frac{1}{2}(\kappa(x-c))^2 t\right) - \frac{T}{T-t}\left(\frac{1}{2}(\kappa(x-c))^2\right)\right) V_t^*\,\mathrm{d}t \\ &= \frac{V_t^*}{(T-t)^2}\left[T\kappa(x-c)\xi_t^* - \frac{1}{2}T^2(\kappa(x-c))^2\right]\mathrm{d}t. \end{aligned}\tag{5.105}$$

It also follows that

$$\frac{\partial g}{\partial \xi_t^*}\,\mathrm{d}\xi_t^* = \frac{V_t^* T\kappa(x-c)}{(T-t)}\,\mathrm{d}\xi_t^* \quad \text{and} \quad \frac{1}{2}\frac{\partial^2 g}{\partial (\xi_t^*)^2}(\mathrm{d}\xi_t^*)^2 = \frac{V_t^*(T\kappa(x-c))^2}{2(T-t)^2}\,\mathrm{d}t, \tag{5.106}$$

for $0 \leq t < T$, since $(\mathrm{d}\xi_t^*)^2 = \mathrm{d}t$ due to the fact that $(\mathrm{d}B_{tT}^*)^2 = \mathrm{d}t$. Thus,

$$\frac{\mathrm{d}V_t^*(x)}{V_t^*(x)} = \left(\frac{T\kappa(x-c)\xi_t^*}{(T-t)^2}\,\mathrm{d}t + \frac{T\kappa(x-c)}{(T-t)}\,\mathrm{d}\xi_t^*\right), \tag{5.107}$$

for $0 \leq t < T$. Note that $Y_t^* = \int_{\mathbb{X}} V_t^*(x)\,\mathrm{d}x$ and from the Lebesgue Dominated Convergence,

$$\mathrm{d}Y_t^* = \int_{\mathbb{X}}\left[V_t^*(x)\left(\frac{T\kappa(x-c)\xi_t^*}{(T-t)^2}\,\mathrm{d}t + \frac{T\kappa(x-c)}{(T-t)}\,\mathrm{d}\xi_t^*\right)\right]\mathrm{d}x, \tag{5.108}$$

for $0 \leq t < T$. We have

$$\frac{\mathrm{d}Y_t^*}{Y_t^*} = \frac{\int_{\mathbb{X}}\left[V_t^*(x)\left(\frac{T\kappa(x-c)\xi_t^*}{(T-t)^2}\,\mathrm{d}t + \frac{T\kappa(x-c)}{(T-t)}\,\mathrm{d}\xi_t^*\right)\right]\mathrm{d}x}{\int_{\mathbb{X}} V_t^*(x)\,\mathrm{d}x}, \tag{5.109}$$

for $0 \leq t < T$. By definition,

$$\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \xi_t^*\right] = \frac{\int_{\mathbb{X}} x V_t^*(x)\,\mathrm{d}x}{\int_{\mathbb{X}} V_t^*(x)\,\mathrm{d}x}. \tag{5.110}$$

Then, the following can be written:

$$\frac{\mathrm{d}Y_t^*}{Y_t^*} = \frac{T\kappa\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \xi_t^*\right]\xi_t^*}{(T-t)^2}\,\mathrm{d}t + \frac{T\kappa\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \xi_t^*\right]}{(T-t)}\,\mathrm{d}\xi_t^* - \frac{T\kappa c\xi_t^*}{(T-t)^2}\,\mathrm{d}t - \frac{T\kappa c}{(T-t)}\,\mathrm{d}\xi_t^*. \tag{5.111}$$

This is the second term in the Ito quotient bracket. The third term in the bracket can be written as

$$\frac{\mathrm{d}\left\langle Y_t^*, Y_t^* \right\rangle}{(Y_t^*)^2} = \frac{\left(T\kappa\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \xi_t^*\right]\right)^2}{(T-t)^2}\,\mathrm{d}t + \frac{(T\kappa c)^2}{(T-t)^2}\,\mathrm{d}t + \frac{T^2\kappa^2 c\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \xi_t^*\right]}{(T-t)^2}\,\mathrm{d}t, \tag{5.112}$$

for $0 \leq t < T$. The last term in the Ito quotient bracket is

$$\frac{\mathrm{d}\left\langle V_t^*, Y_t^* \right\rangle}{V_t^* Y_t^*} = \frac{T^2\kappa^2(x-c)\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \xi_t^*\right]}{(T-t)^2}\,\mathrm{d}t - \frac{T^2\kappa^2(x-c)c}{(T-t)^2}\,\mathrm{d}t, \tag{5.113}$$

for $0 \leq t < T$. Then, putting all the terms together and rearranging, we can write the following:

$$\frac{\mathrm{d}\psi_t^*(x)}{\psi_t^*(x)} = \frac{T\kappa}{(T-t)}\left[L_t(x)\,\mathrm{d}\xi_t^* + \left(\frac{L_t(x)\xi_t^*}{(T-t)} - \frac{T\kappa}{(T-t)}L_t(x)\mathbb{E}^{\mathbb{Q}}\left[X_T - c \,|\, \xi_t^*\right]\right)\mathrm{d}t\right], \tag{5.114}$$

for $0 \leq t < T$, where $L_t(x) = x - \mathbb{E}^{\mathbb{Q}}[X_T \,|\, \xi_t^*]$. This statement can be rewritten as

$$\mathrm{d}\psi_t^*(x) = \sigma_t^*\psi_t^*(x)\,\mathrm{d}W_t^* + \sigma_t^*\psi_t^*(x)\frac{T\kappa c}{(T-t)}\,\mathrm{d}t, \tag{5.115}$$

for $0 \leq t < T$, where $\{W_t^*\}$ is defined by

$$W_t^* = \xi_t^* + \int_0^t \frac{1}{T-s}\xi_s^*\,\mathrm{d}s - T\kappa\int_0^t \frac{1}{T-s}\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \xi_s^*\right]\,\mathrm{d}s, \tag{5.116}$$

and $\sigma_t^*(x) = T\kappa\left(x - \mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \xi_t^*\right]\right)/(T-t)$.

We need to show that $\{W_t^*\}$ is a $\mathbb{Q}$-Brownian motion with drift. We follow similar steps as done in Brody *et al.* (2008a). For $0 \leq t \leq u < T$, note that we can write

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[W_u^* \,|\, \xi_t^*\right] &= \mathbb{E}^{\mathbb{Q}}\left[(W_u^* - W_t^*)\,|\xi_t^*\right] + W_t^* \\ &= W_t^* + \mathbb{E}^{\mathbb{Q}}\left[\xi_u^* - \xi_t^* \,|\, \xi_t^*\right] - T\kappa\mathbb{E}^{\mathbb{Q}}\left[\int_t^u \frac{1}{T-s}\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \xi_s^*\right] ds \;|\, \xi_t^*\right] \\ &+ \mathbb{E}^{\mathbb{Q}}\left[\int_t^u \frac{1}{T-s}\xi_s^*\,\mathrm{d}s \;|\, \xi_t^*\right].\end{aligned} \tag{5.117}$$

Then, by the tower property,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[W_u^* \,|\, \xi_t^*\right] = W_t^* &+ \mathbb{E}^{\mathbb{Q}}\left[\kappa X_T u + B_{uT}^* \,|\, \xi_t^*\right] - \kappa c u - \mathbb{E}^{\mathbb{Q}}\left[\kappa X_T t + B_{tT}^* \,|\, \xi_t^*\right] + \kappa c t \\ &+ \kappa\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \xi_t^*\right]\int_t^u \frac{s}{T-s}\,\mathrm{d}s + \mathbb{E}^{\mathbb{Q}}\left[\int_t^u \frac{1}{T-s}B_{sT}^*\,\mathrm{d}s \,|\, \xi_t^*\right] - \kappa c\int_t^u \frac{s}{T-s}\,\mathrm{d}s \\ &- \kappa\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \xi_t^*\right]\int_t^u \frac{T}{T-s}\,\mathrm{d}s.\end{aligned} \tag{5.118}$$

Note that all the terms involving X_T disappear from (5.118):

$$\mathbb{E}^{\mathbb{Q}}\left[W_u^* \,|\, \xi_t^*\right] = W_t^* + \mathbb{E}^{\mathbb{Q}}\left[B_{uT}^* \,|\, \xi_t^*\right] - \mathbb{E}^{\mathbb{Q}}\left[B_{tT}^* \,|\, \xi_t^*\right] + \int_t^u \frac{1}{T-s}\mathbb{E}^{\mathbb{Q}}\left[B_{sT}^* \,|\, \xi_t^*\right] \mathrm{d}s$$
$$+ \kappa c\left(t - u - \int_t^u \frac{s}{T-s}\,\mathrm{d}s\right). \tag{5.119}$$

Using the independence of X_T and $\{B_{tT}^*\}$, and the tower property, we can write

$$\mathbb{E}^{\mathbb{Q}}\left[B_{uT}^* \,|\, \xi_t^*\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[B_{uT}^* \,|\, X_T, B_{tT}^*\right] |\, \xi_t^*\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[B_{uT}^* \,|\, B_{tT}^*\right] |\, \xi_t^*\right]$$
$$= \frac{T-u}{T-t}\mathbb{E}^{\mathbb{Q}}\left[B_{tT}^* \,|\, \xi_t^*\right]. \tag{5.120}$$

When (5.120) is inserted in (5.119), we can see that

$$\mathbb{E}^{\mathbb{Q}}[B_{uT}^* \,|\, \xi_t^*] - \mathbb{E}^{\mathbb{Q}}[B_{tT}^* \,|\, \xi_t^*] + \int_t^u \frac{1}{T-s}\mathbb{E}^{\mathbb{Q}}[B_{sT}^* \,|\, \xi_t^*]\,\mathrm{d}s = 0, \tag{5.121}$$

which proves

$$\mathbb{E}^{\mathbb{Q}}\left[W_u^* \,|\, \xi_t^*\right] = W_t^* + \kappa c\left(t - u - \int_t^u \frac{s}{T-s}\,\mathrm{d}s\right). \tag{5.122}$$

Since $t \leq u$ and $\kappa > 0$, the second term is negative if $c > 0$ and positive if $c < 0$. Note that $\{W_t^*\}$ is continuous and $(\mathrm{d}W_t^*)^2 = \mathrm{d}t$. Then, by Lévy's characterisation, $\{W_t^*\}$ is a $\mathbb{Q}$-Brownian motion with negative drift if $c > 0$, or with positive drift if $c < 0$. □

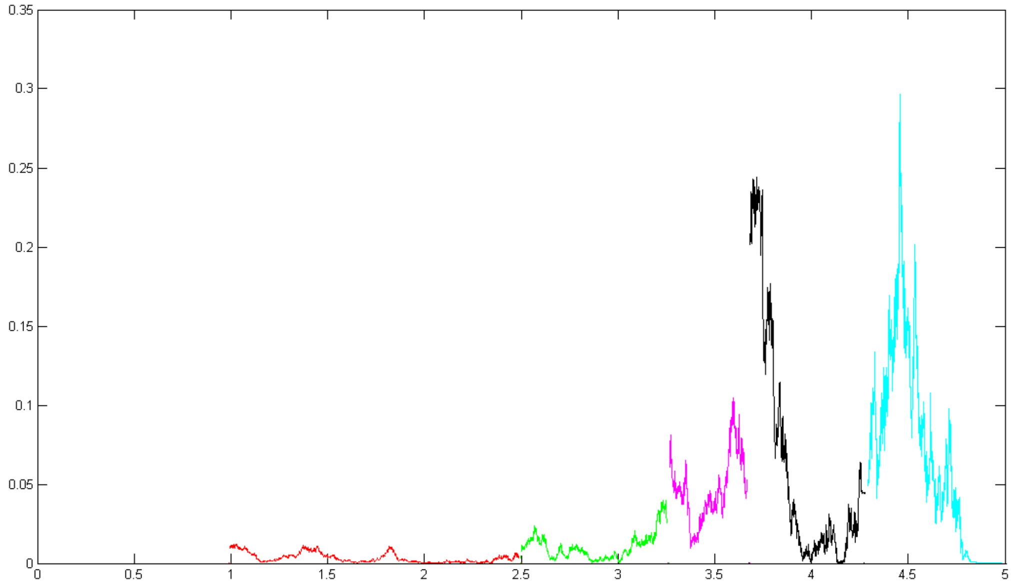

Figure 5.1: A KL asymmetry process between the informed trader and the market. The informed trader receives five additional sources of information when compared to the market. The asymmetry is zero before the informed trader receives its first additional information source. Parameters: $T = 5$, $\kappa^i = 1/T$ and $\rho^i = 0.25$. Stopping times are uniformly distributed on $[0, T]$.

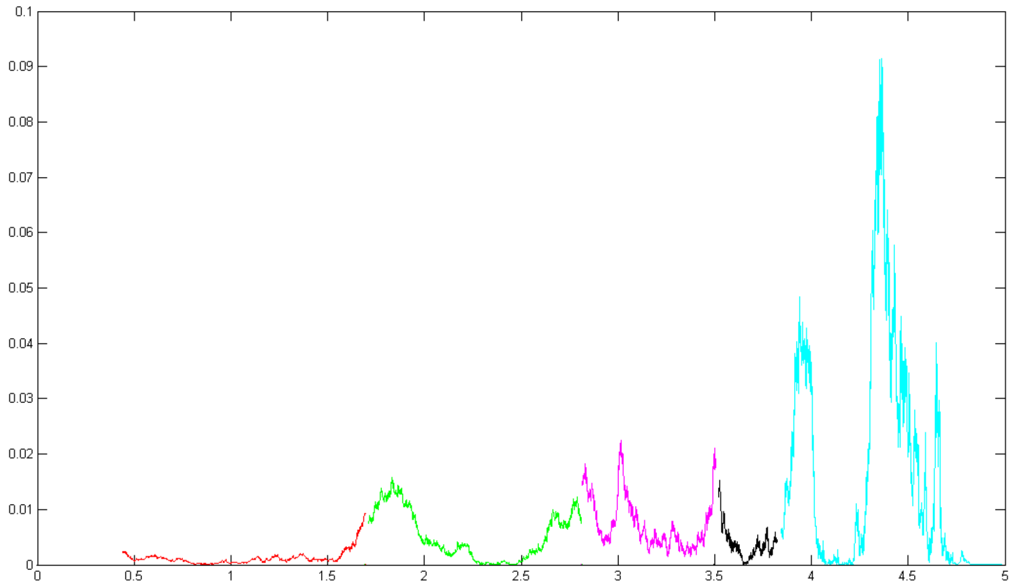

Figure 5.2: A SH asymmetry process between the informed trader and the market. The informed trader receives five additional sources of information when compared to the market. Parameters: $T = 5$, $\kappa^i = 1/T$ and $\rho^i = 0.25$. Stopping times are uniformly distributed on $[0, T]$.

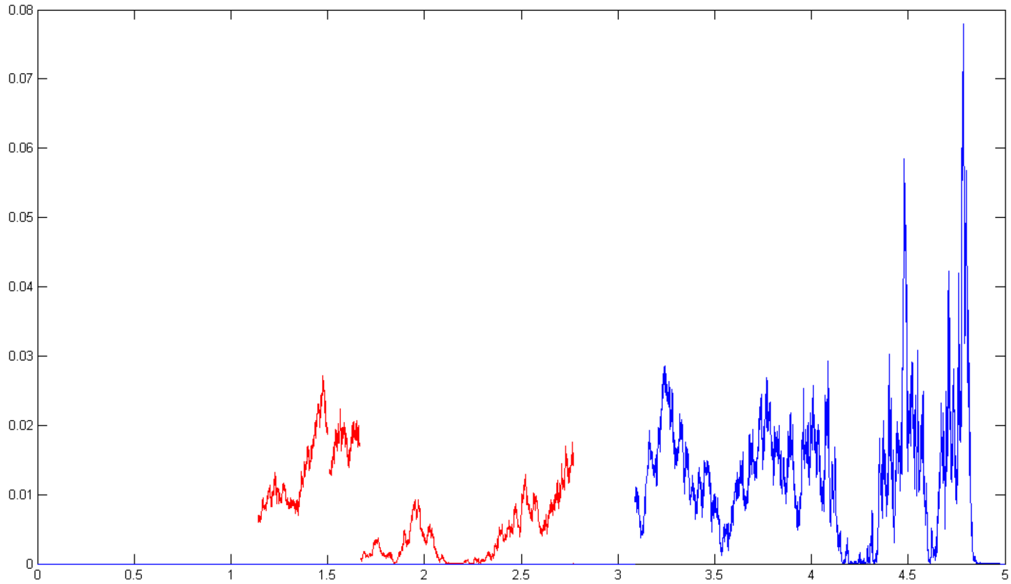

Figure 5.3: A KL asymmetry process between two agents. The process is zero when both agents have equal sets of information. Red shows when Agent 1 is more advantegous and blue shows the opposite. In this plot, Agent 1 gains the advantage by receiving the first additional information source and another one after that. Then, Agent 2 receives two sources in succession which brings the asymmetry back to zero. Finally, Agent 2 recevies yet another information source, hence gains the advantage, and sustains this advantage until the end. Parameters: $T = 5$, $\kappa^i = 1/T$ and $\rho^i = 0.25$. Stopping times are uniformly distributed on $[0, T]$.

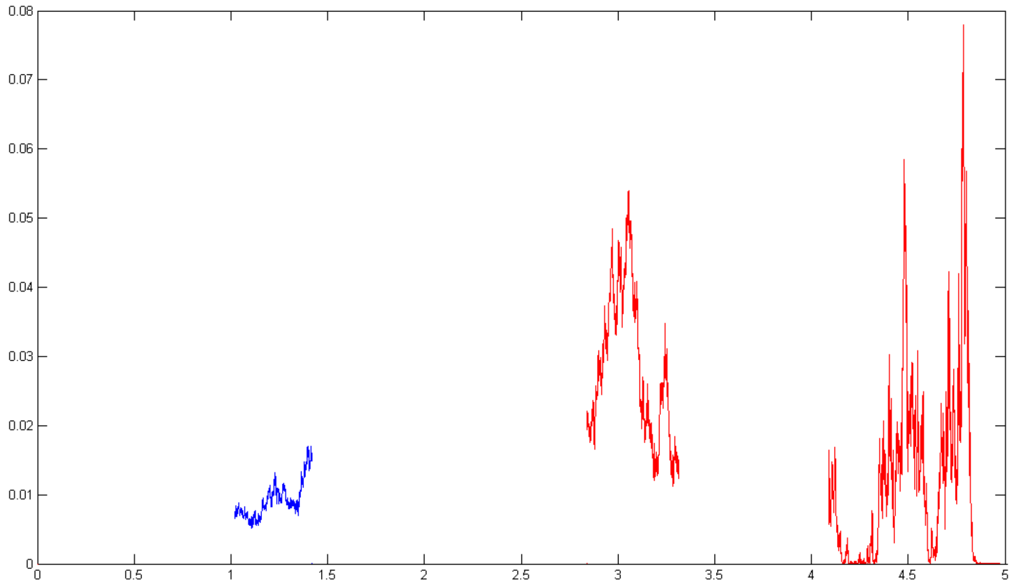

Figure 5.4: A KL asymmetry process between two agents. In this plot, although Agent 2 gains the first advantage, Agent 1 gains and sustains the final advantage with one more source of information compared to Agent 2. Parameters: $T = 5$, $\kappa^i = 1/T$ and $\rho^i = 0.25$. Stopping times are uniformly distributed on $[0, T]$.

Chapter 6

Geometric Quantification of Information Asymmetry

We use differential geometry to quantify information asymmetry. We assume that the filtration of an informed trader is an n-order piecewise enlargement of the market filtration.

One main aim of this chapter is to address the question: How can one dynamically quantify the impact of changes in information sources about a cash flow X_T using geometry? We are partly motivated to ask this question, since the SH asymmetry can be characterised by the spherical distance between two points on the unit sphere, determined by square-roots of two conditional probability densities. Following the setting we discuss in Chapter 5, this angle provides a geometric measurement on information asymmetry, and we aim to find other geometric measures to quantify it. This chapter indicates how differential geometry interacts with information, and introduces the use of various geometric objects to bring an alternative perspective on information asymmetry. In this respect, we aim to analyze the geometric evolution of the informational advantage of a small trader who is relatively more informed than the market. Since $\{\mathcal{G}_t\}$ as shown in (3.58) is an explicit example of an n-order piecewise enlargement of the market filtration $\{\mathcal{F}_t^{\xi^1}\}$, we focus on the case where the informed trader's filtration is $\{\mathcal{G}_t\}$. We shall introduce an asymmetry process that we call the Fisher-Rao (FR) asymmetry process on a hyperbolic space, as an alternative to the SH asymmetry process on a sphere. Similar to Chapter 5, we are also interested in quantifying the competitive advantage between two informed traders with different piecewise enlargements of the same market filtration, and consider financial mispricing with a geometric standpoint.

Geometry is becoming increasingly popular in stochastic analysis, since it may shed light on sophisticated relations that may be hidden from a purely probabilistic point of view. Accordingly, there is a growing interest of using geometry in mathematical finance. For instance, Hughston (1994) uses stochastic differential geometry (see, for example, Emery, 1989, Ikeda and Watanabe, 1989) and formulates a no-arbitrage asset price model when the underlying state-space is a Riemannian manifold. The work interprets geometric objects such as torsion and curvature in a financial context. Nunes and Webber (1997) build interest rate models on two-dimensional manifolds, and Kuruc (2003) applies differential geometry to hedging problems and risk management. Labordère (2008) provides a detailed synthesis of the use of differential geometry in financial problems such as option pricing, stochastic volatility models and portfolio optimization. Brody and Hughston (2001) construct geometric measures to quantify the difference of two term structures. Our work is perhaps most closely related to the stream of literature concerned in measuring distances between distributions. Rao (1945) introduces a method of measuring distances between distributions using Riemannian geometry. It seems that the work of Rao (1945) received little attention at first. However, the interest is re-established with the works of Efron (1975), Atkinson and Mitchell (1981), Reverter and Oller (2003), Arwini and Dodson (2008), and many others.

We shall give a brief overview of this chapter. First, when we discuss the SH asymmetry on the unit sphere $\mathcal{S}$, we don't specify a distribution for the cash flow X_T. Later, we assume that X_T is a Gaussian random variable. Then we can parameterise the conditional probability distributions to form a parametric class of Gaussian distributions, in which the parameters (the mean and the variance) are functions of Brownian information processes. Based on the work of Rao (1945), this induces a natural Riemannian geometry on a manifold of which the points are determined by Gaussian distributions, and where the parameters are the local coordinates of the manifold. In particular, the manifold is a hyperbolic space, which we denote by $\mathcal{P}$, endowed with the Fisher metric tensor. It follows that for each fixed time $t < T$, a Brownian information process determines a point on this hyperbolic space. We include the boundary of this space by using Dirac measures as limits of Gaussian distributions, and define a manifold with boundary that we denote by $\mathcal{M}$. Then we are able to construct what we call the FR asymmetry process on $[0, T]$ using points on $\mathcal{M}$ that are determined by different numbers of information sources. We shall see that the FR asymmetry between points on the boundary takes the value zero at $t = T$, and the FR asymmetry process for $t < T$ jumps when a new information source appears. The jumps of the SH and the FR asymmetry processes induce spherical triangles and hyperbolic triangles on $\mathcal{S}^+$ and $\mathcal{P}$, respectively. The surfaces enable us to measure the jump sizes of conditional probability densities using angles between geodesics and the curvatures of the underlying manifolds, and offer alternative ways of quantifying the impact of appearances of new information sources. Also in a way, these surfaces allow us to view information asymmetry as a geometric shape instead of just a quantity. We introduce an analogy between the SH asymmetry and an isometric invariant of the Poincaré disc under the action of the general Möbius group. The analogy motivates us to propose the use of the isometric invariant as an alternative measure of information asymmetry in the Gaussian setting. The isometric invariant is zero if there is no information asymmetry, and is strictly positive otherwise.

This chapter is organised as follows: Section 1 is a brief background on Riemannian geometry. Section 2 is the geometric perspective gained from the SH asymmetry on the unit sphere. Section 3 is the geometric modelling of information asymmetry on a hyperbolic space. Section 4 is the geometric quantification of the competitive advantage between two informed traders, and also of financial mispricing.

6.1 Preliminaries on Riemannian Geometry

We provide a brief preliminary background on Riemannian geometry (see for example, Do Carmo, 1992 and O'Neill, 2006). We focus on some concepts that we shall later refer to when we discuss information asymmetry. The definitions and notations given below are mostly based on Do Carmo (1992).

We first require the notion of a differentiable manifold to generalise differential calculus to spaces generalising $\mathbb{R}^n$.

Definition 6.1.1. *An n-dimensional* differentiable manifold *is a set $\mathcal{M}$ and a family of injective transformations $\varphi_\alpha : V_\alpha \subset \mathbb{R}^n \to \mathcal{M}$ of open sets V_α, such that*

1. *The union $\bigcup_\alpha \varphi_\alpha(V_\alpha) = \mathcal{M}$,*
2. *Given any pair α and β such that $\varphi_\alpha(V_\alpha) \bigcap \varphi_\beta(V_\beta) = G \neq \varnothing$, the sets $\varphi_\alpha^{-1}(G)$ and $\varphi_\beta^{-1}(G)$ are open sets in $\mathbb{R}^n$, and the transformations $\varphi_\beta^{-1} \circ \varphi_\alpha$ are differentiable,*
3. *The family $\{(V_\alpha, \varphi_\alpha)\}$ is maximal relative to the first two conditions.*

An n-dimensional differentiable manifold $\mathcal{M}$ is locally diffeomorphic to the Euclidean space $\mathbb{R}^n$. There is a natural topology induced by $\mathcal{M}$, if a set $A \subset \mathcal{M}$ is open if and only if for all α, $\varphi_\alpha^{-1}(\varphi_\alpha(V_\alpha) \bigcap A)$ is open in $\mathbb{R}^n$. We further impose topological restrictions on differentiable manifolds such that they are Hausdorff spaces with countable bases (this is to ensure uniqueness of limits of convergent sequences and existence of a differentiable partition of unity). Based on Whitney's theorem, any n-dimensional Hausdorff differentiable manifold $\mathcal{M}$ with a countable basis can be embedded in $\mathbb{R}^{2n+1}$. From this point on, when we use the term differentiable, we mean smooth, or of class C^∞.

Definition 6.1.2. *A* Riemannian metric *on a differentiable manifold $\mathcal{M}$ is a differentiable family of transformations:*

$$g\left\langle \, , \, \right\rangle_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R} \;\; for \;\; x \in \mathcal{M}, \tag{6.1}$$

that is a bilinear, symmetric, positive-definite form on each tangent space $T_p\mathcal{M}$ where $p \in \mathcal{M}$. A differentiable manifold with a Riemannian metric is called a Riemannian manifold*.*

Hence, a Riemannian manifold is a differentiable manifold where each tangent space is equipped with an inner product. It can be shown that any differentiable manifold $\mathcal{M}$, which is Hausdorff with a countable basis, has a Riemannian metric (see Do Carmo, 1992, pp. 43).

The Riemannian metric can be represented in the coordinate system as $g_{ij} = g_{ji}$. Locally, at each point $p \in \mathcal{M}$, each $g \langle , \rangle_p$ can be written as an $n \times n$ matrix $[g_{ij}]$. For each vector $\mathbf{x} \in T_p\mathcal{M}$, the norm of $\mathbf{x}$, denoted by $||\mathbf{x}||$ can be written as: $||\mathbf{x}|| = \sqrt{g \langle \mathbf{x}, \mathbf{x} \rangle}$. The angle between any two vectors $\mathbf{x}$ and $\mathbf{y}$ on the same tangent space is

$$\cos \Theta = \frac{g \langle \mathbf{x}, \mathbf{y} \rangle}{\sqrt{g \langle \mathbf{x}, \mathbf{x} \rangle \, g \langle \mathbf{y}, \mathbf{y} \rangle}}. \tag{6.2}$$

Therefore, the Riemannian metric allows one to define lengths, angles and volumes on a differentiable manifold.

Differentiating vector fields on an Euclidean space is straightforward, since nearby tangent spaces can be identified by translation. However, differentiation of vector fields on a manifold is less clear, since nearby tangent spaces cannot be identified in such a natural way. As a remedy, an *affine connection* allows vector fields to be differentiated by connecting nearby tangent spaces.

Formally, let $\Lambda(\mathcal{M})$ be the set of differentiable vector fields on $\mathcal{M}$ and let $R(\mathcal{M})$ denote the ring of real valued differentiable functions on $\mathcal{M}$. Then, an affine connection ∇ on $\mathcal{M}$ is a transformation

$$\begin{aligned} \nabla : \Lambda(\mathcal{M}) \times \Lambda(\mathcal{M}) &\longrightarrow \Lambda(\mathcal{M}) \\ (X, Y) &\mapsto \nabla_X Y, \end{aligned} \tag{6.3}$$

which satisfies: 1) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$, and 2) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$, and 3) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$, given that $X, Y, Z \in \Lambda(\mathcal{M})$ and $f, g \in R(\mathcal{M})$.

In particular, let $\alpha_i : V \to \mathbb{R}$ be a function. If X is a vector field and $\mathbf{x} : V \subset \mathbb{R}^n \to \mathcal{M}$ (where V is an open set), the following can be written:

$$X(p) = \sum_{i=1}^{n} \alpha_i(p) \frac{\partial}{\partial x_i}, \tag{6.4}$$

where $\partial/\partial x_i$ is the basis for $i = 1, \ldots, n$. Then, with X and Y being vector fields, $\nabla_X Y$ can be calculated as follows:

$$\nabla_X Y = \sum_m (\sum_{i,j} x_i y_j \Gamma_{ij}^m + X(y_m)) X_m, \tag{6.5}$$

provided that $X = \sum_i x_i X_i$ where $X_i = \partial/\partial x_i$. Here, Γ_{ij}^m is called the Christoffel symbol of

∇, defined by

$$\Gamma_{ij}^{m} = \frac{1}{2}\sum_{k}\left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right) g^{km}, \tag{6.6}$$

where $g^{km} g_{mj} = \chi_{\{k=j\}}$, or where $[g^{km}]$ is the inverse of the matrix $[g_{km}]$. An affine connection is symmetric if $\nabla_X Y - \nabla_Y X = XY - YX$.

We now need to define the so-called *covariant derivative* to give a formal definition of a geodesic on $\mathcal{M}$. We make use of geodesics quite extensively in our analysis of information asymmetry. Briefly, a covariant derivative is a generalisation of the directional derivative (from Euclidean geometry), which identifies a derivative of vector fields on a differentiable manifold.

Formally, the correspondence which associates a vector field X with another vector field $\mathrm{D}X/\,\mathrm{d}u$ along the differentiable curve $c : I \to \mathcal{M}$, is called the covariant derivative of X along c, which satisfies: 1) $\frac{\mathrm{D}}{\mathrm{d}u}(X+Y) = \frac{\mathrm{D}X}{\mathrm{d}u} + \frac{\mathrm{D}Y}{\mathrm{d}u}$, and 2) $\frac{\mathrm{D}}{\mathrm{d}u}(fX) = \frac{\mathrm{d}f}{\mathrm{d}u}X + f\frac{\mathrm{D}X}{\mathrm{d}u}$, f being a differentiable function on I, and 3) If $X(u) = Y(c(u))$, then $\frac{\mathrm{D}X}{\mathrm{d}u} = \nabla_{\mathrm{d}c/\,\mathrm{d}u} Y$, given that ∇ is the affine connection on $\mathcal{M}$.

An affine connection is compatible with the Riemannian metric if for any pair of parallel vector fields X and Y (parallel means $\frac{\mathrm{D}X}{\mathrm{d}u} = \frac{\mathrm{D}Y}{\mathrm{d}u} = 0$) along any differentiable curve on $\mathcal{M}$, the metric $g\langle X, Y\rangle = c$ for some constant c. If the affine connection is also symmetric, then we call such connections Levi-Civita connections.

Definition 6.1.3. *Let $\mathcal{M}$ be a Riemannian manifold with its Levi-Civita connection ∇. A parametrized curve $\gamma : I \to \mathcal{M}$ is a* geodesic *at $u \in I$, if $\frac{\mathrm{D}}{\mathrm{d}u}(\frac{\mathrm{d}\gamma}{\mathrm{d}u}) = 0$, for all $u \in I$.*

Therefore, geodesics of $\mathcal{M}$ are the curves with zero acceleration, or more intuitively, they are the curves which locally minimize the distance between two points on $\mathcal{M}$. If $[a, b] \subset I$, then the restriction of γ to $[a, b]$ connects $\gamma(a)$ to $\gamma(b)$, which is a segment of the geodesic γ. Geodesics are the generalisation of straight lines defined on an Euclidean space.

In our analysis, we make use of what are called Riemannian curvatures of the underlying manifolds. Riemannian curvature is a special type of a curvature of a Riemannian manifold.

Formally, the curvature K of a Riemannian manifold $\mathcal{M}$ with its Levi-Civita connection ∇ is a correspondence that associates each pair of vector fields $X, Y \in \Lambda(\mathcal{M})$ a transformation $K(X, Y) : \Lambda(\mathcal{M}) \to \Lambda(\mathcal{M})$ such that

$$K(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[XY]} Z, \tag{6.7}$$

for $Z \in \Lambda(\mathcal{M})$, where the bracket $[X, Y] = XY - YX$.

We can view the curvature as a measure of how far $\mathcal{M}$ is from being $\mathbb{R}^n$, since $\mathbb{R}^n$ is flat with zero curvature. The bracket $[X, Y]$ is also known as the Lie bracket.

Definition 6.1.4. *Let $p \in \mathcal{M}$, and E be a two-dimensional subspace $E \subset T_p\mathcal{M}$. Given that $x, y \in E$ are linearly independent vectors, the real number $R(x, y) = R(E)$ defined by*

$$R(x,y) = \frac{g\left\langle K(x,y)x, y\right\rangle}{|x|^2|y|^2 - g\left\langle x, y\right\rangle^2}, \tag{6.8}$$

is called the Riemannian curvature *of E at the point p.*

The Riemannian manifolds with constant curvatures are quite important for our purposes. For example, SH asymmetry characterises points on the unit sphere $\mathcal{S}$, which is a Riemannian manifold (if endowed with the metric) with constant curvature $R = +1$. We shall see later in Section 6.3 that we can also analyze information asymmetry on a Riemannian manifold with constant negative curvature $R = -1$.

6.2 Squared-Hellinger Asymmetry and the Sphere

Using the bijection between probability densities and square-roots of probability densities, the SH asymmetry can be represented as the squared-norm between the square-roots of two conditional probability densities in the Hilbert space $\mathcal{L}^2$. Since probability densities are non-negative functions and their integral is unity, taking the square-root of conditional probability densities determines points on the positive orthant of the unit sphere $\mathcal{S} \subset \mathcal{L}^2$. As done in Chapter 3, we denote the positive orthant of $\mathcal{S}$ by $\mathcal{S}^+$.

Any two points on $\mathcal{S}$ can be defined on a great circle with its center coinciding with the center of $\mathcal{S}$. Then, denoting $||.||_{\mathcal{L}^2}$ as the $\mathcal{L}^2$-norm, we can define the SH asymmetry as

$$\begin{aligned} \mathrm{SH}_t(q(x|\mathcal{Y}_t)||q(x|\mathcal{G}_t)) &= \frac{1}{2}||\sqrt{q(x|\mathcal{Y}_t)} - \sqrt{q(x|\mathcal{G}_t)}||^2_{\mathcal{L}^2} \\ &= 1 - \cos\left(\vartheta_t(\sqrt{q(x|\mathcal{Y}_t)}, \sqrt{q(x|\mathcal{G}_t)})\right), \end{aligned} \tag{6.9}$$

for $0 \leq t < T$, which follows from (3.7), (3.8), (3.97) and (5.29).

In equation (6.9), the geometric quantity $\vartheta_t(\sqrt{q(x|\mathcal{Y}_t)}, \sqrt{q(x|\mathcal{G}_t)})$ is the Bhattacharyya angle (the angle from the center of $\mathcal{S}$, subtended to the endpoints on $\mathcal{S}^+$) between the square-roots of the conditional probability densities $q(x|\mathcal{Y}_t)$ and $q(x|\mathcal{G}_t)$. Measured in radians, the Bhattacharyya angle equals the spherical distance (arc length) between the points determined by the square-roots of the conditional densities, since $\mathcal{S}$ is the unit sphere. In addition, since the points are on the positive orthant $\mathcal{S}^+$, the Bhattacharyya angle can vary in the interval $[0, \pi/2]$.

The unit sphere $\mathcal{S}$ is a differentiable manifold in the Hilbert space $\mathcal{L}^2$. When equipped with the Riemannian metric, it is a Riemannian manifold. Note that the Riemannian metric $g\langle\,,\,\rangle$ on $\mathcal{S} \subset \mathcal{L}^2$ can be defined as an inner product on $\mathcal{L}^2$. Then one can see from (3.7), (6.2) and (6.9) that

$$
\begin{aligned}
\cos\left(\vartheta_t(\sqrt{q(x|\mathcal{Y}_t)}, \sqrt{q(x|\mathcal{G}_t)})\right) &= \frac{g\left\langle \sqrt{q(x|\mathcal{Y}_t)}, \sqrt{q(x|\mathcal{G}_t)} \right\rangle}{\sqrt{g\left\langle \sqrt{q(x|\mathcal{Y}_t)}, \sqrt{q(x|\mathcal{Y}_t)} \right\rangle g\left\langle \sqrt{q(x|\mathcal{G}_t)}, \sqrt{q(x|\mathcal{G}_t)} \right\rangle}} \\
&= \int_{\mathbb{X}} \sqrt{q(x|\mathcal{Y}_t)}\sqrt{q(x|\mathcal{G}_t)}\,\mathrm{d}x \\
&= 1 - \frac{1}{2}||\sqrt{q(x|\mathcal{Y}_t)} - \sqrt{q(x|\mathcal{G}_t)}||^2_{\mathcal{L}^2},
\end{aligned}
$$

since $||\sqrt{q(x|\mathcal{Y}_t)}||_{\mathcal{L}^2} = ||\sqrt{q(x|\mathcal{G}_t)}||_{\mathcal{L}^2} = 1$. The geodesics on $\mathcal{S}$ are great circles. Hence, the length of a geodesic γ between points on $\mathcal{S}^+$ is the spherical distance. Since the spherical distance equals the Bhattacharya angle on $\mathcal{S}$, the SH asymmetry induces a natural Riemannian geometry when represented in terms of Bhattacharyya angles.

We are now in the position to provide a geometric remark on Proposition 5.2.4:

Remark 6.2.1. *The Bhattacharyya angle (or the spherical distance) process* $\{\vartheta_t(\sqrt{\psi_t}, \sqrt{\overline{\psi}_t})\}$ *on* $\mathcal{S}^+$ *is the inverse cosine of a jump-diffusion process for* $0 \leq t < T$.

6.3 Geometry and Information Asymmetry

From this point on, we assume that X_T is a Gaussian random variable. We can then parametrise conditional distributions in a way that allows us to work on a Riemannian manifold other than $\mathcal{S}$. In particular, each point on this new manifold is determined by a Gaussian distribution with parameters as functions of the Brownian information processes. We assume X_T has the parameter set $\Theta_X = \{\mu_X, \sigma_X^2\}$, where the mean satisfies $-\infty < \mu_X < \infty$ and the variance satisfies $0 < \sigma_X^2 < \infty$. We write $A \rightleftharpoons B$ to denote that B is the parameterization of A.

Lemma 6.3.1. *The information process* $\{\widehat{\xi}_t^{(i)}\}$ *is Gaussian with mean and variance:*

$$
\mu_t^{(i)} = \widehat{\kappa}^{(i)}\mu_X t \qquad \textit{and} \qquad \sigma_t^{(i)} = \sqrt{(\widehat{\kappa}^{(i)}\sigma_X t)^2 + \frac{t(T-t)}{T}}. \tag{6.10}
$$

Proof. If X_T is Gaussian with mean μ_X and variance σ_X^2, the information process $\{\widehat{\xi}_t^{(i)}\}$ is also Gaussian (note that X_T is independent from the Brownian bridge), where

$$
\mathbb{E}^{\mathbb{Q}}[\widehat{\xi}_t^{(i)}] = \mu_t^{(i)} = \widehat{\kappa}^{(i)}\mu_X t, \tag{6.11}
$$

$$
\mathrm{Var}^{\mathbb{Q}}[\widehat{\xi}_t^{(i)}] = (\sigma_t^{(i)})^2 = (\widehat{\kappa}^{(i)}\sigma_X t)^2 + \frac{t(T-t)}{T}, \tag{6.12}
$$

which gives the statement. ☐

Proposition 6.3.2. *The conditional density $\psi_t^{(i)}$ can be parametrically represented as*

$$\psi_t^{(i)} \rightleftharpoons q(x, \widehat{\mu}_t^{(i)}, (\widehat{\sigma}_t^{(i)})^2) = \frac{1}{\sqrt{2\pi}\widehat{\sigma}_t^{(i)}} \exp\left(-\frac{(x-\widehat{\mu}_t^{(i)})^2}{2(\widehat{\sigma}_t^{(i)})^2}\right), \tag{6.13}$$

where the parameters $\widehat{\mu}_t^{(i)}$ and $\widehat{\sigma}_t^{(i)}$ are given by

$$\widehat{\mu}_t^{(i)} = \mu_X + \frac{\sigma_X}{\sigma_t^{(i)}}\rho_t^{(i)}(\widehat{\xi}_t^{(i)} - \mu_t^{(i)}), \qquad \textit{and} \qquad \widehat{\sigma}_t^{(i)} = \sqrt{(1-(\rho_t^{(i)})^2)\sigma_X^2}, \tag{6.14}$$

and the function $\rho_t^{(i)} = \widehat{\kappa}^{(i)}\sigma_X t/\sigma_t^{(i)}$ is the correlation between X_T and $\widehat{\xi}_t^{(i)}$.

Proof. Using Lemma 6.3.1, the conditional distribution of X_T given $\widehat{\xi}_t^{(i)}$ is

$$X_T|\widehat{\xi}_t^{(i)} \sim \Phi\left(\mu_X + \frac{\sigma_X}{\sigma_t^{(i)}}\rho_t^{(i)}(\widehat{\xi}_t^{(i)} - \mu_t^{(i)}), (1-(\rho_t^{(i)})^2)\sigma_X^2\right) \sim \Phi\left(\widehat{\mu}_t^{(i)}, (\widehat{\sigma}_t^{(i)})^2\right), \tag{6.15}$$

where $\Phi(.)$ is the Gaussian distribution, and $\rho_t^{(i)}$ is the correlation between X_T and $\widehat{\xi}_t^{(i)}$. □

At each time t, $\psi_t^{(i)}$ is a density belonging to a parametric family of Gaussian distributions on $\mathbb{R}$. Note that the parameters of the Gaussian distributions are functions of $\widehat{\xi}_t^{(i)}$.

It can be shown that the parameter space of Gaussian distributions with the parameter set $\Theta = \{\mu, \sigma^2\}$, satisfying $-\infty < \mu < \infty$ and $0 < \sigma < \infty$, is a 2-dimensional differentiable manifold (see, for example, Arwini and Dodson, 2008), say $\mathcal{P}$, which is locally diffeomorphic to $\mathbb{R}^2$. The parameters μ and σ are the local coordinates of the manifold, and the points of the manifold are determined by Gaussian distributions with varying parameters. When $\mathcal{P}$ is endowed with a Riemannian metric, it is a Riemannian manifold. More specifically, $\mathcal{P}$ is a hyperbolic space with constant curvature $R = -1/2$, where the Riemannian metric tensor on $\mathcal{P}$ is what is called the Fisher information metric g_{ij} (see Fisher, 1925 and Rao, 1945), given by

$$g_{ij}(\Theta) = \int_{\mathbb{X}} q(x,\Theta)\frac{\partial \log q(x,\Theta)}{\partial \Theta_i}\frac{\partial \log q(x,\Theta)}{\partial \Theta_j}\,\mathrm{d}x. \tag{6.16}$$

On $\mathcal{P}$, the 2×2 matrix $[g_{ij}(\Theta)]$ is a positive-definite matrix given by

$$[g_{ij}(\Theta)] = \begin{bmatrix} \sigma^2 & 2\mu\sigma \\ 2\mu\sigma^2 2\sigma^2 & 2\mu^2+\sigma^2 \end{bmatrix}. \tag{6.17}$$

Setting $X_T = x$, we can write the following limits: $\lim_{t\to T}\rho_t^{(i)} = 1$ which implies $\lim_{t\to T}\widehat{\sigma}_t^{(i)} = 0$ and $\lim_{t\to T}\widehat{\mu}_t^{(i)} = x$. It can be observed that the limits for $i = 1,\ldots,n+1$ are the same, regardless of the number of information processes. It can also be seen that the limits are not on the Riemannian manifold $\mathcal{P}$. However, by allowing $0 \leq \sigma < \infty$, we can form a manifold with boundary (see for example, Lafferty and Lebanon, 2005, and Tu, 2010). The points on the boundary $\partial\mathcal{P}$ are determined by Dirac measures centered at a point mass, which, by taking $\sigma \to 0$, are the limits of Gaussian distributions. Also, $\partial\mathcal{P}$ is itself a 1-dimensional manifold, and is flat with zero curvature. Therefore, we can construct a manifold $\mathcal{M} = \mathcal{P} \bigcup \partial\mathcal{P}$, where $\mathcal{P} = \text{Int}(\mathcal{M})$ is the interior of $\mathcal{M}$ and $\partial\mathcal{P} = \partial\mathcal{M}$ is the boundary of $\mathcal{M}$. This ensures that the limits are included on the manifold $\mathcal{M}$.

The Fisher information metric g_{ij} can be used to define a distance metric between two distributions by integrating the infinitesimal line element along the geodesic connecting the two points on the manifold $\mathcal{P}$. We call this distance the Fisher-Rao distance (also see, Brody and Hughston, 2001). The geodesics $\gamma : I \to \mathcal{P}$ with respect to g_{ij} are the solutions of the following Euler-Lagrange differential equation:

$$\sum_{i=1}^{2} g_{ik} \frac{\mathrm{d}^2\Theta_i(u)}{\mathrm{d}u^2} + \sum_{i,j=1}^{2} \Gamma_{ijk} \frac{\mathrm{d}\Theta_i(u)}{\mathrm{d}u} \frac{\mathrm{d}\Theta_j(u)}{\mathrm{d}u} = 0, \tag{6.18}$$

for $k = 1, 2$, where $\Theta(u)$ is a curve on $\mathcal{P}$ between the given two end points, which are the boundary conditions. Γ_{ijk} is the Christoffel symbol of the first kind:

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial \Theta_i} + \frac{\partial g_{ik}}{\partial \Theta_j} - \frac{\partial g_{ij}}{\partial \Theta_k} \right), \qquad \text{for } k = 1, 2. \tag{6.19}$$

We shall find the distance between the Gaussian distributions with densities $\psi_t^{(1)} = \psi_t$ and any given $\psi_t^{(i)}$ for $i = 2, \ldots, n+1$, at any given time $t \leq T$. The boundary $\partial\mathcal{M}$ has zero curvature and its geodesics are linear curves. Also, since $\pi_T^{(i)}(\mathrm{d}x) = \delta_{X_T}(\mathrm{d}x)$ for all $i = 1, \ldots, n+1$, the limits are the same point on $\partial\mathcal{M}$, and hence the distance between them is zero.

Therefore, we confine ourselves in finding the distance between points at times $t < T$. To do so, one should calculate integrals of infinitesimal line elements along the geodesics on $\mathcal{P}$. The infinitesimal line element $\mathrm{d}s$ on $\mathcal{P}$ is given by

$$\mathrm{d}s^2 = \sum_{i,j} g_{ij}(\Theta)\, \mathrm{d}\Theta_i\, \mathrm{d}\Theta_j, \tag{6.20}$$

which is also called the squared local distance. The length of the curve $\Theta(u)$, connecting two points $\Theta(u_1)$ and $\Theta(u_2)$ is given by the following:

$$\overline{D} = \int_{u_1}^{u_2} \sum_{i,j} \sqrt{g_{ij}(\Theta(u))\, \mathrm{d}\Theta_i(u)\, \mathrm{d}\Theta_j(u)}. \tag{6.21}$$

As discussed previously, the geodesic γ is the curve which minimizes the length $\overline{D}$. It follows from (6.16) that $\mathrm{d}s^2 = (\,\mathrm{d}\mu^2 + 2\,\mathrm{d}\sigma^2)\,/\sigma^2$. Then, at each fixed time $t < T$, the geodesic distance on $\mathrm{Int}(\mathcal{M})$ between points determined by ψ_t^1 and $\psi_t^{(i)}$ for $i = 2, \ldots, n+1$ is given by

$$\overline{D}_t(\psi_t^{(1)}, \psi_t^{(i)}) = \sqrt{2}\left|\log\left(\frac{1+\zeta_t^{1,i}}{1-\zeta_t^{1,i}}\right)\right| = 2\sqrt{2}\tanh^{-1}(\zeta_t^{1,i}), \tag{6.22}$$

where the function $\zeta_t^{1,i}$ is defined as follows:

$$\zeta_t^{1,i} = \left(\frac{(\widehat{\mu}_t^{(i)} - \widehat{\mu}_t^{(1)})^2 + 2(\widehat{\sigma}_t^{(i)} - \widehat{\sigma}_t^{(1)})^2}{(\widehat{\mu}_t^{(i)} - \widehat{\mu}_t^{(1)})^2 + 2(\widehat{\sigma}_t^{(i)} + \widehat{\sigma}_t^{(1)})^2}\right)^{\frac{1}{2}}. \tag{6.23}$$

The functional form of the Fisher-Rao geodesic distance (6.22) for Gaussian distributions can be found in Atkinson and Mitchell (1981), and Burbea and Rao (1982). This is the metric when both parameters are different. The metric takes alternative forms when the mean or variance is fixed (we omit these formulas since, in our setting, the probability of such events is zero $\mathbb{Q}$-a.s., and we refer to Atkinson and Mitchell, 1981).

We define what we call the *Fisher-Rao asymmetry process* $\{\mathrm{FR}_t(\psi_t^{(1)}||\psi_t^{(i)})\}_{t\in[0,T]}$ on $\mathcal{M}$, as follows:

$$\mathrm{FR}_t(\psi_t^{(1)}||\psi_t^{(i)}) = \begin{cases} \overline{D}_t(\psi_t^{(1)}, \psi_t^{(i)}) & \text{if } t < T, \\ 0 & \text{if } t = T. \end{cases} \tag{6.24}$$

The FR asymmetry process takes the value zero at the boundary $\partial\mathcal{M}$, where by definition, $\mathrm{FR}_T(\psi_T^{(1)}||\psi_T^{(i)}) = 0$. This holds on $\partial\mathcal{M}$ since the limits of the Gaussian distributions under consideration is the Dirac measure $\delta_{X_T}(\mathrm{d}x) \in \partial\mathcal{M}$ at $t = T$, irrespective of the number of information sources.

The next proposition shows that the FR asymmetry process jumps at every entry of a new source of information. Since X_T is Gaussian, at a fixed time $t < T$, each Brownian information parameterises a point on $\mathrm{Int}(\mathcal{M})$. Evolving the time in between information entries, a continuous trajectory is determined on $\mathrm{Int}(\mathcal{M})$ by the information processes. If the FR asymmetry process jumps, the new information source parameterises a new point in a discontinuous way. Hence, each jump of the FR asymmetry process measures the impact of the appearance of a new information source geometrically on the hyperbolic space $\mathrm{Int}(\mathcal{M})$.

Proposition 6.3.3. *The dynamics of* $\{FR_t(\psi_t^{(1)}||\overline{\psi}_t)\}$ *on* $\mathrm{Int}(\mathcal{M})$ *are governed by*

$$\begin{aligned} \mathrm{d}FR_t(\psi_t^{(1)}||\overline{\psi}_t) &= \sum_{i=1}^{n+1} \mathrm{d}\overline{D}_t(\psi_t^{(1)}, \psi_t^{(i)})\mathbf{I}_t(i) \\ &\quad + \sum_{i=2}^{n+1}\left(\overline{D}_t(\psi_t^{(1)}, \psi_t^{(i)}) - \overline{D}_t(\psi_t^{(1)}, \psi_t^{(i-1)})\right)\delta_{\tau_{i-1}}(\mathrm{d}t). \end{aligned} \tag{6.25}$$

Proof. On $\mathrm{Int}(\mathcal{M})$, at each fixed time t for $0 \leq t < T$, we can write

$$\overline{D}_t(\psi_t^{(1)}, \overline{\psi}_t) = \overline{D}_t(\psi_t^{(1)}, \sum_{i=1}^{n+1} \psi_t^{(i)} \mathbf{I}_t(i)) = \sum_{i=1}^{n+1} \overline{D}_t(\psi_t^{(1)}, \psi_t^{(i)}) \mathbf{I}_t(i), \tag{6.26}$$

since the elements of $\mathbf{I}_t$ are orthogonal such that $\mathbf{I}_t(i) = 1$ implies $\mathbf{I}_t(j) = 0$ for all $i \neq j$. □

We can also show that the FR asymmetry between $\psi_t^{(1)}$ and $\overline{\psi}_t$ coincides with the KL asymmetry between $\psi_t^{(1)}$ and $\overline{\psi}_t$ at points on $\mathrm{Int}(\mathcal{M})$ infinitesimally close to each other. Assume an open neighborhood $\mathcal{E}_r(p)$ around $p \in \mathrm{Int}(\mathcal{M})$ for some $r > 0$, such that for a fixed time t, the FR distance $\overline{D}_t(\psi_t^{(1)}, \overline{\psi}_t)$ between the points inside $\mathcal{E}_r(p)$ can be approximated to an arbitrary precision by the squared infinitesimal line element $\mathrm{d}s^2$. In the parametric case, the KL asymmetry between $\psi_t^{(1)}$ and $\overline{\psi}_t$ can then be written as

$$\begin{aligned} \mathrm{KL}_t(\psi_t^{(1)} || \overline{\psi}_t) &\rightleftharpoons \mathrm{KL}_t(q(x, \widehat{\mu}_t^{(1)}, \widehat{\sigma}_t^{(1)}) || \sum_{i=1}^{n+1} q(x, \widehat{\mu}_t^{(i)}, \widehat{\sigma}_t^{(i)}) \mathbf{I}_t(i)) \\ &= \mathrm{KL}_t(q(x, \widehat{\mu}_t^{(1)}, \widehat{\sigma}_t^{(1)}) || \sum_{i=1}^{n+1} q(x, \widehat{\mu}_t^{(1)} + \epsilon_\mu^{(i)}(t), \widehat{\sigma}_t^{(1)} + \epsilon_\sigma^{(i)}(t)) \mathbf{I}_t(i)), \end{aligned} \tag{6.27}$$

for small $\epsilon_\mu^{(i)}(t) > 0$ and $\epsilon_\sigma^{(i)}(t) > 0$. Therefore, the KL asymmetry between points belonging to $\mathcal{E}_r(p)$ can be represented with the following functional form:

$$\mathrm{KL}(q(x, \Theta) || q(x, \Theta + \mathrm{d}\Theta)). \tag{6.28}$$

As discussed in Brigo *et al.* (1995), using the Taylor series expansion:

$$\begin{aligned} \mathrm{KL}(q(x, \Theta) || q(x, \Theta + \mathrm{d}\Theta)) &= -\sum_{i=1}^{2} \int \left(\frac{\partial \log q(x, \Theta)}{\partial \Theta_i} q(x, \Theta) \, \mathrm{d}x \right) \mathrm{d}\Theta_i \\ &\quad - \sum_{i=1}^{2} \int \left(\frac{\partial^2 \log q(x, \Theta)}{\partial \Theta_i \partial \Theta_j} q(x, \Theta) \, \mathrm{d}x \right) \mathrm{d}\Theta_i \, \mathrm{d}\Theta_j + O(|\mathrm{d}\Theta|^3) \\ &= \sum_{i,j=1}^{2} g_{ij}(\Theta) \, \mathrm{d}\Theta_i \, \mathrm{d}\Theta_j + O(|\mathrm{d}\Theta|^3) \\ &= \mathrm{d}s^2 + O(|\mathrm{d}\Theta|^3), \end{aligned} \tag{6.29}$$

where g_{ij} is the Fisher information metric on $\mathrm{Int}(\mathcal{M})$. Hence, the KL asymmetry coincides with the FR asymmetry at points in $\mathcal{E}_r(p)$.

In fact, the FR asymmetry between $\psi_t^{(1)}$ and $\overline{\psi}_t$ coincides with any f-asymmetry between $\psi_t^{(1)}$ and $\overline{\psi}_t$ at points infinitesimally close to each other on $\mathrm{Int}(\mathcal{M})$. From Amari and Cichocki (2010, Theorem 5), any f-divergence induces a unique Riemannian metric, which is the Fisher information metric g_{ij}. Also, by the Taylor series expansion, any f-divergence $\Delta_f(q(x,\Theta)||q(x,\Theta+\,\mathrm{d}\Theta))$ for some small $\mathrm{d}\Theta$ can be written as

$$\Delta_f(q(x,\Theta)||q(x,\Theta+\,\mathrm{d}\Theta)) \approx \sum_{i,j} g_{ij}(\Theta)\,\mathrm{d}\Theta_i\,\mathrm{d}\Theta_j = \,\mathrm{d}s^2. \tag{6.30}$$

The statement that the FR asymmetry coincides infinitesimally with any f-asymmetry follows since the FR asymmetry is defined by the Fisher information metric g_{ij}. The same line of argument holds for the KL (or the SH) asymmetry, since KL (or SH) is an f-divergence.

6.3.1 Surfaces of Information Asymmetry

At each entry of a new information source, the jump sizes of the SH and the FR asymmetry processes quantify the sudden impact of a new source of information. However, note that these processes alone do not directly provide the jump sizes of the conditional density process $\{\overline{\psi}_t\}$ itself. We shall show that at each entry of a new information source, both the SH and the FR asymmetries characterise triangles on $\mathcal{S}^+$ and $\mathcal{P}$, respectively. These triangles allow us to represent the jump sizes of $\{\sqrt{\overline{\psi}_t}\}$ on $\mathcal{S}^+$ and of $\{\overline{\psi}_t\}$ on $\mathcal{P}$, using geodesics and curvatures of the underlying manifolds. We call these triangles surfaces of information asymmetry.

Spherical Surfaces of Information Asymmetry

We have shown that at each entry of a new source of information, the SH asymmetry process $\{\mathrm{SH}_t(\psi_t||\overline{\psi}_t)\}$ jumps. The jumps of the SH asymmetry by themselves do not tell much about the actual jump sizes of $\{\sqrt{\overline{\psi}_t}\}$, but instead, tell us about the jumps of the distances between $\{\sqrt{\psi_t}\}$ and $\{\sqrt{\overline{\psi}_t}\}$. Although, we can still bring forth a geometrical machinery in determining the jump sizes of $\{\sqrt{\overline{\psi}_t}\}$ on $\mathcal{S}^+$ from the SH asymmetry process.

First, for a fixed time $t<T$, we write

$$\mathrm{SH}_t(\psi_t||\overline{\psi}_t) = \mathrm{SH}_t(\psi_t||\sum_{i=1}^{n+1}\psi_t^{(i)}\mathbf{I}_t(i)) = \sum_{i=1}^{n+1}\mathrm{SH}_t(\psi_t||\psi_t^{(i)})\mathbf{I}_t(i), \tag{6.31}$$

since $\mathbf{I}(i)=1$ implies $\mathbf{I}(j)=0$ for $i\neq j$. This allows us to write

$$\begin{aligned}\mathrm{dSH}_t(\psi_t||\overline{\psi}_t) &= \sum_{i=1}^{n+1}\mathbf{I}_t(i)\,\mathrm{dSH}_t(\psi_t||\psi_t^{(i)}) + \sum_{i=2}^{n+1}\left(\mathrm{SH}_t(\psi_t||\psi_t^{(i)})-\mathrm{SH}_t(\psi_t||\psi_t^{(i-1)})\right)\delta_{\tau_{i-1}}(\mathrm{d}t)\\ &= -\sum_{i=1}^{n+1}\mathbf{I}_t(i)\,\mathrm{d}\cos\left(\vartheta_t\left(\sqrt{\psi_t},\sqrt{\psi_t^{(i)}}\right)\right)\\ &+\sum_{i=2}^{n+1}\left[\cos\left(\vartheta_t\left(\sqrt{\psi_t},\sqrt{\psi_t^{(i-1)}}\right)\right)-\cos\left(\vartheta_t\left(\sqrt{\psi_t},\sqrt{\psi_t^{(i)}}\right)\right)\right]\delta_{\tau_{i-1}}(\mathrm{d}t),\end{aligned} \tag{6.32}$$

for $0 \leq t < T$ on $\mathcal{S}^+$. From this point on, we shall denote $\rho_t = \sqrt{\psi_t}$ and $\rho_t^{(i)} = \sqrt{\psi_t^{(i)}}$.

Note that at $\tau_i = t$, we can identify 3 points on $\mathcal{S}^+$ determined by ρ_t, $\rho_t^{(i)}$ and $\rho_t^{(i+1)}$ for $i = 2, \ldots, n$ (ignoring the case of $\rho_t^{(1)}$, since $\rho_t = \rho_t^{(1)}$). Then, it can be seen from (6.32) that each jump of the SH asymmetry process is characterised by two spherical distances: $\vartheta_t(\rho_t, \rho_t^{(i)})$ and $\vartheta_t(\rho_t, \rho_t^{(i+1)})$, and not directly by $\vartheta_t(\rho_t^{(i)}, \rho_t^{(i+1)})$. At each jump, two main events may occur: (i) the new point determined by $\rho_t^{(i+1)}$ may be on the same geodesic connecting the points determined by ρ_t and $\rho_t^{(i)}$, or (ii) otherwise.

At $\tau_i = t$, we denote the jump size of $\sqrt{\psi_t}$ by $\vartheta_t(\rho_t^{(i)}, \rho_t^{(i+1)})$ on $\mathcal{S}^+$. Then, for case (i)

$$\vartheta_t(\rho_t^{(i)}, \rho_t^{(i+1)}) = \left[\vartheta_t\left(\rho_t, \rho_t^{(i)}\right) - \vartheta_t\left(\rho_t, \rho_t^{(i+1)}\right)\right] \quad \text{or} \tag{6.33}$$

$$= \left[\vartheta_t\left(\rho_t, \rho_t^{(i+1)}\right) - \vartheta_t\left(\rho_t, \rho_t^{(i)}\right)\right] \quad \text{or} \tag{6.34}$$

$$= \left[\vartheta_t\left(\rho_t, \rho_t^{(i)}\right) + \vartheta_t\left(\rho_t, \rho_t^{(i+1)}\right)\right], \tag{6.35}$$

if, on the same geodesic: $\rho_t^{(i+1)}$ determines a point between the points determined by ρ_t and $\rho_t^{(i)}$, or $\rho_t^{(i)}$ determines a point between the points determined by ρ_t and $\rho_t^{(i+1)}$, or $\rho_t^{(i+1)}$ determines a point on the opposite direction from the points determined by ρ_t to $\rho_t^{(i)}$, respectively.

If at $\tau_i = t$, the new point determined by $\rho_t^{(i+1)}$ is not on the same geodesic connecting the points determined by ρ_t and $\rho_t^{(i)}$, this induces a compact spherical triangle on $\mathcal{S}^+$. In particular, we can identify three pairs from three points determined by ρ_t, $\rho_t^{(i)}$ and $\rho_t^{(i+1)}$ on three distinct geodesics on $\mathcal{S}^+$. This forms a spherical triangle of which the three points are the vertices. This characterises a geometric surface at each entry of a new information source, which we call a spherical surface of information asymmetry. We can now exploit more rules from spherical geometry. We use the term geodesic angles, which in the usual notion, are the angles between the tangent lines of the geodesics.

Proposition 6.3.4. *At $\tau_i = t$, the jump size of $\sqrt{\psi_t}$ on $\mathcal{S}^+$, denoted by $\vartheta_t(\rho_t^{(i)}, \rho_t^{(i+1)})$ can be represented as*

$$\begin{aligned}\cos\left(\vartheta_t(\rho_t^{(i)}, \rho_t^{(i+1)})\right) &= \cos\left(\vartheta_t(\rho_t, \rho_t^{(i)})\right)\cos\left(\vartheta_t(\rho_t, \rho_t^{(i+1)})\right) \\ &+ \sin\left(\vartheta_t(\rho_t, \rho_t^{(i)})\right)\sin\left(\vartheta_t(\rho_t, \rho_t^{(i+1)})\right) \\ &\times \cos\left(\beta_t\left(\gamma(\rho_t, \rho_t^{(i)}), \gamma(\rho_t, \rho_t^{(i+1)})\right)\right),\end{aligned} \tag{6.36}$$

where $\beta_t\left(\gamma\left(\rho_t, \rho_t^{(i)}\right), \gamma\left(\rho_t, \rho_t^{(i+1)}\right)\right)$ is the geodesic angle in radians between the geodesics denoted by $\gamma\left(\rho_t, \rho_t^{(i)}\right)$ and $\gamma\left(\rho_t, \rho_t^{(i+1)}\right)$, connecting the associated points on $\mathcal{S}^+$.

Proof. The statement follows from the spherical law of cosines. □

Hence, spherical surfaces of information asymmetry allow us to represent the jump sizes of $\{\sqrt{\psi_t}\}$ using the angles between geodesics, which are determined by the points characterising the SH asymmetry at each entry of a new information source. Obviously, any other length of the triangle can be found by the lengths of the remaining two sides and the corresponding geodesic angle. We can also analyse the areas of these surfaces. The areas of the spherical triangles offer an alternative way of quantifying the impact of a new source of information.

Proposition 6.3.5. *Denote a spherical triangle on $\mathcal{S}^+$ as Ξ and its surface area as $\Pi(\Xi)$. The area of the spherical surface of information asymmetry at $\tau_i = t$ is*

$$\begin{aligned}\Pi(\Xi_t) = {} & \beta_t\left(\gamma\left(\rho_t, \rho_t^{(i)}\right), \gamma\left(\rho_t, \rho_t^{(i+1)}\right)\right) + \alpha_t\left(\gamma\left(\rho_t, \rho_t^{(i)}\right), \gamma\left(\rho_t^{(i)}, \rho_t^{(i+1)}\right)\right) \\ & + \phi_t\left(\gamma\left(\rho_t, \rho_t^{(i+1)}\right), \gamma\left(\rho_t^{(i)}, \rho_t^{(i+1)}\right)\right) - \pi, \end{aligned} \tag{6.37}$$

where β, α and ϕ are the corresponding geodesic angles in radians on $\mathcal{S}^+$.

Proof. The sum of the geodesic angles of a spherical triangle on $\mathcal{S}$ always exceeds the sum of the angles of an Euclidean triangle, which is called the spherical excess. Girard's theorem (a special case of Gauss-Bonnet theorem) states that spherical excess alone determines the surface area of any spherical triangle on $\mathcal{S}$. The expression in (6.37) follows. □

The sum of the angles at $\tau_i = t$ is $\Pi(\Xi_t) + \pi$. Using the geodesic angles on $\mathcal{S}^+$ and the spherical areas, we can represent the jump size of $\sqrt{\psi_t}$ at $\tau_i = t$, in an alternative way. First, we define

$$Q_t = \frac{(\Pi(\Xi_t) + \pi)}{2}. \tag{6.38}$$

Proposition 6.3.6. *At $\tau_i = t$, the jump size of $\sqrt{\psi_t}$ on $\mathcal{S}^+$, denoted by $\vartheta_t(\rho_t^{(i)}, \rho_t^{(i+1)})$ can be represented as*

$$\begin{aligned}\tan\left(\frac{\vartheta_t(\rho_t^{(i)}, \rho_t^{(i+1)})}{2}\right) = {} & \left[-\cos(Q_t)\cos\left(Q_t - \beta_t\left(\gamma(\rho_t, \rho_t^{(i)}), \gamma(\rho_t, \rho_t^{(i+1)})\right)\right)\right]^{\frac{1}{2}} \\ & \times 1/\left[\cos\left(Q_t - \alpha_t\left(\gamma(\rho_t, \rho_t^{(i)}), \gamma(\rho_t^{(i)}, \rho_t^{(i+1)})\right)\right)\right]^{\frac{1}{2}} \\ & \times 1/\left[\cos\left(Q_t - \phi_t\left(\gamma(\rho_t, \rho_t^{(i+1)}), \gamma(\rho_t^{(i)}, \rho_t^{(i+1)})\right)\right)\right]^{\frac{1}{2}}, \end{aligned} \tag{6.39}$$

where β, α and ϕ are the corresponding geodesic angles in radians on $\mathcal{S}^+$.

Proof. The statement follows from the half-side formula in spherical geometry. □

The Riemannian curvature of $\mathcal{S}$ is $R = +1$, and hence, it does not explicitly appear in equations (6.36)-(6.39). However, (6.36)-(6.39) are implicitly determined by the curvature of $\mathcal{S}$. The Riemannian curvature will explicitly appear in the following section.

Hyperbolic Surfaces of Information Asymmetry

Similar to the SH asymmetry, the jumps of the FR asymmetry process by themselves do not directly provide the jump sizes of $\{\overline{\psi}_t\}$. Then again, we can adopt a geometrical approach in determining the jump sizes of $\{\overline{\psi}_t\}$ on $\mathcal{P}$ from the FR asymmetry process. First, to make more sense of the geometry, we further specify the underlying model of the hyperbolic space $\mathcal{P}$. In particular, we define the more general hyperbolic space $\mathcal{W}$ by using the Poincaré upper-half-plane model, and exploit the associated trigonometry on $\mathcal{P}$. For a detailed account of hyperbolic geometry, see for example, Anderson (2005).

Define the Riemannian sphere $\overline{\mathbb{C}}$ as follows:

$$\overline{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}, \tag{6.40}$$

where $\mathbb{C}$ is the complex plane. The construction of $\overline{\mathbb{C}}$ is an example of a more general topological construction called the one-point compactification. The underlying hyperbolic space of Poincaré upper-half-plane model is the upper-half plane $\mathcal{W}$ in the complex plane $\mathbb{C}$:

$$\mathcal{W} = \{z \in \mathbb{C} : \Im(z) > 0\}, \tag{6.41}$$

where $\Im(z)$ is the imaginary part of z. $\mathcal{W}$ is an open subset of the Riemannian sphere $\overline{\mathbb{C}}$, and the angles between curves in $\mathcal{W}$ are the angles between the tangent lines of curves in $\mathbb{C}$. As a 2-dimensional space, $\mathcal{W}$ has negative constant Riemannian curvature. We can now use the trigonometric rules of $\mathcal{W}$ on the hyperbolic space $\mathcal{P}$. First, recall that

$$\mathrm{dFR}_t(\psi_t||\overline{\psi}_t) = \sum_{i=1}^{n+1} \mathrm{d}\overline{D}_t(\psi_t, \psi_t^{(i)})\mathbf{I}_t(i) + \sum_{i=2}^{n+1} \left(\overline{D}_t(\psi_t, \psi_t^{(i)}) - \overline{D}_t(\psi_t, \psi_t^{(i-1)})\right) \delta_{\tau_{i-1}}(\mathrm{d}t), \tag{6.42}$$

for $0 \leq t < T$ on $\mathcal{P}$. Hence, at $\tau_i = t$, we can identify 3 points on $\mathcal{P}$ determined by ψ_t, $\psi_t^{(i)}$ and $\psi_t^{(i+1)}$ for $i = 2, \ldots, n$ (again, ignoring the case of $\psi_t^{(1)}$, since $\psi_t = \psi_t^{(1)}$). If the new point determined by $\psi_t^{(i+1)}$ is on the same geodesic that connects the points determined by ψ_t and $\psi_t^{(i)}$ on $\mathcal{P}$, the jump size of $\overline{\psi}_t$ at $\tau_i = t$, denoted by $\overline{D}_t(\psi_t^{(i)}, \psi_t^{(i+1)})$ is

$$\begin{aligned} \overline{D}_t(\psi_t^{(i)}, \psi_t^{(i+1)}) &= \left[\overline{D}_t\left(\psi_t, \psi_t^{(i)}\right) - \overline{D}_t\left(\psi_t, \psi_t^{(i+1)}\right)\right] \quad \text{or} && (6.43)\\ &= \left[\overline{D}_t\left(\psi_t, \psi_t^{(i+1)}\right) - \overline{D}_t\left(\psi_t, \psi_t^{(i)}\right)\right] \quad \text{or} && (6.44)\\ &= \left[\overline{D}_t\left(\psi_t, \psi_t^{(i)}\right) + \overline{D}_t\left(\psi_t, \psi_t^{(i+1)}\right)\right], && (6.45) \end{aligned}$$

given that: $\psi_t^{(i+1)}$ determines a point between the points determined by ψ_t and $\psi_t^{(i)}$, or $\psi_t^{(i)}$

determines a point between the points determined by ψ_t and $\psi_t^{(i+1)}$, or $\psi_t^{(i+1)}$ determines a point on the opposite direction from the points determined by ψ_t to $\psi_t^{(i)}$, respectively.

If at $\tau_i = t$, the new point determined by $\psi_t^{(i+1)}$ is not on the same geodesic connecting the points determined by ψ_t and $\psi_t^{(i)}$, this induces a compact hyperbolic triangle on $\mathcal{P}$. Similar to the spherical case, we can identify three pairs from the three points determined by ψ_t, $\psi_t^{(i)}$ and $\psi_t^{(i+1)}$ on three distinct geodesics on $\mathcal{P}$. This forms a hyperbolic triangle of which the three points are the vertices. This characterises a geometric surface at each entry of a new information source, which we call a hyperbolic surface of information asymmetry.

The links between angles and sides of hyperbolic triangles are analagous to those of spherical triangles. It is more convenient to state the following results when the lengths on $\mathcal{P}$ are adjusted such that they are measured in an alternative unit that we denote by r (which is a unit that is analogous to the radian on $\mathcal{S}$), where

$$r = -\frac{\sqrt{-R}}{R} = \sqrt{2}, \tag{6.46}$$

since the Riemannian curvature is $R = -1/2$ on $\mathcal{P}$. Note that if $\mathcal{W}$ has curvature $R = -1$, then $r = 1$ on $\mathcal{W}$.

Proposition 6.3.7. *At $\tau_i = t$, the jump size of $\overline{\psi}_t$ on $\mathcal{P}$, denoted by $\overline{D}_t(\psi_t^{(i)}, \psi_t^{(i+1)})$ can be represented as*

$$\begin{aligned}\cosh\left(\frac{\overline{D}_t(\psi_t^{(i)}, \psi_t^{(i+1)})}{\sqrt{2}}\right) &= \cosh\left(\frac{\overline{D}_t\left(\psi_t, \psi_t^{(i)}\right)}{\sqrt{2}}\right)\cosh\left(\frac{\overline{D}_t\left(\psi_t, \psi_t^{(i+1)}\right)}{\sqrt{2}}\right) \\ &\quad - \sinh\left(\frac{\overline{D}_t\left(\psi_t, \psi_t^{(i)}\right)}{\sqrt{2}}\right)\sinh\left(\frac{\overline{D}_t\left(\psi_t, \psi_t^{(i+1)}\right)}{\sqrt{2}}\right) \\ &\quad \times \cos\left(\beta_t\left(\gamma(\psi_t, \psi_t^{(i)}), \gamma(\psi_t, \psi_t^{(i+1)})\right)\right),\end{aligned} \tag{6.47}$$

where $\beta_t\left(\gamma\left(\psi_t, \psi_t^{(i)}\right), \gamma\left(\psi_t, \psi_t^{(i+1)}\right)\right)$ is the geodesic angle in radians between the geodesics shown as $\gamma\left(\psi_t, \psi_t^{(i)}\right)$ and $\gamma\left(\psi_t, \psi_t^{(i+1)}\right)$, connecting the associated points on $\mathcal{P}$.

Proof. The statement follows from the hyperbolic law of cosines. Note that the denominator $\sqrt{2}$ comes from (6.46). □

Any other length of the hyperbolic triangle can be found by the lengths of the remaining two sides and the corresponding geodesic angle. The areas of the hyperbolic triangles provide an alternative way of quantifying the impact of a new information source:

Proposition 6.3.8. *Denoting a hyperbolic triangle on $\mathcal{P}$ as Ξ and its surface area as $\Pi(\Xi)$,* *the area of the hyperbolic surface of information asymmetry at $\tau_i = t$ is*

$$\begin{aligned}\Pi(\Xi_t) = 2\pi &- 2\beta_t \left(\gamma\left(\psi_t, \psi_t^{(i)}\right), \gamma\left(\psi_t, \psi_t^{(i+1)}\right)\right) \\ &- 2\alpha_t \left(\gamma\left(\psi_t, \psi_t^{(i)}\right), \gamma\left(\psi_t^{(i)}, \psi_t^{(i+1)}\right)\right) \\ &- 2\phi_t \left(\gamma\left(\psi_t, \psi_t^{(i+1)}\right), \gamma\left(\psi_t^{(i)}, \psi_t^{(i+1)}\right)\right),\end{aligned} \tag{6.48}$$

where β, α and ϕ are the corresponding geodesic angles on $\mathcal{P}$.

Proof. The sum of the geodesic angles of a hyperbolic triangle is always less than the sum of the angles of an Euclidean triangle, which may be called the hyperbolic defect. By the Gauss-Bonnet theorem, the surface area of any hyperbolic triangle is $r^2(\pi - \alpha - \beta - \phi)$, given that β, α and ϕ are the geodesic angles. On $\mathcal{P}$, the unit r is as shown in (6.46). □

The sum of the angles at $\tau_i = t$ is $\pi - \Pi(\Xi_t)/2$. It can be seen that hyperbolic surfaces of information asymmetry allow us to represent the jump sizes of $\{\overline{\psi}_t\}$ using the angles between geodesics, which are determined by the FR asymmetry at each entry of a new information source. Their areas are explicitly determined by the curvature of $\mathcal{P}$.

Remark 6.3.9. *Suppose we are interested in multiple asymmetries between the probability density ψ_t and any other $m \in \mathbb{N}_+$ densities $\overline{\psi}_t^j$ for $j = 1 \ldots, m$ at $t < T$. Let's further assume that the time of each jump of $\{\overline{\psi}_t^j\}$'s coincide. Then, for times in between the stopping times, we can identify $m+1$ points on the associated manifolds $\mathcal{S}$ and $\mathcal{P}$.*

At every appearance of a new information source, we can identify $2m+1$ points on $\mathcal{S}$ and $\mathcal{P}$. Hence, each appearance may induce spherical polygons on $\mathcal{S}$ and hyperbolic polygons on $\mathcal{P}$, which can be represented as unions of spherical triangles and hyperbolic triangles, respectively. Spherical and hyperbolic surfaces form geometrical bases to characterise higher dimensional asymmetries with polygons.

6.3.2 Squared-Hellinger Asymmetry and Isometric Invariant of Poincaré Disc Under the Action of General Möbius Group

We shall show a geometric relationship between the SH asymmetry and an isometric invariant of the Poincaré disc under the action of the general Möbius group (for groups and actions of groups, see for example, Allenby, 1991, or Beachy and Blaire, 2006). This relationship motivates us to suggest the use of an alternative geometric measure to quantify information asymmetry.

First, we shall explain what Möbius transformations are, and what the general Möbius group is. We then discuss the Poincaré disc model of the hyperbolic space, and using Möbius transformations from the upper-half-plane $\mathcal{W}$ to the Poincaré disc, we shall highlight the relationship between the SH asymmetry and an isometric invariant of the Poincaré disc.

There exists an important group of transformations of $\overline{\mathbb{C}}$, called the general Möbius group (the group operation being composition), where geometric quantities such as hyperbolic lengths and angles are invariant under its action. A Möbius transformation is a holomorphic function $\eta^* : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, with the following functional form:

$$\eta^*(z) = \frac{az+b}{cz+d}, \tag{6.49}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. The set of all Möbius transformations forms a group under composition, and we denote the group of Möbius transformations by Möb*. The general Möbius group is generated by the set of Möbius transformations and the set of complex conjugations. Denoting Möb as the general Möbius group, $\eta \in$ Möb is the composition:

$$\eta = C \circ \eta_k^* \circ \ldots \circ C \circ \eta_1^*, \tag{6.50}$$

for some $k \geq 1$, each η_j^* being a Möbius transformation, and where

$$C(z) = \overline{z} \qquad \text{given} \qquad C(\infty) = \infty, \tag{6.51}$$

for $z \in \mathbb{C}$. We note that C is a homeomorphism of $\overline{\mathbb{C}}$. It can be shown that Möb is equal to the set of homeomorphisms of $\overline{\mathbb{C}}$ that take circles in $\overline{\mathbb{C}}$ to circles in $\overline{\mathbb{C}}$. In fact, it can be shown that elements of Möb are conformal homeomorphisms of $\overline{\mathbb{C}}$ (homeomorphisms that preserve angles), and Möb$(\mathcal{W}) = \{\eta \in \text{Möb} | \eta(\mathcal{W}) = \mathcal{W}\}$ is equal to the group of isometries (homeomorphisms that preserve distances) of $\mathcal{W}$ given its metric. A hyperbolic area in $\mathcal{W}$ is invariant under the action of Möb$(\mathcal{W})$. For the proofs, refer to Anderson (2005).

Remark 6.3.10. *The FR asymmetry is invariant under the action of Möb$(\mathcal{P})$. Also if $\eta \in$ Möb$(\mathcal{P})$ acts on a hyperbolic triangle, the transformed points induce an equivalent hyperbolic triangle, since η is a conformal isometry on $\mathcal{P}$.*

The underlying hyperbolic space of Poincaré disc model is the unit disc $\mathcal{D}$ in the complex plane $\mathbb{C}$ such that

$$\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}. \tag{6.52}$$

Since both $\mathcal{D}$ and $\mathcal{W}$ are in $\overline{\mathbb{C}}$, it is possible to find a broad class of $\eta \in$ Möb, such that $\eta : \mathcal{D} \to \mathcal{W}$. Therefore, Möb allows to use $\mathcal{D}$ and $\mathcal{W}$ interchangeably when modelling hyperbolic spaces. In particular, if z and α are points in $\mathcal{W}$ and z^* is a point in $\mathcal{D}$, then

$$z^* = e^{i\theta} \frac{z - \alpha}{z - \overline{\alpha}}, \tag{6.53}$$

is a Möbius transformation that maps $\mathcal{W}$ to $\mathcal{D}$ conformally. The point $z^* \in \mathcal{D}$ is the corresponding point of $z \in \mathcal{W}$, and $\alpha \in \mathcal{W}$ is an arbitrary point mapped to the center of the disk $\mathcal{D}$, where θ rotates the disk.

Without loss of generality, we let the curvature of $\mathcal{W}$ be -1. From Poincaré uniformization theorem, we can transform the metric on $\mathcal{P}$ to the metric on $\mathcal{W}$, since $\mathcal{P}$ and $\mathcal{W}$ are conformally equivalent. We can do this by multiplying the metric on $\mathcal{P}$ with a positive constant. Then the distance between the points determined by ψ_t and $\psi_t^{(i)}$ mapped on $\mathcal{W}$ is

$$\mathrm{d}_{\mathcal{W}}(\psi_t^{(1)}, \psi_t^{(i)}) = 2\tanh^{-1}(\zeta_t^{1,i}), \tag{6.54}$$

at each fixed $t < T$, where $\mathrm{d}_{\mathcal{W}}$ is the distance on $\mathcal{W}$, and the function $\zeta_t^{1,i}$ is defined in (6.23). Note that the distance between the points determined by $\psi_t^{(1)} = \psi_t$ and $\psi_t^{(i)}$ on $\mathcal{P}$ is given in (6.22), and the expression in (6.54) follows by multiplying the metric on $\mathcal{P}$ by $1/\sqrt{2}$.

Now, let x, y be points on $\mathcal{W}$, and $A(x, y)$ denote the nonempty collection of hyperbolic paths $h : [a, b] \to \mathcal{W}$ satisfying $h(a) = x$ and $h(b) = y$. Also, having x, y be points on $\mathcal{D}$, $B(x, y)$ denotes the nonempty collection of hyperbolic paths $f : [a, b] \to \mathcal{D}$ satisfying $f(a) = x$ and $f(b) = y$. Let $\mathrm{d}_{\mathcal{W}}$ and $\mathrm{l}_{\mathcal{W}}$ be the distance and the length on $\mathcal{W}$, respectively. Let $\mathrm{d}_{\mathcal{D}}$ and $\mathrm{l}_{\mathcal{D}}$ be the distance and the length on $\mathcal{D}$, respectively. Then, using $\eta \in \text{Möb}(\overline{\mathbb{C}})$, such that $\eta : \mathcal{D} \to \mathcal{W}$, the following can be written:

$$\begin{aligned}
\mathrm{d}_{\mathcal{D}}(x, y) &= \inf\{\mathrm{l}_{\mathcal{D}}(f_t) | f_t \in B(x, y)\} \\
&= \inf\{\mathrm{l}_{\mathcal{H}}(\eta \circ f_t) | f_t \in B(x, y), \eta \in \text{Möb}(\overline{\mathbb{C}})\} \\
&= \inf\{\mathrm{l}_{\mathcal{D}}(\eta^{-1} \circ g_t) | g_t = \eta \circ f_t, g_t \in A(x, y), f_t \in B(x, y), \eta \in \text{Möb}(\overline{\mathbb{C}})\}.
\end{aligned} \tag{6.55}$$

Hence, using Möb, we can find the distances between two points on $\mathcal{D}$ starting from distances on $\mathcal{W}$ (or from distances on $\mathcal{P}$). One can also conformally map the points on $\mathcal{W}$ to points on $\mathcal{D}$ by (6.53), and calculate the distances on $\mathcal{D}$. We shall denote the distance between the points determined by ψ_t and $\psi_t^{(i)}$ mapped on $\mathcal{D}$ by $\mathrm{d}_{\mathcal{D}}(\psi_t, \psi_t^{(i)})$.

We can now define an isometric invariant of $\mathcal{D}$ under the action of Möb($\mathcal{D}$), which we denote by G:

$$G(x||y) = \frac{2||x - y||^2}{(1 - ||x||^2)(1 - ||y||^2)}, \tag{6.56}$$

where x, y are points on $\mathcal{D}$, and $||.||$ is the Euclidean norm. G characterises the distance $\mathrm{d}_{\mathcal{D}}$ on $\mathcal{D}$ (see for example, Anderson, 2005, Proposition 4.3, pp. 126), such that

$$G_t(\psi_t||\psi_t^{(i)}) = \cosh\Big(\mathrm{d}_{\mathcal{D}}(\psi_t, \psi_t^{(i)})\Big) - 1. \tag{6.57}$$

From (6.9), the SH asymmetry can be written as $\mathrm{SH}_t(\psi_t||\psi_t^{(i)}) = 1 - \cos\left(\vartheta_t(\sqrt{\psi_t}, \sqrt{\psi_t^{(i)}})\right)$.

Also, note that we can replace ϑ with $\mathrm{d}_{\mathcal{S}}$, provided that $\mathrm{d}_{\mathcal{S}}$ is the distance on the unit sphere $\mathcal{S}$. This follows since the Bhattacharyya angle ϑ is equal to the spherical distance on $\mathcal{S}^{+}$:

$$\mathrm{SH}_t(\psi_t||\psi_t^{(i)}) = 1 - \cos\left(\mathrm{d}_{\mathcal{S}}(\sqrt{\psi_t}, \sqrt{\psi_t^{(i)}})\right). \tag{6.58}$$

It can be seen from (6.57) and (6.58) that the SH asymmetry is closely related to the isometric invariant G. The curvature of $\mathcal{D}$ and the curvature of $\mathcal{S}$ are opposite in sign: -1 for $\mathcal{D}$ and +1 for $\mathcal{S}$. In addition, the cosine on the sphere $\mathcal{S}$ is replaced by the hyperbolic cosine on the hyperbolic space $\mathcal{D}$. In particular, G is the hyperbolic analogue of SH, and SH is the spherical analogue of G.

Note that $\mathrm{d}_{\mathcal{D}}(\psi_t, \psi_t^{(i)}) \geq 0$, since it is a metric. Also, since $\cosh(0) = 1$ and $\cosh(x)$ is monotonically increasing in $x \in \mathbb{R}_+$, the isometric invariant $G(.||.) \geq 0$. Given the analogy with the SH asymmetry, we are encouraged to propose the use of $G(.||.)$ as an alternative measure of divergence between Gaussian distributions. That is, $G_t(\psi_t||\overline{\psi}_t)$ can be used to quantify the information asymmetry between ψ_t and $\overline{\psi}_t$ geometrically, given that both are Gaussian. Note that the measure $G(.||.) = 0$ when there is no information asymmetry since $\cosh(0) = 1$, and is strictly positive otherwise.

6.4 Competitive Edge and Financial Mispricing

6.4.1 Geometry and Competitive Edge in Information

We assume the same financial setting as discussed in Chapter 5, where there are two informed traders who are unaware of each others' actions. The trader who has access to more information sources has a competitive edge with respect to the other. We let X_T be Gaussian.

We assume that the filtration of Agent 1 is $\{\mathcal{G}_t\}$ as shown in (3.58), and the filtration of Agent 2 is $\{\mathcal{G}_t^*\}$ as shown in (5.40). Recall that $\psi_t^*(x) = q(x|\mathcal{G}_t^*)$. Also, $\tau_i \neq \tau_j^*$ for any i, j for $i, j = 1 \ldots, n+1$.

Proposition 6.4.1. *The dynamics of the competition between Agent 1 and Agent 2 in terms of $\{FR_t(\psi_t^*||\overline{\psi}_t\}$ on Int($\mathcal{M}$) are governed by*

$$\begin{aligned}
\mathrm{d}FR_t(\psi_t^*||\overline{\psi}_t) = &\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} \mathrm{d}\overline{D}_t(\psi_t^{(j)}, \psi_t^{(i)})\mathbf{I}_t(i)\mathbf{I}_t^*(j) \\
&+ \sum_{j=1}^{n+1}\sum_{i=2}^{n+1} \left(\overline{D}_t(\psi_t^{(j)}, \psi_t^{(i)}) - \overline{D}_t(\psi_t^{(j)}, \psi_t^{(i-1)})\right)\delta_{\tau_{i-1}}(\mathrm{d}t)\mathbf{I}_t^*(j) \\
&+ \sum_{i=1}^{n+1}\sum_{j=2}^{n+1} \left(\overline{D}_t(\psi_t^{(j)}, \psi_t^{(i)}) - \overline{D}_t(\psi_t^{(j-1)}, \psi_t^{(i)})\right)\delta_{\tau_{i-1}^*}(\mathrm{d}t)\mathbf{I}_t(i).
\end{aligned} \tag{6.59}$$

Proof. On $\mathrm{Int}(\mathcal{M})$, at each fixed time t for $0 \leq t < T$, we have

$$\mathrm{FR}_t(\psi_t^* || \overline{\psi}_t) = \sum_{i=1}^{n+1} \overline{D}_t\Big(\sum_{j=1}^{n+1} \psi_t^{(j)} \mathbf{I}_t^*(j), \psi_t^{(i)}\Big) \mathbf{I}_t(i) = \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} \overline{D}_t(\psi_t^{(j)}, \psi_t^{(i)}) \mathbf{I}_t(i) \mathbf{I}_t^*(j), \tag{6.60}$$

since $\mathbf{I}_t(i) = 1$ implies $\mathbf{I}_t(j) = 0$ and $\mathbf{I}_t^*(i) = 1$ implies $\mathbf{I}_t^*(j) = 0$ for all $i \neq j$. □

At each appearance of a new information source, a new hyperbolic triangle may be formed that quantifies the competition between the agents.

6.4.2 Geometry and Financial Mispricing

We consider the same financial setting as discussed in Chapter 5, and model financial mispricing as a special type of information asymmetry. The only difference is that we assume X_T is Gaussian.

We define the filtration $\{\mathcal{J}_t\}$ as in (5.58), the information process $\{\xi_t^*\}$ as in (5.59), and denote $\psi_t^* = q(x|\mathcal{J}_t) = q(x|\xi_t^*)$. We assume that $\{\mathcal{Z}_t\}$ given by (5.70) is the market filtration, and denote $\psi_t^a(x) = q(x|\mathcal{Z}_t)$. We refer the reader to Chapter 5.5 to recall other notations.

We define the parameters

$$\widehat{\mu}_t^* = \mu_X + \frac{\sigma_X}{\sigma_t^*}\rho_t^*(\xi_t^* - \mu_t^*), \qquad \text{and} \qquad (\widehat{\sigma}_t^*)^2 = (1 - (\rho_t^*)^2)\sigma_X^2, \tag{6.61}$$

where $\rho_t^* = \kappa\sigma_X t/\sigma_t^*$ is the correlation between X_T and ξ_t^*, also

$$\widehat{\mu}_t^b = \mu_X + \frac{\sigma_X}{\sigma_t^b}\rho_t^b(\xi_t^b - \mu_t^b), \qquad \text{and} \qquad (\widehat{\sigma}_t^b)^2 = (1 - (\rho_t^b)^2)\sigma_X^2, \tag{6.62}$$

where $\rho_t^b = \widehat{\kappa}\sigma_X t/\sigma_t^b$ is the correlation between X_T and ξ_t^b. In (6.61) and (6.62),

$$\mu_t^* = \kappa(\mu_X - c)t, \qquad \text{and} \qquad \sigma_t^* = \sqrt{(\kappa\sigma_X t)^2 + \frac{t(T-t)}{T}}, \tag{6.63}$$

are the mean and variance of ξ_t^*, and

$$\mu_t^b = \widehat{\kappa}\mu_X t, \qquad \text{and} \qquad \sigma_t^b = \sqrt{(\widehat{\kappa}\sigma_X t)^2 + \frac{t(T-t)}{T}}, \tag{6.64}$$

are the mean and variance of ξ_t^b.

Lemma 6.4.2. *The conditional densities ψ_t^* and ψ_t^a can be parametrically represented as follows:*

$$\psi_t^* \rightleftharpoons q(x, \widehat{\mu}_t^*, (\widehat{\sigma}_t^*)^2) = \frac{1}{\sqrt{2\pi}\widehat{\sigma}_t^*} \exp\left(-\frac{(x - \widehat{\mu}_t^*)^2}{2(\widehat{\sigma}_t^*)^2}\right), \tag{6.65}$$

and also,

$$
\begin{aligned}
\psi_t^a &\rightleftharpoons q(x, \widehat{\mu}_t^b, (\widehat{\sigma}_t^b)^2)\mathbf{I}_t(1) + q(x, \widehat{\mu}_t^*, (\widehat{\sigma}_t^*)^2)\mathbf{I}_t(2) \\
&= \frac{\mathbf{I}_t(1)}{\sqrt{2\pi}\widehat{\sigma}_t^b} \exp\left(-\frac{(x-\widehat{\mu}_t^b)^2}{2(\widehat{\sigma}_t^b)^2}\right) + \frac{\mathbf{I}_t(2)}{\sqrt{2\pi}\widehat{\sigma}_t^*} \exp\left(-\frac{(x-\widehat{\mu}_t^*)^2}{2(\widehat{\sigma}_t^*)^2}\right).
\end{aligned}
\tag{6.66}
$$

Proof. Note that $\psi_t^a = q(x|\xi_t^b)\mathbf{I}_t(1) + q(x|\xi_t^*)\mathbf{I}_t(2)$. Then due to the independence of τ, the proof is very similar to that of Proposition 6.3.2. □

For a fixed time $t < T$, we now define the distance:

$$
\overline{D}_t(\psi_t^b, \psi_t^*) = 2\sqrt{2}\tanh^{-1}(\zeta_t^{b,*}), \tag{6.67}
$$

where the function $\zeta_t^{b,*}$ is

$$
\zeta_t^{b,*} = \left(\frac{(\widehat{\mu}_t^* - \widehat{\mu}_t^b)^2 + 2(\widehat{\sigma}_t^* - \widehat{\sigma}_t^b)^2}{(\widehat{\mu}_t^* - \widehat{\mu}_t^b)^2 + 2(\widehat{\sigma}_t^* + \widehat{\sigma}_t^b)^2}\right)^{\frac{1}{2}}. \tag{6.68}
$$

Proposition 6.4.3. *The dynamics of the mispricing in terms of $\{FR_t(\psi_t^a||\psi_t^*)\}$ on Int($\mathcal{M}$) are governed by*

$$
\mathrm{d}FR_t(\psi_t^a||\psi_t^*) = \mathrm{d}\overline{D}_t(\psi_t^b, \psi_t^*)\mathbf{I}_t(1) - \overline{D}_t(\psi_t^b, \psi_t^*)\delta_\tau(\mathrm{d}t). \tag{6.69}
$$

Proof. The statement follows since, for a fixed time t for $0 \le t < T$, $\mathrm{FR}_t(\psi_t^a||\psi_t^*) = \overline{D}_t(\psi_t^b, \psi_t^*)\mathbf{I}_t(1)$ on Int($\mathcal{M}$). □

Note that the FR asymmetry process provides a geometric perspective on financial mispricing. When the fundamental information appears in the market, the FR mispricing process becomes zero and remains zero. The jump represents a sudden market correction at $\tau = t$, determined by the distance $2\sqrt{2}\tanh^{-1}(\zeta_t^{b,*})$.

Remark 6.4.4. *The SH and the FR asymmetries offer a geometric view on quantifying market corrections by providing geodesic distances on $\mathcal{S}^+$ and $\mathcal{P}$, respectively.*

Chapter 7

Archimedean Survival Processes

This chapter introduces a family of multivariate stochastic processes that we call Archimedean survival processes (ASPs). ASPs are constructed in such a way that they are naturally linked to Archimedean copulas.

At this point, we would like to note that an ASP is a multivariate extension of what we call a gamma random bridge (see Hoyle *et al.*, 2011 for Lévy random bridges), and hence, it can be viewed as a multivariate information process within the information-based framework. In this respect, if an ASP is assumed to generate the market filtration where each marginal process carries partial information about an asset, the law of the ASP determines the dependence structure of a vector of assets at a given time. We do not focus on the information-based application of ASPs in this chapter (we provide an information-based account in Chapter 8), and instead, we provide a detailed analysis of the stochastic properties of such processes. As an overview, an ASP is defined over a finite time horizon, and, a priori, its terminal value has an ℓ_1-norm symmetric distribution. This implies that the terminal value of an ASP has an Archimedean survival copula. Indeed, there is a bijection from the class of Archimedean copulas to the class of ASPs. The results presented in this chapter can also be found in Hoyle and Mengütürk (2012).

The use of copulas has become commonplace for dependence modelling in finance, insurance, and risk management (see, for example, Cherubini *et al.*, 2004, Freez and Valdez, 1998, and McNeil *et al.*, 2005). The Archimedean copulas, a subclass of copulas, have received particular attention in the literature for both their tractability and practical convenience. An n-dimensional Archimedean copula $C : [0,1]^n \to [0,1]$ can be written as

$$C(\mathbf{u}) = h(h^{-1}(u_1) + \cdots + h^{-1}(u_n)), \tag{7.1}$$

where h is the *generator function* of C.

Schönbucher and Schubert (2001), and Rogge and Schönbucher (2003) describe continuous-time processes that have Archimedean copulas at all times, and model default times in credit-risk applications. By construction, these processes are limited to have copulas with completely monotone generating functions. Although they bear the link to stochastic processes with Archimedean copulas, these processes are otherwise not closely related to the present work.

A random vector $\mathbf{X}$ has a multivariate Liouville distribution if

$$\mathbf{X} \overset{\text{law}}{=} R \frac{\mathbf{G}}{\sum_{i=1}^{n} G_i}, \tag{7.2}$$

where R is a non-negative random variable, $\mathbf{G}$ is a vector of n independent gamma random variables with identical scale parameters, and G_i is an element of $\mathbf{G}$ (see, for example, Fang *et al.*, 1990). In the special case where $\mathbf{G}$ is a vector of identical exponential random variables, $\mathbf{X}$ has an ℓ_1-norm symmetric distribution. McNeil and Nešlehová (2009) give an account of how Archimedean copulas coincide with survival copulas of ℓ_1-norm symmetric distributions which have no point-mass at the origin. This particular relationship relies on the characterization of n-monotone functions through an integral transform introduced by Williamson (1956), which is analogous to the Laplace transform characterisation of completely monotone functions (see, for example, Widder, 1946). McNeil and Nešlehová (2010) generalise Archimedean copulas to so-called Liouville copulas, which are defined by the survival copulas of multivariate Liouville distributions.

Norberg (1999) suggests using a randomly-scaled gamma bridge for modelling the cumulative payments made on insurance claims (also see, Brody *et al.*, 2008b). The process is an increasing process $\{\xi_{tT}\}_{0\leq t\leq T}$ constructed as

$$\xi_{tT} = X\gamma_{tT}, \tag{7.3}$$

where X is a positive random variable and $\{\gamma_{tT}\}$ is an independent gamma bridge satisfying $\gamma_{0T} = 0$ and $\gamma_{TT} = 1$ for some $T \in (0, \infty)$. Such a process is useful in modelling of cumulative gains or losses. The random variable X is the total, final gain. We can interpret X as a signal and the gamma bridge $\{\gamma_{tT}\}$ as independent multiplicative noise. Brody *et al.* (2008b) shows that $\{\xi_{tT}\}$ is a Markov process, and that

$$\mathbb{E}^{\mathbb{Q}}[X \mid \xi_{tT} = x] = \frac{\int_x^\infty z^{2-mT}(z-x)^{m(T-t)-1}\,\nu(\mathrm{d}z)}{\int_x^\infty z^{1-mT}(z-x)^{m(T-t)-1}\,\nu(\mathrm{d}z)}, \tag{7.4}$$

where ν is the law of X, and $m > 0$ is a parameter.

The process $\{\xi_{tT}\}$ as shown in (7.3) can be considered to be a gamma process conditioned to have the marginal law ν at time T, and so belongs to the class of Lévy random bridges. As such, we call a process that can be decomposed as in (7.3) a gamma random bridge (GRB). In the information-based framework, GRBs model the flow of market information about an aggregate claim determined by the terminal value of a cumulative gains process (for details, see Brody *et al.*, 2008b).

ASPs are an n-dimensional extension of GRBs. That is, each one-dimensional marginal process $\{\xi_t^{(i)}\}$ of an ASP $\{(\xi_t^{(1)}, \xi_t^{(2)}, \ldots, \xi_t^{(n)})^\top\}_{0\leq t\leq T}$ is a GRB. This is the reason why an ASP may be viewed as a multivariate information process, where each marginal process carries partial information about an aggregate claim. At this point, we should clarify the notation we shall use for this chapter and the next: $\{\xi_t^{(i)}\}$ denotes a marginal process, and the integer in the bracketed superscript is *not* used in the same sense as in the previous chapters. We can write

$$\xi_t^{(i)} = X_i \gamma_{tT}^{(i)}, \tag{7.5}$$

for some gamma bridge $\{\gamma_{tT}^{(i)}\}$ and some $X_i > 0$ independent of $\{\gamma_{tT}^{(i)}\}$. The X_i's are identically distributed but in general not independent, and the $\{\gamma_{tT}^{(i)}\}$'s are identically distributed but in general not independent.

We shall construct each $\{\xi_t^{(i)}\}$ by splitting a master GRB into n non-overlapping subprocesses. This method of splitting a Lévy random bridge into subprocesses (which are themselves Lévy random bridges) is used by Hoyle *et al.* (2010b) to develop a bivariate insurance reserving model based on random bridges of the stable-1/2 subordinator. A remarkable feature of the proposed construction is that the terminal vector $(\xi_T^{(1)}, \xi_T^{(2)}, \ldots, \xi_T^{(n)})^\top$ has an ℓ_1-norm symmetric distribution, and hence an Archimedean survival copula. In particular, we shall show that

$$\mathbb{Q}\left(\bar{F}(\xi_T^{(1)}) > u_1, \bar{F}(\xi_T^{(2)}) > u_2, \ldots, \bar{F}(\xi_T^{(n)}) > u_n\right) = \bar{F}\left(\sum_{i=1}^{n} \bar{F}^{-1}(u_i)\right), \tag{7.6}$$

where

$$\bar{F}(u) = \mathbb{Q}\left(\xi_T^{(i)} > u\right), \qquad \text{for } i = 1, 2, \ldots, n. \tag{7.7}$$

In (7.6) and (7.7), $\bar{F}(x)$ is the marginal survival function of the $\xi_T^{(i)}$'s, and $\bar{F}^{-1}(u)$ is its generalised inverse. The right-hand side of (7.6) is an Archimedean copula with generator function $\bar{F}(x)$.

A direct application of ASPs is to the modelling of multivariate cumulative gain (or loss) processes. Consider, for example, an insurance company that underwrites several lines of motor business (such as personal motor, fleet motor or private-hire vehicles) for a given accident year. A substantial payment made on one line of business is unlikely to coincide with a substantial payment made on another line of business (e.g. a large payment is unlikely to be made on a personal motor claim at the same time as a large payment is made on a fleet motor claim). However, the total sums of claims arising from the lines of business will depend on certain common factors such as prolonged periods of adverse weather or the quality of the underwriting process at the company. Such common factors will produce dependence across the lines. An ASP might be a suitable model for the cumulative paid-claims processes of the lines of motor business, if the terminal claims have an Archimedean survival copula.

ASPs can also be used to interpolate the dependence structure when using Archimedean copulas in discrete-time models. Consider a risk model where the marginal distributions of the returns on n assets are fitted for the future dates $t_1 < \cdots < t_n < T < \infty$. An Archimedean copula C is used to model the dependence of the returns to time T. At this stage we have a model for the joint distribution of returns to time T, but we have only the one-dimensional marginal distributions at the intertemporal times $t_1, \ldots, t_n$. The problem then is to choose copulas to complete the joint distributions of the returns to the times $t_1, \ldots, t_n$ in a way that is consistent with the time-T joint distribution. For each time t_i, this can be achieved by using the time-t_i survival copula implied by the ASP with survival copula C at terminal time T.

Our analysis of ASPs also motivates our next chapter, where we generalise ASPs to what we call Generalised Liouville Processes (GLPs). We do this by splitting Lévy random bridges into n pieces, where we allow more flexibility in the splitting mechanism and employ some deterministic time changes. This extension allows us to work with a much larger class of dependence structures under generalised Liouville distributions.

This chapter is organized as follows: Section 1 provides some preliminaries including ℓ_1-norm symmetric distributions, Archimedean copulas and GRBs. In Section 2, we define ASPs and provide various characterisations of their law. Finally, we construct a multivariate process such that each one-dimensional marginal is uniformly distributed.

7.1 Preliminaries

This chapter draws together ideas from the theory of stochastic processes and the theory of multivariate distributions. The preliminary section gives relevant background results from both subjects.

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and assume that all filtrations are right-continuous and complete. We let f^{-1} denote the generalised inverse of a monotonic function f, i.e.,

$$f^{-1}(y) = \begin{cases} \inf\{x : f(x) \geq y\} & f \text{ increasing}, \\ \inf\{x : f(x) \leq y\} & f \text{ decreasing}. \end{cases} \tag{7.8}$$

We denote the ℓ_1-norm of a vector $\mathbf{x} \in \mathbb{R}^n$ by $\|\mathbf{x}\|$, i.e.,

$$\|\mathbf{x}\| = \sum_{i=1}^{n} |x_i|. \tag{7.9}$$

7.1.1 Multivariate Distributions

In this subsection we present some definitions and results from the theory of multivariate distributions. We refer the reader to the thorough exposition of Fang *et al.* (1990) for further details.

Multivariate ℓ_1-norm Symmetric Distributions

The multivariate ℓ_1-norm symmetric distributions form a family of distributions that are closely related to Archimedean copulas. The n-dimensional ℓ_1-norm symmetric distribution is defined in terms of a random variable $\mathbf{U}$ which is uniformly distributed on the simplex

$$S = \{\mathbf{u} \in [0,1]^n : \|\mathbf{u}\| = 1\}. \tag{7.10}$$

Such a random variable $\mathbf{U}$ has the following representation:

$$\mathbf{U} \stackrel{\text{law}}{=} \frac{\mathbf{E}}{\|\mathbf{E}\|}, \tag{7.11}$$

where $\mathbf{E}$ is a vector of n independent, identically-distributed, exponential random variables. Note that this representation holds for any value of the rate parameter $\lambda > 0$ of the exponential random variables, and that the random variable $\|\mathbf{E}\|$ has a gamma distribution with shape parameter n, and scale parameter λ^{-1}. Each marginal variable U_i has a beta distribution with parameters $\alpha = 1$ and $\beta = n - 1$; thus the survival funciton of U_i is

$$\mathbb{Q}(U_i > u) = (1 - u)^{n-1}, \tag{7.12}$$

for $0 \leq u \leq 1$.

Definition 7.1.1. *A random variable $\mathbf{X}$ taking values in $\mathbb{R}^n$ has a* multivariate ℓ_1-norm symmetric distribution *if*

$$\mathbf{X} \stackrel{\text{law}}{=} R\mathbf{U}, \tag{7.13}$$

where R is a non-negative random variable, and $\mathbf{U}$ is a random vector uniformly distributed on the simplex S. We call the law of R the generating law*.*

Remark 7.1.2. *The construction of multivariate ℓ_1-norm symmetric random variables is similar to the construction of spherical random variables. To be precise, in* (7.13) *if $\mathbf{U}$ was uniformly distributed on the unit sphere in $\mathbb{R}^n$, then $\mathbf{X}$ would have a spherical distribution (a special case of elliptical distribution).*

Note that if R admits a density, then $\mathbf{X}$ satisfying (7.13) admits a density, and this density is simplectically contoured. This is analogous to the elliptical contours of elliptical distributions.

If $\mathbf{X}$ is a multivariate ℓ_1-norm symmetric random variable with generating law ν, then the survival function of each one-dimensional marginal of $\mathbf{X}$ is

$$\begin{aligned}\bar{F}(x) &= \mathbb{Q}(X_i > x) \\ &= \int_x^\infty (1 - x/r)^{n-1}\, \nu(\mathrm{d}r),\end{aligned} \tag{7.14}$$

for $x \geq 0$. The survival function $\bar{F}$ determines the law ν. Indeed, using the results of Williamson (1956), McNeil and Nešlehová (2009) show that

$$\nu([0,x]) = 1 - \sum_{k=0}^{n-2} \frac{(-1)^k x^k \bar{F}_0^{(k)}(x)}{k!} - \frac{(-1)^{n-1} x^{n-1} \max[0, \bar{F}_0^{(n-1)}(x)]}{(n-1)!}, \tag{7.15}$$

where $\bar{F}^{(k)}$ is the kth derivative of $\bar{F}$, and

$$\bar{F}_0(x) = \begin{cases} \bar{F}(x) & x > 0, \\ 1 - \bar{F}(0) & x = 0. \end{cases} \tag{7.16}$$

The following theorem provides the multivariate version of (7.14); the proof can be found in Fang *et al.* (1990, Theorem 5.4).

Theorem 7.1.3. *If $\mathbf{X}$ has a multivariate ℓ_1-norm symmetric distribution with generating law ν, then the joint survival function of $\mathbf{X}$ is*

$$\begin{aligned}\mathbb{Q}(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) &= \int_{\|\mathbf{x}\|}^\infty (1 - \|\mathbf{x}\|/r)^{n-1}\, \nu(\mathrm{d}r) \\ &= \bar{F}\left(\|\mathbf{x}\|\right),\end{aligned} \tag{7.17}$$

for $\mathbf{x} \in \mathbb{R}_+^n$.

Multivariate Liouville Distributions

The multivariate Liouville distribution is an extension of the multivariate ℓ_1-norm symmetric distribution. Before defining the multivariate Liouville distribution, it is convenient to first define the Dirichlet distribution. The n-dimensional Dirichlet distribution is a distribution on the simplex S defined in (7.10).

Definition 7.1.4. *Let $\mathbf{G}$ be vector of independent random variables such that G_i is a gamma random variable with shape parameter $\alpha_i > 0$ and scale parameter unity. Then the random vector*

$$\mathbf{D} = \frac{\mathbf{G}}{\|\mathbf{G}\|}, \tag{7.18}$$

has a Dirichlet distribution *with* parameter vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)^\top$.

Remark 7.1.5. *The scaling property of the gamma distribution implies that* $\kappa\mathbf{G}$*,* $\kappa > 0$*, is a vector of gamma random variables each with scale parameter* κ*. Since* (7.18) *holds, if we replace* $\mathbf{G}$ *with* $\kappa\mathbf{G}$*, we could have used an arbitrary positive scale parameter in Definition 7.1.4.*

In two dimensions, a Dirichlet random variable can be written as $(B, 1-B)^\top$, where B is a beta random variable. If all the elements of the parameter vector $\boldsymbol{\alpha}$ are identical, then $\mathbf{D}$ is said to have a symmetric Dirichlet distribution. Notice that if $\alpha_i = 1$ for $i = 1, 2, \ldots, n$, then $\mathbf{D}$ is uniformly distributed on the simplex S. The density of $(D_1, D_2, \ldots, D_{n-1})^\top$ is

$$\mathbf{x} \mapsto \frac{\prod_{i=1}^n \Gamma[\alpha_i]}{\Gamma\left[\|\boldsymbol{\alpha}\|\right]} \prod_{i=1}^n x_i^{\alpha_i - 1}, \tag{7.19}$$

for $\mathbf{x} \in [0,1]^{n-1}$, $\|\mathbf{x}\| \leq 1$, where $x_n = 1 - \sum_{i=1}^{n-1} x_i$, and $\Gamma[z]$ is the gamma function, defined as usual for $x > 0$ by

$$\Gamma[x] = \int_0^\infty u^{x-1} \mathrm{e}^{-u}\, \mathrm{d}u. \tag{7.20}$$

The first- and second-order moments of the Dirichlet distribution are given by

$$\mathbb{E}^{\mathbb{Q}}[D_i] = \frac{\alpha_i}{\|\boldsymbol{\alpha}\|}, \tag{7.21}$$

$$\mathrm{Var}^{\mathbb{Q}}[D_i] = \frac{\alpha_i(\|\boldsymbol{\alpha}\| - \alpha_i)}{\|\boldsymbol{\alpha}\|^2(\|\boldsymbol{\alpha}\| + 1)}, \tag{7.22}$$

$$\mathrm{Cov}^{\mathbb{Q}}[D_i, D_j] = -\frac{\alpha_i \alpha_j}{\|\boldsymbol{\alpha}\|^2(\|\boldsymbol{\alpha}\| + 1)}, \qquad \text{for } i \neq j. \tag{7.23}$$

The Dirichlet distribution is an extension of a random variable uniformly distributed on a simplex. The multivariate Liouville distribution is a similar extension of the multivariate ℓ_1-norm symmetric distribution:

Definition 7.1.6. *A random variable* $\mathbf{X}$ *has a* multivariate Liouville distribution *if*

$$\mathbf{X} \overset{\text{law}}{=} R\mathbf{D}, \tag{7.24}$$

for $R \geq 0$ *a random variable, and* $\mathbf{D}$ *a Dirichlet random variable with parameter vector* $\boldsymbol{\alpha}$*. We call the law of* R *the* generating law*, and* $\boldsymbol{\alpha}$ *the* parameter vector *of the distribution.*

In the case where R has a density p, the density of $\mathbf{X}$ exists and can be written as

$$\mathbf{x} \mapsto \Gamma[\|\boldsymbol{\alpha}\|] \frac{p\left(\|\mathbf{x}\|\right)}{\left(\|\mathbf{x}\|\right)^{\|\boldsymbol{\alpha}\| - 1}} \prod_{i=1}^n \frac{x_i^{\alpha_i - 1}}{\Gamma[\alpha_i]}, \tag{7.25}$$

for $\mathbf{x} \in \mathbb{R}^n$. Writing $\mu_1 = \mathbb{E}^{\mathbb{Q}}[R]$ and $\mu_2 = \mathbb{E}^{\mathbb{Q}}[R^2]$ (when these moments exist), the first- and second-order moments of $\mathbf{X}$ are given by

$$\mathbb{E}^{\mathbb{Q}}[X_i] = \mu_1 \frac{\alpha_i}{\|\boldsymbol{\alpha}\|}, \tag{7.26}$$

$$\mathrm{Var}^{\mathbb{Q}}[X_i] = \frac{\alpha_i}{\|\boldsymbol{\alpha}\|}\left(\mu_2 \frac{\alpha_i + 1}{\|\boldsymbol{\alpha}\| + 1} - \mu_1^2 \frac{\alpha_i}{\|\boldsymbol{\alpha}\|}\right), \tag{7.27}$$

$$\mathrm{Cov}^{\mathbb{Q}}[X_i, X_j] = \frac{\alpha_i \alpha_j}{\|\boldsymbol{\alpha}\|}\left(\frac{\mu_2}{\|\boldsymbol{\alpha}\| + 1} - \frac{\mu_1^2}{\|\boldsymbol{\alpha}\|}\right), \qquad \text{for } i \neq j. \tag{7.28}$$

7.1.2 Archimedean Copulas

A copula is a distribution function on the unit hypercube with the added property that each one-dimensional marginal distribution is uniform. For further details, we refer to Nelsen (2006). We define a copula as follows:

Definition 7.1.7. *An* n-copula *defined on the* n*-dimensional unit hypercube* $[0,1]^n$ *is a function* $C : [0,1]^n \to [0,1]$*, which satisfies the following:*

1. $C(\boldsymbol{u}) = 0$ *whenever* $u_j = 0$ *for at least one* $j = 1, 2, .., n$.

2. $C(\boldsymbol{u}) = u_j$ *if* $u_i = 1$ *for all* $i \neq j$.

3. C *is* n*-increasing on* $[0,1]^n$*, that is*

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C(u_{1,i_1}, \ldots, u_{n,i_n}) \geq 0, \tag{7.29}$$

for all $(u_{1,1}, u_{2,1}, \ldots, u_{n,1})^\top$ *and* $(u_{1,2}, u_{2,2}, \ldots, u_{n,2})^\top$ *in* $[0,1]^n$ *with* $u_{j,1} \leq u_{j,2}$.

Condition 3 is necessary to ensure that the function C is a well-defined distribution function. The theory of copulas is founded upon a theorem of Sklar. This theorem is reformulated in terms of survival functions by McNeil and Nešlehová (2009) as follows:

Theorem 7.1.8. *Let* $\bar{H}$ *be an* n*-dimensional survival function with margins* $\bar{F}_i$*,* $i = 1, 2, \ldots, n$*. Then there exists a copula* C*, referred to as the* survival copula *of* $\bar{H}$*, such that, for any* $\mathbf{x} \in \mathbb{R}^n$*,*

$$\bar{H}(\mathbf{x}) = C(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)). \tag{7.30}$$

Furthermore, C *is uniquely determined on*

$$D = \left\{\mathbf{u} \in [0,1]^n : u \in \mathrm{ran}\bar{F}_1 \times \cdots \times \mathrm{ran}\bar{F}_n\right\}, \tag{7.31}$$

where $\mathrm{ran} f$ *denotes the range of* f*. In addition, for any* $\mathbf{u} \in D$*,*

$$C(\mathbf{u}) = \bar{H}(\bar{F}_1^{-1}(u_1), \ldots, \bar{F}_n^{-1}(u_n)). \tag{7.32}$$

Conversely, given a copula C and univariate survival functions $\bar{F}_i$, $i = 1, \ldots, n$, $\bar{H}$ defined by (7.30) *is an n-dimensional survival function with marginals $\bar{F}_1, \ldots, \bar{F}_n$ and survival copula C.*

From a modelling perspective, one of the attractive features of copulas is that they allow the fitting of one-dimensional marginal distributions to be performed separately from the fitting of cross-sectional dependence. Although, this two-step approach of modelling multivariate phenomena by first specifying marginals, and then choosing a copula is not suited to all situations (for criticism see, for example, Mikosch, 2006).

Archimedean copulas are copulas that take a particular functional form. The following definition given in McNeil and Nešlehová (2009) is convenient for the present work:

Definition 7.1.9. *A decreasing and continuous function $h : [0, \infty) \to [0, 1]$ which satisfies the conditions $h(0) = 1$ and $\lim_{x\to\infty} h(x) = 0$, and is strictly decreasing on $[0, \inf\{x : h(x) = 0\}]$ is called an* Archimedean generator. *An n-dimensional copula C is called an* Archimedean copula *if it permits the representation*

$$C(\mathbf{u}) = h(h^{-1}(u_1) + \cdots + h^{-1}(u_n)), \qquad \mathbf{u} \in [0, 1]^n, \tag{7.33}$$

for some Archimedean generator h with inverse $h^{-1} : [0, 1] \to [0, \infty)$, where we set $h(\infty) = 0$ and $h^{-1}(0) = \inf\{u : h(u) = 0\}$.

If $\mathbf{X}$ is a random vector with a multivariate ℓ_1-norm symmetric distribution such that $\mathbb{Q}(\mathbf{X} = \mathbf{0}) = 0$, then its marginal survival function $\bar{F}$ given in (7.14) is continuous. Hence, it follows from Theorem 7.1.3 that

$$\mathbb{Q}(\bar{F}(X_1) > u_1, \bar{F}(X_2) > u_2, \ldots, \bar{F}(X_n) > u_n) = \bar{F}\left(\sum_{i=1}^{n} \bar{F}^{-1}(u_i)\right). \tag{7.34}$$

In other words, $\mathbf{X}$ has an Archimedean survival copula with generating function $h(x) = \bar{F}(x)$. McNeil and Nešlehová (2009) show that the converse is also true:

Theorem 7.1.10. *Let $\mathbf{U}$ be a random vector whose distribution function is an n-dimensional Archimedean copula C with generator h. Then $(h^{-1}(U_1), h^{-1}(U_2), \ldots, h^{-1}(U_n))^\top$ has a multivariate ℓ_1-norm symmetric distribution with survival copula C and generating law ν. Furthermore, ν is uniquely determined by*

$$\nu([0, x]) = 1 - \sum_{k=0}^{n-2} \frac{(-1)^k x^k h^{(k)}(x)}{k!} - \frac{(-1)^{n-1} x^{n-1} \max[0, h^{(n-1)}(x)]}{(n-1)!}. \tag{7.35}$$

Remark 7.1.11. *There is one-to-one mapping from distribution functions on the positive half-line to the class of n-dimensional Archimedean copulas through the invertible transformation $\nu \leftrightarrow h$.*

7.1.3 Gamma Random Bridges

A gamma random bridge is an increasing stochastic process, and both the gamma process and gamma bridge are special cases.

Gamma Process

A gamma process is an increasing Lévy process (see, for example, Sato, 1999) with gamma distributed increments. Let $\{\gamma_t\}$ denote a gamma process with mean and variance $m > 0$ at $t = 1$. The law of $\{\gamma_t\}$ is determined by its mean and variance at $t = 1$, and the density of γ_t is

$$f_t(x) = \mathbf{1}_{\{x>0\}} \frac{x^{mt-1}}{\Gamma[mt]} \mathrm{e}^{-x}, \tag{7.36}$$

where $\mathbf{1}_{\{.\}}$ is the indicator function (or the Heaviside function). For notational convenience, we shall use $\mathbf{1}_{\{.\}}$ in this chapter instead of $H_{.}(.)$ that we used in previous chapters. The mean and variance of a gamma process are

$$\mathbb{E}^{\mathbb{Q}}[\gamma_t] = mt, \quad \text{and} \quad \mathrm{Var}^{\mathbb{Q}}[\gamma_t] = mt. \tag{7.37}$$

The gamma distribution has scaling property. Therefore, for some $\kappa > 0$, the process $\{\kappa\gamma_t\}$ is also a gamma process, but with mean $m\kappa$, and variance $m\kappa^2$, at $t = 1$. The characteristic function of γ_t is

$$\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{\mathrm{i}\lambda\gamma_t}] = (1 - \mathrm{i}\lambda)^{-mt}. \tag{7.38}$$

As noted in Brody *et al.* (2008b), the parameter m has units of inverse time, and so $\{\gamma_t\}$ is dimensionless. Taking $\kappa = 1/m$, the scaled process $\{\kappa\gamma_t\}$ has units of time, making this alternative parameterisation suitable as a basis for a stochastic time change (see, for example, Madan and Seneta, 1990). The characteristic function of $\kappa\gamma_t$ is then

$$\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{\mathrm{i}\lambda\kappa\gamma_t}] = (1 - \mathrm{i}\lambda/m)^{-mt}. \tag{7.39}$$

It can be shown that $\kappa\gamma_t \stackrel{\text{law}}{=} t$ in the limit $m \to \infty$, since the characteristic function of $\kappa\gamma_t$ coincides with the characteristic function of the Dirac measure centered at t (which is $\mathrm{e}^{\mathrm{i}\lambda t}$) in the limit $m \to \infty$.

Gamma Bridge

A gamma bridge is a gamma process conditioned to have a fixed value at a fixed future time. A gamma bridge is a Lévy bridge, and hence a Markov process (see, for example, Hoyle, 2010a). Emery and Yor (2004) present some remarkable similarities between gamma bridges and Brownian bridges. Let $\{\gamma_{tT}\}_{0\le t\le T}$ denote a gamma bridge identical in law to the gamma process $\{\gamma_t\}$ pinned to the value 1 at time T. Using the Bayes formula,

$$\begin{aligned}
\mathbb{Q}\left(\gamma_{tT} \in \mathrm{d}y \,|\, \gamma_{sT} = x\right) &= \mathbb{Q}\left(\gamma_t \in \mathrm{d}y \,|\, \gamma_s = x, \gamma_T = 1\right) \\
&= \frac{f_{t-s}(y-x) f_{T-t}(1-y)}{f_{T-s}(1-x)} \\
&= \mathbf{1}_{\{x<y<1\}} \frac{\left(\frac{y-x}{1-x}\right)^{m(t-s)-1} \left(\frac{1-y}{1-x}\right)^{m(T-t)-1}}{(1-x)\mathrm{B}[m(t-s), m(T-t)]} \, \mathrm{d}y, \qquad (7.40)
\end{aligned}$$

for $0 \leq s < t \leq T$ and $x \geq 0$. We say that m is the *activity parameter* of $\{\gamma_{tT}\}$. In (7.40), $\mathrm{B}[\alpha, \beta]$ is the beta function, given by

$$\begin{aligned}
\mathrm{B}[\alpha, \beta] &= \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \, \mathrm{d}x \\
&= \frac{\Gamma[\alpha]\Gamma[\beta]}{\Gamma[\alpha+\beta]}, \qquad (7.41)
\end{aligned}$$

for $\alpha, \beta > 0$. If $\gamma_{sT} = x$, then the gamma bridge will complete a distance of $1 - x$ during $(s, T]$, where the proportion of this distance over $(s, t]$ has a beta distribution with parameters $\alpha = m(t-s)$ and $\beta = m(T-t)$, which can be seen from (7.40).

The characteristic function of γ_{tT} given γ_{sT} is

$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}\lambda\gamma_{tT}} \,\middle|\, \gamma_{sT} = x\right] = M[m(t-s), m(T-s), \mathrm{i}(1-x)\lambda], \qquad (7.42)$$

where $M[\alpha, \beta, z]$ is Kummer's confluent hypergeometric function of the first kind (see Hoyle, 2010a, and Abramowitz and Stegun, 1964):

$$M[\alpha, \beta, z] = 1 + \frac{\alpha}{\beta} z + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} \frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} \frac{z^3}{3!} + \cdots . \qquad (7.43)$$

In the limit $m \to \infty$, $\gamma_{tT} \overset{\text{law}}{=} t/T$. This follows from the Markovian property of $\{\gamma_{tT}\}$ and also since, in the limit $m \to \infty$ in (7.42), the characteristic function of γ_{tT} given γ_{sT} is

$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{\mathrm{i}\lambda\gamma_{tT}} \,\middle|\, \gamma_{sT} = x\right] = \sum_{k=0}^{\infty} \left(\frac{t-s}{T-s}\right)^k \frac{(\mathrm{i}(1-x)\lambda)^k}{k!} = \exp\left(\mathrm{i}\frac{t-s}{T-s}(1-x)\lambda\right). \qquad (7.44)$$

This coincides with the characteristic function of the Dirac measure centered at $(1-x)(t-s)/(T-s)$. In other words, since $\gamma_{0T} = 0$, the characteristic function of γ_{tT} coincides with the Dirac measure centered at t/T in the limit $m \to \infty$.

It can be shown that the process $\{\gamma_t/\gamma_T\}_{0 \leq t \leq T}$ is independent of γ_T, and that the following holds:

$$\{\gamma_{tT}\} \overset{\text{law}}{=} \left\{\frac{\gamma_t}{\gamma_T}\right\}. \qquad (7.45)$$

The process $\{\gamma_t/\gamma_T\}_{0\leq t\leq T}$ is a Markov process with transition law as shown in (7.40).

Equation (7.45) implies that the joint distribution of increments of a gamma bridge is Dirichlet. To see this fact, fix times $0 = t_0 < t_1 < \cdots < t_n = T$ and define

$$\bar{\Delta}_i = \gamma_{t_i} - \gamma_{t_{i-1}}, \tag{7.46}$$

$$\Delta_i = \gamma_{t_i,T} - \gamma_{t_{i-1},T}. \tag{7.47}$$

Then $\bar{\Delta}_i$ has a gamma distribution with shape parameter $\alpha_i = m(t_i - t_{i-1})$ and scale parameter unity. Hence

$$(\Delta_1, \Delta_2, \ldots, \Delta_n) \stackrel{\text{law}}{=} \frac{(\bar{\Delta}_1, \bar{\Delta}_2, \ldots, \bar{\Delta}_n)}{\|(\bar{\Delta}_1, \bar{\Delta}_2, \ldots, \bar{\Delta}_n)\|}. \tag{7.48}$$

From Definition 7.1.4, the joint distribution of increments of a gamma bridge is Dirichlet. Equation (7.45) also implies that the bridge of the gamma process $\{\kappa\gamma_t\}$ for some $\kappa > 0$, is equal in law to the bridge of $\{\gamma_t\}$. Also, the bridge of $\{\gamma_t\}$ to some value $a > 0$ at time T, is equal in law to $\{a\gamma_{tT}\}$.

Gamma Random Bridge

A gamma random bridge (GRB) is identical in law to a gamma process conditioned to have a fixed marginal law at some finite future time. Brody *et al.* (2008b) use a GRB to model an information process that generates the market filtration and that provides noisy information about a future cumulative claim.

We define a gamma random bridge as follows:

Definition 7.1.12. *The process* $\{\Gamma_t\}_{0\leq t\leq T}$ *is a* gamma random bridge *if*

$$\{\Gamma_t\} \stackrel{\text{law}}{=} \{R\gamma_{tT}\}, \tag{7.49}$$

for $R > 0$ *a random variable, and* $\{\gamma_{tT}\}$ *a gamma bridge, independent of* R*. We say that* $\{\Gamma_t\}$ *has* generating law ν *and* activity parameter m*, where* ν *is the law of* R *and* m *is the activity parameter of* $\{\gamma_{tT}\}$*.*

Remark 7.1.13. *Suppose that* $\{\Gamma_t\}$ *is a GRB satisfying* (7.49)*. If* $\mathbb{Q}(R = z) = 1$ *for some* $z > 0$*, then* $\{\Gamma_t\}$ *is a gamma bridge. If* R *is gamma random variable with shape parameter* mT *and scale parameter* κ*, then* $\{\Gamma_t\}$ *is a gamma process such that* $\mathbb{E}^{\mathbb{Q}}[\Gamma_t] = m\kappa t$ *and* $\mathrm{Var}^{\mathbb{Q}}[\Gamma_t] = m\kappa^2 t$*, for* $t < T$*.*

Gamma random bridges (GRBs) fall within the class of Lévy random bridges described by Hoyle *et al.* (2011). The process $\{\Gamma_t\}$ is identical in law to a gamma process defined over $[0, T]$, and conditioned to have the law of R at time T. The bridges of a GRB are gamma bridges. GRBs are Markov processes with stationary increments, and the transition law of $\{\Gamma_t\}$ is given by (see Hoyle *et al.*, 2011)

$$\begin{aligned}
&\mathbb{Q}(\Gamma_t \in \mathrm{d}y \,|\, \Gamma_s = x) \\
&\quad = \frac{\mathbf{1}_{\{y>x\}}}{\mathrm{B}[m(T-t), m(t-s)]} \frac{\int_y^\infty (z-y)^{m(T-t)-1} z^{1-mT}\, \nu(\mathrm{d}z)}{\int_x^\infty (z-x)^{m(T-s)-1} z^{1-mT}\, \nu(\mathrm{d}z)} (y-x)^{m(t-s)-1}\, \mathrm{d}y, \quad (7.50)
\end{aligned}$$

and

$$\mathbb{Q}(\Gamma_T \in \mathrm{d}y \,|\, \Gamma_s = x) = \frac{\mathbf{1}_{\{y>x\}} (y-x)^{m(T-s)-1} y^{1-mT}\, \nu(\mathrm{d}y)}{\int_x^\infty (z-x)^{m(T-s)-1} z^{1-mT}\, \nu(\mathrm{d}z)}, \tag{7.51}$$

where $\mathrm{B}[\alpha, \beta]$ is the Beta function.

Since increments of a gamma bridge have a Dirichlet distribution, it follows from Definition 7.1.6 that the increments of a gamma random bridge have multivariate Liouville distributions.

The following proposition, stated as a corollary in Hoyle *et al.* (2011) for a general Lévy random bridge, is a key result for the construction of ASPs:

Proposition 7.1.14. *Let $\{\Gamma_t\}$ be a GRB with generating law ν and activity parameter m.*

(**A**) *Fix times s_1, T_1 satisfying $0 < T_1 \leq T - s_1$. The time-shifted, space-shifted partial process*

$$\xi_t^{(1)} = \Gamma_{s_1+t} - \Gamma_{s_1}, \qquad (0 \leq t \leq T_1), \tag{7.52}$$

is a gamma random bridge with activity parameter m, and with generating law

$$\nu^{(1)}(\mathrm{d}x) = \frac{x^{mT_1-1}}{\mathrm{B}[mT_1, m(T-T_1)]} \int_{z=x}^{\infty} z^{mT-1} (z-x)^{m(T-T_1)-1} \nu(\mathrm{d}z)\, \mathrm{d}x. \tag{7.53}$$

(**B**) *Construct partial processes $\{\xi_t^{(i)}\}_{0 \leq t \leq T_i}$, $i = 1, \ldots, n$, from non-overlapping portions of $\{\Gamma_t\}$, in a similar way to that above. The intervals $[s_i, s_i + T_i]$, $i = 1, \ldots, n$, are non-overlapping except possibly at the endpoints. Set $\xi_t^{(i)} = \xi_{T_i}^{(i)}$ when $t > T_i$.*

If $u > t$,

$$\begin{aligned}
&\mathbb{Q}\left(\xi_u^{(1)} - \xi_t^{(1)} \leq x_1, \ldots, \xi_u^{(n)} - \xi_t^{(n)} \leq x_n \,\middle|\, \mathcal{F}_t^{\xi} \right) = \\
&\qquad \mathbb{Q}\left(\xi_u^{(1)} - \xi_t^{(1)} \leq x_1, \ldots, \xi_u^{(n)} - \xi_t^{(n)} \leq x_n \,\middle|\, \sum_{i=1}^{n} \xi_t^{(i)} \right), \quad (7.54)
\end{aligned}$$

where the filtration $\{\mathcal{F}_t^{\xi}\}$ is is given by

$$\mathcal{F}_t^{\xi} = \sigma\left(\{\xi_s^{(i)}\}_{0 \leq s \leq t}, i = 1, 2, \ldots, n \right). \tag{7.55}$$

Remark 7.1.15. *Define the process* $\{R_t\}$ *by*

$$R_t = \sum_{i=1}^{n} \xi_t^{(i)}, \tag{7.56}$$

for $t \in [0, \max_i T_i]$*. Then* $\{R_t\}$ *is a GRB with generating law* ν*, and time-dependent activity parameter*

$$M(t) = m \sum_{i=1}^{n} \mathbf{1}_{\{t \leq T_i\}}. \tag{7.57}$$

The proof of this result is similar to the proof that appears later in Proposition 7.2.6.

We can construct an n-dimensional Markov process $\{\boldsymbol{\xi}_t\}$ from the partial processes of Proposition 7.1.14, part (B), by setting

$$\boldsymbol{\xi}_t = (\xi_t^{(1)}, \dots, \xi_t^{(n)})^\top. \tag{7.58}$$

The Markov property means that, for any fixed time $s \geq 0$, the $\mathcal{F}_s^{\xi}$-conditional law of $\{\boldsymbol{\xi}_t\}_{s\leq t}$ is identical to the $\boldsymbol{\xi}_s$-conditional law of $\{\boldsymbol{\xi}_t\}_{s\leq t}$. The remarkable feature of Proposition 7.1.14 part (B), together with Remark 7.1.15, is that the $\mathcal{F}_s^{\xi}$-conditional law of $\{\boldsymbol{\xi}_t - \boldsymbol{\xi}_s\}_{s\leq t}$ is identical to the R_s-conditional law of $\{\boldsymbol{\xi}_t - \boldsymbol{\xi}_s\}_{s\leq t}$. Hence the increment probabilities of the n-dimensional process $\{\boldsymbol{\xi}_t\}$ can be described by the one-dimensional state process $\{R_t\}$. In financial modelling, working with R_t is quite convenient when one works with total claims.

7.2 Archimedean Survival Processes

We construct an Archimedean survival process (ASP) by splitting a gamma random bridge into n non-overlapping subprocesses. We start with a master GRB $\{\Gamma_t\}_{0\leq t\leq n}$ with activity parameter $m = 1$ and generating law ν, where $n \in \mathbb{N}_+$, $n \geq 2$. In this section, we write f_t for the gamma density with shape parameter unity and scale parameter unity (in (7.36), we set $m = 1$). That is,

$$f_t(x) = \frac{x^{t-1}\mathrm{e}^{-x}}{\Gamma[t]}. \tag{7.59}$$

Definition 7.2.1. *The process* $\{\boldsymbol{\xi}_t\}_{0\leq t\leq 1}$ *is an* n-dimensional Archimedean survival process *if*

$$\{\boldsymbol{\xi}_t\}_{0\leq t\leq 1} = \left\{ \begin{bmatrix} \xi_t^{(1)} \\ \vdots \\ \xi_t^{(i)} \\ \vdots \\ \xi_t^{(n)} \end{bmatrix} \right\}_{0\leq t\leq 1} \overset{\text{law}}{=} \left\{ \begin{bmatrix} \Gamma_t - \Gamma_0 \\ \vdots \\ \Gamma_{(i-1)+t} - \Gamma_{i-1} \\ \vdots \\ \Gamma_{(n-1)+t} - \Gamma_{n-1} \end{bmatrix} \right\}_{0\leq t\leq 1}, \tag{7.60}$$

where $\{\Gamma_t\}_{0\le t\le n}$ *is a gamma random bridge with activity parameter* $m=1$*. We say that the generating law of* $\{\Gamma_t\}$ *is the* generating law *of* $\{\boldsymbol{\xi}_t\}$*.*

Note that, from Definition 7.1.12, $\mathbb{Q}(\Gamma_n=0)=0$, and so $\mathbb{Q}(\boldsymbol{\xi}_1=\mathbf{0})=0$. Each one-dimensional marginal process of an ASP is a subprocess of a GRB, and hence a GRB. Then, ASPs are a multivariate generalisation of GRBs.

We defined ASPs over the time interval $[0,1]$; it is straightforward to restate the definition to cover an arbitrary closed interval.

Proposition 7.2.2. *The terminal value of an ASP has an Archimedean survival copula.*

Proof. Let $\{\boldsymbol{\xi}_t\}$ be an n-dimensional ASP with generating law ν. Then we have

$$\begin{aligned}\mathbb{Q}(\boldsymbol{\xi}_1\in \mathrm{d}\mathbf{x}) &= \mathbb{Q}\left(\Gamma_1\in \mathrm{d}x_1, \Gamma_2-\Gamma_1\in \mathrm{d}x_2,\dots,\Gamma_n-\Gamma_{n-1}\in \mathrm{d}x_n\right)\\ &= \mathbb{Q}\left(R\frac{\gamma_1}{\gamma_n}\in \mathrm{d}x_1, R\frac{\gamma_2-\gamma_1}{\gamma_n}\in \mathrm{d}x_2,\dots,R\frac{\gamma_n-\gamma_{n-1}}{\gamma_n}\in \mathrm{d}x_n\right),\end{aligned} \tag{7.61}$$

for $\mathbf{x}\in\mathbb{R}^n$, R a random variable with law ν, and $\{\gamma_t\}$ a gamma process such that γ_t has the density (7.59). Each increment $\gamma_i-\gamma_{i-1}$ has an exponential distribution (with unit rate). Thus,

$$\mathbb{Q}(\boldsymbol{\xi}_1\in \mathrm{d}\mathbf{x}) = \mathbb{Q}\left(R\frac{\mathbf{E}}{\|\mathbf{E}\|}\in \mathrm{d}\mathbf{x}\right), \tag{7.62}$$

for $\mathbf{E}$ an n-vector of independent, identically-distributed, exponential random variables. From Definition 7.1.1, $\boldsymbol{\xi}_1$ has a multivariate ℓ_1-norm symmetric distribution. Therefore, it has an Archimedean survival copula. □

Remark 7.2.3. *Let* $g_i:\mathbb{R}_+\to\mathbb{R}$ *be strictly decreasing for* $i=1,\dots,n$*, and let* $\{\boldsymbol{\xi}_t\}$ *be an ASP. Then, a priori, the vector-valued process*

$$\left\{\left(g_1(\xi_t^{(1)}),\dots,g_i(\xi_t^{(i)}),\dots,g_n(\xi_t^{(n)})\right)^\top\right\}_{0\le t\le 1}, \tag{7.63}$$

has an Archimedean copula at time $t=1$*.*

Figure 7.1 at the end of this chapter is a simulation of a 10-dimensional ASP, and Figure 7.2 is a simulation of a 20-dimensional ASP. The time horizon is $[0,1]$, and we fix $R=1$. In these simulations, each different colour represents a marginal process of the ASP, where each marginal process is a GRB.

7.2.1 Characterisations

In this subsection we shall characterize ASPs first through their finite-dimensional distributions, and then through their transition probabilities.

Finite-Dimensional Distributions

The finite-dimensional distributions of the master process $\{\Gamma_t\}$ are given by

$$\mathbb{Q}(\Gamma_{t_1} \in \mathrm{d}x_1, \dots, \Gamma_{t_k} \in \mathrm{d}x_k, \Gamma_n \in \mathrm{d}z) = \mathbb{Q}(\Gamma_{t_1} \in \mathrm{d}x_1, \dots, \Gamma_{t_k} \in \mathrm{d}x_k \,|\, \Gamma_n = z)\, \nu(\mathrm{d}z), \quad (7.64)$$

where $x_0 = 0$, for all $k \in \mathbb{N}_+$, all partitions $0 = t_0 < t_1 < \cdots < t_k < n$, all $z \in \mathbb{R}_+$, and all $(x_1, \dots, x_k)^\top = \mathbf{x} \in \mathbb{R}_+^k$. It was mentioned earlier that the bridges of a GRB are gamma bridges. (In fact, this is the basis of the definition of Lévy random bridges given in Hoyle *et al.*, 2011). Hence, for $\{\gamma_t\}$ a gamma process such that $\mathbb{E}^{\mathbb{Q}}[\gamma_1] = 1$ and $\mathrm{Var}^{\mathbb{Q}}[\gamma_1] = 1$, we have

$$\begin{aligned}\mathbb{Q}(\Gamma_{t_1} &\in \mathrm{d}x_1, \dots, \Gamma_{t_k} \in \mathrm{d}x_k, \Gamma_n \in \mathrm{d}z) \\ &= \mathbb{Q}(\gamma_{t_1} \in \mathrm{d}x_1, \dots, \gamma_{t_k} \in \mathrm{d}x_k \,|\, \gamma_n = z)\, \nu(\mathrm{d}z). \end{aligned} \quad (7.65)$$

From (7.45) and (7.49), we have

$$(\Gamma_{t_1} - \Gamma_{t_0}, \dots, \Gamma_{t_k} - \Gamma_{t_{k-1}}, \Gamma_n - \Gamma_{t_k}) \overset{\text{law}}{=} \frac{R}{\gamma_n}(\gamma_{t_1} - \gamma_{t_0}, \dots, \gamma_{t_k} - \gamma_{t_{k-1}}, \gamma_n - \gamma_{t_k}). \quad (7.66)$$

Hence, from Definition 7.1.6, $(\Gamma_{t_1} - \Gamma_{t_0}, \dots, \Gamma_{t_k} - \Gamma_{t_{k-1}}, \Gamma_n - \Gamma_{t_k})^\top$ has a multivariate Liouville distribution with generating law ν and parameter vector $(t_1 - t_0, \dots, t_k - t_{k-1}, n - t_k)^\top$.

We can use these results to characterise the law of the ASP $\{\boldsymbol{\xi}_t\}$ through the joint distribution of its increments. Fix $k_i \geq 1$ and the partitions

$$0 = t_0^i < t_1^i < \cdots < t_{k_i}^i = 1, \quad (7.67)$$

for $i = 1, \dots, n$. Then define the non-overlapping increments $\{\Delta_{ij}\}$ by

$$\Delta_{ij} = \xi^{(i)}_{t^i_j} - \xi^{(i)}_{t^i_{j-1}}, \quad (7.68)$$

for $j = 1, \dots, k_i$ and $i = 1, \dots, n$. The distribution of the vector

$$\begin{aligned}\boldsymbol{\Delta} = (&\Delta_{11}, \Delta_{12}, \dots, \Delta_{1k_1}, \\ &\Delta_{21}, \Delta_{22}, \dots, \Delta_{2k_2}, \\ &\vdots \\ &\Delta_{n1}, \Delta_{n2}, \dots, \Delta_{nk_n})^\top \end{aligned} \quad (7.69)$$

characterises the finite-dimensional distributions of the ASP $\{\boldsymbol{\xi}_t\}$. Thus it follows from the Kolmogorov extension theorem that the distribution of $\boldsymbol{\Delta}$ characterises the law of $\{\boldsymbol{\xi}_t\}$. Note that $\boldsymbol{\Delta}$ contains non-overlapping increments of the master GRB $\{\Gamma_t\}$ such that $\|\boldsymbol{\Delta}\| = \Gamma_n$.

Hence $\mathbf{\Delta}$ has a multivariate Liouville distribution with parameter vector

$$\begin{aligned}\boldsymbol{\alpha} = (&t_1^1 - t_0^1, t_2^1 - t_1^1, \ldots, t_{k_1}^1 - t_{k_1-1}^1,\\ &t_1^2 - t_0^2, t_2^2 - t_1^2, \ldots, t_{k_2}^2 - t_{k_2-1}^2,\\ &\vdots\\ &t_1^n - t_0^n, t_2^n - t_1^n, \ldots, t_{k_n}^n - t_{k_n-1}^n)^\top,\end{aligned} \tag{7.70}$$

and the generating law ν.

Transition Law

We denote the filtration generated by $\{\boldsymbol{\xi}_t\}_{0\leq t\leq 1}$ by $\{\mathcal{F}_t^{\boldsymbol{\xi}}\}$. From Proposition 7.1.14, $\{\boldsymbol{\xi}_t\}$ is a Markov process with respect to $\{\mathcal{F}_t^{\boldsymbol{\xi}}\}$. We shall calculate the transition probabilities of $\{\boldsymbol{\xi}_t\}$ after introducing some further notation.

For a set $B \in \mathcal{B}(\mathbb{R})$ and a constant $x \in \mathbb{R}$, we write $B + x$ for the shifted set such that

$$B + x = \{y \in \mathbb{R} : y - x \in B\}. \tag{7.71}$$

In what follows, we assume that $\{\boldsymbol{\xi}_t\}$ is an n-dimensional ASP with generating law ν, and that $\{\Gamma_t\}$ is a master process of $\{\boldsymbol{\xi}_t\}$. We define the process $\{R_t\}_{0\leq t\leq 1}$ by setting

$$R_t = \sum_{i=1}^{n} \xi_t^{(i)} = \|\boldsymbol{\xi}_t\|. \tag{7.72}$$

The terminal value of $\{R_t\}$ is the terminal value of the master process $\{\Gamma_t\}$, i.e., $R_1 = \Gamma_n$. We define a family of unnormalised measures, indexed by $t \in [0, 1)$ and $x \in \mathbb{R}_+$, as follows:

$$\theta_0(B; x) = \nu(B), \tag{7.73}$$

$$\begin{aligned}\theta_t(B; x) &= \int_B \frac{f_{n(1-t)}(z - x)}{f_n(z)}\,\nu(\mathrm{d}z)\\ &= \frac{\Gamma[n]\mathrm{e}^x}{\Gamma[n(1-t)]}\int_B \mathbf{1}_{\{z>x\}} z^{1-n}(z - x)^{n(1-t)-1}\,\nu(\mathrm{d}z),\end{aligned} \tag{7.74}$$

for $B \in \mathcal{B}(\mathbb{R})$. We also write $\Psi_t(x) = \theta_t([0, \infty); x)$. It follows from (7.65) and the independent increments of gamma processes that

$$\begin{aligned}\mathbb{Q}(\Gamma_{t_1} \in \mathrm{d}x_1, \ldots, \Gamma_{t_k} \in \mathrm{d}x_k, \Gamma_n \in \mathrm{d}z) &= \prod_{i=1}^{k}[f_{t_i - t_{i-1}}(x_i - x_{i-1})\,\mathrm{d}x_i]\frac{f_{n-t_k}(z - x_k)}{f_n(z)}\nu(\mathrm{d}z)\\ &= \prod_{i=1}^{k}[f_{t_i - t_{i-1}}(x_i - x_{i-1})\,\mathrm{d}x_i]\theta_{t_k/n}(\mathrm{d}z; x_k).\end{aligned} \tag{7.75}$$

Proposition 7.2.4. *The ASP $\{\boldsymbol{\xi}_t\}$ is a Markov process with the transition law given by*

$$\mathbb{Q}\left(\xi_1^{(1)} \in \mathrm{d}z_1, \ldots, \xi_1^{(n-1)} \in \mathrm{d}z_{n-1}, \xi_1^{(n)} \in B \,\middle|\, \boldsymbol{\xi}_s = \mathbf{x}\right) = \frac{\theta_{\tau(s)}(B + \sum_{i=1}^{n-1} z_i; x_n + \sum_{i=1}^{n-1} z_i)}{\Psi_s(\|\mathbf{x}\|)} \prod_{i=1}^{n-1} \frac{(z_i - x_i)^{-s} \mathrm{e}^{-(z_i - x_i)}}{\Gamma[1-s]} \mathrm{d}z_i, \quad (7.76)$$

and

$$\mathbb{Q}\left(\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{y} \,|\, \boldsymbol{\xi}_s = \mathbf{x}\right) = \frac{\Psi_t(\|\mathbf{y}\|)}{\Psi_s(\|\mathbf{x}\|)} \prod_{i=1}^{n} \frac{(y_i - x_i)^{(t-s)-1} \mathrm{e}^{-(y_i - x_i)}}{\Gamma[t-s]} \mathrm{d}y_i, \quad (7.77)$$

where $\tau(t) = 1 - (1-t)/n$, $0 \le s < t < 1$, and $B \in \mathcal{B}(\mathbb{R})$.

Proof. We begin by verifying (7.76). From the Bayes formula we have

$$\mathbb{Q}\left(\xi_1^{(1)} \in \mathrm{d}z_1, \ldots, \xi_1^{(n-1)} \in \mathrm{d}z_{n-1}, \xi_1^{(n)} \in B \,\middle|\, \boldsymbol{\xi}_s = \mathbf{x}\right) = \frac{\mathbb{Q}\left(\xi_1^{(1)} \in \mathrm{d}z_1, \ldots, \xi_1^{(n-1)} \in \mathrm{d}z_{n-1}, \|\boldsymbol{\xi}_1\| \in B + \sum_{i=1}^{n-1} z_i, \boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x}\right)}{\mathbb{Q}\left(\boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x}\right)}. \quad (7.78)$$

The a priori law of $R_1 = \|\boldsymbol{\xi}_1\|$ is ν; hence using (7.75) the numerator of (7.78) is

$$\int_{u \in B + \sum_{i=1}^{n-1} z_i} \mathbb{Q}\left(\xi_1^{(1)} \in \mathrm{d}z_1, \ldots, \xi_1^{(n-1)} \in \mathrm{d}z_{n-1}, \boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x} \,\middle|\, R_1 = u\right) \nu(\mathrm{d}u) = \prod_{i=1}^{n} [f_s(x_i)\,\mathrm{d}x_i] \prod_{i=1}^{n-1} [f_{1-s}(z_i - x_i)\,\mathrm{d}z_i] \int_{u \in B + \sum_{i=1}^{n-1} z_i} \frac{f_{1-s}(u - \sum_{i=1}^{n-1} z_i - x_n)}{f_n(u)} \nu(\mathrm{d}u), \quad (7.79)$$

and the denominator is

$$\begin{aligned} \mathbb{Q}(\boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x}) &= \mathbb{Q}(\Gamma_s \in \mathrm{d}x_1, \Gamma_{1+s} - \Gamma_1 \in \mathrm{d}x_2, \ldots, \Gamma_{n-1+s} - \Gamma_{n-1} \in \mathrm{d}x_n) \\ &= \mathbb{Q}(\Gamma_s \in \mathrm{d}x_1, \Gamma_{2s} - \Gamma_s \in \mathrm{d}x_2, \ldots, \Gamma_{ns} - \Gamma_{(n-1)s} \in \mathrm{d}x_n) \quad (7.80) \\ &= \prod_{i=1}^{n} [f_s(x_i)\,\mathrm{d}x_i] \int_{u=0}^{\infty} \frac{f_{n(1-s)}(u - \|\mathbf{x}\|)}{f_n(u)} \nu(\mathrm{d}u). \quad (7.81) \end{aligned}$$

In (7.79) we have used the fact that, given $\|\boldsymbol{\xi}_1\| = R_1$, $\{\boldsymbol{\xi}_t\}$ is a vector of subprocesses of a gamma bridge. Equation (7.80) follows from the stationary increments property of GRBs and (7.81) follows from (7.75). Dividing (7.79) by (7.81) yields

$$\frac{\int_{u \in B + \sum_{i=1}^{n-1} z_i} \frac{1}{f_n(u)} f_{1-s}(u - \sum_{i=1}^{n-1} z_i - x_n)\,\nu(\mathrm{d}u)}{\int_{u=0}^{\infty} \frac{1}{f_n(u)} f_{n(1-s)}(u - \|\mathbf{x}\|)\,\nu(\mathrm{d}u)} \prod_{i=1}^{n-1} [f_{1-s}(z_i - x_i)\,\mathrm{d}z_i] = \frac{\theta_{\tau(s)}(B + \sum_{i=1}^{n-1} z_i; x_n + \sum_{i=1}^{n-1} z_i)}{\theta_s([0,\infty); \|\mathbf{x}\|)} \prod_{i=1}^{n-1} \frac{(z_i - x_i)^{-s} \mathrm{e}^{-(z_i - x_i)}}{\Gamma[1-s]} \mathrm{d}z_i, \quad (7.82)$$

as required. We shall now verify (7.77) following similar steps. From the Bayes formula we have

$$\mathbb{Q}(\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{y} \,|\, \boldsymbol{\xi}_s = \mathbf{x}) = \frac{\mathbb{Q}(\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{y}, \boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x})}{\mathbb{Q}(\boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x})}. \tag{7.83}$$

The numerator of (7.83) is

$$\int_{z=0}^{\infty} \mathbb{Q}\left(\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{y}, \boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x} \,|\, R_1 = z\right) \nu(\mathrm{d}z) = \prod_{i=1}^{n}[f_s(x_i)\,\mathrm{d}x_i] \prod_{i=1}^{n}[f_{t-s}(y_i - x_i)\,\mathrm{d}y_i] \int_{z=0}^{\infty} \frac{f_{n(1-t)}(z - \|\mathbf{y}\|)}{f_n(z)} \,\nu(\mathrm{d}z), \tag{7.84}$$

and the denominator is given in (7.81). Dividing (7.84) by (7.81) yields

$$\frac{\int_{z=0}^{\infty} \frac{1}{f_n(z)} f_{n(1-t)}(z - \|\mathbf{y}\|)\,\nu(\mathrm{d}z)}{\int_{z=0}^{\infty} \frac{1}{f_n(z)} f_{n(1-s)}(z - \|\mathbf{x}\|)\,\nu(\mathrm{d}z)} \prod_{i=1}^{n}[f_{t-s}(y_i - x_i)\,\mathrm{d}y_i] = \frac{\theta_t([0,\infty); \|\mathbf{y}\|)}{\theta_s([0,\infty); \|\mathbf{x}\|)} \prod_{i=1}^{n} \frac{(y_i - x_i)^{(t-s)-1}\mathrm{e}^{-(y_i - x_i)}}{\Gamma[t-s]}\,\mathrm{d}y_i, \tag{7.85}$$

which completes the proof. □

Remark 7.2.5. *When the generating law ν admits a density p,* (7.78) *is equivalent to the following:*

$$\mathbb{Q}\left(\boldsymbol{\xi}_1 \in \mathrm{d}\mathbf{z} \,|\, \boldsymbol{\xi}_s = \mathbf{x}\right) = \frac{\Gamma[n]\mathrm{e}^{\|\mathbf{x}\|} p(\|\mathbf{z}\|)}{\Psi_s(\|\mathbf{x}\|)\|\mathbf{z}\|^{n-1}} \prod_{i=1}^{n} \frac{(z_i - x_i)^{-s}}{\Gamma[1-s]}\,\mathrm{d}z_i. \tag{7.86}$$

Increments of ASPs

We shall now show that the increments of an ASP have n-dimensional Liouville distributions. Indeed, at time $s \in [0,1)$, the increment $\boldsymbol{\xi}_t - \boldsymbol{\xi}_s$, $t \in (s,1]$, has a multivariate Liouville distribution with a generating law that can be expressed in terms of the $\boldsymbol{\xi}_s$-conditional law of the norm variable $R_t = \|\boldsymbol{\xi}_t\|$. Before we show this, we shall first examine the law of the process $\{R_t\}$.

Proposition 7.2.6. *The process $\{R_t\}_{0\le t\le T}$ is a GRB with generating law ν and activity parameter n. That is,*

$$\mathbb{Q}(R_t \in \mathrm{d}r \,|\, \boldsymbol{\xi}_s = \mathbf{x}) = \frac{\Psi_t(r)}{\Psi_s(\|\mathbf{x}\|)} \frac{(r - \|\mathbf{x}\|)^{n(t-s)-1}\exp(-(r - \|\mathbf{x}\|))}{\Gamma[n(t-s)]}\mathrm{d}r, \tag{7.87}$$

and

$$\mathbb{Q}(R_1 \in \mathrm{d}r \,|\, \boldsymbol{\xi}_s = \mathbf{x}) = \frac{\theta_s(\mathrm{d}r; \|\mathbf{x}\|)}{\Psi_s(\|\mathbf{x}\|)}, \tag{7.88}$$

for $0 < s < t < 1$.

Before proceeding the proof, note that, after simplification, (7.87) and (7.88) are consistent with (7.50) and (7.51).

Proof. Since $\{\boldsymbol{\xi}_t\}$ is a Markov process with respect to $\{\mathcal{F}_t^{\boldsymbol{\xi}}\}$, $\{R_t\}$ is a Markov process with respect to $\{\mathcal{F}_t^{\boldsymbol{\xi}}\}$. Thus, to prove the proposition we need only verify that the transition probabilities $\{R_t\}$ match those given in (7.87) and (7.88).

We first verify the $\boldsymbol{\xi}_s$-conditional law of R_1. We can calculate this using the Bayes formula,

$$\begin{aligned}\mathbb{Q}(R_1 \in \mathrm{d}r \,|\, \boldsymbol{\xi}_s = \mathbf{x}) &= \frac{\mathbb{Q}(\boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x} \,|\, R_1 = r)\mathbb{Q}(R_1 \in \mathrm{d}r)}{\int_{r=0}^{\infty} \mathbb{Q}(\boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x} \,|\, R_1 = r)\mathbb{Q}(R_1 \in \mathrm{d}r)} \\ &= \frac{\frac{1}{f_n(r)} f_{n(1-s)}(r - \|\mathbf{x}\|)\, \nu(\mathrm{d}r)}{\int_{r=0}^{\infty} \frac{1}{f_n(r)} f_{n(1-s)}(r - \|\mathbf{x}\|)\, \nu(\mathrm{d}r)} \\ &= \frac{\theta_s(\mathrm{d}r; \|\mathbf{x}\|)}{\Psi_s(\|\mathbf{x}\|)}. \end{aligned} \tag{7.89}$$

The $\boldsymbol{\xi}_s$-conditional law of R_t for $t \in (s, 1)$ can be derived by the use of the Bayes formula,

$$\begin{aligned}\mathbb{Q}(R_t \in \mathrm{d}r \,|\, \boldsymbol{\xi}_s = \mathbf{x}) &= \frac{\int_{z=0}^{\infty} \mathbb{Q}(\boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x}, R_t \in \mathrm{d}r \,|\, R_1 = z)\mathbb{Q}(R_1 \in \mathrm{d}z)}{\int_{z=0}^{\infty}\int_{r=0}^{\infty} \mathbb{Q}(\boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x}, R_t \in \mathrm{d}r \,|\, R_1 = z)\mathrm{d}r\mathbb{Q}(R_1 \in \mathrm{d}z)} \\ &= \frac{\int_{z=0}^{\infty} \frac{1}{f_n(z)} f_{n(t-s)}(r - \|\mathbf{x}\|) f_{n(1-t)}(z - r)\mathrm{d}r\, \nu(\mathrm{d}z)}{\int_{z=0}^{\infty} \frac{1}{f_n(z)} \int_{r=\|\mathbf{x}\|}^{z} f_{n(t-s)}(r - \|\mathbf{x}\|) f_{n(1-t)}(z - r)\mathrm{d}r\, \nu(\mathrm{d}z)} \\ &= \frac{\Psi_t(r)}{\Psi_s(\|\mathbf{x}\|)} f_{n(t-s)}(r - \|\mathbf{x}\|)\mathrm{d}r. \end{aligned} \tag{7.90}$$

The denominator of (7.90) is simplified using the fact that gamma densities with common scale parameter are closed under convolution. □

For a set $B \in \mathcal{B}(\mathbb{R})$, we define the measure ν_{st}, $0 \leq s < t \leq 1$, by

$$\nu_{st}(B) = \mathbb{Q}(R_t \in B \,|\, \boldsymbol{\xi}_s). \tag{7.91}$$

Thus we have

$$\nu_{s1}(\mathrm{d}r) = \frac{\theta_s(\mathrm{d}r; R_s)}{\Psi_s(R_s)}, \tag{7.92}$$

and

$$\nu_{st}(\mathrm{d}r) = \frac{\Psi_t(r)}{\Psi_s(R_s)} \frac{(r - R_s)^{n(t-s)-1} \exp(-(r - R_s))}{\Gamma[n(t-s)]}\mathrm{d}r, \qquad \text{for} \quad t < 1. \tag{7.93}$$

When ν_{st} admits a density, we denote it by $p_{st}(r) = \nu_{st}(\mathrm{d}r)/\mathrm{d}r$. We see from (7.93) that p_{st} exists for $t < 1$. When $t = 1$, it follows from the definition of θ_t that p_{s1} only exists if ν admits a density.

Note that from Proposition 7.2.6, $\mathbb{Q}(R_t \in \mathrm{d}r \,|\, \boldsymbol{\xi}_s) = \mathbb{Q}(R_t \in \mathrm{d}r \,|\, R_s)$ for $t \in (s, 1]$. This is not surprising since $\{R_s\}$ is a GRB, and hence it is a Markov process with respect to its natural filtration.

Proposition 7.2.7. *Fix $s \in [0, 1)$. Given $\boldsymbol{\xi}_s$, the increment $\boldsymbol{\xi}_t - \boldsymbol{\xi}_s$, $t \in (s, 1]$, has an n-variate Liouville distribution with generating law*

$$\nu^*(B) = \nu_{st}(B + R_s), \tag{7.94}$$

and parameter vector $\alpha = (t - s, \ldots, t - s)^\top$, for a set $B \in \mathcal{B}(\mathbb{R})$.

Proof. First we prove the case $t < 1$. Thus, the density p_{st} exists. From (7.77) and (7.93), we have the following:

$$\begin{aligned} \mathbb{Q}(\boldsymbol{\xi}_t - \boldsymbol{\xi}_s \in \mathrm{d}\mathbf{y} \,|\, \boldsymbol{\xi}_s) &= \frac{\Psi_t(\|\mathbf{y}\| + R_s)}{\Psi_s(R_s)} \prod_{i=1}^n \frac{y_i^{(t-s)-1} \mathrm{e}^{-y_i}}{\Gamma[t-s]} \,\mathrm{d}y_i \\ &= \frac{p_{st}(\|\mathbf{y}\| + R_s)\Gamma[n(t-s)]}{\|\mathbf{y}\|^{n(t-s)-1}} \prod_{i=1}^n \frac{y_i^{(t-s)-1}}{\Gamma[t-s]} \,\mathrm{d}y_i. \end{aligned} \tag{7.95}$$

Comparing (7.95) to (7.25) shows it to be the law of Liouville distribution with generating law $p_{st}(x + R_s)\mathrm{d}x$ and parameter vector $(t-s, \ldots, t-s)^\top$. Noting that $p_{st}(x+R_s)\mathrm{d}x = \nu^*(\mathrm{d}x)$, where ν^* is given by (8.40), yields the required result.

We now consider the case $t = 1$ when ν admits a density p. Thus, the density p_{s1} exists. From (7.86) and (7.92), we have

$$\begin{aligned} \mathbb{Q}(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_s \in \mathrm{d}\mathbf{y} \,|\, \boldsymbol{\xi}_s) &= \frac{\Gamma[n]\mathrm{e}^{R_s} p(\|\mathbf{y}\| + R_s)}{\Psi_s(R_s)(\|\mathbf{y}\| + R_s)^{n-1}} \prod_{i=1}^n \frac{y_i^{-s}}{\Gamma[1-s]} \,\mathrm{d}y_i \\ &= \frac{\Gamma[n(1-t)]p_{s1}(\|\mathbf{y}\| + R_s)}{\|\mathbf{y}\|^{n(1-t)-1}} \prod_{i=1}^n \frac{y_i^{-s}}{\Gamma[1-s]} \,\mathrm{d}y_i. \end{aligned} \tag{7.96}$$

Hence $\boldsymbol{\xi}_t - \boldsymbol{\xi}_s$ has the required density.

For the final case where $t = 1$ and ν has no density, the proof is as follows: Given $\boldsymbol{\xi}_s$, the law of the increment $\boldsymbol{\xi}_1 - \boldsymbol{\xi}_s$ is characterised by (7.76). Then by mixing the Dirichlet density (7.19) with the random scale parameter X, it follows that this law is equal to the law of $X\mathbf{D}$, where X is a random variable with law ν^* which is given by (8.40), and $\mathbf{D}$ is a Dirichlet random variable independent of X, with parameter vector $(1 - s, \ldots, 1 - s)^\top$. The statement follows. □

7.2.2 Moments

In this subsection we fix a time $s \in [0,1)$, and we assume that the first two moments of ν exist and are finite.

Proposition 7.2.8. *The first- and second-order moments of $\boldsymbol{\xi}_t$, $t \in (s,1]$, are*

$$\text{1.} \qquad \mathbb{E}^{\mathbb{Q}}\left[\xi_t^{(i)} \,\middle|\, \boldsymbol{\xi}_s\right] = \frac{1}{n}\mu_1 + \xi_s^{(i)}, \tag{7.97}$$

$$\text{2.} \qquad \mathrm{Var}^{\mathbb{Q}}\left[\xi_t^{(i)} \,\middle|\, \boldsymbol{\xi}_s\right] = \frac{1}{n}\left[\left(\frac{t-s+1}{n(t-s)+1}\right)\mu_2 - \frac{1}{n}\mu_1^2\right], \tag{7.98}$$

$$\text{3.} \qquad \mathrm{Cov}^{\mathbb{Q}}\left[\xi_t^{(i)}, \xi_t^{(j)} \,\middle|\, \boldsymbol{\xi}_s\right] = \frac{t-s}{n}\left[\frac{\mu_2}{n(t-s)+1} - \frac{\mu_1^2}{n(t-s)}\right], \quad (i \neq j), \tag{7.99}$$

where

$$\mu_1 = \frac{t-s}{1-s}\left(\mathbb{E}^{\mathbb{Q}}[R_1 \,|\, R_s] - R_s\right), \tag{7.100}$$

$$\mu_2 = \frac{(t-s)(1+n(t-s))}{(1-s)(1+n(1-s))}\mathbb{E}^{\mathbb{Q}}[(R_1 - R_s)^2 \,|\, R_s]. \tag{7.101}$$

Proof. Given $\boldsymbol{\xi}_s$, the increment $\boldsymbol{\xi}_t - \boldsymbol{\xi}_s$ has an n-dimensional Liouville distribution with generating law

$$\nu^*(A) = \nu_{st}(A + R_s), \tag{7.102}$$

for $t \in (s,1]$, and with parameter vector $(t-s, \dots, t-s)^\top$. We have

$$\mu_1 = \int_0^\infty y\,\nu^*(\mathrm{d}y) = \int_{R_s}^\infty y\,\nu_{st}(\mathrm{d}y) - R_s = \mathbb{E}^{\mathbb{Q}}[R_t \,|\, \boldsymbol{\xi}_s] - R_s, \tag{7.103}$$

$$\mu_2 = \int_0^\infty y^2\,\nu^*(\mathrm{d}y) = \int_{R_s}^\infty (y-R_s)^2\,\nu_{st}(\mathrm{d}y) = \mathbb{E}^{\mathbb{Q}}[(R_t - R_s)^2 \,|\, \boldsymbol{\xi}_s]. \tag{7.104}$$

It then follows from equations (7.26)-(7.28) that

$$\text{1.} \quad \mathbb{E}^{\mathbb{Q}}\left[\xi_t^{(i)} \,\middle|\, \boldsymbol{\xi}_s\right] = \frac{1}{n}\left(\mathbb{E}^{\mathbb{Q}}[R_t \,|\, \boldsymbol{\xi}_s] - R_s\right) + \xi_s^{(i)}, \tag{7.105}$$

$$\text{2.} \quad \mathrm{Var}^{\mathbb{Q}}\left[\xi_t^{(i)} \,\middle|\, \boldsymbol{\xi}_s\right] = \frac{1}{n}\left[\left(\frac{t-s+1}{n(t-s)+1}\right)\mathbb{E}^{\mathbb{Q}}[(R_t - R_s)^2 \,|\, \boldsymbol{\xi}_s] - \frac{1}{n}\left(\mathbb{E}^{\mathbb{Q}}[R_t \,|\, \boldsymbol{\xi}_s] - R_s\right)^2\right], \tag{7.106}$$

$$\text{3.} \quad \mathrm{Cov}^{\mathbb{Q}}\left[\xi_t^{(i)}, \xi_t^{(j)} \,\middle|\, \boldsymbol{\xi}_s\right] = \frac{t-s}{n}\left[\frac{\left(\mathbb{E}^{\mathbb{Q}}[(R_t - R_s)^2 \,|\, \boldsymbol{\xi}_s]\right)}{n(t-s)+1} - \frac{\left(\mathbb{E}^{\mathbb{Q}}[R_t \,|\, \boldsymbol{\xi}_s] - R_s\right)^2}{n(t-s)}\right], \quad (i \neq j). \tag{7.107}$$

To compute $\mathbb{E}^{\mathbb{Q}}[R_t \,|\, \boldsymbol{\xi}_s]$ and $\mathbb{E}^{\mathbb{Q}}[(R_t - R_s)^2 \,|\, \boldsymbol{\xi}_s]$, we use two results about Lévy random bridges found in Hoyle *et al.* (2011). First, we can write

$$\mathbb{E}^{\mathbb{Q}}[R_t \,|\, R_s] = \frac{t-s}{1-s}\mathbb{E}^{\mathbb{Q}}[R_1 \,|\, R_s] + \frac{1-t}{1-s}R_s. \tag{7.108}$$

The expression for μ_1 then follows directly. Second, given R_s, the process $\{R_t - R_s\}_{s\le t\le 1}$ is a GRB with generating law $\bar{\nu}(B) = \nu_{s1}(B + R_s)$ and activity parameter n. Hence, given R_s, we can write

$$\{R_t - R_s\}_{s\le t\le 1} \overset{\text{law}}{=} \{X\gamma_{t1}\}_{s\le t\le 1}, \tag{7.109}$$

where X is a random variable with law $\bar{\nu}$, and $\{\gamma_{t1}\}_{s\le t\le 1}$ is a gamma bridge with activity parameter n, independent of X, satisfying $\gamma_{s1} = 0$ and $\gamma_{11} = 1$. Note that γ_{t1}, $t \in (s, 1)$, is a beta random variable with parameters $\alpha = n(t-s)$ and $\beta = n(1-t)$. Therefore, it follows that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[(R_t - R_s)^2 \,|\, R_s] &= \mathbb{E}^{\mathbb{Q}}[\gamma_{t1}^2]\mathbb{E}^{\mathbb{Q}}[X^2] = \mathbb{E}^{\mathbb{Q}}[\gamma_{t1}^2]\int_0^\infty x^2\,\bar{\nu}(\mathrm{d}x) \\
&= \mathbb{E}^{\mathbb{Q}}[\gamma_{t1}^2]\int_{R_s}^\infty (y - R_s)^2\,\nu_{s1}(\mathrm{d}x) \\
&= \frac{(t-s)(1+n(t-s))}{(1-s)(1+n(1-s))}\mathbb{E}^{\mathbb{Q}}[(R_1 - R_s)^2 \,|\, R_s],
\end{aligned} \tag{7.110}$$

which completes the proof. □

7.2.3 Measure Change

In this section we shall show that the law of an n-dimensional ASP is equivalent to a vector of n independent gamma processes. To demonstrate this result, we begin by assuming that under some probability measure $\widetilde{\mathbb{Q}}$ the process $\{\boldsymbol{\xi}_t\}$ is a vector of n independent gamma processes, and then show that $\{\boldsymbol{\xi}_t\}$ is an ASP under an equivalent measure $\mathbb{Q}$.

In particular, under $\widetilde{\mathbb{Q}}$, we assume that $\{\boldsymbol{\xi}_t\}$ is a vector of n independent gamma processes such that

$$\widetilde{\mathbb{Q}}(\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{x}) = \prod_{i=1}^n \frac{x_i^{t-1}}{\Gamma[t]}\mathrm{e}^{-x_i}\,\mathrm{d}x_i. \tag{7.111}$$

Hence, the gamma processes $\{\xi_t^{(i)}\}$, $i = 1, 2, \ldots, m$, are independent, and they are identical in law. The process $\{R_t\}_{0\le t\le 1}$, defined as above by $R_t = \|\boldsymbol{\xi}_t\|$, is a one-dimensional gamma process and satisfies the following:

$$\widetilde{\mathbb{Q}}(R_t \in \mathrm{d}x) = \frac{x^{nt-1}}{\Gamma[nt]}\mathrm{e}^{-x}\,\mathrm{d}x. \tag{7.112}$$

As before, the filtration $\{\mathcal{F}_t^{\boldsymbol{\xi}}\}$ is generated by $\{\boldsymbol{\xi}_t\}$.

We shall show that the process $\{\Psi_t(R_t)\}_{0\leq t<1}$ is a martingale, where

$$\begin{aligned}\Psi_t(R_t) &= \int_{R_t}^{\infty} \frac{f_{n(1-t)}(z-R_t)}{f_n(z)}\,\nu(\mathrm{d}z)\\ &= \frac{\Gamma[n]\exp(R_t)}{\Gamma[n(1-t)]}\int_{R_t}^{\infty} z^{1-n}(z-R_t)^{n(1-t)-1}\,\nu(\mathrm{d}z). \qquad (7.113)\end{aligned}$$

For times $0\leq s<t<1$, we have

$$\begin{aligned}\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\Psi_t(R_t)\,\middle|\,\mathcal{F}_s^{\boldsymbol{\xi}}\right] &= \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\int_{R_t}^{\infty}\frac{f_{n(1-t)}(z-R_t)}{f_n(z)}\,\nu(\mathrm{d}z)\,\middle|\,\mathcal{F}_s^{\boldsymbol{\xi}}\right]\\ &= \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\int_{R_t}^{\infty}\frac{f_{n(1-t)}(z-R_s-(R_t-R_s))}{f_n(z)}\,\nu(\mathrm{d}z)\,\middle|\,\boldsymbol{\xi}_s\right]\\ &= \int_{y=0}^{\infty}\int_{z=R_s+y}^{\infty}\frac{f_{n(1-t)}(z-R_s-y)}{f_n(z)}\,\nu(\mathrm{d}z)\,f_{n(t-s)}(y)\,\mathrm{d}y\\ &= \int_{z=R_s}^{\infty}\frac{1}{f_n(z)}\int_{y=0}^{z-R_s} f_{n(1-t)}(z-R_s-y)f_{n(t-s)}(y)\,\mathrm{d}y\,\nu(\mathrm{d}z)\\ &= \int_{R_s}^{\infty}\frac{f_{n(1-s)}(z-R_s)}{f_n(z)}\,\nu(\mathrm{d}z)\\ &= \Psi_s(R_s). \qquad (7.114)\end{aligned}$$

Since $\Psi_0(R_0)=1$ and $\Psi_t(R_t)>0$, the process $\{\Psi_t(R_t)\}_{0\leq t<1}$ is a Radon-Nikodym density process.

Proposition 7.2.9. *Define a measure* $\mathbb{Q}$ *by*

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\widetilde{\mathbb{Q}}}\right|_{\mathcal{F}_t^{\boldsymbol{\xi}}} = \Psi_t(R_t). \qquad (7.115)$$

Under $\mathbb{Q}$*,* $\{\boldsymbol{\xi}_t\}_{0\leq t<1}$ *is an ASP with generating law* ν*.*

Proof. We prove the proposition by verifying that the transition law of $\{\boldsymbol{\xi}_t\}$ under $\mathbb{Q}$ is that of an ASP.

$$\begin{aligned}\mathbb{Q}\left(\boldsymbol{\xi}_t\in\mathrm{d}\mathbf{x}\,|\,\mathcal{F}_s^{\boldsymbol{\xi}}\right) &= \mathbb{E}^{\mathbb{Q}}[\mathbf{1}\{\boldsymbol{\xi}_t\in\mathrm{d}\mathbf{x}\}\,|\,\mathcal{F}_s^{\boldsymbol{\xi}}]\\ &= \frac{1}{\Psi_s(R_s)}\,\mathbb{E}^{\widetilde{\mathbb{Q}}}[\Psi_t(R_t)\mathbf{1}\{\boldsymbol{\xi}_t\in\mathrm{d}\mathbf{x}\}\,|\,\boldsymbol{\xi}_s]\\ &= \frac{\Psi_t(R_t)}{\Psi_s(R_s)}\prod_{i=1}^{n} f_{t-s}(x_i-\xi_s^{(i)})\,\mathrm{d}x_i\\ &= \frac{\Psi_t(R_t)}{\Psi_s(R_s)}\prod_{i=1}^{n}\frac{(x_i-\xi_s^{(i)})^{(t-s)-1}\mathrm{e}^{-(x_i-\xi_s^{(i)})}}{\Gamma[t-s]}\,\mathrm{d}x_i. \qquad (7.116)\end{aligned}$$

Comparing equations (7.116) and (7.77) completes the proof. □

We can restate the results of this subsection by the following:

Proposition 7.2.10. *Suppose that* $\{\boldsymbol{\xi}_t\}_{0\le t\le 1}$ *is an ASP with generating law* ν *under some measure* $\mathbb{Q}$. *Then*

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\right|_{\mathcal{F}_t^{\boldsymbol{\xi}}} = \Psi_t(R_t)^{-1}, \tag{7.117}$$

defines a probability measure $\widetilde{\mathbb{Q}}$ *for* $t \in [0,1)$. *Furthermore, under* $\widetilde{\mathbb{Q}}$, $\{\boldsymbol{\xi}_t\}_{0\le t<1}$ *is a vector of* n *independent gamma processes such that*

$$\widetilde{\mathbb{Q}}(\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{x}) = \prod_{i=1}^{n} \frac{x_i^{t-1}}{\Gamma[t]} \mathrm{e}^{-x_i}\, \mathrm{d}x_i. \tag{7.118}$$

7.2.4 Independent Gamma Bridges Representation

In this section, we shall show that the increments of an n-dimensional ASP are identical in law to a positive random variable multiplied by the Hadamard product of an n-dimensional Dirichlet random variable and a vector of n independent gamma bridges.

For vectors $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^n$, we denote their Hadamard product by $\mathbf{X} \circ \mathbf{Y}$. Recall from Chapter 3 that we can write

$$\mathbf{X} \circ \mathbf{Y} = (x_1 y_1, \dots, x_n y_n)^{\top}. \tag{7.119}$$

Proposition 7.2.11. *Given the value of* $\boldsymbol{\xi}_s$, *the ASP process* $\{\boldsymbol{\xi}_t\}$ *satisfies the following identity in law:*

$$\{\boldsymbol{\xi}_t - \boldsymbol{\xi}_s\}_{s\le t\le 1} \overset{\text{law}}{=} \{R^*\, \mathbf{D} \circ \boldsymbol{\gamma}_{t1}\}_{s\le t\le 1}, \tag{7.120}$$

where

1. $\mathbf{D} \in [0,1]^n$ *is a symmetric Dirichlet random variable with parameter vector* $(1-s, \dots, 1-s)^{\top}$;
2. $\{\boldsymbol{\gamma}_{t1}\}$ *is a vector of* n *independent gamma bridges, each with activity parameter* $m = 1$, *starting at the value 0 at time* s, *and terminating with unit value at time 1;*
3. $R^* > 0$ *is a random variable with law* ν^* *given by*

$$\nu^*(A) = \nu_{s1}(A + R_s), \qquad \textit{for} \quad A \in \mathcal{B}(\mathbb{R}); \tag{7.121}$$

4. R^*, $\mathbf{D}$, *and* $\{\boldsymbol{\gamma}_{t1}\}$ *are mutually independent.*

Proof. Fix $k_i \ge 1$ and the partition

$$s = t_0^i < t_1^i < \dots < t_{k_i}^i = 1, \tag{7.122}$$

for $i = 1, \ldots, n$. Define the non-overlapping increments $\{\Delta_{ij}\}$ by

$$\Delta_{ij} = \xi^{(i)}_{t^i_j} - \xi^{(i)}_{t^i_{j-1}}, \tag{7.123}$$

for $j = 1, \ldots, k_i$ and $i = 1, \ldots, n$. The distribution of the vector

$$\begin{aligned}\boldsymbol{\Delta} = (&\Delta_{11}, \Delta_{12}, \ldots, \Delta_{1k_1},\\ &\Delta_{21}, \Delta_{22}, \ldots, \Delta_{2k_2},\\ &\vdots\\ &\Delta_{n1}, \Delta_{n2}, \ldots, \Delta_{nk_n})^{\top},\end{aligned} \tag{7.124}$$

characterises the finite-dimensional distributions of the process $\{\boldsymbol{\xi}_t - \boldsymbol{\xi}_s\}_{s \le t \le 1}$. It follows from the Kolmogorov extension theorem that the distribution of $\boldsymbol{\Delta}$ characterises the law of $\{\boldsymbol{\xi}_t - \boldsymbol{\xi}_s\}$. Note that $\boldsymbol{\Delta}$ are non-overlapping increments of the master GRB $\{\Gamma_t\}$. Thus, given $\boldsymbol{\xi}_s$, $\boldsymbol{\Delta}$ has a multivariate Liouville distribution with parameter vector

$$\begin{aligned}\boldsymbol{\alpha} = (&t^1_1 - t^1_0, t^1_2 - t^1_1, \ldots, t^1_{k_1} - t^1_{k_1-1},\\ &t^2_1 - t^2_0, t^2_2 - t^2_1, \ldots, t^2_{k_2} - t^2_{k_2-1},\\ &\vdots\\ &t^n_1 - t^n_0, t^n_2 - t^n_1, \ldots, t^n_{k_n} - t^n_{k_n-1})^{\top},\end{aligned} \tag{7.125}$$

and generating law

$$\nu^*(A) = \nu_{s1}(A + R_s), \tag{7.126}$$

for $t \in (s, 1]$ and $A \in \mathcal{B}(\mathbb{R})$.

It follows from (Fang *et al.* 1990, Theorem 6.9) that

$$(\Delta_{i1}, \ldots, \Delta_{ik_i})^{\top} \overset{\text{law}}{=} R^*\, D_i \mathbf{Y}_i, \qquad \text{for } i = 1, \ldots, n, \tag{7.127}$$

where (i) R^* has law ν^*, (ii) $\mathbf{D} = (D_1, \ldots, D_n)^{\top}$ has a Dirichlet distribution with parameter vector $(1-s, \ldots, 1-s)^{\top}$, (iii) $\mathbf{Y}_i \in [0,1]^{k_i}$ has a Dirichlet distribution with parameter vector $(t^i_1 - t^i_0, \ldots, t^i_{k_i} - t^i_{k_i-1})^{\top}$, (iv) $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$, R^*, and $\mathbf{D}$ are mutually independent.

Let $\{\gamma(t)\}_{s \le t \le 1}$ be a gamma bridge with activity parameter $m = 1$ such that $\gamma(s) = 0$ and $\gamma(1) = 1$. Then the increment vector

$$(\gamma(t^i_1) - \gamma(t^i_0), \ldots, \gamma(t^i_{k_i}) - \gamma(t^i_{k_i-1}))^{\top}, \tag{7.128}$$

has a Dirichlet distribution with parameter vector $(t^i_1 - t^i_0, \ldots, t^i_{k_i} - t^i_{k_i-1})^{\top}$. Hence the increment vector (7.128) is identical in law to $\mathbf{Y}_i$. From the Kolmogorov extension theorem, this identity characterises the law of $\{\gamma(t)\}$. It follows that

$$\{\xi_t^{(i)} - \xi_s^{(i)}\}_{s\leq t\leq 1} \overset{\text{law}}{=} \{R^* \, D_i \gamma_{t1}\}_{s\leq t\leq 1}, \qquad \text{for } i = 1, \ldots, n, \tag{7.129}$$

which completes the proof. □

7.2.5 Uniform Process

We construct a multivariate process from the ASP $\{\boldsymbol{\xi}_t\}$ such that each one-dimensional marginal is a priori uniformly distributed for every time $t \in (0, 1]$.

Fix a time $t \in (0, 1]$. Each $\xi_t^{(i)}$ is a scale-mixed beta random variable with survival function

$$\begin{aligned} \bar{F}_t(x) &= \int_x^\infty \big(1 - I_{x/y}[t, n-t]\big) \, \nu(\mathrm{d}y) \\ &= \int_x^\infty I_{1-x/y}[n-t, t] \, \nu(\mathrm{d}y), \end{aligned} \tag{7.130}$$

where $I_z[\alpha, \beta]$ is the regularized incomplete Beta function, defined as usual for $z \in [0, 1]$ by

$$I_z[\alpha, \beta] = \frac{\int_0^z u^{\alpha-1}(1-\beta)^{\beta-1} \, \mathrm{d}u}{\int_0^1 u^{\alpha-1}(1-\beta)^{\beta-1} \, \mathrm{d}u} \qquad (\alpha, \beta > 0). \tag{7.131}$$

The random variables

$$Y_t^{(i)} = \bar{F}_t(\xi_t^{(i)}), \qquad i = 1, \ldots, n, \tag{7.132}$$

are then uniformly distributed.

We now define a process $\{\mathbf{Y}\}_{0\leq t\leq 1}$ by

$$\mathbf{Y}_t = \left(\bar{F}_t(\xi_t^{(1)}), \ldots, \bar{F}_t(\xi_t^{(n)})\right)^\top. \tag{7.133}$$

By construction, each one-dimensional marginal $Y_t^{(i)}$ is uniform for $t > 0$. For fixed t, $\mathbf{Y}_t$ is a draw from the survival copula of the Liouville distribution, and $\mathbf{Y}_1$ is a draw from an Archimedean survival copula.

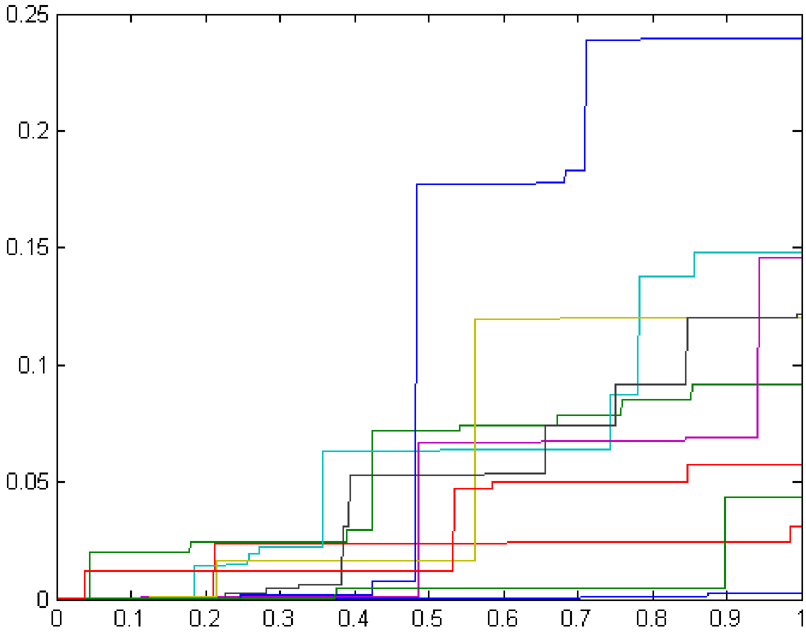

Figure 7.1: A 10-dimensional Archimedean survival process. An Archimedean survival process is a multivariate gamma random bridge, since each marginal process is a gamma random bridge. Time horizon: $[0, 1]$.

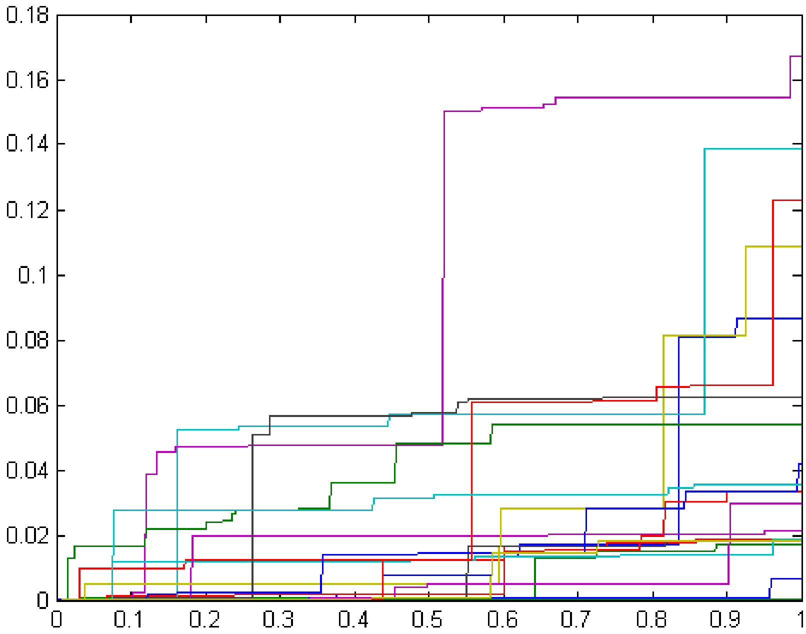

Figure 7.2: A 20-dimensional Archimedean survival process. Each marginal process is a gamma random bridge. Time horizon: $[0, 1]$.

Chapter 8

Generalised Liouville Processes

We introduce a class of Markovian multivariate stochastic processes that we call Generalised Liouville Processes (GLPs). We construct GLPs by splitting Lévy random bridges into n non-overlapping pieces. We allow more flexibility in the splitting mechanism when compared to the way ASPs are constructed, and employ some deterministic time changes. GLPs generalise ASPs.

We have seen in Chapter 7 that ASPs are n-dimensional extensions of gamma random bridges. Hence, an ASP can be viewed as a multivariate gamma information process about a vector of dependent claims determined by the terminal values of cumulative gains processes. We shall show below that we can view GLPs as multivariate information processes as well. This interpretation follows from the fact that GLPs are a natural multivariate extension of Lévy random bridges, and one-dimensional Lévy random bridges are used in Hoyle *et al.* (2011) as market information processes.

This chapter is organized as follows: Section 1 is a brief review of Lévy processes, Lévy bridges and Lévy random bridges. In Section 2, we define GLPs and provide various characterisations of their law. As an example, we introduce what we call Liouville processes as a subclass of GLPs, and show that ASPs are special cases of Liouville processes. We also introduce what we call Standard Variance Gamma Liouville Processes (SVGLPs), and show that SVGLPs can be represented in terms of Liouville processes. Section 3 is an information-based perspective of GLPs.

8.1 Lévy Random Bridges

8.1.1 Lévy Processes and Lévy Bridges

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{0\leq t\leq\infty}$. We fix a finite time horizon $[0, T]$ and assume that all filtrations are right-continuous and complete. An n-dimensional càdlàg process $\{Y_t\}_{t\geq 0}$ with $Y_0 = 0$ is a Lévy process if it is stochastically continuous, and has independent and stationary increments. The characteristic function of a Lévy process satisfies $\mathbb{E}^{\mathbb{Q}}[e^{izY_t}] = e^{t\widetilde{\mu}(z)}$, for $z \in \mathbb{R}^n$, where the characteristic exponent $\widetilde{\mu}(z) : \mathbb{R}^n \to \mathbb{C}^n$ can be written as

$$\widetilde{\mu}(z) = i\langle \gamma, z\rangle - \frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^n} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbf{1}_{\{|x|<1\}})\Lambda(\mathrm{d}x). \tag{8.1}$$

Equation (8.1) is the Lévy-Khintchine representation, where $\langle \, , \rangle$ is the inner product, $\gamma \in \mathbb{R}^n$, A is a symmetric positive-definite $n \times n$ matrix, and Λ is the Lévy measure which satisfies

$$\Lambda(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} (|x|^2 \wedge 1)\Lambda(\mathrm{d}x) < \infty. \tag{8.2}$$

Let $\{Y_t\}_{t\in[0,T]}$ be a one-dimensional Lévy process defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and assume that the density of Y_t exists for every $t \in (0, T]$. For the density to exist, the law of Y_t must be absolutely continuous with respect to the Lebesgue measure.

We denote the density of Y_t by $f_t : \mathbb{R} \to \mathbb{R}_+$. The densities of a Lévy process satisfy the Chapman-Kolmogorov convolution identity

$$f_t(x) = \int_{\mathbb{R}} f_{t-s}(x - y)f_s(y)\, \mathrm{d}y, \tag{8.3}$$

and the finite-dimensional laws of $\{Y_t\}$ are given by

$$\mathbb{Q}(Y_{t_1} \in \mathrm{d}y_1, \ldots, Y_{t_n} \in \mathrm{d}y_n) = \prod_{i=1}^{n} f_{t_i - t_{i-1}}(y_i - y_{i-1})\, \mathrm{d}y_i, \tag{8.4}$$

for $n \in \mathbb{N}_+$, $0 < t_1 < \ldots < t_n < T$ and $(y_1, \ldots, y_n) \in \mathbb{R}^n$. Lévy processes are Markovian.

A Lévy bridge is a Lévy process conditioned to take some fixed value at a fixed future time. See, for example, Fitzsimmons *et al.* (1993) for an analysis of bridges of Markov processes.

If $\{Y_{tT}^{(z)}\}$ is a bridge of $\{Y_t\}$ to the value $z \in \mathbb{R}$ at time T, then

$$\mathbb{Q}(Y_{tT}^{(z)} \in \mathrm{d}y | Y_{sT}^{(z)} = x) = \frac{f_{t-s}(y - x)f_{T-t}(z - y)}{f_{T-s}(z - x)}\, \mathrm{d}y, \tag{8.5}$$

is its transition probability for $0 \le s < t < T$ and $0 < f_T(z) < \infty$. It is shown in Hoyle *et al.* (2011) that Lévy bridges are Markovian.

8.1.2 Lévy Random Bridges

Hoyle *et al.* (2011) define Lévy random bridges (LRBs) as follows:

Definition 8.1.1. $\{L_t\}_{t\in[0,T]}$ *is a Lévy random bridge with law* $LRB_{\mathcal{C}}([0, T], \{f_t\}, \upsilon)$ *if the* *following conditions are satisfied:*

1. *L_T has marginal law ν.*

2. *There exists a Lévy process $\{Y_t\}$ such that Y_t has density $f_t(x)$ for all $t \in (0, T]$.*

3. *ν concentrates mass where $f_T(z)$ is positive and finite, i.e. $0 < f_T(z) < \infty$ ν-a.s.*

4. *For every $n \in \mathbb{N}_+$, every $0 < t_1 < \ldots < t_n < T$, every $(x_1, \ldots, x_n) \in \mathbb{R}^n$, and ν-a.e. z,*

$$\mathbb{Q}(L_{t_1} \leq x_1, \ldots, L_{t_n} \leq x_n | L_T = z) = \mathbb{Q}(Y_{t_1} \leq x_1, \ldots, Y_{t_n} \leq x_n | Y_T = z). \tag{8.6}$$

The finite-dimensional distributions of an LRB $\{L_t\}$ are given by

$$\mathbb{Q}(L_{t_1} \in \mathrm{d}x_1, \ldots, L_{t_n} \in \mathrm{d}x_n, L_T \in \mathrm{d}z) = \prod_{i=1}^{n} (f_{t_i - t_{i-1}}(x_i - x_{i-1})\,\mathrm{d}x_i)\theta_{t_n}(\mathrm{d}z; x_n), \tag{8.7}$$

where the measure $\theta_t(\mathrm{d}z; y)$ is defined by

$$\theta_0(\mathrm{d}z; y) = \nu(\mathrm{d}z) \quad \text{and} \quad \theta_t(\mathrm{d}z; y) = \frac{f_{T-t}(z - y)}{f_T(z)} \nu(\mathrm{d}z), \tag{8.8}$$

for $t \in (0, T)$. The transition law of $\{L_t\}$ is

$$\mathbb{Q}(L_T \in \mathrm{d}z | L_s = y) = \frac{\theta_s(\mathrm{d}z; y)}{\theta_s(\mathbb{R}; y)} \text{ and } \mathbb{Q}(L_t \in \mathrm{d}x | L_s = y) = \frac{\theta_t(\mathbb{R}; x)}{\theta_s(\mathbb{R}; y)} f_{t-s}(x - y)\,\mathrm{d}x. \tag{8.9}$$

Hoyle *et al.* (2011) introduce LRBs to model the flow of market information within the information-based framework. An LRB (or what one may call a Lévy information process) is identical in law to a Lévy process conditioned to have a fixed marginal law (say, the a priori law of the future cash flow) at a finite future time. It is proven in Hoyle *et al.* (2011) that LRBs are Markov processes with stationary increments. Note that GRBs form a subclass of LRBs.

8.2 Generalised Liouville Processes

We are now in the position to introduce what we call Generalised Liouville Processes (GLPs). To construct a GLP, we start with a master LRB $\{L_t\}_{0 \leq t \leq u_n}$ where L_{u_n} has marginal law ν for $u_n \in \mathbb{N}_+$ and $n \geq 2$. We assume that ν has no continuous singular part (see Sato, 1999). Then we split $\{L_t\}_{0 \leq t \leq u_n}$ into n non-overlapping subprocesses. At this point, we would like to note that one can also construct GLPs from Lévy processes which have discrete state-spaces. Refer to Hoyle *et al.* (2011) for details on LRBs where their finite-dimensional distributions are given in terms of probability mass functions.

Definition 8.2.1. *Fix $n \in \mathbb{N}_+$, $n \geq 2$, and let $\{u_i\}_{i=1}^n$ be a strictly increasing sequence with $u_0 = 0$. Then a process $\{\boldsymbol{\xi}_t\}_{0\leq t\leq 1}$ is an n-*dimensional Generalised Liouville Process (GLP) *if*

$$\{\boldsymbol{\xi}_t\}_{0\leq t\leq 1} = \left\{ \begin{bmatrix} \xi_t^{(1)} \\ \vdots \\ \xi_t^{(i)} \\ \vdots \\ \xi_t^{(n)} \end{bmatrix} \right\}_{0\leq t\leq 1} \overset{\text{law}}{=} \left\{ \begin{bmatrix} L_{t(u_1)} - L_0 \\ \vdots \\ L_{t(u_i-u_{i-1})+u_{i-1}} - L_{u_{i-1}} \\ \vdots \\ L_{t(u_n-u_{n-1})+u_{n-1}} - L_{u_{n-1}} \end{bmatrix} \right\}_{0\leq t\leq 1}, \tag{8.10}$$

where $\{L_t\}_{0\leq t\leq u_n}$ is an LRB. We say that the marginal law of L_{u_n} is the generating law *of $\{\boldsymbol{\xi}_t\}$.*

Each one-dimensional marginal process of a GLP is a subprocess of an LRB. Hoyle *et al.* (2011) prove that subprocesses of LRBs are themselves LRBs. Hence, GLPs are a multivariate generalisation of LRBs.

We define GLPs over the time interval $[0, 1]$ for parsimony. It is straightforward to generalise the definition for GLPs to arbitrary closed time horizons.

Proposition 8.2.2. *The law of a GLP is characterised by a generalised multivariate Liouville distribution.*

Proof. Since ν has no continuous singular part, we can write $\nu(\mathrm{d}z) = \sum_{j=-\infty}^{\infty} c_i \delta_{z_i}(z)\,\mathrm{d}z + p(z)\,\mathrm{d}z$, where $c_i \in \mathbb{R}$ is a point mass of ν located at $z_i \in \mathbb{R}_+$, and $p : \mathbb{R} \to \mathbb{R}_+$ is the density of the continuous part of ν (see, Sato, 1999). Then from (8.7), the joint density of an LRB $\{L_t\}$ is given by

$$\mathbb{Q}(L_{t_1} \in \mathrm{d}x_1, \ldots, L_{t_k} \in \mathrm{d}x_k, L_{u_n} \in \mathrm{d}x_n) = \\ = \prod_{i=1}^{n} [f_{t_i-t_{i-1}}(x_i - x_{i-1})\,\mathrm{d}x_i] \frac{\sum_{j=-\infty}^{\infty} c_i \delta_{z_i}(x_n) + p(x_n)}{f_n(x_n)}, \tag{8.11}$$

where $x_0 = 0$, for all $k \in \mathbb{N}_+$, all partitions $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = u_n$, all $x_n \in \mathbb{R}$, and all $(x_1, \ldots, x_k)^\top = \mathbf{x} \in \mathbb{R}^k$. Let $\boldsymbol{\alpha} \in \mathbb{R}_+^n$ be the vector of time increments $\alpha_i = t_i - t_{i-1}$, and $\alpha = ||\boldsymbol{\alpha}|| = u_n$. The Jacobian of the transformation $y_1 = x_1, y_2 = x_2 - x_1, \ldots y_n = x_n - x_{n-1}$ is 1, and it follows that

$$\mathbb{Q}(L_{t_1} - L_{t_0} \in \mathrm{d}y_1, \ldots, L_{u_n} - L_{t_k} \in \mathrm{d}y_n) = \\ = \prod_{i=1}^{n} f_{\alpha_i}(y_i)\,\mathrm{d}y_i \frac{\sum_{j=-\infty}^{\infty} c_i \delta_{z_i}(\sum_{i=1}^n y_i) + p(\sum_{i=1}^n y_i)}{f_\alpha(\sum_{i=1}^n y_i)}. \tag{8.12}$$

From the definition given in Gupta and Richards (1995), $(L_{t_1}-L_{t_0},\dots,L_{t_k}-L_{t_{k-1}},L_{u_n}-L_{t_k})^\top$ has a generalised multivariate Liouville distribution. Fix $k_i \geq 1$ and the partitions $0 = t_0^i < t_1^i < \cdots < t_{k_i}^i = 1$, for $i = 1,\dots,n$. Then define the non-overlapping increments $\{\Delta_{ij}\}$ by $\Delta_{ij} = \xi^{(i)}_{t^i_j} - \xi^{(i)}_{t^i_{j-1}}$, for $j = 1,\dots,k_i$ and $i = 1,\dots,n$. The distribution of the $k_1 \times \cdots \times k_n$-element vector $\boldsymbol{\Delta} = (\Delta_{11},\dots,\Delta_{1k_1},\dots,\Delta_{n1},\dots,\Delta_{nk_n})^\top$ characterises the finite-dimensional distributions of the GLP $\{\boldsymbol{\xi}_t\}$. It follows from the Kolmogorov extension theorem that the distribution of $\boldsymbol{\Delta}$ characterises the law of $\{\boldsymbol{\xi}_t\}$. Note that $\boldsymbol{\Delta}$ contains non-overlapping increments of the master LRB $\{L_t\}$ such that $\|\boldsymbol{\Delta}\| = L_{u_n}$. Hence, $\boldsymbol{\Delta}$ has a generalised multivariate Liouville distribution. □

From Definition 8.2.1 and Proposition 8.2.2, we can see that the terminal value $\boldsymbol{\xi}_1$ has a generalised multivariate Liouville distribution.

8.2.1 Transition Laws

In what follows, we let $\{\boldsymbol{\xi}_t\}$ be an n-dimensional GLP with generating law ν, and $\{L_t\}$ is a master process of $\{\boldsymbol{\xi}_t\}$. We denote the filtration generated by $\{\boldsymbol{\xi}_t\}_{0\leq t\leq 1}$ by $\{\mathcal{F}_t^{\boldsymbol{\xi}}\}$. Note that $\{\boldsymbol{\xi}_t\}$ may be viewed as an n-dimensional LRB, so $\{\boldsymbol{\xi}_t\}$ is Markov with respect to $\{\mathcal{F}_t^{\boldsymbol{\xi}}\}$.

We define a family of unnormalised measures, indexed by $t \in [0,1)$ and $x \in \mathbb{R}$, as follows:

$$\theta_0(B;x) = \nu(B), \tag{8.13}$$

$$\theta_t(B;x) = \int_B \frac{f_{u_n(1-t)}(z-x)}{f_{u_n}(z)}\,\nu(\mathrm{d}z), \tag{8.14}$$

for $B \in \mathcal{B}(\mathbb{R})$. We also denote $\Psi_t(x) = \theta_t(\mathbb{R};x)$. We define the sum of marginals of $\boldsymbol{\xi}_t$ as

$$R_t = \sum_{i=1}^n \xi_t^{(i)}. \tag{8.15}$$

Note that $R_1 = L_{u_n}$.

Proposition 8.2.3. *The GLP $\{\boldsymbol{\xi}_t\}$ is a Markov process with the transition law given by*

$$\mathbb{Q}\left(\xi_1^{(1)} \in \mathrm{d}z_1,\dots,\xi_1^{(n-1)} \in \mathrm{d}z_{n-1},\xi_1^{(n)} \in B \,\middle|\, \boldsymbol{\xi}_s = \mathbf{x}\right) = \frac{\theta_{\tau(s)}(B+\sum_{i=1}^{n-1} z_i; x_n + \sum_{i=1}^{n-1} z_i)}{\Psi_s(\sum_{i=1}^n x_i)} \prod_{i=1}^{n-1}[f_{(1-s)(u_i-u_{i-1})}(z_i - x_i)\,\mathrm{d}z_i], \tag{8.16}$$

and

$$\mathbb{Q}\left(\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{y} \,|\, \boldsymbol{\xi}_s = \mathbf{x}\right) = \frac{\Psi_t(\sum_{i=1}^n y_i)}{\Psi_s(\sum_{i=1}^n x_i)} \prod_{i=1}^{n}[f_{(t-s)(u_i-u_{i-1})}(y_i - x_i)\,\mathrm{d}y_i], \tag{8.17}$$

where $\tau(t) = 1-(u_n-u_{n-1})(1-t)/u_n$, $0 \leq s < t < 1$, and $B \in \mathcal{B}(\mathbb{R})$.

Proof. The proof is similar to that of Proposition 7.2.4. We begin by verifying (8.16). From the Bayes formula we have

$$\mathbb{Q}\left(\xi_1^{(1)} \in \mathrm{d}z_1, \dots, \xi_1^{(n-1)} \in \mathrm{d}z_{n-1}, \xi_1^{(n)} \in B \,\middle|\, \boldsymbol{\xi}_s = \mathbf{x}\right) = \\ = \frac{\mathbb{Q}\left(\xi_1^{(1)} \in \mathrm{d}z_1, \dots, \xi_1^{(n-1)} \in \mathrm{d}z_{n-1}, \sum_{i=1}^n \xi_1^{(i)} \in B + \sum_{i=1}^{n-1} z_i, \boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x}\right)}{\mathbb{Q}\left(\boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x}\right)}. \tag{8.18}$$

The law of $R_1 = \sum_{i=1}^n \xi_1^{(i)}$ is ν; hence the numerator of (8.18) is

$$\int_{r \in B + \sum_{i=1}^{n-1} z_i} \mathbb{Q}\left(\xi_1^{(1)} \in \mathrm{d}z_1, \dots, \xi_1^{(n-1)} \in \mathrm{d}z_{n-1}, \boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x} \,\middle|\, R_1 = r\right) \nu(\mathrm{d}r) = \\ \prod_{i=1}^n [f_{s(u_i-u_{i-1})}(x_i)\,\mathrm{d}x_i] \prod_{i=1}^{n-1} [f_{(1-s)(u_i-u_{i-1})}(z_i - x_i)\,\mathrm{d}z_i] \\ \times \int_{r \in B + \sum_{i=1}^{n-1} z_i} \frac{f_{(1-s)(u_n-u_{n-1})}(r - \sum_{i=1}^{n-1} z_i - x_n)}{f_{u_n}(r)} \nu(\mathrm{d}r), \tag{8.19}$$

and the denominator is

$$\mathbb{Q}\left(\boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x}\right) = \prod_{i=1}^n [f_{s(u_i-u_{i-1})}(x_i)\,\mathrm{d}x_i] \int_{-\infty}^{\infty} \frac{f_{u_n(1-s)}(r - \sum_{i=1}^n x_i)}{f_{u_n}(r)} \nu(\mathrm{d}r). \tag{8.20}$$

Equation (8.19) follows from the fact that, given $\sum_{i=1}^n \xi_1^{(i)} = R_1$, $\{\boldsymbol{\xi}_t\}$ is a vector of subprocesses of a Lévy bridge. Equation (8.20) follows from the stationary increments property of LRBs and (8.7). Dividing (8.19) by (8.20) yields (8.16).

We shall now verify (8.17). From the Bayes formula we have

$$\mathbb{Q}(\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{y} \,|\, \boldsymbol{\xi}_s = \mathbf{x}) = \frac{\mathbb{Q}(\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{y}, \boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x})}{\mathbb{Q}(\boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x})}. \tag{8.21}$$

The numerator of (8.21) is

$$\int_{-\infty}^{\infty} \mathbb{Q}\left(\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{y}, \boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x} \,|\, R_1 = z\right) \nu(\mathrm{d}z) = \\ \prod_{i=1}^n [f_{s(u_i-u_{i-1})}(x_i)\,\mathrm{d}x_i] \prod_{i=1}^n [f_{(t-s)(u_i-u_{i-1})}(y_i - x_i)\,\mathrm{d}y_i] \int_{-\infty}^{\infty} \frac{f_{u_n(1-t)}(z - \sum_{i=1}^n y_i)}{f_{u_n}(z)} \nu(\mathrm{d}z), \tag{8.22}$$

and the denominator is given in (8.20). Dividing (8.22) by (8.20) yields (8.17). □

8.2.2 Sum of Marginals

We shall now show that the one-dimensional process $\{R_t\}_{0 \le t \le 1}$ is an LRB.

Proposition 8.2.4. *The process* $\{R_t\}_{0\le t\le 1}$ *is an LRB with law* $LRB_{\mathcal{C}}([0,1],\{f_{tu_n}\},\upsilon)$.

Proof. Since $\{\boldsymbol{\xi}_t\}$ is a Markov process with respect to $\{\mathcal{F}_t^{\boldsymbol{\xi}}\}$, $\{R_t\}$ is a Markov process with respect to $\{\mathcal{F}_t^{\boldsymbol{\xi}}\}$. Thus, we need to verify whether the transition probabilities of $\{R_t\}$ match those of LRBs. We first verify the $\boldsymbol{\xi}_s$-conditional law of R_1. From the Bayes formula,

$$\begin{aligned}\mathbb{Q}(R_1\in \mathrm{d}r\,|\,\boldsymbol{\xi}_s=\mathbf{x}) &= \frac{\frac{1}{f_{u_n}(r)}f_{u_n(1-s)}(r-\sum_{i=1}^n x_i)\,\nu(\mathrm{d}r)}{\int_{-\infty}^{\infty}\frac{1}{f_{u_n}(r)}f_{u_n(1-s)}(r-\sum_{i=1}^n x_i)\,\nu(\mathrm{d}r)}\\ &= \frac{\theta_s(\mathrm{d}r;\sum_{i=1}^n x_i)}{\Psi_s(\sum_{i=1}^n x_i)}.\end{aligned}\tag{8.23}$$

Similarly, from the Bayes formula, the $\boldsymbol{\xi}_s$-conditional law of R_t for $t\in(s,1)$ is

$$\begin{aligned}\mathbb{Q}(R_t\in \mathrm{d}r\,|\,\boldsymbol{\xi}_s=\mathbf{x}) &= \frac{\int_{-\infty}^{\infty}\frac{1}{f_{u_n}(z)}f_{u_n(t-s)}(r-\sum_{i=1}^n x_i)f_{u_n(1-t)}(z-r)\mathrm{d}r\,\nu(\mathrm{d}z)}{\int_{-\infty}^{\infty}\frac{1}{f_{u_n}(z)}\int_{r=-\infty}^{\infty}f_{u_n(t-s)}(r-\sum_{i=1}^n x_i)f_{u_n(1-t)}(z-r)\mathrm{d}r\,\nu(\mathrm{d}z)}\\ &= \frac{\Psi_t(r)}{\Psi_s(\sum_{i=1}^n x_i)}f_{u_n(t-s)}(r-\sum_{i=1}^n x_i)\mathrm{d}r.\end{aligned}\tag{8.24}$$

The denominator of (8.24) is simplified since the densities of Lévy processes are closed under convolution. The transition probabilities match those of LRBs given in (8.9). □

8.2.3 Measure Change

We shall show that the law of an n-dimensional GLP is equivalent to a vector of n independent Lévy processes. First, we assume that under some measure $\widetilde{\mathbb{Q}}$, the process $\{\boldsymbol{\xi}_t\}$ is a vector of n independent Lévy processes such that $\mathbb{Q}(\boldsymbol{\xi}_t\in \mathrm{d}\mathbf{x})=\prod_{i=1}^n f_{t(u_i-u_{i-1})}(x_i)\,\mathrm{d}x_i$. Under $\widetilde{\mathbb{Q}}$, the process $\{R_t\}_{0\le t\le 1}$ is a Lévy process, since the sum of independent Lévy processes is itself a Lévy process. In particular, $\mathbb{Q}(R_t\in \mathrm{d}x)=f_{tu_n}(x)\,\mathrm{d}x$. The filtration $\{\mathcal{F}_t^{\boldsymbol{\xi}}\}$ is generated by $\{\boldsymbol{\xi}_t\}$. We shall show that the process $\{\Psi_t(R_t)\}_{0\le t<1}$ is a martingale, where

$$\Psi_t(R_t)=\int_{-\infty}^{\infty}\frac{f_{u_n(1-t)}(z-R_t)}{f_{u_n}(z)}\,\nu(\mathrm{d}z).\tag{8.25}$$

For times $0\le s<t<1$, we have

$$\begin{aligned}\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\Psi_t(R_t)\,|\,\mathcal{F}_s\right] &= \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\int_{-\infty}^{\infty}\frac{f_{u_n(1-t)}(z-R_s-(R_t-R_s))}{f_{u_n}(z)}\,\nu(\mathrm{d}z)\,\middle|\,\boldsymbol{\xi}_s\right]\\ &= \int_{z=-\infty}^{\infty}\frac{1}{f_{u_n}(z)}\int_{y=-\infty}^{\infty}f_{u_n(1-t)}(z-R_s-y)f_{u_n(t-s)}(y)\,\mathrm{d}y\,\nu(\mathrm{d}z)\\ &= \int_{-\infty}^{\infty}\frac{f_{u_n(1-s)}(z-R_s)}{f_{u_n}(z)}\,\nu(\mathrm{d}z)\\ &= \Psi_s(R_s).\end{aligned}\tag{8.26}$$

Since $\Psi_0(R_0) = 1$ and $\Psi_t(R_t) > 0$, the process $\{\Psi_t(R_t)\}_{0\leq t<1}$ is a Radon-Nikodym density process.

Proposition 8.2.5. *Define a measure* $\mathbb{Q}$ *by*

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\widetilde{\mathbb{Q}}}\right|_{\mathcal{F}_t^{\boldsymbol{\xi}}} = \Psi_t(R_t). \tag{8.27}$$

Under $\mathbb{Q}$*, the process* $\{\boldsymbol{\xi}_t\}_{0\leq t<1}$ *is a GLP with generating law* ν*.*

Proof. We prove by verifying that under $\mathbb{Q}$, the transition law of $\{\boldsymbol{\xi}_t\}$ is that of a GLP:

$$\begin{aligned}\mathbb{Q}\left(\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{x} \,|\, \mathcal{F}_s^{\boldsymbol{\xi}}\right) &= \mathbb{E}^{\mathbb{Q}}[\mathbf{1}\{\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{x}\} \,|\, \mathcal{F}_s^{\boldsymbol{\xi}}] \\ &= \frac{1}{\Psi_s(R_s)}\,\mathbb{E}^{\widetilde{\mathbb{Q}}}[\Psi_t(R_t)\mathbf{1}\{\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{x}\} \,|\, \boldsymbol{\xi}_s] \\ &= \frac{\Psi_t(R_t)}{\Psi_s(R_s)}\prod_{i=1}^{n} f_{(t-s)(u_i-u_{i-1})}(x_i - \xi_s^{(i)})\,\mathrm{d}x_i. \end{aligned} \tag{8.28}$$

Comparing equations (8.28) and (8.17) completes the proof. □

Proposition 8.2.6. *Suppose that* $\{\boldsymbol{\xi}_t\}_{0\leq t\leq 1}$ *is a GLP with generating law* ν *under some measure* $\mathbb{Q}$*. Then*

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\right|_{\mathcal{F}_t^{\boldsymbol{\xi}}} = \Psi_t(R_t)^{-1}, \tag{8.29}$$

defines a probability measure $\widetilde{\mathbb{Q}}$ *for* $t \in [0, 1)$*. Under* $\widetilde{\mathbb{Q}}$*, the process* $\{\boldsymbol{\xi}_t\}_{0\leq t<1}$ *is a vector of* n *independent Lévy processes.*

8.2.4 Liouville Processes

We now introduce a subclass of GLPs that we call Liouville processes and show that ASPs are special cases of Liouville processes. Most of the results presented here about Liouville processes can also be found in Hoyle and Mengütürk (2012). A Liouville process is a Markov process whose increments have multivariate Liouville distributions. Liouville processes display a broader range of dynamics than ASPs. This generalisation comes at the expense of losing the direct connection to Archimedean copulas. However, a Liouville process has a natural link to a Liouville copula, which is defined by the survival copula of a Liouville distribution (see, McNeil and Ne̋slehová, 2010).

Liouville processes are a natural multivariate extension of GRBs, and thus are a flexible tool in the modelling of cumulative processes. Their one-dimensional marginal processes are in general not identically distributed. Also, the marginal processes are increasing and do not exhibit simultaneous large jumps, but they can display strong correlation.

Definition 8.2.7. *Fix $n \in \mathbb{N}_+$, $n \geq 2$, and the vector $\mathbf{m} \in \mathbb{R}_+^n$ satisfying $m_i > 0$, $i = 1, \ldots, n$. Define the strictly increasing sequence $\{u_i\}_{i=1}^n$ by $u_0 = 0$ and $u_i = u_{i-1} + m_i$ for $i = 1, \ldots, n$. Then a process $\{\boldsymbol{\xi}_t\}_{0 \leq t \leq 1}$ is an n-*dimensional Liouville process *if*

$$\{\boldsymbol{\xi}_t\}_{0 \leq t \leq 1} \stackrel{\text{law}}{=} \left\{ \left[\Gamma_{t(u_1)} - \Gamma_0, \cdots, \Gamma_{t(u_n - u_{n-1}) + u_{n-1}} - \Gamma_{u_{n-1}} \right]^\top \right\}_{0 \leq t \leq 1}, \tag{8.30}$$

where $\{\Gamma_t\}_{0 \leq t \leq u_n}$ is a GRB with activity parameter $m = 1$. We say that the generating law of $\{\Gamma_t\}$ is the generating law *of $\{\boldsymbol{\xi}_t\}$ and the* activity parameter *of $\{\boldsymbol{\xi}_t\}$ is $\boldsymbol{m}$.*

Note that allowing the activity parameter of the master GRB to differ from unity in Definition 8.2.7 is equivalent to multiplying the vector $\mathbf{m}$ by a scale factor. Each one-dimensional marginal process of $\{\boldsymbol{\xi}_t\}$ is a GRB with activity parameter m_i, and Definition 8.2.7 ensures that $\boldsymbol{\xi}_t$ is well-defined for each $t \in [0, 1]$.

From Definition 7.1.6, it can be seen that $\boldsymbol{\xi}_1$ has a Liouville distribution. Hence, in the language of McNeil and Nešlehová (2010), $\boldsymbol{\xi}_1$ has a Liouville copula.

We shall provide the transition law, moments, distribution of increments and an independent gamma bridge representation of a Liouville process. Since the proofs are very similar to those of ASPs, we omit them.

First, we define a family of unnormalised measures, indexed by $t \in (0, 1)$ and $x \in \mathbb{R}_+$, as follows:

$$\theta_t(B; x) = \frac{\Gamma[u_n] \mathrm{e}^x}{\Gamma[u_n(1-t)]} \int_B \mathbf{1}_{\{z > x\}} z^{1-u_n} (z - x)^{u_n(1-t)-1} \, \nu(\mathrm{d}z), \tag{8.31}$$

for $B \in \mathcal{B}(\mathbb{R})$ where $u_n = \|\mathbf{m}\|$. We write $\Psi_t(x) = \theta_t([0, \infty); x)$, and also $R_t = \|\boldsymbol{\xi}_t\|$. The process $\{R_t\}$ is a GRB with activity parameter u_n. Given $\boldsymbol{\xi}_s$, the law of R_1 is given in (8.23), and the law of R_t for $t \in (s, 1)$ is

$$\nu_{st}(\mathrm{d}r) = \frac{\Psi_t(r)}{\Psi_s(\|\mathbf{x}\|)} \frac{(r - \|\mathbf{x}\|)^{u_n(t-s)-1} \exp(-(r - \|\mathbf{x}\|))}{\Gamma[u_n(t-s)]} \mathrm{d}r. \tag{8.32}$$

The Liouville process $\{\boldsymbol{\xi}_t\}$ is a Markov process with the transition law given by

$$\mathbb{Q}\left(\xi_1^{(1)} \in \mathrm{d}z_1, \ldots, \xi_1^{(n-1)} \in \mathrm{d}z_{n-1}, \xi_1^{(n)} \in B \,\middle|\, \boldsymbol{\xi}_s = \mathbf{x} \right) =$$
$$\frac{\theta_{\tau(s)}(B + \sum_{i=1}^{n-1} z_i; x_n + \sum_{i=1}^{n-1} z_i)}{\Psi_s(\|\mathbf{x}\|)} \prod_{i=1}^{n-1} \frac{(z_i - x_i)^{m_i(1-s)-1} \mathrm{e}^{-(z_i - x_i)}}{\Gamma[m_i(1-s)]} \mathrm{d}z_i, \tag{8.33}$$

and

$$\mathbb{Q}\left(\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{y} \,|\, \boldsymbol{\xi}_s = \mathbf{x} \right) = \frac{\Psi_t(\|\mathbf{y}\|)}{\Psi_s(\|\mathbf{x}\|)} \prod_{i=1}^{n} \frac{(y_i - x_i)^{m_i(t-s)-1} \mathrm{e}^{-(y_i - x_i)}}{\Gamma[m_i(t-s)]} \mathrm{d}y_i, \tag{8.34}$$

where $\tau(t) = 1 - m_n(1-t)/u_n$, $0 \leq s < t < 1$, and $B \in \mathcal{B}(\mathbb{R})$.

Fix $s \in [0,1)$. The first- and second-order moments of $\boldsymbol{\xi}_t$, $t \in (s,1]$, are

$$\textbf{1.} \quad \mathbb{E}^{\mathbb{Q}}\left[\xi_t^{(i)} \,\middle|\, \boldsymbol{\xi}_s\right] = \frac{m_i}{u_n}\mu_1 + \xi_s^{(i)}, \tag{8.35}$$

$$\textbf{2.} \quad \mathrm{Var}^{\mathbb{Q}}\left[\xi_t^{(i)} \,\middle|\, \boldsymbol{\xi}_s\right] = \frac{m_i}{u_n}\left[\left(\frac{m_i(t-s)+1}{u_n(t-s)+1}\right)\mu_2 - \frac{m_i}{u_n}\mu_1^2\right], \tag{8.36}$$

$$\textbf{3.} \quad \mathrm{Cov}^{\mathbb{Q}}\left[\xi_t^{(i)}, \xi_t^{(j)} \,\middle|\, \boldsymbol{\xi}_s\right] = \frac{m_i m_j(t-s)}{u_n}\left[\frac{\mu_2}{u_n(t-s)+1} - \frac{\mu_1^2}{u_n(t-s)}\right], \quad (i \neq j), \tag{8.37}$$

where we have

$$\mu_1 = \frac{t-s}{1-s}(\mathbb{E}^{\mathbb{Q}}[R_1 \,|\, R_s] - R_s), \tag{8.38}$$

$$\mu_2 = \frac{(t-s)(1+u_n(t-s))}{(1-s)(1+u_n(1-s))}\mathbb{E}^{\mathbb{Q}}[(R_1 - R_s)^2 \,|\, R_s]. \tag{8.39}$$

Fix $s \in [0,1)$. Given $\boldsymbol{\xi}_s$, the increment $\boldsymbol{\xi}_t - \boldsymbol{\xi}_s$, $t \in (s,1]$, has an n-variate Liouville distribution with generating law

$$\nu^*(B) = \nu_{st}(B + R_s), \tag{8.40}$$

and parameter vector $\alpha = (t-s)\mathbf{m}$ for a set $B \in \mathcal{B}(\mathbb{R})$.

Given the value of $\boldsymbol{\xi}_s$, the Liouville process $\{\boldsymbol{\xi}_t\}$ satisfies the following identity in law:

$$\{\boldsymbol{\xi}_t - \boldsymbol{\xi}_s\}_{s \le t \le 1} \overset{\text{law}}{=} \{R^* \, \mathbf{D} \circ \boldsymbol{\gamma}_{t1}\}_{s \le t \le 1}, \tag{8.41}$$

where

1. $\mathbf{D} \in [0,1]^n$ has a Dirichlet distribution with parameter vector $(1-s)\mathbf{m}$;

2. $\{\boldsymbol{\gamma}_{t1}\}$ is a vector of n independent gamma bridges, such that the ith marginal process is a gamma bridge with activity parameter m_i, starting at the value 0 at time s, and terminating with unit value at time 1;

3. $R^* > 0$ is a random variable with law ν^* given by

$$\nu^*(A) = \nu_{s1}(A + R_s), \qquad \text{for} \quad A \in \mathcal{B}(\mathbb{R}); \tag{8.42}$$

4. R^*, $\mathbf{D}$, and $\{\boldsymbol{\gamma}_{t1}\}$ are mutually independent.

Note that in Definition 8.2.7, if we set $m_i = 1$ so that $u_i = u_{i-1} + 1$ for $i = 1, \ldots, n$, this implies that $u_i = i$ for $i = 1, \ldots, n$, since $u_0 = 0$. In other words, setting $u_i - u_{i-1} = 1$ means splitting the time interval $[0, u_n]$ into n equal pieces. Then, comparing Definition 7.2.1 and

Definition 8.2.7, it is clear to see that an n-dimensional Liouville process is an n-dimensional ASP if $m_i = 1$ for $i = 1, \dots, n$.

8.2.5 Standard Variance Gamma Liouville Processes

We shall show the relationship between a particular class of GLPs, which we call Standard Variance Gamma Liouville Processes (SVGLPs), and Liouville processes. Before doing so, we shall briefly provide some background on variance gamma processes, variance gamma bridges and variance gamma random bridges (see Hoyle, 2010).

Let $\{W_t\}$ be a standard Brownian motion, and $\{\gamma_t\}$ be an independent gamma process where $\mathbb{E}^{\mathbb{Q}}[\gamma_1] = \mathrm{Var}^{\mathbb{Q}}[\gamma_1] = m$. A variance gamma (VG) process $\{V_t\}$ is a Brownian motion subordinated with an independent gamma process:

$$V_t = \sigma W_{\gamma_t} + \beta\gamma_t, \tag{8.43}$$

for $\sigma > 0$ and $\beta \in \mathbb{R}$. From this point on, we assume $\{V_t\}$ is a standard VG process with $\sigma = 1$ and $\beta = 0$. That is, $\{V_t\} \overset{\text{law}}{=} \{W_{\gamma_t}\}$. We denote the density of V_t by $f_t^{(m)}$, which is given by (see Madan *et al.*, 1998):

$$f_t^{(m)}(y) = \sqrt{\frac{2}{\pi}} \frac{m^{mt}}{\Gamma[mt]} \left(\frac{y^2}{2m}\right)^{\frac{mt}{2}-\frac{1}{4}} K_{mt-\frac{1}{2}}\left[\sqrt{2y^2m}\right], \tag{8.44}$$

where $K_y[x]$ is the modified Bessel function of the third kind (see Abramowitz and Stegun, 1964).

Let $\{V_{tT}^{(a)}\}$ be the bridge of a standard VG process to the value $a \in \mathbb{R} \setminus \{0\}$ at time T. Then, we have

$$\mathbb{Q}\left[V_{tT}^{(a)} \in \mathrm{d}y \,\middle|\, V_{sT}^{(a)} = x\right] = \frac{f_{t-s}^{(m)}(y-x) f_{T-t}^{(m)}(a-y)}{f_{T-s}^{(m)}(a-x)} \,\mathrm{d}y. \tag{8.45}$$

Following the arguments presented in Hoyle (2010), we can write the following identity in law:

$$\{V_{tT}^{(a)}\} \overset{\text{law}}{=} \{a\widehat{\gamma}_{tT} + H_T\mu(\widehat{\gamma}_{tT} - \gamma_{tT})\}, \tag{8.46}$$

where $\{\widehat{\gamma}_{tT}\}$ and $\{\gamma_{tT}\}$ are identical gamma bridges (with parameter $m > 0$), independent from each other and independent of H_T. The parameter $\mu = \sqrt{m/2}$ and $H_T > 0$ is a random variable with density

$$h \mapsto \mathbf{1}_{\{h>0\}} \frac{m^{2mt}}{\Gamma[mT]^2 f_T^{(m)}(a)} (ha + h^2)^{mT-1} \mathrm{e}^{-m(2h+a)}. \tag{8.47}$$

Now let $\{L_t\}_{0\leq t\leq T}$ be a standard VG random bridge, with terminal law ν, where $\nu(\{0\})=0$. Then, from (8.46), the following can be written:

$$\{L_t\} \overset{\text{law}}{=} \{L_T\widehat{\gamma}_{tT} + H_T\mu(\widehat{\gamma}_{tT}-\gamma_{tT})\} \overset{\text{law}}{=} \{(L_T+H_T\mu)\widehat{\gamma}_{tT} - H_T\mu\gamma_{tT})\}, \tag{8.48}$$

where given L_T, $H_T>0$ is a random variable with density

$$h \mapsto \mathbf{1}_{\{h>(-L_T,0)^+\}}\frac{m^{2mt}}{\Gamma[mT]^2 f_T^{(m)}(L_T)}(hL_T+h^2)^{mT-1}\mathrm{e}^{-m(2h+L_T)}. \tag{8.49}$$

Note that if $L_T+H_T\mu>0$, statement (8.48) suggests that a standard variance gamma bridge is equal in law to the difference of two dependent gamma random bridges. More specifically, if $L_T>0$, from Definition 7.1.12, we can see that

$$(L_T+H_T\mu)\widehat{\gamma}_{tT} = Z\widehat{\gamma}_{tT} \qquad \text{and} \qquad H_T\mu\gamma_{tT} = R\gamma_{tT} \tag{8.50}$$

are dependent gamma random bridges, where $Z=(L_T+H_T\mu)$ and $R=H_T\mu$ are dependent non-negative random variables. This observation motivates us to represent SVGLPs in terms of Liouville processes.

First, we define a SVGLP:

Definition 8.2.8. *Fix $n\in\mathbb{N}_+$, $n\geq 2$, and let $\{u_i\}_{i=1}^n$ be a strictly increasing sequence with $u_0=0$. Then a process $\{\boldsymbol{\xi}_t\}_{0\leq t\leq 1}$ is an n-*dimensional standard VG Liouville process (SVGLP) *if*

$$\{\boldsymbol{\xi}_t\}_{0\leq t\leq 1} \overset{\text{law}}{=} \left\{\left[L_{t(u_1)}-L_0,\cdots,L_{t(u_n-u_{n-1})+u_{n-1}}-L_{u_{n-1}}\right]^\top\right\}_{0\leq t\leq 1}, \tag{8.51}$$

where $\{L_t\}_{0\leq t\leq u_n}$ is a standard VG random bridge.

Set $T=u_n$ and let $L_{u_n}+H_{u_n}\mu>0$. Denote by ν_Z the law of Z, and ν_R the law of R. Also let $\{\boldsymbol{\xi}_t^Z\}_{0\leq t\leq 1}$ and $\{\boldsymbol{\xi}_t^R\}_{0\leq t\leq 1}$ be Liouville processes with generating laws ν_Z and ν_R, respectively.

Proposition 8.2.9. *Fix $n\in\mathbb{N}_+$, $n\geq 2$, and the vector $\mathbf{m}\in\mathbb{R}_+^n$ satisfying $m_i>0$, $i=1,\ldots,n$. Define the strictly increasing sequence $\{u_i\}_{i=1}^n$, where $u_0=0$ and $u_i=u_{i-1}+m_i$ for $i=1,\ldots,n$. Then, the n-dimensional SVGLP $\{\boldsymbol{\xi}_t\}_{0\leq t\leq 1}$ satisfies*

$$\{\boldsymbol{\xi}_t\}_{0\leq t\leq 1} \overset{\text{law}}{=} \{\boldsymbol{\xi}_t^Z-\boldsymbol{\xi}_t^R\}_{0\leq t\leq 1}, \tag{8.52}$$

where $\{L_t\}_{0\leq t\leq u_n}$ is a standard VG random bridge.

Proof. Note that for each of the marginals of the Standard VG Liouville process $\{\boldsymbol{\xi}_t\}$, the following can be written:

$$
\begin{aligned}
\{\xi_t^{(i)}\} &\overset{\text{law}}{=} \{L_{t(u_i-u_{i-1})+(u_{i-1})} - L_{u_{i-1}}\} \\
&\overset{\text{law}}{=} \{(Z\widehat{\gamma}_{t(u_i-u_{i-1})+(u_{i-1}),u_n} - R\gamma_{t(u_i-u_{i-1})+(u_{i-1}),u_n}) - (Z\widehat{\gamma}_{u_{i-1},u_n} - R\gamma_{u_{i-1},u_n})\} \\
&\overset{\text{law}}{=} \{Z(\widehat{\gamma}_{t(u_i-u_{i-1})+(u_{i-1}),u_n} - \widehat{\gamma}_{u_{i-1},u_n}) - R(\gamma_{t(u_i-u_{i-1})+(u_{i-1}),u_n} - \gamma_{u_{i-1},u_n})\} \\
&\overset{\text{law}}{=} \{(\xi_t^{(i)})^Z - (\xi_t^{(i)})^R\},
\end{aligned}
\tag{8.53}
$$

for $0 \leq t \leq 1$, where by Definition 8.2.7, $\{(\xi_t^{(i)})^Z\}$ and $\{(\xi_t^{(i)})^R\}$ are the marginals of the Liouville processes $\{\boldsymbol{\xi}_t^Z\}$ and $\{\boldsymbol{\xi}_t^R\}$, with generating laws ν_Z and ν_R, respectively. The statement follows. □

Since ASPs are special cases of Liouville processes, SVGLPs can also be represented in terms of ASPs:

Remark 8.2.10. *Fix $n \in \mathbb{N}_+$, $n \geq 2$. Set $u_i = u_{i-1} + 1$ for $i = 1, \ldots, n$ with $u_0 = 0$. Then, the n-dimensional SVGLP $\{\boldsymbol{\xi}_t\}_{0\leq t\leq 1}$ satisfies*

$$
\{\boldsymbol{\xi}_t\}_{0\leq t\leq 1} \overset{\text{law}}{=} \{\boldsymbol{\xi}_t^Z - \boldsymbol{\xi}_t^R\}_{0\leq t\leq 1}, \tag{8.54}
$$

where $\{L_t\}_{0\leq t\leq n}$ is a standard VG random bridge, and $\{\boldsymbol{\xi}_t^Z\}_{0\leq t\leq 1}$ and $\{\boldsymbol{\xi}_t^R\}_{0\leq t\leq 1}$ are ASPs with generating laws ν_Z and ν_R, respectively.

8.3 An Information-Based Perspective

GLPs allow us to model a rich class of dependence structures between cash flows that have a generalised multivariate Liouville distribution. Hence, one can model an information-driven dependence structure for a vector of assets, where the law of a GLP determines the distribution of the asset prices at a given time.

We shall briefly demonstrate the use of GLPs in an information-based model. First, on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{Q})$, we let the probability measure $\mathbb{Q}$ be the pricing measure. We introduce $\mathbf{X}_1 \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{Q})$ as an n-dimensional random vector with state-space $(\mathbb{X}^n, \mathcal{B}(\mathbb{X}^n))$, where $\mathbb{X}^n \subset \mathbb{R}^n$.

We assume that

$$
\mathbf{X}_1 = [X_1^{(1)}, \ldots, X_1^{(n)}]^\top \tag{8.55}
$$

is a vector of n cash flows with values $X_1^{(1)}, \ldots, X_1^{(n)}$ at time $T = 1$. One can introduce $\mathbf{w} = [w_1, \ldots, w_n] \in \mathbb{R}_+^n$ as a vector of number of shares associated to each cash flow, and view $\mathbf{w}\mathbf{X}_1$ as a portfolio of assets.

We assume that $\mathbf{X}_1$ has a generalised Liouville distribution and the market receives partial information about $\mathbf{X}_1$. We let $\{\mathcal{F}_t^{\boldsymbol{\xi}}\}$ be the market filtration generated by a GLP $\{\boldsymbol{\xi}_t\}_{0\leq t\leq 1}$, such that $\xi_1^{(1)} = X_1^{(1)}, \ldots, \xi_1^{(n)} = X_1^{(n)}$.

The prices of the cash flows, which we denote by $\mathbf{X}_t$, are given by

$$\begin{aligned}\mathbf{X}_t &= P_{tT}\mathbb{E}^{\mathbb{Q}}\left[\mathbf{X}_1 \,\middle|\, \mathcal{F}_t^{\boldsymbol{\xi}}\right] \\ &= P_{tT}\left[\mathbb{E}^{\mathbb{Q}}[X_1^{(1)}\,|\,\boldsymbol{\xi}_t], \ldots, \mathbb{E}^{\mathbb{Q}}[X_1^{(n)}\,|\,\boldsymbol{\xi}_t]\right]^{\top}, \end{aligned} \tag{8.56}$$

for $0 \leq t < 1$. In order to proceed further, we define $\mathcal{Q}(\mathbb{X})$-valued stochastic processes $\{\pi_t^i\}_{t\in[0,1]}$, $i = 1, \ldots, n$, by

$$\pi_t^i(A) = \mathbb{Q}\left(X_1^{(i)} \in A \,\middle|\, \mathcal{F}_t^{\boldsymbol{\xi}}\right) = \mathbb{Q}(X_1^{(i)} \in A\,|\,\boldsymbol{\xi}_t), \tag{8.57}$$

for $A \in \mathcal{B}(\mathbb{X})$. Using the random probability measure π_t^i, the time-t price of $X_1^{(i)}$ is given by

$$X_t^{(i)} = P_{tT}\mathbb{E}^{\mathbb{Q}}\left[X_1^{(i)}\,|\,\boldsymbol{\xi}_t\right] = P_{tT}\int_{\mathbb{X}} z^i \pi_t^i(\mathrm{d}z_i), \tag{8.58}$$

for $0 \leq t < 1$ and $i = 1, \ldots, n$.

Note that the measure-valued processes $\{\pi_t^i\}$ and $\{\pi_t^j\}$ are dependent. It follows that the law of the multivariate information $\boldsymbol{\xi}_t$ determines the distribution of asset prices at time t. Hence, GLPs allow us to model a broad range of information-driven dependence structures between assets.

Many subclasses of GLPs can be analyzed in more detail. As an example, one may study the properties of what one may call Brownian Liouville processes constructed from Brownian information processes. Perhaps another interesting process to analyze is what one may call a Poisson Liouville process constructed from a Poisson random bridge. We leave a formal analysis of such processes for further research.

Chapter 9

Conclusion

This final section presents a brief summary of the thesis and provides a general overview of the work by including the objectives, approaches and some of the achievements. Also, we briefly discuss some ideas for future research.

This work is comprised of three main themes within the information-based asset pricing framework of Brody, Hughston and Macrina (BHM): (i) regime-switching information, (ii) information asymmetry, and (iii) multivariate dependence modelling. We shall consider each theme seperately:

9.1 Regime-Switching

Our objective is to develop an information-driven regime-switching framework that allows us to derive a rich class of asset price dynamics and to price financial derivatives. It is our aim to build a framework that is both analytically tractable and financially interpretable. Our motivation arises from the fact that sudden changes in market information may cause asset prices to jump. Also, significant changes in market information may coincide with regime switches. Hence, we extend the BHM framework by considering filtrations driven by regime-switching information sources. In this extended framework: (i) there may be regimes where no new information enters the market, (ii) at the point of switching jumps may appear in the asset price, (iii) jumps can propagate into the volatility of asset returns, and (iv) the effective flow rate of information into the market may increase or decrease.

As an example, we are able to show that under switching Brownian information processes, the asset price process has jump-diffusion dynamics. We see that during each regime, the price process is governed by a different Brownian motion and a different stochastic volatility process. In fact, it is a natural outcome of our framework that the stochastic volatility of the price process may jump at regime switches. We also extend our regime-switching framework to the multiple market factor setting. More precisely, we allow the possibility that each economic variable which determines the value of an asset is subject to different regime switches, and provide mathematical expressions for the asset price processes. In addition, we price European options and credit-based products under regime-switching economies. For example, when regime switches coincide with jumps of a Poisson process, we show that the option price takes a form very similar to what Merton (1976) presents in his jump-diffusion model. Since our pricing formula admits any reasonable distribution for regime switches, we are able to generate a large class of option prices. We also show that CDS prices may jump at every regime switch, which means that the probability of default that the market assigns to a risky bond changes in a discontinuous way.

9.2 Information Asymmetry

Our aim is to quantify the impact of changes in information sources. This includes measuring the information asymmetry between the market and an informed trader, the information asymmetry between two informed traders, and the information gap between the market-implied view of an asset and its fundamentals. In order to achieve our objective, we develop the concept of an n-order piecewise enlargement of the market filtration to model the information set of an informed trader. Then we use information-theoretic and geometric measures to quantify information asymmetry, which in turn quantify the impact of changes in information sources. We also consider a single information-based model where the view of the market towards the value of an asset is different from the fundamental value of that asset.

We derive the dynamics of information asymmetry processes in various models. These processes jump at every activation of a new information source. We construct the information asymmetry processes based on the following measures: (i) Kullback-Leibler, (ii) Squared-Hellinger, and (iii) Fisher-Rao. The reasons we choose these measures are as follows: In information theory, the Kullback-Leibler divergence is widely used to measure the information gain when passing from a prior distribution to a posterior distribution. Since we have information jumps in our framework, the Kullback-Leibler divergence presents itself as a good candidate in measuring the difference between the information content before and after a jump, thus, quantifying the impact of the activation of an information source. We introduce the use of the Fisher-Rao metric in our analysis due to its mathematical link with the Brownian information process when the value of an asset has a Gaussian distribution. More precisely, when we work with Gaussian distributions, we can determine points on a Riemannian manifold in which the Riemannian metric is the Fisher-Rao metric. Therefore, the Fisher-Rao metric is a natural choice when quantifying the distance between two Gaussian distributions determined by different sets of information. The reason why we use the Squared-Hellinger divergence is two-fold. Not only is it commonly used in information theory to measure the distance between two different distributions, but it also brings forth a geometric perspective due to its link with the unit sphere. Thus, the Squared-Hellinger measure is a smooth transition from an information-theoretic setting to a geometric setting. In fact, motivated by this, we are able to show a relationship between the Squared-Hellinger divergence and an isometric invariant of the Poincaré disc under the action of the general Möbius group.

We are able to provide the dynamics for the asymmetry between two informed traders who have differing access to information. In particular, we consider two informed agents who have additional access to information compared to the market, but they have access at different stopping times. This leads to a dynamic interplay between the amount of information that the two informed agents have until the revelation of the value of an asset. If one of the agents has access to more information sources at a given time, then that agent has an informational advantge over the other. If both agents have equal access, then the information asymmetry between them is zero. We are also able to provide the dynamics of market mispricing and the ensuing correction following the arrival of fundamental information. In order to do this, we assume that the market is initially provided with partial information about a cash-flow that will not be paid. That is, the market has incorrect expectations about the value of an asset. At the time when the market receives the information process about the true value of the asset, the asymmetry between the market and the fundamentals jump to zero. This represents a sudden market correction.

9.3 Multivariate Dependence

One of our main objective is to generalise the gamma random bridges to the multivariate Archimedean survival processes (ASPs). We explore their deep links with Archimedean copulas, and provide various characterisations of ASPs. We then discuss further generalisations under what we call Generalised Liouville Processes (GLPs). Our approach in constructing these multivariate processes relies on splitting Lévy random bridges into non-overlapping subprocesses. Since these subprocesses are themselves Lévy random bridges, GLPs can be regarded as multivariate information processes.

We manage to provide numerous results about ASPs. For example, we show that there is a bijection between ASPs and Archimedean copulas. We characterise ASPs as Markov processes through their transition laws, and through their finite-dimensional distributions. We show that they are processes equivalent in law to multivariate gamma processes, and we detail the associated measure change. We are also able to provide an independent-gamma-bridges representation of ASPs. Then we generalise ASPs to Liouville processes. Liouville processes are also constructed from gamma random bridges, but we allow more flexibility in our splitting mechanism. Finally, we present further generalisations and introduce GLPs, which are constructed from arbitrary Lévy random bridges. We provide several characterisations for GLPs, and discuss their use in multi-factor information-based models. More precisely, we consider a market filtration generated by a GLP, where each marginal process carries partial information about an asset. This allows us to introduce information-driven dependence structures across assets.

9.4 Future Research

This thesis offers future research within the three themes mentioned above. We shall briefly discuss them.

In our work, we consider stopping times that are independent of the information processes. This brings forth a level of parsimony and tractability for deriving the stochastic differential equations of price processes and pricing financial derivatives. One natural extension is to relax the independence assumption, and model economies where regime switches depend on information. For instance, we can allow the stopping times to be dependent on the value of the asset and independent of the market noise. Then we can work with conditional independence, instead of complete independence, and would be able to derive dynamics exhibiting even richer price behaviour. In addition, we mainly detail the case when the stopping times are inaccessible, since we model them by the jump times of Heaviside processes. Hence, jumps in asset prices are sudden and unexpected. However, our framework by construction admits the use of previsible stopping times as well. For example, we can model stopping times as the first hitting times of continuous processes, which would allow us to introduce previsible regime switches.

Another potentially fruitful extension arises from the choice of information processes that generate the market filtration. In our work regarding regime switches, we only consider Brownian information processes. But what if different regimes are characterised by information processes that have different laws? More precisely, what if different Lévy random bridges are active during different regimes? Answering these questions offers the flexibility to represent regime switches as jumps from one law to another. This should lead to a framework that admits the derivation of a large class of asset price dynamics under regime switches.

Some extensions on derivatives pricing can also be made. Hoyle *et al.* (2011) provide pricing formulas for European options when the market filtration is generated by an arbitrary Lévy random bridge. These results are highly promising building blocks to generate a large class of option prices under regime-switching economies. In addition, in CDS pricing, what if the recovery rates are random and depend on the economic regime? In order to answer this, we can model recovery rates as functions of information processes that characterise different regimes. One can then generate recovery rate processes that jump at every regime switch.

Further extensions can be made on our analysis on information asymmetry by the use of n-order piecewise enlargements of filtrations. A detailed analysis of n-order piecewise enlargements and their applications to finance offer a potentially prolific route to follow. For instance, we can apply n-order piecewise enlargements to utility maximization problems for insider trading. In addition, we may find other natural relationships between Riemannian manifolds and information processes to quantify information asymmetry geometrically.

Finally, since GLPs form a large class of stochastic processes, they offer a wide range of special examples. We can analyse these special cases in more detail. For instance, we can introduce Brownian Liouville Processes constructed from Brownian information processes, or Poisson Liouville Processes constructed from Poisson random bridges. This would allow us to introduce many relevant financial applications. It should also be possible to use these processes within the regime-switching framework. Then, we can develop an extensive information-based framework which enables us to start discussing about dependence structures that change under different regimes.

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