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DECOMPOSITION TABLES FOR EXPERIMENTS. II. TWO–ONE RANDOMIZATIONS

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We investigate structure for pairs of randomizations that do not follow each other in a chain. These are unrandomized-inclusive, independent, coincident or double randomizations. This involves taking several structures that satisfy particular relations and combining them to form the appropriate orthogonal decomposition of the data space for the experiment. We show how to establish the decomposition table giving the sources of variation, their relationships and their degrees of freedom, so that competing designs can be evaluated. This leads to recommendations for when the different types of multiple randomization should be used.

1. Introduction. The purpose of this paper, and its prequel [10], is to establish the orthogonal decomposition of the data space for experiments that involve multiple randomizations [9], so that the properties of proposed designs can be evaluated. In [10], this was done for randomizations that follow each other in a chain, as in Figure 1(a). Here, analogous results to those in [10] are obtained for experiments in which the randomizations are two-to-one, as in Figure 1(b). In such randomizations, two different sets of objects are directly randomized to a third, as in Figures 3, 5 and 6. The unrandomized-inclusive, independent and coincident randomizations from [9] are of this type. Also covered are experiments in which the randomization is two-from-one in that two different sets of objects have a single set of objects randomized to them; that is, experiments with double randomizations [9] [see Figure 1(c)].

As in [10], we always denote the set of observational units by Ω, so that the data space is the set V_Ω of all real vectors indexed by Ω. This data space has an orthogonal decomposition into subspaces defined by inherent factors and managerial constraints. We call this decomposition the "structure" on Ω, and identify it with the set $\mathcal{P}$ of mutually orthogonal idempotent matrices which project onto those subspaces. Thus if $\mathbf{P} \in \mathcal{P}$ then $\mathbf{P}$ is an $\Omega \times \Omega$ matrix, because its rows and columns are labelled by the elements of Ω [10].

In the setting of Figure 1(b), there are two other sets, Υ and Γ, which typically contain treatments of different types to be randomized to Ω. For example, in Figure 3, the set of treatments (Γ) and the set of rootstocks (Υ) are randomized to the

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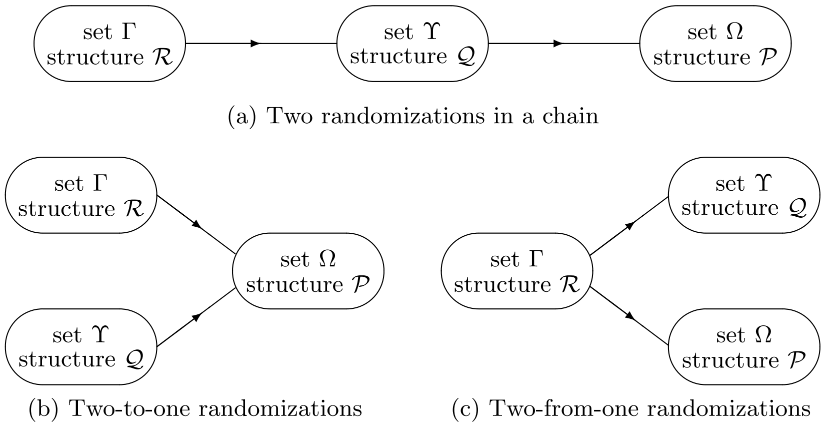

FIG. 1. *The three possibilities for a pair of randomizations.*

set of trees (Ω). Then V_Υ is the space of all real vectors indexed by Υ, and V_Γ is defined similarly. Each of the sets Υ and Γ also has a structure defined on it, the structures being orthogonal decompositions of V_Υ and V_Γ, respectively. These are identified with complete sets $\mathcal{Q}$ and $\mathcal{R}$ of mutually orthogonal idempotent matrices.

There is an immediate technical difficulty. As first defined, a matrix $\mathbf{Q}$ in $\mathcal{Q}$ is not the same size as a matrix $\mathbf{P}$ in $\mathcal{P}$. However, the outcome of the randomization of Υ to Ω is a function f which allocates element $f(\omega)$ of Υ to observational unit ω. This function defines a subspace V_Υ^f of V_Ω isomorphic to V_Υ. Similarly, the outcome of the randomization of Γ to Ω is a function g which allocates element $g(\omega)$ of Γ to observational unit ω. Thus we have a subspace V_Γ^g of V_Ω isomorphic to V_Γ. From now on, we identify V_Υ^f with V_Υ, and V_Γ^g with V_Γ. We also assume that equation (4.1) in [10] holds for both f and g, so that we may regard each matrix $\mathbf{Q}$ in $\mathcal{Q}$ and each matrix $\mathbf{R}$ in $\mathcal{R}$ as an $\Omega \times \Omega$ matrix without losing orthogonality or idempotence. This condition is satisfied for all equi-replicate allocations, and for many others.

In [10] it was seen that a standard two-tiered experiment has just two sets of objects, Ω and Υ say, typically observational units and treatments. To evaluate the design for such an experiment, one needs the decomposition of the data space V_Ω that takes into account both $\mathcal{P}$ and $\mathcal{Q}$. Brien and Bailey [10] introduced the notation $\mathcal{P} \rhd \mathcal{Q}$ for the set of idempotents for this decomposition, and established expressions for its elements under the assumption that $\mathcal{Q}$ is structure balanced in relation to $\mathcal{P}$. They exhibited the decomposition in decomposition tables based on sources corresponding to the elements of $\mathcal{P}$ and $\mathcal{Q}$.

For the idempotents for two sources from different tiers, such as $\mathbf{P}$ in $\mathcal{P}$ and $\mathbf{Q}$ in $\mathcal{Q}$, we follow James and Wilkinson [15] in defining $\mathbf{Q}$ to have *first-order balance* in relation to $\mathbf{P}$ if there is a scalar $\lambda_{\mathbf{PQ}}$ such that $\mathbf{QPQ} = \lambda_{\mathbf{PQ}}\mathbf{Q}$. If this is satisfied and $\lambda_{\mathbf{PQ}} \neq 0$, then $\mathbf{P} \rhd \mathbf{Q}$ is defined in [10] to be $\lambda_{\mathbf{PQ}}^{-1}\mathbf{PQP}$, which is the matrix of orthogonal projection onto $\operatorname{Im}\mathbf{PQ}$, the part of the source $\mathbf{P}$ pertaining to the source $\mathbf{Q}$. The scalar $\lambda_{\mathbf{PQ}}$ is called the *efficiency factor*; it lies in [0, 1] and indicates the proportion of the information pertaining to the source $\mathbf{Q}$ that is (partially) confounded with the source $\mathbf{P}$. Furthermore, a structure $\mathcal{Q}$ is defined in [10] to be *structure balanced* in relation to another structure $\mathcal{P}$ if (i) all idempotents from $\mathcal{Q}$ have first-order balance in relation to all idempotents from $\mathcal{P}$; (ii) all pairs of distinct elements of $\mathcal{Q}$ remain orthogonal when projected onto an element of $\mathcal{P}$, that is, for all $\mathbf{P}$ in $\mathcal{P}$ and all pairs of distinct $\mathbf{Q}_1$ and $\mathbf{Q}_2$ in $\mathcal{Q}$, the product $\mathbf{Q}_1\mathbf{P}\mathbf{Q}_2 = \mathbf{0}$. If $\mathcal{Q}$ is structure balanced in relation to $\mathcal{P}$, and $\mathbf{P} \in \mathcal{P}$, then the residual subspace for $\mathcal{Q}$ in $\operatorname{Im}\mathbf{P}$ is just the orthogonal complement in $\operatorname{Im}\mathbf{P}$ of all the spaces $\operatorname{Im}\mathbf{PQ}$: its matrix of orthogonal projection $\mathbf{P} \vdash \mathcal{Q}$ is given by

$$\mathbf{P} \vdash \mathcal{Q} = \mathbf{P} - {\sum_{\mathbf{Q}\in\mathcal{Q}}}' \mathbf{P} \rhd \mathbf{Q}, \tag{1.1}$$

where $\sum'_{\mathbf{Q}\in\mathcal{Q}}$ means summation over all $\mathbf{Q}$ in $\mathcal{Q}$ with $\lambda_{\mathbf{PQ}} \neq 0$.

This notation was extended in [10] to describe the decomposition for three-tiered experiments where the two randomizations follow each other in a chain, as in composed and randomized-inclusive randomizations [9] [see Figure 1(a)]. This involved combining the three structures $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ defined on three sets of objects to yield the two equivalent decompositions $(\mathcal{P} \rhd \mathcal{Q}) \rhd \mathcal{R}$ and $\mathcal{P} \rhd (\mathcal{Q} \rhd \mathcal{R})$. It was seen that the idempotents of these decompositions could be any of the following forms: $(\mathbf{P} \rhd \mathbf{Q}) \rhd \mathbf{R}$, $\mathbf{P} \rhd (\mathbf{Q} \rhd \mathbf{R})$, $(\mathbf{P} \rhd \mathbf{Q}) \vdash \mathcal{R}$, $\mathbf{P} \rhd (\mathbf{Q} \vdash \mathcal{R})$, and $\mathbf{P} \vdash \mathcal{Q}$, where $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ are idempotents in $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, respectively. In some cases, some idempotents in $(\mathcal{P} \rhd \mathcal{Q}) \rhd \mathcal{R}$ may reduce to idempotents of the form $\mathbf{P}$, $\mathbf{P} \rhd \mathbf{Q}$, $\mathbf{Q}$, $\mathbf{R}$ or $\mathbf{Q} \rhd \mathbf{R}$.

In Sections 2–3 of this paper, corresponding results are obtained for the two-to-one randomizations: unrandomized-inclusive, independent and coincident randomizations. It is shown that, in addition to the decompositions above, the following decompositions occur: $\mathcal{P} \rhd \mathcal{R}$, $(\mathcal{P} \rhd \mathcal{R}) \rhd \mathcal{Q}$ and $(\mathcal{P} \rhd \mathcal{Q}) \square (\mathcal{P} \rhd \mathcal{R})$, where "$\square$" denotes "the combination of compatible decompositions" in a sense defined in Section 3. Also, the list of forms of idempotents is expanded to include: $\mathbf{P} \rhd \mathbf{R}$, $\mathbf{P} \vdash \mathcal{R}$, $(\mathbf{P} \vdash \mathcal{Q}) \rhd \mathbf{R}$ and $(\mathbf{P} \vdash \mathcal{Q}) \vdash \mathcal{R}$.

Section 4 deals with experiments having the only two-from-one randomization: double randomizations.

There are differences between different types of multiple randomization in the reduced forms for the above idempotents and in the efficiency factors. Section 5 gives recommendations for when the different types of multiple randomization should be used. How the results might be applied to experiments with more than three tiers is outlined in Section 6. We finish in Section 7 with a discussion of a number of issues that arise in the decompositions for multitiered experiments.

2. Unrandomized-inclusive randomizations. In an experiment with unrandomized-inclusive randomizations, Υ is randomized to Ω in an initial two-tiered experiment. The unrandomized-inclusive randomization involves a third set, Γ, which is randomized to Ω taking account of the result of the first randomization. As for randomized-inclusive randomizations, the order of the two randomizations is fixed.

Two functions are required to encapsulate the results of these randomizations, say $f\colon\Omega\to\Upsilon$ and $g\colon\Omega\to\Gamma$. For ω in Ω, $f(\omega)$ is the element of Υ assigned to ω by the first randomization, and $g(\omega)$ is the element of Γ assigned to ω by the second randomization. The set-up is represented diagrammatically in Figure 2.

We consider experiments in which the structure $\mathcal{Q}$ on Υ is structure balanced in relation to the structure $\mathcal{P}$ on Ω, so that the first randomization gives the combined decomposition $\mathcal{P} \rhd \mathcal{Q}$ of V_Ω described in [10]. The second randomization takes account of $\mathcal{P} \rhd \mathcal{Q}$, both in the choice of systematic design and in restricting the permutations of Ω to preserve $\mathcal{P} \rhd \mathcal{Q}$, so we assume that the structure $\mathcal{R}$ on Γ is structure balanced in relation to $\mathcal{P} \rhd \mathcal{Q}$.

Put $\mathbf{I}_\mathcal{Q} = \sum_{\mathbf{Q}\in\mathcal{Q}} \mathbf{Q}$, which is the matrix of orthogonal projection onto V_Υ. The condition for $\mathcal{Q}$ to be structure balanced in relation to $\mathcal{P}$ can be written as $\mathbf{I}_\mathcal{Q}\mathbf{P}\mathbf{Q} = \lambda_{\mathbf{PQ}}\mathbf{Q}$ for all $\mathbf{P}$ in $\mathcal{P}$ and all $\mathbf{Q}$ in $\mathcal{Q}$. Similarly, put $\mathbf{I}_\mathcal{R} = \sum_{\mathbf{R}\in\mathcal{R}} \mathbf{R}$, which is the matrix of orthogonal projection onto V_Γ.

THEOREM 2.1. *Let $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ be orthogonal decompositions of the spaces V_Ω, V_Υ and V_Γ, respectively, with $V_\Upsilon \le V_\Omega$ and $V_\Gamma \le V_\Omega$. If $\mathcal{Q}$ is structure balanced in relation to $\mathcal{P}$ with efficiency factors $\lambda_{\mathbf{PQ}}$, and $\mathcal{R}$ is structure balanced in relation to $\mathcal{P} \rhd \mathcal{Q}$ with efficiency factors $\lambda_{\mathbf{P}\rhd\mathbf{Q},\mathbf{R}}$ and $\lambda_{\mathbf{P}\vdash\mathcal{Q},\mathbf{R}}$, then*:

(a) *$\mathcal{R}$ is structure balanced in relation to $\mathcal{P}$ with efficiency matrix $\Lambda_{\mathcal{PR}}$ whose entries are $\lambda_{\mathbf{PR}} = (\lambda_{\mathbf{P}\vdash\mathcal{Q},\mathbf{R}} + \sum'_{\mathbf{Q}\in\mathcal{Q}} \lambda_{\mathbf{P}\rhd\mathbf{Q},\mathbf{R}})$;*
(b) *the decomposition $(\mathcal{P} \rhd \mathcal{Q}) \rhd \mathcal{R}$ is*

$$\begin{aligned}
&\{(\mathbf{P} \rhd \mathbf{Q}) \rhd \mathbf{R} : \mathbf{P}\in\mathcal{P}, \mathbf{Q}\in\mathcal{Q}, \mathbf{R}\in\mathcal{R}, \lambda_{\mathbf{PQ}} \neq 0, \lambda_{\mathbf{P}\rhd\mathbf{Q},\mathbf{R}} \neq 0\}\\
&\quad\cup \{(\mathbf{P} \rhd \mathbf{Q}) \vdash \mathcal{R} : \mathbf{P}\in\mathcal{P}, \mathbf{Q}\in\mathcal{Q}, \lambda_{\mathbf{PQ}} \neq 0\}\\
&\quad\cup \{(\mathbf{P} \vdash \mathcal{Q}) \rhd \mathbf{R} : \mathbf{P}\in\mathcal{P}, \mathbf{R}\in\mathcal{R}, \lambda_{\mathbf{P}\vdash\mathcal{Q},\mathbf{R}} \neq 0\}\\
&\quad\cup \{(\mathbf{P} \vdash \mathcal{Q}) \vdash \mathcal{R} : \mathbf{P}\in\mathcal{P}\}.
\end{aligned}$$

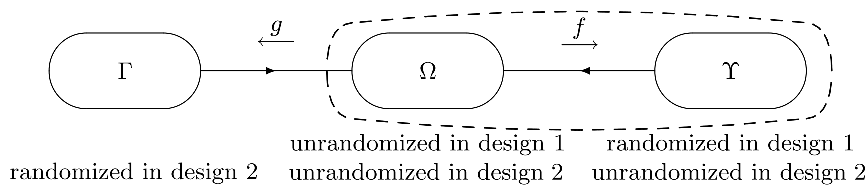

FIG. 2. *Diagram of an experiment with two unrandomized-inclusive randomizations.*

PROOF. (a) Because $\mathcal{R}$ is structure balanced in relation to the decomposition $\mathcal{P} \rhd \mathcal{Q}$, we have $\mathbf{I}_{\mathcal{R}}(\mathbf{P} \rhd \mathbf{Q})\mathbf{R} = \lambda_{\mathbf{P}\rhd\mathbf{Q},\mathbf{R}}\mathbf{R}$ and $\mathbf{I}_{\mathcal{R}}(\mathbf{P} \vdash \mathcal{Q})\mathbf{R} = \lambda_{\mathbf{P}\vdash\mathcal{Q},\mathbf{R}}\mathbf{R}$, for all $\mathbf{P}$ in $\mathcal{P}$, all $\mathbf{Q}$ in $\mathcal{Q}$ with $\lambda_{\mathbf{PQ}} \neq 0$, and all $\mathbf{R}$ in $\mathcal{R}$. Now, $\mathbf{P} = (\mathbf{P} \vdash \mathcal{Q}) + \sum'_{\mathbf{Q}\in\mathcal{Q}} \mathbf{P} \rhd \mathbf{Q}$, so

$$\mathbf{I}_{\mathcal{R}}\mathbf{P}\mathbf{R} = \mathbf{I}_{\mathcal{R}}(\mathbf{P} \vdash \mathcal{Q})\mathbf{R} + \sum_{\mathbf{Q}\in\mathcal{Q}}{}' \mathbf{I}_{\mathcal{R}}(\mathbf{P} \rhd \mathbf{Q})\mathbf{R} = \left(\lambda_{\mathbf{P}\vdash\mathcal{Q},\mathbf{R}} + \sum_{\mathbf{Q}\in\mathcal{Q}}{}' \lambda_{\mathbf{P}\rhd\mathbf{Q},\mathbf{R}}\right)\mathbf{R}.$$

This proves that $\mathcal{R}$ is structure balanced in relation to $\mathcal{P}$ with the given efficiency matrix.

(b) Since $\mathcal{R}$ is structure balanced in relation to $\mathcal{P} \rhd \mathcal{Q}$, we may apply the "$\rhd$" operator to elements of $\mathcal{P} \rhd \mathcal{Q}$ and $\mathcal{R}$, to obtain

$$(\mathbf{P} \rhd \mathbf{Q}) \rhd \mathbf{R} = \lambda^{-1}_{\mathbf{P}\rhd\mathbf{Q},\mathbf{R}}(\mathbf{P} \rhd \mathbf{Q})\mathbf{R}(\mathbf{P} \rhd \mathbf{Q}) = \lambda^{-1}_{\mathbf{P}\rhd\mathbf{Q},\mathbf{R}}(\lambda^{-1}_{\mathbf{PQ}}\mathbf{PQP})\mathbf{R}(\lambda^{-1}_{\mathbf{PQ}}\mathbf{PQP}).$$

Moreover, writing $\sum^*_{\mathbf{R}\in\mathcal{R}}$ to mean summation over $\mathbf{R} \in \mathcal{R}$ with $\lambda_{\mathbf{P}\rhd\mathbf{Q},\mathbf{R}} \neq 0$, applying equation (1.1) to $\mathbf{P} \rhd \mathbf{Q}$ and $\mathcal{R}$ gives

$$(\mathbf{P} \rhd \mathbf{Q}) \vdash \mathcal{R} = \mathbf{P} \rhd \mathbf{Q} - \sum_{\mathbf{R}\in\mathcal{R}}{}^{*} (\mathbf{P} \rhd \mathbf{Q}) \rhd \mathbf{R}.$$

Similarly,

$$(\mathbf{P} \vdash \mathcal{Q}) \rhd \mathbf{R} = \lambda^{-1}_{\mathbf{P}\vdash\mathcal{Q},\mathbf{R}}(\mathbf{P} \vdash \mathcal{Q})\mathbf{R}(\mathbf{P} \vdash \mathcal{Q})$$

and

$$(\mathbf{P} \vdash \mathcal{Q}) \vdash \mathcal{R} = \mathbf{P} \vdash \mathcal{Q} - \sum_{\mathbf{R}\in\mathcal{R}}{}^{*} (\mathbf{P} \vdash \mathcal{Q}) \rhd \mathbf{R}.$$

Thus, using Definition 4 in [10], the decomposition $(\mathcal{P} \rhd \mathcal{Q}) \rhd \mathcal{R}$ is as given. $\square$

The expression for $(\mathcal{P} \rhd \mathcal{Q}) \rhd \mathcal{R}$ in Theorem 2.1(b) differs from that in equation (5.1) of [10] because $(\mathbf{P} \vdash \mathcal{Q}) \rhd \mathbf{R}$ is zero for composed and randomized-inclusive randomizations, but may not be zero for unrandomized-inclusive randomizations.

For simplicity, we write the one-dimensional space for the Mean as V_0, with projector $\mathbf{P}_0 = \mathbf{Q}_0 = \mathbf{R}_0 = n^{-1}\mathbf{J}$, where $n = |\Omega|$ and $\mathbf{J}$ is the $n \times n$ all-1 matrix.

As Brien and Bailey [9] show, unrandomized-inclusive randomizations are common in superimposed experiments. In such an experiment, it may well be the case that $V_\Gamma \cap V_0^\perp$ is orthogonal to every $\mathbf{P} \rhd \mathbf{Q}$ of the decomposition $\mathcal{P} \rhd \mathcal{Q}$. In this case, the decomposition has the simpler form given by Corollary 2.2.

COROLLARY 2.2. *Suppose that $\mathcal{Q}$ is structure balanced in relation to $\mathcal{P}$ and that $\mathcal{R}$ is structure balanced in relation to $\mathcal{P} \rhd \mathcal{Q}$. If $(\mathbf{P} \rhd \mathbf{Q})\mathbf{R} = \mathbf{0}$ for all $\mathbf{P}$ in $\mathcal{P}$, all $\mathbf{Q}$ in $\mathcal{Q}$ and all $\mathbf{R}$ in $\mathcal{R} \setminus \{\mathbf{R}_0\}$, then*

$$\begin{aligned}(\mathcal{P} \rhd \mathcal{Q}) \rhd \mathcal{R} = {} & \{(\mathbf{P} \rhd \mathbf{Q}) : \mathbf{P} \in \mathcal{P}, \mathbf{Q} \in \mathcal{Q}, \lambda_{\mathbf{PQ}} \neq 0\} \\ & \cup \{(\mathbf{P} \vdash \mathcal{Q}) \rhd \mathbf{R} : \mathbf{P} \in \mathcal{P}, \mathbf{R} \in \mathcal{R}, \lambda_{\mathbf{PR}} \neq 0\} \\ & \cup \{(\mathbf{P} \vdash \mathcal{Q}) \vdash \mathcal{R} : \mathbf{P} \in \mathcal{P}\}.\end{aligned}$$

PROOF. If $\lambda_{\mathbf{P}\rhd\mathbf{Q},\mathbf{R}} = 0$ for all $\mathbf{Q} \in \mathcal{Q}$, then $\lambda_{\mathbf{PR}} = \lambda_{\mathbf{P}\vdash\mathcal{Q},\mathbf{R}}$. If this is true for all $\mathbf{R}$, then $(\mathbf{P} \rhd \mathbf{Q}) \vdash \mathcal{R} = \mathbf{P} \rhd \mathbf{Q}$. The result follows. $\square$

LEMMA 2.1. *Suppose that $\mathcal{Q}$ is structure balanced in relation to $\mathcal{P}$, and let $\mathbf{P} \in \mathcal{P} \setminus \{\mathbf{P}_0\}$. The following conditions are equivalent.*

(i) $(\mathbf{P} \rhd \mathbf{Q})\mathbf{I}_{\mathcal{R}} = \mathbf{0}$ *for all* $\mathbf{Q}$ *in* $\mathcal{Q}$ *with* $\lambda_{\mathbf{PQ}} \neq 0$.
(ii) $\mathbf{QPR} = \mathbf{0}$ *for all* $\mathbf{Q}$ *in* $\mathcal{Q}$ *and all* $\mathbf{R}$ *in* $\mathcal{R}$.
(iii) $\mathbf{I}_{\mathcal{Q}}\mathbf{P}\mathbf{I}_{\mathcal{R}} = \mathbf{0}$.

If these are satsified for all $\mathbf{P}$ in $\mathcal{P} \setminus \{\mathbf{P}_0\}$, then $V_\Upsilon \cap V_0^\perp$ is orthogonal to $V_\Gamma \cap V_0^\perp$, and all combinations of elements of Υ with elements of Γ occur on Ω.

PROOF. If $\lambda_{\mathbf{PQ}} = 0$ then $\mathbf{QP} = \mathbf{0}$ so $\mathbf{QPR} = \mathbf{0}$. If $\lambda_{\mathbf{PQ}} \neq 0$ then $\mathbf{QPR} = \lambda_{\mathbf{PQ}}^{-1} \times \mathbf{I}_{\mathcal{Q}}\mathbf{PQPI}_{\mathcal{R}}\mathbf{R} = \mathbf{I}_{\mathcal{Q}}(\mathbf{P} \rhd \mathbf{Q})\mathbf{I}_{\mathcal{R}}\mathbf{R}$. Condition (i) implies that all these terms are zero, which implies condition (ii). Summing $\mathbf{QPR}$ over all $\mathbf{Q}$ and all $\mathbf{R}$ gives $\mathbf{I}_{\mathcal{Q}}\mathbf{P}\mathbf{I}_{\mathcal{R}}$, so condition (ii) implies condition (iii). Finally, if $\lambda_{\mathbf{PQ}} \neq 0$ then $(\mathbf{P} \rhd \mathbf{Q})\mathbf{I}_{\mathcal{R}} = \lambda_{\mathbf{PQ}}^{-1}\mathbf{PQPI}_{\mathcal{R}} = \lambda_{\mathbf{PQ}}^{-1}\mathbf{PQ}(\mathbf{I}_{\mathcal{Q}}\mathbf{P}\mathbf{I}_{\mathcal{R}})$, so condition (iii) implies condition (i).

Summing condition (iii) over all $\mathbf{P}$ in $\mathcal{P} \setminus \{\mathbf{P}_0\}$ gives $\mathbf{0} = \mathbf{I}_{\mathcal{Q}}(\mathbf{I}_{\mathcal{P}} - \mathbf{P}_0)\mathbf{I}_{\mathcal{R}} = \mathbf{I}_{\mathcal{Q}}\mathbf{I}_{\mathcal{R}} - \mathbf{I}_{\mathcal{Q}}\mathbf{P}_0\mathbf{I}_{\mathcal{R}} = (\mathbf{I}_{\mathcal{Q}} - \mathbf{Q}_0)(\mathbf{I}_{\mathcal{R}} - \mathbf{R}_0)$, since $\mathbf{P}_0 = \mathbf{Q}_0 = \mathbf{R}_0$. This shows that $V_\Upsilon \cap V_0^\perp$ is orthogonal to $V_\Gamma \cap V_0^\perp$. This implies that $V_\Upsilon \cap V_\Gamma = V_0$, so Proposition 2 of [2] shows that the Universe is the only partition marginal to both Υ and Γ considered as factors on Ω. Then orthogonality and Proposition 3 of [2] show that all combinations of Υ and Γ occur on Ω. $\square$

The conditions in Lemma 2.1 are a general form of adjusted orthogonality [14].

EXAMPLE 1 (Superimposed experiment in a row-column design). The initial experiment in Example 10 in [9] is a randomized complete-block design to investigate cherry rootstocks: there are three blocks of ten trees each, and there are ten types of rootstock. Many years later, a set of virus treatments is superimposed on this, using the extended Youden square in Table 1. This "square" is a 3×10 rectangle whose rows correspond to Blocks and columns to Rootstocks. Each of

TABLE 1
Extended Youden square showing the Virus Treatment for each Block–Rootstock combination

		Rootstocks									
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Blocks	I	A	B	A	C	D	C	B	E	E	D
	II	D	E	B	D	E	A	C	C	A	B
	III	E	A	C	E	B	D	D	B	C	A

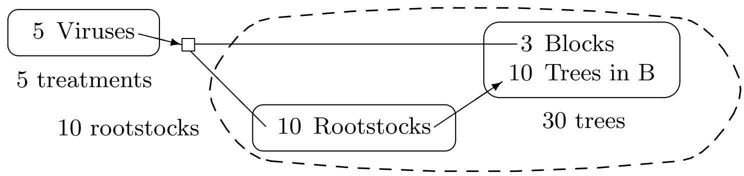

FIG. 3. *Unrandomized-inclusive randomizations in Example 1: rootstocks are randomized to trees in the initial experiment; in the superimposed experiment, treatments are randomized to trees taking account of the allocation of rootstocks;* B *denotes Blocks.*

the five treatments occurs twice in each Block (row), while their disposition in Rootstocks (columns) is that of a balanced incomplete-block design. The sets of objects for this experiment are trees, rootstocks and treatments. Figure 3 shows both randomizations.

For this example, using the notation for sources in [10], but writing $\mathbf{P}_{\mathrm{Mean}}$ as $\mathbf{P}_0$, the three structures are $\mathcal{P} = \{\mathbf{P}_0, \mathbf{P}_{\mathrm{B}}, \mathbf{P}_{\mathrm{T[B]}}\}$, $\mathcal{Q} = \{\mathbf{Q}_0, \mathbf{Q}_{\mathrm{R}}\}$ and $\mathcal{R} = \{\mathbf{R}_0, \mathbf{R}_{\mathrm{V}}\}$. We have $\mathcal{P} \rhd \mathcal{Q} = \{\mathbf{P}_0 \rhd \mathbf{Q}_0, \mathbf{P}_{\mathrm{B}}, \mathbf{P}_{\mathrm{T[B]}} \rhd \mathbf{Q}_{\mathrm{R}}, \mathbf{P}_{\mathrm{T[B]}} \vdash \mathcal{Q}\}$, with $\mathbf{P}_0 \rhd \mathbf{Q}_0 = \mathbf{P}_0$, $\mathbf{P}_{\mathrm{T[B]}} \rhd \mathbf{Q}_{\mathrm{R}} = \mathbf{Q}_{\mathrm{R}}$ and $\mathbf{P}_{\mathrm{T[B]}} \vdash \mathcal{Q} = \mathbf{P}_{\mathrm{T[B]}} - \mathbf{Q}_{\mathrm{R}}$. See the first two columns of Table 2.

The efficiency factors for the structure on treatments in relation to the joint decomposition of trees and rootstocks are derived from the extended Youden square. Viruses are orthogonal to Blocks, which means that $\mathbf{P}_{\mathrm{B}}\mathbf{R}_{\mathrm{V}} = \mathbf{0}$, and hence $\mathbf{P}_{\mathrm{T[B]}}\mathbf{R}_{\mathrm{V}} = \mathbf{R}_{\mathrm{V}}$. Viruses have first-order balance in relation to Rootstocks, with $\lambda_{\mathrm{R,V}} = 1/6$. Hence $\mathbf{R}_{\mathrm{V}}(\mathbf{P}_{\mathrm{T[B]}} \rhd \mathbf{Q}_{\mathrm{R}})\mathbf{R}_{\mathrm{V}} = \mathbf{R}_{\mathrm{V}}\mathbf{Q}_{\mathrm{R}}\mathbf{R}_{\mathrm{V}} = (1/6)\mathbf{R}_{\mathrm{V}}$, and so $\lambda_{\mathrm{T[B]}\rhd\mathrm{R,V}} = 1/6$. Similarly, $\lambda_{\mathrm{T[B]}\vdash\mathcal{Q},\mathrm{V}} = 5/6$. Theorem 2.1 shows that the structure on treatments is orthogonal in relation to the structure on trees since

$$\lambda_{\mathrm{T[B],V}} = \lambda_{\mathrm{T[B]}\rhd\mathrm{R,V}} + \lambda_{\mathrm{T[B]}\vdash\mathcal{Q},\mathrm{V}} = \tfrac{1}{6} + \tfrac{5}{6} = 1.$$

TABLE 2
Decomposition table for Example 1

trees tier		**rootstocks tier**		**treatments tier**		
source	**d.f.**	**source**	**d.f.**	**eff.**	**source**	**d.f.**
Mean	1	Mean	1		Mean	1
Blocks	2					
Trees[Blocks]	27	Rootstocks	9	$\frac{1}{6}$	Viruses	4
					Residual	5
		Residual	18	$\frac{5}{6}$	Viruses	4
					Residual	14

To obtain the full decomposition $(\mathcal{P} \rhd \mathcal{Q}) \rhd \mathcal{R}$, take $\mathcal{P} \rhd \mathcal{Q}$ and refine it by $\mathcal{R}$. In this experiment $V_\Gamma \cap V_0^\perp$ is not orthogonal to $V_\Upsilon \cap V_0^\perp$, because the Viruses source is not orthogonal to Rootstocks. This leads to nonorthogonality between $\mathcal{R}$ and $\mathcal{P} \rhd \mathcal{Q}$. In particular, the Viruses source is not orthogonal to Trees[Blocks] $\rhd$ Rootstocks. Consequently, the decomposition is given by Theorem 2.1(b) rather than Corollary 2.2. The full decomposition of V_{trees}, that contains six elements, one for each line in the decomposition table, is in Table 2:

$$(\mathcal{P} \rhd \mathcal{Q}) \rhd \mathcal{R} = \left\{ \begin{array}{l} (\mathbf{P}_0 \rhd \mathbf{Q}_0) \rhd \mathbf{R}_0, \mathbf{P}_{\mathrm{B}}, \\ (\mathbf{P}_{\mathrm{T[B]}} \rhd \mathbf{Q}_{\mathrm{R}}) \rhd \mathbf{R}_{\mathrm{V}}, (\mathbf{P}_{\mathrm{T[B]}} \rhd \mathbf{Q}_{\mathrm{R}}) \vdash \mathcal{R}, \\ (\mathbf{P}_{\mathrm{T[B]}} \vdash \mathcal{Q}) \rhd \mathbf{R}_{\mathrm{V}}, (\mathbf{P}_{\mathrm{T[B]}} \vdash \mathcal{Q}) \vdash \mathcal{R} \end{array} \right\}$$

with

$$\begin{aligned} (\mathbf{P}_0 \rhd \mathbf{Q}_0) \rhd \mathbf{R}_0 &= \mathbf{P}_0 = \mathbf{Q}_0 = \mathbf{R}_0, \\ (\mathbf{P}_{\mathrm{T[B]}} \rhd \mathbf{Q}_{\mathrm{R}}) \rhd \mathbf{R}_{\mathrm{V}} &= 6(\mathbf{P}_{\mathrm{T[B]}} \rhd \mathbf{Q}_{\mathrm{R}})\mathbf{R}_{\mathrm{V}}(\mathbf{P}_{\mathrm{T[B]}} \rhd \mathbf{Q}_{\mathrm{R}}) \\ &= 6\mathbf{Q}_{\mathrm{R}}\mathbf{R}_{\mathrm{V}}\mathbf{Q}_{\mathrm{R}} = \mathbf{Q}_{\mathrm{R}} \rhd \mathbf{R}_{\mathrm{V}}, \\ (\mathbf{P}_{\mathrm{T[B]}} \rhd \mathbf{Q}_{\mathrm{R}}) \vdash \mathcal{R} &= \mathbf{P}_{\mathrm{T[B]}} \rhd \mathbf{Q}_{\mathrm{R}} - (\mathbf{P}_{\mathrm{T[B]}} \rhd \mathbf{Q}_{\mathrm{R}}) \rhd \mathbf{R}_{\mathrm{V}} \\ &= \mathbf{Q}_{\mathrm{R}} - \mathbf{Q}_{\mathrm{R}} \rhd \mathbf{R}_{\mathrm{V}}, \\ (\mathbf{P}_{\mathrm{T[B]}} \vdash \mathcal{Q}) \rhd \mathbf{R}_{\mathrm{V}} &= \lambda^{-1}_{\mathrm{T[B]} \vdash \mathcal{Q}, \mathrm{V}}(\mathbf{P}_{\mathrm{T[B]}} \vdash \mathcal{Q})\mathbf{R}_{\mathrm{V}}(\mathbf{P}_{\mathrm{T[B]}} \vdash \mathcal{Q}), \\ (\mathbf{P}_{\mathrm{T[B]}} \vdash \mathcal{Q}) \vdash \mathcal{R} &= \mathbf{P}_{\mathrm{T[B]}} \vdash \mathcal{Q} - (\mathbf{P}_{\mathrm{T[B]}} \vdash \mathcal{Q}) \rhd \mathbf{R}_{\mathrm{V}}. \end{aligned}$$

As expected, this decomposition does contain a nontrivial idempotent of the form $(\mathbf{P} \rhd \mathbf{Q}) \rhd \mathbf{R}$. Also, unlike the chain randomizations in [10], it contains an idempotent of the form $(\mathbf{P} \vdash \mathcal{Q}) \rhd \mathbf{R}$.

The efficiency factors are recorded in the decomposition in Table 2, which shows that the Viruses source is partly confounded with both Rootstocks and the part of Trees[Blocks] that is orthogonal to Rootstocks. A consequence of this is that four Rootstocks degrees of freedom cannot be separated from Virus differences. However, there are five Rootstocks degrees of freedom that are orthogonal to Virus differences. Further, while the Viruses source has first-order balance in relation to Rootstocks, the reverse is not true.

3. Independent or coincident randomizations. For independent or coincident randomizations, two sets of objects are randomized to the third; thus we could have Γ and Υ randomized to Ω. Two functions are needed to encapsulate the results of these randomizations, say $f\colon \Omega \to \Gamma$ and $g\colon \Omega \to \Upsilon$. The set-up is represented diagrammatically in Figure 4. A particular feature of these randomizations is that there is no intrinsic ordering of Γ and Υ, because neither randomization takes account of the outcome of the other. Associated with Ω, Υ and Γ are the decompositions $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$. We assume that $\mathcal{Q}$ and $\mathcal{R}$ are both structure balanced in relation to $\mathcal{P}$.

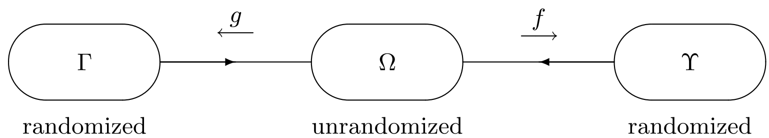

FIG. 4. *Diagram of an experiment with two independent or two coincident randomizations.*

The difference between coincident and independent randomizations is that, for coincident randomizations, there are sources from the two randomized tiers which are both (partly) confounded with the same source in the unrandomized tier. For independent randomizations this does not occur (apart from the Mean).

3.1. *Independent randomizations.* For a pair of independent randomizations, the two functions are randomized by two permutations chosen independently from the same group of permutations of Ω. The precise definition of independence, which we were unable to give in [9], is that the conditions in Lemma 2.1 are satisfied, for all $\mathbf{P}$ in $\mathcal{P} \setminus \{\mathbf{P}_0\}$, for all possible outcomes of the two randomizations. If $\lambda_{\mathbf{PQ}}\lambda_{\mathbf{PR}} \neq 0$ then some outcomes will have $\mathbf{QPR} \neq \mathbf{0}$, violating these conditions. Hence independent randomizations require that $\lambda_{\mathbf{PQ}}\lambda_{\mathbf{PR}} = 0$ for all $\mathbf{Q}$ in $\mathcal{Q}$ and all $\mathbf{R}$ in $\mathcal{R}$ unless $\mathbf{P} = \mathbf{P}_0$. Lemma 2.1 shows that, if $\mathcal{Q}$ and $\mathcal{R}$ are both structure balanced in relation to $\mathcal{P}$, then they are also structure balanced in relation to $\mathcal{P} \rhd \mathcal{R}$ and $\mathcal{P} \rhd \mathcal{Q}$, respectively, with $\lambda_{\mathbf{P}\rhd\mathbf{Q},\mathbf{R}} = \lambda_{\mathbf{P}\rhd\mathbf{R},\mathbf{Q}} = 0$ unless $\mathbf{P} = \mathbf{P}_0$, $\mathbf{Q} = \mathbf{Q}_0$ and $\mathbf{R} = \mathbf{R}_0$. Therefore

$$
\begin{aligned}
(\mathcal{P} \rhd \mathcal{Q}) \rhd \mathcal{R} &= (\mathcal{P} \rhd \mathcal{R}) \rhd \mathcal{Q} \\
&= \{\mathbf{P} \rhd \mathbf{Q} : \mathbf{P} \in \mathcal{P}, \mathbf{Q} \in \mathcal{Q}, \lambda_{\mathbf{PQ}} \neq 0\} \\
&\quad \cup \{\mathbf{P} \vdash \mathcal{Q} : \mathbf{P} \in \mathcal{P}, \mathbf{PI}_{\mathcal{Q}} \neq \mathbf{0}\} \\
&\quad \cup \{\mathbf{P} \rhd \mathbf{R} : \mathbf{P} \in \mathcal{P}, \mathbf{R} \in \mathcal{R}, \lambda_{\mathbf{PR}} \neq 0\} \\
&\quad \cup \{\mathbf{P} \vdash \mathcal{R} : \mathbf{P} \in \mathcal{P}, \mathbf{PI}_{\mathcal{R}} \neq \mathbf{0}\} \\
&\quad \cup \{\mathbf{P} : \mathbf{P} \in \mathcal{P}, \mathbf{PI}_{\mathcal{Q}} = \mathbf{PI}_{\mathcal{R}} = \mathbf{0}\}.
\end{aligned}
\tag{3.1}
$$

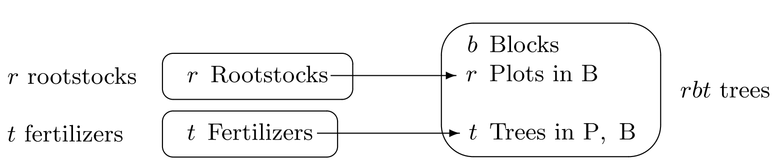

FIG. 5. *Independent randomizations in Example 2: rootstocks are randomized to trees in such a way that all trees in each plot have a single type of rootstock; later, fertilizers are randomized to trees in such a way that each fertilizer is applied to one tree per plot;* B, P *denote Blocks, Plots, respectively.*

TABLE 3
Decomposition table for Example 2

trees tier		**rootstocks tier**		**fertilizers tier**	
source	**d.f.**	**source**	**d.f.**	**source**	**d.f.**
Mean	1	Mean	1	Mean	1
Blocks	$b-1$				
Plots[B]	$b(r-1)$	Rootstocks	$r-1$		
		Residual	$(b-1)(r-1)$		
Trees[P ∧ B]	$br(t-1)$			Fertilizers	$t-1$
				Residual	$(br-1)(t-1)$

As outlined in [9], Section 8.5, wherever possible we reduce two independent randomizations to a single randomization. However, as noted in [9], Section 4.3, this is not always possible—for example, when it is not physically possible to do them simultaneously.

EXAMPLE 2 (Superimposed experiment using split plots). Example 6 in [9] is a superimposed experiment in which the second set of treatments (fertilizers) is randomized to subunits (trees) of the original experimental units (plots). The randomizations are independent, being carried out at different times and with the later one taking no account of the earlier one except to force fertilizers to be orthogonal to rootstocks. See Figure 5. Table 3 shows the decomposition.

In this example the independence of the randomizations implies that $(\mathbf{P}_{\mathrm{P[B]}} \rhd \mathbf{Q}_{\mathrm{R}}) \rhd \mathbf{R}_{\mathrm{F}} = \mathbf{0}$ and so the conditions in Lemma 2.1 are satisfied.

3.2. *Coincident randomizations*. For coincident randomizations, there are idempotents $\mathbf{P}$ in $\mathcal{P} \setminus \{\mathbf{P}_0\}$, $\mathbf{Q}$ in $\mathcal{Q} \setminus \{\mathbf{Q}_0\}$ and $\mathbf{R}$ in $\mathcal{R} \setminus \{\mathbf{R}_0\}$ such that $\mathbf{PQ}$ and $\mathbf{PR}$ are both nonzero. If $\operatorname{Im}\mathbf{PQ}$ and $\operatorname{Im}\mathbf{PR}$ are both proper subspaces of $\operatorname{Im}\mathbf{P}$, then the relationship between $\mathbf{Q}$ and $\mathbf{R}$ depends on the choice of the two independent permutations used in randomizing Υ and Γ to Ω; restricting one of the randomizations to preserve the relationship would make the multiple randomizations unrandomized inclusive rather than coincident. On the other hand, if $\operatorname{Im}\mathbf{PQ} = \operatorname{Im}\mathbf{P}$ then $\operatorname{Im}\mathbf{PR}$ is always contained in $\operatorname{Im}\mathbf{PQ}$. If $\mathcal{Q}$ is structure balanced in relation to $\mathcal{P}$ and $\operatorname{Im}\mathbf{PQ} = \operatorname{Im}\mathbf{P}$, then $\mathbf{P} \rhd \mathbf{Q} = \mathbf{P}$ and the two sources corresponding to $\mathbf{Q}$ and $\mathbf{P}$ have the same number of degrees of freedom. The condition for coincident randomizations hinted at in [9], Section 4.2, is precisely that

(3.2) for all $\mathbf{P}$ in $\mathcal{P}$, $\mathbf{Q}$ in $\mathcal{Q}$ and $\mathbf{R}$ in $\mathcal{R}$, if $\mathbf{PQ}$ and $\mathbf{PR}$ are both nonzero then one of $\mathbf{P} \rhd \mathbf{Q}$ and $\mathbf{P} \rhd \mathbf{R}$ is equal to $\mathbf{P}$.

A special, commonly occurring, case arises when $\mathcal{Q}$ and $\mathcal{R}$ can be assigned to the two randomized sets of objects such that the following condition is satisfied:

(3.3) for all $\mathbf{P}$ in $\mathcal{P}$ and $\mathbf{Q}$ in $\mathcal{Q}$, if $\mathbf{PQ}$ and $\mathbf{PI}_{\mathcal{R}}$ are both nonzero then $\mathbf{P} \rhd \mathbf{Q} = \mathbf{P}$.

THEOREM 3.1. *If $\mathcal{Q}$ and $\mathcal{R}$ are both structure balanced in relation to $\mathcal{P}$ and condition* (3.3) *is satisfied then $\mathcal{R}$ is structure balanced in relation to $\mathcal{P} \rhd \mathcal{Q}$, with $\lambda_{\mathbf{P}\rhd\mathbf{Q},\mathbf{R}} = \lambda_{\mathbf{PR}}$ if $\lambda_{\mathbf{PQ}} \neq 0$ and $\lambda_{\mathbf{P}\vdash\mathcal{Q},\mathbf{R}} = \lambda_{\mathbf{PR}}$ if $\mathbf{P} \vdash \mathcal{Q} \neq \mathbf{0}$. Moreover, the decomposition $(\mathcal{P} \rhd \mathcal{Q}) \rhd \mathcal{R}$ is*

$$\begin{aligned}(3.4)\qquad &\{\mathbf{P} \rhd \mathbf{R} : \mathbf{P} \in \mathcal{P}, \mathbf{R} \in \mathcal{R}, \lambda_{\mathbf{PR}} \neq 0\}\\ &\cup \{(\mathbf{P} \vdash \mathcal{R}) : \mathbf{P} \in \mathcal{P}, \mathbf{PI}_{\mathcal{R}} \neq 0\}\\ &\cup \{(\mathbf{P} \rhd \mathbf{Q}) : \mathbf{P} \in \mathcal{P}, \mathbf{Q} \in \mathcal{Q}, \lambda_{\mathbf{PQ}} \neq 0, \mathbf{PI}_{\mathcal{R}} = 0\}\\ &\cup \{(\mathbf{P} \vdash \mathcal{Q}) : \mathbf{P} \in \mathcal{P}, \mathbf{PI}_{\mathcal{R}} = 0\}.\end{aligned}$$

PROOF. If $\lambda_{\mathbf{PR}} = 0$ then $\mathbf{PR} = \mathbf{0}$, so $(\mathbf{P} \rhd \mathbf{Q})\mathbf{R} = \mathbf{0}$ for all $\mathbf{Q}$ with $\lambda_{\mathbf{PQ}} \neq 0$, and hence $(\mathbf{P} \vdash \mathcal{Q})\mathbf{R} = \mathbf{0}$. Suppose that $\mathbf{PI}_{\mathcal{R}} \neq \mathbf{0}$. Then either $\mathbf{PI}_{\mathcal{Q}} = \mathbf{0}$ or there is a unique $\mathbf{Q}$ in $\mathcal{Q}$ with $\lambda_{\mathbf{PQ}} \neq 0$, which satisfies $\mathbf{P} = \mathbf{P} \rhd \mathbf{Q}$. In the first case, $\mathbf{P} = \mathbf{P} \vdash \mathcal{Q}$: therefore $\mathbf{I}_{\mathcal{R}}(\mathbf{P} \vdash \mathcal{Q})\mathbf{R} = \mathbf{I}_{\mathcal{R}}\mathbf{PR} = \lambda_{\mathbf{PR}}\mathbf{R}$, and $(\mathbf{P} \vdash \mathcal{Q}) \rhd \mathbf{R} = \mathbf{P} \rhd \mathbf{R}$. In the second case, $\mathbf{I}_{\mathcal{R}}(\mathbf{P} \rhd \mathbf{Q})\mathbf{R} = \mathbf{I}_{\mathcal{R}}\mathbf{PR} = \lambda_{\mathbf{PR}}\mathbf{R}$ and $(\mathbf{P} \rhd \mathbf{Q}) \rhd \mathbf{R} = \mathbf{P} \rhd \mathbf{R}$. $\square$

EXAMPLE 3 (A plant experiment). Example 5 in [9] is an experiment to investigate five varieties and two spray regimes. Each bench has one spray regime and two seedlings of each variety. See Figure 6. The sets are positions, seedlings and regimes. The diagram includes the pseudofactor S_1 for Seedlings[Varieties], which indexes the groups of seedlings randomized to the different benches. Although the factor Seedlings is nested in Varieties, S_1 is not, because each of its levels is taken across all levels of Varieties.

The Hasse diagrams displaying the structures for this experiment are in Figure 7. The decomposition is in Table 4, where the source Seedlings[Varieties] $\vdash S_1$ is the part of Seedlings[Varieties] which is orthogonal to the source S_1.

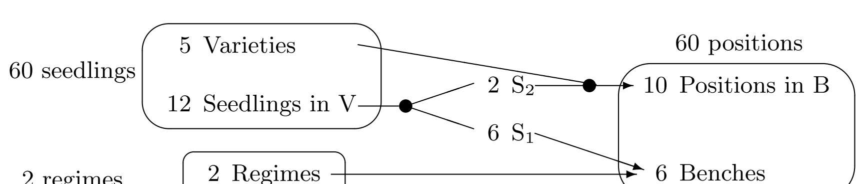

FIG. 6. *Coincident randomizations in Example* 3: *seedlings and regimes are both randomized to positions*; V *denotes Varieties*, B *denotes Benches*; S_1 *and* S_2 *are pseudofactors for Seedlings*.

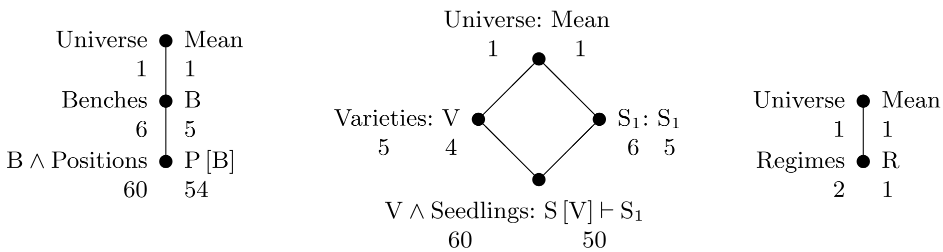

FIG. 7. *Hasse diagrams for Example 3.*

The full decomposition of $V_{\text{positions}}$ in this case contains five elements and is

$$(\mathcal{P} \rhd \mathcal{Q}) \rhd \mathcal{R} = \left\{ \begin{array}{l} (\mathbf{P}_0 \rhd \mathbf{Q}_0) \rhd \mathbf{R}_0, \\ (\mathbf{P}_{\mathrm{B}} \rhd \mathbf{Q}_{\mathrm{S}_1}) \rhd \mathbf{R}_{\mathrm{R}}, (\mathbf{P}_{\mathrm{B}} \rhd \mathbf{Q}_{\mathrm{S}_1}) \vdash \mathcal{R}, \\ \mathbf{P}_{\mathrm{P[B]}} \rhd \mathbf{Q}_{\mathrm{V}}, \mathbf{P}_{\mathrm{P[B]}} \rhd \mathbf{Q}_{\mathrm{S[V]} \vdash \mathrm{S}_1} \end{array} \right\}.$$

This experiment clearly meets condition (3.3), because the only source for positions which is nonorthogonal to sources from both of the randomized tiers is the Benches source, and the five-dimensional pseudosource S_1 is equal to Benches. That is, $\mathbf{P}_{\mathrm{B}} \rhd \mathbf{Q}_{\mathrm{S}_1} = \mathbf{P}_{\mathrm{B}} = \mathbf{Q}_{\mathrm{S}_1}$. The other source nonorthogonal to Benches is the one-dimensional source Regimes, which is a proper subspace of the Benches source, and so $(\mathbf{P}_{\mathrm{B}} \rhd \mathbf{Q}_{\mathrm{S}_1}) \rhd \mathbf{R}_{\mathrm{R}} = \mathbf{P}_{\mathrm{B}} \rhd \mathbf{R}_{\mathrm{R}} = \mathbf{R}_{\mathrm{R}}$.

Consequently, the elements of the full decomposition can be written as follows:

$$(\mathcal{P} \rhd \mathcal{Q}) \rhd \mathcal{R} = \left\{ \begin{array}{l} \mathbf{P}_0, \mathbf{P}_{\mathrm{B}} \rhd \mathbf{R}_{\mathrm{R}}, \mathbf{P}_{\mathrm{B}} \vdash \mathcal{R}, \\ \mathbf{P}_{\mathrm{P[B]}} \rhd \mathbf{Q}_{\mathrm{V}}, \mathbf{P}_{\mathrm{P[B]}} \rhd \mathbf{Q}_{\mathrm{S[V]} \vdash \mathrm{S}_1} \end{array} \right\}.$$

On noting that $\mathbf{P}_{\mathrm{B}}\mathbf{R}_{\mathrm{R}} = \mathbf{R}_{\mathrm{R}}$, $\mathbf{P}_{\mathrm{P[B]}}\mathbf{Q}_{\mathrm{V}} = \mathbf{Q}_{\mathrm{V}}$ and $\mathbf{P}_{\mathrm{P[B]}}\mathbf{Q}_{\mathrm{S[V]} \vdash \mathrm{S}_1} = \mathbf{Q}_{\mathrm{S[V]} \vdash \mathrm{S}_1}$, the decomposition further reduces to

$$(\mathcal{P} \rhd \mathcal{Q}) \rhd \mathcal{R} = \{\mathbf{P}_0, \mathbf{R}_{\mathrm{R}}, \mathbf{P}_{\mathrm{B}} - \mathbf{R}_{\mathrm{R}}, \mathbf{Q}_{\mathrm{V}}, \mathbf{Q}_{\mathrm{S[V]} \vdash \mathrm{S}_1}\}.$$

TABLE 4
Decomposition table for Example 3

positions tier		**seedlings tier**		**regimes tier**	
source	**d.f.**	**source**	**d.f.**	**source**	**d.f.**
Mean	1	Mean	1	Mean	1
Benches	5	S_1	5	Regimes	1
				Residual	4
Positions[Benches]	54	Varieties	4		
		Seedlings[Varieties] $\vdash \mathrm{S}_1$	50		

Decomposition (3.4) is convenient for algorithms, because it is $(\mathcal{P} \rhd \mathcal{Q}) \rhd \mathcal{R}$, like the decompositions in Theorem 5.1(d) in [10], Theorem 2.1(b), Corollary 2.2 and equation (3.1). However, it gives the false impression that the decomposition of V_Ω must have $\mathcal{P}$ refined by $\mathcal{Q}$, then $\mathcal{P} \rhd \mathcal{Q}$ refined by $\mathcal{R}$, suggesting that $\mathcal{Q}$ and $\mathcal{R}$ have different roles. Moreover, condition (3.3) does not hold for all pairs of coincident randomizations. We therefore introduce another joint decomposition that emphasizes the symmetry between $\mathcal{Q}$ and $\mathcal{R}$.

DEFINITION 1. Let $\mathcal{B}$ and $\mathcal{C}$ be orthogonal decompositions of the same space V_Ω. Then $\mathcal{B}$ is *compatible* with $\mathcal{C}$ if $\mathbf{BC} = \mathbf{CB}$ for all $\mathbf{B}$ in $\mathcal{B}$ and all $\mathbf{C}$ in $\mathcal{C}$.

Lemma 2.4 in [3] shows that if $\mathcal{B}$ and $\mathcal{C}$ are compatible then the nonzero products $\mathbf{BC}$, for $\mathbf{B}$ in $\mathcal{B}$ and $\mathbf{C}$ in $\mathcal{C}$, give another orthogonal decomposition of V_Ω, which is a refinement of both $\mathcal{B}$ and $\mathcal{C}$.

DEFINITION 2. If $\mathcal{B}$ and $\mathcal{C}$ are orthogonal decompositions of V_Ω which are compatible with each other, then the decomposition $\mathcal{B} \square \mathcal{C}$ of V_Ω is defined to be

$$\mathcal{B} \square \mathcal{C} = \{\mathbf{BC} : \mathbf{B} \in \mathcal{B}, \mathbf{C} \in \mathcal{C}, \mathbf{BC} \neq \mathbf{0}\}.$$

Thus $\mathcal{B} \square \mathcal{C} = \mathcal{C} \square \mathcal{B}$. Moreover, if $\mathcal{B}$ and $\mathcal{C}$ are also both compatible with $\mathcal{D}$, then $\mathcal{B} \square \mathcal{C}$ is compatible with $\mathcal{D}$, $\mathcal{B}$ is compatible with $\mathcal{C} \square \mathcal{D}$, and $(\mathcal{B} \square \mathcal{C}) \square \mathcal{D} = \mathcal{B} \square (\mathcal{C} \square \mathcal{D})$. Hence if $\mathcal{B}_1, \dots, \mathcal{B}_m$ are pairwise compatible then there is no need for parentheses in defining $\mathcal{B}_1 \square \mathcal{B}_2 \square \cdots \square \mathcal{B}_m$. This decomposition could be referred to as "$\mathcal{B}_1$ combined with $\mathcal{B}_2$ combined with $\cdots$ combined with $\mathcal{B}_m$."

LEMMA 3.1. *If* $\mathbf{PQPRP}$ *is symmetric for all* $\mathbf{P}$ *in* $\mathcal{P}$, *all* $\mathbf{Q}$ *in* $\mathcal{Q}$ *and all* $\mathbf{R}$ *in* $\mathcal{R}$, *then* $\mathcal{P} \rhd \mathcal{Q}$ *is compatible with* $\mathcal{P} \rhd \mathcal{R}$.

PROOF. If $\mathbf{PQPRP}$ is symmetric then $\mathbf{PQPRP} = \mathbf{PRPQP}$. Hence if $\lambda_{\mathbf{PQ}} \times \lambda_{\mathbf{PR}} \neq 0$ then $(\mathbf{P} \rhd \mathbf{Q})(\mathbf{P} \rhd \mathbf{R}) = \lambda_{\mathbf{PQ}}^{-1}\lambda_{\mathbf{PR}}^{-1}\mathbf{PQPPRP} = \lambda_{\mathbf{PQ}}^{-1}\lambda_{\mathbf{PR}}^{-1}\mathbf{PRPPQP} = (\mathbf{P} \rhd \mathbf{R})(\mathbf{P} \rhd \mathbf{Q})$. Thus if $\mathbf{P} \rhd \mathbf{Q}$ is defined then it commutes with $\mathbf{P}$ and with every $\mathbf{P} \rhd \mathbf{R}$, so it commutes with $\mathbf{P} \vdash \mathcal{R}$. Similarly, if $\mathbf{P} \rhd \mathbf{R}$ is defined then it commutes with $\mathbf{P} \vdash \mathcal{Q}$. Now the same argument shows that $\mathbf{P} \vdash \mathcal{Q}$ commutes with $\mathbf{P} \vdash \mathcal{R}$. If $\mathbf{P}_i$ and $\mathbf{P}_j$ are different elements of $\mathcal{P}$ then $\mathbf{P}_i \rhd \mathbf{Q}$ and $\mathbf{P}_i \vdash \mathcal{Q}$ commute with $\mathbf{P}_j \rhd \mathbf{R}$ and $\mathbf{P}_j \vdash \mathcal{R}$, because all products are zero. Hence $\mathcal{P} \rhd \mathcal{Q}$ is compatible with $\mathcal{P} \rhd \mathcal{R}$. $\square$

THEOREM 3.2. *If the conditions in Lemma* 2.1 *are*, *or condition* (3.2) *is*, *satisfied*, *then* $\mathcal{P} \rhd \mathcal{Q}$ *is compatible with* $\mathcal{P} \rhd \mathcal{R}$.

PROOF. The first conditions imply that $\mathbf{QPR} = \mathbf{0}$ or $\mathbf{P} = \mathbf{Q} = \mathbf{R} = \mathbf{P}_0$. The second implies that $\mathbf{QPR} = \mathbf{0}$ or $\mathbf{PQP} = \lambda_{\mathbf{PQ}}\mathbf{P}$ or $\mathbf{PRP} = \lambda_{\mathbf{PR}}\mathbf{P}$. In each case, $\mathbf{PQPRP}$ is symmetric, so Lemma 3.1 completes the proof. □

Thus the decomposition $(\mathcal{P} \rhd \mathcal{Q}) \square (\mathcal{P} \rhd \mathcal{R})$, which is symmetric in $\mathcal{Q}$ and $\mathcal{R}$, can be used for coincident or independent randomizations, or for unrandomized-inclusive randomizations which satisfy the conditions in Lemma 2.1. It is the same as decomposition (3.4) for coincident randomizations when condition (3.3) holds, the same as the decomposition in Corollary 2.2 for unrandomized-inclusive randomizations when the conditions in Lemma 2.1 hold, and the same as decomposition (3.1) for independent randomizations. Condition (3.2) shows that, for a pair of coincident randomizations, each idempotent in $(\mathcal{P} \rhd \mathcal{Q}) \square (\mathcal{P} \rhd \mathcal{R})$ has one of the following forms: $\mathbf{P}$, $\mathbf{P} \rhd \mathbf{Q}$, $\mathbf{P} \rhd \mathbf{R}$, $\mathbf{P} \vdash \mathcal{Q}$ or $\mathbf{P} \vdash \mathcal{R}$.

If a pair of coincident randomizations does not satisfy condition (3.3), then it may be possible to refine $\mathcal{R}$ to, say, $\mathcal{R}_2$ in such a way that $\mathcal{R}_2$ is structure balanced in relation to $\mathcal{P} \rhd \mathcal{Q}$, so that the decomposition in Theorem 2.1(b) can be used. It is possible if $\mathbf{R} = \mathbf{P}$ whenever $\mathbf{P} \rhd \mathbf{R} = \mathbf{P}$.

EXAMPLE 3 (Continued). As already noted, this example satisfies condition (3.3), so $\mathcal{P} \rhd \mathcal{Q}$ is compatible with $\mathcal{P} \rhd \mathcal{R}$. Here

$$\begin{aligned}\mathcal{P} \rhd \mathcal{Q} &= \{\mathbf{P}_0 \rhd \mathbf{Q}_0, \mathbf{P}_{\mathrm{B}} \rhd \mathbf{Q}_{\mathrm{S}_1}, \mathbf{P}_{\mathrm{P[B]}} \rhd \mathbf{Q}_{\mathrm{V}}, \mathbf{P}_{\mathrm{P[B]}} \rhd \mathbf{Q}_{\mathrm{S[V]}\vdash\mathrm{S}_1}\}\\ &= \{\mathbf{P}_0, \mathbf{P}_{\mathrm{B}}, \mathbf{Q}_{\mathrm{V}}, \mathbf{Q}_{\mathrm{S[V]}\vdash\mathrm{S}_1}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{P} \rhd \mathcal{R} &= \{\mathbf{P}_0 \rhd \mathbf{R}_0, \mathbf{P}_{\mathrm{B}} \rhd \mathbf{R}_{\mathrm{R}}, \mathbf{P}_{\mathrm{B}} \vdash \mathcal{R}, \mathbf{P}_{\mathrm{P[B]}}\}\\ &= \{\mathbf{P}_0, \mathbf{R}_{\mathrm{R}}, \mathbf{P}_{\mathrm{B}} - \mathbf{R}_{\mathrm{R}}, \mathbf{P}_{\mathrm{P[B]}}\}.\end{aligned}$$

Then

$$\begin{aligned}(\mathcal{P} \rhd \mathcal{Q}) \square (\mathcal{P} \rhd \mathcal{R}) &= \{\mathbf{P}_0^2, \mathbf{P}_{\mathrm{B}}\mathbf{R}_{\mathrm{R}}, \mathbf{P}_{\mathrm{B}}(\mathbf{P}_{\mathrm{B}} - \mathbf{R}_{\mathrm{R}}), \mathbf{Q}_{\mathrm{V}}\mathbf{P}_{\mathrm{P[B]}}, \mathbf{Q}_{\mathrm{S[V]}\vdash\mathrm{S}_1}\mathbf{P}_{\mathrm{P[B]}}\}\\ &= \{\mathbf{P}_0, \mathbf{R}_{\mathrm{R}}, \mathbf{P}_{\mathrm{B}} - \mathbf{R}_{\mathrm{R}}, \mathbf{Q}_{\mathrm{V}}, \mathbf{Q}_{\mathrm{S[V]}\vdash\mathrm{S}_1}\}\\ &= (\mathcal{P} \rhd \mathcal{Q}) \rhd \mathcal{R}.\end{aligned}$$

4. Double randomizations. Double randomization is the one known type of two-from-one randomizations. In an experiment with double randomization, one set of objects is randomized to two others; thus we could have Γ randomized to Υ and to Ω. We follow the convention that the set of observational units is designated as Ω. Two functions are needed to encapsulate the results of these randomizations, say $f: \Omega \to \Gamma$ and $g: \Upsilon \to \Gamma$. These two functions are randomized independently using two different groups of permutations. The set-up is shown in Figure 8.

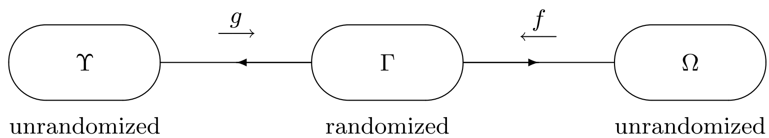

FIG. 8. *Diagram of an experiment with double randomization.*

Now we obtain a subspace V_Γ^f of V_Ω and a subspace V_Γ^g of V_Υ, both isomorphic to V_Γ. If $|\Upsilon| = |\Gamma|$ then $V_\Upsilon = V_\Gamma^g$, so we may effectively identify V_Υ, V_Γ and V_Γ^f. If $|\Upsilon| > |\Gamma|$ then we cannot identify V_Υ with a subspace of V_Ω without further information explicitly assigning an element of Υ to each observational unit in Ω. This may not be possible (see, e.g., Figure 28 in [9]). Thus we shall assume that $|\Upsilon| = |\Gamma|$.

Associated with Ω, Υ and Γ are the decompositions $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$. If $\mathcal{R}$ is structure balanced in relation to $\mathcal{Q}$ and $|\Upsilon| = |\Gamma|$, then Lemma 4.2 in [10] shows that $\mathcal{Q} \rhd \mathcal{R} = \mathcal{R}$. Therefore it suffices to have $\mathcal{R}$ structure balanced in relation to $\mathcal{P}$. Then the overall decomposition is $\mathcal{P} \rhd \mathcal{R} = \mathcal{P} \rhd (\mathcal{Q} \rhd \mathcal{R})$, which must be done from right to left.

EXAMPLE 4 (An improperly replicated rotational grazing experiment). Example 8 in [9] is the rotational grazing trial shown in Figure 9, with Cows substituted for Animals. The double randomization of Availability results in the assignment of Cows to Paddocks, the Cows assigned to an Availability forming a single herd that is used to graze all Paddocks with the same level of Availability. The sets of objects are observational units, paddocks and treatments, and the numbers of paddocks and treatments are equal, as required. The Hasse diagrams for treatments and observational units are like the middle diagram in Figure 7; that for paddocks is trivial.

The structures on observational units, paddocks and treatments are $\mathcal{P} = \{\mathbf{P}_0, \mathbf{P}_{\mathrm{C}}, \mathbf{P}_{\mathrm{R}}, \mathbf{P}_{\mathrm{C\#R}}\}$, $\mathcal{Q} = \{\mathbf{Q}_0, \mathbf{Q}_{\mathrm{P}}\}$ and $\mathcal{R} = \{\mathbf{R}_0, \mathbf{R}_{\mathrm{A}}, \mathbf{R}_{\mathrm{R}}, \mathbf{R}_{\mathrm{A\#R}}\}$, respectively. This leads to the decomposition $\mathcal{P} \rhd (\mathcal{Q} \rhd \mathcal{R})$ in Table 5. It shows that there

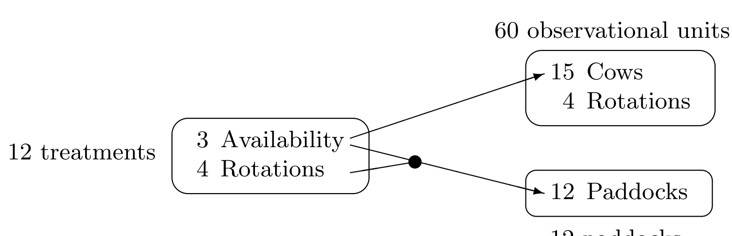

FIG. 9. *Double randomizations in Example 4: treatments are randomized to both observational units and paddocks.*

TABLE 5
Decomposition table for Example 4

observational units tier		paddocks tier		treatments tier	
source	**d.f.**	**source**	**d.f.**	**source**	**d.f.**
Mean	1	Mean	1	Mean	1
Cows	14	Paddocks	2	Availability	2
		Residual	12		
Rotations	3	Paddocks	3	Rotations	3
Cows # Rotations	42	Paddocks	6	Availability # Rotations	6
		Residual	36		

are no residual degrees of freedom for testing any treatment differences—hence the experiment being dubbed *improperly replicated*.

In this case,

$$\mathcal{Q} \rhd \mathcal{R} = \mathcal{R} = \{\mathbf{Q}_0 \rhd \mathbf{R}_0, \mathbf{Q}_\mathrm{P} \rhd \mathbf{R}_\mathrm{A}, \mathbf{Q}_\mathrm{P} \rhd \mathbf{R}_\mathrm{R}, \mathbf{Q}_\mathrm{P} \rhd \mathbf{R}_\mathrm{A\#R}\}$$

with

$$\mathbf{Q}_0 \rhd \mathbf{R}_0 = \mathbf{R}_0, \qquad \mathbf{Q}_\mathrm{P} \rhd \mathbf{R}_\mathrm{A} = \mathbf{R}_\mathrm{A},$$
$$\mathbf{Q}_\mathrm{P} \rhd \mathbf{R}_\mathrm{R} = \mathbf{R}_\mathrm{R}, \qquad \mathbf{Q}_\mathrm{P} \rhd \mathbf{R}_\mathrm{A\#R} = \mathbf{R}_\mathrm{A\#R}.$$

Also, $\mathbf{PR}$ is equal to either $\mathbf{R}$ or $\mathbf{0}$ for all $\mathbf{P} \in \mathcal{P}$ and all $\mathbf{R} \in \mathcal{R}$. That is, $\mathcal{R}$ is orthogonal in relation to $\mathcal{P}$. Therefore the complete decomposition for the experiment is

$$\mathcal{P} \rhd (\mathcal{Q} \rhd \mathcal{R}) = \begin{Bmatrix} \mathbf{P}_0 \rhd (\mathbf{Q}_0 \rhd \mathbf{R}_0), \mathbf{P}_\mathrm{C} \rhd (\mathbf{Q}_\mathrm{P} \rhd \mathbf{R}_\mathrm{A}), \mathbf{P}_\mathrm{C} \vdash (\mathcal{Q} \rhd \mathcal{R}), \\ \mathbf{P}_\mathrm{R} \rhd (\mathbf{Q}_\mathrm{P} \rhd \mathbf{R}_\mathrm{R}), \mathbf{P}_\mathrm{C\#R} \rhd (\mathbf{Q}_\mathrm{P} \rhd \mathbf{R}_\mathrm{A\#R}), \mathbf{P}_\mathrm{C\#R} \vdash (\mathcal{Q} \rhd \mathcal{R}) \end{Bmatrix}$$

with

$$\mathbf{P}_0 \rhd \mathbf{Q}_0 \rhd \mathbf{R}_0 = \mathbf{P}_0, \qquad \mathbf{P}_\mathrm{C} \rhd (\mathbf{Q}_\mathrm{P} \rhd \mathbf{R}_\mathrm{A}) = \mathbf{Q}_\mathrm{P} \rhd \mathbf{R}_\mathrm{A} = \mathbf{R}_\mathrm{A},$$
$$\mathbf{P}_\mathrm{C} \vdash (\mathcal{Q} \rhd \mathcal{R}) = \mathbf{P}_\mathrm{C} - (\mathbf{Q}_\mathrm{P} \rhd \mathbf{R}_\mathrm{A}) = \mathbf{P}_\mathrm{C} - \mathbf{R}_\mathrm{A},$$
$$\mathbf{P}_\mathrm{R} \rhd (\mathbf{Q}_\mathrm{P} \rhd \mathbf{R}_\mathrm{R}) = \mathbf{Q}_\mathrm{P} \rhd \mathbf{R}_\mathrm{R} = \mathbf{R}_\mathrm{R},$$
$$\mathbf{P}_\mathrm{C\#R} \rhd (\mathbf{Q}_\mathrm{P} \rhd \mathbf{R}_\mathrm{A\#R}) = \mathbf{Q}_\mathrm{P} \rhd \mathbf{R}_\mathrm{A\#R} = \mathbf{R}_\mathrm{A\#R},$$
$$\mathbf{P}_\mathrm{C\#R} \vdash (\mathcal{Q} \rhd \mathcal{R}) = \mathbf{P}_\mathrm{C\#R} - (\mathbf{Q}_\mathrm{P} \rhd \mathbf{R}_\mathrm{A\#R}) = \mathbf{P}_\mathrm{C\#R} - \mathbf{R}_\mathrm{A\#R}.$$

In [9] this example was redone as a case of randomized-inclusive randomization, using two pseudofactors $\mathrm{P_A}$ and $\mathrm{P_R}$ for Paddocks, aliased with Availability and Rotations, respectively. These are required if $\mathcal{Q}$ itself is to be structure balanced in relation to $\mathcal{P}$, giving a decomposition from left to right like the one in Section 6 in [10].

5. Summary. We have shown in [10] and here that, under structure balance, the six different types of multiple randomization identified in [9] all lead to orthogonal decompositions of V_Ω using some of the following idempotents: $\mathbf{P}$, $\mathbf{P} \triangleright \mathbf{Q}$, $\mathbf{P} \triangleright \mathbf{R}$, $(\mathbf{P} \triangleright \mathbf{Q}) \triangleright \mathbf{R}$, $\mathbf{P} \triangleright (\mathbf{Q} \triangleright \mathbf{R})$, $\mathbf{P} \vdash \mathcal{Q}$, $\mathbf{P} \vdash \mathcal{R}$, $(\mathbf{P} \triangleright \mathbf{Q}) \vdash \mathcal{R}$, $\mathbf{P} \triangleright (\mathbf{Q} \vdash \mathcal{R})$, $(\mathbf{P} \vdash \mathcal{Q}) \triangleright \mathbf{R}$ and $(\mathbf{P} \vdash \mathcal{Q}) \vdash \mathcal{R}$. The differences between the different multiple randomizations lead to differences in the reduced forms for these elements and in the efficiency factors.

Composed randomizations. If each design is structure balanced then so is the composite; the decompositions $\mathcal{P} \triangleright (\mathcal{Q} \triangleright \mathcal{R})$ and $(\mathcal{P} \triangleright \mathcal{Q}) \triangleright \mathcal{R}$ are equal, and so the decomposition may be done in either order; and there are no idempotents of the form $(\mathbf{P} \vdash \mathcal{Q}) \triangleright \mathbf{R}$ or $(\mathbf{P} \vdash \mathcal{Q}) \vdash \mathcal{R}$.

Randomized-inclusive randomizations. The structures $\mathcal{Q}_1$ and $\mathcal{R}_1$ for design 1 are refined to $\mathcal{Q}$ and $\mathcal{R}$ using the pseudofactors that are necessary for the second randomization, and then the results are the same as for composed randomizations.

Unrandomized-inclusive randomizations. We must have $\mathcal{R}$ structure balanced in relation to $\mathcal{P} \triangleright \mathcal{Q}$; use the decomposition $(\mathcal{P} \triangleright \mathcal{Q}) \triangleright \mathcal{R}$, which is done from left to right; if the conditions in Lemma 2.1 hold then there are no idempotents of the form $(\mathbf{P} \triangleright \mathbf{Q}) \triangleright \mathbf{R}$ apart from the Mean, nor any of the form $(\mathbf{P} \triangleright \mathbf{Q}) \vdash \mathcal{R}$, the decomposition $\mathcal{P} \triangleright \mathcal{Q}$ is compatible with $\mathcal{P} \triangleright \mathcal{R}$, and $(\mathcal{P} \triangleright \mathcal{Q}) \triangleright \mathcal{R} = (\mathcal{P} \triangleright \mathcal{Q}) \,\square\, (\mathcal{P} \triangleright \mathcal{R})$.

Independent randomizations. The conditions in Lemma 2.1 must hold; if both designs are structure balanced then each remains structure balanced after the other has been taken into account; $\mathcal{P} \triangleright \mathcal{Q}$ is compatible with $\mathcal{P} \triangleright \mathcal{R}$; the decompositions $(\mathcal{P} \triangleright \mathcal{Q}) \triangleright \mathcal{R}$, $(\mathcal{P} \triangleright \mathcal{R}) \triangleright \mathcal{Q}$ and $(\mathcal{P} \triangleright \mathcal{Q}) \,\square\, (\mathcal{P} \triangleright \mathcal{R})$ are equal; and there are no idempotents of the form $(\mathbf{P} \triangleright \mathbf{Q}) \triangleright \mathbf{R}$ apart from the Mean, nor any of the form $(\mathbf{P} \triangleright \mathbf{Q}) \vdash \mathcal{R}$.

Coincident randomizations. Condition (3.2) must hold; $\mathcal{P} \triangleright \mathcal{Q}$ is compatible with $\mathcal{P} \triangleright \mathcal{R}$; use the decomposition $(\mathcal{P} \triangleright \mathcal{Q}) \,\square\, (\mathcal{P} \triangleright \mathcal{R})$, whose idempotents have the form $\mathbf{P}$, $\mathbf{P} \triangleright \mathbf{Q}$, $\mathbf{P} \triangleright \mathbf{R}$, $\mathbf{P} \vdash \mathcal{Q}$ or $\mathbf{P} \vdash \mathcal{R}$; if condition (3.3) holds, this is the same as the decomposition $(\mathcal{P} \triangleright \mathcal{Q}) \triangleright \mathcal{R}$, which is done from left to right; otherwise, there may be a refinement of $\mathcal{R}$ giving a left-to-right decomposition.

Double randomizations. We require that $|\Upsilon| = |\Gamma|$ and that $\mathcal{R}$ be structure balanced in relation to both $\mathcal{Q}$ and $\mathcal{P}$, so that the decomposition is $\mathcal{P} \triangleright \mathcal{R} = \mathcal{P} \triangleright$

$(\mathcal{Q} \rhd \mathcal{R})$, which is done from right to left. It appears that they can also be formulated as randomized-inclusive randomizations using pseudofactors to refine $\mathcal{Q}$ to $\mathcal{Q}_2$ for which the left-to-right decomposition $(\mathcal{P} \rhd \mathcal{Q}_2) \rhd \mathcal{R}$ is correct.

6. Structure-balanced experiments with four or more tiers. Each experiment in Sections 2–4 involves only one type of multiple randomization, and so involves three tiers and three structures. However, multitiered experiments are not limited to this configuration. Examples 12–14 in [9] each have four tiers and involve more than one type of multiple randomization. In general, there is the set of observational units, Ω, and each randomization adds another set of objects with its associated tier.

Section 7 in [10] shows how to deal with three or more randomizations which follow each other in a chain. Mixtures of other types of multiple randomization should be amenable to successive decompositions of the sort summarized in Section 5, so long as they are handled in the correct order. Thus we can use a recursive procedure in which each new structure refines the decomposition of V_Ω obtained using structures accounted for previously. All that is required is that each successive structure should be structure balanced in relation to the previous decomposition.

One class of experiments with both two–one randomizations and chain randomizations consists of multiphase experiments in which different treatment factors are applied in different phases, as the following example demonstrates.

EXAMPLE 5 (A two-phase corn seed germination experiment). Example 12 of [9] has the four tiers shown in Figure 10. Here we have taken the opportunity to correct the diagram given in [9]. The 36 Lots of grain within each Plot should be completely randomized to Plates $\wedge$ Containers within each Interval. This will not be achieved by permuting Containers within Intervals and Plates within Intervals $\wedge$ Containers, as implied in the rightmost panel of Figure 10. We introduce pseudofactors L_1 and L_2 for Lots, with nine and four levels, respectively, like

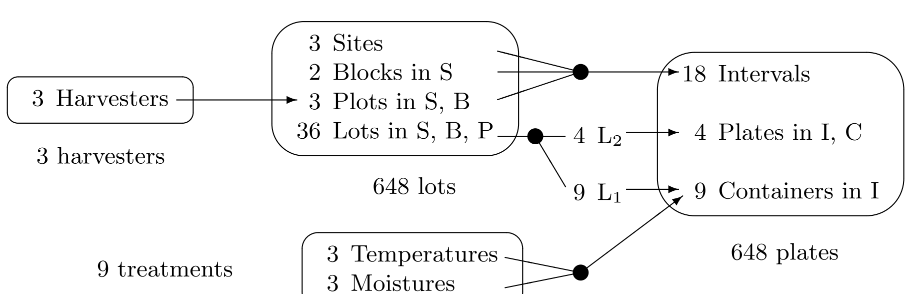

FIG. 10. *Composed and coincident randomizations in Example 5: harvesters are randomized to plots; lots of grain are sampled from each plot and then randomized to plates; and treatments are randomized to plates;* S, B, P, I, C *denote Sites, Blocks, Plots, Intervals, Containers, respectively.*

TABLE 6
Decomposition table for Example 5

plates tier		lots tier		harvesters tier		treatments tier	
source	**d.f.**	**source**	**d.f.**	**source**	**d.f.**	**source**	**d.f.**
Mean	1	Mean	1	Mean	1	Mean	1
Intervals	17	Sites	2				
		Blocks[S]	3				
		Plots[B $\wedge$ S]	12	Harvesters	2		
				Residual	10		
Containers[I]	144	L_1[P $\wedge$ B $\wedge$ S]	144			Temperature	2
						Moistures	2
						T # M	4
						Residual	136
Plates[C $\wedge$ I]	486	Lots[P $\wedge$ B $\wedge$ S]$_{\vdash}$	486				

the pseudofactors for Seedlings in Example 3. The 36 Lots must be randomly allocated to the combinations of levels of L_1 and L_2, independently within each level of Sites $\wedge$ Blocks $\wedge$ Plots, so that neither pseudofactor corresponds to any inherent source of variation.

At each randomization, an orthogonal design is used, so there is no difficulty in constructing the decomposition in Table 6. Here L_1[P $\wedge$ B $\wedge$ S] is the part of the source Lots[P $\wedge$ B $\wedge$ S] which is confounded with Containers[I]. The source Lots[P $\wedge$ B $\wedge$ S]$_{\vdash}$ is the part of Lots[P $\wedge$ B $\wedge$ S] which is orthogonal to L_1[P $\wedge$ B $\wedge$ S]: it is confounded with Plates[C $\wedge$ I].

Bailey [5] suggests an analysis for this example which we reproduce in the first three columns of Table 7(a). In this, the 3-level factors Temperature and Moisture have been combined into a single 9-level Treatment factor, the intertier interactions [9] of Sites, Harvesters and Treatments have been included, and the notation $\times$ is used in place of #. We cannot be sure, but it is plausible that he based this decomposition on the crossing and nesting relationships summarized in the formula

$$\big(\mathrm{T} * \mathrm{H} * (\mathrm{S}/\mathrm{B})\big)/\mathrm{Q}, \tag{6.1}$$

where T, H, S, B and Q represent factors for Treatments, Harvesters, Sites, Blocks and Plates, with 9, 3, 3, 2 and 4 levels, respectively. The sources derived from this are in the final column of Table 7(a), with degrees of freedom matching those in the preceding column.

Revision of Table 6 along similar lines, and with pseudosources replaced with actual sources, yields the skeleton analysis-of-variance table in Table 7(b). Note

TABLE 7

Skeleton analysis-of-variance tables for Example 5(a) *given by Bailey* [5] *and* (b) *from Table 6 with intertier interactions added*

(a)

	source	d.f.	source from (6.1)
Phase I: field study	Site	2	S
	Experimental error (a)	3	B[S]
	Harvester	2	H
	Harvester × Site	4	H#S
	Experimental error (b)	6	H#B[S]
Phase II: laboratory study	Treatment	8	T
	Treatment × Site	16	T#S
	Experimental error (c)	24	T#B[S]
	Treatment × Harvester	16	T#H
	Treatment × Harvester × Site	32	T#H#S
	Experimental error (d)	48	T#H#B[S]
	Residual	486	$Q[T \wedge H \wedge S \wedge B]$

(b)

plates tier		lots tier		harvesters tier		treatments tier	
source	**d.f.**	**source**	**d.f.**	**source**	**d.f.**	**source**	**d.f.**
Mean	1	Mean	1	Mean	1	Mean	1
Intervals	17	Sites	2				
		Blocks[S]	3				
		Plots[$B \wedge S$]	12	Harvesters	2		
				H#S	4		
				Residual	6		
Containers[I]	144	Lots[$P \wedge B \wedge S$]$_1$	144			Treatments	8
						T#S	16
						T#H	16
						T#H#S	32
						Residual	72
Plates[$C \wedge I$]	486	Lots[$P \wedge B \wedge S$]$_{\vdash}$	486				

that, given Step 4 in Table 1 of [8], an intertier interaction will generally occur in the right-most tier that contains a main effect in the interaction. Table 7(b) differs from Table 7(a) in the following ways.

1. The rationale for the sources in Table 7(a) is unclear. We had to reverse-engineer it by producing formula (6.1). On the other hand, the sources in Table 7(b) are

based on the relationships between factors within each tier and on the confounding between sources from different tiers.

2. Table 7(a) does not show, as Table 7(b) does, the successive decomposition of the vector space indexed by the observational units. The impression given is that there is a set of sources that arise from the field phase and another set that arises from the laboratory phase.
3. Table 7(a) has four sources called "experimental error" and does not mention plates, containers, intervals, blocks, plots or lots. Hence, there is no indication of the sources of error variation. By contrast, each source called "Residual" in Table 7(b) is unambiguously identified; and the labelling shows that all terms are affected by variation from both phases. For example, the Residual for Plots[$B \wedge S$], labelled Experimental error (b) in Table 7(a), clearly arises from variability associated with Plots within the Sites-Blocks combinations and variability associated with Intervals. Similarly, it can be seen from Table 7(b) that the Residual in Table 7(a) arises from variability associated with Plates and Lots.
4. As discussed in [9], Section 7.1, the usual default is that there are no intertier interactions because such inclusions would mean that the analysis cannot be justified by the randomization used. It parallels the assumption of unit-treatment additivity in single-randomization experiments. The approach using Table 6 forces the statistician to to consult the researcher about whether intertier interactions should be included, and, if so, to justify them. Tables 7(a) and (b) include the intertier interactions of Sites, Harvesters and Treatments, which suggests that it is anticipated that Harvesters and Treatments will perform differently at different Sites.
5. Even with the addition of intertier interactions, the decompositions in Tables 7(a) and (b) are not equivalent, and so neither are the mixed models underlying them. Experimental errors (c) and (d) from Table 7(a) are combined into the Residual with 72 degrees of freedom for Lots[$P \wedge B \wedge S]_1$ in Table 7(b). To justify an analysis based on Table 7(a), one would need to argue that unit-treatment interaction of Treatments with Blocks within Sites can be anticipated in this experiment.

7. Discussion.

7.1. *Implications of incoherent unrandomized-inclusive randomizations.* The phenomenon of incoherent unrandomized-inclusive randomizations is described in [9], Section 5.2.1. Essentially, when there has been a randomization to factors that are crossed, one or more of these factors become nested in the second randomization.

Consider the cherry rootstock experiment in Example 1. The trees tier gives an orthogonal decomposition of V_Ω into sources Mean, Blocks and Trees[Blocks] of dimensions 1, 2 and 27, respectively, in the left-hand column of Table 2. Similarly, the rootstocks tier decomposes V_Υ into sources Mean and Rootstocks of dimensions 1 and 9. The result of the first randomization is to make the Mean sources equal and to place the Rootstocks source inside Trees[Blocks], thus giving the finer decomposition of V_Ω shown in the middle column of Table 2.

The result of the unrandomized-inclusive randomization should be to further decompose the decomposition resulting from the first two tiers. In the extended Youden square, the source Viruses is orthogonal to Blocks but partially confounded with Rootstocks, so the Viruses source defines the decomposition in the right-hand column in Table 2. That is, the source Viruses further decomposes the sources Rootstocks and the Residual for Trees[Blocks], as required.

In [9] we discussed the possibility that the designer of the superimposed experiment ignores the inherent crossing of the factors Blocks and Rootstocks and randomizes Viruses to Blocks in Rootstocks in a balanced incomplete-block design. Then the randomizations are incoherent. The permutation group for the second randomization does not preserve the structure arising from the first two tiers, exhibited by the two left-most columns in Table 2. We can see immediately that this randomization is senseless because it destroys the Blocks subspace preserved by the first randomization. This randomization might have some appeal if no block effects had been detected during the 20 years of the original experiment, but then the analysis of the second experiment would be based on an assumed model rather than on the intratier structures.

Other examples of incoherent unrandomized-inclusive randomizations are more complicated, and perhaps less easily detected. One is the design proposed by several authors for a split-plot experiment in which the subplot treatments are to be assigned using a row-column design. Example 6 illustrates how consideration of the decomposition table for the proposed design facilitates the design process and helps the detection of incoherence.

EXAMPLE 6 (Split-plots in a row–column design). Example 11 in [9] is based on the design with split-plots in a row–column design given in Cochran and Cox [13], Section 7.33. Diagrams for the two randomizations are given in Figure 11, with leaf treatments named as viruses for clarity, soil treatments designated

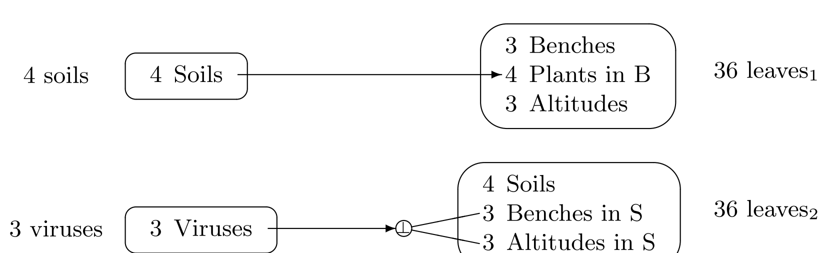

FIG. 11. *Incoherent randomizations in Example 6: both soils and viruses are randomized to leaves, but with different structures on leaves*; B *denotes Benches*; S *denotes Soils*.

as different soils for brevity, and Altitude substituted for Layer so that no two factors begin with the same letter. Two diagrams are needed, because the assumed structure on leaves changes between the randomizations, as shown in the two right-hand panels. At first sight, this experiment seems to involve unrandomized-inclusive randomizations, because soils are randomized to leaves in the first randomization, and then viruses are randomized to leaves, taking into account the location of the soils. However, the change in the assumed structure on the leaves between the two randomizations makes them incoherent rather than unrandomized-inclusive.

Table 8 shows an attempt to build up a decomposition table for this design. The first two columns follow directly from the randomization in the top half of Figure 11. The third column corresponds to the leaves tier in the bottom half of Figure 11. When we use it to refine the decomposition given by the first two tiers, we find that the Soils source occurs in two tiers. Although this can happen in special circumstances like those in Example 4, this is already a signal that something may be wrong. We also find that the nesting, in this tier, of Benches within Soils

TABLE 8
Attempted decomposition table for Example 6

leaves$_1$ tier		**soils tier**		**leaves$_2$ tier**†		**viruses tier**		
source	**d.f.**	**source**	**d.f.**	**source**	**d.f.**	**eff.**	**source**	**d.f.**
Mean	1	Mean	1	Mean	1		Mean	1
Benches	2			B[S]$_{\mathrm{B}}$	2			
Plants[B]	9	Soils	3	Soils	3			
		Residual	6	B[S]$_{\vdash}$	6			
Altitudes	2			A[S]$_{\mathrm{A}}$	2			
A # B	4			A # B[S]$_{\mathrm{A\#B}}$	4	?	Viruses‡	2
						?	V # S‡	2
A # P[B]	18			A[S]$_{\vdash}$	6			
				A # B[S]$_{\vdash}$	12	?	Viruses‡	2
						?	V # S‡	6
							Residual	4

†The subscripts on the sources from this tier indicate that they are the part of the source associated with the subscripted source in the first tier, and the subscript "$\vdash$" that this is the part of the source orthogonal to all previous parts.

‡The partial confounding of Viruses and V # S may not have first-order balance.

gives a source B[S] with 8 degrees of freedom. This is the sum of the previous sources Benches and Residual in Plants[B]; these two parts are denoted $B[S]_B$ and $B[S]_{\vdash}$ in Table 8. Similarly, the nesting of Altitudes within Soils, in this tier, gives sources A[S] and A # B[S] which are each the sum of two previous sources.

The real difficulties come when we try to incorporate the column for the viruses tier, because the location of the Viruses source depends on the outcome of the randomizations. For the outcome given in [13], Section 7.33, and [9], Example 11, the Viruses source does not have first-order balance in relation to either Altitudes # Benches or Altitudes # Plants[Benches]. The interaction Viruses # Soils has the same problem.

If Altitudes # Benches is merged with Altitudes # Plants[Benches] in the decomposition table, then the analysis is orthogonal and is equivalent to that given in [13]. However, this does not allow for consistent Altitude differences across Plants, so it removes six spurious degrees of freedom from what Cochran and Cox call "Error (b)" in [13], page 310. The problem is that the design for the Viruses does not respect the factor relationships established in applying the Soils. As Yates showed in [19], if the randomization respects Benches and Altitudes then a randomization-based model must include their interaction.

What is needed is a design for a two-tiered experiment in which the twelve treatments (combinations of levels of Soils and Viruses) are randomized to leaves_1 in such a way that there is a refinement of the natural decomposition of the treatments space which is structure balanced in relation to Altitudes # Benches. For example, one might choose the systematic design in Table 9 and then randomize benches, altitudes, and plants within benches. In this design the twelve treatments are arranged in a $(3 \times 3)/4$ semi-Latin square constructed from a pair of mutually orthogonal Latin squares of order 3. The Viruses are arranged according to one square for soils s_0 and s_1, and according to the other square for s_2 and s_3. Theorem 5.4 in [1] shows that this design is the most efficient with respect to Altitudes # Benches. Let S_1 be a pseudofactor for Soils whose two levels distinguish between the first two and the last two levels of Soils. The design is structure balanced: Viruses and Viruses # S_1 have efficiency factor $1/2$ in Altitudes # Benches, while the rest of the interaction

TABLE 9
Proposed design for Example 6 (columns denote plants; s_0–s_3 are different soils; 0–2 denote viruses)

	Bench I				**Bench II**				**Bench III**			
Soils	s_0	s_1	s_2	s_3	s_0	s_1	s_2	s_3	s_0	s_1	s_2	s_3
Altitude												
Top	0	0	0	0	1	1	1	1	2	2	2	2
Middle	2	2	1	1	0	0	2	2	1	1	0	0
Bottom	1	1	2	2	2	2	0	0	0	0	1	1

Viruses # Soils is orthogonal to Altitudes # Benches. It has the advantage of having 10 degrees of freedom for the Residual for Altitudes # Plants[Benches], two more than for the Cochran and Cox [13] design.

Thus construction of the decomposition table when designing the experiment can help to detect problems with a proposed design. In this case, it helped to draw attention to the incoherent randomizations, to highlight the associated problems and to give insight into how they might be redressed.

7.2. *Other structures.* All the examples in [10] and this paper are poset block structures, being defined by some factors and their nesting relationships, as explained in [3, 10]. More generally, a structure may be a Tjur structure that is defined by a family of mutually orthogonal partitions or generalized factors (see [17] or [2]). Again, the generalized factors are summarized in the Hasse diagram that depicts their marginality relations. There is one projector $\mathbf{P}$ for each generalized factor F, obtained from the Hasse diagram just as in Section 3 in [10], so that the effect of $\mathbf{P}$ on any vector is still achieved by a straightforward sequence of averaging operations and subtractions. It is possible for some of these projectors to be zero. Structures derived from tiers belong to this class.

Another common source of structure is an association scheme [3, 7]: for example, the triangular scheme for all unordered pairs from a set of parental types, which is appropriate in a diallel experiment with no self-crosses when the cross (i, j) is regarded as the same as the cross (j, i). Then the matrices $\mathbf{P}$ are the minimal idempotents of the association algebra [6], and the corresponding subspaces are its common eigenspaces [3], Chapter 2. The effect of $\mathbf{P}$ is a linear combination of the operations of taking sums over associate classes. In the case of the triangular association scheme with n parental types, the subspaces have dimensions 1, $n - 1$ and $n(n - 3)/2$; they correspond to the Mean, differences between parental types and differences orthogonal to parental types, respectively. The decomposition $\mathcal{R}_3$ in Example 5 in [10] comes from an association scheme with two associate classes.

The set of treatments in a rectangular lattice design exhibits yet another kind of structure [4]. Although this structure derives neither from partitions nor from an association scheme, the effect of each $\mathbf{P}$ is achieved by averaging and subtracting.

The results here and in [10] apply to any structure that is an orthogonal decomposition of the relevant vector space, so long as each structure can be regarded as a decomposition of V_Ω. For a Tjur structure $\mathcal{Q}$ on a set Υ randomized to Ω, condition (4.1) in [10] must hold in order for $\mathcal{Q}$ to be regarded as an orthogonal decomposition of V_Ω. For structures not defined by partitions, it seems that we need $\mathbf{Q}_i\mathbf{X}'\mathbf{X}\mathbf{Q}_j$ to be zero whenever $\mathbf{Q}_i \neq \mathbf{Q}_j$, where $\mathbf{X}$ is the $\Omega \times \Upsilon$ design matrix. For an association scheme, this implies that the design must be equireplicate. The analogue of Theorem 5.1(a) in [10] for association schemes is given in [3], Section 7.7.

We admit that there are relevant experimental structures, such as neighbour relations in a field or increasing quantities of dose, that are not adequately described by an orthogonal decomposition of the space. Nonetheless, a theory which covers designed experiments where all the structures are orthogonal decompositions has wide applicability, and we limit ourselves to such structures here and in [10].

7.3. *Multiphase experiments*. Multiphase experiments are one of the commoner types of multitiered experiment. As outlined in [9], Section 8.1, two-phase experiments may involve almost any of the different types of multiple randomizations and, as is evident from Section 5, these differ in their assumptions.

If treatments are introduced only in the first phase, then the randomizations form a chain, as in [10]. In [18], Wood, Williams and Speed consider a class of such two-phase designs for which $\mathcal{R}$ is orthogonal in relation to the natural structure $\mathcal{Q}_1$ on the middle tier, and there is a refinement $\mathcal{Q}_2$ of $\mathcal{Q}_1$ such that $\mathcal{Q}_2 \rhd \mathcal{R}$ is structure balanced in relation to $\mathcal{P}$. The results there are less general than ours. First, the assumptions for the second phase are in the nature of those for randomized-inclusive randomizations only. Second, the designs are restricted to those for which the design for the first phase is orthogonal.

If treatments are introduced after the first phase, as in Example 5, then some form of two-to-one randomization is needed. Similarly, Brien and Demétrio [11] describe a three-phase experiment involving composed and coincident randomizations.

7.4. *Further work*. While obtaining mixed model analyses of multitiered experiments has been described in [9], Section 7, and [11], it remains to establish their randomization analysis. The effects of intertier interactions on the analysis need to be investigated. We would like to establish conditions under which closed-form expressions are available for the Residual or Restricted Maximum Likelihood (REML) estimates of the variance components [16] and Estimated Generalized Least Squares (EGLS) estimates of the fixed effects. Also required is a derivation of the extended algorithm described in [12] for obtaining the ANOVA for a multitiered experiment.

Furthermore, we have provided the basis for assessing a particular design for a multitiered experiment, yet general principles for designing them are still needed.

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