Robust observer design under measurement noise with gain adaptation and saturated estimates ⋆

Stefano Battilotti [a]

[a] *Dipartimento di Ingegneria Informatica, Automatica e Gestionale "Antonio Ruberti" (e-mail: battilotti@dis.uniromH1.it).*

Abstract

We use incremental homogeneity, gain adaptation and incremental observability for proving new results on robust observer design for systems with noisy measurement and bounded trajectories. A state observer is designed by dominating the incrementally homogeneous nonlinearities of the observation error system with its linear approximation, while gain adaptation and incremental observability guarantee an asymptotic upper bound for the estimation error depending on the limsup of the norm of the measurement noise. A characteristic and innovative feature of this observer is the mixed low/high-gain structure in combination with saturated state estimates and dynamically tuned gains and saturation levels. The gain adaptation is implemented as the output of a stable filter using the squared norm of the measured output estimation error and the mismatch between each estimate and its saturated value.

Key words: measurement noise, robust observers, gain adaptation, saturated estimates.

1 Introduction

Homogeneity and homogeneous approximations have been investigated by many authors for the stability analysis of an equilibrium point: see e.g. the first contributions Massera (1956) and, more recently, Kawski (1989) and Rosier (1998). The homogeneity property has been exploited in the design of global state observers (Qian (2005), Qian & Lin (2006), Yang & Lin (2003), Andrieu et al. (2008)): the idea is to design a state observer for the homogeneous approximation of the system and convergence to zero of the estimation error is preserved under any perturbation which does not change the homogeneous approximation. The class of systems for which an observer can be designed by domination techniques has been enlarged by adding dynamic gain adaptation (Khalil & Saberi (1987), Bullinger & Allgower (1997), Lei et al. (2005), Astolfi & Praly (2006), Andrieu et al. (2009)). The class of homogeneous systems has been enlarged by introducing (incremental) homogeneity in the upper bound in Battilotti (2014) and used together with gain adaptation and self-tuned saturations for designing global observers in Battilotti (2015a) for systems with bounded trajectories. Homogeneity in the upper bound gives enough a general framework for including triangular structures (feedback and feedforward systems), homogeneous and interlaced structures. Self-tuned saturations were previously used in Lei et al. (2005) in the observer design for feedback-linearizable systems with bounded trajectories. However, the gain adaptation is such that the dynamically adapted gain is non-decreasing along solutions. As known, this may lead to serious growth problems in the presence of measurement disturbance (Egardt (1979, Example 4.2), Peterson & Narendra (1982), Mareels (1984), Khalil & Saberi (1987)). This problem has been addressed by several authors (Egardt (1979), Mareels (1984), Peterson & Narendra (1982), Ioannou & Kokotovic (1984)), trying to reduce the adapted gain instead to let it grow with no bound, for example when the measured output estimation error is decreasing. In Vasilijevic & Khalil (2006) it is shown that measurement disturbance introduces an upper bound on the gain when good estimation performances are required. In this direction, we find the works of Ahrens & Khalil (2006), which relies on the knowledge of a bound for the nonlinearities of the system, and Boizot et al. (2010), which relies on the knowledge of a bound for the dynamic gain and the Lipschitz constant of the nonlinearities of the system. The effect of measurement disturbance on observer design has been studied, following Boizot et al. (2010), for a class of lower triangular systems with bounded trajectories and for a given class of observers in Sanfelice & Praly (2011), satisfying additional properties on the mismatch between the vector fields of the system and of

⋆ This work was supported by MIUR. This paper was not presented at any IFAC meeting.

the observer, by proving an upper bound (depending on the measurement noise) for the estimation error in the mean and an upper bound on the limsup of the estimation error in the mean. In the absence of measurement noise, this last bound can be made arbitrarily small by setting properly the parameters of the class of observers. This, however, does not discard a potential oscillatory behavior of the estimates (Mareels et al. (1999)).

In this paper, we prove new results on robust observer design in the presence of measurement disturbance for systems with bounded trajectories by using incremental homogeneity in the upper bound (Battilotti (2014)) and gain adaptation (Andrieu et al. (2008), Bullinger & Allgower (1997), Khalil & Saberi (1987), Lei et al. (2005)) with saturated estimates and dynamically tuned saturation levels (Lei et al. (2005))). A state observer is designed by dominating the incrementally homogeneous (in the upper bound) nonlinearities of the observation error system with its linear approximation. The gain adaptation and updating of the saturation levels is implemented through a stable filter which regulates its output by using a suitable function of the squared norm of the measured output estimation error. Our observer guarantees an upper bound on the limsup of the norm of the estimation error depending on the limsup of the norm of the measurement noise. As a particular case, if the measurement disturbance tends asymptotically to zero the estimation error itself tends to zero.

The paper is organized as follows. In section 2 some notation is introduced. In section 3 the class of system is described and the problem is formulated. In section 4 an observer is presented together with the main result and the parameter observer design is discussed in section 4.1. In section 4.2 example and simulation are given and in section 4.3 the main result is proved. In the appendix the notion of incremental generalized homogeneity is shortly recalled together with some of its properties and related results.

2 Notation

(N1) $\mathbb{R}^n$ (resp. $\mathbb{R}^{n\times n}$) is the set of n-dimensional real column vectors (resp. $n\times n$ matrices). $\mathbb{R}_\geqslant$ (resp. $\mathbb{R}^n_\geqslant$, $\mathbb{R}^{n\times n}_\geqslant$) denotes the set of real non-negative numbers (resp. vectors in $\mathbb{R}^n$, matrices in $\mathbb{R}^{n\times n}$, with real non-negative entries). $\mathbb{R}_>$ (resp. $\mathbb{R}^n_>$) denotes the set of real positive numbers (resp. vectors in $\mathbb{R}^n$ with real positive entries). $\lambda_{min}(A)$ (resp. $\lambda_{max}(A)$) denotes the minimum (resp. maximum) eigenvalue of $A\in\mathbb{R}^{n\times n}$.

(N2) For any matrix $V\in\mathbb{R}^{p\times n}$ we denote by V_{ij} the (i,j)-th entry of V and for any vector $v\in\mathbb{R}^n$ we denote by v_i the i-th element of v. We retain a similar notation for functions. For any $v\in\mathbb{R}^n$ we denote by $\mathrm{diag}\{v\}$ the diagonal $n\times n$ matrix with diagonal elements $v_1,\ldots,v_n$. Also, $|a|$ denotes the absolute value of $a\in\mathbb{R}$, $\|a\|$ (resp. $\|a\|_P$) denotes the euclidean (resp. weighted by P) norm of $a\in\mathbb{R}^n$, $\|A\|$ denotes the norm of $A\in\mathbb{R}^{n\times n}$ induced from the euclidean norm $\|\cdot\|$ and $\langle\!\langle a\rangle\!\rangle$ the column vector of the absolute values of the elements of $a\in\mathbb{R}^n$, i.e. $\langle\!\langle a\rangle\!\rangle := (|a_1|\cdots|a_n|)^T$.

(N3) We denote by $\mathbf{C}^j(\mathscr{X},\mathscr{Y})$, with $j\geqslant 0$, $\mathscr{X}\subset\mathbb{R}^n$ and $\mathscr{Y}\subset\mathbb{R}^p$, the set of j-times continuously differentiable functions $f:\mathscr{X}\to\mathscr{Y}$, $\mathbf{C}^0_0(\mathscr{X},\mathscr{Y})$ the set of uniformly continuous functions $f:\mathscr{X}\to\mathscr{Y}$, by $\mathbf{L}^\infty(\mathbb{R}_\geqslant,\mathscr{Y})$ the set of functions $f\in\mathbf{C}^0(\mathbb{R}_\geqslant,\mathscr{Y})$ such that $\sup_{\theta\geqslant 0}\|f(\theta)\|<+\infty$ and by $\mathbf{L}^j(\mathbb{R}_\geqslant,\mathscr{Y})$, with $j\geqslant 1$, the set of $f\in\mathbf{C}^0(\mathbb{R}_\geqslant,\mathscr{Y})$ such that $\int_0^\infty\|f(\theta)\|^j d\theta<+\infty$. For each $d\in\mathbf{L}^\infty(\mathbb{R}_\geqslant,\mathscr{Y})$, we have the sup norm of d defined as $\|d\|_\infty := \sup_{t\geqslant 0}\|d(t)\|$. Moreover, $\mathcal{K}_0$ denotes the set of functions $f\in\mathbf{C}^0(\mathbb{R}_\geqslant,\mathbb{R}_\geqslant)$, strictly increasing with $f(0)\geqslant 0$ and $\mathcal{K}$ denotes the set of functions $f\in\mathcal{K}_0$ such that $f(0)=0$.

(N4) A *saturation function* $\mathbf{sat}_h(\cdot)$ *with levels* $h\in\mathbb{R}^n_>$ is a function $\mathbf{sat}_h(x) := (\mathbf{sat}_{h_1}(x_1),\ldots,\mathbf{sat}_{h_n}(x_n))^T$ such that for each $i=1,\ldots,n$ and $x_i\in\mathbb{R}$:

$$\mathbf{sat}_{h_i}(x_i)\begin{cases} x_i & |x_i|\leqslant h_i\\ \mathrm{sign}(x_i)h_i & \text{otherwise.}\end{cases} \tag{1}$$

(N5) For any vectors $x\in\mathbb{R}^n,\mathfrak{r}\in\mathbb{R}^n_>$ and $\epsilon\in\mathbb{R}_>$, we define

$$\epsilon^{\mathfrak{r}} := (\epsilon^{\mathfrak{r}_1},\cdots,\epsilon^{\mathfrak{r}_n})^T,\ \epsilon^{\mathfrak{r}}\diamond x := (\epsilon^{\mathfrak{r}_1}x_1,\cdots,\epsilon^{\mathfrak{r}_n}x_n)^T \tag{2}$$

viz. $\epsilon^{\mathfrak{r}}\diamond x$ is the dilation of a vector x with weights $\mathfrak{r}$. Note that for any $x,y\in\mathbb{R}^n,\mathfrak{r}_1,\mathfrak{r}_2\in\mathbb{R}^n_>$ and $\epsilon\in\mathbb{R}_>$

$$\epsilon^{\mathfrak{r}_1}\diamond\epsilon^{\mathfrak{r}_2}\diamond x=\epsilon^{\mathfrak{r}_2}\diamond\epsilon^{\mathfrak{r}_1}\diamond x=\epsilon^{\mathfrak{r}_1+\mathfrak{r}_2}\diamond x, \tag{3}$$

$$(\epsilon^{\mathfrak{r}_1}\diamond x)^T(\epsilon^{\mathfrak{r}_2}\diamond y)=(\epsilon^{\mathfrak{r}_2}\diamond x)^T(\epsilon^{\mathfrak{r}_1}\diamond y)$$
$$=(\epsilon^{\mathfrak{r}_1+\mathfrak{r}_2}\diamond x)^Ty=x^T(\epsilon^{\mathfrak{r}_1+\mathfrak{r}_2}\diamond y) \tag{4}$$

(N6) for any vectors $x,y\in\mathbb{R}^n$ we write $x\preceq y$ if and only if $x_i\leqslant y_i$ for all $i=1,\ldots,n$. We retain the same notation for matrices $A,B\in\mathbb{R}^{n\times n}$: $A\preceq B$ if and only if $A_{ij}\leqslant B_{ij}$ for all $i,j=1,\ldots,n$. On the other hand $A\geqslant B$ (resp. $A>B$) for matrices $A,B\in\mathbb{R}^{n\times n}$ if and only if $A-B$ is positive semidefinite (resp. positive definite).

3 Main assumptions and problem statement

Consider the system

$$\dot{x}=f(x):=[A+BF+HC]x+\phi(x), \tag{5}$$
$$y=h(x,d):=Cx+\psi(x)+d \tag{6}$$

with state $x \in \mathbb{R}^n$, measurement $y \in \mathbb{R}$ and disturbance $d \in \mathbb{R}$. The triple (A, B, C) is in prime form:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{7}$$

$$C = (1 \quad 0 \quad \cdots \quad 0 \quad 0) \tag{8}$$

with $F \in \mathbb{R}^{1\times n}$ and $H \in \mathbb{R}^{n\times 1}$. Moreover, ϕ and ψ are locally Lipschitz continuous with $\phi(0) = 0$, $\psi(0) = 0$, $\frac{\partial \phi}{\partial x}(0) = 0$ and $\frac{\partial \psi}{\partial x}(0) = 0$ so that $\dot{x} = [A + BF + HC]x$, $y = Cx + d$, represents the linear approximation of (5)-(6) around the origin. Motivations for considering $\dot{x} = [A + BF + HC]x$, $y = Cx + d$ as the linear approximation of (5)-(6) around the origin rely in the fact that any linear single-output system is equivalent under coordinate transformations to $\dot{x}_1 = (A + BF_1 + H_1C)x_1 + BF_2x_2, \dot{x}_2 = H_2Cx_1 + Gx_2$, $y = Cx_1$ where (A, B, C) is in prime form and $\dot{x}_2 = Gx_2$ is the zero-dynamics. Therefore, for simplicity and to focus on main results we are neglecting in (5)-(6) the zero dynamics of its linear approximation around the origin. We can also assume without loss of generality that $B^T H = 0$.

We consider in (5)-(6) the class $\mathcal{D}(\Delta)$ of disturbances $d \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R})$ such that $\|d\|_\infty \leqslant \Delta$ and uniformly continuous on their domain. The problem is to give an estimate of the state of (5) using only the noisy measurement (6). Our assumptions on the class of systems (5)-(6) are the following ones (see the appendix for few recalls on incremental homogeneity in the upper bound which we will abbreviate as i.h.u.b. throughout the paper):

(H0) (*incremental homogeneity*)

(i) $C^T\psi$ and $A^T(\phi + HC)$ are incrementally homogeneous in the upper bound (i.h.u.b.) with quadruples $(\mathfrak{r}, \mathfrak{r} - \mathfrak{g}, \mathfrak{g}, C^T\psi_U)$ and, respectively, $(\mathfrak{r}, \mathfrak{r} - \mathfrak{g}, \mathfrak{g}, A^T(\phi_U + H_UC))$, with $\phi_U(0, 0) = 0$ and $\psi_U(0, 0) = 0$ for some $H_U \in \mathbb{R}^{n\times 1}$,

(ii) $(I - AA^T)(\phi + BF)$ is i.h.u.b. with quadruple $(\mathfrak{r}, (I - AA^T)(\mathfrak{r} + \mathfrak{g}), \mathfrak{g}, (I - AA^T)(\phi_U + BF_U)$ for some $F_U \in \mathbb{R}^{1\times n}$,

(iii) the degrees $\mathfrak{g}$ and weights $\mathfrak{r}$ satisfy for each $j = 2, \ldots, n$

$$2(\mathfrak{g}_j - \mathfrak{g}_{j-1}) + \mathfrak{g}_{j-1} + \mathfrak{r}_{j-1} \leqslant \mathfrak{r}_j - \mathfrak{g}_j \leqslant \mathfrak{g}_{j-1} + \mathfrak{r}_{j-1},$$

(H1) (*state boundedness*) $x(\cdot, x_0) \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R})$ for each $x_0 \in \mathbb{R}^n$, where $x(t, x_0)$ is the solution of (5) with initial condition x_0.

Remark 1 *Assumption (H0) captures a large class of nonlinear systems (5)-(6) and it is suitable for characterizing at the same time triangular and interlaced structures, in particular:*

(i) lower triangular vector fields ϕ:

$$\phi(x) := (\phi_1(x_1), \cdots, \phi_n(x_1, x_2, \ldots, x_n))^T$$

with $\psi(x) = 0$, where each ϕ_j, $j = 1, \ldots, n$, is a sum of terms having the form $x_{j_1}^{t_{j_1}} \cdots x_{j_l}^{t_{j_l}}$ for some reals $t_{j_i} \geqslant 0$ such that $\sum_i t_{j_i} > 1$. For example, in the case of $\phi(x) := (ax_1^{\frac{3}{2}}, bx_1^2x_2^3)^T$, $a, b \in \mathbb{R}$, (H0) is met with $\mathfrak{r} = (1, 1)^T$, $\mathfrak{g} = (6, 2)^T$;

(ii) strict upper triangular vector fields ϕ:

$$\phi(x) := (\phi_1(x_3, \ldots, x_n), \cdots, \phi_{n-2}(x_n), 0, 0)^T$$

with $\psi(x) = \psi_0(x_2, \ldots, x_n)$, where each ϕ_j, $j = 1, \ldots, n-2$, and ψ_0 is a sum of terms having the form $x_{j_1}^{t_{j_1}} \cdots x_{j_l}^{t_{j_l}}$ for some reals $t_{j_i} \geqslant 0$ such that $\sum_i t_{j_i} > 1$. For example in the case of $\phi(x) := (ax_3x_4, bx_4^2, 0, 0)^T$ and $\psi(x) := cx_2x_4$, $a, b, c \in \mathbb{R}$, (H0) is met with $\mathfrak{r} = (8, 6, 4, 1)^T$, $\mathfrak{g} = (-1, -1, -1, -2)^T$;

(iii) homogeneous (in the classical sense: Rosier (1998)) vector fields ϕ , resp. functions ψ, with weights $\mathfrak{r}$ such that $\mathfrak{r}_{j+1} - \mathfrak{r}_j = 2\mathfrak{g}_0$ for all $j = 1, \ldots, n-1$ and homogeneity degree $2\mathfrak{g}_0$, resp. 0. For example in the case of $\phi(x) := (x_1^{3/2}, ax_2^{4/3})^T$ and $\psi(x) := 0$, $a \in \mathbb{R}$, (H0) is met with $\mathfrak{r} = (1, 3/2)^T$, $\mathfrak{g} = (1/4, 1/4)^T$.

(iv) non-triangular vector fields ϕ, for example $\phi(x) := (0, x_1^2x_4 + x_3, 0, 0)^T$.

It is not difficult to check for assumption (H0). In general, it amounts to solve a set of algebraic inequalities in the unknowns $\mathfrak{r} \in \mathbb{R}^n_>$ and $\mathfrak{g} \in \mathbb{R}^n$. For example, if $\phi(x) := (ax_1^{\frac{3}{2}}, bx_1^2x_2^3)^T$, $a, b \in \mathbb{R}$, $\psi(x) := 0$, $F = 0$ and $H = 0$ we must have with $\mathfrak{r}_1, \mathfrak{r}_2 > 0$

(a) $\frac{3}{2}\mathfrak{r}_1 \leqslant \mathfrak{r}_2 - \mathfrak{g}_2 + \mathfrak{g}_1$ for (i) of (H0), which amounts to satisfy $|(\epsilon^{\mathfrak{r}_1}w_1')^{\frac{3}{2}} - (\epsilon^{\mathfrak{r}_1}w_1'')^{\frac{3}{2}}| \leqslant |(w_1')^{\frac{3}{2}} - (w_1'')^{\frac{3}{2}}|\epsilon^{\mathfrak{r}_2 - \mathfrak{g}_2 + \mathfrak{g}_1}$ for all $w_1', w_1'' \in \mathbb{R}$ and $\epsilon \geqslant 1$;

(b) $2\mathfrak{r}_1 + 3\mathfrak{r}_2 \leqslant \mathfrak{r}_2 + \mathfrak{g}_2 + \mathfrak{g}_1$ and $2\mathfrak{r}_1 + 3\mathfrak{r}_2 \leqslant \mathfrak{r}_2 + 2\mathfrak{g}_2$ for (ii) of (H0), which amounts to satisfy $|(\epsilon^{\mathfrak{r}_1}w_1')^2(\epsilon^{\mathfrak{r}_2}w_2')^3 - (\epsilon^{\mathfrak{r}_1}w_1'')^2(\epsilon^{\mathfrak{r}_2}w_2'')^3| \leqslant \epsilon^{\mathfrak{r}_2+\mathfrak{g}_2+\mathfrak{g}_1}|(w_1')^2 - (w_1'')^2||w_2'|^3 + \epsilon^{\mathfrak{r}_2+2\mathfrak{g}_2}|w_1''|^2|(w_2')^3 - (w_2'')^3|$ for all $w_1', w_1'', w_2', w_2'' \in \mathbb{R}$ and $\epsilon \geqslant 1$;

(c) $3\mathfrak{g}_2 - \mathfrak{g}_1 \leqslant \mathfrak{r}_2 - \mathfrak{r}_1 \leqslant \mathfrak{g}_2 + \mathfrak{g}_1$ for (iii) of (H0).

Also, we should notice that if $(\mathfrak{r}, \mathfrak{g})$ is a solution of the above set of inequalities, then $(k\mathfrak{r}, k\mathfrak{g})$ is another solution for any $k > 0$. Moreover, (iii) of (H0) implies that the degrees $\{\mathfrak{g}_j\}_{j=1,\ldots,n}$ are non-increasing.

A technical motivation for (H0) is the following: under (H0), for each fixed compact set $\Omega \subset \mathbb{R}^n$, containing the origin, there exist $L \in \mathbb{R}^n$, symmetric and positive definite $P \in \mathbb{R}^{n\times n}$ and $\alpha > 0$ such that the characteristic polynomial of $A + BF + HC - LC$ is Hurwitz and

$$(x - \xi)^T P[f(x) - f(\xi) - L(h(x, 0) - h(\xi, 0))]$$

$$\leqslant -\alpha\|x-\xi\|_P^2 \quad (9)$$

for all $x, \xi \in \Omega$. This means that, for each fixed compact set $\Omega \subset \mathbb{R}^n$,$\dot{\xi} = f(\xi) + L(y - h(\xi, 0))$ is an observer for any state trajectory $x(t)$ of (5)-(6) with $d(t) \equiv 0$ as long as $x(t)$ and $\xi(t)$ remain in Ω for all times, i.e. a semiglobal observer for (5)-(6) in the absence of measurement disturbance. Condition (H0) is very close to be necessary for solving an inequality of the form (9).

Remark 2 *We want to stress the fact that globally convergent observers designed in the absence of measurement noise may show instability when used in the presence of measurement noise. This implies that standard observer design tools cannot be used for designing observers in the presence of measurement noise. For example, consider the system with two outputs and one input*

$$\begin{aligned}
&\dot{x}_1 = -x_1 + 2,\ y_1 = x_1 + d_1\\
&\dot{x}_2 = x_1 x_3,\ y_1 = x_2 + d_2 && (10)\\
&\dot{x}_3 = -x_1 x_2 + u,\ u = \sin t. && (11)
\end{aligned}$$

If x_1 is constant, (10)-(11) is an harmonic oscillator with a periodic input. Any solution $x(t)$ is bounded and the input u does not cause resonance. An observer of the form

$$\begin{aligned}
&\dot{\xi}_1 = -\xi_1 + 2 + (y_1 - \xi_1)\\
&\dot{\xi}_2 = \xi_1\xi_3 + (y_2 - \xi_2) && (12)\\
&\dot{\xi}_3 = -\xi_1\xi_2 + u + (y_2 - \xi_2) && (13)
\end{aligned}$$

is globally convergent to $x(t)$ with $d_1(t) = d_2(t) \equiv 0$. However, if $d_1(t) \equiv -x_1(t)$ and $d_2(t) \equiv 0$, with $\xi_1(0) = 1$ (the same result is obtained for any $\xi_1(0)$ with $\lim_{t\to+\infty} d_1(t) = -2$ and $\lim_{t\to+\infty} d_2(t) = 0$), we get that $\xi_1(t) \equiv 1$ and $\|(\xi_2(t), \xi_3(t))\| \to +\infty$ as $t \to +\infty$ (we have a resonance condition for (12)-(13) at the frequency of 1 rad/sec with the input at the same frequency).

Therefore, tools for observer design in the absence of (measurement) disturbances cannot be directly extended to a noisy measurement environment. To our knowledge, only Ahrens & Khalil (2006) and, more recently, Prasov & Khalil (2013) has considered the problem of semiglobal observer design (i.e. state trajectories in a fixed compact set) in the presence of measurement disturbances for systems (5)-(6) with $\phi(x) = Bp(x)$ and $\psi(x) = 0$. Following Boizot et al. (2010), the effect of measurement disturbance on global observer design has been studied for a class of lower triangular systems with bounded trajectories and for a given class of observers in Sanfelice & Praly (2011).

Remark 3 *Homogeneity in the upper bound, while implied by homogeneity (Rosier (1998)) as pointed out in (iii) of remark 2, is conceptually different from homogeneity in the ∞-limit (Andrieu et al. (2009)). Indeed, this last notion characterizes homogeneous approximations (when x is large) while homogeneity in the upper bound (Battilotti (2014)) characterizes homogeneous upper bounds (for large x). Moreover, homogeneity in the upper bound allows for more flexibility in the choice of the degrees (we have two vector degrees $(\mathfrak{d}, \mathfrak{h})$ instead of the same degree $\mathfrak{d}_\infty$ for each coordinate function). For this specific reason triangular vector fields are homogeneous in the upper bound while not all triangular vector fields are homogeneous in the ∞-limit. For example, $\phi(x) = (x_2, -x_1 + x_2(1 - x_1^2 x_2^2))^T$ is not homogeneous in the ∞-limit but it is homogeneous in the upper bound with weights $\mathfrak{r} = (1, 2)^T$ and degrees $(\mathfrak{d}, \mathfrak{h}) = ((9, 5)^T, (8, 3)^T)$. Similar remarks can be repeated for local homogeneity (Efimov & Perruquetti (2016)), which characterizes local homogeneous approximations.•*

Remark 4 *Assumption (H1) is somewhat restrictive. However, many physical systems have this property (Van Der Pol and Fitzhugh-Nagumo oscillators, Lorentz systems, ...). A very simple relaxation of (H1) is obtained for example by assuming additionally that ϕ and ψ are globally Lipschitz. Significant relaxations of (H1) will be more naturally considered when the estimate of the state is used for global stabilization of the system via output feedback. In other words, for a system with input u which is globally stabilizable by state feedback $u = \alpha(x)$ and after applying a feedback law $u = \alpha(\xi)$, where $\xi(t)$ is an estimate of the state trajectory $x(t)$, the closed-loop system is expected to satisfy assumption (H1) when $\|x(t) - \xi(t)\|$ is bounded by some $\mathcal{K}_\infty$-class function γ of $\|x(t)\|$, i.e. $\|x(t) - \xi(t)\| \leqslant \gamma(\|x(t)\|)$. The upper bound $\|x(t)\| < \alpha(\|x_0\|) + \gamma^{-1}(\|x(t) - \xi(t)\|)$, for some $\mathcal{K}_0$-class function α and for $t \geqslant 0$, characterizes how the state grows unbounded when we apply a feedback law $u = \alpha(\xi)$ instead of $u = \alpha(x)$: we relax (H1) exactly in this sense. •*

4 The structure of the observer and main result

The observer we propose for (5)-(6) has the following interconnected structure. The first part of the filter is devoted to the estimation of x

$$\begin{aligned}
\dot{\xi} &= A\xi + (BF + HC)\mathbf{sat}_{cz^{\mathfrak{r}}}(\xi) + \phi(\mathbf{sat}_{cz^{\mathfrak{r}}}(\xi))\\
&+ L_z[y - C\xi - \psi(\mathbf{sat}_{cz^{\mathfrak{r}}}(\xi))],\ \xi(0) := \xi_0, \qquad (14)
\end{aligned}$$

where

$$L_z := kz^{2\mathfrak{g}_1}(I - A^T G_z)^{-1} C^T,\ G_z := \mathrm{diag}\{\Gamma z^{2A\mathfrak{g}}\} \quad (15)$$

with $c, k > 0$ and diagonal positive definite $\Gamma \in \mathbb{R}^{n\times n}$ (specified in section 4.1), while the second part of the filter is devoted to the gain adaptation and tuning of the saturation levels

$$\begin{aligned}
\dot{z} &= z^{-2|\mathfrak{g}_n|}\sigma\Big(z^{2(\mathfrak{g}_1 - \mathfrak{r}_1)}\max\Big\{q_z(\xi, y) \qquad (16)\\
&- h(\Delta) z^{2(\mathfrak{r}_1 + \mathfrak{g}_1 - \mathfrak{g}_n) + 1}\Big), 0\Big\}\Big),\ z(0) := z_0 \geqslant 1,
\end{aligned}$$

where

$$
\begin{aligned}
\sigma(s) &\coloneqq s/\sqrt{1+s^2}, \\
q_z(\xi, y) &\coloneqq |y - C\xi - \psi(\mathbf{sat}_{cz^{\mathfrak{r}}}(\xi))|^2 - 2\Delta^2 \\
&\quad + z^{2(\mathfrak{r}_1-\mathfrak{g}_1)}\|z^{\mathfrak{g}-\mathfrak{r}} \diamond (\xi - \mathbf{sat}_{cz^{\mathfrak{r}}}(\xi))\|^2.
\end{aligned} \tag{17}
$$

and $h \in \mathcal{K}$ (specified in section 4.1).

The estimator (14) is a copy of the system equations (5), except for saturating estimates inside the terms $BF+HC+\phi$ and ψ, plus an innovation term $L_z[y-C\xi-\psi(\mathbf{sat}_{cz^{\mathfrak{r}}}(\xi))]$. Note also that the gain matrix L_z and the saturation levels are adapted according to the values of z. The dynamics of z is implemented as a stable filter forced by the term q_z, which depends on the squared norm of the output estimation error $y-C\xi-\psi(\mathbf{sat}_{cz^{\mathfrak{r}}}(\xi))$ and the mismatch between ξ and its saturated value $\mathbf{sat}_{cz^{\mathfrak{r}}}(\xi)$, dynamically weighted by adaptation of z. As we will see using the incremental properties in the upper bound of $\phi + BF + HC$ and ψ, the trajectories of (5)-(6)-(14)-(16) are shown to be defined and bounded (the state x is bounded by (H1)) for all times, in particular the behavior of z is bounded in time from above and away from zero (actually, it has a finite limit). Due to the uniform continuity of the solutions and boundedness of their time derivatives, the right-hand side of (16) tends asymptotically to zero (by Barbalat's lemma), which implies that the limsup of $q_z(\xi, y)$, as time tends to infinity, is bounded by some $\mathcal{K}$-class function of Δ, the upper bound for $\|d\|_\infty$, possibly depending on the limit value of z. This leads to establish that also the limsup of the norm of the estimation error is bounded by some $\mathcal{K}$-class function of Δ, possibly depending on the limit value of z. More precisely, the main result of this paper is the following. Let ϕ_0 denote the vector of initial conditions x_0, ξ_0 and $z_0 \geqslant 1$. Also, let d_t denote the measurement disturbance and let $x_t(x_0)$, resp. $\xi_t(\phi_0, d), z_t(\phi_0, d)$, denote the solution of (5), resp. (14)-(16), ensuing from initial condition x_0, resp. ϕ_0 with measurement disturbance $d \in \mathcal{D}(\Delta)$.

Theorem 5 *Assume (H0) and (H1). There exist $c, k > 0$, $h \in \mathcal{K}$ and diagonal positive definite $\Gamma \in \mathbb{R}^{n\times n}$ such that the solution $x_t(x_0)$, $\xi_t(\phi_0, d), z_t(\phi_0, d)$ of (5)-(6)-(14)-(16) is defined and bounded for all $t \geqslant 0$, initial conditions ϕ_0 and measurement disturbance $d \in \mathcal{D}(\Delta)$. In addition,*

$$\lim_{t\to+\infty} z_t(\phi_0, d) = z_\infty, \tag{18}$$

$$\limsup_{t\to+\infty} \|x_t(x_0)-\xi_t(\phi_0, d)\|^2 \leqslant \frac{\lambda_{max}(P)\chi^2_{z_\infty}(\Delta)}{\alpha^2\lambda_{min}(P)} \tag{19}$$

with

$$
\begin{aligned}
\chi_{z_\infty}(\Delta) &:= \Big(\|BF + HC\| \\
&+ \sup_{\substack{\|w_1\|\leqslant 2nc\|z_\infty^{\mathfrak{r}}\| \\ \|w_2\|\leqslant 2n(\sqrt{\nu_{z_\infty}(\Delta)}+c\|z_\infty^{\mathfrak{r}}\|)}} \|\Phi_U(w_1, w_2)\| \\
&+ L_{z_\infty} \sup_{\substack{\|w_1\|\leqslant 2nc\|z_\infty^{\mathfrak{r}}\| \\ \|w_2\|\leqslant 2n(\sqrt{\nu_{z_\infty}(\Delta)}+c\|z_\infty^{\mathfrak{r}}\|)}} \|\Psi_U(w_1, w_2)\|\Big)\sqrt{\nu_{z_\infty}(\Delta)} \\
&+ \sqrt{\mu_{z_\infty}(\Delta)}\|L_{z_\infty} - L\| + \|L\|\Delta \\
\mu_{z_\infty}(\Delta) &:= h(\Delta)z_\infty^{2(\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)+1} + 2\Delta^2, \\
\nu_{z_\infty}(\Delta) &:= \mu_{z_\infty}(\Delta)z_\infty^{2(\max_i \mathfrak{r}_i-\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)}
\end{aligned} \tag{20}
$$

and $L \in \mathbb{R}^n$, symmetric and positive definite $P \in \mathbb{R}^{n\times n}$ and $\alpha > 0$ such that

$$
\begin{aligned}
&(x - \xi)^T P[f(x) - f(\xi) - L(h(x, 0) - h(\xi, 0))] \\
&\leqslant -\alpha\|x - \xi\|^2_P
\end{aligned} \tag{21}
$$

for all $x, \xi \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is any compact set for which $x_t(x_0), \xi_t(\phi_0, d) \in \Omega$ for all $t \geqslant 0$.□

Remark 6 *The inequality (21), which is exactly (9), is instrumental only to obtain the bound (19) on the estimation error and it is not needed in the observer design (see next section). Under assumption (H0) and according to Battilotti (2014), theorem V.1, there indeed exist $L \in \mathbb{R}^n$, symmetric and positive definite $P \in \mathbb{R}^{n\times n}$ and $\alpha > 0$ (all depending on Ω) such that (21) holds for all $x, \xi \in \Omega$.*

Remark 7 *As it results from (19) the limsup of the norm of the estimation error is bounded by a $\mathcal{K}$-class function of Δ, which is an upper bound for the supremum norm of d. The limsup of the norm of the estimation error can be further reduced by replacing, in the equations of (16), Δ with some Δ_∞ such that $\limsup_{t\to+\infty}|d_t| < \Delta_\infty$: it can be shown, following the same lines of the proof of theorem 5, that the conclusions of theorem 5 remain true with Δ replaced by Δ_∞. In other words, the sup norm of the disturbance may be large, but the limsup of its norm may be smaller, so that the limsup of the norm of the estimation error is also smaller. Since we can take $\Delta_\infty := \limsup_{t\to+\infty}|d_t|+\varepsilon$ for arbitrary $\epsilon > 0$, it follows by letting $\epsilon \to 0$ that $\limsup_{t\to+\infty}\|x_t(x_0)-\xi_t(\phi_0, d)\| = 0$ when $\limsup_{t\to+\infty}|d_t| = 0$.*

4.1 Choice of the observer parameters

The observer (14)-(16) is characterized by the parameters $c, k > 0$, $h \in \mathcal{K}$ and diagonal positive definite Γ. These quantities are chosen as follows. Let ϕ_U, ψ_U, F_U, H_U, $\mathfrak{r}$ and $\mathfrak{g}$ be as in assumption (H0) and let Δ be the upper bound for the sup norm of the measurement disturbance d. Towards the filter definition, the following calculations should be accomplished:

(i) find k and Γ such that for some $a > 0$

$$2aI \leqslant \mathcal{X}(k, \Gamma) \coloneqq 2(kC^TC + A^T\Gamma A) \tag{22}$$

$$-\Big[2(I+A^T\Gamma)(BF_U+H_UC)+A+A^T\Gamma^2$$
$$+2\max_{i\geqslant 2}|\mathfrak{g}_i|A^T\Gamma\Big](I-A^T\Gamma)^{-1}$$
$$-(I-A^T\Gamma)^{-T}\Big[2(I+A^T\Gamma)(BF_U+H_UC)+A+A^T\Gamma^2$$
$$+2\max_{i\geqslant 2}|\mathfrak{g}_i|A^T\Gamma\Big]^T-2\mathrm{diag}\{\mathfrak{r}_1,\ldots,\mathfrak{r}_n\}.$$

Inequality (22) is alway solvable in the unknowns c, k and Γ, on account of the fact that $\mathcal{X}(k,\Gamma)$ can be obtained recursively as follows (recall that $\Gamma_{i,i}$ denotes the i-th diagonal entry of Γ)

$$\mathcal{X}^{(n-1)} := 2\Gamma_{n-1,n-1},$$
$$\mathcal{X}^{(n-j)} := \left[\begin{array}{c|c} 2\Gamma_{n-j,n-j}+\mathcal{Z}_1^{(n-j)} & (\mathcal{Z}_2^{(n-j)})^T \\ \hline \mathcal{Z}_2^{(n-j)} & \mathcal{X}^{(n-j+1)} \end{array}\right], = 2,\ldots,n,$$
$$\mathcal{X}^{(0)} = \mathcal{X}(k,\Gamma) \tag{23}$$

with $\Gamma_{00} := k$ and $\mathcal{Z}_2^{(n-j)}, \mathcal{Z}_1^{(n-j)}$, $j = 2,\ldots,n$, are suitable functions of $\Gamma_{n-j+1,n-j+1},\ldots,\Gamma_{n-1,n-1}$. Therefore, it is sufficient to pick any $\Gamma_{n-1,n-1} > 0$ and for each increasing $j = 2,\ldots,n$ select $\Gamma_{n-j,n-j} > 0$ such that $\mathcal{X}^{(n-j)} > 0$. Finally, set $a := \frac{\lambda_{min}(\mathcal{X}^{(0)})}{2}$.

(ii) define $c > 0$ as follows: if $\Phi \in \mathbf{C}^0(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant^{n\times n})$ and $\Psi \in \mathbf{C}^0(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant^{1\times n})$ are matrices for which $\Phi(0) = 0$, $\Psi(0) = 0$ and for all $s \geqslant 0$

$$\phi_U(w,z) \preceq \Phi(s), \forall w,z\in\mathbb{R}^n : \|z\|\leqslant ns,\ \|w\|\leqslant ns, \tag{24}$$
$$\psi_U(w,z) \preceq \Psi(s), \forall w,z\in\mathbb{R}^n : \|z\|\leqslant ns,\ \|w\|\leqslant ns \tag{25}$$

(we recall that $\preceq$ for matrices means $\leqslant$ for each entry), calculate $c > 0$ such that

$$aI \leqslant \Upsilon(c,k,\Gamma) := \mathcal{X}(k,\Gamma)$$
$$-2[(I+A^T\Gamma)\Phi(c)+kC^T\Psi(c)](I-A^T\Gamma)^{-1}$$
$$-2(I-A^T\Gamma)^{-T}[(I+A^T\Gamma)\Phi(c)+kC^T\Psi(c)]^T. \tag{26}$$

The number c always exists on account of (26) and continuity of ϕ_U and ψ_U with $\phi_U(0,0) = 0$ and $\psi_U(0,0) = 0$.

(iii) define $h \in \mathcal{K}$ as follows:

$$\Theta := 9\|(I-A^T\Gamma)^{-1}\|^2$$
$$+\ 2\Big\|C^T(C+2\Psi(c))(I-A^T\Gamma)^{-1}\Big)\Big\|^2 \tag{27}$$
$$h(\Delta) := 10k^2\Delta^2\Theta/a^2. \tag{28}$$

4.2 Example and simulations

The system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + (1-x_1^2x_2^2)x_2,\ y = x_1 + d \tag{29}$$

with measurement disturbance $d_t \in [-4,4]$ satisfies assumptions (H0) and (H1) of theorem 5 with $\mathfrak{r}_1 = 1$, $\mathfrak{r}_2 = 2$, $\mathfrak{g}_1 = 8$ and $\mathfrak{g}_2 = 3$. Notice that $\phi(x) := (x_2, -x_1+(1-x_1^2x_2^2)x_2)^T$ is neither homogeneous nor homogeneous in the ∞-limit.

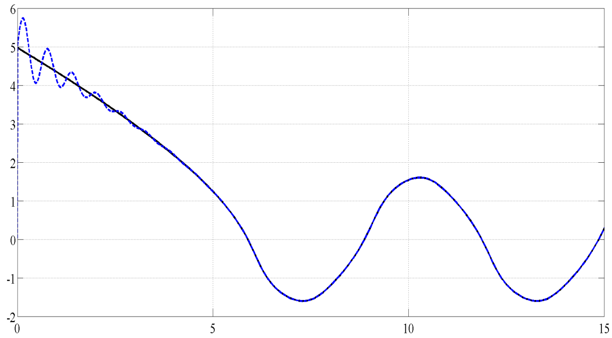

Fig. 1. State $x_1(t)$ (continuous line) and its estimate (dotted line) versus time with $d_t = e^{-t}sin(10t)$.

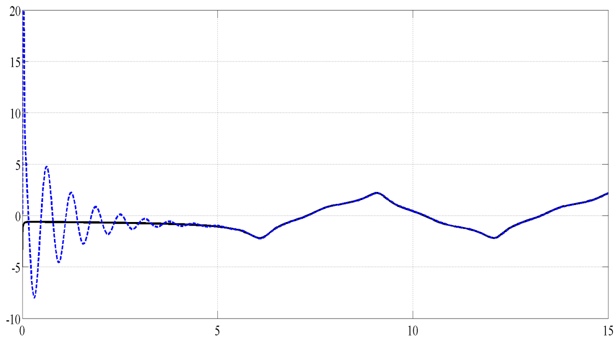

Fig. 2. State $x_2(t)$ (continuous line) and its estimate (dotted line) versus time with $d_t = e^{-t}sin(10t)$ (tail).

An observer has been designed according to our procedure and a simulation has been worked out with initial conditions $x(0) = (5,-5)^T$, $\xi(0) = (0,0)^T$, $z(0) = 1$ and $\Delta = 4$. The saturation levels of the estimates are set with $c = 0.1$, the diagonal elements of Γ are respectively 8 and 30 and $k = 100$. The states $x_{1,t}, x_{2,t}$ together with their estimates are shown versus time in Figs. 1,2 with vanishing disturbance $d_t = e^{-(1/2)t}sin(10t)$, in Figs. 3,4 with persistent disturbance $d_t = sin(10t)$ and in Figs. 5,6 with $d_t = sin(10t) + 3sin(2t) - sin(4t) + sin(20t)$. The last disturbance configuration has a structure which tends to that of a general periodic disturbance as the number of harmonics tends to infinity. Moreover, Figs. 1,2 refer to the case in which $\limsup_{t\to+\infty}|d_t| = 0$.

In Figs. 7 and 8 we have shown the effect on the estimation errors of a disturbance $d_t = (0.1+3.9e^{-2t})sin(10t)$ with $\|d\|_\infty = 4$ but $\limsup_{t\to+\infty}|d_t| = 0.1 << 4$. In our observer we replaced $\Delta = 4$ with $\Delta_\infty = 0.2$, which is a tighter upper bound for $\limsup_{t\to+\infty}|d_t|$. The simulations show that the estimation error is significantly

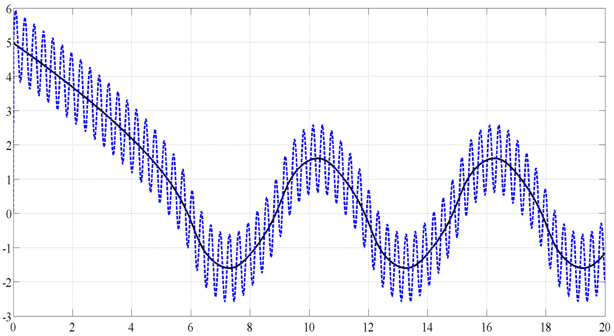

Fig. 3. State $x_1(t)$ (continuous line) and its estimate (dotted line) versus time with $d_t = sin(10t)$.

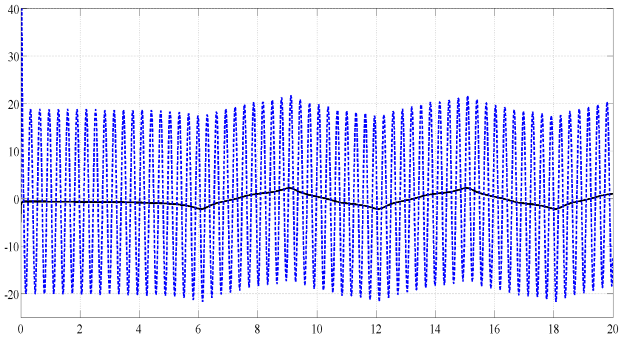

Fig. 4. State $x_2(t)$ (continuous line) and its estimate (dotted line) with $d_t = sin(10t)$.

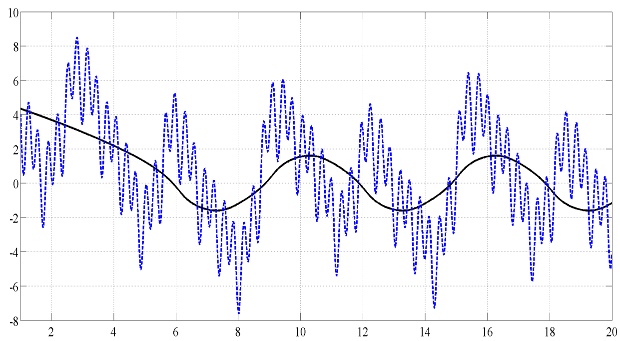

Fig. 5. State $x_1(t)$ (continuous line) and its estimate (dotted line) versus time with $d_t = sin(10t) + 3cos(2t) - sin(4t) + cos(20t)$ (tail).

improved with respect to the case of Figs. 3,4 in which $\|d\|_\infty = \limsup_{t\to+\infty} |d_t| = 1$: the estimation error for x_1 is reduced by a factor 2 while the estimation error for x_2 is reduced by a factor 20.

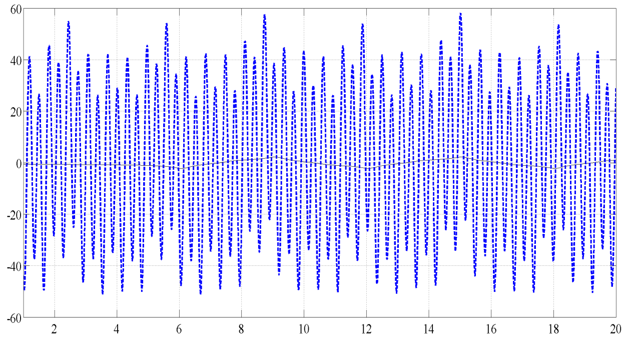

Fig. 6. State $x_2(t)$ (continuous line) and its estimate (dotted line) with $d_t = sin(10t) + 3cos(2t) - sin(4t) + cos(20t)$ (tail).

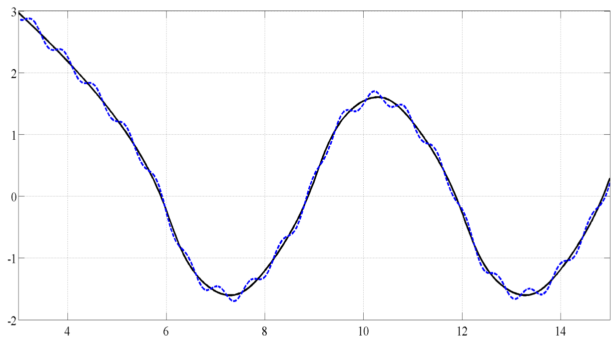

Fig. 7. State $x_1(t)$ (continuous line) and its estimate (dotted line) versus time with $d_t = (0.1 + 3.9e^{-2t})sin(10t)$ (tail).

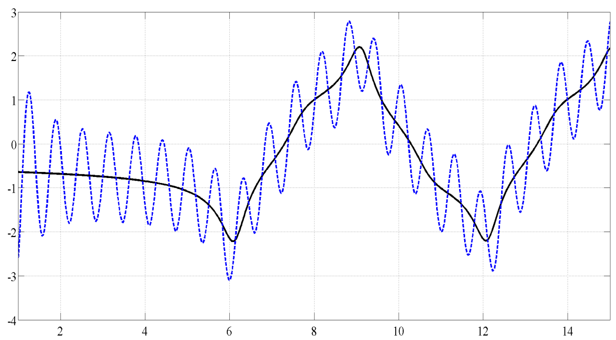

Fig. 8. State $x_2(t)$ (continuous line) and its estimate (dotted line) with $d_t = (0.1 + 3.9e^{-2t})sin(10t)$ (tail).

4.3 Proof of the theorem 5

For simplifying the main passages of the proof, we will consider $\psi \equiv 0$ in (6) (this term can be treated in the same way as ϕ) which requires to set $\Psi \equiv 0$ in (26). Let c, k, $h(\Delta)$, $\Phi(c)$ and Γ be selected as in section 4.1. Consider the following coordinate transformation

$$(x, \xi, z) \mapsto (x, \eta, z) : \eta \coloneqq X_z^{-1}(x - \xi), \tag{30}$$

with $X_z := (I - A^T G_z)^{-1}$ (the identity matrix is $n \times n$). Recalling that $L_z := kz^{2\mathfrak{g}_1} X_z C^T$ and using the identities

$$CC^T = 1,\ A^T G_z A C^T = 0,\ CX_z = C,$$
$$A^T G_z A A^T = A^T G_z,\ X_z - I = A^T G_z X_z$$

and $\frac{d}{dz}(A^T G_z) = \frac{2}{z}\mathrm{diag}\{A^T A \mathfrak{g}\} A^T G_z$, after few passages (5)-(6)-(14)-(16) reads out in η-coordinates as

$$\begin{aligned}
\dot{x}_t &= (A + BF + HC)x_t + \phi(x_t),\\
\dot{\eta}_t &= -\Sigma_{z_t}\eta_t\\
&\quad + \pi_{z_t}\Big(X_z^{-1}x_t\Big) - \pi_{z_t}\Big(-\eta_t + X_z^{-1}x_t\Big)\\
&\quad - 2\frac{\dot{z}_t}{z_t}\mathrm{diag}\{A^T A\mathfrak{g}\} A^T G_{z_t} X_{z_t}\eta_t + \rho_{z_t}(x_t, d_t) \qquad (31)\\
\dot{z}_t &= z_t^{-2|\mathfrak{g}_n|}\sigma\Big(z_t^{2(\mathfrak{g}_1-\mathfrak{r}_1)} \max\Big\{q_{z_t}(-X_{z_t}\eta_t + x_t, y_t)\\
&\quad - h(\Delta) z_t^{2(\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)+1}, 0\Big\}\Big),
\end{aligned}$$

with

$$\Sigma_z := kz^{2\mathfrak{g}_1} C^T C + A^T G_z A \qquad (32)$$
$$\pi_z(w_1) := [I - A^T G_z]\Big[(BF + HC)(\mathbf{sat}(cz^{\mathfrak{r}}, X_z w_1)) + \phi(\mathbf{sat}(cz^{\mathfrak{r}}, X_z w_1))\Big] + (A - A^T G_z^2) X_z w_1, \qquad (33)$$

$$\rho_z(w_1, w_2) := [I - A^T G_z]\Big[(BF + HC)(w_1 - \mathbf{sat}(cz^{\mathfrak{r}}, w_1)) + \phi(w_1) - \phi\Big(\mathbf{sat}(cz^{\mathfrak{r}}, w_1)\Big)\Big] + kz^{2\mathfrak{g}_1} C^T w_2 \qquad (34)$$

We split up the proof into five steps.

(A) The solutions η_t and z_t have infinite escape time.

The solutions η_t and z_t of (31) are defined over some maximal extension intervals $[0, T_\eta)$ and, respectively, $[0, T_z)$, where $T_\eta, T_z \leqslant +\infty$. Notice that $\dot{z}_t \geqslant 0$ at each $t \in [0, T_z)$. From $z_0 \geqslant 1$ it follows that

$$z_t \geqslant 1 \qquad (35)$$

for all $t \in [0, T_z)$.

Since $\sigma(s) \leqslant 1$ for all $s \geqslant 0$, we have $0 \leqslant \dot{z}_t \leqslant z_t^{-2|\mathfrak{g}_n|}$ for each $t \in [0, T_z)$. It follows that for each $t \in [0, T_z)$

$$1 \leqslant z_t \leqslant \Big((2|\mathfrak{g}_n| + 1)t + z_0^{2|\mathfrak{g}_n|+1}\Big)^{\frac{1}{2|\mathfrak{g}_n|+1}} \qquad (36)$$
$$0 \leqslant \dot{z}_t \leqslant z_t^{-2|\mathfrak{g}_n|} \leqslant 1. \qquad (37)$$

By letting $t \to T_z^-$ in (36), it follows that $T_z = +\infty$ and (35) and (36)-(37) hold for all $t \geqslant 0$.

Also, by integration over $[0, t]$ and subsequent majorization of the second equation in (31) we can see that also $T_\eta = +\infty$, with $[0, T_\eta)$ being the maximal extension interval of η_t. Indeed, as a consequence of (i) and (ii) of lemma 12 and the definition of incremental homogeneity in the upper bound,

$$\Sigma_z(z^{\mathfrak{r}} \diamond w) = z^{\mathfrak{r}+\mathfrak{g}} \diamond \Big(\Sigma(z^{\mathfrak{g}} \diamond w)\Big),\ \forall w \in \mathbb{R}^n,\ z \geqslant 1, \quad (38)$$
$$\Big\langle\!\Big\langle \pi_z\Big(z^{\mathfrak{r}} \diamond w'\Big) - \pi_z\Big(z^{\mathfrak{r}} \diamond w''\Big)\Big\rangle\!\Big\rangle \qquad (39)$$
$$\preceq z^{\mathfrak{r}+\mathfrak{g}} \diamond \Big(\Pi_U \Big\langle\!\Big\langle z^{\mathfrak{g}} \diamond (w' - w'')\Big\rangle\!\Big\rangle\Big), \forall w', w'' \in \mathbb{R}^n,\ z \geqslant 1,$$

(recall that $\preceq$ means $\leqslant$ componentwise and $\langle\!\langle\cdot\rangle\!\rangle$ means $|\cdot|$ componentwise: see notation section) where

$$\Sigma := kC^T C + A^T \Gamma A, \qquad (40)$$
$$\Pi_U := \Big[2(I + A^T\Gamma)(BF_U + H_U C + \Phi(c)) + A + A^T\Gamma^2\Big] X_U$$
$$X_U =: (I - A^T\Gamma)^{-1}. \qquad (41)$$

Since $x \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}^n)$ (consequence of (H1)) and $\|d\|_\infty < +\infty$ with (36)-(39) and on account of lemma 10, by integrating the second equation of (31) over $[0, t]$ for each $t \in [0, T_\eta)$ we have

$$\begin{aligned}
\|\eta_t\| &\leqslant \|\eta_0\| + \Big(\sup_{0\leqslant s\leqslant t}\|\Sigma_{z_s}\|\Big)\int_0^t \|\eta_s\| ds\\
&+ \Big(\sup_{0\leqslant s\leqslant t}\|z_s^{\mathfrak{r}+\mathfrak{g}}\|\Big)\Big(\sup_{0\leqslant s\leqslant t}\|z_s^{\mathfrak{g}}\|\Big)\|\Pi_U\|\int_0^t \|\eta_s\| ds\\
&+ \|\rho_z(x, d)\|_\infty t + 2\|A^T A\mathfrak{g}\|\Big(\sup_{0\leqslant s\leqslant t}\|A^T G_{z_s} X_{z_s}\|\Big)\int_0^t \|\eta_s\| ds\\
&:= \delta_1(t) + \delta_2(t)\int_0^t \|\eta_s\| ds \qquad (42)
\end{aligned}$$

with $\delta_1, \delta_2 \in \mathcal{K}_0$. It follows by a generalized Gronwall inequality (Beesack (1975)) that

$$\|\eta_t\| \leqslant \delta_1(t) + \delta_2(t)\int_0^t \delta_1(s) e^{\int_s^t \delta_2(r) dr} ds$$

for all $t \in [0, T_\eta)$. By letting $t \to T_\eta^-$ above, we conclude that $T_\eta = +\infty$.

(B) A Lyapunov function for the estimation error system.

Let $V_z(\eta) := \|z^{-\mathfrak{r}} \diamond \eta\|^2$. We evaluate the time derivative of V_z along the trajectories of (31):

$$\dot{V}_{z_t}\Big|_{(31)} = \overbrace{-\frac{\partial V_z}{\partial \eta}\Big|_{\substack{\eta=\eta_t\\ z=z_t}} \Sigma_{z_t}\eta_t}^{(1)}$$

$$+\overbrace{\frac{\partial V_z}{\partial \eta}\Big|_{\substack{\eta=\eta_t\\ z=z_t}}\Big\{\pi_{z_t}\Big((X_{z_t})^{-1}x_t\Big)-\pi_{z_t}\Big(-\eta_t+(X_{z_t})^{-1}x_t\Big)\Big\}}^{(2)}$$

$$-\overbrace{2\frac{\dot z_t}{z_t}\frac{\partial V_z}{\partial \eta}\Big|_{\substack{\eta=\eta_t\\ z=z_t}}\mathrm{diag}\{A^T A\mathfrak{g}\}A^T G_{z_t}X_{z_t}\eta_t}^{(3)}$$

$$+\overbrace{\frac{\partial V_z}{\partial \eta}\Big|_{\substack{\eta=\eta_t\\ z=z_t}}\rho_z(x_t,d_t)}^{(4)}+\overbrace{\frac{\partial V_z}{\partial z}\Big|_{\substack{\eta=\eta_t\\ z=z_t}}(\eta_t)\dot z_t}^{(5)} \tag{43}$$

In many occasions, we will exploit the monotonicity of the degrees $\{\mathfrak{g}_j\}_{j=1,\dots,n}$, i.e. $\mathfrak{g}_{i+1}\leqslant\mathfrak{g}_i$ for all $i=1,\dots,n-1$, which is a consequence of (iii) in (H0). We begin with majorizing the term (1) in (43). On account of (38) with $w\coloneqq z^{-\mathfrak{r}}\diamond\eta$, for all $t\geqslant 0$

$$\begin{aligned}
&-\frac{\partial V_z}{\partial \eta}\Big|_{\substack{\eta=\eta_t\\ z=z_t}}\Sigma_{z_t}\eta_t\\
&=-2(z_t^{-\mathfrak{r}}\diamond z_t^{-\mathfrak{r}}\diamond\eta_t)^T\Sigma_{z_t}(z_t^{\mathfrak{r}}\diamond z_t^{-\mathfrak{r}}\diamond\eta_t) \qquad (44)\\
&=-2(z_t^{-\mathfrak{r}}\diamond\eta_t)^T\Big(z_t^{-\mathfrak{r}}\diamond\Sigma_{z_t}(z_t^{\mathfrak{r}}\diamond z_t^{-\mathfrak{r}}\diamond\eta_t)\Big)\\
&\leqslant -2\langle\!\langle z_t^{\mathfrak{g}-\mathfrak{r}}\diamond\eta_t\rangle\!\rangle^T\Sigma\langle\!\langle z_t^{-\mathfrak{r}+\mathfrak{g}}\diamond\eta_t\rangle\!\rangle=-2\|z_t^{\mathfrak{g}-\mathfrak{r}}\diamond\eta_t\|_\Sigma^2
\end{aligned}$$

(in the second and third passages we used properties (3)-(4)). Next, we majorize the term (2) in (43). On account of (39) with $w'\coloneqq z^{-\mathfrak{r}}\diamond\Big((X_z)^{-1}x\Big)$ and $w''\coloneqq z^{-\mathfrak{r}}\diamond\Big(-\eta+(X_z)^{-1}x\Big)$, for all $t\geqslant 0$

$$\begin{aligned}
&\frac{\partial V_z}{\partial \eta}\Big|_{\substack{\eta=\eta_t\\ z=z_t}}\Big\{\pi_{z_t}\Big((X_{z_t})^{-1}x_t\Big)-\pi_{z_t}\Big(-\eta_t+(X_{z_t})^{-1}x_t\Big)\Big\}\\
&=2(z_t^{-\mathfrak{r}}\diamond z_t^{-\mathfrak{r}}\diamond\eta_t)^T\Big\{\pi_{z_t}\Big(z_t^{\mathfrak{r}}\diamond z_t^{-\mathfrak{r}}\diamond\Big((X_{z_t})^{-1}x_t\Big)\Big)\\
&-\pi_{z_t}\Big(z_t^{\mathfrak{r}}\diamond z_t^{-\mathfrak{r}}\diamond\Big(-\eta_t+(X_{z_t})^{-1}x_t\Big)\Big)\Big\} \qquad (45)\\
&\leqslant 2\langle\!\langle z_t^{-\mathfrak{r}}\diamond\eta_t\rangle\!\rangle^T\Big\{z_t^{-\mathfrak{r}}\diamond\Big\langle\!\Big\langle\pi_{z_t}\Big(z_t^{\mathfrak{r}}\diamond z_t^{-\mathfrak{r}}\diamond\Big((X_{z_t})^{-1}x_t\Big)\Big)\\
&-\pi_{z_t}\Big(z_t^{\mathfrak{r}}\diamond z_t^{-\mathfrak{r}}\diamond\Big(-\eta_t+(X_{z_t})^{-1}x_t\Big)\Big)\Big\rangle\!\Big\rangle\Big\}\\
&\leqslant 2\langle\!\langle z_t^{\mathfrak{g}-\mathfrak{r}}\diamond\eta_t\rangle\!\rangle^T\Pi_U\langle\!\langle z_t^{\mathfrak{g}-\mathfrak{r}}\diamond\eta_t\rangle\!\rangle=\|z_t^{\mathfrak{g}-\mathfrak{r}}\diamond\eta_t\|^2_{\Pi_U+\Pi_U^T}
\end{aligned}$$

(in the second and third passages we used properties (3)-(4)). Next, we majorize the term (3) in (43). Notice that by monotonicity of the degrees $\{\mathfrak{g}_j\}_{j=1,\dots,n}$

$$\langle\!\langle z^{-\mathfrak{r}-\mathfrak{g}}\diamond\eta\rangle\!\rangle\preceq z^{-2\mathfrak{g}_n}\langle\!\langle z^{\mathfrak{g}-\mathfrak{r}}\diamond\eta\rangle\!\rangle\preceq z^{2|\mathfrak{g}_n|}\langle\!\langle z^{\mathfrak{g}-\mathfrak{r}}\diamond\eta\rangle\!\rangle \tag{46}$$

$$z^{-2|\mathfrak{g}_n|}\langle\!\langle z^{-\mathfrak{r}}\diamond\eta\rangle\!\rangle\preceq\langle\!\langle z^{\mathfrak{g}-\mathfrak{r}}\diamond\eta\rangle\!\rangle \tag{47}$$

for all $\eta\in\mathbb{R}^n$ and $z\geqslant 1$ and, moreover, by (v) of lemma 12, $A^TG_zX_z$ is i.h.u.b. with quadruple $(\mathfrak{r},\mathfrak{r}-\mathfrak{g},\mathfrak{g},A^T\Gamma X_U)$:

$$\begin{aligned}
&\Big\langle\!\Big\langle A^TG_zX_z\Big(z^{\mathfrak{r}}\diamond w'-z^{\mathfrak{r}}\diamond w''\Big)\Big\rangle\!\Big\rangle\\
&\preceq z^{\mathfrak{r}-\mathfrak{g}}\diamond\Big(A^T\Gamma X_U\Big\langle\!\Big\langle z^{\mathfrak{g}}\diamond(w'-w'')\Big\rangle\!\Big\rangle\Big),\\
&\forall w',w''\in\mathbb{R}^n,\ z\geqslant 1.
\end{aligned}$$

Using these facts, for all $t\geqslant 0$

$$\begin{aligned}
&2\Big|\frac{\dot z_t}{z_t}\Big|\Big|\frac{\partial V_{z_t}}{\partial \eta}\Big|_{\substack{\eta=\eta_t\\ z=z_t}}\mathrm{diag}\{A^TA\mathfrak{g}\}A^TG_{z_t}X_{z_t}\eta_t\Big|\\
&\leqslant 4\max_{i\geqslant 2}|\mathfrak{g}_i|\Big|\frac{\dot z_t}{z_t}\Big|\Big|(z_t^{-\mathfrak{r}}\diamond z_t^{-\mathfrak{r}}\diamond\eta_t)^TA^TG_{z_t}X_{z_t}(z_t^{\mathfrak{r}}\diamond z^{-\mathfrak{r}}\diamond\eta_t)\Big|\\
&\leqslant 4\Big|\frac{\dot z_t}{z_t}\Big|\max_{i\geqslant 2}|\mathfrak{g}_i|\langle\!\langle z_t^{-\mathfrak{r}-\mathfrak{g}}\diamond\eta\rangle\!\rangle^TA^T\Gamma X_U\langle\!\langle z_t^{-\mathfrak{r}+\mathfrak{g}}\diamond\eta_t\rangle\!\rangle\\
&\leqslant 2\Big|\frac{\dot z_t}{z_t}\Big|\max_{i\geqslant 2}|\mathfrak{g}_i|z_t^{2|\mathfrak{g}_n|}\|z_t^{\mathfrak{g}-\mathfrak{r}}\diamond\eta\|^2_{A^T\Gamma X_U+X_U^T\Gamma A}\\
&\leqslant 2\max_{i\geqslant 2}|\mathfrak{g}_i|\|z_t^{-\mathfrak{r}+\mathfrak{g}}\diamond\eta\|^2_{A^T\Gamma X_U+X_U^T\Gamma A} \qquad (48)
\end{aligned}$$

(in the second and third passages we used properties (3)-(4), in the third we used (46) while (37) in the fourth passage). Next, we majorize the term (4) in (43). By Young's inequality and using properties (3)-(4) together with $\|d\|_\infty\leqslant\Delta$

$$\begin{aligned}
&\Big|\frac{\partial V_{z_t}}{\partial \eta}\Big|_{\substack{\eta=\eta_t\\ z=z_t}}\rho_{z_t}(x_t,d_t)\Big| \qquad (49)\\
&\leqslant\frac{a}{2}\|z_t^{\mathfrak{g}-\mathfrak{r}}\diamond\eta_t\|^2+\frac{4}{a}\|z_t^{-\mathfrak{g}-\mathfrak{r}}\diamond\delta_{z_t}(x_t)\|^2+\frac{4k^2\Delta^2}{a}z_t^{2(\mathfrak{g}_1-\mathfrak{r}_1)}
\end{aligned}$$

with

$$\begin{aligned}
\delta_z(x)\coloneqq[I-A^TG_z]\Big[&(BF+HC)(x-\mathbf{sat}(cz^{\mathfrak{r}},x))\\
&+\phi(x)-\phi\Big(\mathbf{sat}(cz^{\mathfrak{r}},x)\Big)\Big].
\end{aligned}$$

Finally, we majorize the term (5) in (43). On account of (37) and (47) we also have

$$\begin{aligned}
&\Big|\frac{\partial V_z}{\partial z}\Big|_{\substack{\eta=\eta_t\\ z=z_t}}\dot z_t\Big|=2\Big|\frac{\dot z_t}{z_t}\Big|\|z_t^{-\mathfrak{r}}\diamond\eta_t\|^2_{\mathrm{diag}\{\mathfrak{r}_1,\dots,\mathfrak{r}_n\}}\\
&\leqslant 2\|z_t^{\mathfrak{g}-\mathfrak{r}}\diamond\eta_t\|^2_{\mathrm{diag}\{\mathfrak{r}_1,\dots,\mathfrak{r}_n\}}. \qquad (50)
\end{aligned}$$

Collecting (44)-(45) and (48)-(50), upon noting that with Σ, Π_U in (40) and Υ in (26) (with $\Psi\equiv 0$ by our simplifying assumption $\psi\equiv 0$)

$$\begin{aligned}
aI\leqslant\Upsilon=2\Sigma-2\max_{i\geqslant 2}|\mathfrak{g}_i|[A^T\Gamma X_U+X_U^T\Gamma A]\\
-[\Pi_U+\Pi_U^T]-2\mathrm{diag}\{\mathfrak{r}_1,\dots,\mathfrak{r}_n\},
\end{aligned}$$

we obtain for all $t\geqslant 0$

$$\begin{aligned}
\dot V_{z_t}\Big|_{(31)}\leqslant&-\frac{a}{2}\|z_t^{\mathfrak{g}-\mathfrak{r}}\diamond\eta_t\|^2+\frac{4}{a}\|z_t^{-\mathfrak{g}-\mathfrak{r}}\diamond\delta_{z_t}(x_t)\|^2\\
&+\frac{4k^2\Delta^2}{a}z_t^{2(\mathfrak{g}_1-\mathfrak{r}_1)} \qquad (51)
\end{aligned}$$

and using the monotonicity of the degrees $\{\mathfrak{g}_j\}_{j=1,\dots,n}$,

$$\dot V_{z_t}\Big|_{(31)} \leqslant -\frac{a}{2} z_t^{2\mathfrak{g}_n} V_{z_t} + \frac{4}{a}\|z_t^{-\mathfrak{g}-\mathfrak{r}} \diamond \delta_{z_t}(x_t)\|^2 + \frac{4k^2\Delta^2}{a} z_t^{2(\mathfrak{g}_1-\mathfrak{r}_1)} \tag{52}$$

(C) We claim $z \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$. Since z_t is non-decreasing for $t \geqslant 0$, we have either $\lim_{t\to+\infty} z_t < +\infty$ or $\lim_{t\to+\infty} z_t = +\infty$. Assume by absurd that

$$\lim_{t\to+\infty} z_t = +\infty \tag{53}$$

Pick $\bar z > 1$ and $\bar T > 0$ such that $z_t \geqslant \bar z$ for all $t \geqslant \bar T$ and

$$\mathbf{sat}(cz_t^{\mathfrak{r}}, x_t) = x_t, \ \forall t \geqslant \bar T \tag{54}$$

and, consequently,

$$\delta_{z_t}(x_t) = 0, \ \forall t \geqslant \bar T. \tag{55}$$

Directly from (i) and (iii) of lemma 12, for all $x, \eta \in \mathbb{R}^n$ and $z \geqslant 1$

$$\Big\langle\!\Big\langle \kappa_z\Big((X_z)^{-1}x\Big) - \kappa_z\Big(-\eta + (X_z)^{-1}x\Big)\Big\rangle\!\Big\rangle \preceq z^{\mathfrak{r}-\mathfrak{g}} \diamond \Big\langle\!\Big\langle 3X_U(z^{\mathfrak{g}-\mathfrak{r}} \diamond \eta)\Big\rangle\!\Big\rangle, \tag{56}$$

where

$$\kappa_z(w) := -X_z w + \mathbf{sat}(cz^{\mathfrak{r}}, X_z w).$$

Throughout the remaining proof we will denote $q_z(-X_z\eta + x, y)$ simply by q_z. Recalling the definition of q_z in (17) and Θ in (27) (with $\Psi \equiv 0$ by our simplifying assumption $\psi \equiv 0$ and taking into account that $CX_U = C$), we have for all $t \geqslant \bar T$

$$\begin{aligned} & z_t^{2(\mathfrak{g}_1-\mathfrak{r}_1)} q_{z_t} \\ &\leqslant \Big\| z_t^{\mathfrak{g}-\mathfrak{r}} \diamond \Big[-X_{z_t}\eta_t + x_t - \mathbf{sat}\Big(cz_t^{\mathfrak{r}}, -X_{z_t}\eta_t + x_t\Big)\Big]\Big\|^2 \\ &+ z_t^{2(\mathfrak{g}_1-\mathfrak{r}_1)}\Big(\|C\eta_t + d_t\|^2 - 2\Delta^2\Big) \\ &= \Big\| z_t^{\mathfrak{g}-\mathfrak{r}} \diamond \Big[-X_{z_t}\eta_t + \mathbf{sat}\Big(cz_t^{\mathfrak{r}}, x_t\Big) \\ &\qquad - \mathbf{sat}\Big(cz_t^{\mathfrak{r}}, -X_{z_t}\eta_t + x_t\Big)\Big]\Big\|^2 \\ &+ z_t^{2(\mathfrak{g}_1-\mathfrak{r}_1)}\Big(\|C\eta_t + d_t\|^2 - 2\Delta^2\Big) \\ &\leqslant \Big\| z_t^{\mathfrak{g}-\mathfrak{r}} \diamond \Big\langle\!\Big\langle \kappa_z\Big((X_{z_t})^{-1}x_t\Big) \\ &\qquad - \kappa_z\Big(-\eta_t + (X_{z_t})^{-1}x_t\Big)\Big\rangle\!\Big\rangle\Big\|^2 \\ &+ 2z_t^{2(\mathfrak{g}_1-\mathfrak{r}_1)}\|C\eta_t\|^2 \leqslant (9\|X_U\|^2 + 2\|C^TC\|^2)\|z_t^{\mathfrak{g}-\mathfrak{r}} \diamond \eta_t\|^2 \\ &:= \Theta\|z_t^{\mathfrak{g}-\mathfrak{r}} \diamond \eta_t\|^2 \end{aligned} \tag{57}$$

(in the second passage we used (54), in the third (56) and properties (3)-(4)). From here, on account of the monotonicity of the degrees $\{\mathfrak{g}_j\}_{j=1,\dots,n}$

$$z_t^{2(\mathfrak{g}_1-\mathfrak{r}_1)} q_{z_t} \leqslant \Theta z_t^{2\mathfrak{g}_1} V_{z_t} \tag{58}$$

Now, let

$$\begin{aligned} W_z &:= z^{2(\mathfrak{g}_1-\mathfrak{g}_n)+1}, \\ \mathcal{R} &:= \Big\{(z,\eta) \in [1,+\infty) \times \mathbb{R}^n : V_z < \frac{h(\Delta)W_z}{\Theta}\Big\}. \end{aligned} \tag{59}$$

Also, let $s \geqslant \bar T$ any time at which $(z_s, \eta_s) \in \mathcal{R}$ and $t_s := \inf\{t \geqslant s : (z_t,\eta_t) \notin \mathcal{R}\}$ (i.e. the first exit time of (z_t,η_t) from $\mathcal{R}$). We claim that $t_s = +\infty$. Indeed, assume that $t_s < +\infty$. Since $V_{z_t} < \frac{h(\Delta)W_{z_t}}{\Theta}$ for all $t \in [s, t_s)$ (by definition of t_s) and using (58), for all $t \in [s, t_s)$ we have

$$\frac{z_t^{-2\mathfrak{r}_1} q_{z_t}}{\Theta} \leqslant V_{z_t} < \frac{h(\Delta)W_{z_t}}{\Theta}$$

and, consequently, $q_{z_t} z_t^{-2(\mathfrak{g}_1-\mathfrak{g}_n+\mathfrak{r}_1)-1} < h(\Delta)$. From this with $(z_t,\eta_t) \in \mathcal{R}$ for all $t \in [s, t_s)$, it follows $\dot z_t = 0$ and, therefore, $\dot W_{z_t} = 0$ for all $t \in [s, t_s)$. Moreover, directly from (52) with (55) and by definition of $h(\Delta)$ in (28),

$$\frac{h(\Delta)}{\Theta} W_{z_t} > V_{z_t} \geqslant \frac{4h(\Delta)}{5\Theta} W_{z_t} \Rightarrow \dot V_{z_t}|_{(31)} \leqslant 0. \tag{60}$$

It follows from (60) that $V_{z_t} \leqslant \max\{V_{z_s}, \frac{4h(\Delta)}{5\Theta} W_{z_s}\}$ for all $t \in [s, t_s)$. This implies that

$$V_{z_t} \leqslant \max\{V_{z_s}, \frac{4h(\Delta)}{5\Theta} W_{z_s}\} < \frac{h(\Delta)W_{z_s}}{\Theta} = \frac{h(\Delta)W_{z_t}}{\Theta}$$

for all $t \in [s, t_s)$. By letting $t \to t_s^-$ and by continuity, we obtain $V_{z_{t_s}} \leqslant \max\{V_{z_s}, \frac{4h(\Delta)}{5\Theta} W_{z_s}\} < \frac{h(\Delta)W_{z_{t_s}}}{\Theta}$ which contradicts the definition of t_s.

We conclude that the set $\mathcal{R}$ is forward invariant for (31) and that z_t remains constant when (z_t,η_t) enters the set $\mathcal{R}$. If $(z_{t_0}, \eta_{t_0}) \notin \mathcal{R}$ at some $t_0 \geqslant \bar T$, either $(z_{t_1}, \eta_{t_1}) \in \mathcal{R}$ for some $t_1 > t_0$ (and therefore for all $t \geqslant t_1$ by forward invariance of $\mathcal{R}$) or $(z_t,\eta_t) \notin \mathcal{R}$ for all $t \geqslant t_0$. If $(z_{t_1}, \eta_{t_1}) \in \mathcal{R}$ for some $t_1 > t_0$ then we get a contradiction with (53) since $z_t = z_{t_1} < +\infty$ for all $t \geqslant t_1$. We remain with discussing the case $(z_t,\eta_t) \notin \mathcal{R}$ for all $t \geqslant t_0$. In this case, $V_{z_t} \geqslant \frac{hW_{z_t}}{\Theta}$ for all $t \geqslant t_0$ and directly from (51) with (55)

$$\begin{aligned} \dot V_{z_t}\Big|_{(31)} &\leqslant -\frac{a}{2}\|z_t^{\mathfrak{g}-\mathfrak{r}} \diamond \eta_t\|^2 + \frac{2z_t^{\mathfrak{g}_n}}{5} h(\Delta)W_{z_t} \\ &\leqslant -\frac{a}{10}\|z_t^{\mathfrak{g}-\mathfrak{r}} \diamond \eta_t\|^2 + \frac{2z_t^{\mathfrak{g}_n}}{5}\Big(-V_{z_t} + \frac{h(\Delta)W_{z_t}}{\Theta}\Big) \\ &\leqslant -\frac{a}{10}\|z_t^{\mathfrak{g}-\mathfrak{r}} \diamond \eta_t\|^2. \end{aligned} \tag{61}$$

which by integration over $[t_0, +\infty)$ gives for all $t \geqslant t_0$

$$\int_{t_0}^{t} \|z_\tau^{\mathfrak{g}-\mathfrak{r}} \diamond \eta_\tau\|^2 d\tau \leqslant \frac{10}{a} V_{z_{t_0}} \tag{62}$$

On the other hand, since σ is monotone increasing, with $\sigma(s) \leqslant s$ for all $s \geqslant 0$, and on account of (57),

$$\dot{z}_t \leqslant \Theta \|z_t^{\mathfrak{g}-\mathfrak{r}} \diamond \eta_t\|^2 \tag{63}$$

which, together with (62), gives for all $t \geqslant t_0$

$$z_t \leqslant z_{t_0} + \frac{10\Theta}{a} V_{z_{t_0}} \tag{64}$$

which contradicts (53). Therefore, $\lim_{t\to+\infty} z_t := z_\infty < +\infty$ which proves claim (C).

(D) We claim $\limsup_{t\to+\infty} q_{z_t}(\xi_t, y_t) \leqslant \delta(\Delta) z_\infty^{2(\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)+1}$.

From (52) and $z \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$ and $x \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}^n)$ (claim (C) and assumption (H1)), we have

$$\dot{V}_{z_t}|_{(31)} \leqslant -\frac{a\|z\|_\infty^{-2|\mathfrak{g}_n|}}{4} V_{z_t} + N_{\|x\|_\infty, \|z\|_\infty, \Delta}$$

for all $t \geqslant 0$ and for some $N_{\|x\|_\infty, \|z\|_\infty, \Delta} > 0$ which depends only on the sup norms $\|x\|_\infty, \|z\|_\infty$ and Δ. This implies that

$$V_{z_t} \leqslant \max\{V_{z_0}, \frac{4\|z\|_\infty^{2|\mathfrak{g}_n|}}{a} N_{\|x\|_\infty, \|z\|_\infty, \Delta}\} \tag{65}$$

for all $t \geqslant 0$ and, therefore, $V_z(\eta) \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$. Since $z \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$ (claim (C)), we conclude that $\eta \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}^n)$ and, therefore, $\xi \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}^n)$ (see the change of coordinates (30)).

Set

$$\alpha_z(\xi, y) := \max\Big\{q_z(\xi, y) - h(\Delta) z_t^{2(\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)+1}, 0\Big\}.$$

On account of the fact that $\xi, x \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}^n)$ and $z \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$, also $\dot{x}, \dot{\xi} \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}^n)$ with $\dot{z} \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R})$ so that $x, \xi \in \mathbf{C}_0^0(\mathbb{R}_\geqslant, \mathbb{R}^n)$ and $z \in \mathbf{C}_0^0(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$. Since $z \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$, by integration of the $\dot{z}_t$ equation we get $\alpha_z(\xi, y) \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant) \cap \mathbf{L}^1(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$. If we prove that also $\alpha_z(\xi, y) \in \mathbf{C}_0^0(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$ it follows

$$\lim_{t\to+\infty} \alpha_{z_t}(\xi_t, y_t) = 0 \tag{66}$$

by virtue of Barbalat's lemma. In order to prove that $\alpha_z(\xi, y) \in \mathbf{C}_0^0(\mathbb{R}_\geqslant, \mathbb{R}_\geqslant)$ it is sufficient to prove that $q_z(\xi, y) \in \mathbf{C}_0^0(\mathbb{R}_\geqslant, \mathbb{R})$ (z^γ, $\gamma \in \mathbb{R}$, is uniformly continuous since z is uniformly continuous and bounded from above and below with bounded derivative and $\max\{\cdot, 0\}$ is uniformly continuous since it is globally Lipschitz). First of all, $\mathbf{sat}(cz^\mathfrak{r}, \xi) \in \mathbf{C}_0^0(\mathbb{R}_\geqslant, \mathbb{R}^n)$: indeed, using (i) of lemma 10, lemma 11 and the triangle inequality with the uniform continuity of ξ and $z^\mathfrak{r}$, for each $\varepsilon > 0$ we always find $\delta, \eta_1, \eta_2 > 0$ such that for all $t_2, t_1 \geqslant 0 : |t_2 - t_1| \leqslant \delta$ we have

$$\begin{aligned}
&\|\mathbf{sat}(cz_{t_2}^\mathfrak{r}, \xi_{t_2}) - \mathbf{sat}(cz_{t_1}^\mathfrak{r}, \xi_{t_1})\| \\
&\leqslant \|\mathbf{sat}(cz_{t_2}^\mathfrak{r}, \xi_{t_2}) - \mathbf{sat}(cz_{t_2}^\mathfrak{r}, \xi_{t_1})\| \\
&+ \|\mathbf{sat}(cz_{t_2}^\mathfrak{r}, \xi_{t_1}) - \mathbf{sat}(cz_{t_1}^\mathfrak{r}, \xi_{t_1})\| \\
&\leqslant 2\|\xi_{t_2} - \xi_{t_1}\| + c\|z_{t_2}^\mathfrak{r} - z_{t_1}^\mathfrak{r}\| \leqslant 2\eta_1 + c\eta_2 < \varepsilon
\end{aligned}$$

which proves that $\mathbf{sat}(cz^\mathfrak{r}, \xi) \in \mathbf{C}_0^0(\mathbb{R}_\geqslant, \mathbb{R}^n)$ and also $z^{\mathfrak{g}-\mathfrak{r}} \diamond \mathbf{sat}(cz^\mathfrak{r}, \xi) \in \mathbf{C}_0^0(\mathbb{R}_\geqslant, \mathbb{R}^n)$. Finally, $y \in \mathbf{C}_0^0(\mathbb{R}_\geqslant, \mathbb{R})$ since $d \in \mathbf{C}_0^0(\mathbb{R}_\geqslant, \mathbb{R})$ and $\xi, x \in \mathbf{L}^\infty(\mathbb{R}_\geqslant, \mathbb{R}^n)$. This proves that $q_z(\xi, y) \in \mathbf{C}_0^0(\mathbb{R}_\geqslant, \mathbb{R})$ being $q_z(\xi, y)$ the product of uniformly continuous functions.

From (66) (recall that $\limsup_{t\to+\infty} a_t \leqslant \limsup_{t\to+\infty}(a_t - b_t) + \limsup_{t\to+\infty} b_t$ when $\limsup_{t\to+\infty}(a_t - b_t)$, $\limsup_{t\to+\infty} b_t < +\infty$) it follows

$$\begin{aligned}
0 &\geqslant \limsup_{t\to+\infty} \Big\{q_{z_t}(\xi_t, y_t) - \delta(\Delta) z_t^{2(\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)+1}\Big\} \\
&\geqslant \limsup_{t\to+\infty} q_{z_t}(\xi_t, y_t) - \delta(\Delta) \limsup_{t\to+\infty} z_t^{2(\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)+1}
\end{aligned} \tag{67}$$

This gives (recall that z_t is nondecreasing with $\lim_{t\to+\infty} z_t < +\infty$ by claim (C))

$$0 \geqslant \limsup_{t\to+\infty} q_{z_t}(\xi_t, y_t) - \delta(\Delta) z_\infty^{2(\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)+1} \tag{68}$$

which finally implies (D).

(E) An upper bound for the limsup of $\|x_t - \xi_t\|^2$. As a consequence of claim (D) and definition of q_z, since $\lim_{t\to+\infty} z_t = z_\infty \in [1, +\infty)$ and using the monotonicity of the degrees $\{\mathfrak{g}_j\}_{j=1,\ldots,n}$

$$\begin{aligned}
&h(\Delta) z_\infty^{2(\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)+1} \geqslant \limsup_{t\to+\infty} q_z(\xi, y) \\
&= -2\Delta^2 + \limsup_{t\to+\infty} |y - C\xi|^2 \\
&+ \limsup_{t\to+\infty} z^{2(\mathfrak{r}_1-\mathfrak{g}_1)} \|z^{\mathfrak{g}-\mathfrak{r}} \diamond (\xi - \mathbf{sat}_{cz^\mathfrak{r}}(\xi))\|^2 \\
&\geqslant -2\Delta^2 + \limsup_{t\to+\infty} |y - C\xi|^2 \\
&+ z_\infty^{2(\mathfrak{r}_1 - \max_i \mathfrak{r}_i - \mathfrak{g}_1 + \mathfrak{g}_n)} \limsup_{t\to+\infty} \|\xi - \mathbf{sat}_{cz^\mathfrak{r}}(\xi)\|^2
\end{aligned}$$

so that

$$\limsup_{t\to+\infty} |y - C\xi|^2$$

$$\leqslant h(\Delta) z_\infty^{2(\mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)+1} + 2\Delta^2 = \mu_{z_\infty}(\Delta),$$
$$\limsup_{t\to+\infty} \|\xi - \mathbf{sat}_{cz^{\mathfrak{r}}}(\xi)\|^2$$
$$\leqslant \mu_{z_\infty}(\Delta) z_\infty^{2(\max_i \mathfrak{r}_i - \mathfrak{r}_1+\mathfrak{g}_1-\mathfrak{g}_n)} = \nu_{z_\infty}(\Delta) \quad (69)$$

Let be $\Omega \subset \mathbb{R}^n$ be a compact set including the origin such that $x_t, \xi_t \in \Omega$ for all $t \geqslant 0$. Under assumption (ii) and according to Battilotti (2014), theorem V.1, there exist $L \in \mathbb{R}^n$, symmetric and positive definite $P \in \mathbb{R}^{n\times n}$ and $\alpha > 0$ (all depending on Ω) such that (21) (with $\psi \equiv 0$) holds for all $x, \xi \in \Omega$. On the other hand, (5)-(6) can be rewritten as follows

$$\dot{x}_t = (A + BF + HC)x_t + \phi(x_t), \quad (70)$$
$$\dot{\xi}_t = (A + BF + HC)\xi_t + \phi(\xi_t) + LC(x_t - \xi_t) + \mathcal{W}_t,$$

where

$$\mathcal{W} := (BF + HC)(\mathbf{sat}_{cz^{\mathfrak{r}}}(\xi) - \xi)$$
$$+\phi(\mathbf{sat}_{cz^{\mathfrak{r}}}(\xi)) - \phi(\xi) + (L_z - L)(y - C\xi) + Ld.$$

Using (69) and the incremental properties of ϕ (assumption (i) of (H0)), recalling that $\limsup_{t\to+\infty} a_t b_t \leqslant \limsup_{t\to+\infty} a_t \limsup_{t\to+\infty} b_t$ and $\limsup_{t\to+\infty} f(b_t) \leqslant \sup_{\|b\|\leqslant 2n \limsup_{t\to+\infty}\|c_t\|} f(b)$ if $\|b_t\| \leqslant n\|c_t\|$ for all $t \geqslant 0$ with $b_t, c_t \in \mathbb{R}^n$),

$$\limsup_{t\to+\infty} \|\mathcal{W}_t\| \leqslant \Big(\|BF + HC\|$$
$$+ \sup_{\substack{\|w_1\|\leqslant 2nc\|z_\infty^{\mathfrak{r}}\| \\ \|w_2\|\leqslant 2n(\sqrt{\nu_{z_\infty}(\Delta)}+c\|z_\infty^{\mathfrak{r}}\|)}} \|\Phi_U(w_1, w_2)\|\Big)\sqrt{\nu_{z_\infty}(\Delta)}$$
$$+\sqrt{\mu_{z_\infty}(\Delta)}\|L_{z_\infty} - L\| + \|L\|\Delta = \chi_{z_\infty}(\Delta) \quad (71)$$

Pick $\epsilon > 0$ and let $T_\epsilon > 0$ be such that

$$\|\mathcal{W}_{t+T_\epsilon}\| \leqslant \chi_{z_\infty}(\Delta) + \epsilon,\ \forall t \geqslant 0 \quad (72)$$

(which always exists by (71)). With $V_t = \|x_t - \xi_t\|_P^2$ we have from (21) and (70) and for all $t \geqslant 0$

$$\dot{V}_{t+T_\epsilon}|_{(70)} \leqslant -\alpha V_{t+T_\epsilon} + \frac{\lambda_{max}(P)}{\alpha}\|\mathcal{W}_{t+T_\epsilon}\|^2,$$

so that, on account of (72),

$$\lambda_{min}(P)\|x_{t+T_\epsilon} - \xi_{t+T_\epsilon}\|^2 \leqslant V_{t+T_\epsilon}$$
$$\leqslant V_{T_\epsilon} e^{-\alpha t} + \frac{\lambda_{max}(P)(\chi_{z_\infty}(\Delta)+\epsilon)^2}{\alpha}\int_{T_\epsilon}^{t+T_\epsilon} e^{-\alpha(t+T_\epsilon - s)} ds$$
$$= V_{T_\epsilon} e^{-\alpha t} + \frac{\lambda_{max}(P)(\chi_{z_\infty}(\Delta)+\epsilon)^2}{\alpha^2}[1 - e^{-\alpha t}]$$

Passing to the limsup on both sides of the above inequality, we get

$$\limsup_{t\to+\infty} \|x_t - \xi_t\|^2 = \limsup_{t\to+\infty} \|x_{t+T_\epsilon} - \xi_{t+T_\epsilon}\|^2$$
$$\leqslant \frac{\lambda_{max}(P)(\chi_{z_\infty}(\Delta)+\epsilon)^2}{\alpha^2 \lambda_{min}(P)} \quad (73)$$

which, on account of ϵ being arbitrary, gives the conclusions of theorem 5.

5 Conclusions

We have presented a class of nonlinear observers for systems with noisy measurements and bounded trajectories. The main ingredients are: domination techniques of the incrementally homogeneous (in the upper bound) nonlinearities of the observation error system with its linear approximation, gain adaptation and estimate saturations with dynamically tuned saturation levels. The adaptation of the gains and saturation levels is implemented through a stable filter which regulates its output according to a suitable function of the squared norm of the measured output estimation error. Our observer guarantees an upper bound for the limsup of the norm of the estimation error depending on the limsup of the norm of the measurement noise. In future research we will consider disturbances affecting also the state equations and unbounded state trajectories.

A Incremental homogeneity in the generalized sense: a review

The notion of (incremental) homogeneity has been introduced in Battilotti (2014) in the context of semi-global stabilization and observer design problems. Here we recall this notion in a slightly more general form.

A.1 Definitions

Definition 8 *A parametrized function $\phi_z \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}^l)$, $z \in \mathbb{R}_>$, is said to be incrementally homogeneous (i.h.) with quadruple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \phi)$ if there exist $\mathfrak{d} \in \mathbb{R}^l, \mathfrak{h} \in \mathbb{R}^n$, $\mathfrak{r} \in \mathbb{R}^n_>$ and $\phi \in \mathbf{C}^0(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^{l\times n})$ such that for all $\epsilon > 0$ and $w', w'' \in \mathbb{R}^n$*

$$\phi_\epsilon(\epsilon^{\mathfrak{r}} \diamond w') - \phi_\epsilon(\epsilon^{\mathfrak{r}} \diamond w'')$$
$$= \epsilon^{\mathfrak{d}} \diamond \Big(\phi(w', w'')\Big(\epsilon^{\mathfrak{h}} \diamond (w' - w'')\Big)\Big)$$

In few words, the increment of ϕ_ϵ between two dilated points $\epsilon^{\mathfrak{r}} \diamond w'$ and $\epsilon^{\mathfrak{r}} \diamond w''$ behaves "homogeneously" in the sense that it is equal to the image of a linear operator $\phi(w', w'') \in \mathbb{R}^{l\times n}$ under the increment between the two dilated points $\epsilon^{\mathfrak{h}} \diamond w'$ and $\epsilon^{\mathfrak{h}} \diamond w''$, followed by a componentwise dilation by $\epsilon^{\mathfrak{d}}$. The vector $\mathfrak{d} \in \mathbb{R}^l$ describes the "vertical" degrees and the vector $\mathfrak{h} \in \mathbb{R}^n$ describes the "horizontal" degrees. The notion of incremental homogeneity incapsulates as a particular case the notion of homogeneity (see for example Rosier (1998)). When w'' is set to 0 in definition 8 we say that ϕ_z *is homogeneous with quadruple* $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \phi)$.

Note that the function ϕ_z may be parametrized by the dilating parameter itself. The function $\phi_z(x) := x_1 + x_2^3$ (in this case ϕ_z does not depend on the dilating parameter) is i.h. with quadruple $(\mathfrak{r}, 0, \mathfrak{h}, \phi)$, where $\mathfrak{r} := (1,2)^T$, $\mathfrak{h} := (1,6)^T$ and $\phi(w', w'') := (1, (w_2')^2 + (w_2'')^2 + w_2''w_2')$. The function $\phi_z(x) := z(x_1 + x_2^3)$ (here ϕ does depend on the dilating parameter) is i.h. with quadruple $(\mathfrak{r}, 1, \mathfrak{h}, \phi)$ and the same ϕ above.

There are functions, like $\sin x$, which are not i.h. but behaves in the upper bound as an i.h. function. This motivates the following definition ($\langle\!\langle a \rangle\!\rangle$ denotes the column vector of the absolute values of the elements of $a \in \mathbb{R}^n$).

Definition 9 *A parametrized function $\phi \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}^l)$, $z \in \mathbb{R}_>$, is said to be incrementally homogeneous in the upper bound (i.h.u.b.) with quadruple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \phi_U)$ if there exist $\mathfrak{d} \in \mathbb{R}^l$, $\mathfrak{h} \in \mathbb{R}^n$, $\mathfrak{r} \in \mathbb{R}^n_>$, $\phi_U \in \mathbf{C}^0(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^{l\times n}_\geqslant)$ such that for all $\epsilon \geqslant 1$ and $w', w'' \in \mathbb{R}^n$*

$$\langle\!\langle \phi_\epsilon(\epsilon^{\mathfrak{r}} \diamond w') - \phi_\epsilon(\epsilon^{\mathfrak{r}} \diamond w'') \rangle\!\rangle \leq \epsilon^{\mathfrak{d}} \diamond \left(\phi_U(w', w'') \left\langle\!\!\left\langle \epsilon^{\mathfrak{h}} \diamond (w' - w'') \right\rangle\!\!\right\rangle \right)$$

When w'' is set to 0 in definition 9 we will simply say that ϕ_z *is homogeneous in the upper bound with quadruple* $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \phi_U)$.

The function $\phi_z(x) := z\,(\,x_2 \quad x_2^3 g(x_1)\,)^T$, $g \in \mathbf{C}^0(\mathbb{R}, \mathbb{R})$ any bounded and globally Lipschitz function, is i.h.u.b. with triple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \phi_U)$, where $\mathfrak{r} := (1,2)^T$, $\mathfrak{d} := (3,7)^T$, $\mathfrak{h} := (1,0)^T$ and the matrix $\phi_U(w', w'')$ defined as

$$\begin{aligned}
&[\phi_U(w', w'')]_{11} := 0, [\phi_U(w', w'')]_{12} := 1,\\
&[\phi_U(w', w'')]_{21} := (w_2'')^3 \frac{|g(w_1') - g(w_1'')|}{|w_1' - w_1''|},\\
&[\phi_U(w', w'')]_{22} := |(w_2')^2 + (w_2'')^2 + w_2'w_2''||g(w_1')|.
\end{aligned}$$

A.2 Properties of incrementally homogeneous functions

The proof of the following properties can be found in Battilotti (2014).

(P0) *For any i.h.u.b. (resp. i.h.) functions $\phi_z \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}^l)$ with quadruple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \phi_U)$ and $\psi_z \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}^l)$ with quadruple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \psi_U)$, the function $\phi_z + \psi_z$ is i.h.u.b. (resp. i.h.) with quadruple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \phi_U + \psi_U)$.*

(P1) *Any i.h.u.b. (resp. i.h.) function $\phi_z \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}^l)$ with quadruple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \phi_U)$ and diagonal ϕ_U is also i.h.u.b. (resp. i.h.) with quadruple $(\mathfrak{r}, \mathfrak{d}', \mathfrak{h}', \phi_U)$ for all pairs $(\mathfrak{d}', \mathfrak{h}')$ such that $\mathfrak{d}+\mathfrak{h} \leq \mathfrak{d}'+\mathfrak{h}'$ (resp. $\mathfrak{d}+\mathfrak{h} = \mathfrak{d}'+\mathfrak{h}'$).*

In particular, we can replace the degrees $(\mathfrak{d}, \mathfrak{h})$ with some upper bounds $(\mathfrak{d}', \mathfrak{h}')$ or swap them: $(\mathfrak{d}', \mathfrak{h}') = (\mathfrak{h}, \mathfrak{d})$.

(P2) *For any i.h.u.b. functions $\phi_z \in \mathbf{C}^0(\mathbb{R}^s, \mathbb{R}^l)$ with quadruple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \phi_U)$ and $\psi_z \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}^s)$ with quadruple $(\mathfrak{r}, -\mathfrak{h} + \mathfrak{r}, \mathfrak{p}, \psi_U)$ if there exists $\pi_U \in \mathbf{C}^0(\mathbb{R}^s \times \mathbb{R}^s, \mathbb{R}^{l\times s}_\geqslant)$ such that for all $\epsilon \geqslant 1$ and $w, z \in \mathbb{R}^n$*

$$\phi_U(w', z')\Big|_{\substack{w' = \epsilon^{-\mathfrak{r}} \diamond \psi_\epsilon(\epsilon^{\mathfrak{r}} \diamond w)\\ z' = \epsilon^{-\mathfrak{r}} \diamond \psi_\epsilon(\epsilon^{\mathfrak{r}} \diamond z)}} \leq \pi_U(w, z) \tag{A.1}$$

then $\phi_z \circ \psi_z$ is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{p}, \pi_U\psi_U)$.

In particular, for ϕ with constant ϕ_U (A.1) is trivially satisfied with $\Pi = \phi_U$.

Let $\mathrm{Im}\{W\}$ denote the vector space generated by the columns of the matrix W.

(P3.1) *given any i.h.u.b. (resp.i.h.) $\phi_z \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}^l)$ with quadruple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \phi_U)$, $A\phi_z$ (resp. $A^T\phi_z$) is i.h.u.b. (resp.i.h.) with quadruple $(\mathfrak{r}, A\mathfrak{d} + \mathfrak{z}, \mathfrak{h}, A\phi_U)$ (resp. $(\mathfrak{r}, A^T\mathfrak{d} + \mathfrak{z}, \mathfrak{h}, A^T\phi_U)$), for any $\mathfrak{z} \in \mathrm{Im}\{I - AA^T\}$ (resp. $\mathfrak{z} \in \mathrm{Im}\{I - A^TA\}$).*

(P3.2) *given any i.h.u.b. (resp.i.h.) $\phi_z \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}^l)$, $(z, x) \mapsto \phi_z(x)$, with quadruple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \phi_U)$ and constant ϕ_U, $\phi_z \circ A$ (resp. $\phi_z \circ A^T$) is i.h.u.b. (resp.i.h.) with quadruple $(\mathfrak{r}, \mathfrak{d}, A^T(\mathfrak{h} - \mathfrak{r}) + \mathfrak{r} + \mathfrak{z}, \phi_U A)$ (resp. $(\mathfrak{r}, \mathfrak{d}, A(\mathfrak{h} - \mathfrak{r}) + \mathfrak{r} + \mathfrak{z}, \phi_U A^T)$), for any $\mathfrak{z} \in \mathrm{Im}\{I - A^TA\}$ (resp. $\mathfrak{z} \in \mathrm{Im}\{I - AA^T\}$).*

B Auxiliary results

Lemma 10 *If $\mathbf{sat}_h$ is a saturation function with levels $h \in \mathbb{R}^n_>$, for all $w, z \in \mathbb{R}^n$*

(i) $\langle\!\langle \mathbf{sat}_h(w) - \mathbf{sat}_h(z) \rangle\!\rangle \preceq 2\langle\!\langle w - z \rangle\!\rangle$
(ii) $\langle\!\langle \mathbf{sat}_h(w) \rangle\!\rangle \preceq \langle\!\langle w \rangle\!\rangle$, *(iii)* $\langle\!\langle \mathbf{sat}_h(w) \rangle\!\rangle \preceq \langle\!\langle h \rangle\!\rangle$. □

Lemma 11 *If $\mathbf{sat}_h$ and $\mathbf{sat}_k$ are saturation functions with levels $h \in \mathbb{R}^n_>$ and, respectively, $k \in \mathbb{R}^n_>$, $\langle\!\langle \mathbf{sat}_h(x) - \mathbf{sat}_k(x) \rangle\!\rangle \preceq \langle\!\langle k - h \rangle\!\rangle$ for all $x \in \mathbb{R}^n$.* □

For the proof of lemmas 10 and 11 see lemmas 9 and 10 of Battilotti (2015b).

We usually identify matrices A with linear applications $A : \eta \mapsto A\eta$. In this sense we mean that A is i.h.u.b. (or i.h.) with some quadruple.

Lemma 12 *With assumption (H0) and for each $c, k > 0$ and diagonal positive definite Γ,*

- *(i) Σ_z, defined in (32), is i.h. with quadruple $(\mathfrak{r}, \mathfrak{r} + \mathfrak{g}, \mathfrak{g}, \Sigma)$, where Σ is defined in (40),*
- *(ii) π_z, defined in (33), is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r} + \mathfrak{g}, \mathfrak{g}, \Pi_U)$, where Π_U is defined in (41) and $\Phi(c) \in \mathbb{R}^{n\times n}$ is a matrix satisfying (24),*

(iii) κ_z*, defined in (56), is i.h.u.b. with quadruple* $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, 3X_U)$*,*
(iv) ϕ *is i.h.u.b. with quadruple* $(\mathfrak{r}, \mathfrak{r}+\mathfrak{g}, \mathfrak{g}, \phi_U)$*,*
(v) $A^T G_z X_z$ *is i.h.u.b. with quadruple* $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, A^T\Gamma X_U)$*.*□

PROOF. Proof of part (iv). Notice that $\phi = AA^T\phi + (I - AA^T)\phi$ and that $A(\mathfrak{r}-\mathfrak{g}) \preceq AA^T(\mathfrak{r}+\mathfrak{g})$ (by (B.3) since $AA^TA = A$) and $AA^T(\mathfrak{r}+\mathfrak{g}) + (I - AA^T)(\mathfrak{r}+\mathfrak{g}) = \mathfrak{r}+\mathfrak{g}$. From (H0) and (P0), (P1) and (P3.1) we get the desired result.

Proof of parts (i), (ii) and (v). We break up the proof in several claims. Condition (iii) of assumption (H0) reads out as

$$2A\mathfrak{g} + AA^T(\mathfrak{r}-\mathfrak{g}) \preceq A(\mathfrak{r}-\mathfrak{g}) \preceq AA^T(\mathfrak{r}+\mathfrak{g}) \qquad \text{(B.1)}$$

and notice the following ensuing inequalities

$$A^T(\mathfrak{r}+2A\mathfrak{g}-\mathfrak{g}) \preceq A^TA(\mathfrak{r}-\mathfrak{g}) \qquad \text{(B.2)}$$
$$AA^T(A\mathfrak{r}-\mathfrak{r}) \preceq AA^T(A\mathfrak{g}+\mathfrak{g}) \qquad \text{(B.3)}$$

(the first by multiplying the first inequality of (B.1) by A^T and using $A^TAA^T = A^T$, the second by multiplying the second inequality of (B.1) by AA^T and using $AA^TAA^T = AA^T$).

Claim I. A^TG_z *(resp.* $A^TG_z^2$*) is i.h.u.b. with quadruple* $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, A^T\Gamma)$ *(resp.* $(\mathfrak{r}, \mathfrak{r}+\mathfrak{g}, \mathfrak{g}, A^T\Gamma^2)$*).* Since by its definition G_z is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}, 2A\mathfrak{g}, \Gamma)$ and Γ is diagonal, by property (P1) with $\mathfrak{d}' := \mathfrak{r}+2A\mathfrak{g}-\mathfrak{g}$ and $\mathfrak{h}' := \mathfrak{g}$, G_z is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}+2A\mathfrak{g}-\mathfrak{g}, \mathfrak{g}, \Gamma)$. By (P3.1) with $\mathfrak{z} := (I - A^TA)(\mathfrak{r}-\mathfrak{g})$, A^TG_z is i.h.u.b. with quadruple $(\mathfrak{r}, A^T(\mathfrak{r}+2A\mathfrak{g}-\mathfrak{g}) + (I - A^TA)(\mathfrak{r}-\mathfrak{g}), \mathfrak{g}, A^T\Gamma)$. On account of (B.2) and (P1) we get that A^TG_z is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, A^T\Gamma)$, i.e. the first part of the claim. On the other hand, since by its definition G_z^2 is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}, 4A\mathfrak{g}, \Gamma^2)$ and Γ is diagonal, by (P1) with $\mathfrak{d}' := \mathfrak{r}+4A\mathfrak{g}-\mathfrak{g}$ and $\mathfrak{h}' := \mathfrak{g}$, G_z is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}+4A\mathfrak{g}-\mathfrak{g}, \mathfrak{g}, \Gamma)$. By (P3.1) with $\mathfrak{z} := (I - A^TA)(\mathfrak{r}+\mathfrak{g})$, A^TG_z is i.h.u.b. with quadruple $(\mathfrak{r}, A^T(\mathfrak{r}+4A\mathfrak{g}-\mathfrak{g}) + (I - A^TA)(\mathfrak{r}+\mathfrak{g}), \mathfrak{g}, A^T\Gamma^2)$. On account of (B.2) and (P1) we get that $A^TG_z^2$ is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}+\mathfrak{g}, \mathfrak{g}, A^T\Gamma^2)$, i.e. the second part of the claim.

Claim II. $X_z := (I - A^TG_z)^{-1}$*, is i.h.u.b. with quadruple* $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, X_U)$, $X_U := (I - A^T\Gamma)^{-1}$. Notice that the identity function ι is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}, 0, I)$. Therefore, since I is diagonal and invoking (P1) with $\mathfrak{d}' := \mathfrak{r}+\mathfrak{g}$ and $\mathfrak{h}' := -\mathfrak{g}$, ι is is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, I)$. On the other hand, notice that $X_z := (I - A^TG_z)^{-1} = \sum_{j=0}^{n-1}(A^TG_z)^j$ (notice that $(I - A^TG_z)\sum_{j=0}^{n-1}(A^TG_z)^j = I$ since $(A^TG_z)^n = 0$). As already established, $(A^TG_z)^0 = I$ is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, I)$. We proceed by induction. Assume that $(A^TG_z)^j$ for some $j = 1, \ldots, n-1$, is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, (A^T\Gamma)^j)$. Since $(A^TG_z)^{j+1} = (A^TG_z)^jA^TG_z$ and both $(A^TG_z)^j$ (induction step) and A^TG_z (claim I) are i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, (A^T\Gamma)^j)$ and, respectively, $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, A^T\Gamma)$, by property (P2) it follows that $(A^TG_z)^{j+1}$ is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, (A^T\Gamma)^{j+1})$. By induction and property (P0), since $X_U := (I - A^T\Gamma)^{-1} = \sum_{j=0}^{n-1}(A^T\Gamma)^j$, it follows that X_z is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, X_U)$.

Claim III. H_z *is i.h. with quadruple* $(\mathfrak{r}, \mathfrak{r}+\mathfrak{g}, \mathfrak{g}, H)$. Since by its definition G_z is i.h. with quadruple $(\mathfrak{r}, \mathfrak{r}, 2A\mathfrak{g}, \Gamma)$, by using property (P3.1) with $\mathfrak{z} := 0$ and (P3.2) with $\mathfrak{z} := (I - A^TA)2\mathfrak{g}$, A^TG_zA is i.h. with quadruple $(\mathfrak{r}, A^T\mathfrak{r}, 2\mathfrak{g} - A^T\mathfrak{r} + \mathfrak{r}, A^T\Gamma A)$. Since A^TG_zA is diagonal, by (P1) with $\mathfrak{d}' := \mathfrak{r}+\mathfrak{g}$ and $\mathfrak{h}' := \mathfrak{g}$, A^TG_zA is i.h. with quadruple $(\mathfrak{r}, \mathfrak{r}+\mathfrak{g}, \mathfrak{g}, A^T\Gamma A)$. Similarly, $kz^{2C\mathfrak{g}}C^TC$ is i.h. with quadruple $(\mathfrak{r}, \mathfrak{r}+\mathfrak{g}, \mathfrak{g}, kC^TC)$. By (P0) the claim follows.

Claim IV. $\mathbf{sat}_{cz^{\mathfrak{r}}}$ *(resp.* $\mathbf{sat}_{cz^{\mathfrak{r}}} \circ X_z$*) are i.h.u.b. with quadruple* $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, 2I)$ *(resp.* $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, 2X)$*).* On account of (i) of lemma 10 with $h := cz^{\mathfrak{r}}$, $(z, \eta) \mapsto \mathbf{sat}_{cz^{\mathfrak{r}}}(\eta)$ is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}, 0, 2I)$. By (P1) with $\mathfrak{d}' := \mathfrak{r}-\mathfrak{g}$ and $\mathfrak{h}' := \mathfrak{g}$, $\mathbf{sat}_{cz^{\mathfrak{r}}}$ is also i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, 2I)$, i.e. the first part of the claim. Finally, by virtue of (P2) and claim II we obtain the second part of the claim.

Claim V. AX_z *(resp.* $A^TG_z\,X_z$*)) is i.h.u.b. with quadruple* $(\mathfrak{r}, \mathfrak{r}+\mathfrak{g}, \mathfrak{g}, A)$ *(resp.* $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, A^T\Gamma X_U)$*).* Note that the identity function ι is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}, 0, I)$, therefore by (P1) with $\mathfrak{d}' := \mathfrak{r}-\mathfrak{g}$ and $\mathfrak{h}' := \mathfrak{g}$, $(z, \eta) \mapsto z$ is also i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, I)$. Using (P3.1) with $\mathfrak{z} := (I - AA^T)(\mathfrak{g}+\mathfrak{r})$, A is i.h.u.b. with quadruple $(\mathfrak{r}, A(\mathfrak{r}-\mathfrak{g}) + (I - AA^T)(\mathfrak{g}+\mathfrak{r}), \mathfrak{g}, A)$. Upon noticing that $A\mathfrak{r} - AA^T\mathfrak{r} \preceq A\mathfrak{g} + AA^T\mathfrak{g}$ (from (B.3) since $AA^TA = A$) and on account of (B.3), we get by (P1) that A is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}+\mathfrak{g}, \mathfrak{g}, A)$. From claim II and (P2) it follows that AX_z is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}+\mathfrak{g}, \mathfrak{g}, A)$. The second part of the claim follows directly from claims I and II and (P2).

Claims III and V prove (i) and (v) of the lemma. Let us prove part (ii). Since $\|z^{-\mathfrak{r}} \diamond \mathbf{sat}_{cz^{\mathfrak{r}}}(w)\| \leqslant cn$ for all $w \in \mathbb{R}^n$ and $z \geqslant 1$, we find out that any matrix $\Phi(c) \in \mathbb{R}^{n\times n}$ for which (24) holds true is such that $A^T\phi(z^{-\mathfrak{r}} \diamond \mathbf{sat}_{cz^{\mathfrak{r}}}(X_zw), z^{-\mathfrak{r}} \diamond \mathbf{sat}_{cz^{\mathfrak{r}}}(X_z\eta)) \preceq A^T\Phi(c)$ for all $w, \eta \in \mathbb{R}^n$ and $z \geqslant 1$. By virtue of (H0), claim IV and property (P2), it follows that $A^T((BF+HC)\mathbf{sat}_{cz^{\mathfrak{r}}} \circ X_z + \phi \circ \mathbf{sat}_{cz^{\mathfrak{r}}} \circ X_z)$ is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}-\mathfrak{g}, \mathfrak{g}, 2A^T[BF_U + H_UC + \Phi(c)]X_U)$. Finally, from claim III, (P2) and (P0) and on account of part (iv) of the lemma it follows that $(I - H_zA^T)((BF+HC)\mathbf{sat}_{cz^{\mathfrak{r}}} \circ X_z + \phi \circ \mathbf{sat}_{cz^{\mathfrak{r}}} \circ X_z)$ is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}+\mathfrak{g}, \mathfrak{g}, 2(I + HA^T)[BF_U + H_UC + \Phi(c)]X_U)$.

On the other hand, by claims I, V and (P2) and (P0),

$[A - A^T G_z^2]X_z$ is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r} + \mathfrak{g}, \mathfrak{g}, [A + A^T \Gamma^2]X_U)$. Using (P0) we obtain part (ii) of our lemma.

Proof of part (iii). As part (ii) using (H0), claims II and IV and (P0), (P2). •

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