EFFECTS OF BOUNDARY CURVATURE ON SURFACE SUPERCONDUCTIVITY

MICHELE CORREGGI AND NICOLAS ROUGERIE

Abstract. We investigate, within 2D Ginzburg-Landau theory, the ground state of a type-II superconducting cylinder in a parallel magnetic field varying between the second and third critical values. In this regime, superconductivity is restricted to a thin shell along the boundary of the sample and is to leading order constant in the direction tangential to the boundary. We exhibit a correction to this effect, showing that the curvature of the sample affects the distribution of superconductivity.

Contents

1. Introduction and Main Result

The response of type-II superconductors to external magnetic fields is a rich source of fascinating mathematical problems [BBH, FH3, SS, Sig]. Physically, this is because of the occurrence of mixed states where normal and superconducting regions may coexist. If the sample is a very long cylinder and the applied magnetic field is parallel to it, one may adopt a 2D description on a cross-section of the cylinder. One then distinguishes mainly two types of mixed phases:

- The vortex lattice (Abrikosov lattice [Abr]) where the normal regions take the form of vortices, and are arranged on a triangular lattice embedded in a sea of superconducting material;
- The surface superconductivity state where the whole bulk of the sample is in the normal state, and superconductivity only survives close to the boundary.

The second situation shall concern us here. In this regime, the magnetic field is very large, varies between two critical values H_{c2} and H_{c3}, and mostly penetrates the sample. That superconducting electrons may still exist because of boundary effects is a highly non-trivial fact, first derived in [SJdG]. At leading order, the phenomenon may be completely understood by considering the case of an infinite half-plane sample, with a straight boundary. It thus has some universal features: although superconducting electrons concentrate along the boundary, the geometry of the latter does not affect their distribution much. This is because the density of superconducting electrons essentially only varies in the direction

Date: November, 2015.

normal to the boundary, as we proved rigorously in [CR1, CR2], following several earlier contributions [Alm1, AH, FHP, FH1, FH2, FK, LP, Pan] (see [FH3] for a review).

However, in order to prove some of the most refined results in [CR1, CR2], we had to precisely estimate subleading order contributions to the energy. These do not share the universal character of the leading order, in that they depend on the sample, via the curvature of its boundary. This is reminiscent of earlier works, e.g., [FH1] (see [FH3, Chapters 13 and 15] for extensive references), where it has been proved that, when decreasing the magnetic field just below H_{c3}, superconductivity appears first where the curvature of the boundary is maximum. In this paper we aim at evaluating the effect of sample curvature in the whole regime of magnetic fields comprised between H_{c2} and H_{c3}. We shall give a simple expression of the curvature dependent contribution to surface superconductivity, which, again, appears only at subleading order in the energy, and thus requires rather refined estimates.

Our setting is the following: we consider the Ginzburg-Landau functional (in convenient units whose relation to other conventions is discussed[1] in [CR1, CR2])

$$\mathcal{G}_{\varepsilon}^{\mathrm{GL}}[\Psi,\mathbf{A}] = \int_{\Omega} \mathrm{d}\mathbf{r} \left\{ \left| \left(\nabla + i \frac{\mathbf{A}}{\varepsilon^2} \right) \Psi \right|^2 - \frac{1}{2b\varepsilon^2} \left(2|\Psi|^2 - |\Psi|^4 \right) + \frac{1}{\varepsilon^4} \left| \mathrm{curl} \mathbf{A} - 1 \right|^2 \right\}. \quad (1.1)$$

The domain $\Omega \subset \mathbb{R}^2$ represents the cross-section of an infinitely long cylinder of superconducting material. We assume that it is bounded, simply connected and that its boundary is smooth. The applied magnetic field is perpendicular to Ω.

We shall denote

$$E_{\varepsilon}^{\mathrm{GL}} := \min_{(\Psi,\mathbf{A}) \in \mathscr{D}^{\mathrm{GL}}} \mathcal{G}_{\varepsilon}^{\mathrm{GL}}[\Psi,\mathbf{A}], \quad (1.2)$$

with

$$\mathscr{D}^{\mathrm{GL}} := \left\{ (\Psi,\mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1(\Omega;\mathbb{R}^2) \right\}, \quad (1.3)$$

and denote by $(\Psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}})$ a minimizing pair. We recall that $|\Psi^{\mathrm{GL}}|^2$ gives the local relative density of superconducting electrons (bound in Cooper pairs) and that curl $\mathbf{A}^{\mathrm{GL}}$ is the induced magnetic field in the sample.

We are interested in the behavior of $|\Psi^{\mathrm{GL}}|$ in the surface superconductivity regime

$$1 < b < \Theta_0^{-1}, \quad (1.4)$$

where Θ_0 is the minimal ground state energy of the shifted harmonic oscillator on the half-line:

$$\Theta_0 := \min_{\alpha \in \mathbb{R}} \min_{\|u\|_2 = 1} \int_0^{+\infty} \mathrm{d}t \left\{ |\partial_t u|^2 + (t+\alpha)^2 |u|^2 \right\}. \quad (1.5)$$

This corresponds to asking that the applied magnetic field varies between H_{c2} and H_{c3}, and we shall also assume that ε is a small parameter, in order to prove asymptotic results in the limit $\varepsilon \to 0$. Physically this means we consider an "extreme" type-II superconductor.

In the parameter regime of our interest, the GL order parameter is concentrated near the boundary of the sample and the induced magnetic field is very close to the (constant) applied one. To leading order, superconducting electron pairs are uniformly distributed along the boundary as a function of the tangential variable. The main variations are in the direction normal to the boundary. Our main result in this note is an asymptotic estimate

[1] Note that in [CR1, CR2], an extra factor of b has been mistakenly inserted in front of the last term. This does not have any incidence on the results since this term is negligible in the regime of our interest.

for $|\Psi^{\mathrm{GL}}|^4$ which exhibits a subleading (in ε) curvature-dependent correction. This shows that variations of the boundary's curvature influence the superconductivity distribution.

In our previous papers [CR1, CR2], we have emphasized the role played by the simplified functional

$$\mathcal{E}^{\mathrm{1D}}_{k,\alpha}[f] := \int_0^{c_0|\log\varepsilon|} \mathrm{d}t(1-\varepsilon k t)\left\{|\partial_t f|^2 + V_{k,\alpha}(t) f^2 - \tfrac{1}{2b}\left(2f^2 - f^4\right)\right\} \tag{1.6}$$

where

$$V_{k,\alpha}(t) := \frac{(t+\alpha-\frac{1}{2}\varepsilon k t^2)^2}{(1-\varepsilon k t)^2} \tag{1.7}$$

and c_0 is a (somewhat arbitrarily) fixed, large enough, constant. Note that in the limit $\varepsilon \to 0$ the above expression reduces to a 1D nonlinear energy independent of the curvature, which is known to provide the leading order contribution to the GL asymptotics (see [CR1] and references therein).

This functional should be thought of as giving the GL energy to subleading order in the case of a sample Ω which is a "disc" of curvature k, i.e., either a disc of radius $R = k^{-1}$ when $k > 0$ or the exterior of such a disc when $k < 0$. The variable t corresponds to the coordinate normal to the boundary (in units of ε). Denoting by s the tangential coordinate, $f(t)e^{-i\alpha s}$ gives an ansatz for the GL order parameter in boundary coordinates. To get an optimal energy, this functional should be minimized with respect to both the function f and the number α, leading to an optimal profile f_k, an optimal phase $\alpha(k)$ and an optimal energy $E^{\mathrm{1D}}_\star(k)$.

Since we work in the regime $\varepsilon \to 0$, it is natural to try to consider a perturbative expansion of $E^{\mathrm{1D}}_\star(k(s))$. We then get that the leading order is given by the $k = 0$ functional (corresponding to a half-plane sample and extensively studied in the literature [AH, FHP, Pan]):

$$\mathcal{E}^{\mathrm{1D}}_{0,\alpha}[f] := \int_0^{+\infty} \mathrm{d}t\left\{|\partial_t f|^2 + (t+\alpha)^2 f^2 - \tfrac{1}{2b}\left(2f^2 - f^4\right)\right\}, \tag{1.8}$$

and the first correction by $-\varepsilon k$ times

$$\mathcal{E}^{\mathrm{corr}}_{\alpha}[f] := \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, t\left\{|\partial_t f|^2 + f^2\left(-\alpha(t+\alpha) - \frac{1}{b} + \frac{1}{2b} f^2\right)\right\} \tag{1.9}$$

which is obtained by retaining only linear terms in εk when expanding (1.6). We shall denote E^{1D}_0 the minimum of $\mathcal{E}^{\mathrm{1D}}_{0,\alpha}[f]$, and α_0, f_0 a minimizing pair (α_0 is unique, f_0 also is, up to a sign). Note the mild abuse of notation: the $k = 0$ functional is well-defined even for $c_0 = +\infty$, and we take this convention. Due to known decay estimates for f_0 (see, e.g., [CR1, Proposition 3.3]), this creates only an exponentially small discrepancy in ε, provided c_0 is a large enough constant.

We previously proved energy estimates relating the full GL energy to the infimum of the above 1D, curvature-dependent functional. Since our method was local, and the GL energy density is related to $|\Psi^{\mathrm{GL}}|^4$, it is natural to expect a result about the distribution of the latter quantity from the energy estimate. The goal of this paper is to provide this estimate.

Let us first introduce scaled boundary coordinates: the surface superconductivity layer

$$\tilde{\mathcal{A}}_\varepsilon := \left\{\mathbf{r} \in \Omega \mid \tau \leq c_0 \varepsilon |\log\varepsilon|\right\}, \tag{1.10}$$

where

$$\tau := \mathrm{dist}(\mathbf{r}, \partial\Omega), \tag{1.11}$$

can be mapped to

$$\mathcal{A}_\varepsilon := \{(s,t) \in [0, |\partial\Omega|] \times [0, c_0 |\log \varepsilon|]\} \tag{1.12}$$

via a diffeomorphism Φ.

For technical reasons (see Remark 1.2 below), we can only evaluate $\int_D |\Psi^{\mathrm{GL}}|^4$ with the desired precision in the case that the set D looks "rectangular" in boundary coordinates. Let then $D \subset \Omega$ be a measurable set *independent of* ε such that

$$\Phi(D \cap \tilde{\mathcal{A}}_\varepsilon) = [s_D, s_D'] \times [0, c_0 |\log \varepsilon|] \tag{1.13}$$

for some $s_D, s_D' \in [0, |\partial\Omega|]$. Notice that this implies that the boundary of D intersects $\partial\Omega$ with $\pi/2$ angles. Our main result is the following:

Theorem 1.1 (**Curvature dependence of the order parameter**).
Let Ψ^{GL} be a GL minimizer and $D \subset \Omega$ be a measurable set such that (1.13) *holds. Denote $s \mapsto k(s)$ the curvature of $\partial\Omega$ as a (smooth) function of the tangential coordinate s. For any $1 < b < \Theta_0^{-1}$, in the limit $\varepsilon \to 0$,*

$$\int_D \mathrm{d}\mathbf{r}\, |\Psi^{\mathrm{GL}}|^4 = \varepsilon\, C_1(b) |\partial\Omega \cap \partial D| + \varepsilon^2 C_2(b) \int_{\partial D \cap \partial\Omega} \mathrm{d}s\, k(s) + o(\varepsilon^2), \tag{1.14}$$

where $\mathrm{d}s$ *stands for the 1D Lebesgue measure along $\partial\Omega$ and*

$$C_1(b) = -2b E_0^{\mathrm{1D}} = \int_0^{+\infty} \mathrm{d}t\, f_0^4 > 0 \tag{1.15}$$

$$C_2(b) = 2b\, \mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] = \tfrac{2}{3} b f_0^2(0) - 2b\alpha_0 E_0^{\mathrm{1D}}. \tag{1.16}$$

Moreover, for $|b - \Theta_0^{-1}|$ small enough (independently of ε), $C_2(b) > 0$.

The leading order term in (1.14) had previously been computed [FK, Kac, Pan], with a less explicit expression of the constant $C_1(b)$ however.

Remark 1.1. (Concentration of Cooper pairs.)
For obvious physical reasons it would be preferable to have an estimate of $\int_D |\Psi^{\mathrm{GL}}|^2$, which would directly give information on the distribution of Cooper pairs close to the boundary of the sample. Unfortunately, our method, which is mostly energy-based, does not provide this. An estimate on $\int_D |\Psi^{\mathrm{GL}}|^4$ is easier to obtain because more directly linked to the concentration of the energy density

$$e^{\mathrm{GL}}_\varepsilon(\mathbf{r}) := \left| \left(\nabla + i \frac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2} \right) \Psi^{\mathrm{GL}} \right|^2 - \frac{1}{2b\varepsilon^2} \left(2|\Psi^{\mathrm{GL}}|^2 - |\Psi^{\mathrm{GL}}|^4 \right) + \frac{b}{\varepsilon^4} \left| \mathrm{curl} \mathbf{A}^{\mathrm{GL}} - 1 \right|^2, \tag{1.17}$$

as we shall see below. The above theorem still indicates that, to leading order, $|\Psi^{\mathrm{GL}}|$ is concentrated evenly along the boundary of Ω, and that the first correction to this effect is directly proportional to the curvature function $k(s)$. Notice that, in order to exploit (1.17), the use of the variational equation solved by the GL minimizer is crucial and therefore our result does not extend directly to low energy configurations, although one would expect it.

Remark 1.2. (Convergence as measures.)
It would be natural to reformulate the result in terms of convergence of measures, i.e., by stating that

$$\frac{1}{\varepsilon} \left(\frac{1}{\varepsilon} |\Psi^{\mathrm{GL}}|^4 \mathrm{d}\mathbf{r} - C_1(b) \mathrm{d}s(\mathbf{r}) \right) \underset{\varepsilon \to 0}{\longrightarrow} C_2(b) k(s) \mathrm{d}s(\mathbf{r}), \tag{1.18}$$

in the sense of measure, where $\mathrm{d}s(\mathbf{r})$ is the 1D Lebesgue measure along the boundary of Ω. However, due to the restriction (1.13) on the shape of the set D, the statement (1.14) is weaker than (1.18).

We in fact expect (1.14) to be wrong as stated if the set D intersects the boundary with an angle $\neq \pi/2$. Indeed, the result is based on integrating the energy density on lines normal to the boundary that cover the full extent of the physical region. These integrals give the energy of the 1D model (1.6) that enters the main formula. One can easily decompose $D \cap \mathcal{A}_\varepsilon$ into the union of sets of the form (1.13) and some remainder which is more triangular-like. In the latter regions we cannot follow the proof procedure to obtain a simple expression. This does not affect the leading order of the result, as noted in [FK, Pan], since the energy contained in the triangular regions is small relatively to the leading order. It is however *not* small compared to the correction we isolate in (1.14) because the area of the triangular regions can easily be seen to be $O(\varepsilon^2)$. Integrating $|\Psi^{\mathrm{GL}}|^4$ on such an area gives a $O(\varepsilon^2)$ contribution and thus ruins the result.

Technically speaking, Assumption (1.13) enters the proof in the estimate of the boundary integral appearing in Lemma 4.1. It allows to exploit a new pointwise estimate of the tangential derivative of $|\Psi^{\mathrm{GL}}|^2$ that we prove in Lemma 4.2. It is important to remark that such an estimate does not hold for the normal component of the gradient (see the discussion preceding Lemma 4.2).

Remark 1.3. (Limiting regimes.)
In [FK, Corollary 1.3] a similar result about the boundary behavior of surface superconductivity is proven for magnetic fields slightly below the second critical one $H_{\mathrm{c}2}$, i.e., for $b \to 1^-$, $b \leq 1$. In this estimate the leading order is the same as in (1.14) but the first correction is proportional to the area of the set $|D|$ and is due to the bulk behavior of the superconductor.

The regime where $b \to \Theta_0^{-1}$ at the same time as $\varepsilon \to 0$ was studied in details in [FH1] (see in particular [FH1, Theorem 1.4] and the discussion thereafter). The behavior of the GL functional becomes approximately linear in this limit, and therefore the whole machinery of spectral theory of linear operators can be exploited to extract a lot of details about the boundary behavior. In particular, according to the asymptotics of $\Theta_0^{-1} - b$ when $\varepsilon \to 0$, superconductivity can be either uniformly distributed all over the boundary or concentrated close to the points of maximal curvature. The first order correction to the boundary behavior is however not known, although thanks to the approximate linearity of the problem, estimates can be formulated in terms of the integral of $|\Psi^{\mathrm{GL}}|^2$ instead of $|\Psi^{\mathrm{GL}}|^4$.

Remark 1.4. (Sign of the curvature correction)
Unfortunately we are not able to determine the sign of the correction in (1.14). Based on the results of [FH1] we have just recalled, we conjecture that $\mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] > 0$ for any $1 < b < \Theta_0^{-1}$. This would mean that points with large curvature attract more superconductivity. We can prove this conjecture only when b is close (independently of ε) to Θ_0^{-1}, see Lemma 2.3.

Note that the 2D setting we consider here corresponds to an infinite 3D cylinder with a magnetic field parallel to the axis. In a more general 3D setting, the angle between the magnetic field and the surface of the sample also plays a role in the distribution of surface superconductivity, see [FKP] and references therein.

The rest of the paper contains the proof of Theorem 1.1. In Section 2 we first discuss the perturbative expansion of the 1D ground state energy, obtain the expression (1.16) and prove that it is positive close to the third critical field. Section 3 contains a result of the same form as Theorem 1.1 where $|\Psi^{\mathrm{GL}}|^4$ is replaced by the GL energy density (1.17). Finally, in Section 4, which is the more involved of the three, we deduce our main result from the estimate of the energy density.

Acknowledgments. M.C. acknowledges the support of MIUR through the FIR grant 2013 "Condensed Matter in Mathematical Physics (Cond-Math)" (code RBFR13WAET). N.R. acknowledges the support of the ANR project Mathostaq (ANR-13-JS01-0005-01). We are indebted to one of the anonymous referees of our previous paper [CR2], whose remarks motivated the present investigation.

2. Correction to the 1D Energy

The asymptotic expansion of the 1D energy that is behind (1.14) is a direct consequence of first-order perturbation theory. We first prove that the functional (1.9) gives the subleading order of the 1D energy.

Let us recall the notation: for any $k \in \mathbb{R}$, $E^{\mathrm{1D}}_{\star}(k)$ is the infimum of (1.6) w.r.t. both f and α, while f_0 and α_0 stand for the minimizing pair of (1.8).

Lemma 2.1 (**Perturbative expansion of the 1D energy**).
As $\varepsilon \to 0$

$$\begin{aligned} E^{\mathrm{1D}}_{\star}(k) &= E^{\mathrm{1D}}_{0} - \varepsilon k \mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] + O(\varepsilon^{3/2}|\log\varepsilon|^{\gamma}) \\ &= -\frac{1}{2b}\int_0^{+\infty} f_0^4 - \varepsilon k \mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] + O(\varepsilon^{3/2}|\log\varepsilon|^{\gamma}) \end{aligned} \tag{2.1}$$

for some fixed $\gamma > 0$ *where* $\mathcal{E}^{\mathrm{corr}}$ *is defined as in* (1.9).

Proof. We first take f_0, α_0 as a trial pair for the functional (1.6), which yields

$$E^{\mathrm{1D}}_{\star}(k) \leq E^{\mathrm{1D}}_{0} - \varepsilon k \mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] + O(\varepsilon^2)$$

by expanding $\mathcal{E}^{\mathrm{1D}}_{k,\alpha_0}[f_0]$ and using that α_0, f_0 do not depend on ε. In order to estimate the remainders it suffices to use the exponential decay of f_0 [CR1, Proposition 3.3], which implies that

$$\int_0^{\infty} \mathrm{d}t\, t^n f_0^2(t) = O(1),$$

for any $n \in \mathbb{N}$. The decay estimate also implies that we make no significant error by considering the $k = 0$ functional as being defined on the whole half-line, provided c_0 is large enough.

Next we write

$$E^{\mathrm{1D}}_{\star}(k) = \mathcal{E}^{\mathrm{1D}}_{k,\alpha(k)}[f_k] = \mathcal{E}^{\mathrm{1D}}_{0,\alpha(k)}[f_k] - \varepsilon k \mathcal{E}^{\mathrm{corr}}_{\alpha(k)}[f_k] + O(\varepsilon^2) \geq E^{\mathrm{1D}}_{0} - \varepsilon k \mathcal{E}^{\mathrm{corr}}_{\alpha(k)}[f_k] + O(\varepsilon^2),$$

where we use the variational principle defining E^{1D}_{0}. Finally it easily follows from the estimates of [CR2, Proposition 1] that

$$\mathcal{E}^{\mathrm{corr}}_{\alpha(k)}[f_k] = \mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] + O(\varepsilon^{1/2}|\log\varepsilon|^{\gamma})$$

for some $\gamma > 0$. Gathering the previous inequalities, we get an upper and a lower bound to $E^{\mathrm{1D}}_{\star}(k)$ which together give (2.1). That

$$E_0^{\mathrm{1D}} = -\frac{1}{2b}\int_0^{+\infty} \mathrm{d}t\, f_0^4 \tag{2.2}$$

follows by mutliplying the variational equation

$$-\partial_t^2 f_0 + (t+\alpha_0)^2 f_0 = \frac{1}{b}(1-f_0^2)f_0 \tag{2.3}$$

by f_0 and integrating. □

The expression (1.16) is somewhat unwieldy, but can be simplified a lot:

Lemma 2.2 (**Simple formula for the 1D energy correction**).
Let α_0, f_0 *be a minimizing pair for the 1D functional* (1.8) *at* $k = 0$. *Then, for all* $1 < b < \Theta_0^{-1}$,

$$\begin{aligned}\mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] &= \frac{2}{3}(1-\alpha_0^2 b) + \frac{\alpha_0}{2b}\int_0^{+\infty}\mathrm{d}t\, f_0^4 \\ &= \frac{1}{3}f_0^2(0) - \alpha_0 E_0^{\mathrm{1D}}.\end{aligned} \tag{2.4}$$

Proof. Let us first recall two useful identities:

$$\int_0^{+\infty}\mathrm{d}t\,(t+\alpha_0)f_0^2(t) = 0 \tag{2.5}$$

expresses the optimality of α_0, see, e.g., [FHP, Eq. (3.20)] or [CR1, Lemma 3.1], while

$$\int_0^{+\infty}\mathrm{d}t\, t(t+\alpha_0)f_0^2(t) = \int_0^{+\infty}\mathrm{d}t\,\left(|\partial_t f_0|^2 + \frac{1}{4b}f_0^4\right) \tag{2.6}$$

is a virial identity, equivalent to [FHP, Eq. (3.22)]. It is obtained by noting that

$$E_{\alpha_0}(\ell) := \inf_f \mathcal{E}_0^{\mathrm{1D}}\left[\frac{1}{\sqrt{\ell}}f(\cdot/\ell)\right] = \inf_f \mathcal{E}_0^{\mathrm{1D}}[f]$$

for all ℓ and thus

$$\partial_\ell E_{\alpha_0}(\ell)|_{\ell=1} = 0.$$

Using the Feynman-Hellmann principle to evaluate the latter expression one gets (2.6).

We now start the computation:

Step 1. We claim that

$$\mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] = \int_0^{+\infty}\mathrm{d}t\,\left(\frac{t}{b} - \frac{3\alpha_0}{4b}\right)f_0^4(t) + \int_0^{+\infty}\mathrm{d}t\,\left(2t^3 - \frac{2t}{b} - 2\alpha_0^2 t + \frac{\alpha_0}{b}\right)f_0^2(t). \tag{2.7}$$

First, using the virial identity (2.6) we obtain

$$\mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] = \int_0^{+\infty}\mathrm{d}t\left\{(t-\alpha_0)|\partial_t f_0|^2 + \left(\frac{t}{2b} - \frac{\alpha_0}{4b}\right)f_0^4(t) - \frac{t}{b}f_0^2\right\}.$$

Next, integrating by parts and using the Neumann boundary condition for f_0,

$$\int_0^{+\infty}\mathrm{d}t\,(t-\alpha_0)|\partial_t f_0|^2 = -\int_0^{+\infty}\mathrm{d}t\,(t-\alpha_0)^2\partial_t f_0 \partial_t^2 f_0.$$

Inserting the variational equation (2.3) and integrating by parts again we deduce

$$\int_0^{+\infty} \mathrm{d}t\, (t-\alpha_0)|\partial_t f_0|^2 = \int_0^{+\infty} \mathrm{d}t \left\{ (t-\alpha_0)^2(t+\alpha_0) + (t+\alpha_0)^2(t-\alpha_0) \right.$$
$$\left. - \frac{t-\alpha_0}{b} + \frac{t-\alpha_0}{2b} f_0^2(t) \right\} f_0^2(t) + \frac{\alpha_0^4}{2} f_0^2(0) - \frac{\alpha_0^2}{2b} f_0^2(0) + \frac{\alpha_0^2}{4b} f_0^4(0).$$

The claim then follows from the identity (see, e.g., [CR1, Proof of Lemma 3.3])

$$f_0^2(0) = 2 - 2\alpha_0^2 b \tag{2.8}$$

and a bit of algebra.

Step 2. Next we compute that

$$\int_0^{+\infty} \mathrm{d}t\, (t+\alpha_0)^3 f_0^2 = -\frac{1}{2b} \int_0^{+\infty} \mathrm{d}t\, (t+\alpha_0) f_0^4 + \frac{1}{3}\left(1 - \alpha_0^2 b\right), \tag{2.9}$$

following [FH3, Proof of Lemma 3.2.7]. Let us denote

$$H_{\alpha_0} := -\partial_t^2 + (t+\alpha_0)^2 + \tfrac{1}{b} f_0^2 - \tfrac{1}{b}$$

and recall that f_0 is a ground state for this Schrödinger operator, with eigenvalue 0. For any function v we therefore have, integrating by parts

$$\langle f_0, H_{\alpha_0} v \rangle = v'(0) f_0(0).$$

If we apply this with $v = (t+\alpha_0)^2 f_0' - (t+\alpha_0) f_0$, we find

$$\langle f_0, H_{\alpha_0} v \rangle = \left(\alpha_0^4 + \frac{\alpha_0^2}{b}(f_0^2(0) - 1) - 1 \right) f_0^2(0).$$

On the other hand, using the variational equation,

$$-\partial_t^2 v = -3(t+\alpha_0) f_0'' - (t+\alpha_0)^2 f_0'''$$
$$= -(t+\alpha_0)\left(5(t+\alpha_0)^2 + \tfrac{3}{b}(f_0^2 - 1)\right) f_0 - (t+\alpha_0)^2 \left((t+\alpha_0)^2 + \tfrac{3}{b} f_0^2 - \tfrac{1}{b}\right) f_0',$$

so that

$$H_{\alpha_0} v = -6(t+\alpha_0)^3 f_0 - \tfrac{4}{b}(t+\alpha_0)(f_0^2 - 1) f_0 - \tfrac{2}{b}(t+\alpha_0)^2 f_0^2 f_0'.$$

Multiplying this by f_0 and integrating yields

$$\langle f_0, H_{\alpha_0} v \rangle = -6 \int_0^{+\infty} \mathrm{d}t\, (t+\alpha_0)^3 f_0^2 - \frac{3}{b} \int_0^{+\infty} \mathrm{d}t\, (t+\alpha_0) f_0^4 + \frac{\alpha_0^2}{2b} f_0^4(0)$$

where we use also (2.5) and an integration by parts. Equating the two different expressions of $\langle f_0, H_{\alpha_0} v \rangle$ we obtained and noting that, since $f_0^2(0) = 2 - 2\alpha_0^2 b$,

$$f_0^2(0) \left(-\alpha_0^4 - \frac{\alpha_0^2}{b}(f_0^2(0) - 1) + 1 \right) + \frac{\alpha_0^2}{2b} f_0^4(0) = f_0^2(0) = 2\left(1 - \alpha_0^2 b\right)$$

yields (2.9).

Step 3. From (2.9), (2.5) and (2.6) we get

$$\int_0^{+\infty} \mathrm{d}t\, t^3 f_0^2 = \int_0^{+\infty} \mathrm{d}t \left\{ (t+\alpha_0)^3 - 3\alpha_0^2 t - 3\alpha_0 t^2 - \alpha_0^3 \right\} f_0^2$$
$$= \frac{1}{6} f_0^2(0) + \int_0^{+\infty} \mathrm{d}t \left\{ -\frac{1}{2b}(t+\alpha_0) f_0^4 - 3\alpha_0 |\partial_t f_0|^2 - \frac{3\alpha_0}{4b} f_0^4 - \alpha_0^3 f_0^2 \right\}.$$

Inserting in (2.7) and using (2.5) again we find

$$\mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] = \frac{1}{3} f_0^2(0) + \int_0^{+\infty} \mathrm{d}t \left\{ -\frac{13\alpha_0}{4b} f_0^4 - 6\alpha_0 |\partial_t f_0|^2 + \left(\frac{\alpha_0}{b} - \frac{2t}{b} - 2\alpha_0^2 t - 2\alpha_0^3 \right) f_0^2 \right\}$$
$$= \frac{1}{3} f_0^2(0) + \int_0^{+\infty} \mathrm{d}t \left\{ -\frac{13\alpha_0}{4b} f_0^4 - 6\alpha_0 |\partial_t f_0|^2 + 3\frac{\alpha_0}{b} f_0^2 \right\}. \tag{2.10}$$

Finally we note that

$$\int_0^{+\infty} \mathrm{d}t\, |\partial_t f_0|^2 = E_0^{\mathrm{1D}} - \frac{1}{2b} \int_0^{+\infty} \mathrm{d}t\, f_0^4 + \int_0^{+\infty} \mathrm{d}t \left(\frac{1}{b} - (t+\alpha_0)^2 \right) f_0^2$$
$$= -\frac{1}{b} \int_0^{+\infty} \mathrm{d}t\, f_0^4 + \int_0^{+\infty} \mathrm{d}t \left(\frac{1}{b} - (t+\alpha_0)^2 \right) f_0^2$$

whereas (2.5) and (2.6) together imply

$$\int_0^{+\infty} \mathrm{d}t\, (t+\alpha_0)^2 f_0^2 = \int_0^{+\infty} \mathrm{d}t \left\{ |\partial_t f_0|^2 + \frac{1}{4b} f_0^4 \right\}$$

so that, combining the two identities,

$$\int_0^{+\infty} \mathrm{d}t\, |\partial_t f_0|^2 = \int_0^{+\infty} \mathrm{d}t \left\{ \frac{1}{2b} f_0^2 - \frac{5}{8b} f_0^4 \right\}.$$

Inserting this in (2.10) and recalling once more (2.8) and (2.2) this yields the final expressions (2.4). □

Unfortunately we are not able to determine the sign of the energy correction from the expressions we found. However, when $b \to \Theta_0^{-1}$ we have more information: it is known that f_0^2 scales as $(1 - b\Theta_0)^{1/2}$. It then immediately follows from Lemma 2.2 that the correction must be positive for b close enough to Θ_0^{-1}:

Lemma 2.3 (**Sign of the correction close to** H_{c3}**).**
There exists $1 < b_0 < \Theta_0^{-1}$ *such that, for all* $b_0 < b < \Theta_0^{-1}$,

$$\mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] > 0. \tag{2.11}$$

Proof. We recall from [FH3, Section 3.2] that the minimum in (1.5) is achieved by a unique pair $u_0, \alpha_{\mathrm{opt}}$ with $\alpha_{\mathrm{opt}} = -\sqrt{\Theta_0}$ and u_0 normalized in L^2. At $b > \Theta_0^{-1}$ it is easy to see that $f_0 \equiv 0$. When $b \to \Theta_0^{-1}$ one should therefore expect $f_0 \to 0$, in which case the quartic term becomes a second order correction. One should thus expect that the solution is close to that of the linear problem, the only effect of the quartic term being to fix the overall normalization. More precisely, following the techniques of [FH3, Section 14.2.2] one can show that

$$\alpha_0 \underset{b \to \Theta_0^{-1}}{\longrightarrow} -\sqrt{\Theta_0}$$

and that

$$\left(\frac{\|u_0\|_4^4}{1 - b\Theta_0} \right)^{1/2} f_0 \underset{b \to \Theta_0^{-1}}{\longrightarrow} u_0.$$

It is easy to see that the latter convergence holds in the quadratic form domain of the harmonic oscillator. Using standard elliptic estimates, one can upgrade this to any Sobolev or Hölder norm. In particular, convergence holds in L^∞ and L^4, so that

$$f_0^2(0) = \frac{(1-b\Theta_0)}{\|u_0\|_4^4} u_0^2(0)(1+o_b(1))$$

where we denoted by $o_b(1)$ a quantity going to 0 as $b \to \Theta_0^{-1}$, and

$$\int_0^{+\infty} \mathrm{d}t \, f_0^4 = \frac{(1-b\Theta_0)^2}{\|u_0\|_4^4}(1+o_b(1)).$$

From Lemma 2.2 we have that

$$\mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] = \frac{1}{3} f_0^2(0) + \frac{\alpha_0}{2b} \int_0^{+\infty} \mathrm{d}t \, f_0^4 = \frac{1-b\Theta_0}{3\,\|u_0\|_4^4} \left[u_0^2(0) + \frac{3\alpha_0}{2b}(1-b\Theta_0) \right]$$
$$= \frac{1-b\Theta_0}{3\,\|u_0\|_4^4} \left[u_0^2(0) + o_b(1) \right] > 0$$

since $u_0(0) > 0$ is independent of b. □

3. Estimates of the Energy Density

Our main estimate on the order parameter is obtained by exploiting the link between $|\Psi^{\mathrm{GL}}|^4$ and the GL energy density. We discuss first the asymptotics for the latter.

Proposition 3.1 (**Estimates for the energy density**).
Let $e_\varepsilon^{\mathrm{GL}}$ be the GL energy density defined in (1.17) *and $D \subset \Omega$ be a measurable set satisfying* (1.13)*. Under the same assumptions and with the same notation as in Theorem 1.1 we have, as $\varepsilon \to 0$*

$$\int_D \mathrm{d}\mathbf{r} \, e_\varepsilon^{\mathrm{GL}} = \varepsilon^{-1} E_0^{\mathrm{1D}} |\partial\Omega \cap \partial D| - \mathcal{E}^{\mathrm{corr}}_{\alpha_0}[f_0] \int_{\partial D \cap \partial\Omega} \mathrm{d}s \, k(s) + O(\varepsilon^{1/2} |\log \varepsilon|^\gamma). \tag{3.1}$$

Proof. This is an adaptation of the method developed in [CR1, CR2]. First we recall from [CR2, Lemma 4] the energy lower bound

$$E_\varepsilon^{\mathrm{GL}} \geq \frac{1}{\varepsilon} \mathcal{G}_{\mathcal{A}_\varepsilon}[\psi] - C\varepsilon^2 |\log \varepsilon|^2, \tag{3.2}$$

where ψ is, up to a phase factor, the GL order parameter in boundary coordinates and the reduced functional is

$$\mathcal{G}_{\mathcal{A}_\varepsilon}[\psi] := \int_0^{|\partial\Omega|} \mathrm{d}s \int_0^{c_0|\log\varepsilon|} \mathrm{d}t \, (1-\varepsilon k(s) t) \left\{ |\partial_t \psi|^2 + \frac{1}{(1-\varepsilon k(s)t)^2} \left| (\varepsilon \partial_s + i a_\varepsilon(s,t)) \psi \right|^2 - \frac{1}{2b} \left[2|\psi|^2 - |\psi|^4 \right] \right\} \tag{3.3}$$

with

$$a_\varepsilon(s,t) := -t + \tfrac{1}{2} \varepsilon k(s) t^2 + \varepsilon \delta_\varepsilon, \tag{3.4}$$

and

$$\delta_\varepsilon := \frac{\gamma_0}{\varepsilon^2} - \left\lfloor \frac{\gamma_0}{\varepsilon^2} \right\rfloor, \qquad \gamma_0 := \frac{1}{|\partial\Omega|} \int_\Omega \mathrm{d}\mathbf{r} \, \mathrm{curl} \, \mathbf{A}^{\mathrm{GL}}, \tag{3.5}$$

$\lfloor \cdot \rfloor$ standing for the integer part.

This lower bound may in fact (with identical proof) be localized, yielding

$$\varepsilon \int_D \mathrm{d}\mathbf{r}\, e_\varepsilon^{\mathrm{GL}} \geq \int_{s_D}^{s_D'} \mathrm{d}s \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, (1-\varepsilon k(s)t) \Big\{ |\partial_t \psi|^2 + \frac{1}{(1-\varepsilon k(s)t)^2} \left| (\varepsilon\partial_s + i a_\varepsilon(s,t))\, \psi \right|^2 - \frac{1}{2b}\left[2|\psi|^2 - |\psi|^4 \right] \Big\} - C\varepsilon^3 |\log\varepsilon|^2. \quad (3.6)$$

We next split the interval $[s_D, s_D']$ into $N_\varepsilon = O(\varepsilon^{-1})$ sub-intervals $[s_n, s_{n+1}]$, $n = 1, \ldots, N_\varepsilon$, of side length $O(\varepsilon)$. The convention is that $s_D = s_1$ and $s_D' = s_{N_\varepsilon+1}$. This gives a decomposition of $[s_D, s_D'] \times [0, c_0|\log\varepsilon|]$ into N_ε rectangular cells $\mathcal{C}_n$, $n = 1, \ldots, N_\varepsilon$.

Arguing as in [CR2, Lemma 6] we then deduce

$$\varepsilon \int_D \mathrm{d}\mathbf{r}\, e_\varepsilon^{\mathrm{GL}} \geq \int_{s_D}^{s_D'} \mathrm{d}s\, E_\star^{\mathrm{1D}}(k(s)) + \sum_{n=1}^{N_\varepsilon} \mathcal{E}_n[u_n] - C\varepsilon^2 |\log\varepsilon|^\infty \quad (3.7)$$

where, within the n-th cell,

$$\psi(s,t) =: u_n(s,t) f_n(t) \exp\left\{ -i \left(\tfrac{\alpha_n}{\varepsilon} + \delta_\varepsilon \right) s \right\}, \quad (3.8)$$

and the reduced functionals $\mathcal{E}_n$ are defined as

$$\mathcal{E}_n[u] := \int_{\mathcal{C}_n} \mathrm{d}s\mathrm{d}t\, (1-\varepsilon k_n t)\, f_n^2 \Big\{ |\partial_t u|^2 + \tfrac{1}{(1-\varepsilon k_n t)^2} |\varepsilon\partial_s u|^2 - 2\varepsilon b_n(t) J_s[u] + \tfrac{1}{2b} f_n^2 \left(1 - |u|^2\right)^2 \Big\}, \quad (3.9)$$

with

$$b_n(t) := \frac{t + \alpha_n - \frac{1}{2}\varepsilon k_n t^2}{(1-\varepsilon k_n t)^2}, \quad (3.10)$$

and

$$J_s[u] := (iu, \partial_s u) = \tfrac{i}{2}\left(u^* \partial_s u - u \partial_s u^* \right). \quad (3.11)$$

The mean curvature in the n-th cell is denoted k_n, with f_n and α_n the minimizing profile and phase for the associated functional (1.6). Inserting the result of Lemma 2.1 into (3.7) we find

$$\int_D \mathrm{d}\mathbf{r}\, e_\varepsilon^{\mathrm{GL}} \geq \varepsilon^{-1} E_0^{\mathrm{1D}} |\partial\Omega \cap \partial D| - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0] \int_{\partial D \cap \partial\Omega} \mathrm{d}s\, k(s) + \varepsilon^{-1} \sum_{n=1}^{N_\varepsilon} \mathcal{E}_n[u_n] - C\varepsilon |\log\varepsilon|^\infty. \quad (3.12)$$

We next adapt the strategy of [CR2, Proof of Lemma 7] to estimate the reduced functionals from below:

$$\sum_{n=1}^{N_\varepsilon} \mathcal{E}_n[u_n] \geq -C\varepsilon^{3/2} |\log\varepsilon|^\gamma, \quad (3.13)$$

thereby concluding the proof of the lower bound corresponding to (3.1). This is a long procedure that we will not recall in details. We shall emphasize the only point that has to be modified, due to the fact that we now bound from below the energy density in the set D instead of the full GL energy.

Step 1 of the proof of [CR2, Lemma 7] adapts with no modification, leading to

$$\mathcal{E}_n[u_n] \geq \varepsilon \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t F_n(t) \left[J_t[u_n](s_{n+1},t) - J_t[u_n](s_n,t)\right] + d_\varepsilon \int_{\mathcal{C}_n} \mathrm{d}s\mathrm{d}t \; (1-\varepsilon k_n t)\, f_n^2 \left[|\partial_t u_n|^2 + \tfrac{1}{(1-\varepsilon k_n t)^2} |\varepsilon \partial_s u_n|^2 \right] + O(\varepsilon^\infty), \quad (3.14)$$

for some $d_\varepsilon \sim |\log\varepsilon|^{-4}$, denoting

$$J_t[u] := (iu, \partial_t u) = \tfrac{i}{2}\left(u^*\partial_t u - u\partial_t u^*\right)$$

and

$$F_n(t) = 2\int_0^t \mathrm{d}\eta \; (1-\varepsilon k_n \eta) b_n(\eta) f_n^2(\eta).$$

We then follow Step 2 of the same proof to combine and estimate the boundary terms produced by the use of Stokes' formula in Step 1 (terms on the first line of the above formula). In this procedure it is crucial to sum boundary terms living on the same cell boundary. Since here $s_1 \neq s_{N_\varepsilon+1}$ there is obviously a need for a different estimate of the terms located on the corresponding boundaries, i.e., those of the original set D. This is the only point where we depart slightly from the method of [CR2] and rely on more refined inequalities.

We proceed as follows (say for the $n=1$ term, located on the boundary corresponding to $s_1 = s_D$): let χ be a smooth cut-off function depending only on s with

$$\chi(s_1) = 1, \quad |\chi| \leq 1, \quad \mathrm{supp}(\chi) \subset \mathcal{C}_1, \quad |\nabla\chi| \leq C\varepsilon^{-1}.$$

Since our cells have side-length $O(\varepsilon)$ in the s direction, the last two requirements are obviously compatible. Intergrating by parts in the s variable we get

$$\int_0^{\bar{t}_{1,\varepsilon}} \mathrm{d}t\; F_1(t) J_t[u_1](s_1,t) = \int_0^{\bar{t}_{1,\varepsilon}} \mathrm{d}t\; \chi(s_1) F_1(t) J_t[u_1](s_1,t) = \int_{\mathcal{C}_1} \mathrm{d}s\mathrm{d}t\; \chi F_1 \partial_s J_t[u_1] + \int_{\mathcal{C}_1} \mathrm{d}s\mathrm{d}t\; F_1 \partial_s \chi J_t[u_1]. \quad (3.15)$$

We drop the subscripts 1 for shortness. To handle the first term in the above we note that

$$\partial_s J_t[u] = \frac{i}{2}\left(\partial_s u \partial_t u^* - \partial_s u^* \partial_t u\right) + \frac{i}{2}\left(u\partial_s\partial_t u^* - u^*\partial_s\partial_t u\right)$$

and hence a further integration by parts in t yields

$$\int_{\mathcal{C}} \mathrm{d}s\mathrm{d}t\; \chi F \partial_s J_t[u] = -\frac{i}{2}\int_{\mathcal{C}} \mathrm{d}s\mathrm{d}t\; \chi \partial_t F (u\partial_s u^* - u^*\partial_s u).$$

Note that the boundary terms vanish by definition of F. We also have $|\partial_t F| \leq C|\log\varepsilon| f^2$ and thus

$$\varepsilon \left| \int_{\mathcal{C}} \mathrm{d}s\mathrm{d}t\; \chi F \partial_s J_t[u] \right| \leq C\varepsilon |\log\varepsilon| \int_{\mathcal{C}} \mathrm{d}s\mathrm{d}t\; f^2 |u||\partial_s u| \leq C\delta|\log\varepsilon| \int_{\mathcal{C}} \mathrm{d}s\mathrm{d}t\; f^2|u|^2 + C\delta^{-1}|\log\varepsilon| \int_{\mathcal{C}} \mathrm{d}s\mathrm{d}t\; f^2 |\varepsilon\partial_s u|^2 \leq C\delta\varepsilon|\log\varepsilon|^2 + C\delta^{-1}\varepsilon^2|\log\varepsilon|^\gamma \leq C\varepsilon^{3/2}|\log\varepsilon|^\gamma$$

where we use that $f^2|u|^2 = |\psi|^2 \leq 1$ plus the fact that $|\mathcal{C}| = O(\varepsilon|\log\varepsilon|)$, recall the estimate [CR2, Eq. (6.15)] and have chosen $\delta = \varepsilon^{1/2}|\log\varepsilon|^\gamma$ for the final optimization.

For the second term in (3.15) we write, using essentially the same ingredients (in particular [CR2, Eq. (6.15)]),

$$\begin{aligned}\left|\varepsilon \int_{\mathcal{C}} \mathrm{d}s\mathrm{d}t\, F\partial_s\chi J_t[u]\right| &\leq C\int_{\mathcal{C}} \mathrm{d}s\mathrm{d}t\, f^2|u||\partial_t u| \\ &\leq C\delta \int_{\mathcal{C}} \mathrm{d}s\mathrm{d}t\, f^2|u|^2 + C\delta^{-1}\int_{\mathcal{C}} \mathrm{d}s\mathrm{d}t\, f^2|\partial_t u|^2 \\ &\leq C\delta\varepsilon|\log\varepsilon| + C\delta^{-1}\varepsilon^2|\log\varepsilon|^\gamma \leq C\varepsilon^{3/2}|\log\varepsilon|^\gamma.\end{aligned}$$

Combining the previous estimates, we obtain

$$\varepsilon \int_0^{\bar{t}_{1,\varepsilon}} \mathrm{d}t\, F(t) J_t[u](s_1,t) = O(\varepsilon^{3/2}|\log\varepsilon|^\gamma)$$

and a similar estimate for the term located on the boundary $s = s_{N_\varepsilon+1}$. Dealing with the other boundary terms as in [CR2] concludes the proof of (3.13).

At this stage we have the lower bound corresponding to (3.1),

$$\int_D \mathrm{d}\mathbf{r}\, e_\varepsilon^{\mathrm{GL}} \geq \varepsilon^{-1}E_0^{\mathrm{1D}}|\partial\Omega\cap\partial D| - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0]\int_{\partial D\cap\partial\Omega} \mathrm{d}s\, k(s) + O(\varepsilon|\log\varepsilon|^\gamma) \tag{3.16}$$

and by the same method also

$$\int_{D^c} \mathrm{d}\mathbf{r}\, e_\varepsilon^{\mathrm{GL}} \geq \varepsilon^{-1}E_0^{\mathrm{1D}}|\partial\Omega\cap\partial D^c| - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0]\int_{\partial D^c\cap\partial\Omega} \mathrm{d}s\, k(s) + O(\varepsilon^{1/2}|\log\varepsilon|^\gamma). \tag{3.17}$$

On the other hand, combining the energy estimate of [CR2, Theorem 1] and Lemma 2.1 we have the global estimate

$$\int_\Omega \mathrm{d}\mathbf{r}\, e_\varepsilon^{\mathrm{GL}} = \varepsilon^{-1}E_0^{\mathrm{1D}}|\partial\Omega| - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0]\int_{\partial\Omega} \mathrm{d}s\, k(s) + O(\varepsilon|\log\varepsilon|^\gamma).$$

Combining with (3.17) we deduce

$$\begin{aligned}\int_D \mathrm{d}\mathbf{r}\, e_\varepsilon^{\mathrm{GL}} &= \int_\Omega \mathrm{d}\mathbf{r}\, e_\varepsilon^{\mathrm{GL}} - \int_{D^c} \mathrm{d}\mathbf{r}\, e_\varepsilon^{\mathrm{GL}} \\ &\leq \varepsilon^{-1}E_0^{\mathrm{1D}}|\partial\Omega\cap\partial D| - \mathcal{E}_{\alpha_0}^{\mathrm{corr}}[f_0]\int_{\partial D\cap\partial\Omega} \mathrm{d}s\, k(s) + O(\varepsilon^{1/2}|\log\varepsilon|^\gamma)\end{aligned}$$

which we combine with (3.16) to complete the proof. □

4. From the Energy Density to the Order Parameter

We now conclude the proof of Theorem 1.1 by adding the following to Proposition 3.1:

Proposition 4.1 (Energy density versus order parameter).
Under the assumptions and with the notation of Theorem 1.1 and Proposition 3.1, we have

$$\int_D \mathrm{d}\mathbf{r}\, e_\varepsilon^{\mathrm{GL}} = -\frac{1}{2b\varepsilon^2}\int_D \mathrm{d}\mathbf{r}\, |\Psi^{\mathrm{GL}}|^4 + o(1). \tag{4.1}$$

The proof is split in two lemmas. First we have a general result which does not require the set under consideration to be rectangular:

Lemma 4.1 (**Reduction to a boundary term**).
Let $S \subset \Omega$ be a measurable set. Then, with the previous notation

$$\int_S \mathrm{d}\mathbf{r}\, e_\varepsilon^{\mathrm{GL}} + \frac{1}{2b\varepsilon^2}\int_S \mathrm{d}\mathbf{r}\, |\Psi^{\mathrm{GL}}|^4 = \frac{1}{2}\int_{\partial S} \mathrm{d}\sigma\, \nabla|\Psi^{\mathrm{GL}}|^2 \cdot \boldsymbol{\nu} + o(1) \tag{4.2}$$

with $\boldsymbol{\nu}$ the outward-pointing normal to ∂S.

Proof. We first note that

$$\frac{1}{\varepsilon^4}\int_S \mathrm{d}\mathbf{r}\, \left|\mathrm{curl}\mathbf{A}^{\mathrm{GL}} - 1\right|^2 = O(\varepsilon|\log\varepsilon|^3). \tag{4.3}$$

Indeed, using the elliptic estimate (see [FH3, Eq. (11.51)])

$$\left\|\mathrm{curl}\mathbf{A}^{\mathrm{GL}} - 1\right\|_{C^1(\Omega)} = O(\varepsilon)$$

and the fact that $\mathrm{curl}\mathbf{A}^{\mathrm{GL}} = 1$ on $\partial\Omega$, we deduce that in the full boundary layer (1.10) we have

$$\left|\mathrm{curl}\mathbf{A}^{\mathrm{GL}} - 1\right| = O(\varepsilon^2|\log\varepsilon|)$$

and thus

$$\int_{S\cap\tilde{\mathcal{A}}_\varepsilon} \mathrm{d}\mathbf{r}\, \left|\mathrm{curl}\mathbf{A}^{\mathrm{GL}} - 1\right|^2 = O(\varepsilon^5|\log\varepsilon|^3).$$

The part of the integral located in $S \cap \tilde{\mathcal{A}}_\varepsilon^c$ is of much lower order, as follows from the usual Agmon estimates, for instance [FH3, Eq. (12.10)], and we deduce (4.3).

At the level of precision we aim at we may thus neglect the magnetic kinetic energy:

$$\int_S \mathrm{d}\mathbf{r}\, e_\varepsilon^{\mathrm{GL}} = \int_S \mathrm{d}\mathbf{r}\left\{\left|\left(\nabla + i\frac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2}\right)\Psi^{\mathrm{GL}}\right|^2 - \frac{1}{2b\varepsilon^2}\left(2|\Psi^{\mathrm{GL}}|^2 - |\Psi^{\mathrm{GL}}|^4\right)\right\} + O(\varepsilon|\log\varepsilon|^3).$$

Next we recall that since Ψ^{GL} is a minimizer for $\mathcal{G}_\varepsilon^{\mathrm{GL}}$ we have the first Ginzburg-Landau variational equation:

$$-\left(\nabla + i\frac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2}\right)^2\Psi^{\mathrm{GL}} + \frac{1}{b\varepsilon^2}\Psi^{\mathrm{GL}}\left(|\Psi^{\mathrm{GL}}|^2 - 1\right) = 0. \tag{4.4}$$

Combining with the identity

$$\tfrac{1}{2}\Delta|\Psi^{\mathrm{GL}}|^2 = \Re(\overline{\Psi^{\mathrm{GL}}}\Delta\Psi^{\mathrm{GL}}) + |\nabla\Psi^{\mathrm{GL}}|^2$$

we deduce

$$\frac{1}{2}\Delta|\Psi^{\mathrm{GL}}|^2 = \left|\left(\nabla + i\frac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2}\right)\Psi^{\mathrm{GL}}\right|^2 + \frac{1}{b\varepsilon^2}|\Psi^{\mathrm{GL}}|^2\left(|\Psi^{\mathrm{GL}}|^2 - 1\right). \tag{4.5}$$

Integrating over S we obtain

$$\int_S \mathrm{d}\mathbf{r}\, e_\varepsilon^{\mathrm{GL}} = -\frac{1}{2b\varepsilon^2}\int_S |\Psi^{\mathrm{GL}}|^4 + \frac{1}{2}\int_S \mathrm{d}\mathbf{r}\, \Delta|\Psi^{\mathrm{GL}}|^2 \tag{4.6}$$

and the proof is complete since of course

$$\int_S \mathrm{d}\mathbf{r}\, \Delta|\Psi^{\mathrm{GL}}|^2 = \int_{\partial S} \mathrm{d}\sigma\, \nabla|\Psi^{\mathrm{GL}}|^2 \cdot \boldsymbol{\nu}.$$

□

Applying the previous lemma with $D = S$, our task should now be to estimate the boundary term in the right-hand side of (4.2). It is similar to terms showing up in [FK, Proof of Lemma 6.1], but using the estimates therein shows at best that it is of order $O(1)$, a remainder that we cannot afford. A technical novelty in the present paper is thus to show that this term is in fact $o(1)$, provided the set D is rectangular in boundary coordinates.

We certainly have

$$\int_{\partial D} \mathrm{d}\sigma \, \nabla |\Psi^{\mathrm{GL}}|^2 \cdot \boldsymbol{\nu} = \int_{(\partial D) \cap \mathcal{A}_\varepsilon} \mathrm{d}\sigma \, \nabla |\Psi^{\mathrm{GL}}|^2 \cdot \boldsymbol{\nu} + O(\varepsilon^\infty)$$

by the usual decay estimates, where we recall that $\mathcal{A}_\varepsilon$ is defined in (1.12). Splitting the curve $\mathcal{L}_D := \Phi(\partial D \cap \mathcal{A}_\varepsilon)$ (which is a rectangle in boundary coordinates) in a part $\mathcal{L}_D^s$ parallel to the boundary of $\partial\Omega$ and a part $\mathcal{L}_D^t$ normal to the boundary of $\partial\Omega$ we have

$$\int_{\partial D} \mathrm{d}\sigma \, \nabla |\Psi^{\mathrm{GL}}|^2 \cdot \boldsymbol{\nu} = \int_{\mathcal{L}_D^t} \mathrm{d}\sigma \, \nabla |\Psi^{\mathrm{GL}}|^2 \cdot \boldsymbol{\nu} + O(\varepsilon^\infty). \tag{4.7}$$

Indeed, on the part of $\mathcal{L}_D^s$ which coincides with $\partial\Omega$ we have

$$\nabla |\Psi^{\mathrm{GL}}| \cdot \boldsymbol{\nu} = 0$$

by taking the real part of the Neumann boundary condition

$$\left(\nabla \Psi^{\mathrm{GL}} + i \frac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2} \right) \cdot \boldsymbol{\nu} = 0, \qquad \text{on } \partial\Omega$$

satisfied by Ψ^{GL}. The other part of $\mathcal{L}_D^s$ is deep in the region where the order parameter decays exponentially and may thus be neglected. The new key ingredient is that on $\mathcal{L}_D^t$ we can prove

$$\left| \nabla |\Psi^{\mathrm{GL}}|^2 \cdot \boldsymbol{\nu} \right| = \left| \partial_s |\Psi^{\mathrm{GL}}|^2 \right| \leq C \varepsilon^{a-1} \tag{4.8}$$

for some $a > 0$. This is natural in view of the results of [CR1, CR2]: the variations in the s-direction should be much smaller than those in the t-direction, which happen on a scale ε. Combining this with the fact that the length of this part of the boundary is $O(\varepsilon |\log \varepsilon|)$ we get the desired estimate.

We are in fact not able to prove (4.8) in all the boundary region $\mathcal{A}_\varepsilon$ and will thus have to split the line integral on $\mathcal{L}_D^t$ into two pieces. Let us introduce the following subset of $\mathcal{A}_\varepsilon$ where the estimate (4.8) is proven in the next Lemma 4.2:

$$\tilde{\mathcal{A}}_> := \left\{ \mathbf{r} \in \Omega \mid f_0(\tau/\varepsilon) \geq \varepsilon^{1/6} \right\}. \tag{4.9}$$

Any power of ε strictly smaller than 1/4 would do the job but we fix it equal to 1/6 for concreteness. Recall that $\tau = \mathrm{dist}(\mathbf{r}, \partial\Omega)$. As before we denote by $\mathcal{A}_> = \Phi(\tilde{\mathcal{A}}_>)$, the set $\tilde{\mathcal{A}}_>$ in boundary coordinates and it is easy to see that

$$\mathcal{A}_> = [0, \partial\Omega] \times [0, t_>], \qquad t_> \gg 1. \tag{4.10}$$

In fact, exploiting the available pointwise bounds on f_0 (see for example [CR2, Eq. (A.6)]), one immediately verifies that

$$t_> \geq \tfrac{1}{\sqrt{3}} \sqrt{|\log \varepsilon|} (1 + o(1)). \tag{4.11}$$

Before stating the pointwise estimate mentioned above, let us stress that a similar bound cannot hold for the normal component of the gradient of Ψ^{GL}: the estimate $\partial_t |\Psi^{\mathrm{GL}}|^2 \propto \varepsilon^{-1}$

is optimal. The different behavior of the s and t derivatives will be apparent in the proof of the following Lemma.

Lemma 4.2 (**Estimate of the tangential derivative**).
As $\varepsilon \to 0$

$$\left\| \partial_s \left| \Psi^{\mathrm{GL}}(\Phi(s,\tau)) \right|^2 \right\|_{L^\infty(\mathcal{A}_>)} = O(\varepsilon^{-5/6} |\log \varepsilon|^\infty). \tag{4.12}$$

Proof. The starting point is the variational equation (4.4) satisfied by Ψ^{GL}: setting as in [CR2, Sect. 5]

$$\psi(s,t) = \Psi^{\mathrm{GL}}(\Phi(s,\varepsilon t)) e^{-i\phi_\varepsilon(s,t)}, \tag{4.13}$$

where the explicit expression of the gauge phase is given in [CR2, Eq. (5.4)], one gets

$$-\partial_t^2 \psi + \tfrac{1}{(1-\varepsilon k(s) t)^2} \left(-i\varepsilon \partial_s + \tfrac{\tilde{A}}{\varepsilon} \right)^2 \psi = \tfrac{1}{b} \left(1 - |\psi|^2 \right) \psi, \tag{4.14}$$

i.e., thanks to the choice of the gauge the magnetic field is now purely tangential. The explicit expression of $\tilde{A}$ can be easily recovered in terms of ϕ_ε (see, e.g., [CR2, Eq. (5.6)]), but the most important point is the estimate

$$\left\| \tilde{A}(s,t) + \varepsilon t \right\|_{L^\infty(\mathcal{A}_\varepsilon)} = O(\varepsilon^2 |\log \varepsilon|^2),$$

which follows from a priori estimates on $\mathbf{A}^{\mathrm{GL}}$ as [CR1, Eq. (4.23)]. In addition the explicit expression [CR2, Eq. (5.6)] also implies that $\left\| \partial_s \tilde{A} \right\|_\infty = O(\varepsilon |\log \varepsilon|)$. Notice that here we are exploiting the smoothness of $\partial\Omega$ and the fact that the curvature is infinitely differentiable. Plugging the ansatz

$$\psi(s,t) = f_0(t) e^{-i \frac{\alpha_0}{\varepsilon} s} u(s,t), \tag{4.15}$$

for some unknown function u and with f_0 and α_0 the minimizing density and phase of the half-plane functional (1.8), we get

$$-\partial_t^2 (f_0 u) + \tfrac{f_0}{(1-\varepsilon k(s) t)^2} \left(-i\varepsilon \partial_s + \alpha_0 + t + O(\varepsilon |\log \varepsilon|) \right)^2 u = \tfrac{1}{b} \left(1 - f_0^2 |u|^2 \right) f_0 u.$$

Exploiting now the variational equation (2.3) for f_0 and dividing by $f_0 > 0$, we obtain

$$-\partial_t^2 u - 2 \tfrac{f_0'}{f_0} \partial_t u - \tfrac{1}{(1-\varepsilon k(s) t)} \varepsilon^2 \partial_s^2 u - 2i\varepsilon(\alpha_0 + t) \partial_s u = \tfrac{f_0^2}{b} \left(1 - |u|^2 + O(\varepsilon |\log \varepsilon|^2) \right) u.$$

Since (see [CR1, Lemma A.1])

$$\frac{f_0'}{f_0} = O(|\log \varepsilon|^3),$$

inside $\mathcal{A}_>$ the above equation yields the estimate

$$\begin{aligned} \left| \left(\partial_t^2 + \varepsilon^2 \partial_s^2 \right) u \right| &\leq C \left[|\log \varepsilon|^3 \left| \left(\partial_t + \varepsilon \partial_s \right) u \right| + |1 - |u|| \, |u| \right] \\ &\leq C \left[|\log \varepsilon|^3 \left| (\partial_t, \varepsilon \partial_s) u \right| + \varepsilon^{1/12} |\log \varepsilon|^b \right] \end{aligned}$$

where we have used the upper bound $|u| \leq f_0^{-1} \leq \varepsilon^{-1/6}$ and the estimate

$$|1 - |u|| = O(\varepsilon^{1/4} |\log \varepsilon|^\infty) \text{ in } \mathcal{A}_>$$

which follows from [CR2, Theorem 2]. Rescaling now also the s variable by setting $\xi = s/\varepsilon$ and denoting $v(\xi,t) = u(\varepsilon\xi, t)$, we can apply the Gagliardo-Nirenberg inequality [Nir, p. 125]

$$\left\| \nabla_{\xi,t} v \right\|_\infty \leq C \left(\left\| \Delta_{\xi,t} v \right\|_\infty^{1/2} \left\| 1 - |v| \right\|_\infty^{1/2} + \left\| 1 - |v| \right\|_\infty \right),$$

which implies $\|\nabla v\|_\infty = O(\varepsilon^{1/6}|\log\varepsilon|^\infty)$ and therefore

$$\|(\partial_t + \varepsilon\partial_s)\,|u|\|_{L^\infty(\mathcal{A}_>)} \leq \|\nabla v\|_\infty = O(\varepsilon^{1/6}|\log\varepsilon|^\infty). \tag{4.16}$$

The result on Ψ^{GL} then follows from the identities (4.13) and (4.15). □

Putting together the results of Lemma 4.1 and Lemma 4.2, we are now in position to complete the proof of the main result of this Section:

Proof of Proposition 4.1. The estimate (4.2) and the properties of the set D yield (4.7) and therefore

$$\int_S \mathrm{d}\mathbf{r}\, e_\varepsilon^{\mathrm{GL}} + \frac{1}{2b\varepsilon^2}\int_S \mathrm{d}\mathbf{r}\,|\Psi^{\mathrm{GL}}|^4 = \frac{1}{2}\int_{\partial S}\mathrm{d}\sigma\,\nabla|\Psi^{\mathrm{GL}}|^2\cdot\boldsymbol{\nu} + o(1)$$
$$= \int_{\mathcal{L}_D^t}\mathrm{d}\sigma\,\nabla|\Psi^{\mathrm{GL}}|^2\cdot\boldsymbol{\nu} + o(1) = \int_{\mathcal{L}_D^t\cap\tilde{\mathcal{A}}_>}\mathrm{d}\sigma\,\nabla|\Psi^{\mathrm{GL}}|^2\cdot\boldsymbol{\nu} + o(1), \tag{4.17}$$

since

$$\int_{\mathcal{L}_D^t\cap\tilde{\mathcal{A}}_>^c}\mathrm{d}\sigma\,\nabla|\Psi^{\mathrm{GL}}|^2\cdot\boldsymbol{\nu} = 2\int_{\Phi(\mathcal{L}_D^t\cap\tilde{\mathcal{A}}_>^c)}\mathrm{d}\tau\,|\psi(s,\tau/\varepsilon)|\partial_s|\psi(s,\tau/\varepsilon)|$$
$$= 2\int_{\Phi(\mathcal{L}_D^t\cap\tilde{\mathcal{A}}_>^c)}\mathrm{d}\tau\,|\psi(s,\tau/\varepsilon)|f_0(\tau/\varepsilon)\partial_s|u(s,\tau/\varepsilon)| \leq C\varepsilon^{-1}\,\|\psi\|_{L^\infty(\mathcal{A}_>^c)}\left|\Phi(\mathcal{L}_D^t\cap\tilde{\mathcal{A}}_>^c)\right|$$

where we have used [CR2, Eq. (6.2)]. Now the Agmon estimate for ψ stated, e.g., in [CR2, Eq. (5.5)] yields

$$\|\psi\|_{L^\infty(\mathcal{A}_>^c)} \leq Ce^{-At_>} \leq C\exp\left\{-\tfrac{1}{\sqrt{3}}\sqrt{|\log\varepsilon|}\right\} \ll |\log\varepsilon|^{-1},$$

thanks to (4.11). Hence we conclude that

$$\int_{\mathcal{L}_D^t\cap\tilde{\mathcal{A}}_>^c}\mathrm{d}\sigma\,\nabla|\Psi^{\mathrm{GL}}|^2\cdot\boldsymbol{\nu} \leq C|\log\varepsilon|\,\|\psi\|_{L^\infty(\mathcal{A}_>^c)} = o(1),$$

and (4.17) is proven. For the rest of the boundary integral it suffices to apply (4.12):

$$\int_{\mathcal{L}_D^t\cap\tilde{\mathcal{A}}_>}\mathrm{d}\sigma\,\nabla|\Psi^{\mathrm{GL}}|^2\cdot\boldsymbol{\nu} = \int_{\Phi(\mathcal{L}_D^t)\cap\mathcal{A}_>}\mathrm{d}\sigma\,\partial_s\left|\Psi^{\mathrm{GL}}(\Phi(s,\tau))\right|^2 = O(\varepsilon^{1/6}|\log\varepsilon|^\infty).$$

□

References

[Abr] A. Abrikosov, On the magnetic properties of superconductors of the second type, *Soviet Phys. JETP* **5**, 1174–1182 (1957).

[Alm1] Y. Almog, Nonlinear Surface Superconductivity in the Large κ Limit, *Rev. Math. Phys.* **16**, 961–976 (2004).

[Alm2] Y. Almog, Abrikosov lattices in finite domains, *Commun. Math. Phys.* **262**, 677-702 (2006).

[AH] Y. Almog, B. Helffer, The Distribution of Surface Superconductivity along the Boundary: on a Conjecture of X.B. Pan, *SIAM J. Math. Anal.* **38**, 1715–1732 (2007).

[BBH] F. Béthuel, H. Brézis, F. Hélein, *Ginzburg-Landau Vortices*, Progress in Nonlinear Differential Equations and their Applications **13**, Birkhäuser, Basel, 1994.

[CR1] M. Correggi, N. Rougerie, On the Ginzburg-Landau Functional in the Surface Superconductivity Regime, *Commun. Math. Phys.* **332**, 1297–1343 (2014). Erratum *ibid.* **338**, 1451–1452 (2015).

[CR2] M. Correggi, N. Rougerie, Boundary behavior of the Ginzburg-Landau order parameter in the surface superconductivity regime, *Archive for Rational Mechanics and Analysis* **219** (1), 553-606 (2016).

[FH1] S. Fournais, B. Helffer, Energy asymptotics for type II superconductors, *Calc. Var. Partial Differential Equations* **24**, 341–376 (2005).

[FH2] S. Fournais, B. Helffer, On the third critical field in Ginzburg-Landau theory, *Commun. Math. Phys.* **266**, 153–196 (2006).

[FH3] S. Fournais, B. Helffer, *Spectral Methods in Surface Superconductivity*, Progress in Nonlinear Differential Equations and their Applications **77**, Birkhäuser, Basel, 2010.

[FHP] S. Fournais, B. Helffer, M. Persson, Superconductivity between H_{c_2} and H_{c_3}, *J. Spectr. Theory* **1**, 273–298 (2011).

[FK] S. Fournais, A. Kachmar, Nucleation of bulk superconductivity close to critical magnetic field, *Adv. Math.* **226**, 1213–1258 (2011).

[FKP] S. Fournais, A. Kachmar, M. Persson, The ground state energy of the three dimensional Ginzburg-Landau functional. Part II: surface regime, *J. Math. Pures. App.* **99**, 343–374 (2013).

[Kac] A. Kachmar, The Ginzburg-Landau order parameter near the second critical field, *SIAM J. Math. Anal.* **46**, 572–587 (2014).

[LP] K. Lu, X.B. Pan, Estimates of the upper critical field for the Ginzburg-Landau equations of superconductivity, *Physica D* **127**, 73–104 (1999).

[Nir] L. Nirenberg, On Elliptic Partial Differential Equations, *Ann. Scuola Norm. Sup. Pisa* **13** (1959), 115–162.

[Pan] X. B. Pan, Surface Superconductivity in Applied Magnetic Fields above H_{c2}, *Commun. Math. Phys.* **228**, 327–370 (2002).

[SS] E. Sandier, S. Serfaty, *Vortices in the Magnetic Ginzburg-Landau Model*, Progress in Nonlinear Differential Equations and their Applications **70**, Birkhäuser, Basel, 2007.

[Sig] I.M. Sigal, Magnetic Vortices, Abrikosov Lattices and Automorphic Functions, preprint *arXiv:1308.5446* (2013).

[SJdG] D. Saint-James, P.G. de Gennes, Onset of superconductivity in decreasing fields, *Phys. Lett.* **7**, 306–308 (1963).

Dipartimento di Matematica e Fisica, Università degli Studi Roma Tre, L.go San Leonardo Murialdo, 1, 00146, Rome, Italy.

CNRS & Université Grenoble Alpes, LPMMC (UMR 5493), B.P. 166, F-38042 Grenoble, France

E-mail address: `nicolas.rougerie@lpmmc.cnrs.fr`