# ENDOMORPHISMS OF FRAÏSSÉ LIMITS AND AUTOMORPHISM GROUPS OF ALGEBRAICALLY CLOSED RELATIONAL STRUCTURES

**Jillian Dawn McPhee**

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# Endomorphisms of Fraïssé Limits and Automorphism Groups of Algebraically Closed Relational Structures.

**Jillian Dawn McPhee**

University of
St Andrews

This thesis is submitted in partial fulfilment for the degree of PhD at the University of St Andrews

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# Abstract

Let $\Omega$ be the Fraïssé limit of a class of relational structures. We seek to answer the following semigroup theoretic question about $\Omega$.

What are the group $\mathscr{H}$-classes, *i.e.* the maximal subgroups, of $\mathrm{End}(\Omega)$?

Fraïssé limits for which we answer this question include the random graph $R$, the random directed graph $D$, the random tournament $T$, the random bipartite graph $B$, Henson's graphs $G_n$ ($n \geq 3$) and the total order $\mathbb{Q}$.

The maximal subgroups of $\mathrm{End}(\Omega)$ are closely connected to the automorphism groups of the relational structures induced by the images of idempotents from $\mathrm{End}(\Omega)$. In [BD00] and [Dol12] it was shown that the relational structure induced by the image of an idempotent from $\mathrm{End}(\Omega)$ is algebraically closed. Accordingly, we investigate which groups can be realised as the automorphism group of an algebraically closed relational structure in order to determine the maximal subgroups of $\mathrm{End}(\Omega)$ in each case.

In particular, we show that if $\Gamma$ is a countable graph and $\Omega = R, D, B$, then there exist $2^{\aleph_0}$ maximal subgroups of $\mathrm{End}(\Omega)$ which are isomorphic to $\mathrm{Aut}(\Gamma)$. Additionally, we provide a complete description of the subsets of $\mathbb{Q}$ which are the image of an idempotent from $\mathrm{End}(\mathbb{Q})$. We call these subsets retracts of $\mathbb{Q}$ and show that if $\Omega$ is a total order and $f : \Omega \to \mathbb{Q}$ is an embedding such that $\operatorname{im} f$ is a retract of $\mathbb{Q}$, then there exist $2^{\aleph_0}$ maximal subgroups of $\mathrm{End}(\mathbb{Q})$ isomorphic to $\mathrm{Aut}(\Omega)$. We also show that any countable maximal subgroup of $\mathrm{End}(\mathbb{Q})$ must be isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$.

As a consequence of the methods developed, we are also able to show that when $\Omega = R, D, B, \mathbb{Q}$ there exist $2^{\aleph_0}$ regular $\mathscr{D}$-classes of $\mathrm{End}(\Omega)$ and when $\Omega = R, D, B$ there exist $2^{\aleph_0}$ $\mathscr{J}$-classes of $\mathrm{End}(\Omega)$. Additionally we show that if $\Omega = R, D$ then all regular $\mathscr{D}$-classes contain $2^{\aleph_0}$ group $\mathscr{H}$-classes. On the other hand, we show that when $\Omega = B, \mathbb{Q}$ there exist regular $\mathscr{D}$-classes which contain countably many group $\mathscr{H}$-classes.

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# Contents

# Chapter 1

# Motivation and Introduction

In 1953 Roland Fraïssé defined under which circumstances a relational structure could be approximated by its finitely generated substructures, [Fra53]. He was able to provide a full set of axioms which determined precisely when the class of finitely generated substructures defined the relational structure up to isomorphism.

Since then, Fraïssé limits have been the subject of much interest and study. In particular, group theorists have asked many natural questions about the automorphism groups of such structures. For example, the Fraïssé limit of the class of finite graphs was first introduced by Erdős and Rényi in 1963, and is now commonly known as the random graph $R$, [ER63]. Subsequently, in 1985, Truss proved that the automorphism group of the random graph was simple and also gave a characterisation of the possible cycle decomposition types of automorphisms of $R$, [Tru85]. In 2011, this was generalised by Macpherson and Tent who showed that if $\Omega$ is any homogeneous Fraïssé limit of a class which has the free amalgamation property and $\mathrm{Aut}(\Omega)$ is transitive on $\Omega$ but not equal to $\mathrm{Sym}(\Omega)$, then $\mathrm{Aut}(\Omega)$ is simple, [MT11].

Only more recently have semigroup theorists started to study the natural semigroup of endomorphisms, $\mathrm{End}(\Omega)$, for various Fraïssé limits $\Omega$. In 2000, Bonato and Delić started this task by providing many semigroup theoretic properties of $\mathrm{End}(R)$, [BD00]. For example, they showed that $\mathrm{End}(R)$ is non-regular and provided a description of the subgraphs of $R$ induced by the images of idempotent endomorphisms. Furthermore, in [Dol07] and [BDD10] it was shown that all countable monoids embed into the endomorphism monoid of the random graph and random poset.

In this thesis we seek to answer further semigroup theoretic questions about the semigroup of endomorphisms of a selection of Fraïssé limits. These are: the random graph $R$, the random directed graph $D$, the random tournament $T$, the random bipartite graph $B$, Henson's graphs $G_n$ and the total order $\mathbb{Q}$. These are examined in Chapters 3 through 8. In particular, we determine the group $\mathscr{H}$-classes (and therefore maximal subgroups) of the semigroup of endomorphisms of each of the Fraïssé limits mentioned.

It is known that if $\Omega$ is a relational structure, then the maximal subgroups of $\mathrm{End}(\Omega)$ are in one-one correspondence with the $\mathscr{H}$-classes of idempotents from $\mathrm{End}(\Omega)$. It can further be shown that the group $\mathscr{H}$-class of an idempotent from $\mathrm{End}(\Omega)$ is isomorphic to the automorphism group of the relational structure induced by the image of the idempotent (see Theorem 2.7). In [BD00] and [Dol12] a characterisation is provided of the relational structures induced by the image of an idempotent from $\mathrm{End}(\Omega)$ when $\Omega$ is a Fraïssé limit of a class of relational structures satisfying certain conditions. They show that the structures induced by the image of such idempotents are algebraically closed. Accordingly, we investigate which groups can be realised as the automorphism group of an algebraically closed relational structure in order to determine the maximal subgroups of $\mathrm{End}(\Omega)$ when $\Omega$ is any of the Fraïssé limits mentioned above.

We show that if $\Gamma$ is a countable graph and $\Omega = R, D, B$, then there exist $2^{\aleph_0}$ maximal subgroups of $\mathrm{End}(\Omega)$ which are isomorphic to $\mathrm{Aut}(\Gamma)$. By using Frucht's Theorem (see Theorem 3.12), this leads us to the conclusion that every countable group is a maximal subgroup of $\mathrm{End}(\Omega)$ in these cases. On the other hand we show that if $\Omega = T, G_n$, then $\mathrm{End}(\Omega)$ has exactly one maximal subgroup, namely $\mathrm{Aut}(\Omega)$. When considering the total order $\mathbb{Q}$, we take a slightly different approach and provide a complete description of the subsets of $\mathbb{Q}$ which are the image of an idempotent from $\mathrm{End}(\mathbb{Q})$. We call these subsets retracts of $\mathbb{Q}$ and show that if $\Omega$ is a total order and $f : \Omega \to \mathbb{Q}$ is an embedding such that $\mathrm{im}\, f$ is a retract of $\mathbb{Q}$, then there exist $2^{\aleph_0}$ maximal subgroups of $\mathrm{End}(\mathbb{Q})$ isomorphic to $\mathrm{Aut}(\Omega)$. When investigating $\mathrm{End}(\mathbb{Q})$, we found ourselves asking the question: which countable groups can be realised as the automorphism groups of a countable total order? As it turned out, the answer to this question was not straightforward and finding an answer proved to be particularly complicated. The answer to this question is developed in Chapter 9, where we show that if $\Lambda$ is a countable total order and $\mathrm{Aut}(\Lambda)$ is countable, then $\mathrm{Aut}(\Lambda)$ is isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$. Thus an analogue of Frucht's Theorem does not hold in the setting of total orders and consequently, if $H$ is a countable maximal subgroup of $\mathrm{End}(\mathbb{Q})$, then $H \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}$.

As a consequence of the methods developed throughout the thesis, we are also able to gain information on the regular $\mathscr{D}$-classes and $\mathscr{J}$-classes of End($\Omega$) for each of the Fraïssé limits $\Omega$. In particular we can determine the cardinality of the set of regular $\mathscr{D}$-classes and (for some of the Fraïssé limits) the cardinality of the set of $\mathscr{J}$-classes of End($\Omega$). We show that when $\Omega = R, D, B, \mathbb{Q}$ there exist $2^{\aleph_0}$ regular $\mathscr{D}$-classes of End($\Omega$) and when $\Omega = R, D, B$ there exist $2^{\aleph_0}$ $\mathscr{J}$-classes of End($\Omega$). Additionally we show that if $\Omega = R, D$ then all regular $\mathscr{D}$-classes contain $2^{\aleph_0}$ group $\mathscr{H}$-classes. On the other hand, we show that when $\Omega = B, \mathbb{Q}$ there exist regular $\mathscr{D}$-classes which contain countably many group $\mathscr{H}$-classes.

In the final chapter, Chapter 10, we provide some open questions and possible areas for further work which arise from this thesis.

# Chapter 2

# Preliminaries

In this preliminary chapter, we provide the necessary notation and preliminary results that will be referred to continually throughout this thesis.

## 2.1 Sets and Functions

Throughout this thesis, we will assume the axiom of choice. Unless otherwise stated the *natural numbers* will be defined as the set $\mathbb{N} = \{0, 1, 2, \dots\}$. A set $V$ will be called *countable* if there exists an injective function from $V$ to $\mathbb{N}$. Under this definition finite sets are countable.

Throughout, maps will be written on the right of their argument so that functions are composed from left to right. As is standard, the *domain* of a function $f$ will be denoted by $\operatorname{dom} f$ and its *image* by $\operatorname{im} f$. Thus if $f : V \to W$ is a function of sets, then $\operatorname{dom} f = V$ and $\operatorname{im} f = \{vf : v \in V\} \subseteq W$. If $U$ is a subset of the domain of $f$ then we will denote the image of the set $U$ under $f$ by $Uf$. That is, we let $Uf = \{uf : u \in U\}$. The *kernel* of $f$ is, as usual, $\ker f = \{(u, v) \in V \times V : uf = vf\}$. Given $u \in V$ we call the set $\{v \in V : (u, v) \in \ker f\}$ the *kernel class* of $u$. It is easy to see that $\ker f$ is an equivalence relation and hence the kernel classes of $f$ partition $V$.

If $f : V \to W$ is a function and $U \subseteq V$ then $f$ *restricted* to $U$, denoted $f|_U$ is the function $f|_U : U \to W$, where $uf|_U = uf$ for all $u \in U$. On the other hand, if $f : U \to W$ is a function, then an *extension* of $f$ is a function $\tilde{f} : V \to W$ where $U \subseteq V$ and $\tilde{f}|_U = f$.

Since a function $f : U \to V$ can be thought of as a subset of $U \times V$ we can take the union of a set of functions $\{f_i : i \in I\}$ for some index set $I$. In general $\bigcup f_i$ is not a function. However if $f_{i+1}$ is an extension of $f_i$ for all $i$, or if $\operatorname{dom} f_i \cap \operatorname{dom} f_j = \emptyset$ for all $i \neq j$, then $\bigcup f_i$ is itself a function.

## 2.2 Relational Structures

A *binary relation* $E$ on a set $V$ is just a subset of $V \times V$. The binary relation $E$ is said to be:

> *Reflexive* if for all $v \in V$ it is true that $(v, v) \in V$.
>
> *Irreflexive* if for all $v \in V$ it is true that $(v, v) \notin V$.
>
> *Symmetric* if for all $u, v \in V$ such that $(u, v) \in E$, it is true that $(v, u) \in E$.
>
> *Antisymmetric* if for all $u, v \in V$, whenever $(u, v) \in E$ and $(v, u) \in E$ it can be deduced that $u = v$.
>
> *Transitive* if for all $u, v, w \in V$ such that $(u, v) \in E$ and $(v, w) \in E$, it is true that $(u, w) \in E$.

For the purpose of this thesis a *(binary) relational structure* $\Omega = (V, \mathcal{E})$ is a non-empty[1] set $V$ together with a sequence $\mathcal{E} = (E_i)_{i \in I}$ of one or more binary relations. If $\mathcal{E}$ consists of a finite sequence of binary relations $E_1, \ldots E_n$ say, we may simply write $\Omega = (V, E_1, \ldots, E_n)$. A relational structure $\Omega = (V, \mathcal{E})$ is said to be countable if $V$ is a countable set. Likewise, if $V$ is finite we say that $\Omega$ is finite. Notation may be abused slightly and $v \in \Omega$ may often be written to mean $v \in V$. A *relational substructure* of $\Omega$ is a relational structure $(U, \mathcal{D})$, where $U$ is a subset of $V$ and where $\mathcal{D}$ is the sequence $(D_i)_{i \in I}$ with $D_i = E_i \cap (U \times U)$ for all $i \in I$. Clearly, each binary relation $D_i$ will inherit the above properties (*i.e.* reflexivity, symmetry and so forth) that the parent relation $E_i$ possesses. Often $(U, \mathcal{D})$ is called the *relational substructure induced* by $U$ and denoted by $\langle U \rangle$. If $U$ is finite then $\langle U \rangle$ is said to be a *finitely generated* substructure (of $\Omega$).

Let $\Omega = (V_\Omega, (E_i)_{i \in I})$ and $\Lambda = (V_\Lambda, (F_i)_{i \in I})$ be relational structures. A *homomorphism* $f : \Omega \to \Lambda$ is a function $f : V_\Omega \to V_\Lambda$ such that $(uf, vf) \in F_i$

[1] We could, of course, permit $V$ to be empty and allow the structure consisting of an empty set together with an empty set of binary relations to be a relational structure. However, since this makes our lives more complicated whilst not adding anything interesting to the discussion, this structure will be omitted from the definition.

whenever $(u,v) \in E_i$ for $i \in I$. If $\Omega = \Lambda$ then $f$ is said to be an *endomorphism.* The set of all endomorphisms of a relational structure $\Omega$ forms a monoid under composition of functions and is denoted by $\mathrm{End}(\Omega)$ or $\mathrm{End}(V_\Omega, \mathcal{E}_\Omega)$. Clearly, if $f : V_\Omega \to V_\Lambda$ is a homomorphism of relational structures then the image of $f$ induces a relational structure on $\Lambda$. Where there can be no confusion, we may abuse the notation to write $\mathrm{im}\, f$ to mean the relational structure induced by the image set (*i.e.* $\langle \mathrm{im}\, f \rangle$) as well as simply the image set itself.

If $f : V_\Omega \to V_\Lambda$ is an injective function such that $(u,v) \in E_i$ if and only if $(uf, vf) \in F_i$, then $f$ is said to be an *embedding* (of $\Omega$ into $\Lambda$). If instead $f$ is bijective and $(u,v) \in E_i$ if and only if $(uf, vf) \in F_i$, then $f$ is said to be an *isomorphism.* Thus an embedding $\Omega \to \Lambda$ is a map $f : V_\Omega \to V_\Lambda$ which defines an isomorphism between $\Omega$ and the relational substructure of $\Lambda$ induced by $\mathrm{im}\, f$. If $\Omega$ and $\Lambda$ are relational structures and there exists an isomorphism $f : \Omega \to \Lambda$, then $\Omega$ and $\Lambda$ are said to be *isomorphic* and this is denoted by $\Omega \cong \Lambda$. When $\Omega = \Lambda$, we call $f$ an *automorphism.* The set of all automorphisms of a relational structure $\Omega$ forms a group under composition of functions and is denoted by $\mathrm{Aut}(\Omega)$ or $\mathrm{Aut}(V_\Omega, \mathcal{E}_\Omega)$. The function $\mathbf{1} : V_\Omega \to V_\Omega$ defined by $v\mathbf{1} = v$ for all $v \in V_\Omega$ is an automorphism which we call the *identity* automorphism. Where the domain needs to be made clear we may write $\mathbf{1}_{V_\Omega}$ to mean the identity function $V_\Omega \to V_\Omega$. If $f : \Omega \to \Lambda$ is an isomorphism we define the *inverse* of $f$ to be the map $f^{-1} : V_\Lambda \to V_\Omega$ such that $vf^{-1} = u$, where $u \in V_\Omega$ is the unique element such that $uf = v$. Since $f$ is an isomorphism $(v,w) \in E_i$ if and only if $(vf, wf) \in F_i$. In other words, $(x,y) \in F_i$ if and only if $(xf^{-1}, yf^{-1}) \in E_i$. Thus the inverse map $f^{-1}$ is also an isomorphism.

A relational structure $\Omega$ is said to be *homogeneous* if any isomorphism between finitely generated substructures of $\Omega$ can be extended to an automorphism of $\Omega$. That is, whenever $\Delta_1$ and $\Delta_2$ are isomorphic finitely generated substructures of $\Omega$ via the isomorphism $f : \Delta_1 \to \Delta_2$, then there exists an automorphism $g : \Omega \to \Omega$ such that $g|_{\Delta_1} = f$.

A *graph* $\Gamma = (V_\Gamma, E_\Gamma)$ is a relational structure with an irreflexive and symmetric binary relation $E_\Gamma$. As is standard practice we call the set $V_\Gamma$ the set of *vertices* and $E_\Gamma$ the set of *edges.* If $\Gamma$ is a graph and $U \subseteq V_\Gamma$, we call the relational substructure induced by $U$ a *subgraph* of $\Gamma$. If $(u,v) \in E_\Gamma$ (and hence $(v,u) \in E_\Gamma$ also), we say that there exists an edge between $u$ and $v$ or that $u$ is *adjacent* to $v$. The *degree* of a vertex $v \in V_\Gamma$ (in $\Gamma$) is the number of vertices to which $v$ is adjacent in $\Gamma$. Graph isomorphisms preserve degree, in the sense that if $f : \Gamma \to \Lambda$ is an isomorphism, then the degree of $vf$ in $\Lambda$ is equal to the degree of $v$ in $\Gamma$ for all $v \in V_\Gamma$. A graph $\Gamma$ is said to be *locally finite* if every vertex of $\Gamma$ has finite degree. We call a graph $\Gamma$ *connected* if for any pair of vertices $u, v \in V_\Gamma$, such that $u \neq v$, there exists a sequence of edges $(u, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, v)$ in $E_\Gamma$. The sequence of edges $(u, x_1), \ldots, (x_n, v)$ will be called a *path* from $u$ to $v$ of length $n$. A path from a vertex $u$ to itself is called a *cycle*.

A graph $\Gamma = (V_\Gamma, E_\Gamma)$ is said to satisfy the *bipartite condition* if there exists a function $c : V_\Gamma \to \{0, 1\}$ such that $uc \neq vc$ whenever $(u, v) \in E_\Gamma$. In other words we can write $V_\Gamma = V_0 \cup V_1$ where $V_0 \cap V_1 = \emptyset$ and where $(u, v) \in E_\Gamma$ implies that $u \in V_0$ and $v \in V_1$ or *vice versa*. We call the decomposition of $V$ into the sets $V_0$ and $V_1$ a *bipartition* of $\Gamma$. It is well known that a graph can satisfy the bipartite condition if and only if it contains no odd cycles. For $m, n \in \mathbb{N} \setminus \{0\}$, let $K_{m,n}$ denote the graph with vertex set

$$V_{K_{m,n}} = \{u_i, v_j : i = 1, \ldots, m,\ j = 1, \ldots, n\}$$

and edge set

$$E_{K_{m,n}} = \{(u_i, v_j), (v_j, u_i) : i = 1, \ldots m,\ j = 1, \ldots n\}.$$

Then $K_{m,n}$ satisfies the bipartite condition with bipartition $V_{K_{m,n}} = V_0 \cup V_1$ where $V_0 = \{u_i : i = 1, \ldots, m\}$ and $V_1 = \{v_j : j = 1, \ldots, n\}$. It must be stressed that a graph satisfying the bipartite condition will *not* be called a bipartite graph and a separate definition will be made (see Definition 7.1). The reason for this will become clear in Chapter 7.

A *directed graph* (or *digraph*) is a relational structure $\Gamma = (V_\Gamma, E_\Gamma)$ in which the binary relation $E_\Gamma$ is irreflexive. We continue to call $V_\Gamma$ the vertices and $E_\Gamma$ the edges of $\Gamma$ in this setting. Of course, every graph is a directed graph but the converse does not always hold. If $\Gamma$ is a directed graph and $(u, v) \in E_\Gamma$, we say that there is an edge from $u$ to $v$ in $\Gamma$. Substructures in this setting will be called *directed subgraphs*. Given a directed graph $\Gamma$, the relational structure $\hat{\Gamma} = (V_\Gamma, E_\Gamma \cup \{(v, u) : (u, v) \in E_\Gamma\})$ is a graph. A directed graph $\Gamma$ is said to be *connected* if the graph $\hat{\Gamma}$ is connected. A *tournament* is a directed graph $\Gamma$ in which for every pair of vertices $u, v \in V_\Gamma$, exactly one of $(u, v) \in E_\Gamma$ or $(v, u) \in E_\Gamma$ holds.

If $\Gamma$ is a directed graph and $U$ is a subset of $V_\Gamma$, then we call $U$ a *connected component* of $\Gamma$ if the induced directed subgraph $\langle U \rangle$ is connected and there exist no edges between any vertex in $U$ and any vertex in $V_\Gamma \setminus U$. It is easy to show that any automorphism of $\Gamma$ must map a connected component to a connected component. If $\Gamma$ is a directed graph and $U$ is a non-empty subset of $V_\Gamma$ such that $\langle U\rangle = (U, \emptyset)$, *i.e.* the directed subgraph induced by $U$ has no edges, then $U$ is said to be an *independent set* in $\Gamma$. An independent set $U \subseteq V_\Gamma$ is said to be *maximal* if for any $v \in V_\Gamma \setminus U$, $U \cup \{v\}$ is not an independent set. Clearly if $v \in V_\Gamma$, then $v$ is contained in the independent set $\{v\}$. By Zorn's Lemma, every vertex of $V_\Gamma$ is contained in at least one maximal independent set and $V_\Gamma$ is the union of its maximal independent sets. The (directed) graph $(V_\Gamma, (V_\Gamma \times V_\Gamma) \setminus \{(v, v) : v \in V_\Gamma\})$ is called the *complete graph* on $|V_\Gamma|$ vertices and is denoted by $K_{|V_\Gamma|}$. If $\Gamma$ and $\Lambda$ are directed graphs then $\Gamma$ is said to be $\Lambda$-*free* if there exists no directed subgraph of $\Gamma$ which is isomorphic to $\Lambda$. A graph which is $K_3$-free is said to be *triangle-free* for obvious reasons.

Given two directed graphs $\Gamma$ and $\Lambda$, we can produce a new directed graph by taking the *union* of $\Gamma$ and $\Lambda$. That is, we define $\Gamma \cup \Lambda$ to be the directed graph with vertex set $V_\Gamma \cup V_\Lambda$ and edge set $E_\Gamma \cup E_\Lambda$. Clearly, if $\Gamma$ and $\Lambda$ are both graphs, then $\Gamma \cup \Lambda$ is also a graph. If the vertex sets of $\Gamma$ and $\Lambda$ are disjoint we call their union a *disjoint union* and denote this by $\Gamma \,\dot{\cup}\, \Lambda$. These definitions can be extended to form the union of an arbitrary set of directed graphs.

A *partial order* $\Omega = (V_\Omega, E_\Omega)$ is a relational structure with a single reflexive, antisymmetric and transitive binary relation. When $\Omega$ is a partial order, the binary relation $E_\Omega$ is called the *order* on $\Omega$. Often, when $\Omega = (V_\Omega, E_\Omega)$ is a partial order, the order is denoted by $\leq$ and $(u, v) \in \leq$ is replaced by $u \leq v$. In the case where $u \leq v$ but $u \neq v$ we can denote this by $u < v$. If $U_1$ and $U_2$ are non-empty subsets of $V_\Omega$ we write

$$U_1 \leq U_2 \text{ to mean } u \leq v \text{ for all } u \in U_1 \text{ and for all } v \in U_2.$$

If $U_1$ contains only one element, $x$ say, then we may abuse notation and write $x \leq U_2$ rather than $\{x\} \leq U_2$. Likewise if $U_2$ contains only one element, $y$ say, we will often write $U_1 \leq y$ to mean $U_1 \leq \{y\}$. An element $u \in V_\Omega$ is said to be *maximal* if whenever $v \in V_\Omega$ and $u \leq v$, then $u = v$. A *total order* is a partial order $\Omega$ satisfying *totality*. That is, for all $u, v \in V_\Omega$ at least one of $(u, v) \in E_\Omega$ or $(v, u) \in E_\Omega$ holds. If $\Omega$ is a partial order, then a *chain* in $\Omega$ is a subset $U \subseteq V_\Omega$, such that $\langle U\rangle$ is a total order. If $\Omega$ is a total order, then clearly $V_\Omega$ is chain itself. By an *interval* in a total order $\Omega$, we mean a non-empty subset $U \subseteq V_\Omega$ such that for all $u, v \in U$ with $u < v$, whenever $x \in V_\Omega$ and $u < x < v$ it follows that $x \in U$. It is easy to see that if $T$

and $U$ are intervals in the total order $\Omega$ such that $T \cap U = \emptyset$, then either $T < U$ or $U < T$. Furthermore, if $U$ is an interval in a total order $\Omega$ and $f : \Omega \to \Lambda$ is an isomorphism of total orders, then $Uf$ is an interval in $\Lambda$. An automorphism of a total order $\Omega = (V_\Omega, \leq)$ is defined (as expected) to be a function $f : V_\Omega \to V_\Omega$ such that $u \leq v$ if and only if $uf \leq vf$. If $f \in \mathrm{Aut}(\Omega)$ and $f \neq \mathbf{1}_{V_\Omega}$, then we can show that $f$ has infinite order as follows. For if $f \neq \mathbf{1}_{V_\Omega}$, then there exists $u \in V_\Omega$ such that $u < uf$ or $uf < u$. If $u < uf$, then $f^n = \mathbf{1}_{V_\Omega}$ implies that $u < uf^{n-1} < uf^n = u$, and if $uf < u$, then $f^n = \mathbf{1}_{V_\Omega}$ implies that $u = uf^n < uf^{n-1} < u$, both contradictions. Thus $f$ has infinite order as claimed. A *well order* is a total order $\Omega$ in which for every non-empty subset $U \subseteq V_\Omega$, there exists $u \in U$ such that $u \leq U$. It is not hard to show that a countable well order must have trivial automorphism group (see [Cam08, Lemmas 2.1 and 2.3] for example).

## 2.3 Archimedean Groups

Let $(G, \cdot)$ be a group and suppose that $\leq$ is a binary relation on the underlying set $G$ such that $(G, \leq)$ is a total order. The total order $(G, \leq)$ is said to be *translation invariant* if for all $f, g, h \in G$, if $f \leq g$ then $f \cdot h \leq g \cdot h$ and $h \cdot f \leq h \cdot g$. If $(G, \cdot)$ is a group and $(G, \leq)$ is a translation invariant total order then we say that $(G, \cdot, \leq)$ is a *totally ordered group*. If 1 is the identity element of $G$, then an element $g \in G$ is said to be *positive* if $1 < g$ and *negative* if $g < 1$. An *Archimedean group* is a totally ordered group $(G, \cdot, \leq)$ such that whenever $g, h \in G$ are positive elements with $g < h$, then there exists $n \in \mathbb{N}$ such that $h < g^n$. It is not hard to show that if $(G, \cdot, \leq)$ is an Archimedean group then $g < h$ implies that $g^n < h^n$ and $h^{-n} < g^{-n}$ for all $n \in \mathbb{N}$. Archimedean groups will be of importance in Chapter 9. The following theorem, originally proved by Hölder in 1910, will be particularly useful. For a proof see for example, [Dar97, Theorem 24.16].

**Theorem 2.1** (Hölder's Theorem)**.** *Any Archimedean group is isomorphic to a subgroup of the additive group of real numbers.*

Helpfully, the additive subgroups of the real numbers can be categorised in the following manner.

**Theorem 2.2** ([Goo86, Lemma 4.21])**.** *Any additive subgroup of the real numbers is either cyclic, or is a dense subset of the reals.*

*Proof.* Suppose that $(G, +)$ is a subgroup of $(\mathbb{R}, +)$ that is not dense. Then there exists $g, h \in G$, $g < h$, such that there exists no $f \in G$ with $g < f < h$. Thus it must be the case that there exists no $k \in G$ such that $1 < k < h - g$, for otherwise $g < k + g < h$, a contradiction. Hence $h - g$ is the minimal positive element of $G$. So let $h - g = n$. Using the Euclidean algorithm, we can express any element $a \in G$ as $a = nq + r$, where $q \in \mathbb{Z}$ and $0 \leq r < n$. But since $a, qn \in G$ it follows that $r \in G$ and hence $r = 0$ since $n$ is minimal positive. Thus $a = nq$ and it follows that $G = n\mathbb{Z}$ and $G$ is cyclic. $\square$

## 2.4 Fraïssé Limits

Roughly speaking, a *class* of relational structures is defined to be a collection of relational structures, such that each structure consists of a set with a defined number of binary relations each having identical properties. For example, the class of graphs consists of all relational structures which are formed from a set and a irreflexive symmetric relation on that set. A more precise definition can be made through model theory using words such as 'signature' or 'type'. However, we do not require the full generality of the model theoretic definition and so the description above will suffice for our needs.

Let $\Omega$ be a countable relational structure. The *age* of $\Omega$ is the class of all finite structures embeddable in $\Omega$; that is, the class of all finite structures which are isomorphic to a finitely generated substructure of $\Omega$. The following three properties will be of importance.

A class $K$ of structures is said to have the:

*Hereditary property* if for all $A \in K$ and for all finitely generated substructures $B$ of $A$, $B$ is isomorphic to a structure in $K$.

*Joint embedding property* if for all $A, B \in K$ there exists a structure $C \in K$ such that $A$ and $B$ are both embeddable in $C$.

*Amalgamation property* if whenever $A, B_1, B_2 \in K$ and there exist embeddings $f_1 : A \to B_1$ and $f_2 : A \to B_2$, then there exists a structure $C \in K$ and embeddings $g_1 : B_1 \to C$ and $g_2 : B_2 \to C$ such that $f_1 \cdot g_1 = f_2 \cdot g_2$.

It can be shown that if $A$ is the age of a relational structure $\Omega$ then $A$ has both the hereditary and joint embedding properties. Furthermore, Fraïssé [Fra53] showed that if $K$ is a non-empty countable class of finitely generated structures which has both the hereditary and joint embedding properties, then $K$ is the age of some countable structure $\Omega_K$.

Additionally Fraïssé proved that if $K$ is a non-empty countable class of finitely generated structures with the hereditary, joint embedding *and* amalgamation properties then not only is $K$ the age of some countable structure $\Omega_K$, but $\Omega_K$ is homogeneous and unique up to isomorphism. The unique countable structures arising in this way are known as *Fraïssé Limits* (of the relevant class of structures).

## 2.5 Semigroups, Green's Relations and Relational Structures

Let $S$ be a semigroup and let $S^1$ denote the semigroup $S$ with an identity adjoined if necessary. We define *Green's $\mathscr{L}$-, $\mathscr{R}$-, $\mathscr{H}$-, $\mathscr{D}$-* and $\mathscr{J}$- (binary equivalence) *relations* on $S^1$ as follows.

For $s, t \in S^1$ we say that $s$ is $\mathscr{L}$-related to $t$ and write $s\mathscr{L}t$, if there exist $u, v \in S^1$ such that $s = ut$ and $t = vs$ (or in other words $s$ and $t$ generate the same principal left ideals). Similarly $s\mathscr{R}t$ if there exist $x, y \in S^1$ such that $s = tx$ and $t = sy$ (equivalently $s$ and $t$ generate the same principal right ideals).

Two elements $s$ and $t$ are $\mathscr{H}$-related if they are both $\mathscr{L}$- and $\mathscr{R}$-related, that is $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$. Green's $\mathscr{D}$-relation is likewise formed from the $\mathscr{L}$- and $\mathscr{R}$-relations. The relation $\mathscr{D}$ is the smallest equivalence containing both $\mathscr{L}$ and $\mathscr{R}$. Equivalently, two elements $s, t \in S^1$ are $\mathscr{D}$-related, written $s\mathscr{D}t$, if there exists $x \in S^1$ such that $s\mathscr{L}x$ and $x\mathscr{R}t$ (see [How95, page 46]). It should be easy to see that since $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$, $\mathscr{H} \subseteq \mathscr{D}$.

The last relation is the two sided analogue of Green's $\mathscr{L}$- and $\mathscr{R}$-relations and is known as Green's $\mathscr{J}$-relation. Two elements $s, t \in S^1$ are $\mathscr{J}$-related, written $s\mathscr{J}t$, if there exists $u, v, x, y \in S^1$ such that $usv = t$ and $xty = s$ (that is, $s$ and $t$ generate the same principal two-sided ideals). It should be clear that $\mathscr{L}, \mathscr{R} \subseteq \mathscr{J}$ and hence it follows that $\mathscr{D} \subseteq \mathscr{J}$. When $S$ is a finite semigroup we can show that $\mathscr{D} = \mathscr{J}$ (see [How95, Proposition 2.4.1]).

For $s \in S^1$ we let $H_s = \{t \in S : s\mathscr{H}t\}$ and call $H_s$ the *$\mathscr{H}$-class* of $s$. Likewise we can define $D_s = \{t \in S : s\mathscr{D}t\}$, the *$\mathscr{D}$-class* of $s$, and $J_s = \{t \in S : s\mathscr{J}t\}$, the *$\mathscr{J}$-class* of $s$. Since each of Green's relations is an equivalence relation, the set of classes under each of the relations provides a partition of the semigroup $S$. Furthermore the $\mathscr{D}$-classes are a union of

$\mathscr{L}$-classes of equal size, a union of $\mathscr{R}$-classes of equal size and a union of $\mathscr{H}$-classes of equal size (see [How95, Lemmas 2.21–2.23]).

For a semigroup $S$ we let $E(S)$ denote the set of idempotents of $S$, that is the set of elements $s \in S$ such that $s^2 = s$. It can be shown that if $s \in E(S)$ then $H_s$ is a (maximal) subgroup of the semigroup $S$ with identity $s$ (see [CP61, Chapter 2] for details). As a consequence, no $\mathscr{H}$-class of $S$ can contain more than one idempotent. Thus if $s \in E(S)$, $H_s$ is often called the *group $\mathscr{H}$-class* of $s$. Conversely any maximal subgroup of the semigroup $S$ must be the group $\mathscr{H}$-class of an idempotent (the identity of the subgroup). Thus the idempotents of $S$ are in one-one correspondence with the maximal subgroups of $S$.

If we let $S$ above be the semigroup $\mathrm{End}(\Omega)$ for some relational structure $\Omega$, the $\mathscr{H}$-classes of functions $f \in E(\mathrm{End}(\Omega))$ provide us with the maximal subgroups of $\mathrm{End}(\Omega)$. With that said, understanding the properties of the idempotents in $\mathrm{End}(\Omega)$ is then essential. In particular, the following results are key to proving a theorem linking the $\mathscr{H}$-classes of an idempotent $f \in \mathrm{End}(\Omega)$ with its image.

**Lemma 2.3.** *Let $\Omega = (V, \mathcal{E})$ be a relational structure. Then $f \in E(\mathrm{End}(\Omega))$ if and only if $f|_{\mathrm{im}\, f} = \mathbf{1}_{\mathrm{im}\, f}$.*

*Proof.* Suppose $f \in E(\mathrm{End}(\Omega))$ and let $v \in \mathrm{im}\, f$. Then there exists $u \in V$ such that $v = uf$. Now since $f$ is idempotent $vf = uf^2 = uf = v$ and hence $f|_{\mathrm{im}\, f} = \mathbf{1}_{\mathrm{im}\, f}$. Conversely if $f|_{\mathrm{im}\, f} = \mathbf{1}_{\mathrm{im}\, f}$ and $u \in V$ then $uf \in \mathrm{im}\, f$ and so $uf^2 = (uf)f = (uf)\mathbf{1}_{\mathrm{im}\, f} = uf$. Thus indeed $f \in E(\mathrm{End}(\Omega))$. $\square$

For the next result we require the notion of a *regular* element of a semigroup $S$. An element $s \in S$ is said to be regular if there exists an element $t \in S$ such that $sts = s$. If $e \in E(S)$ then $e$ is regular since $eee = e$. Regular elements have many interesting properties. For example if $s$ is a regular element of a semigroup $S$, then every element of $D_s$ is also regular [How95, Proposition 2.3.1]. Such $\mathscr{D}$-classes are thus called regular themselves. It can be shown that in a regular $\mathscr{D}$-class, every $\mathscr{L}$-class and every $\mathscr{R}$-class contains an idempotent and hence a group $\mathscr{H}$-class [How95, Proposition 2.3.2]. Additionally, if $H$ and $K$ are two group $\mathscr{H}$-classes contained in the same regular $\mathscr{D}$-class then $H$ and $K$ are isomorphic groups [How95, Proposition 2.3.6].

**Theorem 2.4** ([RS09, Proposition A.1.16])**.** *Let $S$ be a subsemigroup of a semigroup $T$ and suppose that $s$ and $t$ are regular elements of $S$. Then $s\mathscr{L}t$ in $S$ if and only if $s\mathscr{L}t$ in $T$ and similarly, $s\mathscr{R}t$ in $S$ if and only if $s\mathscr{R}t$ in $T$.*

*Proof.* It should be clear that $s\mathscr{L}t$ in S implies that $s\mathscr{L}t$ in T. Now suppose that $s\mathscr{L}t$ in $T$. Since $s$ is regular in $S$ there exists $h \in S$ such that $shs = s$. Let $d = hs$ so that $d \in S$. Then $d^2 = d$ and since $sd = s$ and $hs = d$ we deduce that $s\mathscr{L}d$ in $S$. In a similar fashion we can construct an idempotent $e$ such that $e\mathscr{L}t$ in $S$. Since $s\mathscr{L}t$ in $T$, transitivity of Green's $\mathscr{L}$-relation on $T$ implies that $d\mathscr{L}e$ in $T$. Thus there exists $d', e' \in T$ such that $d'd = e$ and $e'e = d$. Then $ed = d'd^2 = d'd = e$ and $de = e'e^2 = e'e = d$. Hence $d\mathscr{L}e$ in $S$ and by transitivity of Green's $\mathscr{L}$-relation on $S$ it follows that $s\mathscr{L}t$ in S. A similar argument for Green's $\mathscr{R}$-relation completes the proof. $\square$

In order to apply Theorem 2.4 to regular elements of $\mathrm{End}(\Omega)$, for a relational structure $\Omega$, we consider the following lemma.

**Lemma 2.5.** *Let $V$ be a set and let $\mathcal{T}_V$ denote the monoid of all functions from $V$ to $V$ under composition. Then for $f, g \in \mathcal{T}_V$, $f\mathscr{L}g$ if and only if* $\operatorname{im} f = \operatorname{im} g$ *and $f\mathscr{R}g$ if and only if* $\ker f = \ker g$.

For a proof, see for example [CP61, Lemmas 2.5 and 2.6]. The following corollary now tells us under which circumstances regular elements of $\mathrm{End}(\Omega)$, for a relational structure $\Omega$, are related.

**Corollary 2.6.** *Let $\Omega = (V, \mathcal{E})$ be a relational structure and let $f, g \in$* $\mathrm{End}(\Omega)$. *If $f$ and $g$ are regular elements of* $\mathrm{End}(\Omega)$ *then $f\mathscr{L}g$ if and only if* $\operatorname{im} f = \operatorname{im} g$ *and analogously $f\mathscr{R}g$ if and only if* $\ker f = \ker g$.

*Proof.* First we note that $\mathrm{End}(\Omega)$ is a subsemigroup of $\mathcal{T}_V$. By Lemma 2.5, if $f, g \in \mathcal{T}_V$, then $f\mathscr{L}g$ in $\mathcal{T}_V$ if and only if $\operatorname{im} f = \operatorname{im} g$ and $f\mathscr{R}g$ in $\mathcal{T}_V$ if and only if $\ker f = \ker g$. Thus, it follows from Theorem 2.4 that if $f$ and $g$ are regular elements of $\mathrm{End}(\Omega)$, then $f\mathscr{L}g$ in $\mathrm{End}(\Omega)$ if and only if $\operatorname{im} f = \operatorname{im} g$ and $f\mathscr{R}g$ in $\mathrm{End}(\Omega)$ if and only if $\ker f = \ker g$. $\square$

Now, given the above corollary, we are able to prove the following important theorem. This theorem is one of the main tools which will be repeatedly used throughout this thesis.

**Theorem 2.7.** *Let $\Omega = (V, \mathcal{E})$ be a relational structure and suppose that $f \in E(\mathrm{End}(\Omega))$. Then $H_f \cong \mathrm{Aut}(\operatorname{im} f)$ as groups.*

*Proof.* Define $\phi : \mathrm{Aut}(\operatorname{im} f) \to H_f$ by $\pi\phi = f\pi$ for all $\pi \in \mathrm{Aut}(\operatorname{im} f)$. We note foremost that $\phi$ is indeed well defined since $f\pi$ is clearly an endomorphism of $\Omega$ and by Lemma 2.3 we have:

$$\begin{aligned} f \cdot f\pi = f^2\pi &= f\pi, \\ f\pi \cdot f\pi^{-1} &= f, \\ f\pi \cdot f &= f\pi, \\ f\pi^{-1} \cdot f\pi &= f. \end{aligned}$$

Thus $f\pi \mathscr{H} f$ and so $f\pi \in H_f$.

Also for $\pi, \rho \in \mathrm{Aut}(\mathrm{im}\, f)$:

$$\begin{aligned}
(\pi\rho)\phi &= f\pi\rho \\
&= f\pi f\rho \qquad \text{(by use of Lemma 2.3)} \\
&= (\pi)\phi(\rho)\phi,
\end{aligned}$$

so that $\phi$ is indeed a group homomorphism. It should be clear by construction that $\phi$ is injective so it only remains to show that $\phi$ is surjective.

So let $g \in H_f$. Define $h := g|_{\mathrm{im}\, f}$ ($= g|_{\mathrm{im}\, g}$ by recalling $\mathrm{im}\, f = \mathrm{im}\, g$ by Corollary 2.6). We claim that $h \in \mathrm{Aut}(\mathrm{im}\, f)$ and $(h)\phi = g$. First we prove that $h \in \mathrm{Aut}(\mathrm{im}\, f)$. Since $g \mathscr{L} f$, Corollary 2.6 allows us to deduce that $\mathrm{im}\, g = \mathrm{im}\, f$. Then $\mathrm{im}\, h = \mathrm{im}\, g|_{\mathrm{im}\, f} \subseteq \mathrm{im}\, f$ so that $h$ does indeed define a function $\mathrm{im}\, f \to \mathrm{im}\, f$. Let $\mathrm{im}\, f = \{v_i : i \in I\}$ for some index set $I$. Note that each $v_i$ lies in a unique kernel class of $f$. Furthermore, since $f$ is idempotent each kernel class of $f$ has a unique point $v \in V_\Omega$ such that $vf = v$ (namely $v_i$ for some unique $i \in I$). To see that $h$ is injective, suppose that $v_i h = v_j h$ . Then by definition, $v_i g = v_j g$ or equivalently, $(v_i, v_j) \in \ker g$. Now, since $g \in H_f$, it holds that $g \mathscr{R} f$ and thus by Corollary 2.6, $\ker g = \ker f$. Hence $(v_i, v_j) \in \ker f$ and by our observation above it must then be the case that $i = j$ and hence $v_i = v_j$. To show that $h$ is surjective we suppose that $v_i \in \mathrm{im}\, f$. Then by definition there exist some $u \in V_\Omega$ such that $ug = v_i$. Now by our previous comments there is a unique $v_j$ such that $(u, v_j) \in \ker f = \ker g$. Then,

$$v_j h = v_j g = ug = v_i.$$

We have thus shown that $h$ defines a bijective function $\mathrm{im}\, f \to \mathrm{im}\, f$. The last condition to check is that $h$ defines an automorphism of the relational structure $\langle \mathrm{im}\, f \rangle$. So let $E \in \mathcal{E}$ and suppose that $v_i, v_j \in \mathrm{im}\, f$ with $(v_i, v_j) \in E$. Then, since $g$ is a endomorphism of the relational structure $\Omega$ and $h$ is simply the restriction of $g$ to $\mathrm{im}\, f$, we deduce that,

$$(v_i h, v_j h) = (v_i g, v_j g) \in E.$$

On the other hand suppose that we have $v_i, v_j \in \mathrm{im}\, f$ with $(v_i, v_j) \notin E$. Seeking a contradiction, suppose that $(v_i h, v_j h) \in E$. Since $f \mathscr{R} g$ there exists some $g' \in \mathrm{End}(\Omega)$ such that $gg' = f$. Thus if $(v_i h, v_j h) = (v_i g, v_j g) \in E$ then $(v_i gg', v_j gg') \in E$ since $g'$ is a homomorphism. Now, as $gg' = f$, we can conclude that $(v_i f, v_j f) \in E$. However $f$ is idempotent and so, by Lemma 2.3,

$f$ acts as the identity on its image points. But this means that $(v_i, v_j) \in E$ which is a contradiction to our original assumption. We can thus conclude that $(v_i, v_j) \in E$ if and only if $(v_i h, v_j h) \in E$ for all $E \in \mathcal{E}$ and hence $h$ is an automorphism of $\operatorname{im} f$.

Finally, to finish the proof, we show that $(h)\phi = g$. For each $v \in V_\Omega$ there exists some $v_i \in \operatorname{im} f$ such that $(v_i, v) \in \ker f (= \ker g)$ and $v_i f = v_i$. Hence,

$$(v)(h)\phi = (v)fh = (v)fg = (v_i)fg = (v_i)g = (v)g.$$

Thus $(h)\phi = g$ and the proof is complete. $\square$

Note that if $f = \mathbf{1}_\Omega$ then the map $\phi$ defined in the proof of Theorem 2.7 is just the identity map and hence $H_{\mathbf{1}} = \operatorname{Aut}(\Omega)$.

In view of Theorem 2.7, the problem of understanding the group $\mathscr{H}$-classes of the endomorphism monoid of a relational structure reduces to understanding the images of idempotent endomorphisms. Additionally, the following theorems, originally proved in [MS74, Theorem 2.6, Theorem 2.8], are useful for gaining information on the cardinality of the set of regular $\mathscr{D}$- and $\mathscr{J}$-classes of $\operatorname{End}(\Omega)$.

**Theorem 2.8.** *Let $\Omega = (V, \mathcal{E})$ be a relational structure and let $f, g \in \operatorname{End}(\Omega)$ be regular. Then $f \mathscr{D} g$ if and only if $\langle \operatorname{im} f \rangle$ and $\langle \operatorname{im} g \rangle$ are isomorphic relational substructures of $\Omega$.*

**Theorem 2.9.** *Let $\Omega = (V, \mathcal{E})$ be a relational structure and let $f, g \in \operatorname{End}(\Omega)$ be regular. Then $f \mathscr{J} g$ if and only if there exist embeddings $\phi : \langle \operatorname{im} f \rangle \to \langle \operatorname{im} g \rangle$ and $\theta : \langle \operatorname{im} g \rangle \to \langle \operatorname{im} f \rangle$.*

In this thesis we will only apply Theorems 2.8 and 2.9 in the case where $f$ and $g$ are idempotents of $\operatorname{End}(\Omega)$. Accordingly, we state and prove the theorems in this case as follows.

**Theorem 2.10.** *Let $\Omega = (V, \mathcal{E})$ be a relational structure and let $f, g \in E(\operatorname{End}(\Omega))$. Then $f \mathscr{D} g$ if and only if $\langle \operatorname{im} f \rangle$ and $\langle \operatorname{im} g \rangle$ are isomorphic relational substructures of $\Omega$.*

*Proof.* Suppose that $f \mathscr{D} g$. Then there exists $h \in \operatorname{End}(\Omega)$ such that $f \mathscr{R} h$ and $h \mathscr{L} g$. Since $f \mathscr{R} h$ we deduce that $h = fs$ and $f = ht$ for some $s, t \in \operatorname{End}(\Omega)$. We will show that $s|_{\operatorname{im} f}$ provides an isomorphism from $\operatorname{im} f$ onto $\operatorname{im} g$. First we show that $s|_{\operatorname{im} f}$ defines a map $\operatorname{im} f \to \operatorname{im} g$. To see this note that for all $v \in V_\Omega$, $(v)fs = vh$. Moreover since $h \mathscr{L} g$ we know from Corollary 2.6 that

$\operatorname{im} h = \operatorname{im} g$ and hence $vh \in \operatorname{im} g$. It is easy to see that $s|_{\operatorname{im} f}$ is injective since if there exists $x, y \in V_\Omega$ such that $(xf)s = (yf)s$ then $(x)fst = (y)fst$. Hence $(x)ht = (y)ht$ and thus $xf = yf$. To see that $s|_{\operatorname{im} f}$ is surjective let $v \in \operatorname{im} g$. Since $\operatorname{im} g = \operatorname{im} h$ there exists $u \in V_\Omega$ such that $uh = v$. Now $uf \in \operatorname{im} f$ and $(uf)s|_{\operatorname{im} f} = uh = v$. Now, since $s$ is a homomorphism the restriction of $s$ to the subset $\operatorname{im} f$ of $V_\Omega$ is also a homomorphism. To see that $s|_{\operatorname{im} f}$ is a isomorphism, suppose that $(xfs, yfs) \in E$ for some $x, y \in V_\Omega$ and for some $E \in \mathcal{E}_\Omega$. Then since $t$ is a homomorphism $(xfst, yfst) \in E$. But since $fst = ht = f$ we can conclude that $(xf, yf) \in E$ and hence that $f$ is a isomorphism.

Conversely suppose that $\langle \operatorname{im} f \rangle$ and $\langle \operatorname{im} g \rangle$ are isomorphic substructures. Then there exists an isomorphism $\phi : \operatorname{im} f \to \operatorname{im} g$. The composition $f\phi$ is a homomorphism since both $f$ and $\phi$ are homomorphisms. We will show that $f \mathscr{R} f\phi$ and $f\phi \mathscr{L} g$ so that $f \mathscr{D} g$. First we claim that $f\phi$ is regular. To see this recall that since $f$ is idempotent, $f$ is regular. Hence every element of $D_f$, the $\mathscr{D}$-class of $f$, is regular. Since $f\phi$ is $\mathscr{R}$-related to $f$ and since $\mathscr{R} \subseteq \mathscr{D}$, it immediately follows that $f\phi$ is $\mathscr{D}$-related to $f$ and thus $f\phi$ is regular. By Corollary 2.6 it then suffices to show that $\ker f = \ker f\phi$ and $\operatorname{im} f\phi = \operatorname{im} g$. We begin with the former. If $(u, v) \in \ker f$ then $uf = vf$. Hence $uf\phi = vf\phi$ and $(u, v) \in \ker f\phi$. If on the other hand $(u, v) \in \ker f\phi$ then $uf\phi = vf\phi$. Since $\phi$ is injective this implies that $uf = vf$ and hence $(u, v) \in \ker f$. Thus $\ker f = \ker f\phi$. Finally it should be easy to see that $\operatorname{im} f\phi = \operatorname{im} g$ since $\phi$ is an isomorphism from $\operatorname{im} f$ to $\operatorname{im} g$. □

**Theorem 2.11.** *Let $\Omega = (V, \mathcal{E})$ be a relational structure and let $f, g \in E(\operatorname{End}(\Omega))$. Then $f \mathscr{J} g$ if and only if there exist embeddings $\phi : \langle \operatorname{im} f \rangle \to \langle \operatorname{im} g \rangle$ and $\theta : \langle \operatorname{im} g \rangle \to \langle \operatorname{im} f \rangle$.*

*Proof.* Let $f, g \in E(\operatorname{End}(\Omega))$ and suppose that $f \mathscr{J} g$. Then there exists $s, t \in \operatorname{End}(\Omega)$ such that $f = sgt$. Moreover since $f$ is idempotent, Lemma 2.3 allows us to deduce that $(sgt)|_{\operatorname{im} f} = \mathbf{1}|_{\operatorname{im} f}$. We will show that $(sg)|_{\operatorname{im} f}$ defines an embedding of $\langle \operatorname{im} f \rangle$ into $\langle \operatorname{im} g \rangle$. Clearly $(sg)|_{\operatorname{im} f}$ is a homomorphism $\operatorname{im} f \to \operatorname{im} g$. To see that it is injective suppose that $v, w \in \operatorname{im} f$ are such that $vsg = wsg$. Then $vsgt = wsgt$ and hence since $(sgt)|_{\operatorname{im} f} = \mathbf{1}|_{\operatorname{im} f}$ it follows that $v = w$. Now suppose that $E \in \mathcal{E}$ and that $v, w \in \operatorname{im} f$ are such that $(vsg, wsg) \in E$. Then $(vsgt, wsgt) = (v, w) \in E$ and hence we can conclude that $(sg)|_{\operatorname{im} f}$ defines an embedding of $\langle \operatorname{im} f \rangle$ into $\langle \operatorname{im} g \rangle$. Similarly since $f \mathscr{J} g$, there also exists $s', t' \in \operatorname{End}(\Omega)$ such that $g = s'ft'$ and dual argument shows that if $(s'f)|_{\operatorname{im} g}$ defines an embedding of $\langle \operatorname{im} g \rangle$ into $\langle \operatorname{im} f \rangle$.

For the converse suppose that $f, g \in E(\operatorname{End}(\Omega))$ and there exist embeddings $\phi : \langle \mathrm{im}\, f \rangle \to \langle \mathrm{im}\, g \rangle$ and $\theta : \langle \mathrm{im}\, g \rangle \to \langle \mathrm{im}\, f \rangle$. Then $\langle \mathrm{im}\, \phi \rangle \cong \langle \mathrm{im}\, f \rangle$ and $\langle \mathrm{im}\, \theta \rangle \cong \langle \mathrm{im}\, g \rangle$. Thus let $d : \mathrm{im}\, f \to \mathrm{im}\, \phi$ and $e : \mathrm{im}\, g \to \mathrm{im}\, \theta$ be the resulting isomorphisms of the induced relational structures. Since $\mathrm{im}\, \phi \subseteq \mathrm{im}\, g$, $g|_{\mathrm{im}\, \phi} = \mathbf{1}|_{\mathrm{im}\, \phi}$ and similarly since $\mathrm{im}\, \theta \subseteq \mathrm{im}\, f$, $f|_{\mathrm{im}\, \theta} = \mathbf{1}|_{\mathrm{im}\, \theta}$. Hence,

$$f = (fd)g(gd^{-1}) \quad \text{and} \quad g = (ge)f(fe^{-1}).$$

Since $f$ and $g$ are endomorphisms of $\Omega$ and since $d$ and $e$ are automorphisms between substructures of $\Omega$ it follows $fd, fe^{-1}, ge$ and $gd^{-1}$ all lie in $\mathrm{End}(\Omega)$. Hence $f \mathscr{J} g$ and the result is complete. $\square$

It is worth observing that if $\langle \mathrm{im}\, f \rangle$ or $\langle \mathrm{im}\, g \rangle$ is finite, then the existence of embeddings $\langle \mathrm{im}\, f \rangle \to \langle \mathrm{im}\, g \rangle$ and $\langle \mathrm{im}\, g \rangle \to \langle \mathrm{im}\, f \rangle$ implies that $\mathrm{im}\, f$ and $\mathrm{im}\, g$ are isomorphic. Hence if $f$ and $g$ are regular then $f \mathscr{J} g$ implies $f \mathscr{D} g$ in this case. However, if $\Gamma$ and $\Lambda$ are infinite relational structures, then it is possible for there to exist embeddings $\Gamma \to \Lambda$ and $\Lambda \to \Gamma$ even when $\Gamma$ and $\Lambda$ are not isomorphic. For example consider the graph $\Gamma = \dot{\bigcup}_{n \in \mathbb{N}} K_n$, the disjoint union of the graphs $K_n$ for all $n \in \mathbb{N}$, and the graph $\Lambda = \dot{\bigcup}_{n \in \mathbb{N}} (K_n \,\dot{\cup}\, K_n)$, the disjoint union of two copies of the graph $K_n$ for all $n \in \mathbb{N}$. Then $\Gamma$ and $\Lambda$ are clearly not isomorphic, but $\Lambda$ can be embedded into $\Gamma$ by embedding $K_n \,\dot{\cup}\, K_n$ into $K_{2n+1} \,\dot{\cup}\, K_{2(n+1)}$ for all $n \in \mathbb{N}$ and $\Gamma$ can be embedded into $\Lambda$ by embedding $K_n$ into $K_n \,\dot{\cup}\, K_n$ for all $n \in \mathbb{N}$.

# Chapter 3

# The Random Graph

In this section we introduce the well-known random graph $R$ and consider the maximal subgroups of its endomorphism monoid. We will show that if $\Gamma$ is any countable graph, then there are $2^{\aleph_0}$ maximal subgroups of $\mathrm{End}(R)$ isomorphic to $\mathrm{Aut}(\Gamma)$. As a consequence to the methods developed, we will also show that there are $2^{\aleph_0}$ regular $\mathscr{D}$-classes of $\mathrm{End}(R)$ and that each regular $\mathscr{D}$-class contains $2^{\aleph_0}$ group $\mathscr{H}$-classes. Additionally, we show that there are $2^{\aleph_0}$ $\mathscr{J}$-classes of $\mathrm{End}(R)$.

## 3.1 Defining Properties and Constructions

It can easily be shown that the class of all finite graphs has the hereditary, joint embedding and amalgamation properties (see [Hod97, Lemma 6.4.3] for example). Consequently, the class of finite graphs has a unique homogenous Fraïssé Limit, which we will call $R$. Now consider the following definition.

Let us say that a graph $\Gamma = (V_\Gamma, E_\Gamma)$ is *existentially closed*[1] (in the class of graphs) if, for any two finite and disjoint subsets $U_1$ and $U_2$ of $V_\Gamma$, there exists a vertex $v \in V_\Gamma \setminus (U_1 \cup U_2)$ such that $v$ is adjacent to all vertices in $U_1$ but to no vertices of $U_2$. Since we will only be considering the class of graphs in this chapter, we will simply call such a graph existentially closed with the setting assumed.

In fact, if a graph $\Gamma$ is existentially closed then for any two finite and

[1] The term 'existentially closed' can be defined in general for relational structures using model theoretic language (see [Hod97, Chapter 7] for example). However the general definition is not particularly useful in this thesis and so existential closure will be defined in each setting separately.

disjoint subsets $U_1$, $U_2$ of $V_\Gamma$ there must exist infinitely many vertices $v \in V_\Gamma \setminus (U_1 \cup U_2)$ such that $v$ is adjacent to all vertices of $U_1$ but to no vertices of $U_2$. For suppose that there were only finitely many such vertices, $v_1, \ldots, v_n$ say. Then $U_1 \cup \{v_1, \ldots, v_n\}$ would be a finite set of vertices for which there exists no vertex in $V_\Gamma$ adjacent to every member: a contradiction. Perhaps a somewhat surprising result about existential closure is the following theorem.

**Theorem 3.1.** *Let $\Gamma$ be an existentially closed graph. Then every finite graph can be embedded into $\Gamma$.*

For details of the proof see for example [Cam97]. Alternatively, the construction described in Definition 3.3 will make this clear. In other words, Theorem 3.1 says that if $\Gamma$ is an existentially closed graph, then the age of $\Gamma$ is exactly the class of all finite graphs. Furthermore, it is easy to show that any existentially closed graph must be homogeneous (see [Hod97]). Since we observed that the class of all finite graphs has unique homogeneous Fraïssé limit $R$, it follows that if $\Gamma$ is an existentially closed graph, then $\Gamma \cong R$. We could thus have equally defined $R$ to be the unique existentially closed graph.

It should be clear that not all graphs are existentially closed, for example no finite graph can be, but an example of an existentially closed graph may not be immediately obvious. However in [ER63] Erdős and Rényi made the following observation.

**Theorem 3.2.** *Let $\Lambda$ be a graph with vertices $V_\Lambda = \{v_i : i \in \mathbb{N}\}$ and edge set $E_\Lambda$ formed by selecting 2-element subsets independently and with probability $\frac{1}{2}$, from the set of all 2-element subsets of the vertex set. Then with probability 1, $\Lambda$ is existentially closed and hence $\Lambda \cong R$.*

Notice that Erdős and Rényi defined a graph to be a set $V$ with a set of two element subsets of $V$. In order to correlate this with the definition of a graph given earlier we need only identify the 2–element subsets $\{v_j, v_k\}$, $j \neq k$, with both the ordered pairs $(v_j, v_k)$ and $(v_k, v_j)$, so that the edge set $E_R$ becomes an irreflexive, symmetric binary relation on $V_R$. The proof of their result then follows through in this setting.

Erdős and Rényi's probabilistic, or 'random' construction of $R$ has led to $R$ being commonly referred to as *the random graph*. Note that we can truly use the word *the* since $R$, being the Fraïssé limit of the class of graphs, is unique up to isomorphism.

An explicit construction of the random graph was not given by Erdős and Rényi – instead this was first achieved by Rado who was able to construct a countable graph and exhibit explicitly that it was existentially closed. See [Rad64, Theorem 1] for details. Since then many other constructions of the random graph have been exhibited. The following is a standard construction of the random graph which will be used throughout this chapter.

**Definition 3.3.** Let $\Gamma = (V_\Gamma, E_\Gamma)$ be any countable graph. Construct a new graph $\mathcal{G}(\Gamma)$ from $\Gamma$ by adding, for each finite subset $U$ of $V_\Gamma$, a vertex $v$ adjacent to every member of $U$ but to no other vertices. That is if we enumerate the finite subsets of $V_\Gamma$ as $\{U_i : i \in \mathbb{N}\}$ (replacing the natural numbers with some finite subset if $\Gamma$ is finite) then we let,

$$V_{\mathcal{G}(\Gamma)} = V_\Gamma \cup \{v_i : i \in \mathbb{N}\},$$

and

$$E_{\mathcal{G}(\Gamma)} = E_\Gamma \cup \{(v_i, u), (u, v_i) : u \in U_i,\ i \in \mathbb{N}\}.$$

If $\Gamma$ is finite, $|V_{\mathcal{G}(\Gamma)}| = 2^{|V_\Gamma|} + |V_\Gamma|$ and hence $\mathcal{G}(\Gamma)$ is a finite graph. Likewise, if $\Gamma$ is countably infinite then so is $\mathcal{G}(\Gamma)$, since the set of all finite sets of a countably infinite set is itself countably infinite.

Now inductively define a sequence of graphs by setting $\Gamma_0 = \Gamma$ and $\Gamma_{n+1} = \mathcal{G}(\Gamma_n)$ for $n \in \mathbb{N}$. Define $\Gamma_\infty$ to be the limit of this process, in the sense that,

$$\Gamma_\infty = \bigcup_{n\in\mathbb{N}} \Gamma_n = \left( \bigcup_{n\in\mathbb{N}} V_{\Gamma_n}, \bigcup_{n\in\mathbb{N}} E_{\Gamma_n} \right).$$

Then it is easy to see that $\Gamma_\infty$ is a graph since the edge set consists of a union of irreflexive and symmetric binary relations and therefore possesses these properties itself.

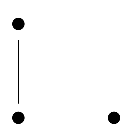
**Example 3.4.** [Construction of $\Gamma_0$, $\Gamma_1$ and $\Gamma_2$ when $\Gamma = (\{v\}, \emptyset)$].

Dots represent vertices and continuous lines represent edges.

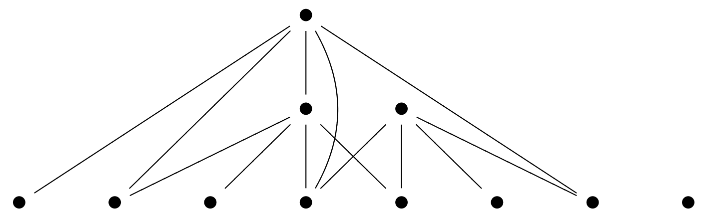
$\Gamma_0 = \Gamma$



$\Gamma_1$



$\Gamma_2$



**Lemma 3.5.** *For any countable graph* $\Gamma$, $\Gamma_\infty$ *is existentially closed and thus* $\Gamma_\infty \cong R$.

*Proof.* Let $U$ and $V$ be finite disjoint subsets of $V_{\Gamma_\infty}$. Then $U, V \subseteq V_{\Gamma_k}$ for some $k \in \mathbb{N}$. Then by construction of $\Gamma_{k+1}$, there exists a vertex $v \in V_{\Gamma_{k+1}} \setminus V_{\Gamma_k}$ adjacent to every member of $U$ but to no member of $V_{\Gamma_{k+1}} \setminus U$. In particular this means that $v$ is adjacent to every member of $U$ but to no member of $V$ in $\Gamma_{k+1}$. Since the construction of $\Gamma_\infty$ makes no changes to edge set of the induced subgraph $\Gamma_{k+1}$, it follows that $v$ is adjacent to every member of $U$ but to no member of $V$ in $\Gamma_\infty$. $\square$

The construction of $\Gamma_\infty$ from any countable graph $\Gamma$ makes it easy to see that any finite graph can be embedded into $R$. For we can easily define an embedding $\Gamma \to \Gamma_\infty$ by identifying $\Gamma$ with $\Gamma_0 \subseteq \Gamma_\infty$.

## 3.2 Group $\mathscr{H}$-classes of End($R$)

By Theorem 2.7, the group $\mathscr{H}$-class of an endomorphism $f \in E(\operatorname{End} R)$ is isomorphic to the automorphism group of its induced image subgraph. In [BD00], Bonato and Delić provided some insight into the structure of the image graphs of such endomorphisms. Their main result is encapsulated in Theorem 3.10, although an alternative approach to the proof is given.

**Definition 3.6.** We will say that a graph $\Gamma$ is *algebraically closed*[2] (in the class of graphs), if for each finite subset $U \subseteq V_\Gamma$, there exists a vertex $v \in V_\Gamma$ such that $v$ is adjacent to every member of $U$.

[2]Algebraic closure can also be defined in general for relational structures using model theoretic terms (see for example, [Dol12]). Once again, the general description is not particularly helpful for this thesis and so the definitions will be explicitly made in each setting.

Again we will omit explicitly stating the setting of algebraic closure where it is clear. Note that as a consequence of this definition, any algebraically closed graph must be infinite. For suppose that $\Gamma$ was a finite algebraically closed graph. Then $V_\Gamma$ is finite and so there should exist a vertex $x \in V_\Gamma$ such that $(x, v) \in E_\Gamma$ for all $v \in V_\Gamma$. In particular this would mean that $(x, x) \in E_\Gamma$ which is a contradiction. We can easily show (using the same argument as for existential closure) that if $\Gamma$ is an algebraically closed graph then for a finite subset $U$ of $V_\Gamma$ there must exist infinitely many vertices $v \in V_\Gamma$ such that $v$ is adjacent to all members of $U$. Furthermore, each of these vertices must lie outside of $U$ since $E_\Gamma$ is irreflexive. Additionally, it should be easy to see that $R$ itself is algebraically closed since it is existentially closed.

**Lemma 3.7.** *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a countable algebraically closed graph and let $f \in \mathrm{End}(\Gamma)$. Then $\mathrm{im}\, f$ is a countable algebraically closed graph.*

*Proof.* It is clear that since $\Gamma$ is a countable graph, so is $\mathrm{im}\, f$. Now let $X$ be a finite subset of the vertices of $\mathrm{im}\, f$. Enumerate $X$ as $\{v_i : 1 \leq i \leq n\}$ for some $n \in \mathbb{N}$. Since each $v_i$ lies in the image of $f$ there exists some vertex $u_i \in V_\Gamma$ such that $u_i f = v_i$. So let $U = \{u_i : 1 \leq i \leq n\}$. Then since $\Gamma$ is algebraically closed, there exists a vertex $x \in V_\Gamma \setminus U$ such that $x$ is adjacent to $u_i$ for all $i$. Furthermore since $f$ is a graph homomorphism we can deduce that $xf \in \mathrm{im}\, f \setminus X$ and $xf$ is adjacent to $v_i$ for all $i \in \{1, \ldots, n\}$. □

**Corollary 3.8.** *Let $f \in \mathrm{End}(R)$. Then $\mathrm{im}\, f$ is a countable algebraically closed graph.*

*Proof.* By definition, $R$ is a countable graph. Furthermore, since $R$ is existentially closed, it is algebraically closed. Thus, by Lemma 3.7, $\mathrm{im}\, f$ is algebraically closed. □

**Lemma 3.9.** *Let $\Gamma$ be a countable graph and let $f : \Gamma \to \Gamma$ be a homomorphism such that $\mathrm{im}\, f$ is algebraically closed. Let $\mathcal{G}(\Gamma)$ be the graph constructed from $\Gamma$ as in Definition 3.3. Then there exist $2^{\aleph_0}$ distinct extensions $\tilde{f} : \mathcal{G}(\Gamma) \to \mathcal{G}(\Gamma)$ of $f$ such that $\tilde{f}$ is a homomorphism and $\mathrm{im}\, \tilde{f} = \mathrm{im}\, f$. Furthermore, if $f$ is idempotent, then so is each $\tilde{f}$.*

*Proof.* Since $\mathrm{im}\, f$ is an algebraically closed subgraph of $\Gamma$ it follows that $\Gamma$ is countably infinite. Thus we will enumerate the vertices of $\mathcal{G}(\Gamma) \setminus \Gamma$ as $\{v_i : i \in \mathbb{N}\}$. An important observation is that by construction, each vertex $v_i$ is adjacent in $\mathcal{G}(\Gamma)$ to every member of some finite subset $U_i$ of $V_\Gamma$ and to no other vertices. Now, inductively define a sequence of maps $f_i : \langle \Gamma \cup \{v_1, \ldots, v_i\} \rangle \to \mathcal{G}(\Gamma)$ as follows.

Let $f_0 = f$ and suppose that for $n \in \mathbb{N}$ we can extend $f$ to a homomorphism $f_n : \langle \Gamma \cup \{v_0, \dots, v_n\}\rangle \to \mathcal{G}(\Gamma)$ with $\operatorname{im} f_n = \operatorname{im} f$. Since $\operatorname{im} f$ is algebraically closed there exists a vertex $w \in (\operatorname{im} f_n \backslash U_{n+1}f) = (\operatorname{im} f \backslash U_{n+1}f)$, such that $w$ is adjacent to every member of $U_{n+1}f$. We can even ensure that $w \neq v_i f$ for $i = 0, \dots n$ by adding the requirement that $w$ should be adjacent to $U_{n+1}f \cup \{v_0 f, \dots, v_n f\}$. Now define $f_{n+1} : \langle \Gamma \cup \{v_0, \dots, v_{n+1}\}\rangle \to \mathcal{G}(\Gamma)$ by,

$$v f_{n+1} = \begin{cases} v f_n & \text{if } v \in \Gamma \cup \{v_0, ..., v_n\}, \\ w & \text{if } v = v_{n+1}. \end{cases}$$

It is clear that $f_{n+1}$ defines a map of vertices $V_\Gamma \cup \{v_0, \dots, v_{n+1}\}$ to $V_{\mathcal{G}(\Gamma)}$. It is a graph homomorphism since by hypothesis $f_n$ was a graph homomorphism and additionally if $(v_{n+1}, x) \in E_{\mathcal{G}(\Gamma)}$ then it must be the case that $x \in U_{n+1} \subseteq V_\Gamma$. Hence $x f_{n+1} \in U_{n+1}f$ and so by choice of $w$, $(w, x f_{n+1}) = (v_{n+1} f_{n+1}, x f_{n+1}) \in E_{\mathcal{G}(\Gamma)}$.

Since $f_{n+1}$ is exactly $f_n$ when restricted to the domain of $f_n$ and $v_{n+1} f \in \operatorname{im} f_n$, $\operatorname{im} f_{n+1} = \operatorname{im} f_n$. Furthermore if $f_n$ is idempotent then so is $f_{n+1}$ since by choice $w \in \operatorname{im} f_n$ and,

$$v_{n+1} f_{n+1}^2 = w f_{n+1} = w f_n = w = v_{n+1} f_{n+1}.$$

Now let

$$\tilde{f} = \bigcup_{n=0}^{\infty} f_n.$$

Then $\tilde{f}$ is a graph homomorphism from $\mathcal{G}(\Gamma)$ to $\mathcal{G}(\Gamma)$ extending $f_n$ for all $n \in \mathbb{N}$ and if $f$ is idempotent so is $\tilde{f}$. Since $\operatorname{im} f_n = \operatorname{im} f$ for all $n \in \mathbb{N}$, it follows that $\operatorname{im} \tilde{f} = \operatorname{im} f$.

Finally, we take a moment to notice that since $\operatorname{im} f$ is algebraically closed, for each $n \in \mathbb{N}$ there are actually infinitely many choices for the vertex $w \in \operatorname{im} f \setminus U_{n+1}f$ with $w$ adjacent to every member of $U_{n+1}f$. That is to say, there are infinitely many choices for the image of $v_{n+1}$ when constructing $f_{n+1}$. Since the choice for $v_{n+1} f_{n+1}$ is determined only by the subset $U_{n+1}f$ and since $v_{n+1}$ is not adjacent to $v_m$ for all $m \in \mathbb{N}$, the choice of vertex made for $v_{n+1} f_{n+1}$ is independent from any $v_m f_m$ chosen for $m \leq n$. Consequently, for each $n \in \mathbb{N}$, there are infinitely many distinct extensions $f_{n+1}$ of $f_n$ which differ on $v_{n+1}$. It follows now that there are ${\aleph_0}^{\aleph_0} = 2^{\aleph_0}$ distinct extensions $\tilde{f}$ of $f$. $\square$

**Theorem 3.10.** *Let $\Gamma$ be a graph. Then there exists an idempotent $f \in$ $\operatorname{End}(R)$ with $\operatorname{im} f \cong \Gamma$ if and only if $\Gamma$ is a countable algebraically closed* *graph. Furthermore, for every countable algebraically closed graph* $\Gamma$*, there exist* $2^{\aleph_0}$ *idempotents* $f \in \mathrm{End}(R)$ *with* $\mathrm{im}\, f \cong \Gamma$.

*Proof.* If $f \in E(\mathrm{End}(R))$ then by Corollary 3.8, $\mathrm{im}\, f$ is countable and algebraically closed.

Conversely suppose that $\Gamma$ is a countable algebraically closed graph. Apply the construction in Definition 3.3 to produce the graph $\Gamma_\infty = \bigcup_{n\in\mathbb{N}} \Gamma_n$. Then, by Lemma 3.5, $\Gamma_\infty = R$. Define inductively a sequence of functions $f_n : \Gamma_n \to \Gamma_\infty$ as follows. Let $f_0 : \Gamma_0 \to \Gamma_\infty$ be the identity function on $\Gamma_0 = \Gamma$, *i.e.* let $vf_0 = v$ for all vertices $v \in V_\Gamma$. Then $f_0$ is trivially an idempotent graph homomorphism and $\mathrm{im}\, f = \Gamma$. Now for $n \in \mathbb{N}$, we let $f_{n+1} = \tilde{f}_n$ be an extension of $f_n$ to $\mathcal{G}(\Gamma_n) = \Gamma_{n+1}$ constructed in Lemma 3.9. The proof of Lemma 3.9 ensures that $f_{n+1}$ is an idempotent graph homomorphism from $\Gamma_{n+1}$ to $\Gamma_\infty$ and that $\mathrm{im}(f_n) = \Gamma$. Now let,

$$f = \bigcup_{n=0}^{\infty} f_n.$$

Then as the union of idempotent graph homomorphisms such that each $f_{n+1}$ is an extension of $f_n$, $f$ itself is an idempotent graph homomorphism from $\Gamma_\infty$ to $\Gamma_\infty$. Furthermore, since $\mathrm{im}\, f_n = \Gamma$ for all $n \in \mathbb{N}$, $\mathrm{im}\, f = \Gamma$.

Finally, since Lemma 3.9 tells us that there are $2^{\aleph_0}$ distinct extensions $\tilde{f}_n$ of $f_n$, it follows that there are $2^{\aleph_0}$ distinct extensions $f_{n+1}$ of $f_n$ for all $n \in \mathbb{N}$. Hence there are $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ many distinct idempotents $f \in \mathrm{End}(R)$ with $\mathrm{im}\, f = \Gamma$. □

In light of Theorems 2.7 and 3.10 we can conclude that the group $\mathscr{H}$-classes of $\mathrm{End}(R)$ are exactly the automorphism groups of countable algebraically closed graphs. The question of which groups are automorphism groups of countable algebraically closed graphs now arises. We can easily find an example where such an automorphism groups is uncountable. For example, the complete graph on a countably infinite number of vertices is trivially algebraically closed and has automorphism group isomorphic to the uncountable group $S_{\mathbb{N}}$. But what about countable groups? Does the trivial group arise? How about any given countable group?

Fortunately, to help us, there is the following theorem due to Frucht, see [Fru39] for details.

**Theorem 3.11.** *If* $G$ *is a finite group then there exists a finite connected graph* $\Gamma$ *such that* $G \cong \mathrm{Aut}(\Gamma)$.

This result was later extended to infinite groups by de Groot in [Gro59] and independently by Sabidussi in [Sab60]. In particular we have the following theorem.

**Theorem 3.12.** *Every infinite group $G$ can be realised as the automorphism group of a connected graph with vertex set of size $|G|$.*

Using Frucht's theorem as inspiration, we will show that for any countable group $G$ there exists an algebraically closed graph $\Gamma$ such that $G$ is isomorphic to $\mathrm{Aut}(\Gamma)$. Thus showing that every countable group can be found as a (maximal) subgroup of $\mathrm{End}(R)$. To do this we will need the following definition and subsequent lemmas.

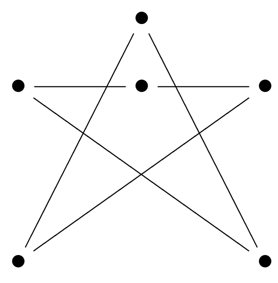
If $\Gamma = (V_\Gamma, E_\Gamma)$ is a directed graph we can define the *complement* of $\Gamma$ to be the relational structure $\Gamma^\dagger = (V_{\Gamma^\dagger}, E_{\Gamma^\dagger})$, where $V_{\Gamma^\dagger} = V_\Gamma$ and

$$E_{\Gamma^\dagger} = (V_\Gamma \times V_\Gamma) \setminus (E_\Gamma \cup \{(v, v) : v \in V_\Gamma\}).$$

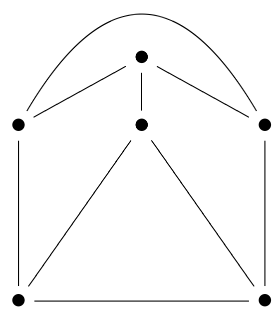
Since $E_{\Gamma^\dagger}$ is irreflexive by construction, it follows that $\Gamma^\dagger$ is a directed graph. If $\Gamma$ is a graph, then $E_\Gamma$ is a symmetric relation and hence the binary relation $E_\Gamma \cup \{(v, v) : v \in V_\Gamma\}$ is symmetric. Thus $E_{\Gamma^\dagger}$, being the complement of a symmetric relation, is symmetric. Consequently, it follows that if $\Gamma$ is a graph, then $\Gamma^\dagger$ is also a graph. If $\Gamma$ is a graph, $\Gamma^\dagger$ can be thought of as the graph formed from $\Gamma$ by replacing all edges with a non-edge and all non-edges with an edge. If $\Gamma$ is a directed graph, then the symmetric edges in $\Gamma$ behave as above and additionally the orientation of the non-symmetric edges in $\Gamma$ are reversed in $\Gamma^\dagger$.

**Example 3.13.** [Example construction of $\Gamma^\dagger$ given a graph $\Gamma$.]

$\Gamma$



$\Gamma^\dagger$



It is not hard to see that for any graph $\Gamma$ it follows that $(\Gamma^\dagger)^\dagger = \Gamma$. Furthermore, the following useful properties hold.

**Lemma 3.14.** *Let $\Gamma$ and $\Lambda$ be directed graphs and suppose that $f : V_\Gamma \to V_\Lambda$ defines an embedding of $\Gamma$ into $\Lambda$. Then $f$ also defines an embedding of the complement $\Gamma^\dagger$ into $\Lambda^\dagger$.*

*Proof.* Since $f$ defines an embedding of $\Gamma$ into $\Lambda$, it is immediate that $f$ is an injective function. Now suppose that $(u, v) \in E_{\Gamma^\dagger}$. Then $(u, v) \notin E_\Gamma$ and since $f$ is an embedding it follows that $(uf, vf) \notin E_\Lambda$. Thus either $uf = vf$ or $(uf, vf) \in E_{\Lambda^\dagger}$. Clearly the injectivity of $f$ rules out the former case and so $(uf, vf) \in E_{\Lambda^\dagger}$. On the other hand if $(u, v) \notin E_{\Gamma^\dagger}$ then either $u = v$ or $(u, v) \in E_\Gamma$. If $u = v$ then $uf = vf$ and hence $(uf, vf) \notin E_{\Lambda^\dagger}$. If instead $(u, v) \in E_\Gamma$ then since $f$ is an embedding $(uf, vf) \in E_\Lambda$ and hence we can still conclude that $(uf, vf) \notin E_{\Lambda^\dagger}$. Thus $f$ defines an embedding $\Gamma^\dagger \to \Lambda^\dagger$ as required. $\square$

**Corollary 3.15.** *Let $\Gamma$ be a directed graph. Then* $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma^\dagger)$.

*Proof.* Let $f \in \mathrm{Aut}(\Gamma)$. Then clearly $f$ is a bijective map $V_\Gamma \to V_\Gamma$ which defines an embedding of $\Gamma$ into itself. Hence by Lemma 3.14 the bijective function $f$ also defines an embedding of $\Gamma^\dagger$ into itself. In other words $f \in \mathrm{Aut}(\Gamma^\dagger)$ and we can conclude that $\mathrm{Aut}(\Gamma) \subseteq \mathrm{Aut}(\Gamma^\dagger)$. Now suppose instead that $g \in \mathrm{Aut}(\Gamma^\dagger)$. Then $g$ is a bijective map $V_\Gamma \to V_\Gamma$ which defines an embedding of $\Gamma^\dagger$ into itself. Another application of Lemma 3.14 allows us to deduce that the bijective function $g$ also defines an embedding of $(\Gamma^\dagger)^\dagger = \Gamma$ into itself. Hence $\mathrm{Aut}(\Gamma^\dagger) \subseteq \mathrm{Aut}(\Gamma)$ and it now follows that $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma^\dagger)$. $\square$

There is also a nice connection between algebraic closure and the complement $\Gamma^\dagger$ for certain graphs $\Gamma$. Recall from Chapter 2, that a graph $\Gamma$ is locally finite if every vertex $v \in V_\Gamma$ is adjacent to only finitely many vertices in $\Gamma$.

**Lemma 3.16.** *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be an infinite, locally finite graph. Then the complement $\Gamma^\dagger$ is algebraically closed.*

*Proof.* Let $U \subset V_\Gamma$ be a finite subset of vertices of $\Gamma^\dagger$. Since $\Gamma$ is locally finite each $u \in U$ is adjacent to only finitely many vertices in $\Gamma$. So let $\Gamma(u) = \{v \in V_\Gamma : (u, v) \in E_\Gamma\}$ and let $W = \bigcup_{u \in U} \Gamma(u)$. Then $U \cup W$ is finite and so there exists $x \in V_\Gamma \setminus (U \cup W)$. Then $x$ is not adjacent to any vertex of $U$ in $\Gamma$ and so $x$ is adjacent to all vertices of $U$ in $\Gamma^\dagger$. $\square$

**Lemma 3.17.** *Let $\Gamma$ be a countable graph and let $\Lambda$ be an infinite locally finite graph. Then $(\Gamma \dot\cup \Lambda)^\dagger$, the complement of the disjoint union of $\Gamma$ and $\Lambda$, is an algebraically closed graph.*

Figure 3.1: The line graph $L$.

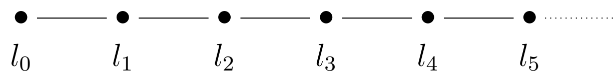
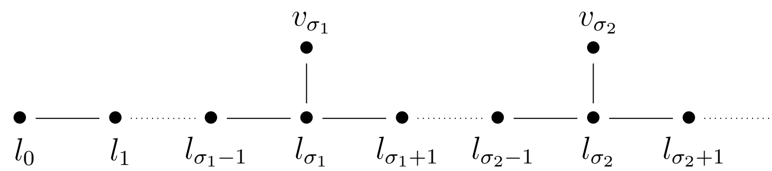



Figure 3.2: The graph $L_\Sigma$.




*Proof.* Let $\Delta = (\Gamma \dot\cup \Lambda)^\dagger$ so that $\Delta$ has vertex set $V_\Delta = V_\Gamma \cup V_\Lambda$. Let $U$ be a finite set of vertices from $V_\Delta$. Since $U$ is finite $U \cap V_\Lambda$ is finite. Furthermore, since $\Lambda$ is infinite and locally finite, $\Lambda^\dagger$ is algebraically closed by Lemma 3.16. Thus there exists a vertex $v \in V_\Lambda$ such that $v$ is adjacent to every member of $U \cap V_\Lambda$ in $\Lambda^\dagger$. Now since $v$ is also adjacent to every member of $V_\Gamma$ in $(\Gamma \dot\cup \Lambda)^\dagger$ it follows that $v$ is adjacent to every member of $U$ in $(\Gamma \dot\cup \Lambda)^\dagger$. $\square$

We now consider the line graph $L = (V_L, E_L)$ with $V_L = \{l_n : n \in \mathbb{N}\}$ and where $(l_i, l_j) \in E_L$ if and only if $j = i+1$ or $i = j+1$. See Figure 3.1 for a pictorial representation. It should be clear that $\mathrm{Aut}(L) = \mathbf{1}$ since any automorphism must fix $l_0$ and thus every vertex in $V_L$.

**Definition 3.18.** Let $\Sigma$ be a subset of $\mathbb{N} \setminus \{0,1\}$. Using $L$ and $\Sigma$ we define a new graph $L_\Sigma$ with vertices $V_{L_\Sigma} = V_L \cup \{v_\sigma : \sigma \in \Sigma\}$ and edges $E_{L_\Sigma} = E_L \cup \{(l_\sigma, v_\sigma), (v_\sigma, l_\sigma) : \sigma \in \Sigma\}$. See Figure 3.2 for a pictorial representation of $L_\Sigma$ when $\Sigma = \{\sigma_n : n \in \mathbb{N}\}$.

**Lemma 3.19.** *Let* $\Sigma \subseteq \mathbb{N} \setminus \{0,1\}$. *Then* $\mathrm{Aut}(L_\Sigma) = \mathbf{1}$.

*Proof.* Since $l_0$ is the only vertex of degree one adjacent to a vertex of degree two, it must be fixed by any automorphism of $L_\Sigma$. Subsequently, since $l_1$ is the sole vertex adjacent to $l_0$ it must also be fixed by any automorphism. Continuing by induction we see that $l_i$ must be fixed for all $i \geq 2$. Finally since each vertex $v_\sigma$ is adjacent to only $l_\sigma$ it must be the case that each $v_\sigma$ is also fixed. Thus any automorphism must act as the identity on all vertices and the result follows. □

For a subset $\Sigma$ of $\mathbb{N} \setminus \{0,1\}$ and for $k \in \mathbb{N}$ we will define $\Sigma + k$ to be the set $\Sigma + k = \{\sigma + k : \sigma \in \Sigma\} \subseteq \mathbb{N} \setminus \{0,1\}$. The set $\Sigma + k$ is said to be a (positive) translation of $\Sigma$.

**Lemma 3.20.** *Let $\Sigma, \Psi \subseteq \mathbb{N} \setminus \{0,1\}$. Then there exists a graph embedding $f : L_\Sigma \to L_\Psi$ if and only if $\Sigma + k \subseteq \Psi$ for some $k \in \mathbb{N}$.*

*Proof.* So suppose that $f : L_\Sigma \to L_\Psi$ is a graph embedding. Since $f$ must map the infinite path $(l_0, l_1)$, $(l_1, l_2)$, $(l_2, l_3), \ldots$ of distinct vertices contained in $L_\Sigma$ to an infinite path of distinct vertices in $L_\Psi$, it must be the case that $l_i f = l_{i+k}$ for all $i \in \mathbb{N}$ and for some $k \in \mathbb{N}$. Now let $\sigma \in \Sigma$. Then $v_\sigma \in L_\Sigma$ and $(v_\sigma, l_\sigma) \in E_{L_\Sigma}$. Hence since $f$ is a graph homomorphism, we can deduce that $(v_\sigma f, l_{\sigma+k}) \in E_{L_\Psi}$. But the only vertices in $L_\Psi$ to which $l_{\sigma+k}$ is adjacent are $l_{\sigma+k-1}$, $l_{\sigma+k+1}$ or $v_{\sigma+k}$ if $\sigma + k \in \Psi$. Since $\sigma + k - 1 \geq k + 1$ and since $f$ is injective it follows that $v_\sigma f \neq l_j$ for $j \in \{\sigma + k - 1, \sigma + k + 1\}$. Thus it must be the case that $\sigma + k \in \Psi$ and $v_\sigma f = v_{\sigma+k}$. Since $\sigma \in \Sigma$ was arbitrary it now follows that $\Sigma + k \subseteq \Psi$.

Now suppose that $\Sigma + k \subseteq \Psi$ for some $k \in \mathbb{N}$. Define a function on $V_{L_\Sigma}$ by

$$uf = \begin{cases} l_{i+k} & \text{if } u = l_i \text{ for some } i \in \mathbb{N}, \\ v_{\sigma+k} & \text{if } u = v_\sigma \text{ for some } \sigma \in \Sigma. \end{cases}$$

Since $\Sigma + k \subseteq \Psi$, $f$ defines a map $V_{L_\Sigma} \to V_{L_\Psi}$. It is obviously injective and to finish the proof we will show that it also defines a graph embedding. So suppose that $(t, u) \in V_{L_\Sigma}$. Then without loss of generality either $t = l_i$ and $u = l_{i+1}$ for some $i \in \mathbb{N}$, or $t = v_\sigma$ and $u = l_\sigma$ for some $\sigma \in \Sigma$. In either case it is easy to see that $(tf, uf) \in V_{L_\Psi}$. If instead $(t, u) \notin V_{L_\Sigma}$, then without loss of generality either $t = l_i$ and $u = l_j$ for some $i, j \in \mathbb{N}$, $j \neq i - 1, i + 1$ or $t = v_\sigma$ and $u = l_n$ for some $n \neq \sigma$. In the former case, if $j \notin \{i - 1, i + 1\}$ then $j + k \notin \{i + k - 1, i + k + 1\}$ and so $(tf, uf) = (l_{i+k}, l_{j+k}) \notin E_{L_\Psi}$. In the latter case $(v_\sigma f, l_j f) = (v_{\sigma+k}, l_{j+k}) \notin E_{L_\Psi}$ since $j \neq \sigma$. Hence $f$ defines a graph embedding as required. □

**Corollary 3.21.** *Let $\Sigma, \Psi \subseteq \mathbb{N} \setminus \{0,1\}$. Then $L_\Sigma \cong L_\Psi$ if and only if $\Sigma = \Psi$.*

*Proof.* It is clear that if $\Sigma = \Psi$ then $L_\Sigma = L_\Psi$. So suppose instead that $L_\Sigma \cong L_\Psi$ but that $\Sigma \neq \Psi$. Let $f : V_{L_\Sigma} \to V_{L_\Psi}$ be an isomorphism of graphs. Since $f$ is an embedding Lemma 3.20 tells us that $\Sigma + k \subseteq \Psi$ for some $k \in \mathbb{N}$ where $l_i f = l_{i+k}$ for all $i \in \mathbb{N}$. But since $l_0$ is the only vertex of degree one adjacent to a vertex of degree two in both $L_\Sigma$ and $L_\Psi$, it must be the case that $l_0 f = l_0$ and hence by induction that $l_i f = l_i$ for all $i \in \mathbb{N}$. Thus $k = 0$ and $\Sigma \subseteq \Psi$. A dual argument with the isomorphism $f^{-1} : V_{L_\Psi} \to V_{L_\Sigma}$ leads us to deduce that $\Psi \subseteq \Sigma$ and hence $\Sigma = \Psi$. $\square$

As we will see in the following lemmas, the graphs $L_\Sigma$, for $\Sigma \subseteq \mathbb{N} \setminus \{0, 1\}$, will play a critical part in the process of constructing algebraically closed graphs with a given countable automorphism group.

**Lemma 3.22.** *Let* $\Gamma$ *be any countable graph and let* $\Delta_\Sigma = (\Gamma \,\dot\cup\, L_\Sigma)^\dagger$*, the complement of the disjoint union of* $\Gamma$ *and the line graph* $L_\Sigma$ *defined in Definition 3.18. Then* $\Delta_\Sigma$ *is algebraically closed for all subsets* $\Sigma \subseteq \mathbb{N} \setminus \{0, 1\}$*.*

*Proof.* In light of Lemma 3.17 we only need to note that $L_\Sigma$ is a countably infinite locally finite graph for all subsets $\Sigma \subseteq \mathbb{N} \setminus \{0, 1\}$. $\square$

**Lemma 3.23.** *Let* $\Gamma$ *and* $\Lambda$ *be countable (directed) graphs with no isomorphic components. Then* $\mathrm{Aut}(\Gamma \,\dot\cup\, \Lambda) \cong \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(\Lambda)$*.*

*Proof.* Define a map $\phi : \mathrm{Aut}(\Gamma \,\dot\cup\, \Lambda) \to \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(\Lambda)$ by $f\phi = (f|_\Gamma, f|_\Lambda)$. First we show that this map is well defined. Recall that a (directed) graph homomorphism must map connected components to connected components. Since $\Gamma$ and $\Lambda$ have no isomorphic connected components it must be the case that any automorphism $f$ of $\Gamma \,\dot\cup\, \Lambda$ is such that $\operatorname{im} f|_\Gamma = \Gamma$ and $\operatorname{im} f|_\Lambda = \Lambda$. Using this observation we can easily deduce that the restrictions $f|_\Gamma$ and $f|_\Lambda$ define automorphisms since $f$ itself is an automorphism. Hence it follows that $(f|_\Gamma, f|_\Lambda) \in \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(\Lambda)$ and the map $\phi$ is well defined.

Now for any pair $f, g \in \mathrm{Aut}(\Gamma \,\dot\cup\, \Lambda)$,

$$(fg)\phi = ((fg)|_\Gamma, (fg)|_\Lambda) = (f|_\Gamma \cdot g|_\Gamma, f|_\Lambda \cdot g|_\Lambda) = (f|_\Gamma, f|_\Lambda) \cdot (g|_\Gamma, g|_\Lambda) = f\phi \cdot g\phi.$$

This shows that $\phi$ is a group homomorphism. To show that the map $\phi$ is injective, suppose that $f$ and $g$ are automorphisms of $\Gamma \,\dot\cup\, \Lambda$ such that $f\phi = g\phi$. Then $f|_\Gamma = g|_\Gamma$ and $f|_\Lambda = g|_\Lambda$. Now since $\operatorname{im} f|_\Gamma = \operatorname{im} g|_\Gamma = \Gamma$ and $\operatorname{im} f|_\Lambda = \operatorname{im} g|_\Lambda = \Lambda$ we deduce that $f = g$. To show that $\phi$ is surjective let $(f, g) \in \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(\Lambda)$. Define the map $h : \Gamma \,\dot\cup\, \Lambda \to \Gamma \,\dot\cup\, \Lambda$ by

$$vh = \begin{cases} vf & \text{if } v \in V_\Gamma, \\ vg & \text{if } v \in V_\Lambda. \end{cases}$$

Since $f$ and $g$ are automorphisms of $\Gamma$ and $\Lambda$ respectively, and since there are no edges between $\Gamma$ and $\Lambda$ in the disjoint union, $h$ defines an automorphism of $\Gamma \,\dot\cup\, \Lambda$. Moreover, $h\phi = (f, g)$ and thus $\phi$ is surjective. We can now conclude that $\phi$ is an isomorphism of groups and the result is complete. $\square$

**Lemma 3.24.** *Let $\Gamma$ be a countable graph and let $\Delta_\Sigma = (\Gamma \dot\cup L_\Sigma)^\dagger$, the complement of the disjoint union of $\Gamma$ and the line graph $L_\Sigma$ defined in Definition 3.18. Then there exist $2^{\aleph_0}$ subsets $\Sigma \subset \mathbb{N} \setminus \{0,1\}$ such that $\mathrm{Aut}(\Delta_\Sigma) \cong \mathrm{Aut}(\Gamma)$.*

*Proof.* Since $\Gamma$ is a countable graph, the number of connected components of $\Gamma$ is countable. As a result at most countably many choices of $\Sigma$ would result in the graph $L_\Sigma$ being isomorphic to some component of $\Gamma$. Since the set of all subsets of the natural numbers has size $2^{\aleph_0}$ this still leaves $2^{\aleph_0}$ distinct choices for the subset $\Sigma$ which ensure that $L_\Sigma$ is isomorphic to no component of $\Gamma$.

For each of these distinct choices we can form the graph $\Delta_\Sigma = (\Gamma \dot\cup L_\Sigma)^\dagger$ and by Corollary 3.15 we can deduce that $\mathrm{Aut}(\Delta_\Sigma) \cong \mathrm{Aut}(\Gamma \dot\cup L_\Sigma)$. Furthermore, since $\Gamma$ and $L_\Sigma$ have no isomorphic components, Lemma 3.23 allows us to conclude that $\mathrm{Aut}(\Delta_\Sigma) \cong \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(L_\Sigma)$. All that remains is to recall our earlier observation that $L_\Sigma$ has no non-trivial automorphisms. Then,

$$\mathrm{Aut}(\Delta_\Sigma) \cong \mathrm{Aut}(\Gamma) \times \mathbf{1} \cong \mathrm{Aut}(\Gamma),$$

as required. □

We now have collected enough machinery to state and prove the main theorem of this chapter.

**Theorem 3.25.** *Let $\Gamma$ be a countable graph. Then there exist $2^{\aleph_0}$ group $\mathscr{H}$-classes $H$ of $\mathrm{End}(R)$ such that $H \cong \mathrm{Aut}(\Gamma)$.*

*Proof.* By applying Lemma 3.24, there exist $2^{\aleph_0}$ sets $\Sigma \subset \mathbb{N} \setminus \{0,1\}$ such that $\mathrm{Aut}(\Delta_\Sigma) \cong \mathrm{Aut}(\Gamma)$. By Lemma 3.22, $\Delta_\Sigma$ is algebraically closed for each choice of $\Sigma$ and so by Theorem 3.10 there exists an idempotent $f_\Sigma \in \mathrm{End}(R)$ such that $\mathrm{im}\, f_\Sigma \cong \Delta_\Sigma$. Now Theorem 2.7 tells us that,

$$H_{f_\Sigma} \cong \mathrm{Aut}(\mathrm{im}\, f_\Sigma) \cong \mathrm{Aut}(\Delta_\Sigma) \cong \mathrm{Aut}(\Gamma).$$

We know from Corollary 3.21 that $L_\Sigma$ is not isomorphic to $L_\Psi$ for $\Sigma \neq \Psi$ and by choice both are isomorphic to no component of $\Gamma$. Hence we can deduce that $\Delta_\Sigma$ and $\Delta_\Psi$ are not isomorphic for $\Sigma \neq \Psi$. In other words $\mathrm{im}\, f_\Sigma \neq \mathrm{im}\, f_\Psi$ for $\Sigma \neq \Psi$ and the idempotents are all distinct. Since no group $\mathscr{H}$-class can contain more than one idempotent, the result now follows. □

**Corollary 3.26.** *Let $G$ be any countable group. Then there exist $2^{\aleph_0}$ group $\mathscr{H}$-classes $H$ of $\mathrm{End}(R)$ such that $H \cong G$.*

*Proof.* By (the extended version of Frucht's) Theorem 3.12, $G$ can be realised as the automorphism group of a countable graph $\Gamma$. Now by applying Theorem 3.25 the result is complete. $\square$

In summary, Theorems 3.10 and 3.25 tell us that if $H$ is a maximal subgroup of $\mathrm{End}(R)$ then $H \cong \mathrm{Aut}(\Gamma)$ for a countable graph $\Gamma$ and conversely, if $\Lambda$ is a countable graph, then there exist $2^{\aleph_0}$ maximal subgroups of $\mathrm{End}(R)$ isomorphic to $\mathrm{Aut}(\Lambda)$. We are also able to deduce the following Corollary.

**Corollary 3.27.** *There exist $2^{\aleph_0}$ non-isomorphic maximal subgroups of* $\mathrm{End}(R)$.

*Proof.* Corollary 3.26 tells us that for any countable group $G$, there exists a group $\mathscr{H}$-class which is isomorphic to $G$. Thus it suffices to show that there are $2^{\aleph_0}$ non-isomorphic countable groups. This is a well known fact but can easily be shown by considering the following groups. Let $S$ be a set of prime numbers and let

$$G_S = \prod_{p \in S} \mathbb{Z}/p\mathbb{Z},$$

with addition component-wise. Clearly if $S$ and $T$ are two sets of prime numbers such that $S \neq T$, then $G_S \ncong G_T$. To see this note that if $p \in S \setminus T$, then $G_S$ contains an element of order $p$, where as $G_T$ does not. Thus since the set of all subsets of the prime numbers has size $2^{\aleph_0}$, the result is complete. $\square$

## 3.3 Regular $\mathscr{D}$-classes and $\mathscr{J}$-classes of $\mathrm{End}(R)$

From the results obtained in the previous subsection we can gain some insight into the structure and cardinality of the set of regular $\mathscr{D}$-classes of $\mathrm{End}(R)$.

**Theorem 3.28.** *There exist $2^{\aleph_0}$ distinct regular $\mathscr{D}$-classes of* $\mathrm{End}(R)$.

*Proof.* Recall from the preliminary chapter that if two group $\mathscr{H}$-classes are contained in the same $\mathscr{D}$-class then they are isomorphic as groups. Thus since, by Corollary 3.27, there exist $2^{\aleph_0}$ non-isomorphic group $\mathscr{H}$-classes of $\mathrm{End}(R)$, there must exist $2^{\aleph_0}$ distinct (regular) $\mathscr{D}$-classes of $\mathrm{End}(R)$. $\square$

Any two group $\mathscr{H}$-classes which are contained in the distinct $\mathscr{D}$-classes provided by the proof of Theorem 3.28 are not isomorphic. The next result shows that there also exist distinct (regular) $\mathscr{D}$-classes whose group $\mathscr{H}$-classes are all isomorphic.

**Theorem 3.29.** *There exists a set of $2^{\aleph_0}$ distinct regular $\mathscr{D}$-classes of* $\mathrm{End}(R)$ *for which any two group $\mathscr{H}$-classes are isomorphic.*

*Proof.* In Theorem 2.10 we have shown that if $f$ and $g$ are two idempotents of $\mathrm{End}(R)$ then $f\mathscr{D}g$ if and only if $\langle \mathrm{im}\, f\rangle$ and $\langle \mathrm{im}\, g\rangle$ are isomorphic. By the details of the proof of Theorem 3.25, if $\Gamma$ is a graph then there exist $2^{\aleph_0}$ sets $\Sigma \subseteq \mathbb{N} \setminus \{0,1\}$ such that $H_{f_\Sigma} \cong \mathrm{Aut}(\Gamma)$ and such that $\langle \mathrm{im}\, f_\Sigma\rangle \ncong \langle \mathrm{im}\, f_\Psi\rangle$ for any $\Psi \subseteq \mathbb{N} \setminus \{0,1\}$ where $\Sigma \neq \Psi$. Therefore each idempotent $f_\Sigma$ is contained in a distinct (regular) $\mathscr{D}$-class, but the group $\mathscr{H}$-classes $H_{f_\Sigma}$ are all isomorphic. Now since any two group $\mathscr{H}$-classes which are contained in the same regular $\mathscr{D}$-class are isomorphic, the result follows. □

**Theorem 3.30.** *Each regular $\mathscr{D}$-class of* $\mathrm{End}(R)$ *contains* $2^{\aleph_0}$ *distinct group $\mathscr{H}$-classes.*

*Proof.* If a $\mathscr{D}$-class is regular it contains at least one group $\mathscr{H}$-class. Let $f \in E(\mathrm{End}(R))$ be the subgroup identity of this $\mathscr{H}$-class. By Corollary 3.7 $\mathrm{im}\, f$ is algebraically closed and so by Theorem 3.10 there exist $2^{\aleph_0}$ distinct idempotents whose image is isomorphic to $\langle \mathrm{im}\, f\rangle$. Thus, by Theorem 2.10 all of these idempotents are $\mathscr{D}$-related. However, since a group $\mathscr{H}$-class can contain at most one idempotent, no two of these idempotents can lie in the same $\mathscr{H}$-class and the result follows. □

As a further consequence of the previous work, we can gain some information on the cardinality of the set of $\mathscr{J}$-classes of $\mathrm{End}(R)$. In order to prove an analogous result to Theorem 3.28 for the set of $\mathscr{J}$-classes of $\mathrm{End}(R)$ we will require the following lemma.

**Lemma 3.31.** *There exists a set $P$ of $2^{\aleph_0}$ distinct subsets of the natural numbers such that for all $\Sigma$, $\Psi \in P$ and for all $k \in \mathbb{N}$, $\Sigma + k \nsubseteq \Psi$ and $\Psi + k \nsubseteq \Sigma$.*

*Proof.* First we will need the following definitions. If $A$ is a finite set, then by a word $A$ we will mean a finite or infinite string (or sequence) $a_1a_2a_3\cdots$ such that $a_i \in A$ for all $i \in \mathbb{N}$. If $w$ is a word consisting of finite string $a_1a_2\cdots a_n$ on $A$, then $w$ is said to have length $n$ and we will denote this by $l(w) = n$. If $w$ consists of an infinite string, then accordingly $w$ is said to have infinite length and we write $l(w) = \infty$. Two words $a_1a_2a_3\cdots$ and $b_1b_2b_3\cdots$ will be equal if and only if they have equal length and $a_i = b_i$ for all $i$. A finite word $b_1b_2\cdots b_n$ is said to be a prefix of a word $w = a_1a_2a_3\cdots$, where $l(w) \geq n$, if $b_i = a_i$ for all $i = 1,\ldots,n$.

Let $\{0,1\}^n$ denote the set of all words on the set $\{0,1\}$ of length at most $n$. So that, for example $\{0,1\}^0 = \emptyset$ and $\{0,1\}^2 = \{\emptyset, 0, 1, 00, 01, 10, 11\}$. Define a sequence of functions inductively as follows. Let $f_0 : \{0,1\}^0 \to \mathbb{N}$ be defined by $\emptyset f = 1$. Now suppose that $f_k : \{0,1\}^k \to \mathbb{N}$ has been defined for all $k < n$ for some $n \in \mathbb{N}$ such that:

(i) $wf - xf > 0$ for all words $w, x$ with $l(x) < l(w) \leq k$.

(ii) If $w, x, y, z$ are words with $w \neq x$ and $y \neq z$ and such that $l(x) \leq l(w) \leq k$, $l(z) \leq l(y) \leq k$, then $wf - xf = yf - zf$ if and only if $y = w$ and $x = z$.

We will show that we can extend $f_{n-1}$ to a function $f_n : \{0,1\}^n \to \mathbb{N}$ which again satisfies the conditions above. We will do this by using a second induction argument. For each $n \in \mathbb{N}$ the number of words of length $n$ is $2^n$. Enumerate the words of length $n$ by $w_1, w_2, \ldots, w_{2^n}$. First set $g_0 = f_{n-1}$. Now for $j \in \mathbb{N}$ define,

$$g_j : \{0,1\}^{n-1} \cup \{w_1, \ldots, w_j\} \to \mathbb{N}$$

by

$$xg_j = \begin{cases} xg_{j-1} & \text{if } x \in \{0,1\}^{n-1} \cup \{w_1, \ldots, w_{j-1}\} \\ p & \text{if } x = w_j \end{cases}$$

where $p \in \mathbb{N}$ is such that

$$p - xf > \max\big\{wf - yf : w, y \in \{0,1\}^{n-1} \cup \{w_1, \ldots, w_j\}\big\},$$

for all $x \in \{0,1\}^{n-1} \cup \{w_1, \ldots, w_j\}$. Clearly $g_j$ satisfies the conditions (i) and (ii) above by construction. Let

$$f_n = \bigcup_{j=0}^{2^n} g_j.$$

Then since $g_{j+1}$ is an extension of $g_j$ for all $j = 0, \ldots 2^n$, it follows that $f_n$ is itself a function. It is not hard to see that $f_n$ also satisfies the conditions (i) and (ii) since each $g_j$ did. Now let

$$f = \bigcup_{n_0}^{\infty} f_n.$$

Then $f : \{0,1\}^\infty \to \mathbb{N}$ is injective and satisfies the conditions (i) and (ii). Now construct a set, $P$ say, of subsets of the natural numbers by setting

$$P = \Big\{\{\emptyset f, a_1 f, a_1 a_2 f, \ldots\} \,:\, a_1 a_2 a_3 \ldots \text{ is an infinite word on } \{0,1\}\Big\}.$$

We claim that for any two subsets $\Sigma, \Psi \in P$, $\Sigma + k \not\subseteq \Psi$ and $\Psi + k \not\subseteq \Sigma$ for all $k \in \mathbb{N}$. To see this let $\Sigma = \{\emptyset f, a_1 f, a_1 a_2 f, \ldots\}$ and $\Psi = \{\emptyset f, b_1 f, b_1 b_2 f, \ldots\}$

for distinct words $a_1a_2a_3\dots$ and $b_1b_2b_3\dots$. Now suppose that $\Sigma+k\subseteq\Psi$ for some $k\in\mathbb{N}\setminus\{0\}$, so that

$$\{\emptyset f+k, a_1f+k, a_1a_2f+k,\dots\}\subset\Psi.$$

Then there exist prefixes $b_1b_2\dots b_r$, $b_1b_2\dots b_s$ with $1<r<s$ such that

$$(a_1f+k)-(\emptyset f+k)=(b_1b_2\dots b_s)f-(b_1b_2\dots b_r)f.$$

But then $a_1f-\emptyset f=(b_1b_2\dots b_s)f-(b_1b_2\dots b_r)f$. But clearly this is a contradiction to property (ii) since $b_1b_2\dots b_r\neq\emptyset$. Thus $\Sigma+k\not\subseteq\Psi$ and a similar argument shows that $\Psi+k\not\subseteq\Sigma$ for all $k\in\mathbb{N}$. Hence $P$ has size $2^{\aleph_0}$ and the result is complete. □

**Theorem 3.32.** *There exist $2^{\aleph_0}$ distinct $\mathscr{J}$-classes of* $\mathrm{End}(R)$*.*

*Proof.* By Lemma 3.31 there exists a set $P$ of $2^{\aleph_0}$ distinct subsets of the natural numbers such that if $\Sigma,\Psi\in P$ then $\Sigma+k\not\subseteq\Psi$ and $\Psi+k\not\subseteq\Sigma$ for all $k\in\mathbb{N}$. Thus by Lemma 3.20 if $\Sigma,\Psi\in P$ then $L_\Sigma$ cannot be embedded into $L_\Psi$ and similarly $L_\Psi$ cannot be embedded into $L_\Sigma$. Using Lemma 3.14, we can hence deduce that $L_\Sigma^\dagger$ cannot be embedded into $L_\Psi^\dagger$ and *vice versa*. Since $L_\Sigma^\dagger$ and $L_\Psi^\dagger$ are algebraically closed graphs by Lemma 3.16, we can apply Lemma 3.10 to conclude that there exist idempotents $f_\Sigma, f_\Psi\in E(\mathrm{End}(R))$ such that $\mathrm{im}\, f_\Sigma\cong L_\Sigma^\dagger$ and $\mathrm{im}\, f_\Psi\cong L_\Psi^\dagger$. But, by Theorem 2.11 and by the previous observations, $f_\Sigma$ and $f_\Psi$ are not $\mathscr{J}$-related. Thus since $P$ contained $2^{\aleph_0}$ sets, it follows that there must indeed exist $2^{\aleph_0}$ distinct $\mathscr{J}$-classes. □

As a consequence of Theorem 3.32, there are $2^{\aleph_0}$ ideals of $\mathrm{End}(R)$. It should be noted that a proof of Theorem 3.32 has been previously provided using an alterative method in [DD04, Theorem 3 and Remark 6].

# Chapter 4

# The Random Directed Graph

In this chapter we describe the random directed graph $D$ and, amongst other things, consider the maximal subgroups of its endomorphism monoid in relation to those of $\mathrm{End}(R)$. Many of the results that held true for $\mathrm{End}(R)$, also hold on $\mathrm{End}(D)$. We will show that if $\Gamma$ is any directed graph, then there are $2^{\aleph_0}$ maximal subgroups of $\mathrm{End}(D)$ isomorphic to $\mathrm{Aut}(\Gamma)$. We will also show that there are $2^{\aleph_0}$ regular $\mathscr{D}$-classes of $\mathrm{End}(D)$, each containing $2^{\aleph_0}$ group $\mathscr{H}$-classes, and that there are $2^{\aleph_0}$ $\mathscr{J}$-classes of $\mathrm{End}(D)$.

## 4.1 Defining Properties and Constructions

It is easily shown that the class of finite directed graphs has the hereditary, joint embedding and amalgamation properties. Thus, the class of finite directed graphs has a unique homogeneous Fraïssé limit which we will call the *random directed graph*, $D$. As with the random graph, we can prove that $D$ has certain properties.

To begin, let us say that a directed graph $\Gamma = (V_\Gamma, E_\Gamma)$ is *existentially closed* (in the class of directed graphs) if for any four finite disjoint sets $U_1, \ldots, U_4 \subseteq V_\Gamma$, there exists a vertex $x \in V_\Gamma \setminus \bigcup_{i=1}^{4} U_i$ such that there exists: an edge from $x$ to every member of $U_1$ but no edge from $U_1$ to $x$, an edge from every member of $U_2$ to $x$ but no edge from $x$ to $U_2$, an edge from $x$ to every member of $U_3$ and from every member of $U_3$ to $x$ and finally no edge between $x$ and $U_4$. More succinctly, $\Gamma$ is existentially closed if the following conditions hold for some vertex $x \in V_\Gamma \setminus \bigcup_{i=1}^{4} U_i$.

(i) $(x, u) \in E_\Gamma$ and $(u, x) \notin E_\Gamma$ for all $u \in U_1$,

(ii) $(x, u) \notin E_\Gamma$ and $(u, x) \in E_\Gamma$ for all $u \in U_2$,

(iii) $(x, u), (u, x) \in E_\Gamma$ for all $u \in U_3$, and

(iv) $(x, u), (u, x) \notin E_\Gamma$ for all $u \in U_4$,

Unless otherwise stated, a directed graph which is said to be existentially closed should be assumed to be existentially closed in the class of directed graphs. Clearly, a directed graph which is existentially closed must be infinite by condition (i). Existentially closed directed graphs also have the following property.

**Theorem 4.1.** *Let $\Gamma$ be an existentially closed directed graph. Then $\Gamma$ is homogeneous and every finite directed graph can be embedded into $\Gamma$.*

For a proof, see for example [Hod97]. Alternatively, a proof will follow easily from the construction described in Definition 4.3. Theorem 4.1 tells us that the age of any existentially closed directed graph is exactly the class of finite directed graphs. Since this class has a unique homogeneous Fraïssé Limit, we can thus conclude that if $\Gamma$ is any existentially closed directed graph, then $\Gamma \cong D$.

In fact, we can theoretically (and probabilistically) construct a countable existentially closed directed graph in a similar manner to that exhibited for the random graph by Erdős and Rényi in [ER63].

**Theorem 4.2.** *Let $\Lambda = (V_\Lambda, E_\Lambda)$ be a countable directed graph with vertices $V_\Lambda = \{v_i : i \in \mathbb{N}\}$ and edge set $E_\Lambda$ constructed by selecting edges independently, with probability $\frac{1}{2}$, from the set $V_\Lambda \times V_\Lambda$. Then with probability 1, $\Lambda$ is existentially closed and hence $\Lambda \cong D$.*

*Proof.* Let $U_1, \ldots, U_4$ be finite and disjoint subsets of $V_\Lambda$ and let $|U_i| = n_i$ for $1 \leq i \leq 4$. We will say that a vertex $x \in V_\Lambda \setminus \bigcup_{i=1}^4 U_i$ is *joined correctly* (to $U_1, U_2, U_3$ and $U_4$) if $x$ satisfies the conditions (i)–(iv) above. We will show that with probability 1, such a vertex exists. Given $u \in U_1$ the probability that $(x, u) \in E_\Lambda$ is $\frac{1}{2}$ and the probability that $(u, x) \notin E_\Lambda$ is also $\frac{1}{2}$. Since these probabilities are independent, the probability that both $(x, u) \in E_\Lambda$ and $(u, x) \notin E_\Lambda$ is $\frac{1}{4}$. Similar arguments can be made for the sets $U_2$, $U_3$ and $U_4$. Consequentially, the probability that $x \in V_\Lambda \setminus \bigcup_{i=1}^4 U_i$ is not joined correctly is

$$1 - \left(\frac{1}{4}\right)^{\sum_{i=1}^4 n_i}.$$

Also, the event that a vertex $x$ is not joined correctly is independent from the event that a distinct vertex $y$ is not joined correctly. Now since the set

$V_\Lambda$ is infinite, the probability that no vertex of $V_\Lambda \setminus \bigcup_{i=1}^4 U_i$ is joined correctly is

$$\lim_{k\to\infty} \left(1 - \left(\frac{1}{4}\right)^{\sum_{i=1}^4 n_i}\right)^k = 0.$$

Since there are only countably many choices for the subsets $U_1, \ldots, U_4$ it follows that the probability that $\Lambda$ is not existentially closed is 0. Thus, $\Lambda$ is existentially closed with probability 1. $\square$

Conveniently, we can also exhibit a standard and explicit construction of the random directed graph as follows. We will use this construction throughout this chapter.

**Definition 4.3.** Let $\Gamma = (V_\Gamma, E_\Gamma)$ be any countable directed graph. We will construct a new graph $\mathcal{H}(\Gamma)$ by the addition of vertices and edges to $\Gamma$. To do this, we will consider ordered triples of subsets $(S, T, U)$, where $S, T, U \subseteq V_\Gamma$ are finite and mutually disjoint. We allow the possibility that one or more of the subsets are empty. If $\Gamma$ is countable then the set of all finite sets of $V_\Gamma$ is countable and so the set of distinct triples $(S, T, U)$ is also countable. Thus we can enumerate all such distinct triples of $V_\Gamma$ as $(S_i, T_i, U_i)_{i\in\mathbb{N}}$, where the natural numbers are replaced by a finite set wherever necessary. For each such ordered triple, we add a vertex $v_i$ and edges from $v_i$ to every vertex in $S_i$ and $U_i$ and from every vertex in $T_i$ and $U_i$ to $v_i$. That is, $\mathcal{H}(\Gamma)$ is the directed graph formed by letting

$$V_{\mathcal{H}(\Gamma)} = V_\Gamma \cup \{v_i : i \in \mathbb{N}\}$$

and

$$E_{\mathcal{H}(\Gamma)} = E_\Gamma \cup \{(v_i, s), (t, v_i), (v_i, u), (u, v_i) : s \in S_i, t \in T_i, u \in U_i, i \in \mathbb{N}\}.$$

If $\Gamma$ is a finite digraph then $|V_{\mathcal{H}(\Gamma)}| = 4^{|V_\Gamma|} + |V_\Gamma|$ so that $\mathcal{H}(\Gamma)$ is a finite graph. If $\Gamma$ is countably infinite then $V_{\mathcal{H}(\Gamma)}$ is a countable union of countably infinite sets and hence $\mathcal{H}(\Gamma)$ is a countably infinite directed graph.

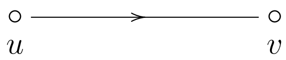
Now that we have described the construction of $\mathcal{H}(\Gamma)$, we can inductively define a sequence of graphs by setting $\Gamma_0 = \Gamma$ and $\Gamma_{n+1} = \mathcal{H}(\Gamma_n)$ for $n \in \mathbb{N}$. Define $\Gamma_\infty$ be the limit of this process in the sense that,

$$\Gamma_\infty = \bigcup_{n\in\mathbb{N}} \Gamma_n = \left(\bigcup_{n\in\mathbb{N}} V_{\Gamma_n}, \bigcup_{n\in\mathbb{N}} E_{\Gamma_n}\right).$$

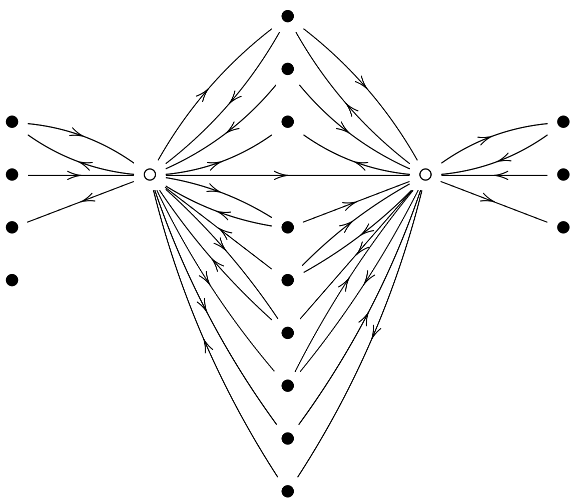
It should be easy to see that $\Gamma_\infty$ is a countable digraph since the vertex set is a union of countably many countable sets and since the edge set consists of a union of irreflexive binary relations and is therefore irreflexive.

**Example 4.4.** [Construction of $\Gamma_1$ when $\Gamma = (\{u, v\}, (u, v))$.]

$\Gamma = \Gamma_0$




$\Gamma_1$



**Lemma 4.5.** *Let $\Gamma$ be a countable digraph. Then $\Gamma_\infty$ is existentially closed and hence $\Gamma_\infty \cong D$.*

*Proof.* Let $U_1, U_2, U_3$ and $U_4$ be finite and disjoint subsets of $V_{\Gamma_\infty}$. Then $U_1, U_2, U_3$ and $U_4$ lie in $\Gamma_k$ for some $k \in \mathbb{N}$. Thus the since triple $\{U_1, U_2, U_3\}$ lies in $\Gamma_k$, the construction of $\Gamma_{k+1}$ guarantees that there exists a vertex $x \in V_{\Gamma_{k+1}} \setminus V_{\Gamma_k}$ such that:

$(x, u), (u, x) \in E_\Gamma$ for all $u \in U_1$,

$(x, u) \in E_\Gamma$ for all $u \in U_2$, and

$(u, x) \in E_\Gamma$ for all $u \in U_3$.

Moreover, these are the only edges between $x$ and the sets $U_1, \ldots, U_4$ in $\Gamma_{k+1}$. Thus, the existential closure property holds for the subsets $U_1, \ldots, U_4$ in $\Gamma_{k+1}$. Since the construction process makes no changes to the edge set of $\Gamma_{k+1}$ it follows that the existential closure property holds for the subsets $U_1, \ldots, U_4$ in $\Gamma_\infty$ and hence $\Gamma_\infty$ is existentially closed. $\square$

The construction of $\Gamma_\infty$ for any finite directed graph $\Gamma$ should make it clear that any directed graph can be embedded into $D$ as claimed in Theorem 4.1.

## 4.2 Group $\mathscr{H}$-classes of End($D$)

An application of Theorem 2.7 in this setting allows the deduction that the group $\mathscr{H}$-classes of $D$ are isomorphic to the automorphism groups of the directed subgraphs of $D$ induced by images of idempotents. We will see that the directed subgraphs of $D$ arising in this way can be characterised in a similar fashion to the result for graphs given in [BD00].

**Definition 4.6.** A directed graph $\Gamma = (V_\Gamma, E_\Gamma)$ is said to be *algebraically closed* (in the class of directed graphs) if for any finite set $U \subseteq V_\Gamma$, there exists a vertex $x \in V_\Gamma$ such that, $(x,u),(u,x) \in E_\Gamma$ for all $u \in U$.

Notice that the condition on the set $U$ above is exactly the definition of algebraic closure given for the class of graphs in the previous chapter. Consequently, it should be easy to see that any graph which is algebraically closed in the class of graphs is also algebraically closed in the class of directed graphs. Alternately, any graph which is algebraically closed in the class of directed graphs is algebraically closed in the class of graphs. But, of course, a directed graph which is algebraically closed in the class of directed graphs is not necessarily a graph and therefore not necessarily an algebraically closed graph. The definition of algebraic closure for directed graphs is thus consistent with the definition of algebraic closure for graphs. Consequently a directed graph can be called algebraically closed (in the class of directed graphs) for the remainder of this chapter with no ambiguity. This will be important for much of the work in this chapter.

It is not hard to see that an algebraically closed directed graph must then be infinite. This follows directly from the same reasoning as for algebraically closed graphs. Similarly we can easily show that if $\Gamma$ is an algebraically closed directed graph then for each finite subset $U \subseteq V_\Gamma$ there actually exist infinitely many vertices $x \in V_\Gamma \setminus U$ such that $(x,u),(u,x) \in E_\Gamma$ for all $u \in U$. Of course, the random directed graph $D$ is an obvious example of an algebraically closed graph since it is existentially closed. For if $W \subseteq V_D$ is any finite subset, let $U_1 = \emptyset$, $U_2 = \emptyset$, $U_3 = W$ and $U_4 = \emptyset$. Then by existential closure there exists a vertex $x \in V_D$ such that $(x,u),(u,x) \in E_\Gamma$ for all $u \in W$.

**Lemma 4.7.** *Let $\Gamma$ be an algebraically closed directed graph and let $f \in \operatorname{End}(\Gamma)$. Then $\operatorname{im} f$ is an algebraically closed directed graph.*

*Proof.* The proof is identical to the argument given for Lemma 3.7 when the statement '$y$ is adjacent to $z$' is replaced by the equivalent statement that $(y, z), (z, y) \in E_\Gamma$, wherever it appears. $\square$

**Corollary 4.8.** *Let $f \in \operatorname{End}(D)$. Then $\operatorname{im} f$ is a countable algebraically closed directed graph.*

*Proof.* It is immediate that $\operatorname{im} f$ is countable since $D$ is a countable directed graph. Now since $D$ is algebraically closed, it follows by Lemma 4.7 that $\operatorname{im} f$ is an algebraically closed directed graph. $\square$

**Lemma 4.9.** *Let $\Gamma$ be a countable directed graph and let $f : \Gamma \to \Gamma$ be a homomorphism such that $\operatorname{im} f$ is algebraically closed. Let $\mathcal{H}(\Gamma)$ be the directed graph formed from $\Gamma$ as in Definition 4.3. Then there exist $2^{\aleph_0}$ extensions $\tilde{f} : \mathcal{H}(\Gamma) \to \mathcal{H}(\Gamma)$ of $f$ such that $\tilde{f}$ is a homomorphism and $\operatorname{im} \tilde{f} = \operatorname{im} f$. Furthermore, if $f$ is idempotent then so is each $\tilde{f}$.*

*Proof.* Since $\operatorname{im} f$ is algebraically closed, $\Gamma$ is countably infinite. So let us enumerate the vertices of $\mathcal{H}(\Gamma) \setminus \Gamma$ as $\{v_i : i \in \mathbb{N}\}$. Then, by construction, for each vertex $v_i$ there exist disjoint sets $S_i, T_i, U_i \subseteq V_\Gamma$ such that $(v_i, s), (t, v_i), (v_i, u), (u, v_i) \in E_{\mathcal{H}(\Gamma)}$ for all $s \in S_i$, $t \in T_i$ and $u \in U_i$. Moreover, there exist no other edges between $v_i$ and $V_{\mathcal{H}(\Gamma)}$. We will inductively define a sequence of maps $f_j : \langle \Gamma \cup \{v_1, \ldots, v_j\} \rangle \to \mathcal{H}(\Gamma)$ as follows.

Let $f_0 = f$ and suppose that for $n \in \mathbb{N}$ we can extend $f$ to a homomorphism $f_n : \langle \Gamma \cup \{v_0, \ldots, v_n\} \rangle \to \mathcal{H}(\Gamma)$ with $\operatorname{im} f_n = \operatorname{im} f$. Since $\operatorname{im} f$ is algebraically closed there exists a vertex $w \in \operatorname{im} f \setminus \{v_0 f, \ldots, v_n f\}$ such that $(w, u), (u, w) \in E_{\mathcal{H}(\Gamma)}$ for all $u \in (S_{n+1} \cup T_{n+1} \cup U_{n+1}) f$. Indeed we can ensure that $w \neq v_i f$ for $i = 1, \ldots, n$ by insisting that there are edges to and from $w$ to every member of $(S_{n+1} \cup T_{n+1} \cup U_{n+1}) f \cup \{v_0 f, \ldots, v_n f\}$. Now define $f_{n+1} : \langle \Gamma \cup \{v_0, \ldots, v_{n+1}\} \rangle \to \mathcal{H}(\Gamma)$ by,

$$v f_{n+1} = \begin{cases} v f_n & \text{if } v \in \Gamma \cup \{v_0, ..., v_n\}, \\ w & \text{if } v = v_{n+1}. \end{cases}$$

Evidently, $f_{n+1}$ defines a map of vertices $V_\Gamma \cup \{v_0, \ldots, v_{n+1}\} \to V_{\mathcal{H}(G)}$. We must check that it is a graph homomorphism. Since $f_n$ is a graph homomorphism, it follows that $f_{n+1}|_{\operatorname{dom} f_n}$ is a graph homomorphism. Now suppose that $(v_{n+1}, y) \in E_{\mathcal{H}(G)}$ then by the observations we made at the start of the proof, $y$ lies in either $S_{n+1}$ or $U_{n+1}$. Thus, by choice of $w$,

$(v_{n+1}f_{n+1}, yf) = (w, yf) \in E_{\mathcal{H}(G)}$. Similarly if $(z, v_{n+1}) \in E_{\mathcal{H}(G)}$ then $z$ lies in either $T_{n+1}$ or $U_{n+1}$ and we can again deduce that $(zf, v_{n+1}f) \in E_{\mathcal{H}(G)}$. Thus, $f_{n+1}$ defines a graph homomorphism $\langle \Gamma \cup \{v_0, \ldots, v_{n+1}\}\rangle \to \mathcal{H}(\Gamma)$. Since $f_{n+1}$ is exactly $f_n$ when restricted to the domain of $f_n$ and since $v_{n+1}f \in \operatorname{im} f$ it is guaranteed that $\operatorname{im} f_{n+1} = \operatorname{im} f$. Additionally, if $f_n$ is idempotent, then so is $f_n$ since $w \in \operatorname{im} f_n$ and thus,

$$v_{n+1}f_{n+1}^2 = wf_{n+1} = wf_n = w = v_{n+1}f_{n+1}.$$

Now let

$$\tilde{f} = \bigcup_{n=0}^{\infty} f_n.$$

Then $\tilde{f}$ is a graph homomorphism $\mathcal{H}(\Gamma) \to \mathcal{H}(\Gamma)$ extending $f_n$ for all $n \in \mathbb{N}$. If $f$ is idempotent then each $f_n$ is idempotent and consequently so is $\tilde{f}$. Furthermore, since $\operatorname{im} f_n = \operatorname{im} f$ for all $n \in \mathbb{N}$, it follows that $\operatorname{im} \tilde{f} = \operatorname{im} f$.

Finally, since $\operatorname{im} f$ is algebraically closed, for each $n \in \mathbb{N}$ there are actually infinitely many choices for the vertex $w \in \operatorname{im} f$ with $(w, u), (u, w) \in E_{\mathcal{H}(\Gamma)}$ for all $u \in (S_{n+1} \cup T_{n+1} \cup U_{n+1})f$. In other words, there are infinitely many choices for the image of $v_{n+1}$ when constructing $f_{n+1}$. Since $v_{n+1}$ is not adjacent to $v_m$ for all $m \in \mathbb{N}$ the choice of vertex made for $v_{n+1}f_{n+1}$ is independent from any $v_m f_m$ chosen for $m \leq n$. Consequently, for each $n \in \mathbb{N}$, there are infinitely many distinct extensions $f_{n+1}$ of $f_n$ which differ on $v_{n+1}$. It follows now that there are ${\aleph_0}^{\aleph_0} = 2^{\aleph_0}$ distinct extensions $\tilde{f}$ of $f$. $\square$

**Theorem 4.10.** *Let $\Gamma$ be a directed graph. Then there exists an idempotent $f \in \operatorname{End}(D)$ with $\operatorname{im} f \cong \Gamma$ if and only if $\Gamma$ is a countable algebraically closed directed graph. Furthermore, for every algebraically closed directed graph $\Gamma$, there exist $2^{\aleph_0}$ idempotents $f \in \operatorname{End}(D)$ with $\operatorname{im} f \cong \Gamma$.*

*Proof.* If $f \in E(\operatorname{End}(D))$, then by Corollary 4.8, $\operatorname{im} f$ is an algebraically closed directed graph.

Conversely suppose that $\Gamma$ is an algebraically closed directed graph. From $\Gamma$, construct $\Gamma_\infty = \bigcup_{n\in\mathbb{N}} \Gamma_n$ as described in Definition 4.3. By Lemma 4.5, we can assume that $\Gamma_\infty = D$. Now, define inductively a sequence of functions $f_n : \Gamma_n \to \Gamma_\infty$ in the following way. We let $f_0 : \Gamma_0 \to \Gamma_\infty$ be the identity function on $\Gamma$. That is, we set $vf_0 = v$ for all $v \in V_\Gamma$. Then $f_0$ is an idempotent directed graph homomorphism such that $\operatorname{im} f = \Gamma$. As $\Gamma$ is algebraically closed we can apply Lemma 4.9. Thus, for $n \in \mathbb{N}$, we define $f_{n+1} = \tilde{f}_n$, where $\tilde{f}_n$ is an extension of $f_n$ to $\mathcal{H}(\Gamma_n) = \Gamma_{n+1}$ constructed in

Lemma 4.9. The proof of the lemma ensures that $f_{n+1}$ is idempotent and that $\operatorname{im} f_{n+1} = \operatorname{im} f = \Gamma$.

Let
$$f = \bigcup_{n=0}^{\infty} f_n.$$
Then as a union of idempotent directed graph homomorphisms such that $f_{n+1}$ is an extension of $f_n$, for all $n \in \mathbb{N}$, $f$ is an idempotent directed graph homomorphism $\Gamma_\infty \to \Gamma_\infty$. Moreover, since $\operatorname{im} f_n = \Gamma$ for all $n \in \mathbb{N}$, $\operatorname{im} f = \Gamma$.

Finally, since Lemma 4.9 tells us that there are $2^{\aleph_0}$ distinct extensions $\tilde{f}_n$ of $f_n$, it follows that there are $2^{\aleph_0}$ distinct extensions $f_{n+1}$ of $f_n$ for all $n \in \mathbb{N}$. Hence there are $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ many distinct idempotents $f \in \operatorname{End}(D)$ with $\operatorname{im} f = \Gamma$. □

The group $\mathscr{H}$-classes of $\operatorname{End}(D)$ are thus exactly the automorphism groups of algebraically closed directed graphs. Interestingly, since the random graph $R$ is an example of an algebraically closed directed graph, Theorem 4.10 guarantees that $\operatorname{Aut}(R)$ appears as a maximal subgroup of $\operatorname{End}(D)$. But what other groups can be realised as the automorphism group of an algebraically closed directed graph?

In Chapter 3 we showed that if a group arose as the automorphism group of a countable graph, then in fact there are $2^{\aleph_0}$ non-isomorphic algebraically closed graphs with the same automorphism group. Thus showing that any automorphism group of a countable graph is isomorphic to $2^{\aleph_0}$ distinct group $\mathscr{H}$-class of $\operatorname{End}(R)$. Since every algebraically closed graph is an algebraically closed directed graph, the following theorems and corollaries can be deduced almost immediately.

**Theorem 4.11.** *Let $\Gamma$ be a countable graph. Then there exist $2^{\aleph_0}$ group $\mathscr{H}$-classes $H$ of $\operatorname{End}(D)$ such that $H \cong \operatorname{Aut}(\Gamma)$.*

*Proof.* Since $\Gamma$ is a countable graph, Lemmas 3.22 and 3.24 tell us that there are $2^{\aleph_0}$ non-isomorphic algebraically closed (symmetric) digraphs $\Delta_\Sigma$, ($\Sigma \subseteq \mathbb{N} \setminus \{0, 1\}$) such that $\operatorname{Aut}(\Delta_\Sigma) \cong \operatorname{Aut}(\Gamma)$. By Theorem 4.10, for each of these digraphs there exists an idempotent $f_\Sigma \in \operatorname{End}(D)$ such that $\operatorname{im} f_\Sigma \cong \Delta_\Gamma$. Theorem 2.7 now ensures that
$$H_{f_\Sigma} \cong \operatorname{Aut}(\operatorname{im} f_\Sigma) \cong \operatorname{Aut}(\Delta_\Sigma) \cong \operatorname{Aut}(\Gamma).$$

Furthermore, since the $\Delta_\Sigma$'s are all non-isomorphic the idempotents $f_\Sigma$ are distinct. Since no $\mathscr{H}$-class can contain more than one idempotent, the result now follows. $\square$

**Corollary 4.12.** *There exist $2^{\aleph_0}$ group $\mathscr{H}$-classes $H$ of* $\mathrm{End}(D)$ *such that* $H \cong \mathrm{Aut}(R)$.

*Proof.* Since the random graph $R$ is an algebraically closed (symmetric) directed graph, the result follows immediately by Theorem 4.11. $\square$

**Corollary 4.13.** *Let $G$ be a countable group. Then there exist $2^{\aleph_0}$ group $\mathscr{H}$-classes $H$ of* $\mathrm{End}(D)$ *such that $H \cong G$.*

*Proof.* By Theorem 3.12, for any countable group $G$ there exists a countable (symmetric) directed graph $\Gamma$ such that $G \cong \mathrm{Aut}(\Gamma)$. Hence by Theorem 4.11 the result follows. $\square$

**Corollary 4.14.** *There exist $2^{\aleph_0}$ non-isomorphic maximal subgroups of* $\mathrm{End}(D)$.

*Proof.* In the proof of Corollary 3.27 we showed that there exist $2^{\aleph_0}$ non-isomorphic groups. The result now follows from Corollary 4.13. $\square$

So far we have shown that the groups which arise as maximal subgroups of $\mathrm{End}(R)$ also appear as maximal subgroups of $\mathrm{End}(D)$ to the same cardinality of repetition. It might, however seem plausible that there exist maximal subgroups of $\mathrm{End}(D)$ which cannot be found in $\mathrm{End}(R)$. In other words, that there exists a group which arises as the automorphism group of an algebraically closed directed graph, but cannot be realised as the group of an algebraically closed graph. We will show that this is not possible through an application of the following construction and accompanying lemmas.

Let $\Gamma$ be a countable directed graph. We will construct a graph $\Gamma^{\dashv}$ as follows. Enumerate the vertices of $\Gamma$ as $V_\Gamma = \{v_i : i \in \mathbb{N}\}$, where we replace $\mathbb{N}$ with a finite set if $\Gamma$ is finite. We then let

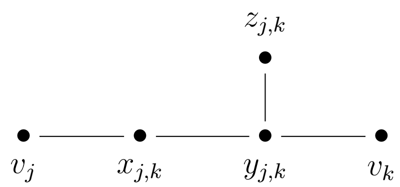
$$V_{\Gamma^{\dashv}} = V_\Gamma \cup \{x_{j,k}, y_{j,k}, z_{j,k} : (v_j, v_k) \in E_\Gamma\}$$

and we let

$$\begin{aligned} E_{\Gamma^{\dashv}} = \{&(v_j, x_{j,k}), (x_{j,k}, v_j), (x_{j,k}, y_{j,k}), (y_{j,k}, x_{j,k}), \\ &(y_{j,k}, z_{j,k}), (z_{j,k}, y_{j,k}), (y_{j,k}, v_k), (v_k, y_{j,k}) : (v_j, v_k) \in E_\Gamma\} \end{aligned}$$

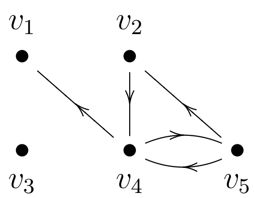
Intuitively we can think of the construction of $\Gamma^{\dashv}$ as taking the directed graph $\Gamma$ and for each pair of vertices $u, v \in V_\Gamma$ with an edge from $u$ to $v$, replacing the edge with a finite graph which, in some sense, still retains some information about the direction of the original edge between $u$ and $v$. See Figure 4.1 for a pictorial representation.

Figure 4.1: The finite graph replacing an edge $(v_j, v_k)$.

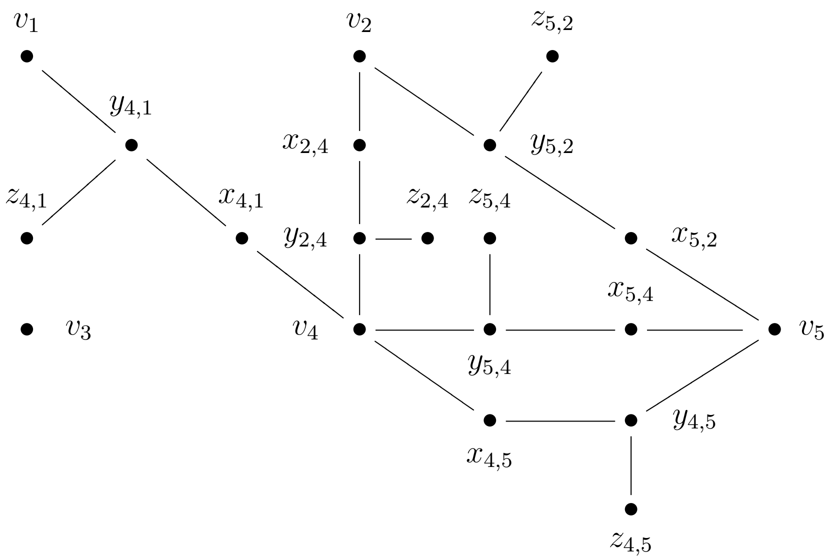



**Example 4.15.** [Construction of the graph $\Gamma^{\dashv}$ given a directed graph $\Gamma$]

$\Gamma$




$\Gamma^{\dashv}$



It is important to note that the degree of a vertex $v \in V_\Gamma$, in $\Gamma^{\dashv}$, is equal to sum of the number of edges which start at $v$ and the number of edges which end at $v$ in $\Gamma$. For a directed graph $\Lambda$ and $v \in V_\Lambda$ we will let,

$$\begin{aligned}\Lambda_-(v) =&\{x : (v, x) \in E_\Lambda\}\\ \Lambda_+(v) =&\{x : (x, v) \in E_\Lambda\}.\end{aligned}$$

Thus, if $\Lambda$ is a directed graph then $|\Lambda^{\dashv}(v)| = |\Lambda_-(v)| + |\Lambda_+(v)|$ for $v \in V_\Lambda$. $|\Lambda_-(v)|$ is often called the *out degree* of $v$, and $|\Lambda_+(v)|$ the *in degree* of $v$. Of course, if $\Lambda$ is a graph then $|\Lambda(v)| = |\Lambda_-(v)| = |\Lambda_+(v)|$.

**Lemma 4.16.** *Let $\Gamma$ be a directed graph. Suppose that $|\Gamma_-(v)| + |\Gamma_+(v)| > 3$ for all $v \in V_\Gamma$. Then $\mathrm{Aut}(\Gamma^{\dashv}) \cong \mathrm{Aut}(\Gamma)$.*

*Proof.* Consider the vertex set of $V_{\Gamma^{\dashv}}$. In $\Gamma^{\dashv}$, every vertex in the set $X = \{x_{jk} : (v_j, v_k) \in E_\Gamma\}$ has degree 2, every vertex in the set $Y = \{y_{jk} : (v_j, v_k) \in E_\Gamma\}$ has degree 3 and every vertex in the set $Z = \{z_{jk} : (v_j, v_k) \in E_\Gamma\}$ has degree 1. Also, by assumption the degree (in $\Gamma^{\dashv}$) of every $v \in V_\Gamma$ is greater than three. Thus if $f \in \mathrm{Aut}(\Gamma^{\dashv})$, then $Xf = X$, $Yf = Y$, $Zf = Z$ and $V_\Gamma f = V_\Gamma$. Now define a map $\phi : \mathrm{Aut}(\Gamma^{\dashv}) \to \mathrm{Aut}(\Gamma)$ by $f\phi = f|_{V_\Gamma}$ for all $f \in \mathrm{Aut}(\Gamma^{\dashv})$. First we check the map is well defined. By the previous observations, $f|_{V_\Gamma}$ is a bijective map $V_\Gamma \to V_\Gamma$. To see that $f|_{V_\Gamma}$ defines a graph homomorphism we note that by construction of $\Gamma^{\dashv}$, $(u, v) \in E_\Gamma$ if and only if there exists $x \in X$, $y \in Y$ such that $(u, x, )(x, y), (y, v) \in E_{\Gamma^{\dashv}}$. Thus if $(u, v) \in E_\Gamma$ then there exists $x \in X$, $y \in Y$ such that $(u, x), (x, y), (y, v) \in E_{\Gamma^{\dashv}}$. Now, since $f \in \mathrm{Aut}(\Gamma^{\dashv})$ it follows that $(uf, xf), (xf, yf), (yf, vf) \in E_{\Gamma^{\dashv}}$. By our previous observations $xf \in X$ and $yf \in Y$ and we can now deduce that $(uf, vf) = (uf|_{V_\Gamma}, vf|_{V_\Gamma}) \in V_\Gamma$. A similar argument shows that $(u, v) \notin E_\Gamma$ implies that $(uf|_{V_\Gamma}, vf|_{V_\Gamma}) \notin E_\Gamma$ and hence completes the proof that $f\phi$ is a graph automorphism. The map $\phi$ is a group homomorphism since clearly if $f, g \in \mathrm{Aut}(\Gamma^{\dashv})$ then

$$(fg)\phi = (fg)|_{V_\Gamma} = f|_{V_\Gamma} \cdot g|_{V_\Gamma} = f\phi \cdot g\phi.$$

The map $\phi$ is clearly injective since if $f\phi = g\phi$ then $f|_{V_\Gamma} = g|_{V_\Gamma}$. Now since the images of $x_{jk}$, $y_{jk}$ and $z_{jk}$, for $(v_j, v_k) \in E_\Gamma$, are determined completely by the image of $v_i$ and $v_j$ under $f$, we can conclude that $f = g$. The map $\phi$ is surjective since if we are given $h \in \mathrm{Aut}(\Gamma)$ then we can extend $h$ to an automorphism $\tilde{h}$ of $\mathrm{Aut}(\Gamma^{\dashv})$ by defining $x_{jk}\tilde{h} = x_{mn}$, $y_{jk}\tilde{h} = y_{mn}$ and $z_{jk}\tilde{h} = z_{mn}$ where $v_j h = v_m$ and $v_k h = v_n$. Now, since $\tilde{h}$ is an extension of $h$, $\tilde{h}\phi = h$ and we are finished. $\square$

**Theorem 4.17.** *Let $\Gamma$ be a countable directed graph. Then there exists a countable graph $\Lambda$ such that $\mathrm{Aut}(\Lambda) \cong \mathrm{Aut}(\Gamma)$.*

*Proof.* Consider the (directed) graphs $L_\Sigma$, $\Sigma \subseteq \mathbb{N} \setminus \{0,1\}$, described in Definition 3.18. Since $\Gamma$ is a countable digraph, there exists a set $\Sigma$ such that $L_\Sigma$ is isomorphic to no component of $\Gamma$. By Corollary 3.15, $\mathrm{Aut}((\Gamma \dot\cup L_\Sigma)^\dagger) \cong \mathrm{Aut}(\Gamma)$ and moreover, every vertex in $(\Gamma \dot\cup L_\Sigma)^\dagger$ has infinite in degree and infinite out degree. Now construct the graph $((\Gamma \dot\cup L_\Sigma)^\dagger)^\dashv$. Then by Lemma 4.16, $\mathrm{Aut}(((\Gamma \dot\cup L_\Sigma)^\dagger)^\dashv) \cong \mathrm{Aut}((\Gamma \dot\cup L_\Sigma)^\dagger) \cong \mathrm{Aut}(\Gamma)$. Taking $\Lambda = ((\Gamma \dot\cup L_\Sigma)^\dagger)^\dashv$ gives the required result. $\square$

As a direct consequence of Lemma 4.17, the following results can now be deduced.

**Corollary 4.18.** *Let $\Gamma$ be a countable directed graph. Then there exist $2^{\aleph_0}$ distinct $\mathscr{H}$-classes $H$ of* $\mathrm{End}(D)$ *such that $H \cong \mathrm{Aut}(\Gamma)$.*

*Proof.* Theorem 4.17 guarantees that there exists a countable graph $\Lambda$ such that $\mathrm{Aut}(\Lambda) \cong \mathrm{Aut}(\Gamma)$. Now by Theorem 4.11 the result follows. $\square$

In summary, Theorem 4.10 and Corollary 4.18 tell us that if $H$ is a maximal subgroup of $\mathrm{End}(D)$ then $H \cong \mathrm{Aut}(\Gamma)$ for a countable directed graph $\Gamma$ and conversely, if $\Lambda$ is a countable directed graph, then there exist $2^{\aleph_0}$ maximal subgroups of $\mathrm{End}(D)$ isomorphic to $\mathrm{Aut}(\Lambda)$. Furthermore, we are also able to deduce the following theorem.

**Theorem 4.19.** *The group $\mathscr{H}$-classes of* $\mathrm{End}(D)$ *are the same (up to isomorphism) as the group $\mathscr{H}$-classes of* $\mathrm{End}(R)$*.*

*Proof.* If $H$ is a group $\mathscr{H}$-class of $\mathrm{End}(R)$ then let $f \in E(\mathrm{End}(R))$ be the idempotent identity of the subgroup. We saw in the previous chapter that $\mathrm{im}\, f$ must be an algebraically closed graph. Thus by Theorem 4.11 there exists an idempotent $g \in \mathrm{End}(D)$ such that $H_g \cong \mathrm{Aut}(\mathrm{im}\, f) \cong H_f = H$.

Now suppose that $K$ is a group $\mathscr{H}$-class of $\mathrm{End}(D)$ then let $g \in \mathrm{End}(D)$ be the idempotent identity of the subgroup. Then by Theorem 4.10, $\mathrm{im}\, g$ is an algebraically closed directed graph. By Theorem 4.17 there exists a graph $\Gamma$ such that $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathrm{im}\, g)$. Now by Theorem 3.25 there exists an idempotent $f \in \mathrm{End}(R)$ such that $H_f \cong \mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathrm{im}\, f) \cong H_g = K$. $\square$

## 4.3 Regular $\mathscr{D}$-classes and $\mathscr{J}$-classes of $\mathrm{End}(D)$

We can also obtain analogous results about the regular $\mathscr{D}$-classes and $\mathscr{J}$-classes of $\mathrm{End}(D)$.

**Corollary 4.20.** *There are* $2^{\aleph_0}$ *distinct regular* $\mathscr{D}$*-classes of* $\operatorname{End}(D)$ *such that no two group* $\mathscr{H}$*-classes from distinct* $\mathscr{D}$*-classes are isomorphic.*

*Proof.* By Corollary 4.14 there exist $2^{\aleph_0}$ non-isomorphic group $\mathscr{H}$-classes of $\operatorname{End}(D)$. Since each of these must lie in a distinct regular $\mathscr{D}$-class the result follows. $\square$

**Corollary 4.21.** *There are* $2^{\aleph_0}$ *distinct regular* $\mathscr{D}$*-classes of* $\operatorname{End}(D)$ *whose group* $\mathscr{H}$*-classes are all isomorphic.*

*Proof.* By the proof of Theorem 4.11, if $\Gamma$ is a (symmetric) directed graph then there exist $2^{\aleph_0}$ idempotents $f_\Sigma \in E\operatorname{End}(D)$ such that $H_{f_\Sigma} \cong \operatorname{Aut}(\Gamma)$ but such that $\langle \operatorname{im} f_\Sigma \rangle \ncong \langle \operatorname{im} f_\Psi \rangle$ for $\Sigma \neq \Psi$. Hence by Theorem 2.10, each $f_\Sigma$ is contained in a distinct regular $\mathscr{D}$-class but the $H_{f_\Sigma}$ are all isomorphic to $\operatorname{Aut}(\Gamma)$. Since every group $\mathscr{H}$-class contained in one of these $\mathscr{D}$-classes must also be isomorphic to $\operatorname{Aut}(\Gamma)$ the result is complete. $\square$

**Corollary 4.22.** *Each regular* $\mathscr{D}$*-class of* $\operatorname{End}(D)$ *contains* $2^{\aleph_0}$ *distinct group* $\mathscr{H}$*-classes.*

*Proof.* If a $\mathscr{D}$-class is regular, it contains at least one group $\mathscr{H}$-class. Let $f \in E(\operatorname{End}(D))$ be the identity of the group $\mathscr{H}$-class. Since $\operatorname{im} f$ is algebraically closed Theorem 4.10 guarantees the existence $2^{\aleph_0}$ distinct idempotents whose images induce subgraphs which are isomorphic to $\langle \operatorname{im} f \rangle$. By Theorem 2.10 these idempotents all lie in the same $\mathscr{D}$-class, but since no $\mathscr{H}$-class can contain more than one idempotent they lie in distinct group $\mathscr{H}$-classes. $\square$

**Corollary 4.23.** *There are* $2^{\aleph_0}$ *distinct* $\mathscr{J}$*-classes of* $\operatorname{End}(D)$ *.*

*Proof.* In the proof of 3.32 we saw that there exist $2^{\aleph_0}$ algebraically closed (symmetric) directed graphs which are mutually non-embeddable. By Theorems 4.10 and 2.11, there thus exist $2^{\aleph_0}$ idempotents in $\operatorname{End}(D)$ which are not $\mathscr{J}$-related and hence the result follows. $\square$

# Chapter 5

# The Random Tournament

In this chapter we will briefly discuss the random tournament and see why, in the context of this thesis, its endomorphism monoid is a somewhat less interesting structure.

## 5.1 Defining Properties and Constructions

Recall that a tournament is a directed graph in which for every pair of distinct vertices there exists exactly one edge between them (in one direction or the other). It is not hard to show that the class of finite tournaments has the hereditary, joint embedding and amalgamation properties. Therefore, the class of finite tournaments has a unique homogeneous Fraïssé limit which we will call *the random tournament*, $T$. We can show that $T$ has the following properties.

We will say that a tournament $\Gamma$ is *existentially closed in the class of tournaments* if for all finite subsets $U_1, U_2 \in V_\Gamma$ there exists a vertex $x \in V_\Gamma \setminus (U_1 \cup U_2)$ such that there exists an edge from $x$ to every vertex in $U_1$ and from every vertex in $U_2$ to $x$. For the remainder of this chapter a tournament which is said to be existentially closed should be assumed to be existentially closed in the class of tournaments.

Clearly any existentially closed tournament must be infinite, for if $\Gamma$ is a finite tournament then $V_\Gamma$ is a finite set for which there exists no vertex $x$ with an edge from $x$ to every member of $V_\Gamma$.

**Theorem 5.1.** *Let $\Gamma$ be an existentially closed tournament. Then $\Gamma$ is homogeneous and every finite tournament can be embedded into $\Gamma$.*

For a proof see for example [Hod97] or alternatively, the construction described in Definition 5.3 will make this clear. Theorem 5.1 tells us that the age of any existentially closed tournament $\Gamma$ is exactly the class of all finite tournaments. Since the class of finite tournaments has a unique homogeneous Fraïssé limit it follows that if $\Gamma$ is an existentially closed tournament then $\Gamma \cong T$. As one might expect, we can probabilistically carry out a construction of an existentially closed tournament as follows.

**Theorem 5.2.** *Let $\Lambda$ be a countable tournament constructed as follows. Let $V_\Lambda$ be a countably infinite set, and for any two distinct vertices $u, v \in V_\Lambda$ chose either $(u, v)$ or $(v, u)$ to be in the edge set (each with probability $\frac{1}{2}$) independently from any other pair of distinct vertices. Then with probability 1, $\Lambda$ is existentially closed.*

*Proof.* Let $U_1$ and $U_2$ be finite subsets of $V_\Lambda$. Suppose that $|U_1| = m$ and $|U_2| = n$ for $m, n \in \mathbb{N}$. We will say that a vertex $x \in V_\Lambda \setminus (U_1 \cup U_2)$ is joined correctly to $U_1$ and $U_2$ if there exists an edge from $x$ to every vertex of $U_1$ and an edge from every vertex of $U_2$ to $x$. The probability that a vertex $x$ is not joined correctly is

$$1 - \frac{1}{2^{m+n}}$$

and is independent from the probability that any other distinct vertex $y$ is not joined correctly. Now since $V_\Lambda$ is infinite, the probability that no vertex of $V_\Lambda \setminus (U_1 \cup U_2)$ is joined correctly to $U_1$ and $U_2$ is,

$$\lim_{k\to\infty} \left(1 - \frac{1}{2^{m+n}}\right)^k = 0.$$

Thus the probability that existential closure is not satisfied for the sets $U_1$ and $U_2$ is 0. Since there are only countably many choices for the sets $U_1$ and $U_2$ it follows that the probability that $\Lambda$ is not existentially closed is 0 and hence it is existentially closed with probability 1. $\square$

As in the other settings, there exists a standard explicit construction of the random tournament from any given tournament.

**Definition 5.3.** Starting with any countable tournament $\Gamma$ we can create a new tournament $\mathcal{J}(\Gamma)$ by the addition of vertices and edges. Since $\Gamma$ is countable we can enumerate the finite subsets of $V_\Gamma$ as $\{U_i\}_{i\in\mathbb{N}}$ where the natural numbers can be replaced by a finite set if $\Gamma$ is finite. Now for each finite set $U_i$ add a vertex $v_i$ and edges from $v_i$ to every vertex in $U_i$ and from every vertex in $V_\Gamma \setminus U_i$ to $v_i$. In order to make the resulting graph a tournament we need to have an edge between each pair $\{v_i, v_j\}$. The direction of these edges turns out to be irrelevant. So we let

$$V_{\mathcal{J}(\Gamma)} = V_\Gamma \cup \{v_i : i \in \mathbb{N}\},$$

and

$$\begin{aligned} E_{\mathcal{J}(\Gamma)} = E_\Gamma &\cup \{(v_i, u), (w, v_i) : u \in U_i, w \in V_\Gamma \setminus U_i\} \\ &\cup \{(v_i, v_j) : i, j \in \mathbb{N}, i < j\}. \end{aligned}$$

If $\Gamma$ is a finite graph then $|V_{\mathcal{J}(\Gamma)}| = 2^{|V_\Gamma|} + |V_\Gamma|$ and hence $\mathcal{J}(\Gamma)$ is also a finite tournament. If $\Gamma$ is in fact countably infinite, then since the set of all finite sets of $V_\Gamma$ is also countably infinite, $\mathcal{J}(\Gamma)$ is countably infinite itself.

Now inductively define a sequence of tournaments by setting $\Gamma_0 = \Gamma$ and $\Gamma_{n+1} = \mathcal{J}(\Gamma_n)$ for all $n \in \mathbb{N} \setminus \{0\}$. Let $\Gamma_\infty$ be the limit of this process so that,

$$\Gamma_\infty = \bigcup_{n \in \mathbb{N}} \Gamma_n = \left( \bigcup_{n \in \mathbb{N}} V_{\Gamma_n}, \bigcup_{n \in \mathbb{N}} E_{\Gamma_n} \right).$$

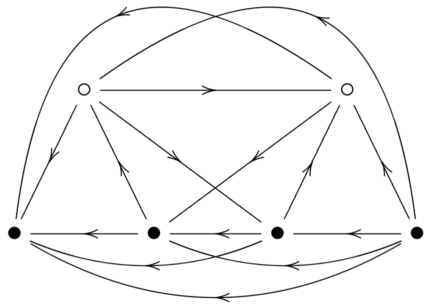
Since $\Gamma_\infty$ is a countable union of tournaments $\Gamma_n$ such that $\Gamma_{n-1}$ is contained in $\Gamma_n$ for all $n \in \mathbb{N} \setminus \{0\}$, it should be easy to see that $\Gamma_\infty$ is a countable tournament itself.

**Example 5.4.** [Construction of $\Gamma_1$ given $\Gamma$.]

$\Gamma_0 = \Gamma$



$\Gamma_1$



Since the construction of $\Gamma_\infty$ is dependent on the enumeration of the finite sets $U \subseteq V_\Gamma$, it may seem plausible that taking a different enumeration would give us a different (non-isomorphic) graph. However the following theorem proves that this is not true.

**Theorem 5.5.** *Let $\Gamma$ be a countable tournament. Then $\Gamma_\infty$ is existentially closed and thus $\Gamma_\infty \cong T$.*

*Proof.* Suppose that $U_1$ and $U_2$ are finite and disjoint subsets of $V_{\Gamma_\infty}$. Then $U_1, U_2 \subseteq \Gamma_k$ for some $k \in \mathbb{N}$. By construction of $\Gamma_{k+1}$ there exists a vertex $v \in V_{\Gamma_{k+1}} \setminus V_{\Gamma_k}$ such that there is an edge from $v$ to every vertex in $U_1$ and from every vertex in $V_\Gamma \setminus U_1$ to $v$. In particular this means that there is an edge from $v$ to every vertex in $U_1$ and from every vertex in $U_2$ to $v$ in $\Gamma_{k+1}$. Since $\Gamma_{k+1}$ is contained as an induced substructure of $\Gamma_\infty$ it follows that there is an edge from $v$ to every vertex in $U_1$ and from every vertex in $U_2$ to $v$ in $\Gamma_\infty$. Thus since $U_1$ and $U_2$ are arbitrary, $\Gamma_\infty$ is existentially closed. $\square$

The construction of $\Gamma_\infty$ from a countable tournament $\Gamma$ should make it clear that any finite tournament can be embedded into $T$.

## 5.2 Group $\mathscr{H}$-classes and Regular $\mathscr{D}$-classes of End($T$)

When considering endomorphisms of a tournament $\Gamma$ we note any endomorphism must be an embedding.

**Lemma 5.6.** *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a tournament and let $f \in \operatorname{End}(\Gamma)$. Then $f$ is an embedding of $\Gamma$ into $\Gamma$.*

*Proof.* Let $u, v \in V_\Gamma$ with $u \neq v$ and assume without loss of generality that $(u, v) \in E_\Gamma$. Now, since $f$ is an endomorphism of $\Gamma$, if $uf = vf$ then $(vf, vf) \in E_\Gamma$ which is a contradiction. Hence $uf \neq vf$ and $f$ is indeed injective. Furthermore if $(uf, vf) \notin E_\Gamma$ then $(vf, uf) \in E_\Gamma$ and so it must be the case that $(u, v) \notin E_\Gamma$ since otherwise we would have a contradiction. Thus it follows that if $(uf, vf) \notin E_\Gamma$ then $(u, v) \notin E_\Gamma$ and hence since $f$ was an injective endomorphism it is an embedding. $\square$

With Lemma 5.6 in mind, the following result is then of no surprise.

**Theorem 5.7.** *Let $f \in \operatorname{End}(T)$. Then $\operatorname{im} f$ is existentially closed and hence $\operatorname{im} f \cong T$.*

*Proof.* Let $U_1$ and $U_2$ be finite and disjoint subsets in $\operatorname{im} f$. Suppose that $|U_1| = m$ and $|U_2| = n$ for $m, n \in \mathbb{N}$. Enumerate $U_1$ as $\{u_i : 1 \leq i \leq m\}$ and $U_2$ as $\{u_j : m+1 \leq j \leq m+n\}$. Since $u_k \in \operatorname{im} f$ for all $1 \leq k \leq m+n$, there exist vertices $v_k$ such that $v_k f = u_k$. Let $V_1 = \{v_i : 1 \leq i \leq m\}$ and let $V_2 = \{v_j : m+1 \leq j \leq m+n\}$. Then $V_1$ and $V_2$ are finite disjoint subsets of $V_T$. Since $T$ is existentially closed it follows that there exists a vertex $x \in V_\Gamma \setminus (V_1 \cup V_2)$ such that there is an edge from $x$ to every member of $V_1$ and from every member of $V_2$ to $x$. Now since $f$ is an endomorphism it follows that $xf \in \operatorname{im} f \setminus (U_1 \cup U_2)$ and that there exists an edge from $xf$ to every member of $U_1$ and from every member of $U_2$ to $xf$. Since $U_1$ and $U_2$ were arbitrary the result is complete. $\square$

**Lemma 5.8.** *Let $f \in E(\mathrm{End}(T))$. Then $f = \mathbf{1}$.*

*Proof.* If $f \in E(\mathrm{End}(T))$ then $f|_{\operatorname{im} f} = \mathbf{1}|_{\operatorname{im} f}$. We also observed that $f$ must be an injective embedding. So suppose that $y \in V_T$. Then $yf = x$ for some $x \in \operatorname{im} f$. Since $f$ is idempotent $xf = x$ and hence by injectivity, $x = y$. Thus $y \in \operatorname{im} f$ and hence $V_T \setminus \operatorname{im} f = \emptyset$. Thus, $f = f|_{V_T} = \mathbf{1}_{V_T}$ as required. $\square$

Consequently, we now have the following results on the group $\mathscr{H}$-classes and regular $\mathscr{D}$-classes of $\mathrm{End}(T)$.

**Theorem 5.9.** *The only group $\mathscr{H}$-class of* $\mathrm{End}(T)$ *is* $\mathrm{Aut}(T)$.

*Proof.* Every group $\mathscr{H}$-class of $\mathrm{End}(T)$ contains an idempotent (the subgroup identity). By Lemma 5.8 the only such idempotent is the identity idempotent $\mathbf{1}$, and $H_{\mathbf{1}} = \mathrm{Aut}(T)$ as required. $\square$

**Corollary 5.10.** $\mathrm{End}(T)$ *has only one regular $\mathscr{D}$-class.*

*Proof.* Every regular $\mathscr{D}$-class contains at least one idempotent. Hence, since Lemma 5.8 told us that the only idempotent in $\mathrm{End}(T)$ is the identity, there can only be one regular $\mathscr{D}$-class. $\square$

We can now conclude that the only maximal subgroup of $\mathrm{End}(T)$ is $\mathrm{Aut}(T)$.

# Chapter 6

# Henson's Graphs

In this section we discuss $K_n$-free graphs for $n \geq 3$ and introduce Henson's graphs, $G_n$. We will see that, much like the random tournament $T$, the graphs $G_n$ are somewhat uninteresting in terms of maximal subgroups.

## 6.1 Defining Properties and Constructions

Recall that a $K_n$-free graph is a graph which has no substructure isomorphic to the graph $K_n$, the complete graph on $n$ vertices. It is not hard to show that for $n \geq 3$, the class of finite $K_n$-free graphs has the hereditary, joint embedding and amalgamation properties (see [Hen71], for example). Thus, the class of finite $K_n$-free graphs has a unique homogeneous Fraïssé limit, which is known as Henson's graph, $G_n$. We will show that $G_n$ can be characterised by the following property.

We will say that a $K_n$-free graph $\Gamma$ is *existentially closed* (in the class of $K_n$-free graphs) if for all finite and disjoint subsets $U, V \in V_\Gamma$ such that $\langle U \rangle$ is $K_{n-1}$-free, there exists a vertex $x \in V_\Gamma \setminus (U \cup V)$ such that $x$ is adjacent to every member of $U$ but to no member of $V$. We will assume for the rest of this chapter that whenever the phrase existentially closed is used for a $K_n$-free graph, we mean existentially closed in the class of $K_n$-free graphs.

It is not hard to see that an existentially closed $K_n$-free graphs must be infinite. For suppose that $\Gamma$ was a such a finite $K_n$-free graph. Then there would exists a maximal and finite $K_{n-1}$-free set of vertices $U$ from $V_\Gamma$. But by the existential closure property, there must exist a vertex $x \in V_\Gamma \setminus U$, such that $x$ is adjacent to no member of $U$. If $|U| = |V_\Gamma|$, then such an $x$ cannot exist. On the other hand if $|U| < |V_\Gamma|$ then $U \cup \{x\}$ is a $K_{n-1}$-free set which contradicts the maximality of $U$.

We can also show that any existentially closed $K_n$-free graph does not satisfy the bipartite condition as follows. For suppose that $\Gamma$ is an existentially closed $K_n$-free graph and let $v \in V_\Gamma$. Since $v$ by itself is an independent set, there must exist a vertex $x_1$ which is adjacent to $v$. Another application of the property ensures the existence of vertices $x_2$ and $x_3$ such that $x_2$ is adjacent to $v$ and not to $x_1$ and such that $x_3$ is adjacent to $x_1$ and not to $v$ nor $x_2$. Finally since $\{x_2, x_3\}$ is then an independent set by construction we can find a vertex $x_4$ adjacent to both $x_2$ and $x_3$. Then the path $(v, x_1), (x_1, x_3), (x_3, x_4), (x_4, x_2), (x_2, v)$ is a cycle of odd length and hence $\Gamma$ cannot satisfy the bipartite condition.

In some sense the existential closure property for $K_n$-free graphs is 'equivalent' to existential closure described for the class of all graphs but which holds only where possible (that is, avoiding the sets for which the property cannot hold due to the graph being $K_n$-free). Existentially closed $K_n$-free graphs also have the following additional property.

**Theorem 6.1.** *Let $\Gamma$ be a countable existentially closed $K_n$-free graphs for some $n \geq 3$. Then $\Gamma$ is homogeneous and every finite $K_n$-free graph can be embedded into $\Gamma$.*

A proof can be found in [Hen71, Theorem 2.3] or alternatively, the theorem will follow easily from the construction described in Definition 6.2. The age of an existentially closed $K_n$-free graph is thus the class of all finite $K_n$-free graphs. Since the class of all finite $K_n$-free graphs has a unique homogeneous Fraïssé limit, it follows that if $\Gamma$ is an existentially closed $K_n$-free graph, then $\Gamma \cong G_n$.

In the case of the random graph and random directed graph, we are able to exhibit a relatively easy probabilistic construction *via* a finitary method in which edges are chosen one at a time with a set probability. However, to exhibit a random or probabilistic construction of an infinite $K_n$-free graph is not so straightforward. For example, consider the triangle-free graph $G_3$. If we start with a countably infinite vertex set $V$ and attempt to construct a triangle-free graph by choosing edges (as symmetric pairs from $V \times V \setminus \{(x,x) : x \in V\}$) one at a time with probability $\frac{1}{2}$ say, then we quickly run in to trouble. For example, if we happen to have started the process by choosing the edge $(u, v)$ and then the edge $(v, w)$, we are not allowed to chose the edge $(u, w)$ in order to ensure the graph remains triangle-free. Clearly, the probability that the edge $(u, v)$ is chosen is $\frac{1}{2}$ and the probability that the edge $(v, w)$ is chosen is also $\frac{1}{2}$ and is independent from the choice of $(u, v)$. However the probability that the edge $(u, w)$ is chosen is dependent upon the choices for $(u, v)$ and $(v, w)$ and is thus (using the law of total probability) equal to $0 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} = \frac{3}{8}$. Furthermore, this type of procedure provides a graph which is dependent on the order in which we decide to choose edges.

As Cameron discusses in [Cam01, Section 4.10] an attempt to bypass this problem by constructing a random triangle-free graph in a finitary way which is not dependent on the particular ordering of the vertices, still does not have the desired outcome. Any such triangle-free graph which is constructed in this way satisfies, by a result of Erdős, Kleitman and Rothschild [EKR76], with probability 1 the bipartite condition, and thus cannot be the Henson graph $G_3$.

At the moment, it is unclear how to construct Henson's graph $G_3$ using a finitary probabilistic construction. However in [PV10, Section 3], Petrov and Vershik construct a measure-theoretic triangle-free graph on the real numbers and, by taking countably many independent samples from a probability distribution on the real numbers, use it to produce a graph which is isomorphic to the Henson graph $G_3$ with probability 1. In effect their method is probabilistic on vertices rather than edges. It has been conjectured that a consequence of some of the stronger results in this paper will show that a finitary probabilistic construction of Henson's graph $G_3$ (and indeed $G_n$ for $n > 3$) is not possible. It is for this reason that the Henson graph $G_n$ not normally said to be 'random' unlike its counterparts $R$, $D$ and $T$.

Even with that all said, we *can* give an explicit construction of Henson's graphs as follows.

**Definition 6.2.** Let $\Gamma$ be a countable $K_n$-free graph. We will construct a new graph $\mathcal{L}_n(\Gamma)$ from $\Gamma$ by adjoining vertices and edges in the following manner. We will consider the set of finite and $K_{n-1}$-free subsets of $V_\Gamma$. If $\Gamma$ is countable, then the set of all finite sets of $V_\Gamma$ is countable. Thus the set of all finite and $K_{n-1}$-free sets of $V_\Gamma$ is a subset of a countable set and hence countable. We can thus enumerate all such finite $K_{n-1}$-free sets from $V_\Gamma$ as $\{U_i : i \in \mathbb{N}\}$ – replacing the natural numbers by a finite set when necessary.

We create $\mathcal{L}_n(\Gamma)$ by adding, for each such finite $K_{n-1}$-free set $U_i$, a vertex $v_i$ such that $v_i$ is adjacent to every member of $U_i$ and to no other vertices.

More precisely, we let,

$$V_{\mathcal{L}_n(\Gamma)} = V_\Gamma \cup \{v_i : i \in \mathbb{N}\}$$

and

$$E_{\mathcal{L}_n(\Gamma)} = E_\Gamma \cup \{(v_i, u), (u, v_i) : u \in U_i,\ i \in \mathbb{N}\}.$$

If $\Gamma$ is a finite graph then $|V_{\mathcal{L}_n(\Gamma)}| \leq 2^{|V_\Gamma|} + |V_\Gamma|$ and so $\mathcal{L}_n(\Gamma)$ is a finite graph. Likewise if $\Gamma$ is countably infinite then the set of all finite $K_{n-1}$-free sets is countably infinite (it cannot be finite for then there would exist an infinite subset of vertices such that any two vertices are adjacent – *i.e.* a complete graph which is impossible since it is not $K_n$-free) and hence $\mathcal{L}_n(\Gamma)$ is countably infinite.

We can inductively define a sequence of graphs by setting $\Gamma_0 = \Gamma$ and $\Gamma_{n+1} = \mathcal{L}_n(\Gamma_n)$. Now define $\Gamma_\infty$ to be the limit of this process by letting:

$$\Gamma_\infty = \left( \bigcup_{n \in \mathbb{N}} V_{\Gamma_n}, \bigcup_{n \in \mathbb{N}} V_{\Gamma_n} \right).$$

It should be clear that $\Gamma_\infty$ is $K_n$-free since it is the union of the $K_n$-free graphs $\Gamma_n$ where $\Gamma_n$ is an induced subgraph of $\Gamma_{n+1}$.

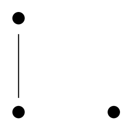
**Example 6.3.** [Construction of $\Gamma_1 = \mathcal{L}_3(\Gamma_0)$ and $\Gamma_2 = \mathcal{L}_3(\Gamma_1)$ when $\Gamma = (\{v\}, \emptyset)$]

Compare with Example 3.3

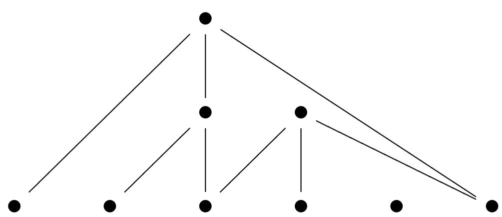
$\Gamma_0 = \Gamma$



$\Gamma_1$



$\Gamma_2$



**Lemma 6.4.** *For any countable $K_n$-free graph $\Gamma$, the $K_n$-free graph $\Gamma_\infty$ is existentially closed and thus $\Gamma_\infty \cong G_n$.*

*Proof.* Let $U$ and $V$ be disjoint subsets from $\Gamma_\infty$ such that $U$ is a $K_{n-1}$-free set. Then there exists $k \in \mathbb{N}$ such that $U, V \subset V_{\Gamma_k}$. By construction of $\Gamma_{k+1}$, there exists a vertex $v \in V_{\Gamma_{k+1}} \backslash V_{\Gamma_k}$ adjacent to every member of $U$. Moreover $x$ is adjacent to no other vertices in $V_{\Gamma_k}$. Thus $v$ is adjacent to every member of $U$, but to no member of $V$ in $\Gamma_{k+1}$. Since the construction of $\Gamma_\infty$ makes no change to the edge set of the induced subgraph $\Gamma_{k+1}$, it follows that $v$ is adjacent to every member of $U$, but to no member of $V$ in $\Gamma_\infty$. □

Since any $K_n$-free graph $\Gamma$ can clearly be embedded into $\Gamma_\infty$ and since $\Gamma_\infty \cong G_n$ by Lemma 6.4, a proof of Theorem 6.1 should now be clear.

## 6.2 Group $\mathscr{H}$-classes and Regular $\mathscr{D}$-classes of End($G_n$)

To make use of Theorem 2.7 on the group $\mathscr{H}$-classes of $\mathrm{End}(G_n)$ we once again seek information on the structure of the subgraphs of $G_n$ induced by the images of idempotents in $\mathrm{End}(G_n)$. However, the following theorem, originally proved in [Mud10, Proposition 1], makes this task trivial.

**Theorem 6.5.** *Let $f \in \mathrm{End}(G_n)$. Then $f$ is an embedding of $G_n$ into $G_n$.*

*Proof.* To show that $f$ is an embedding we must show that $f$ is an injective function and that $(u, v) \in E_{G_n}$ if and only if $(uf, vf) \in E_{G_n}$. We begin with the latter. So suppose that $u, v \in V_{G_n}$ and suppose that $(u, v) \notin E_{G_n}$. We claim that $(uf, vf) \notin E_{G_n}$. To see this suppose for a contradiction that $(uf, vf) \in E_{G_n}$. Then since $\{u, v\}$ is an independent set and since $G_n$ is existentially closed, there exists $w_1 \in V_{G_n}$ such that $w_1$ is adjacent to both $u$ and $v$. Now for $1 < i \leq n-2$ let $w_{i+1}$ be chosen such that $w_{i+1}$ is adjacent to $\{u, v, w_1, \ldots, w_i\}$. Note that this is possible since $u$ and $v$ are not adjacent and so for all $1 \leq i \leq n-2$, $\{u, v, w_1, \ldots, w_i\}$ is $K_{i+1}$-free and hence $K_{n-1}$-free. Hence existential closure of $G_n$ guarantees the existence of the required vertices $w_i$. Now since $w_j$ is adjacent to $w_k$ for all $j \neq k$ $(1 \leq j, k \leq n-2)$, it follows that $w_j f \neq w_k f$ for all $j \neq k$. Similarly since $w_j$ is adjacent to $u$ and $v$ it follows that $w_j f \neq uf$ and $w_j f \neq vf$ for all $1 \leq j \leq n-2$. Thus since $f$ is an endomorphism and since by assumption $(uf, vf) \in E_{G_n}$ it follows that $\langle\{uf, vf, w_1 f, \ldots, w_{n-2} f\}\rangle \cong K_n$. This is clearly a contradiction since $G_n$ is $K_n$-free. Hence $(uf, vf) \notin E_{G_n}$ as claimed. Since $f$ was an endomorphism it now follows that $(u, v) \in E_{G_n}$ if and only if $(uf, vf) \in E_{G_n}$.

To show that $f$ is injective let $u, v \in V_{G_n}$ and suppose that $uf = vf$. If $u \neq v$ then since $\{u\}$ is trivially $K_n$-free and since $\{v\}$ is disjoint from $\{u\}$, it follows from existential closure that there exists a vertex $x \in V_{G_n}$ such that $x$ is adjacent to $u$ but not to $v$. Then since $f$ is an endomorphism $xf$ is adjacent to $uf$. But by our argument above we also know that $xf$ is not adjacent to $vf = uf$, a contradiction. Thus $u = v$ and $f$ is an injective function. $\square$

**Lemma 6.6.** *Let $f \in E(\mathrm{End}(G_n))$. Then $f = \mathbf{1}$.*

*Proof.* If $f \in E(\mathrm{End}(G_n))$ then $f|_{\mathrm{im}\, f} = \mathbf{1}|_{\mathrm{im}\, f}$. We also observed that $f$ must be an injective embedding. So suppose that $y \in V_{G_n}$. Then $yf = x$ for some $x \in \mathrm{im}\, f$. Since $f$ is idempotent $xf = x$ and hence by injectivity, $x = y$. Thus $y \in \mathrm{im}\, f$ and hence $V_T \setminus \mathrm{im}\, G_n = \emptyset$. Thus, $f = f|_{G_n} = \mathbf{1}_{G_n}$ as required. $\square$

Consequently, we now have the following results on the group $\mathscr{H}$-classes and regular $\mathscr{D}$-classes of $\mathrm{End}(G_n)$.

**Theorem 6.7.** *The only group $\mathscr{H}$-class of* $\mathrm{End}(G_n)$ *is* $\mathrm{Aut}(G_n)$.

*Proof.* Every group $\mathscr{H}$-class of $\mathrm{End}(G_n)$ contains an idempotent (the subgroup identity). By Lemma 6.6 the only such idempotent is the identity idempotent $\mathbf{1}$, and $H_{\mathbf{1}} = \mathrm{Aut}(G_n)$ as required. $\square$

Thus we can now conclude that the only maximal subgroup of $\mathrm{End}(G_n)$ is $\mathrm{Aut}(G_n)$. Furthermore, we have the following result as a consequence of Lemma 6.6.

**Corollary 6.8.** $\mathrm{End}(T)$ *has a only one regular $\mathscr{D}$-class.*

*Proof.* Every regular $\mathscr{D}$-class contains at least one idempotent. Hence, since Lemma 6.6 told us that the only idempotent of $\mathrm{End}(G_n)$ is the identity, there can only be one regular $\mathscr{D}$-class. $\square$

## 6.3 A Related Class of Triangle-free Graphs

Earlier we made the observation that, in some sense, the existential closure property for triangle-free graphs is 'equivalent' to existential closure described for the class of all graphs but which holds only where possible (that is, avoiding the sets for which the property cannot hold due to the graph being triangle-free). In a similar manner, we can consider the property of algebraic closure defined for graphs and examine the triangle-free graphs which satisfy algebraic closure wherever possible. More precisely, we make the following definition.

**Definition 6.9.** We will say that a triangle-free graph $\Gamma$ has property $\star$ if for each finite independent set $U \subseteq V_\Gamma$, there exists a vertex $v \in V_\Gamma$ such that $v$ is adjacent to every member of $U$.

Triangle-free graphs with property $\star$ can be finite or infinite. For example, the graph $(\{v_1, v_2\}, \{(v_1, v_2), (v_2, v_1)\})$ has property $\star$ having only the independent sets $\{v_1\}$ and $\{v_2\}$. On the other hand we can easily show that $G_3$ has property $\star$, since $G_3$ is existentially closed.

In the setting of graphs and directed graphs a pivotal mechanism was the idea of taking the complement, in some sense or other, and producing another graph or directed graph. However, when dealing with triangle-free graphs, the complement of a triangle-free graph is not necessarily triangle-free. As a result, examples of triangle-free graphs with property $\star$ (which are not $G_3$) are more difficult to produce since, for example, an analogue of Lemma 3.16 is not available to us. We could easily produce $2^{\aleph_0}$ algebraically closed graphs with trivial automorphism group. It is unclear if even one triangle-free graph with property $\star$ and with trivial automorphism group exists. It is even unclear exactly which groups can arise as automorphism groups of triangle-free graph with property $\star$.

We are, however, able to gain some information on triangle-free graphs with property $\star$ and their automorphism groups. In the remainder of this chapter we will provide partial classification results on the cardinality of the automorphism groups of triangle-free graphs with property $\star$. In particular, we will provide a complete description of finite triangle-free graphs with property $\star$ which have exactly two maximal independent sets. We will also show that if $\Gamma$ is a countably infinite triangle-free graph with property $\star$ which has only finitely many vertices of infinite degree, then there are $2^{\aleph_0}$ automorphisms of $\Gamma$. First we start with some lemmas.

**Lemma 6.10.** *Let $\Gamma$ be a graph. If $T$ and $U$ are maximal independent subsets of $\Gamma$ then either $T = U$ or there exists $x, y \in V_\Gamma$ such that $x \in T \setminus U$ and $y \in U \setminus T$.*

*Proof.* If $U = T$ then we are done. So suppose that $U \neq T$. If $U \setminus T = \emptyset$ then $U \subset T$. Thus $T$ is a maximal independent set containing $U$ as a proper subset. But this is a contradiction to $U$ being maximal. Similarly if $T \setminus U = \emptyset$ then $T \subset U$ and $U$ would be an independent set containing $T$ as a proper subset. Hence $T \setminus U \neq \emptyset$ and $U \setminus T \neq \emptyset$ and we are done. $\square$

**Lemma 6.11.** *If $\Gamma$ is a finite triangle-free graph with property $\star$, then $V_\Gamma$ has at least two maximal independent sets and hence $|V_\Gamma| \geq 2$.*

*Proof.* Since $V_\Gamma \neq \emptyset$ we know that $\Gamma$ has at least one maximal independent set, $U$ say. Since $\Gamma$ is finite, $U$ is finite and so there must exist a vertex $v \in V_\Gamma \setminus U$ such that $v$ is adjacent to every member of $U$. Then $\{v\}$ is an independent set and is contained in a maximal independent set, $T$ say, which cannot contain any vertices from $U$. In other words $T$ and $U$ are distinct maximal independent sets and the result follows. □

**Lemma 6.12.** *Let $\Gamma$ be a finite triangle-free graph with property $\star$ and suppose that $U_1, \ldots, U_n$ is a list of all maximal independent sets in $\Gamma$. Then $U_1 \cap \cdots \cap U_n = \emptyset$.*

*Proof.* Seeking a contradiction, suppose that $v \in U_1 \cap \cdots \cap U_n$. Then since $\Gamma$ has property $\star$ there must exists a vertex $x \in V_\Gamma \setminus \{v\}$ such that $x$ is adjacent to $v$. Since $V_\Gamma = \bigcup_{i=1}^{n} U_i$, $x \in U_i$ for some $i \in \{1, \ldots, n\}$. But by assumption $v \in U_i$ and cannot be adjacent to $x$, a contradiction. Hence $U_1 \cap \cdots \cap U_n = \emptyset$ as required. □

**Theorem 6.13.** *Let $\Gamma$ be a finite triangle-free graph with property $\star$ and suppose that $\Gamma$ has exactly two maximal independent sets $T$ and $U$. Then $T \cap U = \emptyset$ and $\Gamma \cong K_{m,n}$ for some $m, n \in \mathbb{N} \setminus \{0\}$.*

*Proof.* If $\Gamma$ has exactly two maximal independent sets $T$ and $U$, then $V_\Gamma = T \cup U$. Also, by Lemma 6.12, $T \cap U = \emptyset$. Let $x, y \in V_\Gamma$ and suppose without loss of generality that $x \in T$ and $y \in U$. Suppose that $(x, y) \notin E_\Gamma$. The $\{x, y\}$ is an independent set contained in neither $T$ nor $U$ and thus must be contained in a maximal independent set not equal to $T$ or $U$. This is a contradiction and so it follows that $(x, y) \in E_\Gamma$ for all $x \in T$ and for all $y \in U$. Since $|T|$,$|U| > 0$, the result follows. □

**Corollary 6.14.** *Let $\Gamma$ be a finite triangle-free graph with property $\star$ and suppose that $\Gamma$ has exactly two maximal independent sets $T$ and $U$. Then the automorphism group of $\Gamma$ either has cardinality $2(n!)^2$ for some $n \in \mathbb{N} \setminus \{0\}$ or has cardinality $n!m!$ for some $m \in \mathbb{N} \setminus \{0\}$, $m \neq n$.*

*Proof.* It is well known that the automorphism group of the graph $K_{m,n}$, $m, n \in \mathbb{N} \setminus \{0\}$, is $S_m \times S_n$ if $m \neq n$ and $(S_n \times S_n) \rtimes C_2$ if $m = n$, [Ros99, Section 8.10.2 Example 2]. Thus it immediately follows from Theorem 6.13 that the automorphism group of $\Gamma$ has cardinality $2(n!)^2$ or has cardinality $n!m!$. □

It remains an open problem to determine which groups occur as the automorphism group of a finite triangle-free graph with property $\star$ which has three or more maximal independent sets. The following lemma tells us that such a triangle-free graph with property $\star$ cannot contain a set of three or more mutually disjoint maximal independent sets.

**Lemma 6.15.** *There does not exist a finite triangle-free graph with property $\star$ which contains a set of three or more mutually disjoint maximal independent sets.*

*Proof.* Let $S$, $T$ and $U$ be maximal mutually disjoint independent sets in $\Gamma$. Since $V_\Gamma$ is finite $S$, $T$ and $U$ are finite sets. Since $\Gamma$ has property $\star$ there exists a vertex $x \in V_\Gamma \setminus S$ such that $(x, s) \in E_\Gamma$ for all $s \in S$. If $x \notin T$, then since $T$ is a maximal independent set there exists a vertex $t \in T$ such that $(x, t) \in E_\Gamma$. Furthermore, since $S$ is also a maximal independent set and $t \notin S$, there exists a vertex $r \in S$ such that $(r, t) \in E_\Gamma$. But then the induced subgraph $\langle\{x, t, r\}\rangle$ is isomorphic to $K_3$ - a contradiction to $\Gamma$ being triangle-free. Hence $x \in T$. But since $U$ is maximal and $x \notin U$, there exists $u \in U$ such that $(x, u) \in E_\Gamma$. Similarly since $S$ is maximal and $u \notin S$, there exists $q \in S$ such that $(u, q) \in E_\Gamma$. In this case it follows that $\langle\{x, u, q\}\rangle \cong K_3$, another contradiction and the result follows. □

We now consider infinite triangle-free graphs with property $\star$. The following lemma will be of importance in the proof of Theorem 6.17.

**Lemma 6.16.** *If $\Gamma$ is a countably infinite triangle-free graph with property $\star$, then $\Gamma$ contains a least one vertex of infinite degree and every vertex has either infinite degree or is adjacent to a vertex of infinite degree.*

*Proof.* Let $v \in V_\Gamma$. If $v$ has infinite degree then we are done. So suppose that $\Gamma(v) = \{u \in V_\Gamma : (u, v) \in E_\Gamma\}$ is finite and hence that $V_\Gamma \setminus (\Gamma(v) \cup \{v\})$ is infinite. Now for each $w \in V_\Gamma \setminus (\Gamma(v) \cup \{v\})$, $\{v, w\}$ is an independent set and so since $\Gamma$ has property $\star$, there must exist $x \in \Gamma(v)$ such that $x$ is adjacent to both $v$ and $w$. Since $V_\Gamma \setminus (\Gamma(v) \cup \{v\})$ is infinite and $\Gamma(v)$ is finite, it follows by the pigeonhole principle that at least one member of $\Gamma(v)$ has infinite degree. □

The following theorem on countably infinite triangle-free graphs with property $\star$ now follows.

**Theorem 6.17.** *Let $\Gamma$ be a countably infinite triangle-free graph with property $\star$. Suppose that $\Gamma$ has exactly $n \in \mathbb{N} \setminus \{0\}$ vertices of infinite degree. Then the automorphism group of $\Gamma$ has cardinality $2^{\aleph_0}$.*

*Proof.* Let $v_1, \ldots, v_n \in V_\Gamma$ denote the $n$ vertices of infinite degree and (as usual) for $u \in V_\Gamma$, let $\Gamma(u) = \{w \in V_\Gamma : (u, w) \in E_\Gamma\}$. Clearly, since $\Gamma$ is triangle-free, $\Gamma(v_k)$ is an independent set for all $k \in \{1, \ldots, n\}$. Recursively define a sequence of subsets of $V_\Gamma$ as follows. Let,

$$Y_\emptyset = \bigcap_{k=1}^{n} \Gamma(v_k).$$

If $Y_\emptyset \neq \emptyset$, then for all $v \in Y_\emptyset$, $\Gamma(v) = \{v_1, \ldots, v_n\}$. Hence since $\Gamma$ is triangle-free, $Y_\emptyset$ is an independent set. Let $S \subseteq \{1, \ldots, n\}$ such that $|S| \geq 1$. Suppose that for all $R \subseteq \{1, \ldots, n\}$ such that $|R| < |S|$, $Y_R$ has been defined. Now define

$$Y_S = \left( \bigcap_{k \notin S} \Gamma(v_k) \right) \setminus Z_{|S|-1},$$

where for $r \geq 0$,

$$Z_r = \left( \bigcup_{|R| \leq r} Y_R \right) \cup \{v_1, \ldots, v_k\}.$$

If $Y_S \neq \emptyset$, then for all $v \in Y_S$ and for all for all $k \notin S$, $v_k \in \Gamma(v)$ . Hence, since $\Gamma$ is triangle-free, $Y_S$ is an independent set.

Now, by Lemma 6.16,

$$V_\Gamma = \bigcup_{k=1}^{n} \Gamma(v_k) \cup \{v_1 \ldots, v_n\}$$

and thus

$$V_\Gamma \setminus \{v_1, \ldots, v_n\} = \bigcup_{S \subseteq \{1,\ldots,n\}} Y_S.$$

Since $\Gamma$ is infinite and since the set of all subsets of $\{1, \ldots, n\}$ is finite, it follows by the pigeonhole principle that $Y_S$ is infinite for some $S \subseteq \{1, \ldots, n\}$. So fix $T \subseteq \{1, \ldots, n\}$ such $Y_T$ is countably infinite. We will show that $\mathrm{Aut}(\Gamma)$ contains a subgroup isomorphic to $S_{|Y_T|}$. To do this we will first show that

$$V_\Gamma \setminus \left( \bigcup_{k \notin T} \Gamma(v_k) \cup \{v_1 \ldots, v_n\} \right) = \emptyset.$$

Seeking a contradiction, suppose that

$$V_\Gamma \setminus \left( \bigcup_{k \notin T} \Gamma(v_k) \cup \{v_1 \ldots, v_n\} \right) \neq \emptyset.$$

Then there exists a vertex,

$$z \in \Gamma(v_l) \setminus \left( \bigcup_{k \notin T} \Gamma(v_k) \cup \{v_1 \ldots, v_n\} \right),$$

for some $l \in T$. Since $z$ has finite degree and since $Y_T$ is infinite, there exists an infinite subset of distinct vertices $\{y_i : i \in \mathbb{N}\} \subseteq Y_T$ to which $z$ is not adjacent. Now, since $\Gamma$ has property $\star$, for each $i \in \mathbb{N}$ there exists a vertex $x_i$ such that $x_i$ is adjacent to both $y_i$ and to $z$. If all the $x_i$ are distinct then $z$ would have infinite degree which is a contradiction. Hence it must be the case that infinitely many of the $x_i$ are equal. But in this case $x_i$ would have infinite degree and thus must be equal to $v_j$ for some $j \in \{1, \dots, n\}$. However, since $y_i$ is not adjacent to $v_k$ for $k \notin T$ and since $z$ can be adjacent only to $v_k$ for $k \in T$ this is another contradiction. Hence it follows that,

$$V_\Gamma \setminus \left( \bigcup_{k \notin T} \Gamma(v_k) \cup \{v_1 \dots, v_n\} \right) = \emptyset.$$

Now, if $u \in Y_T$ then $u$ is adjacent to $v_k$ for all $k \notin T$. Hence since

$$V_\Gamma = \bigcup_{k \notin T} \Gamma(v_k) \cup \{v_1 \dots, v_n\},$$

and since $\Gamma$ is triangle-free, there can exist no edges between vertices in $Y_T$ and vertices in $v \in V_\Gamma \setminus \{v_1 \dots, v_n\}$. Thus $\Gamma(u) = \{v_k : k \notin T\}$ for all $u \in Y_T$. If we enumerate the vertices of $Y_T$ as $\{u_i : i \in \mathbb{N}\}$, then for each $\pi \in S_{|Y_T|}$ we can define a map $f_\pi$ on $V_\Gamma$ by,

$$vf_\pi = \begin{cases} v & \text{if } v \notin Y_T, \\ u_{(i)\pi} & \text{if } v = u_i \text{ for some } i \in \mathbb{N}. \end{cases}$$

Since $Y_T$ is an independent set, and since $\Gamma(u) = \{v_k : k \notin T\}$ for all $u \in Y_T$ it follows that $f_\pi$ is a graph automorphism for all $\pi \in S_{|Y_T|}$. Furthermore, it should be easy to see that the map $\phi : S_{|Y_T|} \to \mathrm{Aut}(\Gamma)$ defined on $\pi \in S_{|Y_T|}$ by $\pi\phi = f_\pi$ is an injective group homomorphism. Thus $\mathrm{Aut}(\Gamma)$ contains a subgroup isomorphic to the infinite symmetric group. Since the infinite symmetric group has cardinality $2^{\aleph_0}$ the result now follows. $\square$

It remains an open problem to determine the cardinality of the automorphism group of a countably infinite triangle-free graph with property $\star$ which has infinitely many vertices of infinite degree.

# Chapter 7

# The Random Bipartite Graph

In this chapter we will discuss the random bipartite graph, which we denote by $B$. We will show that if $\Gamma$ is a countable graph, then there exist $2^{\aleph_0}$ maximal subgroups of $\mathrm{End}(B)$ isomorphic to $\mathrm{Aut}(\Gamma)$. We will also show that there are $2^{\aleph_0}$ $\mathscr{D}$-classes and $2^{\aleph_0}$ $\mathscr{J}$-classes of $\mathrm{End}(B)$. Furthermore, in contrast with the random graph and random directed graph, we will show that there exist regular $\mathscr{D}$-classes of $\mathrm{End}(B)$ which contain countably many group $\mathscr{H}$-classes as well as regular $\mathscr{D}$-classes which contain $2^{\aleph_0}$ group $\mathscr{H}$-classes. First however, we will observe that we must make a slight adjustment to the standard definition of a bipartite graph.

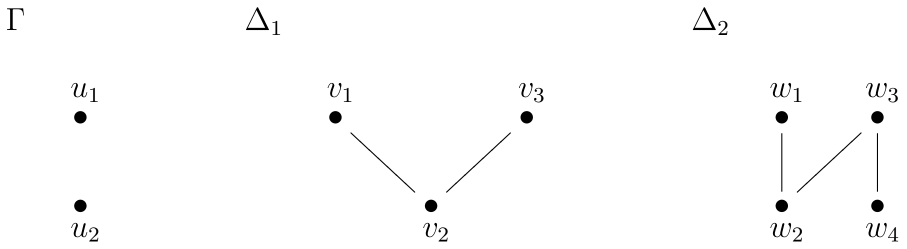
## 7.1 Defining a Bipartite Graph

Recall that a graph $\Gamma = (V_\Gamma, E_\Gamma)$ satisfies the bipartite condition if there exists a function $c : V_\Gamma \to \{0, 1\}$ such that $uc \neq vc$ whenever $(u, v) \in E_\Gamma$. In other words we can write $V_\Gamma = V_0 \cup V_1$ where $(u, v) \in E$ implies that $u \in V_0$ and $v \in V_1$ or *vice versa*.

If we let $K$ be the class of finitely generated graphs which satisfy the bipartite condition, then $K$ has the hereditary and joint embedding properties. However, we will show that $K$ fails the amalgamation property as follows. We consider the following graphs which can be shown to satisfy the bipartite condition, see Figure 7.1.

$$\begin{aligned}
\Gamma &= (\{u_1, u_2\}, \emptyset),\\
\Delta_1 &= (\{v_1, v_2, v_3\}, \{(v_1, v_2), (v_2, v_1), (v_2, v_3), (v_3, v_2)\}),\\
\Delta_2 &= (\{w_1, w_2, w_3, w_4\},\\
&\qquad \{(w_1, w_2), (w_2, w_1), (w_2, w_3), (w_3, w_2), (w_3, w_4), (w_4, w_3)\}).
\end{aligned}$$

Figure 7.1: The graphs $\Gamma$, $\Delta_1$ and $\Delta_2$.




It is easy to see that $\Gamma$ can be embedded into $\Delta_1$ *via* the injective homomorphism which takes $u_1$ to $v_1$ and $u_2$ to $v_3$. Similarly $\Gamma$ can be embedded into $\Delta_2$ *via* the injective homomorphism taking $u_1$ to $w_1$ and $u_2$ to $w_4$. Now suppose that there exists a finite graph $\Lambda$ satisfying the bipartite condition and embeddings $f_1 : \Delta_1 \to \Lambda$ and $f_2 : \Delta_2 \to \Lambda$ such that $v_1 f_1 = w_1 f_2$ and $v_3 f_1 = w_4 f_2$. Since $\Lambda$ satisfies the bipartite condition there exists a function $c : V_\Lambda \to \{0, 1\}$ such that $uc \neq vc$ whenever $(u, v) \in E_\Lambda$. Suppose that $(v_1 f_1)c = 0$ and $(v_3 f_1)c = 1$. Then since $(v_1, v_2) \in E_{\Delta_1}$ it must be the case that $(v_2 f_1)c = 1$. However $(v_2, v_3)$ also lies in $E_{\Delta_1}$ so that $(v_2 f_1)c = 0$ a contradiction to $\Lambda$ satisfying the bipartite condition. Hence we deduce that either $(v_1 f_1)c = 0$ and $(v_3 f_1)c = 0$, or $(v_1 f_1)c = 1$ and $(v_3 f_1)c = 1$. Without loss of generality assume the latter. If $(v_1 f_1)c = 1$ then $(w_1 f_2)c = 1$ and since $(w_1, w_2) \in E_{\Delta_2}$, $(w_2 f_2)c = 0$. Similarly if $(v_3 f_2)c = 1$ then $(w_4 f_2)c = 1$ and since $(w_3, w_4) \in E_{\Delta_2}$, $(w_3 f_2)c = 0$. However $(w_2, w_3)$ also lies in $E_{\Delta_2}$ and so it cannot be the case that $(w_3 f_2)c = (w_2 f_2)c$. Hence again we have a contradiction and there can be no such graph $\Lambda$ which satisfies the bipartite condition.

As a result the class of finite graphs satisfying the bipartite condition does not have the amalgamation property and thus does not have a Fraïssé Limit. However, if we create a relational structure which 'encodes' the bipartite structure of a graph which satisfies the bipartite condition, a Fraïssé Limit can be found. One of the ways in which we can do this is as follows.

**Definition 7.1.** A *bipartite graph* is a relational structure $\Gamma = (V_\Gamma, E_\Gamma, P_\Gamma)$ where the following two conditions are satisfied.

(i) $(V_\Gamma, E_\Gamma)$ is a graph satisfying the bipartite condition with bipartition $V_\Gamma = V_0 \cup V_1$.

(ii) $P_\Gamma = (V_0 \times V_0) \cup (V_1 \times V_1)$.

The bipartite graph $\Gamma$ is said to have bipartition $V_\Gamma = V_0 \cup V_1$ and $V_0$ and $V_1$ will be known as the *parts*. The binary relation $P_\Gamma$ is called the *partition relation*. In order to make the bipartition clear we will sometimes denote such a bipartite graph as $\Gamma = (V_0 \cup V_1,\, E_\Gamma, P_\Gamma)$ where it is assumed that $P_\Gamma = (V_0 \times V_0) \cup (V_1 \times V_1)$.

A bipartite graph can thus be thought of as a graph with an extra binary relation which identifies the bipartition. With this in mind, we will continue to call $V_\Gamma$ the set of vertices and $E_\Gamma$ the set of edges. Two vertices $u, v \in V_\Gamma$ will said to be adjacent in $\Gamma$ if and only if $(u, v) \in E_\Gamma$. Likewise, we will say that a bipartite graph $(V_\Gamma, (E_\Gamma, P_\Gamma))$ is connected, locally finite or has connected component $U$ if the graph $(V_\Gamma, E_\Gamma)$ is connected, locally finite or has connected component $U \subseteq V_\Gamma$ respectively.

Of course if $(V, E)$ is a graph which satisfies the bipartite condition with bipartition $V = V_0 \cup V_1$, then we can produce a bipartite graph $(V, E, P)$ by setting $P = (V_0 \times V_0) \cup (V_1 \times V_1)$. Naturally, the endomorphism monoid of a bipartite graph constructed in this manner will be dependent upon the choice of bipartition. Two bipartite graphs which are formed from a single graph satisfying the bipartite condition but with two distinct bipartitions are, in general, not isomorphic. However, if a graph is connected then its bipartition is unique (see [AG07, Theorem 5.3] for details) and so the bipartite graph formed from this graph is unique up to isomorphism.

It must be noted that if $\Gamma = (V_0 \cup V_1,\, E_\Gamma, P_\Gamma)$ is a bipartite graph, then an endomorphism $f$ can map both $V_0$ and $V_1$ solely to $V_0$ (or indeed $V_1$) only when the edge set $E$ is empty. However, there can exist endomorphisms which allow the partition sets to be interchanged. More precisely, if an endomorphism maps one vertex in $V_0$ to a vertex in $V_1$ then it must in fact map all vertices of $V_0$ to vertices in $V_1$ and *vice versa*. These conditions are enforced by the partition relation $P_\Gamma$. This discussion is summarised by the following two lemmas.

**Lemma 7.2.** *Let $\Gamma = (V_\Gamma, E_\Gamma, P_\Gamma)$ and $\Lambda = (V_\Lambda, E_\Lambda, P_\Lambda)$ be bipartite graphs with bipartitions $V_\Gamma = V_0 \cup V_1$ and $V_\Lambda = W_0 \cup W_1$, respectively. Let $f : V_\Gamma \to V_\Lambda$ be a function. Then $f$ is a bipartite graph homomorphism if and only if $f$ defines a graph homomorphism $(V_\Gamma, E_\Gamma) \to (V_\Lambda, E_\Lambda)$ and one of the following four cases hold:*

(i) $V_0 f \subseteq W_0$ *and* $V_1 f \subseteq W_1$,

(ii) $V_0 f \subseteq W_1$ *and* $V_1 f \subseteq W_0$,

(iii) $E_\Gamma = \emptyset$, $V_0 f \subseteq W_0$ *and* $V_1 f \subseteq W_0$, *or*

(iv) $E_\Gamma = \emptyset$, $V_0 f \subseteq W_1$ *and* $V_1 f \subseteq W_1$.

*Proof.* ($\Rightarrow$) Suppose $f : V_\Gamma \to V_\Lambda$ is a bipartite graph homomorphism and let $u \in V_\Gamma$. Then necessarily $(u, v) \in E_\Gamma$ implies that $(uf, vf) \in E_\Lambda$ and thus $f$ defines a graph homomorphism $(V_\Gamma, E_\Gamma) \to (V_\Lambda, E_\Lambda)$.

Now suppose, without loss of generality, that $u \in V_0$. Then for all $v \in V_0$, $(u, v) \in V_0 \times V_0 \subseteq P_\Gamma$. Since $f$ is a bipartite graph homomorphism, it must be the case that $(uf, vf) \in P_\Lambda$ for all $v \in V_0$. Since $P_\Lambda = (W_0 \times W_0) \cup (W_1 \times W_1)$ we can deduce that if $uf \in W_0$ then $V_0 f \subseteq W_0$. Similarly if $uf \in W_1$ then $V_0 f \subseteq W_1$.

Suppose that $E_\Gamma \neq \emptyset$. Then there exists $(x, y) \in E_\Gamma$, where $x \in V_0$ and $y \in V_1$. Now, since $f$ is an endomorphism we know that $(xf, yf) \in E_\Lambda$. Thus, if $V_0 f \subseteq W_0$ then $xf \in W_0$ and hence $yf \in W_1$ since $E_\Lambda \cap (W_0 \times W_0) = \emptyset$. Now, since $(y, z) \in V_1 \times V_1$ for all $z \in V_1$ it follows that $V_1 f \subseteq W_1$. Similarly if $V_0 f \subseteq W_1$ we can conclude that $V_1 f \subseteq W_0$. Thus if $E_\Gamma \neq \emptyset$ then only cases (i) and (ii) above can occur.

If $E_\Gamma = \emptyset$ and $V_1 = \emptyset$ then there is nothing further to do. So suppose that $E_\Gamma = \emptyset$ and $y \in V_1$. Since $E_\Gamma = \emptyset$, there exist no edges between $y$ and $V_0$ and thus it is possible have either $yf \in W_0$ or $yf \in W_1$. An identical argument to that shown above then leads to the conclusion that $V_1 f \subseteq W_0$ or $V_1 f \subseteq W_0$ independent of whether $V_0 f \subseteq W_0$ or $V_0 f \subseteq W_1$. In particular if $E_\Gamma = \emptyset$ then any of the cases (i)–(iv) above can occur.

($\Leftarrow$) For the converse suppose that $f : V_\Gamma \to V_\Lambda$ defines a graph homomorphism $(V_\Gamma, E_\Gamma) \to (V_\Lambda, E_\Lambda)$. Then $(uf, vf) \in E_\Lambda$ whenever $(u, v) \in E_\Gamma$. Now suppose that $V_0 f \subseteq W_0$ and $V_1 f \subseteq W_1$ (case *(i)* above). Then whenever $(u, v) \in V_0 \times V_0$ it holds that $(uf, vf) \in W_0 \times W_0$. Similarly $(u, v) \in V_1 \times V_1$ implies that $(uf, vf) \in W_1 \times W_1$. Thus together we have that $(u, v) \in P_\Gamma$ implies $(uf, vf) \in P_\Lambda$. Hence $f$ defines a bipartite graph homomorphism $(V_\Gamma, E_\Gamma, P_\Gamma) \to (V_\Lambda, E_\Lambda, P_\Lambda)$. Similar arguments for the remaining three cases completes the proof. $\square$

If $\Gamma$ is a bipartite graph and $f \in \mathrm{End}(\Gamma)$, then $f$ is said to be *part fixing* if $f$ follows case (i) in Lemma 7.2 above. In other words, $f$ maps the sets in the bipartition only to themselves.

**Lemma 7.3.** *Let $\Gamma = (V_\Gamma, E_\Gamma, P_\Gamma)$ and $\Lambda = (V_\Lambda, E_\Lambda, P_\Lambda)$ be bipartite graphs with bipartitions $V_\Gamma = V_0 \cup V_1$ and $V_\Lambda = W_0 \cup W_1$, respectively. Let $f : V_\Gamma \to V_\Lambda$ be a function. Then $f$ is an isomorphism of bipartite graphs if and only if $f$ defines a graph isomorphism $(V_\Gamma, E_\Gamma) \to (V_\Lambda, E_\Lambda)$ and one of the following two cases hold:*

(i) *$V_0 f = W_0$ and $V_1 f = W_1$,*

(ii) *$V_0 f = W_1$ and $V_1 f = W_0$,*

*Proof.* ($\Rightarrow$) Suppose $f : V_\Gamma \to V_\Lambda$ is an isomorphism of bipartite graphs and let $u \in V_\Gamma$. Then, by definition, $(u, v) \in E_\Gamma$ if and only if $(uf, vf) \in E_\Lambda$ and thus $f$ defines a graph isomorphism $(V_\Gamma, E_\Gamma) \to (V_\Lambda, E_\Lambda)$.

Now suppose, without loss of generality, that $u \in V_0$. Since $f$ is an endomorphism Lemma 7.2 tells us that either $V_0 f \subseteq W_0$ or $V_0 f \subseteq W_1$. So, suppose that $V_0 f \subseteq W_0$. If $V_1 = \emptyset$ then the bijectivity of $f$ ensures that $W_1 = \emptyset$ and we are finished by deducing that $V_0 f = W_0$ . Otherwise let $x \in V_1$. Since $(u, x) \notin P_\Gamma$ and $f$ is an isomorphism we can conclude that $(uf, xf) \notin P_\Lambda$. Since this holds for all $x \in V_1$ we see that if $V_0 f \subseteq W_0$ then $V_1 f \subseteq W_1$. Moreover, since $f$ defines a bijection of sets we can conclude that $V_0 f = W_0$ and $V_1 f = W_1$. A similar argument leads us to deduce that if $V_0 f \subseteq W_1$ then in fact $V_0 f = W_1$ and $V_1 f = W_0$.

($\Leftarrow$) For the converse suppose that suppose that $f : V_\Gamma \to V_\Lambda$ defines a graph isomorphism $(V_\Gamma, E_\Gamma) \to (V_\Lambda, E_\Lambda)$. Then $f$ is a bijection and $(uf, vf) \in E_\Lambda$ whenever $(u, v) \in E_\Gamma$. Now suppose that $V_0 f = W_0$ and $V_1 f = W_1$ (case (i) above). Then $(u, v) \in V_0 \times V_0$ if and only if $(uf, vf) \in W_0 \times W_0$. Similarly $(u, v) \in V_1 \times V_1$ if and only if $(uf, vf) \in W_1 \times W_1$. Putting these together allows us to deduce that $(u, v) \in P_\Gamma$ if and only if $(uf, vf) \in P_\Lambda$. Hence $f$ defines an isomorphism of bipartite graphs $(V_\Gamma, E_\Gamma, P_\Gamma) \to (V_\Lambda, E_\Lambda, P_\Lambda)$. A similar argument for case (ii) completes the proof. □

As a consequence of the previous lemma we see that an isomorphism between bipartite graphs $\Gamma = (V_0 \cup V_1, E_\Gamma, P_\Gamma)$ and $\Lambda = (W_0 \cup W_1, E_\Lambda, P_\Lambda)$ is possible only if (i) $|V_0| = |W_0|$ and $|V_1| = |W_1|$, or (ii) $|V_0| = |W_1|$ and $|V_1| = |W_0|$. Additionally, if $f : \Gamma \to \Lambda$, is an embedding then $f$ defines an isomorphism between $\Gamma$ and the bipartite graph induced by $\operatorname{im} f$. As a result any such embedding $f$ must follow either case (i) or (ii) in Lemma 7.2 above.

It will be important to identify those cases where the automorphism group of a bipartite graph $\Gamma$ is isomorphic to that of its underlying graph structure.

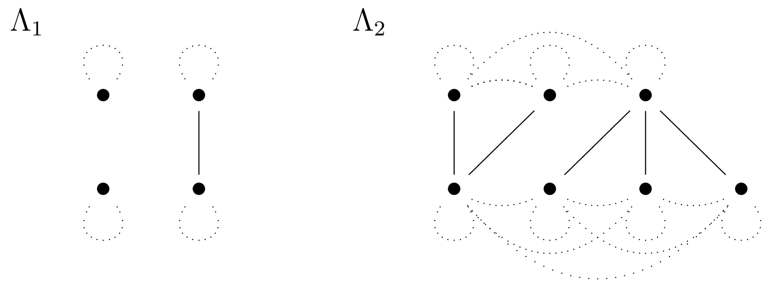
**Lemma 7.4.** *Let* $\Gamma = (V_\Gamma, E_\Gamma, P_\Gamma)$ *be a connected bipartite graph. Then* $\mathrm{Aut}(\Gamma) = \mathrm{Aut}((V_\Gamma, E_\Gamma))$.

*Proof.* Let $V_\Gamma = V_0 \cup V_1$ be the bipartition of $\Gamma$. Since $\Gamma$ is a connected bipartite graph, the graph $(V_\Gamma, E_\Gamma)$ is connected and satisfies the bipartite condition. In particular this means that the bipartition of $(V_\Gamma, E_\Gamma)$ is unique. It should be easy to see that $V_0 f \cup V_1 f$ also provides a bipartition of $(V_\Gamma, E_\Gamma)$ for any $f \in \mathrm{Aut}(\Gamma)$. Thus by the uniqueness of the bipartition we conclude that for any $f \in \mathrm{Aut}(\Gamma)$, either $V_0 f = V_0$ and $V_1 f = V_1$ or $V_0 f = V_1$ and $V_1 f = V_0$. Thus if $f$ is an automorphism of $(V_\Gamma, E_\Gamma)$, it is an automorphism of $\Gamma$ *via* Lemma 7.3. Thus $\mathrm{Aut}((V_\Gamma, E_\Gamma)) \subseteq \mathrm{Aut}(\Gamma)$. Since the reverse inclusion is clear, the result follows. $\square$

Notice that connectivity is a sufficient but not necessary condition in Lemma 7.4. For consider the following example.

**Example 7.5.** [Examples for the converse of Lemma 7.4.]

Let $\Lambda_1$ and $\Lambda_2$ be the bipartite graphs shown in the figure below (where continuous lines represent edges and dotted lines represent elements of the partition relation).




The bipartite graph $\Lambda_1$ is not connected and it is easy to see that $\mathrm{Aut}(\Lambda_1) = C_2$ and $\mathrm{Aut}((V_{\Lambda_1}, E_{\Lambda_1})) = C_2 \times C_2$. On the other hand, the bipartite graph $\Lambda_2$ is not connected but $\mathrm{Aut}(\Lambda_2) = C_2 \times S_3 = \mathrm{Aut}((V_{\Lambda_2}, E_{\Lambda_2}))$.

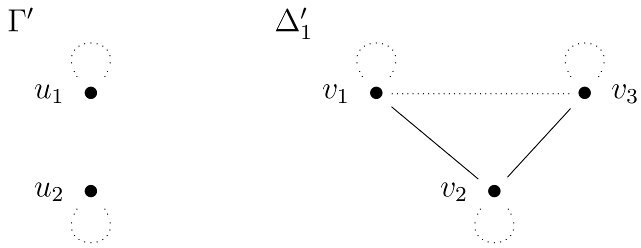
Now let us reconsider the graphs

$$\begin{aligned}
\Gamma &= (\{u_1, u_2\}, \emptyset), \\
\Delta_1 &= (\{v_1, v_2, v_3\}, \{(v_1, v_2), (v_2, v_1), (v_2, v_3), (v_3, v_2)\}),
\end{aligned}$$

from the start of the chapter (see Figure 7.1). Since $\Gamma$ and $\Delta_1$ satisfy the bipartite condition we can produce the corresponding bipartite graphs

$$\begin{aligned}
\hat{\Gamma} &= (\{u_1, u_2\}, \emptyset, P_\Gamma), \\
\hat{\Delta}_1 &= (\{v_1, v_2, v_3\}, \{(v_1, v_2), (v_2, v_1), (v_2, v_3), (v_3, v_2)\}, P_{\Delta_1}),
\end{aligned}$$

Figure 7.2: The bipartite graphs $\Gamma'$ and $\Delta_1'$.




by setting

$$\begin{aligned}P_\Gamma &= (u_1, u_1) \cup (u_2, u_2),\\ P_{\Delta_1} &= \{(v_1, v_1), (v_1, v_3), (v_3, v_1), (v_3, v_3)\} \cup (v_2, v_2).\end{aligned}$$

See Figure 7.2. Notice that since $\Delta_1$ is a connected graph the bipartite graph $\hat{\Delta}_1$ is the unique bipartite graphs formed from $\Delta_1$. The graph embedding described earlier which takes $u_1$ to $v_1$ and $u_2$ to $v_3$ is not an embedding of the bipartite graphs since $(u_1, u_2) \notin P_\Gamma$ but $(u_1 f, u_2 f) = (v_1, v_3) \in P_{\Delta_1}$. In fact by Lemma 7.3, the partition relation prohibits any graph embedding which does not preserve the bipartition from being an embedding of bipartite graphs. Consequently, counterexamples such as the one at the start of the chapter do not occur in the setting of bipartite graphs.

As it turns out we can then easily show that the class of finite bipartite graphs has the hereditary, joint embedding and, unlike the previous class, the amalgamation properties. As a result the class of finite bipartite graphs has a Fraïssé Limit, which we will call the *random bipartite graph* $B$ and which will be the subject of study for the rest of this chapter.

## 7.2 Defining Properties and Constructions

**Definition 7.6.** We will say that a bipartite graph $\Gamma = (V_0 \cup V_1, E, P)$ is *existentially closed* (in the class of bipartite graphs) if for all finite disjoint sets $T_0, U_0 \subseteq V_0$ and $T_1, U_1 \subseteq V_1$ there exists $x \in V_1 \backslash (T_1 \cup U_1)$ and $y \in V_0 \backslash (T_0 \cup U_0)$ such that:

(i) $(x, s) \in E$ for all $s \in T_0$,

(ii) $(x, u) \notin E$ for all $u \in U_0$,

(iii) $(y, t) \in E$ for all $t \in T_1$, and

(iv) $(y, v) \notin E$ for all $v \in U_1$.

For the rest of this chapter, existentially closed should be taken to mean existentially closed in the class of bipartite graphs. If $\Gamma = (V_0 \cup V_1, E, P)$ is an existentially closed bipartite graph, then we can easily show that $\Gamma$ is infinite in the following way. Suppose that $\Gamma$ is finite, that is, $V_0$ and $V_1$ are finite. In order to satisfy the property of existential closure there must exist $x \in V_1$ such that $(x, v) \in E$ for all $v \in V_0$. However this means that every vertex in $V_0$ is adjacent to $x$ and thus there does not exist a vertex $y$ in $V_0$ which is not adjacent to $x$. Consequently no finite bipartite graph can be existentially closed. It is also not hard to show that if $\Gamma$ is existentially closed, then there must in fact exist infinitely many vertices $x \in V_1 \setminus (T_1 \cup U_1)$ and $y \in V_0 \setminus (T_0 \cup U_0)$ as above. Furthermore, we have the following theorem.

**Theorem 7.7.** *Let $\Gamma$ be an existentially closed bipartite graph. Then $\Gamma$ is homogeneous and every finite bipartite graph can be embedded into $\Gamma$.*

For a proof see for example [Hod97], or alternatively the construction given in Definition 7.9 will make this theorem clear. In view of Theorem 7.7, the age of an existentially closed bipartite graph is exactly the class of all finite bipartite graphs. Thus by Fraïssé 's Theorem if $\Gamma$ is any existentially closed bipartite graph, then $\Gamma \cong B$.

As with the other existentially closed relational structures considered in this thesis, we can probabilistically construct an existentially closed bipartite graph $\Lambda$ as follows. Let $V_\Lambda = V_0 \cup V_1$ where $V_0$ and $V_1$ are countably infinite sets and let $P_\Lambda = (V_0 \times V_0) \cup (V_1 \times V_1)$. Construct the edge set $E_\Lambda$ by selecting edges independently with probability $\frac{1}{2}$ from the set $V_0 \times V_1$. In order to ensure symmetry add the edge $(v, u)$ to the edge set $E_\Lambda$ whenever $(u, v)$ is selected. The graph $(V_\Lambda, E_\Lambda)$ satisfies the bipartite condition since, by construction, $V_\Lambda$ contains no edges between vertices in $V_0$ and no edges between vertices in $V_1$. Therefore, $\Lambda = (V_\Lambda, E_\Lambda, P_\Lambda)$ is a bipartite graph with bipartition $V_\Lambda = V_0 \cup V_1$.

**Theorem 7.8.** *Let $\Lambda = (V_\Lambda, E_\Lambda, P_\Lambda)$ be a countable bipartite graph as constructed above. Then $\Lambda$ is existentially closed with probability 1 and hence $\Lambda \cong B$.*

*Proof.* Recall that $V_\Lambda = V_0 \cup V_1$ where $V_0$ and $V_1$ are countably infinite sets. Let $T_0, U_0 \subseteq V_0$ and $T_1, U_1 \subseteq V_1$ be finite and disjoint subsets of $V_0$ and $V_1$, respectively. Let $|T_0| = m$ and $|U_0| = n$ for $m, n \in \mathbb{N}$. We will say that a vertex $x \in V_1 \setminus (T_1 \cup U_1)$ is joined correctly (to $T_0$ and $U_0$) if $x$ is adjacent to all members of $T_0$, but to no member of $U_0$ in $(V_\Lambda, E_\Lambda)$. We will show that with probability 1, such a vertex exists. The probability that a vertex $x \in V_1 \setminus (T_1 \cup U_1)$ is not joined correctly is

$$1 - \frac{1}{2^{m+n}}.$$

Furthermore, if $x$ and $y$ are distinct vertices then the probability that $x$ is not joined correctly is independent of the event that $y$ is not joined correctly. Hence, since $V_1$ is infinite (and $T_1 \cup U_1$ is finite), the probability that no vertex of $V_1 \setminus (T_1 \cup U_1)$ is joined correctly to $T_0$ and $U_0$ is:

$$\lim_{k \to \infty} \left(1 - \frac{1}{2^{m+n}}\right)^k = 0.$$

Similarly, we can show that the probability that there does not exists a vertex of $V_0 \setminus (T_0 \cup U_0)$ joined correctly to $T_1$ and $U_1$ is 0. Thus we have shown that the probability that existential closure is not satisfied for the sets $T_0, U_0 \subseteq V_0$ and $T_1, U_1 \subseteq V_1$ is 0. Since there are only countably many choices for the subsets $U_0, U_1, V_0$ and $V_1$ it follows that the probability that $\Lambda$ is not existentially closed is 0. In other words, $\Lambda$ is existentially closed with probability 1. □

In a similar fashion to the previous chapters, we can exhibit a standard and explicit construction of the random bipartite graph.

**Definition 7.9.** Let $\Gamma = (V_\Gamma, E_\Gamma, P_\Gamma)$ be any countable bipartite graph with bipartition $V_\Gamma = U_0 \cup U_1$. Construct a new graph $\mathcal{I}(\Gamma)$ from $\Gamma$ as follows. Enumerate the finite subsets of $U_0$ as $\{S_i : i \in \mathbb{N}\}$ and enumerate the finite subsets of $U_1$ as $\{T_j : j \in \mathbb{N}\}$ (replacing the natural numbers with some finite subset if $U_0$ or $U_1$ is finite), we let

$$V_0 = U_0 \cup \{y_j : j \in \mathbb{N}\} \text{ and,}$$

$$V_1 = U_1 \cup \{x_i : i \in \mathbb{N}\}.$$

Then $\mathcal{I}(\Gamma) = (V_{\mathcal{I}(\Gamma)}, E_{\mathcal{I}(\Gamma)}, P_{\mathcal{I}(\Gamma)})$ is the bipartite graph formed by letting

$$V_{\mathcal{I}(\Gamma)} = V_0 \cup V_1,$$

$$E_{\mathcal{I}(\Gamma)} = E_\Gamma \cup \{(x_i, s), (s, x_i) : s \in S_i,\ i \in \mathbb{N}\} \cup \{(y_j, t), (t, y_j) : t \in T_j,\ j \in \mathbb{N}\},$$

and

$$P_{\mathcal{I}(\Gamma)} = (V_0 \times V_0) \cup (V_1 \times V_1).$$

Roughly speaking, $\mathcal{I}(\Gamma)$ is the bipartite graph formed from $\Gamma$ by adding, for each finite subset $S$ of $U_0$ and for each finite subset $T$ of $U_1$, a vertex $x$ to the partition set $U_1$ and a vertex $y$ to the partition set $U_0$ such that there is an edge between $x$ and every member of $S$ and an edge between $y$ and every member of $T$.

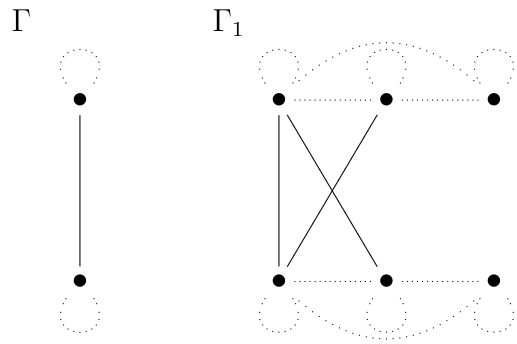
Note that the induced bipartite graph $\langle V_\Gamma \rangle$ of $\mathcal{I}(\Gamma)$ is just $\Gamma$ itself. If $\Gamma$ is finite then $|V_{\mathcal{I}(\Gamma)}| = 2^{|V_0|} + 2^{|V_1|} + |V_\Gamma|$ so that $\mathcal{I}(\Gamma)$ is a finite bipartite graph. If instead $\Gamma$ is countably infinite then, since the set of all finite subsets of a countable set is countable, $\{y_j : j \in \mathbb{N}\}$ and $\{x_i : i \in \mathbb{N}\}$ are countable and hence $\mathcal{I}(\Gamma)$ is a countably infinite bipartite graph.

Now inductively define a sequence of graphs by setting $\Gamma_0 = \Gamma$ and $\Gamma_{n+1} = \mathcal{I}(\Gamma_n)$ for $n \in \mathbb{N}$. Notice that by construction, $P_{\Gamma_n} \subseteq P_{\Gamma_{n+1}}$ for all $n \in \mathbb{N}$. Now define $\Gamma_\infty$ to be the limit of this process in the sense that,

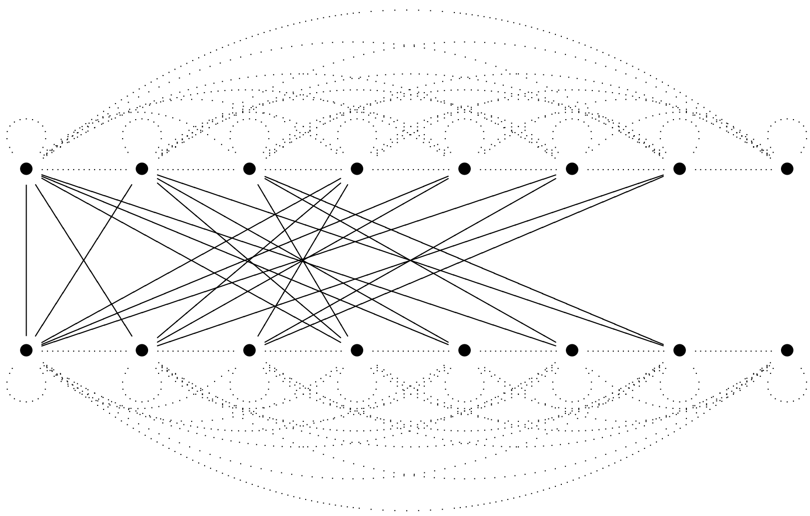
$$\Gamma_\infty = \bigcup_{n\in\mathbb{N}} \Gamma_n = \left( \bigcup_{n\in\mathbb{N}} V_{\Gamma_n}, \bigcup_{n\in\mathbb{N}} E_{\Gamma_n}, \bigcup_{n\in\mathbb{N}} P_{\Gamma_n} \right).$$

Then $\Gamma_\infty$ is a bipartite graph. This should be clear since at each stage $(V_{\Gamma_n}, E_{\Gamma_n})$ is a graph satisfying the bipartite condition, with $P_{\Gamma_{n-1}} \subseteq P_{\Gamma_n}$. Furthermore, since $\Gamma_{n+1}$ contains $\Gamma_n$ as an induced bipartite graph for each $n \in \mathbb{N}$, $\Gamma_\infty$ contains $\Gamma$ as an induced bipartite graph.

**Example 7.10.** [Construction of $\Gamma_1$ and $\Gamma_2$ given $\Gamma$.]



$\Gamma_2$



**Theorem 7.11.** *Let $\Gamma$ be a countable bipartite graph. Then the bipartite graph $\Gamma_\infty$ is existentially closed and thus $\Gamma_\infty \cong B$.*

*Proof.* Let $\Gamma_\infty = (V_0 \cup V_1, E, P)$ and suppose that $T_0, U_0 \subseteq V_0$ and $T_1, U_1 \subseteq V_1$ are finite disjoint subsets. Then $T_0, T_1, U_0, U_1 \subseteq V_{\Gamma_k}$ for some $k \in \mathbb{N}$. By construction of $\Gamma_{k+1}$, there exists a vertex $x \in V_1 \setminus V_{\Gamma_k}$ such that $(x, s) \in E$ for all $s \in T_0$ and such that $(x, s') \notin E$ for all $s' \in V_{\Gamma_k} \setminus T_0$. Similarly there must exist a vertex $y \in V_0 \setminus V_{\Gamma_k}$ such that $(y, t) \in E$ for all $t \in T_1$ and such that $(y, t') \notin E$ for all $t' \in V_{\Gamma_k} \setminus T_1$. In particular this means that the vertices $x$ and $y$ are such that:

(i) $(x, s) \in E$ for all $s \in T_0$,

(ii) $(x, u) \notin E$ for all $u \in U_0$,

(iii) $(y, t) \in E$ for all $t \in T_1$, and

(iv) $(y, v) \notin E$ for all $v \in U_1$.

Furthermore, since $T_0, T_1, U_0, U_1 \subseteq V_{\Gamma_k}$, it follows that $x \in V_1 \setminus (T_1 \cup U_1)$ and $y \in V_0 \setminus (T_0 \cup U_0)$. Thus existential closure holds for the sets $T_0, T_1, U_0$ and $U_1$ inside $\Gamma_{k+1}$. Since the construction process makes no changes to the edge set of $\Gamma_{k+1}$, existential closure holds for the sets $T_0, T_1, U_0$ and $U_1$ inside and in $\Gamma_\infty$ and the result follows. $\square$

Clearly, if $\Gamma$ is a finite bipartite graph, then $\Gamma$ embeds into $\Gamma_\infty$ by identifying $\Gamma$ with $\Gamma_0$. Thus since $\Gamma_\infty \cong B$ for all finite bipartite graphs $\Gamma$, Theorem 7.7 should now be clear.

## 7.3 Group $\mathscr{H}$-classes of End($B$)

As discussed earlier in Theorem 2.7, the group $\mathscr{H}$-classes of End($B$) are isomorphic to the automorphism groups of the bipartite graphs induced by the images of idempotents in End($B$). We will show that if $\Gamma$ is a countable graph, then there exist $2^{\aleph_0}$ group $\mathscr{H}$-classes $H$ of End($B$) such that $H \cong \text{Aut}(\Gamma)$. To do this, we first provide some results on the structure of the bipartite graphs which arise as the images of idempotents from End($B$).

**Definition 7.12.** A bipartite graph $\Gamma = (V_0 \cup V_1, E, P)$ will be called *algebraically closed* (in the class of bipartite graphs) if for all finite sets $U_0 \subseteq V_0$ and $U_1 \subseteq V_1$ there exist vertices $x \in V_1$ and $y \in V_0$ such that $x$ is adjacent to every member of $U_0$ and $y$ is adjacent to every member of $U_1$.

Unless otherwise stated, for the rest of this chapter the phrase 'algebraically closed' will be used to mean algebraically closed in the class of bipartite graphs. Unlike algebraically closed graphs or directed graphs, algebraically closed bipartite graphs can be finite. For example the bipartite graph,

$$\Lambda = (\{u, v\}, \{(u, v), (v, u)\}, \{(u, u), (v, v)\})$$

is algebraically closed. This is a major difference to the previous classes of relational structures. In particular, it means that we are no longer able to deduce that any two finite sets $U_0 \subseteq V_0$ and $U_1 \subseteq V_1$ have an infinite set of vertices to satisfy algebraic closure. Instead we make the following separate definition.

**Definition 7.13.** A bipartite graph $\Gamma = (V_0 \cup V_1, E, P)$ will be called *strongly algebraically closed* (in the class of bipartite graphs) if for all finite sets $U_0 \subseteq V_0$ and $U_1 \subseteq V_1$ there exist infinitely many vertices $x \in V_1$ and $y \in V_0$ such that $x$ is adjacent to every member of $U_0$ and $y$ is adjacent to every member of $U_1$.

We can easily provide an example of a strongly algebraically closed bipartite graph. For consider the bipartite graph $\Omega = (V_\Omega, E_\Omega, P_\Omega)$, with bipartition $V_\Omega = V_0 \cup V_1$ where $|V_0| = |V_1| = \aleph_0$ and $(u, v), (v, u) \in E_\Omega$ for all $u \in V_0$, $v \in V_1$. Then clearly $\Omega$ satisfies the conditions for strong algebraic closure. Additionally $B$ is strongly algebraically closed since it is existentially closed (recall that an existentially closed bipartite graph was defined in Definition 7.6).

**Lemma 7.14.** *Let $\Gamma$ be a countable bipartite graph and let $f \in \text{End}(\Gamma)$. Suppose that* $\operatorname{im} f$ *is strongly algebraically closed. Then both parts of $\Gamma$ are countably infinite.*

*Proof.* Let $\Gamma = (V_0 \cup V_1, E_\Gamma, P_\Gamma)$ and suppose without loss of generality that $v \in V_0$. Then there must exist infinitely many vertices in $V_1$ adjacent to $v$ and hence $V_1$ is infinite. A similar argument shows that $V_0$ is infinite. Since $\Gamma$ is countable the result now follows. $\square$

An important observation for the application of Lemma 7.2 is that for any algebraically closed graph $\Gamma$, $E_\Gamma \neq \emptyset$. To see this suppose that $V_\Gamma = V_0 \cup V_1$ is the bipartition of $\Gamma$. Since $V_\Gamma$ is non-empty we can assume without loss of generality that there exists a vertex $x \in V_0$. Then $\Gamma$ being algebraically closed guarantees the fact that there exists a vertex $y \in V_1$ such that $(x, y) \in E_\Gamma$. Furthermore the following lemma holds.

**Lemma 7.15.** *Let $\Gamma = (V_\Gamma, E_\Gamma, P_\Gamma)$ be an algebraically closed bipartite graph. Then $\Gamma$ is connected and hence $\mathrm{Aut}(\Gamma) = \mathrm{Aut}((V_\Gamma, E_\Gamma))$.*

*Proof.* Let $V_\Gamma = V_0 \cup V_1$ be the bipartition of $\Gamma$ and suppose that $u, v \in V_\Gamma$. If $u, v \in V_0$ then since $\Gamma$ is algebraically closed there exists a vertex $x \in V_1$ such that $(u, x)(x, v) \in E_\Gamma$. Thus there exists a path of length two from $u$ to $v$. If $u, v \in V_1$, then a similar argument leads to the same conclusion. Now suppose without loss of generality that $u \in V_0$ and $v \in V_1$. Once again, the fact that $\Gamma$ is algebraically closed ensures the existence of vertices $x \in V_1$ and $y \in V_0$ such that $(u, x), (x, y), (y, v) \in E_\Gamma$. Hence there is path of length three from $u$ to $v$ in this case. Since $u$ and $v$ were arbitrary we can conclude that $(V_\Gamma, E_\Gamma)$ is connected and hence so is $\Gamma$. Now by Lemma 7.4 $\mathrm{Aut}(\Gamma) = \mathrm{Aut}((V_\Gamma, E_\Gamma))$, as required. $\square$

The next two lemmas give the first step in determining the structure of the bipartite graphs which arise as images of idempotents in $\mathrm{End}(B)$.

**Lemma 7.16.** *Let $\Gamma$ be a countable algebraically closed bipartite graph and let $f \in \mathrm{End}(\Gamma)$. Then $\mathrm{im}\, f$ is a countable algebraically closed bipartite graph.*

*Proof.* Let $\Gamma = (V_0 \cup V_1, E_\Gamma, P_\Gamma)$ and let $f \in \mathrm{End}(\Gamma)$. By definition $\mathrm{im}\, f$ is an induced substructure of $\Gamma$ and hence is a countable bipartite graph. To check that it is algebraically closed, let $U_0 \subseteq V_0 \cap \mathrm{im}\, f$ and $U_1 \subseteq V_1 \cap \mathrm{im}\, f$ be any finite subsets. Enumerate $U_0$ as $\{u_i : 1 \leq i \leq m\}$ and $U_1$ as $\{v_j : 1 \leq j \leq n\}$. Since each $u_i$ $(1 \leq i \leq m)$ lies in the image of $f$ there exists a vertex $s_i \in V_0 \cup V_1$ such that $s_i f = u_i$. Let $S = \{s_i : 1 \leq i \leq m\}$. Since $U_0 \subseteq V_0$, Lemma 7.2 allows the conclusion that either $S \subseteq V_0$ or $S \subseteq V_1$. So suppose, without loss of generality, that $S \subseteq V_0$. Since each $v_j$ $(1 \leq j \leq n)$ also lies in the image of $f$ there exist vertices $t_j \in V_0 \cup V_1$ such that $t_j f = v_j$. If we let $T = \{t_j : 1 \leq j \leq n\}$, then $T \subseteq V_1$ by a further application of Lemma 7.2. Now, since $\Gamma$ is algebraically closed there must exist vertices $x \in V_1$

and $y \in V_0$ such that $(x, s_i) \in E_\Gamma$ and $(y, t_j) \in E_\Gamma$ for all $1 \leq i \leq m$ and for all $1 \leq j \leq n$. Then $xf \in V_1 \cap \operatorname{im} f$, $yf \in V_0 \cap \operatorname{im} f$ and, since $f$ is a bipartite graph endomorphism, $(xf, u_i) \in E_\Gamma$ and $(yf, v_j) \in E_\Gamma$ for all for all $1 \leq i \leq m$ and for all $1 \leq j \leq n$. □

**Corollary 7.17.** *Let $f \in \operatorname{End}(B)$. Then $\operatorname{im} f$ is a countable algebraically closed bipartite graph.*

*Proof.* By definition, $B$ is a countable bipartite graph. Furthermore, since $B$ is existentially closed, it is algebraically closed. Thus, by Lemma 7.16, $\operatorname{im} f$ is an algebraically closed bipartite graph. □

**Lemma 7.18.** *Let $\Gamma = (V_\Gamma, E_\Gamma, P_\Gamma)$ be a bipartite graph and let $f : \Gamma \to \Gamma$ be a homomorphism such that $\operatorname{im} f$ is algebraically closed. Let $\mathcal{I}(\Gamma)$ be the bipartite graph formed from $\Gamma$ as in Definition 7.9. Then there exists an extension $\tilde{f} : \mathcal{I}(\Gamma) \to \mathcal{I}(\Gamma)$ of $f$ such that $\tilde{f}$ is a homomorphism and $\operatorname{im} \tilde{f} = \operatorname{im} f$ and if $f$ is idempotent then so is $\tilde{f}$. Furthermore if $\operatorname{im} f$ is strongly algebraically closed then there exist $2^{\aleph_0}$ such extensions.*

*Proof.* Let $\mathcal{I}(\Gamma) = (V_0 \cup V_1, E, P)$. Enumerate the vertices of $V_{\mathcal{I}(\Gamma)} \setminus V_\Gamma$ as $\{v_i : i \in \mathbb{N}\}$ replacing the natural numbers with a finite set if necessary. Let $T_0 = V_\Gamma \cap V_0$ and let $T_1 = V_\Gamma \cap V_1$ so that $V_\Gamma = T_0 \cup T_1$ is the bipartition of $\Gamma$. For each $i \in \mathbb{N}$ the vertex $v_i$ is such that $(v_i, u) \in E$ if and only if $u$ lies in a specific finite subset $U_i$ of either $T_0$ or $T_1$. We will inductively define a sequence of maps $f_i : \langle V_\Gamma \cup \{v_1, \ldots, v_i\}\rangle \to \mathcal{I}(\Gamma)$ as follows.

First, let $f_0 = f$ and suppose that for $n \in \mathbb{N}$ we can extend $f$ to a homomorphism $f_n : \langle V_\Gamma \cup \{v_1, \ldots, v_n\}\rangle \to \mathcal{I}(\Gamma)$ with $\operatorname{im} f_n = \operatorname{im} f$. Let $V_\Gamma \cup \{v_1, \ldots, v_n\} = S_0 \cup S_1$ be the induced bipartition of $\langle V_\Gamma \cup \{v_1, \ldots, v_i\}\rangle$. That is

$$S_0 = V_0 \cap (V_\Gamma \cup \{v_1, \ldots, v_n\}) = T_0 \cup (V_0 \cap \{v_1, \ldots, v_n\}),$$

and

$$S_1 = V_1 \cap (V_\Gamma \cup \{v_1, \ldots, v_n\}) = T_1 \cup (V_1 \cap \{v_1, \ldots, v_n\}).$$

Suppose without loss of generality that $v_{n+1} \in V_1$ and that $U_{n+1} \subseteq T_0$. Since $f_n$ is a bipartite graph homomorphism and $\operatorname{im} f_n = \operatorname{im} f$, Lemma 7.2 tells us that either $S_0 f_n \subseteq T_0$ or $S_0 f_n \subseteq T_1$. Suppose without loss of generality that $S_0 f_n \subseteq T_0$. Now, since $\Gamma$ is algebraically closed, $E_\Gamma \neq \emptyset$ and a further application of Lemma 7.2 allows the conclusion that $S_1 f_n \subseteq T_1$. Then $(U_{n+1})f \subseteq T_0$ and since $\operatorname{im} f$ is algebraically closed, there exists a vertex $x \in \operatorname{im} f$ such that $(x, vf) \in E$ for all $v \in U_{n+1}$.

Now define $f_{n+1} : \langle V_\Gamma \cup \{v_1, \ldots, v_{n+1}\}\rangle \to \mathcal{I}(\Gamma)$ by

$$uf_{n+1} = \begin{cases} uf & \text{if } u \in V_\Gamma \cup \{v_1, \ldots, v_n\}, \\ x & \text{if } u = v_{n+1}. \end{cases}$$

Then $f_{n+1}$ defines a map of vertices $V_\Gamma \cup \{v_1, \ldots, v_{n+1}\} \to V_{\mathcal{I}(\Gamma)}$. By assumption, $f_n$ is a bipartite graph homomorphism extending $f$. Hence $f_n$ defines a graph homomorphism from the induced subgraph $\langle V_\Gamma \cup \{v_1, \ldots, v_n\}\rangle$ of $(V_{\mathcal{I}(\Gamma)}, E_{\mathcal{I}(\Gamma)})$ to the graph $(V_{\mathcal{I}(\Gamma)}, E_{\mathcal{I}(\Gamma)})$. If $(v_{n+1}, u) \in E$ then by the observation we made at the beginning of the proof, $u \in U_{n+1}$. Thus by choice of $x$, $(v_{n+1}f_{n+1}, uf_{n+1}) = (x, uf) \in E$. Hence $f_{n+1}$ defines a graph homomorphism from the induced subgraph $\langle V_\Gamma \cup \{v_1, \ldots, v_{n+1}\}\rangle$ of $(V_{\mathcal{I}(\Gamma)}, E)$ to the graph $(V_{\mathcal{I}(\Gamma)}, E)$. All that remains is to note that, since $f_n$ is a bipartite graph homomorphism, $(S_0)f_n \subseteq T_0$ and $(S_1)f_n \subseteq T_1$. Hence,

$$(S_0)f_{n+1} = (S_0)f_n \subseteq T_0,$$

and

$$(S_1 \cup \{v_{n+1}\})f_{n+1} = (S_1)f_n \cup \{x\} \subseteq T_1.$$

Thus by Lemma 7.2, $f_{n+1}$ is a homomorphism of bipartite graphs $\langle V_\Gamma \cup \{v_1, \ldots, v_{n+1}\}\rangle \to \mathcal{I}(\Gamma)$. Since $f_{n+1}$ is exactly $f_n$ when restricted to the domain of $f_n$ and since $v_{n+1}f \in \operatorname{im} f$ it is ensured that $\operatorname{im} f_{n+1} = \operatorname{im} f$. Furthermore if $f_n$ is idempotent then so is $f_{n+1}$ since $x \in \operatorname{im} f_n$ and thus,

$$v_{n+1}f_{n+1}^2 = xf_{n+1} = xf_{n+1} = x = v_{n+1}f_{n+1}.$$

Now let,

$$\tilde{f} = \bigcup_{i \in \mathbb{N}} f_i.$$

Then $\tilde{f}$ is a bipartite graph homomorphism $\mathcal{I}(\Gamma) \to \mathcal{I}(\Gamma)$ extending $f_i$ for all $i \in \mathbb{N}$. If $f$ is idempotent then by construction so is $\tilde{f}$. Finally since $\operatorname{im} f_i = \operatorname{im} f$ for all $i \in \mathbb{N}$, $\operatorname{im} \tilde{f} = \operatorname{im} f$ as required.

Now suppose that $\operatorname{im} f$ is in fact strongly algebraically closed. Then in the construction of $f_{n+1}$ above, there exist infinitely many vertices $x \in \operatorname{im} f \setminus U_{n+1}f$ such that $(x, vf) \in E$ for all $v \in U_{n+1}$. That is to say, there are infinitely many choices for the image of $v_{n+1}$ when constructing $f_{n+1}$. As a result, there are infinitely many distinct extensions $f_{n+1}$ of $f_n$ which differ on $v_{n+1}$. By definition $v_{n+1}$ is not adjacent to $v_m$ for all $m \in \mathbb{N}$ and hence the choice of vertex made for $v_{n+1}f_{n+1}$ is independent of any $v_m f_m$ for $m \leq n$. It follows then, that there are ${\aleph_0}^{\aleph_0} = 2^{\aleph_0}$ distinct extensions $\tilde{f}$ of $f$. $\square$

The notable difference in Lemma 7.18, when compared with the analogous results (*i.e.* Lemmas 3.9 and 4.9) in Chapters 3 and 4, is the fact that we can no longer guarantee the existence of $2^{\aleph_0}$ extensions without placing an additional condition on image of the homomorphism $f : \Gamma \to \Gamma$.

It is worth observing that if $\operatorname{im} f$ is algebraically closed but not strongly algebraically closed, then there can still exist $2^{\aleph_0}$ extensions $\tilde{f}$ of $f$, as constructed in Lemma 7.18, provided that there exists at least two distinct extensions $f_{n+1}$ of $f_n$ for infinitely many $n \in \mathbb{N}$. In other words there must exist at least two choices for the image of $v_{n+1}$ when constructing $f_{n+1}$ for infinitely many $n \in \mathbb{N}$.

**Theorem 7.19.** *Let $\Gamma$ be a countable bipartite graph. Then there exists $f \in E(\operatorname{End}(B))$ with $\operatorname{im} f \cong \Gamma$ if and only if $\Gamma$ is a countable algebraically closed bipartite graph. Furthermore, if $\Gamma$ is a countable strongly algebraically closed bipartite graph, then there exist $2^{\aleph_0}$ idempotents $f \in \operatorname{End}(B)$ with $\operatorname{im} f \cong \Gamma$.*

*Proof.* Let $f \in E(\operatorname{End}(B))$. Then by Corollary 7.17, $\operatorname{im} f$ is algebraically closed.

Conversely suppose that $\Gamma$ is a countable algebraically closed bipartite graph. Apply the construction given in Definition 7.9 to produce the bipartite graph $\Gamma_\infty$. By Lemma 7.11, $\Gamma_\infty = B$. Define inductively a sequence of functions $f_n : \Gamma_n \to \Gamma_\infty$ as follows. Let $f_0 : \Gamma_0 \to \Gamma_\infty$ be the identity on $\Gamma = \Gamma_0$. That is we let $vf_0 = v$ for all $v \in V_\Gamma$, so that $f_0$ is an idempotent homomorphism with $\operatorname{im} f_0 = \Gamma$. Now, for $n \in \mathbb{N}$, let $f_{n+1} = \tilde{f_n}$ where $\tilde{f_n}$ is the extension of $f_n$ to $\mathcal{I}(\Gamma_n) = \Gamma_{n+1}$ defined in Lemma 7.18. By the proof of Lemma 7.18, $f_{n+1}$ is an idempotent graph homomorphism and $\operatorname{im} f_{n+1} = \operatorname{im} f_0 = \Gamma$. Now let,

$$f = \bigcup_{n=0}^{\infty} f_n.$$

Then $f$ is a function $\Gamma_\infty \to \Gamma_\infty$. Furthermore, $f$ is a union of a sequence of idempotent bipartite graph homomorphisms each of which extends the previous. Thus $f$ is an idempotent bipartite graph homomorphism and since $\operatorname{im} f_n = \operatorname{im} f_0$ for all $n \in \mathbb{N}$, $\operatorname{im} f = \operatorname{im} f_0 = \Gamma$.

Finally, if $\Gamma$ is strongly algebraically closed, Lemma 7.18 guarantees that there exist $2^{\aleph_0}$ distinct extensions $\tilde{f_n}$ of $f_n$ for each $n \in \mathbb{N}$. Thus since there are $2^{\aleph_0}$ extensions $f_{n+1}$ of $f_n$ for each $n \in \mathbb{N}$, there are $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ many distinct idempotents $f \in \operatorname{End}(B)$ with $\operatorname{im} f = \Gamma$. $\square$

Theorems 2.7 and 7.19 now allow us to deduce that the group $\mathscr{H}$-classes of $\mathrm{End}(B)$ are exactly the automorphism groups of countable algebraically closed bipartite graphs. Once again we now find ourselves asking a familiar question: which groups can be realised as the automorphism group of a countable algebraically closed bipartite graph?

By Theorem 7.15, any group which is the automorphism group of a countable algebraically closed bipartite graph, is also the automorphism group of a graph. We will show that for any countable graph $\Gamma$, there exists a countable bipartite graph with the same automorphism group. As a consequence we will then show that we can construct a countable algebraically closed bipartite graph with the same automorphism group as $\Gamma$. We begin with the following construction and subsequent lemmas.

**Definition 7.20.** Given any countable graph $\Gamma = (V_\Gamma, E_\Gamma)$, we can produce a countable bipartite graph $\Gamma' = (V_{\Gamma'}, E_{\Gamma'}, P_{\Gamma'})$ in the following way. First, enumerate the vertices in $\Gamma$ by $V_\Gamma = \{v_i : i \in \mathbb{N}\}$, replacing the natural numbers by a finite set wherever necessary. We let $X$ be a set disjoint from $V_\Gamma$ with elements $x_{i,j}$ for all $i < j$ such that $(v_i, v_j) \in E_\Gamma$. That is we let

$$X = \{x_{i,j} : i < j \text{ and } (v_i, v_j) \in E_\Gamma\}.$$

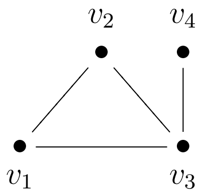
Define $\Gamma'$ by setting $V_{\Gamma'} = V_\Gamma \cup X$, $P_{\Gamma'} = (V_\Gamma \times V_\Gamma) \cup (X \times X)$ and

$$E_{\Gamma'} = \{(v_i, x_{i,j}), (v_j, x_{i,j}), (x_{i,j}, v_i), (x_{i,j}, v_j) : x_{ij} \in X)\}.$$

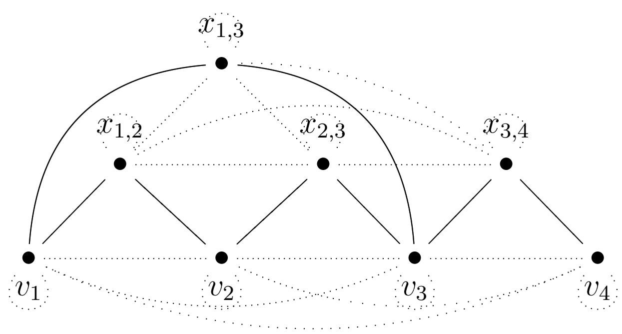
Intuitively, we are adding a vertex 'in the middle' of each edge of the graph $\Gamma$. The partition relation $P_{\Gamma'}$ then specifies that the bipartition consists of the set of vertices from $\Gamma$ and the set of new vertices, $X$, which are added to create $\Gamma'$.

**Example 7.21.** [Construction of $\Gamma'$ given $\Gamma$.]

$\Gamma$




$\Gamma'$



It easy to prove the following lemma using the construction of $\Gamma'$.

**Lemma 7.22.** *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a countable graph and let $\Gamma'$ be the bipartite graph constructed in Definition 7.20. Then the following hold.*

(i) *If $(u, v) \in E_{\Gamma'}$, then either $u \in X$ or $v \in X$ but not both.*

(ii) *If $u, v \in V_\Gamma$ and $(u, x), (x, v), (u, y), (y, v) \in E_{\Gamma'}$, then $x = y$.*

We will make repeated use of the following sets for a bipartite graph $\Lambda = (V_\Lambda, E_\Lambda, P_\Lambda)$. We will let,

$$\Lambda(v) = \{u \in V_\Lambda : (u, v) \in E_\Lambda\}, \text{ and}$$

$$\Lambda^*(v) = \{u \in V_\Lambda : (u, v) \notin E_\Lambda\}.$$

**Lemma 7.23.** *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a countable graph. Then $|\Gamma'(v)| = |\Gamma(v)|$ for all $v \in V_\Gamma$.*

*Proof.* Let $\Gamma' = (V_\Gamma \cup X, E_{\Gamma'}, P_{\Gamma'})$ as in Definition 7.20. If $E_\Gamma = \emptyset$, then $E_{\Gamma'} = \emptyset$ and we are done. So suppose $v \in V_\Gamma$ and suppose that $(u, v) \in E_\Gamma$. Then $(u, x), (x, v) \in E_{\Gamma'}$ for a unique $x \in X \subset V_{\Gamma'}$. Hence $|\Gamma(v)| \leq |\Gamma'(v)|$. Now suppose that $(v, y) \in E_{\Gamma'}$. Then by construction of $\Gamma'$, $y \in X$ and there exists a unique $w \in V_{\Gamma'}$ such that $(y, w) \in E_{\Gamma'}$ and $(v, w) \in E_\Gamma$. Thus, $|\Gamma'(v)| \leq |\Gamma(v)|$ and it follows that $|\Gamma'(v)| = |\Gamma(v)|$. $\square$

**Lemma 7.24.** *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a countable graph such that $|\Gamma(v)| \geq 3$ for all $v \in V_\Gamma$. Then $\mathrm{Aut}(\Gamma') \cong \mathrm{Aut}(\Gamma)$ and every automorphism of $\Gamma'$ is part fixing.*

*Proof.* Let $V_{\Gamma'} = V_\Gamma \cup X$ as in Definition 7.20. If $f \in \mathrm{Aut}(\Gamma')$ then $f : V_{\Gamma'} \to V_{\Gamma'}$ must define an automorphism of the graph $(V_{\Gamma'}, E_{\Gamma'})$. Hence, since each $x \in X$ has degree 2 and each $v \in V_\Gamma$ has degree at least 3, any automorphism of $\Gamma'$ must be such that $Xf = X$ and $V_\Gamma f = V_\Gamma$. In other words, every automorphism of $\Gamma'$ is part fixing. Now define a map $\phi : \mathrm{Aut}(\Gamma') \to \mathrm{Aut}(\Gamma)$ by $f\phi = f|_{V_\Gamma}$ for all $f \in V_{\Gamma'}$. By the previous comment $f|_{V_\Gamma}$ defines a bijection $V_\Gamma \to V_\Gamma$. Suppose that $(u, v) \in E_\Gamma$. Then there exists a unique $x \in X$ such that $(u, x), (x, v) \in E_{\Gamma'}$. Since $f$ is an automorphism of $\Gamma'$ we know that $(uf, xf), (xf, vf) \in E_{\Gamma'}$ with $xf \in X$. But, by construction of $\Gamma'$ this means that $(uf|_{V_\Gamma}, vf|_{V_\Gamma}) \in E_\Gamma$. Similarly it is easy to show that $(u, v) \notin E_\Gamma$ implies that $(uf|_{V_\Gamma}, vf|_{V_\Gamma}) \notin E_\Gamma$. Hence, $f|_{V_\Gamma}$ defines an automorphism on $\Gamma$.

The map $\phi$ defines a homomorphism of groups since $(fg)|_{V_\Gamma} = f|_{V_\Gamma} \cdot g|_{V_\Gamma}$. It remains to check that $\phi$ is both injective and surjective. Since each $x \in X$

is joined to two unique vertices $u, v \in V_\Gamma$, the images of $u$ and $v$ under an isomorphism of $\Gamma'$ determines the image of $x$ completely. Thus, if $f, g \in \mathrm{Aut}(\Gamma')$ are such that $f|_{V_\Gamma} = g|_{V_\Gamma}$ then $f|_X = g|_X$ and so $f = g$ and $\phi$ is injective. On a similar note, $\phi$ is surjective since if $h \in \mathrm{Aut}(\Gamma)$ then we can easily extend $h$ to an automorphism $\tilde{h}$ of $\Gamma'$ as follows. For $x \in X$, we let $x\tilde{h} = y$, where $y$ is the unique vertex in $X$ such that $(uf, y), (y, vf) \in E_{\Gamma'}$ for some $u, v \in V_\Gamma$. Then $\tilde{h}\phi = h$ and $\phi$ is a surjection. As a result we can finish by concluding that $\phi$ is then a group isomorphism. □

We now have the following theorem.

**Theorem 7.25.** *Let $\Gamma$ be a countable graph. Then there exists a countable bipartite graph $\Lambda$ such that $\mathrm{Aut}(\Lambda) \cong \mathrm{Aut}(\Gamma)$ and such that every automorphism of $\Lambda$ is part fixing.*

*Proof.* Let $\Psi \subseteq \mathbb{N} \setminus \{0, 1\}$. Recall that the infinite graph $L_\Psi$, constructed in Definition 3.18, is a countably infinite, locally finite graph. Let $\Delta_\Psi = (\Gamma \,\dot\cup\, L_\Psi)^\dagger$, the complement of the disjoint union of the graph $\Gamma$ and the graph $L_\Psi$. Now let $v \in V_{(\Gamma \dot\cup L_\Psi)^\dagger} = V_\Gamma \cup V_{L_\Psi}$. If $v \in V_\Gamma$, then since $v$ is adjacent to no vertex of $V_{L_\Psi}$ in $\Gamma \,\dot\cup\, L_\Psi$ it follows that $v$ is adjacent to every vertex of $V_{L_\Psi}$ in $(\Gamma \,\dot\cup\, L_\Psi)^\dagger$. Hence since $V_{L_\Psi}$ is infinite, $v$ has infinite degree. If on the other hand $v \in V_{L_\Psi}$, then since $L_\Psi$ is locally finite, $v$ is not adjacent to infinitely many vertices of $V_{L_\Psi}$ in $\Gamma \,\dot\cup\, L_\Psi$. Thus $v$ is adjacent to infinitely many vertices of $V_{L_\Psi}$ in $(\Gamma \,\dot\cup\, L_\Psi)^\dagger$ and hence has infinite degree. Thus every vertex in $\Delta_\Psi = (\Gamma \,\dot\cup\, L_\Psi)^\dagger$ has infinite degree. By Lemma 3.24, there exists $\Psi \subseteq \mathbb{N} \setminus \{0, 1\}$ such that $\mathrm{Aut}(\Delta_\Psi) \cong \mathrm{Aut}(\Gamma)$. Now let $\Delta'_\Psi$ be the bipartite graph constructed from $\Delta_\Psi$ as in Definition 7.20. By Lemma 7.24, $\mathrm{Aut}(\Delta'_\Psi) \cong \mathrm{Aut}(\Delta_\Psi) \cong \mathrm{Aut}(\Gamma)$ and every automorphism of $\Delta'_\Psi$ is part fixing. Taking $\Lambda = \Delta'_\Psi$ completes the proof. □

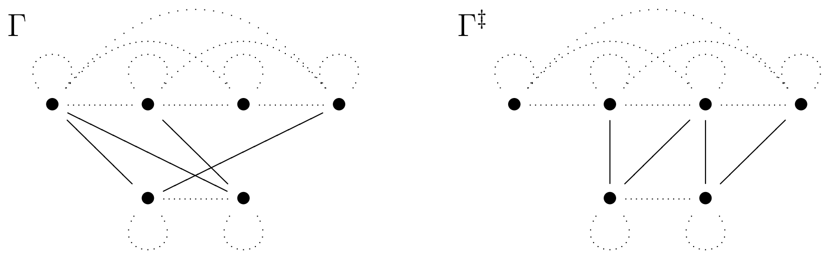
We have thus shown that for any countable graph $\Gamma$, there exists a countable bipartite graph with the same automorphism group. We will now show that if $\Lambda$ is a countable bipartite graph such that all automorphisms of $\Lambda$ are part fixing, then we can construct an algebraically closed bipartite graph with automorphism group isomorphic to $\mathrm{Aut}(\Lambda)$. The main theorem of this chapter, Theorem 7.38, then follows by an application of Theorems 7.19 and 7.25.

First we will define the complement in the setting of bipartite graphs. Let $\Gamma = (V_\Gamma, E_\Gamma, P_\Gamma)$ be a countable bipartite graph with bipartition $V_\Gamma = V_0 \cup V_1$. The *bipartite complement* of $\Gamma$ will be defined to be the relational structure $\Gamma^\ddagger = (V_{\Gamma^\ddagger}, E_{\Gamma^\ddagger}, P_{\Gamma^\ddagger})$, where $V_{\Gamma^\ddagger} = V_\Gamma$, $P_{\Gamma^\ddagger} = P_\Gamma$ and

$$E_{\Gamma^\ddagger} = (V_\Gamma \times V_\Gamma) \setminus \{E_\Gamma \cup (V_0 \times V_0) \cup (V_1 \times V_1)\}.$$

It should be easy to see that $\Gamma^\ddagger$ defines a bipartite graph with bipartition $V_{\Gamma^\ddagger} = V_0 \cup V_1$. The bipartite graph $\Gamma^\ddagger$ contains edges between vertices of $V_0$ and $V_1$ if and only if there are no such edges in $\Gamma$. Hence $(\Gamma^\ddagger)^\ddagger = \Gamma$ and so in this sense, $\Gamma^\ddagger$ is the bipartite graph complement to $\Gamma$.

**Example 7.26.** [Construction of $\Gamma^\ddagger$ given $\Gamma$.]



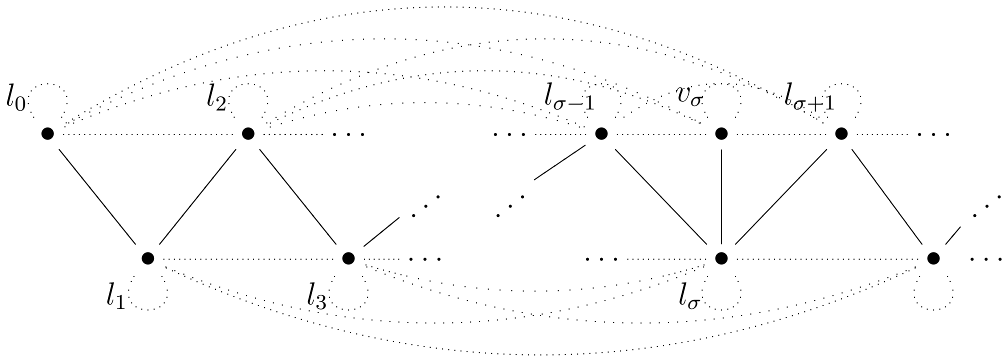

**Lemma 7.27.** *Let $\Gamma$ and $\Lambda$ be bipartite graphs and suppose that $f : \Gamma \to \Lambda$ is an embedding of $\Gamma$ into $\Lambda$. Then $f$ also defines an embedding of the bipartite complement $\Gamma^\ddagger$ into $\Lambda^\ddagger$.*

*Proof.* Let $\Gamma = (U_0 \cup U_1, E_\Gamma, P_\Gamma)$ and let $\Lambda = (V_0 \cup V_1, E_\Lambda, P_\Lambda)$. If $f$ is an embedding, then it is an injective function $V_\Gamma \to V_\Lambda$ and by Lemma 7.3, either $U_0 f \subseteq V_0$ and $U_1 f \subseteq V_1$ or $U_0 f \subseteq V_1$ and $U_1 f \subseteq V_0$. Thus $f$ defines an injective function $V_{\Gamma^\ddagger} \to V_{\Lambda^\ddagger}$ with the same properties. Now suppose that $(u, v) \in E_{\Gamma^\ddagger}$. Then $u$ and $v$ lie in distinct parts of $\Gamma$ and $u \neq v$. Thus $(u, v) \notin E_\Gamma$ and since $f$ is an embedding it follows that $(uf, vf) \notin E_\Lambda$. Since $u$ and $v$ were from distinct parts, $uf$ and $vf$ also lie in distinct parts and hence $(uf, vf) \in E_{\Lambda^\ddagger}$. On the other hand if $(u, v) \notin E_{\Gamma^\ddagger}$, then either $u$ and $v$ lie in the same part of $\Gamma$ or $(u, v) \in E_\Gamma$. In the former case it follows that $uf$ and $vf$ lie in the same part of $\Lambda$ and hence $(uf, vf) \notin E_{\Lambda^\ddagger}$. In the latter, $f$ being an embedding allows us to deduce that $(uf, vf) \in E_\Lambda$ and hence $(uf, vf) \notin E_{\Lambda^\ddagger}$. In either case we have shown that $f$ defines an embedding of $\Gamma^\ddagger$ into $\Lambda^\ddagger$. □

**Corollary 7.28.** *Let $\Gamma$ be a countable bipartite graph. Then* $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma^\ddagger)$.

*Proof.* Let $f \in \mathrm{Aut}(\Gamma)$. Then $f$ is a bijective function $V_\Gamma \to V_\Gamma$ which defines an embedding of $\Gamma$ into $\Gamma$. Hence by Lemma 7.27, $f$ also defines a bijective embedding of $\Gamma^\ddagger$ into $\Gamma^\ddagger$. Thus it follows that $f \in \mathrm{Aut}(\Gamma^\dagger)$. On the other hand suppose that $g \in \mathrm{Aut}(\Gamma^\ddagger)$. Then $g$ is a bijective function $V_{\Gamma^\dagger} \to V_{\Gamma^\dagger}$ which defines an embedding of $\Gamma^\ddagger$ into $\Gamma^\ddagger$. A second application of Lemma 7.27 tells us that $f$ also defines a bijective embedding of $(\Gamma^\ddagger)^\ddagger = \Gamma$ into itself. Hence $g \in \mathrm{Aut}(\Gamma)$ and the result is complete. □

Figure 7.3: The bipartite graph $\Lambda_\Sigma$.




Recall from Chapter 3 that, for $\Sigma \subseteq \mathbb{N} \setminus \{0,1\}$, $L_\Sigma = (V_{L_\Sigma}, E_{L_\Sigma})$ is the graph with,

$$V_{L_\Sigma} = \{l_n : n \in \mathbb{N}\} \cup \{v_\sigma : \sigma \in \Sigma\}$$

and

$$E_{L_\Sigma} = \{(l_i, l_{i+1}), (l_{i+1}, l_i) : i \in \mathbb{N}\} \cup \{(l_\sigma, v_\sigma), (v_\sigma, l_\sigma) : \sigma \in \Sigma\}.$$

We can show that $L_\Sigma$ satisfies the bipartite condition as follows. Define,

$$V_0 = \{l_i : i \in \mathbb{N},\ i \text{ is even}\} \cup \{v_\sigma : \sigma \in \Sigma,\ \sigma \text{ is odd}\},$$

and

$$V_1 = \{l_i : i \in \mathbb{N},\ i \text{ is odd}\} \cup \{v_\sigma : \sigma \in \Sigma,\ \sigma \text{ is even}\}.$$

Then it should be clear by inspection that $(u, v) \in E_{L_\Sigma}$ if and only if $u \in V_0$ and $v \in V_1$ or *vice versa*. Consequently, $P_{\Lambda_\Sigma} = (V_0 \times V_0) \cup (V_1 \times V_1)$ is a bipartition for $L_\Sigma$ and so $\Lambda_\Sigma = (V_{L_\Sigma}, E_{L_\Sigma}, P_{\Lambda_\Sigma})$ is a bipartite graph. See Figure 7.3 for a pictorial representation of $\Lambda_\Sigma$. Since $L_\Sigma$ is a connected graph, the bipartition of $L_\Sigma$ is unique and so $\Lambda_\Sigma$ is the unique bipartite graph constructed from $L_\Sigma$.

**Lemma 7.29.** *Let $\Sigma \subseteq \mathbb{N} \setminus \{0,1\}$ and let $\Lambda_\Sigma = (V_{L_\Sigma}, E_{L_\Sigma}, P_{\Lambda_\Sigma})$ be the bipartite graph defined above. Then* $\mathrm{Aut}(\Lambda_\Sigma) = \mathbf{1}$.

*Proof.* Recall that in Lemma 7.3 we proved that if $f$ is an automorphism of $\Lambda_\Sigma$, then $f$ must also define a graph automorphism on $L_\Sigma = (V_{L_\Sigma}, E_{L_\Sigma})$. In Lemma 3.19 we showed that the only automorphism of the graph $L = (V_{L_\Sigma}, E_{L_\Sigma})$ was the trivial automorphism, $\mathbf{1} : V_{L_\Sigma} \to V_{L_\Sigma}$. Thus, we can immediately conclude that $\mathrm{Aut}(\Lambda_\Sigma) = \mathbf{1}$. $\square$

**Lemma 7.30.** *Let $\Sigma, \Psi \subseteq \mathbb{N} \setminus \{0,1\}$. Then there exists a bipartite graph embedding $f : \Lambda_\Sigma \to \Lambda_\Psi$ if and only if $\Sigma + k \subseteq \Psi$ for some $k \in \mathbb{N}$.*

*Proof.* A consequence of Lemma 7.3 was that if $f$ is an embedding $\Lambda_\Sigma \to \Lambda_\Psi$, then $f$ also defined an embedding of $L_\Sigma$ into $L_\Psi$. But we know from Lemma 3.20, that $f$ defines an embedding of $L_\Sigma$ into $L_\Psi$ if and only if $\Sigma + k \subseteq \Psi$ for some $k \in \mathbb{N}$. On the other hand, if there does exist $k \in \mathbb{N}$ such that $\Sigma + k \subseteq \Psi$ then it can be shown that the embedding $g : V_{L_\Sigma} \to V_{L_\Psi}$ of $L_\Sigma$ into $L_\Psi$ defined in Lemma 3.20 defines a bipartite graph embedding of $\Lambda_\Sigma \to \Lambda_\Psi$. Thus we deduce that $f : \Lambda_\Sigma \to \Lambda_\Psi$ is a bipartite graph embedding if and only if $\Sigma + k \subseteq \Psi$ for some $k \in \mathbb{N}$. □

**Corollary 7.31.** *Let $\Sigma, \Psi \subseteq \mathbb{N} \setminus \{0,1\}$. Then $\Lambda_\Sigma \cong \Lambda_\Psi$ if and only if $\Sigma = \Psi$.*

*Proof.* We will again use the fact that by Lemma 7.3, if $f$ is an isomorphism $\Lambda_\Sigma \to \Lambda_\Psi$, then $f$ also defined an isomorphism of $L_\Sigma$ into $L_\Psi$. But by Corollary 3.21, $f$ defines an isomorphism of $L_\Sigma$ into $L_\Psi$ if and only if $\Sigma = \Psi$. Hence the result follows immediately. □

**Lemma 7.32.** *Let $\Sigma \subseteq \mathbb{N} \setminus \{0,1\}$. Then the bipartite complement $\Lambda_\Sigma^\ddagger$ of $\Lambda_\Sigma$ is a countable algebraically closed bipartite graph.*

*Proof.* Since $\Lambda_\Sigma$ is a bipartite graph by definition, $\Lambda_\Sigma^\ddagger$ is a bipartite graph. Recall that $\Lambda_\Sigma = (V_0 \cup V_1, E_{L_\Sigma}, P)$ where

$$V_0 = \{l_i : i \in \mathbb{N},\ i \text{ is even}\} \cup \{v_\sigma : \sigma \in \Sigma,\ \sigma \text{ is odd}\}, \text{ and}$$
$$V_1 = \{l_i : i \in \mathbb{N},\ i \text{ is odd}\} \cup \{v_\sigma : \sigma \in \Sigma,\ \sigma \text{ is even}\}.$$

Now let $U_0$ be a finite subset of vertices from $V_0$. Let $m = \max\{n \in \mathbb{N} : l_n \text{ or } v_n \in U_0\}$. Then $l_{m+3} \in V_1$ and $l_{m+3}$ is adjacent to no vertices from $U_0$ in $\Lambda_\Sigma$. Hence $l_{m+3}$ is adjacent to every vertex from $U_0$ in $\Lambda_\Sigma^\ddagger$. A similar argument for any finite subset of vertices $U_1$ of $V_1$ completes the result. □

The final construction required for our repertoire is the following. Let $\Gamma = (V_\Gamma, E_\Gamma, P_\Gamma)$ and $\Lambda = (V_\Lambda, E_\Lambda, P_\Lambda)$ be countable bipartite graphs with bipartitions $V_\Gamma = V_0 \cup V_1$ and $V_\Lambda = W_0 \cup W_1$ respectively. From $\Gamma$ and $\Lambda$, we can produce a new bipartite graph in the following way. We let $\Gamma \sqcup \Lambda$ be the bipartite graph with vertex set $V_{\Gamma \sqcup \Lambda} = V_\Gamma \cup V_\Lambda$, edge partition relation,

$$P_{\Gamma \sqcup \Lambda} = \big((V_0 \cup W_0) \times (V_0 \cup W_0)\big) \cup \big((V_1 \cup W_1) \times (V_1 \cup W_1)\big).$$

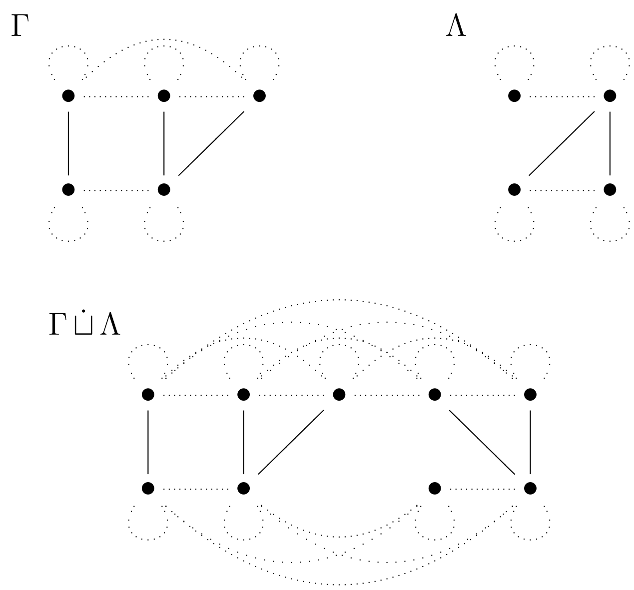
When the vertex sets are disjoint, $\Gamma \sqcup \Lambda$ will be called a *bipartite disjoint union* of $\Gamma$ and $\Lambda$ and this will be denoted by $\Gamma \dot{\sqcup} \Lambda$. It is worth noting that,

$$(V_{\Gamma \dot{\sqcup} \Lambda}, E_{\Gamma \dot{\sqcup} \Lambda}) = (V_\Gamma \cup V_\Lambda, E_\Gamma \cup E_\Lambda) = (V_\Gamma, E_\Gamma) \dot{\cup} (V_\Lambda, E_\Lambda).$$

In other words, $\Gamma \,\dot{\sqcup}\, \Lambda$ is the bipartite graph formed from the disjoint graph union $(V_\Gamma, E_\Gamma) \cup (V_\Lambda, E_\Lambda)$ with partition $(V_0 \cup W_0) \cup (V_1 \cup W_1)$.

There is some ambiguity in this definition since we could interchange the labels on the partition sets $W_0$ and $W_1$ to potentially create a different bipartite graph (with non-isomorphic automorphism group). However, this will not cause any problems for us.

**Example 7.33.** [One possible construction of $\Gamma \,\dot{\sqcup}\, \Lambda$ given $\Gamma$ and $\Lambda$.]




**Lemma 7.34.** *Let $\Gamma = (V_\Gamma, E_\Gamma, P_\Gamma)$ and $\Lambda = (V_\Lambda, E_\Lambda, P_\Lambda)$ be countable bipartite graphs such that the graphs $(V_\Gamma, E_\Gamma)$ and $(V_\Lambda, E_\Lambda)$ have no isomorphic components. Suppose that all elements of $\mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(\Lambda)$ are part fixing. Then, $\mathrm{Aut}(\Gamma \,\dot{\sqcup}\, \Lambda) \cong \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(\Lambda)$.*

*Proof.* Let $V_\Gamma = V_0 \cup V_1$ and let $V_\Lambda = W_0 \cup W_1$ be the partitions of $\Gamma$ and $\Lambda$ respectively. Recall that $V_{\Gamma \dot{\sqcup} \Lambda} = V_\Gamma \cup V_\Lambda$ and $E_{\Gamma \dot{\sqcup} \Lambda} = E_\Gamma \cup E_\Lambda$. If $f \in \mathrm{Aut}(\Gamma \,\dot{\sqcup}\, \Lambda)$, then $f$ defines an automorphism of the graph,

$$(V_{\Gamma \dot{\sqcup} \Lambda}, E_{\Gamma \dot{\sqcup} \Lambda}) = (V_\Gamma, E_\Gamma) \,\dot{\cup}\, (V_\Lambda, E_\Lambda).$$

Since the graphs $(V_\Gamma, E_\Gamma)$ and $(V_\Lambda, E_\Lambda)$ have no isomorphic components it must be the case that $V_\Gamma f = V_\Gamma$ and $V_\Lambda f = V_\Lambda$, for all $f \in \mathrm{Aut}(\Gamma \,\dot{\sqcup}\, \Lambda)$. Hence we can deduce that $f|_{V_\Gamma}$ defines an automorphism of the graph $(V_\Gamma, E_\Gamma)$ and similarly that $f|_{V_\Lambda}$ defines an automorphism of the graph $(V_\Lambda, E_\Lambda)$. Furthermore since $f$ is an automorphism we know from Lemma 7.3 that either,

$$(V_0 \cup W_0)f = V_0 \cup W_0 \text{ or } (V_0 \cup W_0)f = V_1 \cup W_1.$$

Suppose the latter holds true. Then our previous observation implies that $V_0 f = V_1$ and $W_0 f = W_1$ – a contradiction. Hence it must be the case that $(V_0 \cup W_0)f = V_0 \cup W_0$ and we can deduce that $V_0 f = V_0$ and $V_1 f = V_1$. A similar argument shows that $W_0 f = W_0$ and $W_1 f = W_1$. As a consequence, we can conclude that $f|_{V_\Gamma} \in \mathrm{Aut}(\Gamma)$ and $f|_{V_\Lambda} \in \mathrm{Aut}(\Gamma)$. Thus, if we define a map $\phi$ on $\mathrm{Aut}(\Gamma \dot{\sqcup} \Lambda)$ by $f\phi = (f|_{V_\Gamma}, f|_{V_\Lambda})$, for all $f \in \mathrm{Aut}(\Gamma \dot{\sqcup} \Lambda)$. Then $\phi$ defines a map $\mathrm{Aut}(\Gamma \dot{\sqcup} \Lambda) \to \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(\Lambda)$. The map $\phi$ defines a group homomorphism since for any pair $f, g \in \mathrm{Aut}(\Gamma \dot{\sqcup} \Lambda)$,

$$(fg)\phi = (fg|_{V_\Gamma}, fg|_{V_\Lambda}) = (f|_{V_\Gamma}, f|_{V_\Lambda}) \cdot (g|_{V_\Gamma}, g|_{V_\Lambda}) = f\phi \cdot g\phi.$$

The map $\phi$ is clearly injective since if $f$ and $g$ are two automorphisms of $\Gamma \dot{\sqcup} \Lambda$ such that $f\phi = g\phi$, then $f|_{V_\Gamma} = g|_{V_\Gamma}$ and $f|_{V_\Lambda} = g|_{V_\Lambda}$. Since $V_{\Gamma \dot{\sqcup} \Lambda} = V_\Gamma \cup V_\Lambda$ we can immediately conclude that $f = g$. It remains to check that $\phi$ is surjective. So, suppose that $(f, g) \in \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(\Lambda)$. Let $h$ be the map defined by:

$$vh = \begin{cases} vf & \text{if } u \in V_\Gamma, \\ vg & \text{if } u \in V_\Lambda. \end{cases}$$

Then $h$ defines a map $V_{\Gamma \dot{\sqcup} \Lambda} \to V_{\Gamma \dot{\sqcup} \Lambda}$. By assumption $f \in \mathrm{Aut}(\Gamma)$ is such that $V_0 f = V_0$ and hence $V_1 f = V_1$. Similarly we assumed that $W_0 g = W_0$ and hence $W_1 g = W_1$. Thus $(V_0 \cup W_0)h = V_0 \cup W_0$ and $(V_1 \cup W_1)h = V_1 \cup W_1$. Now, $f$ and $g$ are bipartite graph isomorphisms and so they define an automorphism on the graphs $(V_\Gamma, E_\Gamma)$ and $(V_\Lambda, E_\Lambda)$ respectively. Since there exist no edges between $V_\Gamma$ and $V_\Lambda$ in the graph $(V_\Gamma, E_\Gamma) \dot{\cup} (V_\Lambda, E_\Lambda)$ it follows that $h$ defines a bipartite graph isomorphism on $(V_{\Gamma \dot{\sqcup} \Lambda}, E_{\Gamma \dot{\sqcup} \Lambda})$. Now, an application of Lemma 7.3 allows the conclusion that $h$ is a bipartite graph isomorphism $\Gamma \dot{\sqcup} \Lambda \to \Gamma \dot{\sqcup} \Lambda$. Furthermore $h\phi = (f, g)$ and thus $\phi$ is a surjective group homomorphism. All together we have shown that $\phi$ defines a bijective group homomorphism and the result is complete. □

The next lemma exhibits the connection between bipartite complement, bipartite disjoint union and algebraic closure.

**Lemma 7.35.** *Let $\Gamma$ be a countable bipartite graph and let $\Lambda$ be a countable locally finite bipartite graph such that both partition sets are infinite. Then $(\Gamma \dot{\sqcup} \Lambda)^\ddagger$, the bipartite complement of a bipartite disjoint union of $\Gamma$ and $\Lambda$, is a countable strongly algebraically closed bipartite graph.*

*Proof.* Let $V_\Gamma = V_0 \cup V_1$ and $V_\Lambda = W_0 \cup W_1$ be the partitions of $\Gamma$ and $\Lambda$ respectively. Then $W_0$ and $W_1$ are countably infinite sets. Recall that $V_{(\Gamma \dot{\sqcup} \Lambda)^\ddagger} = V_\Gamma \cup V_\Lambda$ and that the partition of $(\Gamma \dot{\sqcup} \Lambda)^\ddagger$ is then given by $V_{(\Gamma \dot{\sqcup} \Lambda)^\ddagger} = (V_0 \cup W_0) \cup (V_1 \cup W_1)$. Since $V_\Gamma$ and $V_\Lambda$ are countable, so is $V_{(\Gamma \dot{\sqcup} \Lambda)^\ddagger}$ and hence $(\Gamma \dot{\sqcup} \Lambda)^\ddagger$ is a countable graph.

Now, let $U$ be a finite subset of $V_0 \cup W_0$. Then $U \cap W_0$ is finite. Since $\Lambda$ is locally finite and since $W_1$ is infinite there exist infinitely many vertices,

$$x \in W_1 \setminus \bigcup_{u \in U \cap W_0} \Lambda(u).$$

Since there does not exist an edge between $x$ and any member of $U \cup W_0$ in $\Lambda$ (and hence in $\Gamma \dot{\sqcup} \Lambda$), there must be an edge between $x$ and every member of $U \cup W_0$ in $(\Gamma \dot{\sqcup} \Lambda)^\ddagger$. Similarly since by construction there exists no edge between $x$ and any vertex in $U \cup V_0$ in $\Gamma \dot{\sqcup} \Lambda$, there must exist an edge between $x$ and every member of $U \cup V_0$ in $(\Gamma \dot{\sqcup} \Lambda)^\ddagger$. In other words, $(x, u) \in E_{(\Gamma \dot{\sqcup} \Lambda)^\ddagger}$ for all $u \in U$. A similar argument shows that for any finite subset $T \subseteq V_1 \cup W_1$, there exist infinitely many vertices $y \in W_0$ such that $(y, t) \in E_{(\Gamma \dot{\sqcup} \Lambda)^\ddagger}$ for all $t \in T$. $\square$

**Corollary 7.36.** *Let $\Gamma$ be any countable bipartite graph and let $\Sigma \subseteq \mathbb{N} \setminus \{0, 1\}$. Then $(\Gamma \dot{\sqcup} \Lambda_\Sigma)^\ddagger$, the bipartite complement of a bipartite disjoint union of $\Gamma$ and $\Lambda_\Sigma$, is a countable strongly algebraically closed bipartite graph.*

*Proof.* Recall that $\Lambda_\Sigma$ is locally finite and that the partition of $\Lambda_\Sigma$ is given by $\Lambda_\Sigma = V_0 \cup V_1$ where,

$$V_0 = \{l_i : i \in \mathbb{N},\ i \text{ is even}\} \cup \{v_\sigma : \sigma \in \Sigma,\ \sigma \text{ is odd}\},$$

$$V_1 = \{l_i : i \in \mathbb{N},\ i \text{ is odd}\} \cup \{v_\sigma : \sigma \in \Sigma,\ \sigma \text{ is even}\}.$$

Thus, both $V_0$ and $V_1$ are infinite sets. Now by Lemma 7.35, $(\Gamma \dot{\sqcup} \Lambda_\Sigma)^\ddagger$ is a countable strongly algebraically closed bipartite graph. $\square$

The next lemma now shows that for any countable bipartite graph which has only part fixing automorphisms, there exist $2^{\aleph_0}$ algebraically closed bipartite graphs with the same automorphism group.

**Lemma 7.37.** *Let $\Gamma$ be a countable bipartite graph such that all automorphisms of $\Gamma$ are part fixing. Then there exist $2^{\aleph_0}$ subsets $\Sigma \subseteq \mathbb{N} \setminus \{0, 1\}$ such that* $\mathrm{Aut}((\Gamma \dot{\sqcup} \Lambda_\Sigma)^\ddagger) \cong \mathrm{Aut}(\Gamma)$.

*Proof.* Let $\Gamma = (V_\Gamma, E_\Gamma, P_\Gamma)$. Since $\Gamma$ is a countable bipartite graph, the graph $(V_\Gamma, E_\Gamma)$ is countable and thus has only countably many connected components. Thus, at most countably many choices for $\Sigma \subseteq \mathbb{N} \setminus \{0,1\}$ would create a $\Lambda_\Sigma$ with $(V_{\Lambda_\Sigma}, E_{\Lambda_\Sigma})$ isomorphic to a component in $(V_\Gamma, E_\Gamma)$. Since the set of all subsets of the natural numbers is uncountable, this leaves $2^{\aleph_0}$ choices of the subset $\Sigma \subseteq \mathbb{N} \setminus \{0,1\}$ which ensure that $\Lambda_\Sigma$ is isomorphic to no components in $\Gamma$.

For each of these distinct choices we can deduce from Corollary 7.28 that $\mathrm{Aut}((\Gamma \,\dot\sqcup\, \Lambda_\Sigma)^\ddagger) = \mathrm{Aut}(\Gamma \,\dot\sqcup\, \Lambda_\Sigma)$. Now, by Lemma 7.29, $\mathrm{Aut}(\Lambda_\Sigma) = \mathbf{1}$, thus the sole automorphism of $\Lambda_\Sigma$ fixes its bipartition. Furthermore, by assumption, any automorphism of $\Gamma$ fixes its bipartition. Thus since $\Gamma$ and $\Lambda_\Sigma$ have no isomorphic components, Lemma 7.34 allows the conclusion that $\mathrm{Aut}(\Gamma \,\dot\sqcup\, \Lambda_\Gamma) \cong \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(\Lambda_\Sigma)$. Hence,

$$\mathrm{Aut}(\Gamma \,\dot\sqcup\, \Lambda_\Gamma) \cong \mathrm{Aut}(\Gamma) \times \mathbf{1} \cong \mathrm{Aut}(\Gamma).$$

Putting this all together gives the required result. □

We can now state and prove the main theorem for this chapter.

**Theorem 7.38.** *Let $\Gamma$ be a countable graph. Then there exist $2^{\aleph_0}$ group $\mathscr{H}$-classes $H$ of* $\mathrm{End}(B)$ *such that $H \cong \mathrm{Aut}(\Gamma)$.*

*Proof.* First, by Theorem 7.25, there exists a bipartite graph $\Delta_\Gamma$ such that $\mathrm{Aut}(\Delta_\Gamma) \cong \mathrm{Aut}(\Gamma)$ and such that all automorphisms of $\Delta_\Gamma$ are part fixing. Now, by Lemma 7.37 there exist $2^{\aleph_0}$ sets $\Sigma \subseteq \mathbb{N}\setminus\{0,1\}$ such that $\Lambda_\Sigma$ is isomorphic to no component of $\Delta_\Gamma$ and such that $\mathrm{Aut}((\Delta_\Gamma \,\dot\sqcup\, \Lambda_\Sigma)^\ddagger) \cong \mathrm{Aut}(\Gamma)$ Furthermore, for each such choice of $\Sigma$, Corollary 7.36 ensures that $(\Delta_\Gamma \,\dot\sqcup\, \Lambda_\Sigma)^\ddagger$ is algebraically closed and so by Theorem 7.19, there exists an idempotent $f_\Sigma \in \mathrm{End}(B)$ such that $\mathrm{im}\, f_\Sigma \cong (\Delta_\Gamma \,\dot\sqcup\, \Lambda_\Sigma)^\ddagger$. Now Theorem 2.7 allows the conclusion that,

$$H_{f_\Sigma} \cong \mathrm{Aut}((\Delta_\Gamma \,\dot\sqcup\, \Lambda_\Sigma)^\ddagger) \cong \mathrm{Aut}(\Gamma).$$

By Corollary 7.31, $\Lambda_\Sigma$ is not isomorphic to $\Lambda_\Psi$ for $\Sigma \neq \Psi$ and since both are isomorphic to no component of $\Delta_\Gamma$. Hence we can deduce that $\Delta_\Gamma \,\dot\sqcup\, \Lambda_\Sigma$ and $\Delta_\Gamma \,\dot\sqcup\, \Lambda_\Psi$ are not isomorphic for $\Sigma \neq \Psi$. In other words $\mathrm{im}\, f_\Sigma \neq \mathrm{im}\, f_\Psi$ for $\Gamma \neq \Psi$ and the idempotents are all distinct. Since no $\mathscr{H}$-class can contain more than one idempotent, the result now follows. □

**Corollary 7.39.** *Let $G$ be a countable group. Then there exist $2^{\aleph_0}$ group $\mathscr{H}$-classes $H$ of* $\mathrm{End}(B)$ *such that $H \cong G$.*

*Proof.* If $G$ is a countable group then, by the extension of Frucht's Theorem (Theorem 3.12), there exists a countable graph $\Gamma$ such that $\mathrm{Aut}(\Gamma) \cong G$. An application of Theorem 7.38 now gives the required result. $\square$

**Corollary 7.40.** *Let $\Gamma$ be a countable algebraically closed bipartite graph. Then there exist $2^{\aleph_0}$ group $\mathscr{H}$-classes $H$ of* $\mathrm{End}(B)$ *such that* $H \cong \mathrm{Aut}(\Gamma)$.

*Proof.* Let $\Gamma = (V_\Gamma, E_\Gamma, P_\Gamma)$. Since $\Gamma$ is algebraically closed, it is both connected and $\mathrm{Aut}((V_\Gamma, E_\Gamma)) = \mathrm{Aut}(\Gamma)$ by Lemma 7.15. Thus by applying Theorem 7.38 to the graph $(V_\Gamma, E_\Gamma)$ the result follows. $\square$

Theorem 7.38 tells us that every group which arises as a group $\mathscr{H}$-class of $\mathrm{End}(R)$ appears as a group $\mathscr{H}$-class of $\mathrm{End}(B)$ and we can now state and prove the following additional theorem.

**Theorem 7.41.** *The groups arising as group $\mathscr{H}$-classes of* $\mathrm{End}(B)$ *are the same (up to isomorphism) as those of* $\mathrm{End}(R)$ *and thus* $\mathrm{End}(D)$.

*Proof.* Let $H$ be a group $\mathscr{H}$-class of $\mathrm{End}(R)$ and so let $f \in E(\mathrm{End}(R))$ be the idempotent identity of $H$. Then $\mathrm{im}\, f$ is an (algebraically closed) graph. Now by Theorem 7.38 there exists an idempotent $g \in \mathrm{End}(B)$ such that $H_g \cong \mathrm{Aut}(\mathrm{im}\, f) \cong \mathscr{H}_f = H$.

Now suppose instead that $K$ is a group $\mathscr{H}$-class of $\mathrm{End}(B)$. Let $g \in E(\mathrm{End}(B))$ be the idempotent identity of $K$. Then by Theorem 7.19, $\mathrm{im}\, g$ is a countable algebraically closed bipartite graph. Let $\mathrm{im}\, g = (V_g, (E_g, P_g))$ where, of course, $V_g \subseteq V_B$, $E_g \subseteq E_B$ and $P_g = P_B \cap (V_g \times V_g)$. By Lemma 7.15, $\mathrm{im}\, g$ is connected and so by Lemma 7.4 the countable graph $(V_g, E_g)$ is such that $\mathrm{Aut}(V_g, E_g) = \mathrm{Aut}(\mathrm{im}\, g)$. Now by Theorem 3.25 there exists an idempotent $f \in \mathrm{End}(R)$ such that $H_f \cong \mathrm{Aut}(V_g, E_g) = \mathrm{Aut}(\mathrm{im}\, g) \cong H_g$.

Thus we have shown that the groups arising as group $\mathscr{H}$-classes of $\mathrm{End}(B)$ are the same (up to isomorphism) as those of $\mathrm{End}(R)$. By combining this with Theorem 4.19, the result is complete. $\square$

## 7.4 Regular $\mathscr{D}$-classes and $\mathscr{J}$-classes of $\mathrm{End}(B)$

The results obtained so far in this section also allows us the liberty of gaining some information about the regular $\mathscr{D}$-classes and number of $\mathscr{J}$-classes of $\mathrm{End}(B)$.

**Lemma 7.42.** *There exist $2^{\aleph_0}$ non-isomorphic group $\mathscr{H}$-classes of* $\mathrm{End}(B)$ *and hence $2^{\aleph_0}$ distinct regular $\mathscr{D}$-classes where the group $\mathscr{H}$-classes in different $\mathscr{D}$-classes are non-isomorphic.*

*Proof.* In the proof of Corollary 3.27 we saw that there exist $2^{\aleph_0}$ non-isomorphic countable groups. Corollary 7.39 tells us that $\operatorname{End}(B)$ has a group $\mathscr{H}$-class isomorphic to each of these and hence there exist $2^{\aleph_0}$ non-isomorphic group $\mathscr{H}$-classes of $\operatorname{End}(B)$. However, we also know that if two group $\mathscr{H}$-classes are contained in the same $\mathscr{D}$-class, then they must be isomorphic. Hence these $2^{\aleph_0}$ non-isomorphic maximal group $\mathscr{H}$-classes must be contained in $2^{\aleph_0}$ distinct regular $\mathscr{D}$-classes of $\operatorname{End}(B)$ and the result follows. $\square$

**Theorem 7.43.** *There exist $2^{\aleph_0}$ distinct regular $\mathscr{D}$-classes of* $\operatorname{End}(B)$ *for which any two group $\mathscr{H}$-classes are isomorphic.*

*Proof.* In Theorem 2.10 we saw that if $f$ and $g$ are two elements of $E(\operatorname{End}(B))$ then $f \mathscr{D} g$ if and only if the induced bipartite graphs $\langle \operatorname{im} f\rangle$ and $\langle \operatorname{im} g\rangle$ are isomorphic. By the details of Theorem 7.38, if $\Gamma$ is a countable graph, then there exist $2^{\aleph_0}$ sets $\Sigma \subseteq \mathbb{N} \setminus \{0, 1\}$ such that $H_{f_\Sigma} \cong \operatorname{Aut}(\Gamma)$ and such that $\langle \operatorname{im} f_\Sigma\rangle \not\cong \langle \operatorname{im} f_\Psi\rangle$ for any $\psi \subseteq \mathbb{N} \setminus \{0, 1\}$ with $\Sigma \neq \Psi$. Therefore these idempotents are contained in $2^{\aleph_0}$ distinct regular $\mathscr{D}$-classes of $\operatorname{End}(B)$ but the group $\mathscr{H}$-classes $H_{f_\Sigma}$ are all isomorphic. Since any other group $\mathscr{H}$-class contained in one of these $\mathscr{D}$-classes must then also be isomorphic to $\operatorname{Aut}(\Gamma)$ the result follows. $\square$

So far all of the results obtained have been analogous to a result proved for the random graph, $R$. The next example illustrates that $\operatorname{End}(B)$ has $\mathscr{D}$-classes which are 'smaller' than any of those of $\operatorname{End}(R)$, thus exhibiting a difference in the semigroup theoretic structure of these two semigroups.

**Example 7.44.** Let $f \in E(\operatorname{End}(B))$ and suppose that $\operatorname{im} f$ is isomorphic to

$$\Lambda = (\{u, v\}, (\{(u, v), (v, u)\}, \{(u, u), (v, v)\})).$$

Then $D_f$ contains exactly countably many group $\mathscr{H}$-classes.

*Proof.* First we note that since $\Lambda$ is algebraically closed an application of Theorem 7.19 guarantees the existence of an idempotent $f \in \operatorname{End}(B)$ with $\operatorname{im} f \cong \Lambda$. By Theorem 2.10, we know that an idempotent $g \in \operatorname{End}(B)$ lies in $D_f$ if and only if $\operatorname{im} g \cong \Lambda$. Since $V_B$ is countable and since $B$ is existentially closed, the number of finite subsets $U \subseteq V_\Gamma$ such that $\langle U\rangle \cong \Lambda$ is $\aleph_0$. Furthermore, we can show that there exists exactly one idempotent $g$ such that $\operatorname{im} g = \langle U\rangle$ for each fixed $U = \{x, y\}$ as follows. First note that we can assume without loss of generality that $x \in V_0$ and $y \in V_1$, where $V_B = V_0 \cup V_1$ is the bipartition of $B$. Thus if such an idempotent $g$ existed then $g|_U = 1_U$ and so by Lemma 7.2 we can conclude that $V_0 g = x$ and $V_1 g = y$. Clearly, this lone map is an idempotent homomorphism of $B$ and is then the unique map such that $\operatorname{im} g = \langle U\rangle$. Thus there exists exactly $\aleph_0$ idempotents $g \in D_f$ and hence exactly $\aleph_0$ group $\mathscr{H}$-classes in $D_f$. $\square$

**Theorem 7.45.** *Let $g \in E(\mathrm{End}(B))$ and suppose that* $\operatorname{im} g$ *is strongly algebraically closed. Then $D_g$ contains $2^{\aleph_0}$ group $\mathscr{H}$-classes.*

*Proof.* Since $\operatorname{im} g$ is algebraically closed we can apply Lemma 7.19 to show that there are $2^{\aleph_0}$ distinct idempotents with image isomorphic to $\operatorname{im} g$. By Theorem 2.10 these idempotents are all $\mathscr{D}$-related and therefore lie in $D_g$. However since no group $\mathscr{H}$-class can contain more than one idempotent they lie in distinct $\mathscr{H}$-classes and the result follows. □

**Theorem 7.46.** *There exist $2^{\aleph_0}$ distinct $\mathscr{J}$-classes of* $\mathrm{End}(B)$*.*

*Proof.* By Lemma 2.11, we know that two maps $g, h \in E(\mathrm{End}(B))$ are $\mathscr{J}$-related, if and only if $\langle \operatorname{im} h \rangle$ can be embedded in to $\langle \operatorname{im} g \rangle$ and *vice versa*. By Lemma 3.31, there exists a set $P$ of $2^{\aleph_0}$ subsets of $\mathbb{N} \setminus \{0, 1\}$ such that if $\Sigma$, $\Psi \in P$ then $\Sigma + k \not\subseteq \Psi$ and $\Psi + k \not\subseteq \Sigma$ for all $k \in \mathbb{N}$. Hence by Lemma 7.30 $\Lambda_\Sigma$ cannot be embedded into $\Lambda_\Psi$ for any $\Sigma$, $\Psi \in P$. Furthermore by Lemma 7.27 it follows that $\Lambda_\Sigma^\ddagger$ cannot be embedded into $\Lambda_\Psi^\ddagger$ for any $\Sigma$, $\Psi \in P$. Now by Lemma 7.32, $\Lambda_\Sigma^\dagger$ is an algebraically closed bipartite graph for all $\Sigma \in P$. Thus for all $\Sigma \in P$ there exists an idempotent $f_\Sigma \in \mathrm{End}(B)$ such that $\operatorname{im} f_\Sigma \cong \Lambda_\Sigma^\ddagger$. By our previous observations, $f_\Sigma$ and $f_\Psi$ cannot be $\mathscr{J}$-related for sets $\Sigma \neq \Psi$ in $P$. Since $P$ had size $2^{\aleph_0}$ it now follows that these idempotents must be contained in $2^{\aleph_0}$ distinct $\mathscr{J}$-classes of $\mathrm{End}(B)$. □

# Chapter 8

# The Total Order $\mathbb{Q}$

In this chapter we consider the well known total order $\mathbb{Q}$. We will find that it is a great deal more complicated to determine the maximal subgroups of $\mathrm{End}(\mathbb{Q})$. We will first show that we can characterise the exact subsets of $\mathbb{Q}$ which are the image of some idempotent from $E(\mathrm{End}(\mathbb{Q}))$. We call these subsets retracts of $\mathbb{Q}$. We will then show that if $\Omega$ is a total order and there exists an embedding $f : \Omega \to \mathbb{Q}$ such that $\operatorname{im} f$ is a retract of $\mathbb{Q}$, then there exist $2^{\aleph_0}$ maximal subgroups of $\mathrm{End}(\mathbb{Q})$ which are isomorphic to $\mathrm{Aut}(\Omega)$. We will also show that there exist regular $\mathscr{D}$-classes of $\mathrm{End}(\mathbb{Q})$ which contain countably many group $\mathscr{H}$-classes as well as regular $\mathscr{D}$-classes which contain $2^{\aleph_0}$ group $\mathscr{H}$-classes.

## 8.1 Defining Properties

It is well known that the class of all finite linear orders has the hereditary, joint embedding and amalgamation properties and thus that this class has a unique Fraïssé limit. We will first show that this Fraïssé limit has particular properties.

Recall that a total order $\Omega = (V_\Omega, \leq_\Omega)$ is called *dense* if for all $u, v \in V_\Omega$ with $u <_\Omega v$ there exists $x \in V_\Omega$ such that $u <_\Omega x <_\Omega v$. Additionally, $\Omega$ is said to be *without endpoints* if for all $u \in V_\Omega$ there exists $y, z \in V_\Omega$ such that $y <_\Omega u <_\Omega z$.

Notice that if $\Omega = (V_\Omega, \leq_\Omega)$ is a dense total order, then for any $u, v \in V_\Omega$ with $u <_\Omega v$ there must exist infinitely many elements $x$ such that $u <_\Omega x <_\Omega v$. Similarly if the total order $\Omega$ is without endpoints, then there must exist infinitely many elements $y$ and infinitely many elements $z$ such that $y <_\Omega u$

and $v <_\Omega z$. Clearly, total orders which are dense or without endpoints must then be infinite. Furthermore, the following theorem is well known.

**Theorem 8.1.** *Let $\Omega = (V_\Omega, \leq)$ be a countable dense total order without endpoints. Then every countable total order can be embedded into $\Omega$.*

*Proof.* First, let $V_\Omega = \{q_j : j \in \mathbb{N}\}$. Now let $\Lambda = (V_\Lambda, \leq_\Lambda)$ be any countable total order. Enumerate the elements in $V_\Lambda$ as $\{v_i : i \in \mathbb{N}\}$, replacing the natural numbers by a finite set when necessary. Now inductively define a sequence of functions as follows. Let $f_0 : \{v_0\} \to V_\Omega$ be defined by $v_0 f_0 = q_0$. Then clearly $f_0$ is an embedding of $\langle v_0 \rangle$ into $\Omega$. Now suppose that for $n \in \mathbb{N}$ we have defined $f_n : \{v_0, \ldots, v_n\} \to \Omega$ which is an embedding of $\langle v_0, \ldots, v_n \rangle$ into $\Omega$. Let

$$N_- = \{v_i : v_i <_\Lambda v_{n+1},\ 0 \leq i \leq n\},$$

and let

$$N_+ = \{v_i : v_{n+1} <_\Lambda v_i,\ 0 \leq i \leq n\}.$$

Suppose that $N_-, N_+ \neq \emptyset$. Then since $\Omega$ is dense there exists $q_j \in V_\Omega$ such that $(N_-)f_n < q_j < (N_+)f_n$. On the other hand if $N_- = \emptyset$ or $N_+ = \emptyset$, then since $\Omega$ is without endpoints there exists $q_j \in V_\Omega$ such that $q_j < (N_+)f_n$ or $(N_-)f_n < q_j$ respectively. In any case define $f_{n+1} : \{v_0, \ldots, v_{n+1}\} \to \Omega$ by,

$$v_k f_{n+1} = \begin{cases} v_k f_n & \text{if } k = 0, \ldots, n \\ q_j & \text{if } k = n+1. \end{cases}$$

Then clearly $f_{n+1}$ is an injective map since by assumption $f_n$ was and since $q_j \neq v_k f_n$ for $k = 0, \ldots, n$. Furthermore, $v_{n+1} <_\Lambda v_k$ for some $k = 0, \ldots, n$ if and only if $q_j = v_{n+1} f_{n+1} < v_k f_{n+1}$ and similarly $v_k <_\Lambda v_{n+1}$ if and only if $v_k f_{n+1} < v_{n+1} f_{n+1} = q_j$. Thus $f_{n+1}$ is an embedding of $\langle v_0, \ldots, v_{n+1} \rangle$ into $\Omega$. Now let

$$f = \bigcup_{n \in \mathbb{N}} f_n.$$

Then as the union of embeddings $f_n$ such that $f_{n+1}$ is an extension of $f_n$ for all $n \in \mathbb{N}$, $f$ is an embedding of $\Lambda$ into $\Omega$. $\square$

By Theorem 8.1, the age of any countable dense total order without endpoints is exactly the class of all finite total orders. It thus follows that any countable total order which is dense and without endpoints is isomorphic to the unique homogeneous Fraïssé limit of the class of finite total orders. Of course $(\mathbb{Q}, \leq)$, the set of rational numbers with the natural ordering, is a countable total order which is both dense and without endpoints. Thus

$(\mathbb{Q}, \leq)$ *is* the Fraïssé limit of the class of finite total orders. For convenience, we will often abuse notation in this chapter and write $\mathbb{Q}$ to mean $(\mathbb{Q}, \leq)$.

As usual $(\mathbb{Q}, \leq)$ can be thought of as a substructure of the relational structure $(\mathbb{R} \cup \{-\infty, \infty\}, \leq)$, the set of affinely extended real numbers with the natural ordering. This allows for the definition of an *interval in* $\mathbb{Q}$ with real or infinite endpoints as follows. For $p, q \in \mathbb{R} \cup \{-\infty, \infty\}$ define the *closed interval* in $\mathbb{Q}$ with closed endpoints $p \leq q$ by

$$[p, q] = \{x \in \mathbb{Q} : p \leq x \leq q\}.$$

The *open interval* in $\mathbb{Q}$ with open endpoints $p$ and $q$ will be defined by

$$(p, q) = \{x \in \mathbb{Q} : p < x < q\}.$$

We similarly define the *right closed interval* in $\mathbb{Q}$ with left open endpoint $p$ and right closed endpoint $q$ by

$$(p, q] = \{x \in \mathbb{Q} : p < x \leq q\}$$

and the *left closed interval* in $\mathbb{Q}$ with left closed endpoint $p$ and right open endpoint $q$ by,

$$[p, q) = \{x \in \mathbb{Q} : p \leq x < q\}.$$

Note that if $p \in (\mathbb{R} \setminus \mathbb{Q}) \cup \{-\infty\}$ then the intervals in $\mathbb{Q}$ given by $(p, q)$ and $[p, q)$ are equal as are the intervals $[p, q]$ and $(p, q]$. Similarly if $q \in (\mathbb{R} \setminus \mathbb{Q}) \cup \{\infty\}$ then $(p, q) = (p, q]$ and $[p, q] = [p, q)$. Also worth mentioning is that if $p \in \mathbb{R} \cup \{-\infty, \infty\}$, then $(p, p] = [p, p) = (p, p) = \emptyset$. Furthermore if $p \in \mathbb{Q}$ then $[p, p] = p$ and if $p \notin \mathbb{Q}$ then $[p, p] = \emptyset$.

The term *non-closed interval* will be used to mean an interval which cannot be written in the form $[p, q]$ where $p, q \in \mathbb{R} \cup \{-\infty, \infty\}$. Thus the empty set is *not* a non-closed interval since $[p, p] = \emptyset$ for all $p \in \mathbb{R} \setminus \mathbb{Q}$. As we will see in the next subsection, non-closed intervals play a key part in the structure of the images of idempotents from $\mathrm{End}(\mathbb{Q})$.

## 8.2 Retracts of End($\mathbb{Q}$)

By Theorem 2.7, the group $\mathscr{H}$-class of an idempotent $f \in \mathrm{End}(\mathbb{Q})$ is isomorphic to the automorphism group of the total order induced by the image of $f$. With the definitions from Section 9.1, we can prove the following theorem on subsets of $\mathbb{Q}$ which are the image of an idempotent from $\mathrm{End}(\mathbb{Q})$.

**Theorem 8.2.** *Let $X \subseteq \mathbb{Q}$. Then there exists $f \in E(\mathrm{End}(\mathbb{Q}))$ such that* $\mathrm{im}\, f = X$ *if and only if $X = \mathbb{Q}$ or $X = \mathbb{Q} \setminus S$ where $S = \bigcup_{i \in I} T_i$ satisfies the following properties.*

(i) *For each $i \in I$, $T_i$ is a non-closed interval in $\mathbb{Q}$.*

(ii) *For $i \neq j$, $T_i \cap T_j = \emptyset$.*

(iii) *If $T_i < T_j$ then there exists $x \in X$ such that $T_i < x < T_j$.*

*Furthermore, if one or more $T_i$ is an open interval with rational endpoints, then there exists $2^{\aleph_0}$ such idempotents $f$ such that* $\mathrm{im}\, f = X$.

To prove Theorem 8.2 we must first make a series of definitions and accompanying lemmas. For the following, fix $f \in E(\mathrm{End}(\mathbb{Q}))$.

**Definition 8.3.** For $x \in \mathbb{Q}$ define

$$xf^{-1} = \{q \in \mathbb{Q} : qf = x\}.$$

If $x \notin \mathrm{im}\, f$ then by definition $xf^{-1} = \emptyset$. On the other hand if $x \in \mathrm{im}\, f$ then, since $f$ is idempotent, $xf = x$ and so $x \in xf^{-1}$. Now let

$$J = \{x \in \mathrm{im}\, f : xf^{-1} \neq \{x\}\}.$$

Note that if $J = \emptyset$, then clearly $f = \mathbf{1}_{\mathbb{Q}}$ and $\mathrm{im}\, f = \mathbb{Q}$.

For each $x \in J$ we will define non-closed intervals $U_x$ and $L_x$ such that $U_x, L_x \subseteq \mathbb{Q} \setminus \mathrm{im}\, f$. From the set $\{U_x, L_x : x \in J\}$ we will construct the set $S$ for the proof of the only if statement in Theorem 8.2.

In the next lemma we will require the concept of the infimum (or greatest lower bound) and supremum (or least upper bound) of a subset of real numbers. Recall that if $R \subseteq \mathbb{R}$, then the infimum of $R$, denoted $\inf(R)$, is an element $x \in \mathbb{R}$ such that $x \leq R$ and such that if there exists $y \in \mathbb{R}$ with $x \leq y \leq R$, then $x = y$. If no such element $x$ exists then we define $\inf(R)$ to be $-\infty$. Dually, the supremum of $R$, denoted $\sup(R)$, is an element $x' \in \mathbb{R}$ such that $R \leq x'$ and such that if there exists $y' \in \mathbb{R}$ with $R \leq y' \leq x'$, then $x' = y'$. If no such element $x'$ exists then we define $\sup(R)$ to be $\infty$.

**Lemma 8.4.** *Let $J = \{x \in \mathrm{im}\, f : xf^{-1} \neq \{x\}\}$ and let $z \in \mathbb{Q}$. For $x \in J$ define $l_x = \inf(xf^{-1})$ and $u_x = \sup(xf^{-1})$ so that $l_x \leq x \leq u_x$. Then the following statements hold.*

(i) *If $l_x < z \leq x$, then $zf = x$.*

(ii) *If* $z < l_x$*, then* $zf \leq l_x$*.*

(iii) *If* $x \leq z < u_x$*, then* $zf = x$*.*

(iv) *If* $u_x < z$*, then* $u_x \leq zf$*.*

*Proof.* Suppose that $l_x < z \leq x$. Then $z$ is not a lower bound for $xf^{-1}$ and so there exists $y \in xf^{-1}$ such that $l_x \leq y < z$. Thus since $f$ is a homomorphism, $yf \leq zf \leq xf$. In other words $x \leq zf \leq x$ and hence it follows that $zf = x$ and statement (i) holds. Now suppose instead that $z < l_x$. Then since $l_x \leq x$ and $f$ is a homomorphism, we deduce that $zf \leq xf$. If $l_x < zf$, it follows that $l_x < zf \leq xf = x$. Thus by case (i), $zf^2 = zf = x$ and so $z \in xf^{-1}$. But this contradicts the assumption that $l_x$ is a lower bound for $xf^{-1}$. Thus $zf \leq l_x$ and statement (ii) holds.

Statements (iii) and (iv) are proved in a dual manner as follows. Suppose that $x \leq z < u_x$. Then $z$ is not an upper bound for $xf^{-1}$ and so there exists $y \in xf^{-1}$ such that $z < y \leq u_x$. Thus since $f$ is a homomorphism, $xf \leq zf \leq yf$. In other words $x \leq zf \leq x$ and hence it follows that $zf = x$ and statement (iii) holds. Now suppose instead that $u_x < z$. Then since $x \leq u_x$ and $f$ is a homomorphism, we deduce that $xf \leq zf$. If $zf < u_x$, then it follows that $x = xf \leq zf < u_x$. Thus by case (iii), $zf^2 = zf = x$ and so $z \in xf^{-1}$. But this contradicts the assumption that $u_x$ is an upper bound for $xf^{-1}$. Thus $u_x \leq zf$ and statement (iv) holds. □

**Definition 8.5.** For $x \in J$ we define

$$\begin{aligned} m_x &= \max\{q \in \operatorname{im} f : q < x\}, \\ n_x &= \min\{q \in \operatorname{im} f : x < q\}, \end{aligned}$$

whenever the maximum or minimum exist.

**Lemma 8.6.** *Let* $J = \{x \in \operatorname{im} f : xf^{-1} \neq \{x\}\}$*. For* $x \in J$ *let* $l_x = \inf(xf^{-1})$*. If* $l_x \notin \operatorname{im} f$ *then one of the following three cases hold.*

*(a)* $m_x = \max\{q \in \operatorname{im} f : q < x\}$ *exists and* $m_x < l_x$*.*

*(b)* $l_x = -\infty$*.*

*(c) There exists a sequence* $\{q_i\}_{i \in \mathbb{N}}$ *in* $\mathbb{Q}$ *such that* $q_i < l_x$*,* $q_i \in \operatorname{im} f$ *and* $\lim_{i \to \infty} q_i = l_x$*.*

*Proof.* If $l_x = -\infty$, there is nothing to do. So suppose that $l_x \neq -\infty$ and let $p \in \mathbb{Q}$ such that $p < l_x$. By Lemma 8.4, $pf \leq l_x$ and since $l_x \notin \operatorname{im} f$ it follows that $pf < l_x$. Let $q_0 = pf$. If $m_x = \max\{q \in \operatorname{im} f : q < x\}$ exists then clearly $q_0 \leq m_x < l_x$ and again there is nothing to do. So suppose that $m_x$ does not exist. We show that there exists $q_1 \in \operatorname{im} f$ such that $l_x - (l_x - q_0)/2 \leq q_1 < l_x$ as follows. Seeking a contradiction, suppose that there exists no $q \in \operatorname{im} f$ such that $l_x - (l_x - q_0)/2 \leq q < l_x$. Let $r \in \mathbb{Q}$ be such that $l_x - (l_x - q_0)/2 < r < l_x$. By Lemma 8.4, $rf \leq l_x$ and since $l_x \notin \operatorname{im} f$ it follows that $rf < l_x$. Furthermore, since there exists no $q \in \operatorname{im} f$ such that $l_x - (l_x - q_0)/2 \leq q < l_x$ it follows that $rf < l_x - (l_x - q_0)/2$. But since $m_x$ does not exist, there exists $s \in \operatorname{im} f$ such that $rf < s \leq l_x - (l_x - q_0)/2$. Thus we have $s < r$ but $rf < s = sf$, a contradiction. Hence there must exist $q_1 \in \operatorname{im} f$ such that $l_x - (l_x - q_0)/2 \leq q_1 < l_x$ as claimed. By repeating this argument we can produce a monotonic increasing sequence $\{q_i\}_{i \in \mathbb{N}}$ in $\mathbb{Q}$ such $q_i \in \operatorname{im} f$ for all $i \in \mathbb{N}$ and such that

$$l_x - \frac{(l_x - q_0)}{2^i} \leq q_i < l_x.$$

Moreover, for any $a \in R$ such that $a < l_x$, there exists $N \in \mathbb{N}$ such that $a < l_x - (l_x - q_0)/2^N$. Thus $a < q_n < l_x$ for all $n \geq N$ and so $\lim_{i \to \infty} q_i = l_x$. □

**Lemma 8.7.** *Let $J = \{x \in \operatorname{im} f : xf^{-1} \neq \{x\}\}$. For $x \in J$ let $u_x = \sup(xf^{-1})$. If $u_x \notin \operatorname{im} f$ then one of the following three cases hold.*

*(a′)* $n_x = \min\{q \in \operatorname{im} f : x < q\}$ *exists and* $u_x < n_x$.

*(b′)* $u_x = \infty$.

*(c′) There exists a sequence $\{p_i\}_{i \in \mathbb{N}}$ in $\mathbb{Q}$ such that $u_x < p_i$, $p_i \in \operatorname{im} f$ and $\lim_{i \to \infty} p_i = u_x$.*

The proof of Lemma 8.7 is dual to that of Lemma 8.6 and is therefore omitted.

We will now make the definition of the interval $L_x$ for each $x \in J$. The definition of $L_x$ will be dependent on whether $l_x \in \operatorname{im} f$ or $l_x \notin \operatorname{im} f$ and therefore which case of Lemma 8.6 holds for $x$.

**Definition 8.8.** Let $J = \{x \in \operatorname{im} f : xf^{-1} \neq \{x\}\}$, as defined in Definition 8.3. For $x \in J$ we define the interval $L_x$ as follows. If $l_x \notin \operatorname{im} f$ then one of cases $(a)$, $(b)$ or $(c)$ in Lemma 8.6 holds. In case:

(a) let $L_x$ be the interval in $\mathbb{Q}$ given by $L_x = (m_x, x)$.

(b) let $L_x$ be the interval in $\mathbb{Q}$ given by $L_x = (-\infty, x)$.

(c) let $L_x$ be the interval in $\mathbb{Q}$ given by $L_x = [l_x, x)$.

On the other hand, if $l_x \in \operatorname{im} f$, then let $L_x$ be the interval in $\mathbb{Q}$ given by $L_x = (l_x, x)$. It is not hard to see that if $l_x \in \operatorname{im} f$, then $l_x = m_x$ and so we could have equally defined $L_x = (m_x, x)$ in this case. Notice that $L_x = \emptyset$ if and only if $l_x = x$.

We dually make the definition of the interval $U_x$ for each $x \in J$ as follows. The definition of $U_x$ will be dependent on wether $u_x \in \operatorname{im} f$ or $u_x \notin \operatorname{im} f$ and therefore which case of Lemma 8.7 holds for $x$.

**Definition 8.9.** Let $J = \{x \in \operatorname{im} f : xf^{-1} \neq \{x\}\}$, as defined in Definition 8.3. For $x \in J$ we define the interval $U_x$ as follows. If $u_x \notin \operatorname{im} f$ then one of cases $(a')$, $(b')$ or $(c')$ in Lemma 8.7 holds. In case:

$(a')$ let $U_x$ be the interval in $\mathbb{Q}$ given by $U_x = (x, n_x)$

$(b')$ let $U_x$ be the interval in $\mathbb{Q}$ given by $U_x = (x, \infty)$.

$(c')$ let $U_x$ be the interval in $\mathbb{Q}$ given by $U_x = (x, u_x]$.

If on the other hand $u_x \in \operatorname{im} f$, then let $U_x$ be the interval in $\mathbb{Q}$ given by $U_x = (x, u_x)$. It is not hard to see that if $u_x \in \operatorname{im} f$, then $u_x = n_x$ and so we could have equally defined $U_x = (x, n_x)$ in this case. Notice that $U_x = \emptyset$ if and only if $u_x = x$.

It is worth observing that since $x \in J$, at least one of $L_x$ and $U_x$ is non empty. For suppose that $L_x = U_x = \emptyset$. Then, $l_x = x = u_x$ and hence $xf^{-1} = \{x\}$. Since this contradicts the assumption that $x \in J$ we find that at least one of $L_x$ and $U_x$ is non empty as required.

**Lemma 8.10.** *Let $J = \{x \in \operatorname{im} f : xf^{-1} \neq \{x\}\}$ and let $x \in J$. Then $L_x$ and $U_x$, the intervals defined in Definitions 8.8 and 8.9, are either empty or are non-closed intervals in $\mathbb{Q}$ which do not meet* $\operatorname{im} f$.

*Proof.* We first consider the interval $L_x$. If $L_x = \emptyset$ then there is nothing to do. So suppose that $L_x \neq \emptyset$ and hence that $l_x < x$. If $l_x \notin \operatorname{im} f$ then $L_x$ is of form $(a), (b)$ or $(c)$ from Definition 8.8. In case $(a)$, $L_x$ is the interval in $\mathbb{Q}$ given by $L_x = (m_x, x)$. It should be clear that in this case $L_x$ is a non-closed interval since $m_x, x \in \mathbb{Q}$. To see that $L_x$ does not meet $\operatorname{im} f$ notice that if there exists $y \in \operatorname{im} f$ such that $y \in L_x$ then $m_x < y$, contradicting the definition of $m_x$. In case $(b)$, $L_x$ is the interval in $\mathbb{Q}$ given by $L_x = (-\infty, x) = (l_x, x)$ and in case $(c)$, $L_x$ is the interval in $\mathbb{Q}$ given by $L_x = [l_x, x)$. In both cases $L_x$ is a non-closed interval since $x \in \mathbb{Q}$ and since $l_x < x$. Furthermore, by Lemma 8.4 any $z \in \mathbb{Q}$ with $l_x < z \leq x$ is such that $zf = x$. Since $yf = y$ for all $y \in \operatorname{im} f$, it thus follows that in both cases $L_x$ is a non-closed interval which does not meet $\operatorname{im} f$. If on the other hand $l_x \in \operatorname{im} f$, then $L_x$ is the interval in $\mathbb{Q}$ given by $L_x = (l_x, x)$. Notice that in this case $l_x = \max\{q \in \operatorname{im} f : q < x\} = m_x$, and so an identical argument to that of case $(a)$ above allows us to deduce that $L_x$ is a non-closed interval and does not meet $\operatorname{im} f$. A dual argument for the interval $U_x$ completes the proof. □

**Lemma 8.11.** *Let $J = \{x \in \operatorname{im} f : xf^{-1} \neq \{x\}\}$. Let $x, y \in J$ and suppose that $x < y$. Then*

$$L_x \cap U_x = L_y \cap U_y = L_x \cap U_y = L_x \cap L_y = U_x \cap U_y = \emptyset,$$

*and either $L_y \cap U_x = \emptyset$ or $L_y = U_x$.*

*Proof.* If, for example, $U_x = \emptyset$ then clearly $L_x \cap U_x = U_x \cap U_y = L_y \cap U_x = \emptyset$. So suppose $U_x, U_y, L_x, L_y \neq \emptyset$. By construction it is true that $L_x < x < U_x$ and $L_y < y < U_y$ and so we can easily deduce that,

$$L_x \cap U_x = L_y \cap U_y = L_x \cap U_y = \emptyset.$$

By Lemma 8.4, if $z \in \mathbb{Q}$ is such that $l_y < z \leq y$, then $zf = y$. Hence since $x < y$ and $x \in \operatorname{im} f$ it must be the case that $x \leq l_y$ and if $m_y$ exists that $x \leq m_y$. Thus by construction, $L_x < x < L_y$. A similar argument shows that $U_x < y < U_y$, and hence it holds that $L_x \cap L_y = U_x \cap U_y = \emptyset$.

It remains to consider the intersection $U_x \cap L_y$. Notice that if $x < y$ then $L_y$ is either of the form $(m_y, y)$ or of the form $[l_y, y)$ where $l_y \notin \operatorname{im} f$ and where there exists a sequence $\{q_i\}_{i \in \mathbb{N}}$ in $\mathbb{Q}$ such that $q_i < l_x$, $q_i \in \operatorname{im} f$ and $\lim_{i \to \infty} q_i = l_x$. Similarly $U_x$ will be of the form $(x, n_x)$ or the form $(x, u_x]$ where $u_x \notin \operatorname{im} f$ and where there exists a sequence $\{p_i\}_{i \in \mathbb{N}}$ in $\mathbb{Q}$ such that $u_x < p_i$, $p_i \in \operatorname{im} f$ and $\lim_{i \to \infty} p_i = u_x$. We will check the intersection $U_x \cap L_y$ for each of these possibilities case by case.

So suppose that $U_x = (x, n_x)$ and $L_y = [l_y, y)$. In this case we know that $l_y \notin \operatorname{im} f$ and there exists a sequence $\{q_i\}_{i \in \mathbb{N}}$ in $\mathbb{Q}$ such that $q_i < l_y$, $q_i \in \operatorname{im} f$ and $\lim_{i \to \infty} q_i = l_y$. Suppose that $l_y \leq n_x$. Since $\lim_{i \to \infty} q_i = l_y$ there exists $j \in \mathbb{N}$ such that $x < q_j < l_y \leq n_x$. But this contradicts the fact that $U_x = (x, n_x)$ does not meet $\operatorname{im} f$ by Lemma 8.10. Hence we conclude that in this case $n_x < l_y$ and thus $L_y \cap U_x = \emptyset$. If we suppose that $U_x = (x, u_x]$ and $L_y = (m_y, y)$ then $u_x \notin \operatorname{im} f$ and there instead exists a sequence $\{p_i\}_{i \in \mathbb{N}}$ in

$\mathbb{Q}$ such that $u_x < p_i$, $p_i \in \operatorname{im} f$ and $\lim_{i \to \infty} p_i = u_x$. By a similar argument to that above we can easily deduce that $u_x < m_y$ and hence $L_y \cap U_x = \emptyset$.

Now suppose instead that $U_x = (x, u_x]$ and $L_y = [l_y, y)$, then since $l_y \notin \operatorname{im} f$ the existence of the sequence $\{q_i\}_{i \in \mathbb{N}}$ converging to $l_y$ ensures that we can find an element $q_j \in \operatorname{im} f$ such that $x < q_j < l_y$. But by Lemma 8.10, $L_y = [l_y, y)$ does not meet $\operatorname{im} f$. Thus $u_x < q_j < l_y$ and hence $L_y \cap U_x = \emptyset$ as required. The last case is to suppose that $U_x = (x, n_x)$ and $L_y = (m_y, y)$. If $n_x = m_y$ there is nothing to do. So suppose that $m_y < n_x$. Then by definition of $n_x$ and $m_y$ it must be the case that $x \leq m_y < n_x \leq y$. If $x < m_y$ then there is a contradiction since $n_x$ was minimal. Similarly we obtain a contradiction to $m_y$ being maximal if we supposed that $n_x < y$. The only remaining possibility is that $x = m_y$ and $y = n_x$. In this case $L_y = U_x$. □

**Lemma 8.12.** *Let $J = \{x \in \operatorname{im} f : xf^{-1} \neq \{x\}\}$. Let $x, y \in J$ and suppose that $x < y$. Now let $S, T \in \{L_x, L_y, U_x, U_y\}$ be such that $S < T$. Then there exists $z \in \operatorname{im} f$ such that $S < z < T$.*

*Proof.* By definition it is true that $L_x < x < U_x$ and $L_y < y < U_y$. Thus since $x, y \in \operatorname{im} f$ it follows that if

$$(S, T) \in \{(L_x, U_x), (L_x, L_y), (L_x, U_y), (U_x, U_y), (L_y, U_y)\},$$

then the result holds. It remains to check that if $S = U_x$ and $T = L_y$ then there exists $z \in \operatorname{im} f$ such that $S < z < T$.

We will again use the observation that if $x < y$ then $L_y$ is either of the form $(m_y, y)$ or $[l_y, y)$ and $U_x$ is either of the form $(x, n_x)$ or $(x, u_x]$. We will check these possibilities case by case. So suppose that $U_x = (x, n_x)$ and $L_y = [l_y, y)$. By the proof of Lemma 8.11, it follows that $n_x < l_y$. Thus $U_x < n_x < L_y$ and since $n_x \in \operatorname{im} f$ by definition, we are finished. If we suppose that $U_x = (x, u_x]$ and $L_y = (m_y, y)$ then a similar argument shows that $U_x < m_y < L_y$ and since $m_y \in \operatorname{im} f$ by definition the result holds.

If we suppose instead that $U_x = (x, u_x]$ with $L_y = [l_y, y)$ then by the proof of Lemma 8.11 shows that there exists $q \in \operatorname{im} f$ such that $u_x < q < l_y$. Hence $U_x \leq q \leq L_y$ as required. The last case is to suppose that $U_x = (x, n_x)$ and $L_y = (m_y, y)$. By the proof of Lemma 8.11, if $n_x \neq m_y$ then $U_x = L_y$. Since by assumption $U_x = S < T = L_y$ it must be the case that $n_x = m_y$. But then $U_x < n_x < L_y$ and since $n_x \in \operatorname{im} f$ by definition, the result is complete. □

We are now able to restate and prove Theorem 8.2.

**Theorem 8.2.** *Let $X \subseteq \mathbb{Q}$. Then there exists $f \in E(\mathrm{End}(\mathbb{Q}))$ such that $\operatorname{im} f = X$ if and only if $X = \mathbb{Q}$ or $X = \mathbb{Q} \setminus S$ where $S = \bigcup_{i \in I} T_i$ satisfies the following properties.*

(i) *For each $i \in I$, $T_i$ is a non-closed interval in $\mathbb{Q}$.*

(ii) *For $i \neq j$, $T_i \cap T_j = \emptyset$.*

(iii) *If $T_i < T_j$ then there exists $x \in X$ such that $T_i < x < T_j$.*

*Furthermore, if one or more $T_i$ is an open interval with rational endpoints, then there exists $2^{\aleph_0}$ such idempotents $f$ such that $\operatorname{im} f = X$.*

*Proof.* Suppose first that $f \in E(\mathrm{End}(\mathbb{Q})$. Let $J = \{x \in \operatorname{im} f : xf^{-1} \neq \{x\}\}$ as defined in Definition 8.3. If $J = \emptyset$ then $f = \mathbf{1}_{\mathbb{Q}}$ and hence $\operatorname{im} f = \mathbb{Q}$. Otherwise consider the non-empty set $\{L_x, U_x : x \in J\}$, where $L_x$ and $U_x$ are the intervals defined in Definitions 8.8 and 8.9. If $x, y \in J$ and $x < y$ then by Lemma 8.11 the list: $L_x, L_y, U_x, U_y$ contains at most one repetition, namely when $U_x = L_y$. Furthermore, since $U_x \cap U_z = L_y \cap L_z = \emptyset$ for all $z \in J$, $z \neq x, y$, we can deduce $y$ is the only element of $J$ for which $L_y = U_x$. So let $\{T_i : i \in I\}$, $|I| \leq 2|J|$, be an enumeration of the set $\{U_x, L_x : x \in J\}$ with the empty set discarded. Then by Lemma 8.10, each $T_i$ is a non-closed interval in $\mathbb{Q}$ which does not meet $\operatorname{im} f$. Furthermore, by Lemma 8.11, $T_i \cap T_j = \emptyset$ for $i \neq j$. We have also shown in Lemma 8.12 that if $T_i < T_j$, then there exists $x \in \operatorname{im} f$ such that $T_i < x < T_j$. Finally since $T_i$ does not meet $\operatorname{im} f$ for each $i \in I$, it holds that $\operatorname{im} f \subseteq \mathbb{Q} \setminus \left(\bigcup_{i \in I} T_i\right)$. If there exists $y \in \mathbb{Q} \setminus \left(\bigcup_{i \in I} T_i\right)$ such that $y \notin \operatorname{im} f$ then $yf = z$ for some $z \in \mathbb{Q}$ where $z \neq y$. If $y < z$ then by construction $y \in L_z \subseteq \bigcup_{i \in I} T_i$, a contradiction. If instead $z < y$ then $y \in U_z \subseteq \bigcup_{i \in I} T_i$ which is another contradiction. Thus $\mathbb{Q} \setminus \left(\bigcup_{i \in I} T_i\right) \subseteq \operatorname{im} f$ and we can deduce that $\operatorname{im} f = \mathbb{Q} \setminus \left(\bigcup_{i \in I} T_i\right)$. Taking $S = \bigcup_{i \in I} T_i$ completes the proof of the only if statement.

For the converse it should be clear that if $X = \mathbb{Q}$, then $\mathbf{1}_{\mathbb{Q}}$ is idempotent and that $\operatorname{im} \mathbf{1}_{\mathbb{Q}} = X$. Now suppose that $X = \mathbb{Q} \setminus S$ where $S = \bigcup_{i \in I} T_i$ satisfies properties (i), (ii) and (iii). Let $i \in I$. By assumption (i), $T_i$ is a non-closed interval in $\mathbb{Q}$ and so must have one of the following forms (since otherwise $T_i$ can be written as $[x, y]$ where $x, y \in \mathbb{R} \cup \{-\infty, \infty\}$).

(a) $[x, y)$ where $x, y \in \mathbb{Q}$.

(b) $(x, y]$ where $x, y \in \mathbb{Q}$.

(c) $(x, y)$ where $x \in \mathbb{Q}$, $y \in (\mathbb{R} \setminus \mathbb{Q}) \cup \{-\infty, \infty\}$.

(d) $(x, y)$ where $y \in \mathbb{Q}$, $x \in (\mathbb{R} \setminus \mathbb{Q}) \cup \{-\infty, \infty\}$.

(e) $(x, y)$ where $x, y \in \mathbb{Q}$.

For each $i \in \mathbb{N}$, we will define a function $f_i : T_i \to \mathbb{Q}$ as follows. If $T_i$ has form $(a)$, $(d)$ or $(e)$, we define $zf_i = y$ for all $z \in T_i$. If instead $T_i$ has form $(b)$ or $(c)$ then we let $zf_i = x$ for all $z \in T_i$. In any case it is trivial to show that we have defined a homomorphism $f_i : T_i \to \mathbb{Q}$ such that $|\operatorname{im} f_i| = 1$. Furthermore, for any $T_i$ of form $(a)$, $(d)$ or $(e)$ it clearly holds that $T_i < \{y\} = \operatorname{im} f_i$ and that there exist no $z \in X$ such that $T_i < z < y$. Thus for any $T_k$ such that $T_i < T_k$ it must follow that $\operatorname{im} f_i = \{y\} < T_k$ since otherwise we contradict property (iii). In other words $\operatorname{im} f_i \notin T_j$ for any $j \neq i$ and $\operatorname{im} f_i \in \mathbb{Q} \setminus S$. Similarly for any $T_i$ of form $(b)$ or $(c)$ we can show that $\operatorname{im} f_i \in \mathbb{Q} \setminus S$ and hence we can conclude that $\operatorname{im} f_i \in \mathbb{Q} \setminus S$ for all $i \in I$.

Now let $f : \mathbb{Q} \to \mathbb{Q}$ be defined by

$$qf = \begin{cases} qf_i & \text{if } q \in T_i, \\ q & \text{otherwise .} \end{cases}$$

First we note that $f$ is a well defined function since by property (ii), $T_i \cap T_j = \emptyset$ for $i \neq j$. Additionally, by our previous observation $\operatorname{im} f_i \in \mathbb{Q} \setminus S$ and since $\mathbb{Q} \setminus S \subseteq \operatorname{im} f$ it follows that $\operatorname{im} f = \mathbb{Q} \setminus S$.

To see that $f$ is an endomorphism let $x \leq y \in \mathbb{Q}$ If $x, y \in \mathbb{Q} \setminus S$ then clearly $xf = x < y = yf$. So suppose that $x \in S$ and $y \in \mathbb{Q} \setminus S$. Then by definition $x \in T_i$ for some $i \in I$. Since $T_i$ is an interval and since $y \in \mathbb{Q} \setminus S$ we can deduce that $T_i < y$. If $T_i$ has form $(b)$ or $(c)$ then clearly $xf = xf_i < T_i$ and hence $xf < y = yf$. If on the other hand $T_i$ is of form $(a)$, $(d)$ or $(e)$, then $T_i < xf_i = xf$ and by definition there exist no $z \in \mathbb{Q} \setminus S$ such that $T_i < z < xf$. Hence $xf \leq y = yf$. A dual argument shows that if $y \in S$ and $x \in \mathbb{Q} \setminus S$ then $xf = x \leq yf$. The last case to consider is the case where $x, y \in S$. If $x, y \in T_i$ for some $i \in I$ then $xf = yf$ and we are done. So suppose that $x \in T_i$ and $y \in T_j$ for some $i, j \in I$ where of course it must be the case that $T_i < T_j$. If $T_i$ has form $(b)$ or $(c)$ and $T_j$ has form $(a)$, $(d)$ or $(e)$ then by definition,

$$xf = xf_i < T_i < T_j < yf_j = yf.$$

Similarly our previous observations allow us to deduce that if $T_i$ and $T_j$ both have form $(a)$, $(d)$ or $(e)$, then

$$T_i < xf_i = xf < T_j < yf_j = yf,$$

if they both have form $(b)$ or $(c)$, then

$$xf = xf_i < T_i < yf_j = yf < T_j.$$

and if $T_i$ has form $(a)$, $(d)$ or $(e)$ and $T_j$ has form $(b)$ or $(c)$ then

$$T_i < xf = xf_i \leq yf_j = yf < T_j.$$

In any case we have shown that $x \leq y$ implies that $xf \leq yf$ and hence $f \in \mathrm{End}(\mathbb{Q})$. Additionally since

$$f|_{\mathrm{im}\, f} = f|_{\mathbb{Q}\setminus S} = \mathbf{1}_{\mathrm{im}\, f},$$

we can conclude by Lemma 2.3 that $f \in E(\mathrm{End}(\mathbb{Q}))$ is idempotent.

To finish we observe that if one of the $T_i$ is of form $(e)$, then there are actually a large number of possible homomorphisms $f_i : T_i \to \mathbb{Q}$ which could have been chosen in the construction of $f$ above. For if $T_i = (x, y)$, where $x, y \in \mathbb{Q}$, then let $a \in \{r \in \mathbb{R} : x < r < y\}$ and define a function $g_{ia}$ on $T_i$ by

$$zg_{ia} = \begin{cases} x & \text{if } z \leq a, \\ y & \text{if } a < z. \end{cases}$$

It should be easy to see that $g_{ia}$ defines a homomorphism $T_i \to \mathbb{Q}$. Furthermore, if $a, b \in \{r \in \mathbb{R} : x < r < y\}$ and $a \neq b$, then we can show that $g_{ia} \neq g_{ib}$ as follows. For suppose without loss of generality that $a < b$. Then for all $c \in (x, y)$ such that $a < c < b$, $cg_{ia} = y$ whereas $cg_{ib} = x$. Thus since there exist $2^{\aleph_0}$ choices for the element $a \in \{r \in \mathbb{R} : x < r < y\}$, there exist $2^{\aleph_0}$ choices for the map $f_i : T_i \to \mathbb{Q}$ (namely $g_{ia}$ for all $x < a < y$). Consequently there exist $2^{\aleph_0}$ distinct idempotents $f$ as required. □

Theorem 8.2 exhibits a result that we are unable to achieve for the other relational structures discussed in this thesis. It identifies exactly which subsets of $\mathbb{Q}$ are the image of an idempotent from $\mathrm{End}(\mathbb{Q})$. In the setting of graphs for example, we saw in Theorem 3.10 that the image of an idempotent $f \in \mathrm{End}(R)$ induces an algebraically closed graph. However, given a subset $U \subseteq V_\Omega$, such that $\langle U \rangle$ is algebraically closed, Theorem 3.25 only guarantees the existence of an idempotent $g \in \mathrm{im}\, f$ such that $\mathrm{im}\, g \cong \langle U \rangle$. With the total order $\mathbb{Q}$, Theorem 8.2 gives an explicit description of the subsets $X$ of $\mathbb{Q}$ for which there exists an idempotent $h \in \mathrm{End}(\mathbb{Q})$ with $\mathrm{im}\, h = X$.

Theorem 8.2 together with Theorem 2.7 tells us that the group $\mathscr{H}$-classes of endomorphisms of $\mathrm{End}(\mathbb{Q})$ are exactly the automorphism groups of the induced total orders $\langle \mathbb{Q} \setminus S \rangle$ of $\mathbb{Q}$ where either $S = \emptyset$ or $S = \bigcup_{i \in I} T_i$ and satisfies properties (i)-(iii) in Theorem 8.2. For convenience we will call $\mathbb{Q}$ and the subsets $\mathbb{Q} \setminus S$ which have properties (i)-(iii) *retracts* of $\mathbb{Q}$. That is, a retract of $\mathbb{Q}$ is a subset $U \subseteq \mathbb{Q}$ which is the image of an idempotent from $\mathrm{End}(\mathbb{Q})$.

If $\Omega$ is a countable total order then clearly, by Theorem 8.1, there exists an embedding $g : \Omega \to \mathbb{Q}$. If $\operatorname{im} g$ is a retract it follows that $\mathrm{Aut}(\Omega) \cong H_f$ for some idempotent $f$ where $\operatorname{im} f = \operatorname{im} g$. Therefore if we could identify the total orders which can be embedded into $\mathbb{Q}$ *via* an embedding $g$ such that $\operatorname{im} g$ is a retract, then we would be able to discover exactly which groups appear as maximal subgroups of $\mathrm{End}(\mathbb{Q})$. Unfortunately, for the moment, it is not clear exactly which total orders can be embedded into $\mathbb{Q}$ in this manner. Of course, the automorphism group of any total order $\Omega$ cannot be non-trivial and finite. Since if $f \in \mathrm{Aut}(\Omega)$ and $f \neq \mathbf{1}$ then $f$ has infinite order. As a result, we can at least deduce that no finite group can be a maximal subgroup of $\mathrm{End}(\mathbb{Q})$.

It should be observed that it can be possible to embed a total order $\Omega$ into $\mathbb{Q}$ *via* two embeddings $f$ and $g$ such that $\operatorname{im} f$ is a retract but such that $\operatorname{im} g$ is not. For consider the following example.

**Example 8.13.** Consider the induced total orders

$$S = \langle (-\infty, -2] \cup (-1, 1) \cup [2, \infty) \rangle,$$

and

$$T = \langle (-\infty, 0) \cup (1, \infty) \rangle,$$

of $\mathbb{Q}$. Notice that both $S$ and $T$ are dense and without endpoints so that $S \cong \mathbb{Q} \cong T$. Thus there exist embeddings $f, g : \mathbb{Q} \to \mathbb{Q}$ such that $\operatorname{im} f = S$ and $\operatorname{im} g = T$. Clearly $\operatorname{im} f = \mathbb{Q} \setminus \big((-2, -1] \cup [1, 2)\big)$ and therefore $\operatorname{im} f$ is a retract of $\mathbb{Q}$. However since $\operatorname{im} g = \mathbb{Q} \setminus [0, 1]$ it fails condition (i) and is therefore not a retract of $\mathbb{Q}$.

## 8.3 $2^{\aleph_0}$ Total Orders with Trivial Automorphism Group

In this section we will exhibit $2^{\aleph_0}$ non-isomorphic total orders which have trivial automorphism group. Furthermore we will show that each of these total orders can be embedded into $\mathbb{Q}$ in such a way that their image under the embedding is a retract of $\mathbb{Q}$.

**Definition 8.14.** Let $X = (x_n : n \in \mathbb{N})$ be any enumeration of $\mathbb{Q}$ so that $\mathbb{Q} = (\{x_n : n \in \mathbb{N}\}, \leq)$. Furthermore let $\mathbb{N} = (\mathbb{N}, \leq)$ be the natural numbers with the natural total order inherited from $(\mathbb{Q}, \leq)$. For $i \in \mathbb{N}$, let $B_i = \{a_{i0}, a_{i1}, \ldots, a_{ii}\}$ so that $|B_i| = i + 1$. Now let,

$$C = \bigcup_{i \in \mathbb{N}} B_n$$

and define the relational structure $\mathscr{C}_X = (C, \leq_X)$ where $a_{ij} \leq_X a_{kl}$ if and only if either $x_i < x_k$ or $i = k$ and $j \leq l$.

Notice then that $B_i <_X B_k$ if and only if $x_i < x_k$ and $B_i = B_k$ if and only if $i = k$. Also, by construction, $\mathscr{C}_X$ satisfies the following lemma.

**Lemma 8.15.** *Let $i, k \in \mathbb{N}$ and let $a_{ij}$, $a_{kl} \in C$. Then there exist infinitely many elements $b \in C$ and infinitely many elements $c \in C$ such that $b <_X a_{ij} <_X c$. Furthermore, if $x_i < x_k$ then there exist infinitely many elements $d \in C$ such that $a_{ij} <_X d <_X a_{kl}$. On the other hand if $i = k$ and $l = j + 1$ then there exists no element $d \in C$ such that $a_{ij} <_X c <_X a_{kl}$.*

*Proof.* Since $\mathbb{Q}$ is without endpoints, there exists a subset $\{i_m : m \in \mathbb{N}\} \subseteq \mathbb{N}$, such that $x_{i_m} < x_i$ for all $m \in \mathbb{N}$. Then by definition of $\leq_X$, $a_{i_m 0} <_X a_{ij}$ for all $m \in \mathbb{N}$. We can thus conclude that there exist infinitely many elements $b \in C$ such that $b <_X a_{ij}$. A similar argument shows that there exist infinitely many elements $c \in C$ such $a_{ij} <_X c$. Now suppose that $x_i < x_k$. Then since $\mathbb{Q}$ is dense, there exists a subset $\{k_n : n \in \mathbb{N}\} \subseteq \mathbb{N}$, such that $x_i < x_{k_n} < x_k$ for all $n \in \mathbb{N}$. Thus by definition of $\leq_X$, it follows that $a_{ij} <_X a_{k_n 0} <_X a_{kl}$ for all $n \in \mathbb{N}$. If on the other hand, $i = k$ and $l = j + 1$ then it should be clear by construction of $\mathscr{C}_X$ that there exists no element $c \in C$ such that $a_{ij} <_X c <_X a_{i(j+1)}$. □

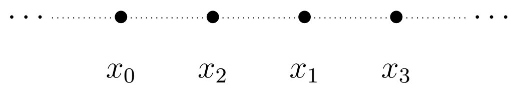
More importantly, we can show that $\mathscr{C}_X$ is a total order as follows.

**Lemma 8.16.** *Let $X$ be an enumeration of $\mathbb{Q}$ and let $\mathscr{C}_X = (C, \leq_X)$ be the relational structure defined in Definition 8.14. Then $\mathscr{C}_X$ is a total order.*

*Proof.* Let $X = (x_n : n \in \mathbb{N})$. We must check that $\leq_X$ defines a reflexive, antisymmetric and transitive binary relation on $C$ and that for all $a_{ij}, a_{kl} \in C$ at least one of $a_{ij} \leq_X a_{kl}$ or $a_{kl} \leq_X a_{ij}$ holds.

It should be clear that $\leq_X$ is reflexive since $i = i$, $j \leq j$ and so $a_{ij} \leq_X a_{ij}$. Now suppose that $a_{ij} \leq_X a_{kl}$ and $a_{kl} \leq_X a_{ij}$. Then since $a_{ij} \leq_X a_{kl}$ we know that either $x_i < x_k$ or $x = k$ and $j \leq l$. But since $a_{kl} \leq a_{ij}$ we also know

Figure 8.1: Part of $\mathbb{Q} = (\{x_n : n \in \mathbb{N}\}, \leq)$ if $x_0 < x_2 < x_1 < x_3$.



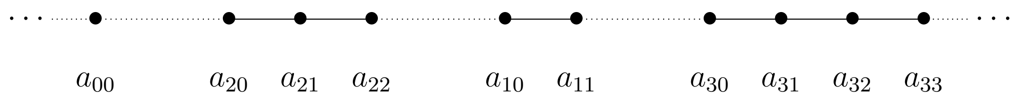

that either $x_k < x_i$ or $i = k$ and $l \leq j$. If $x_i < x_k$ then it clearly cannot be the case that $x_k \leq x_i$. Similarly if $x_k < x_i$ then it cannot be the case that $x_i \leq x_k$. The remaining possibility is that $i = k$, $j \leq l$ and $l \leq j$. But since $\mathbb{N}$ is a total order it follows that $j = l$ and hence $a_{ij} = a_{kl}$. Thus antisymmetry is satisfied.

To check transitivity we suppose that $a_{ij} \leq_X a_{kl}$ and $a_{kl} \leq_X a_{mn}$. Now since $a_{ij} \leq a_{kl}$ we know that either $x_i < x_k$ or $i = k$ and $j \leq l$ and since $a_{kl} \leq_X a_{mn}$ we know that either $x_k < x_m$ or $k = m$ and $j \leq l$. So suppose that $x_i < x_k$ and $x_k < x_m$. Then by transitivity of $\mathbb{Q}$, $x_i < x_m$ and hence $a_{ij} \leq_X a_{mn}$. If $x_i < x_k$ and $k = m$ then clearly $x_i < x_m$ and hence $a_{ij} \leq_X a_{mn}$. Similarly if $i = k$, $j \leq l$ and $x_k < x_m$ then $x_i < x_m$ and once again $a_{ij} \leq_X a_{mn}$. Finally if $i = k$, $j \leq l$, $k = m$ and $l \leq n$, then clearly $i = m$ and by transitivity of $\mathbb{N}$, $j \leq n$. Thus $a_{ij} \leq_X a_{mn}$ and $\leq_X$ is indeed transitive. To finish we note that totality of $\leq_X$ follows from totality of the natural order $\leq$ on $\mathbb{Q}$ and $\mathbb{N}$. $\square$

It can be helpful to have a pictorial representation of the total orders $\mathbb{Q}$ and $\mathscr{C}_X$ (and in fact any total order). Since total orders are antisymmetric and transitive we can view the total order as arranging elements into an ordered line from left to right. Thus an element $b$ to the left of an element $c$ will signify that $b < c$. We will also use a continuous line between two elements $b < c$ to represent the fact that there exists no element $d$ such that $b < d < c$. Dotted lines between two elements $b < c$ will represent the existence of one or more element $e$ such that $b < e < c$. See Figure 8.1 for a representation the total order $\mathbb{Q}$ and Figure 8.2 for a representation of the total order $\mathscr{C}_X$.

Clearly the order on $\mathscr{C}_X$ is dependent upon the enumeration $X = (x_n : n \in \mathbb{N})$ of $\mathbb{Q}$. Two enumerations $X = (x_n : n \in \mathbb{N})$ and $Y = (y_m : m \in \mathbb{N})$ of $\mathbb{Q}$ are said to be equal, written $X \equiv Y$, if $x_n = y_n$ for all $n \in \mathbb{N}$. In general, if $X \not\equiv Y$ then $\mathscr{C}_X = (C, \leq_X)$ and $\mathscr{C}_Y = (C, \leq_Y)$ will be non-isomorphic total orders. In particular, the identity map $1_C : C \to C$, may not define an isomorphism from $\mathscr{C}_X$ to $\mathscr{C}_Y$ (see Theorem 8.20). However for any two

Figure 8.2: Part of $\mathscr{C}_X$ if $x_0 < x_2 < x_1 < x_3$




enumerations $X$ and $Y$ of $\mathbb{Q}$, we find that the following results hold.

**Lemma 8.17.** *Let $X$ and $Y$ be any two enumerations of $\mathbb{Q}$. Suppose that $\phi : \mathscr{C}_X \to \mathscr{C}_Y$ is an isomorphism and let $i \in \mathbb{N}$. Then $B_i\,\phi = B_i$.*

*Proof.* Let $X = (x_n : n \in \mathbb{N})$ and let $Y = (y_m : m \in \mathbb{N})$. First we will show that $B_i\,\phi \subseteq B_k$ for some $k \in \mathbb{N}$. Seeking a contradiction, suppose that $B_i\phi \not\subseteq B_k$ for all $k \in \mathbb{N}$. Then

$$m = \max\{n \in \mathbb{N} : a_{in}\,\phi \in B_j\,,\, a_{i(n+1)}\,\phi \notin B_j,\ \text{for some } j \in \mathbb{N}\}$$

exists and $m < i$. Since $a_{im} <_X a_{i(m+1)}$ we must have that $a_{im}\,\phi \in B_j$ and $a_{i(m+1)}\,\phi \in B_k$ for some $j, k \in \mathbb{N}$ such that $y_j < y_k$. Now, by Lemma 8.15 there exists $c \in C$ such that $a_{im}\,\phi <_Y c <_Y a_{i(m+1)}\,\phi$. But since $\phi$ is an automorphism this means that $a_{im} <_X c\phi^{-1} <_X a_{i(m+1)}$, a contradiction to Lemma 8.15. Thus it must be the case that $B_i\,\phi \subseteq B_k$ for some $k \in \mathbb{N}$ as required. Furthermore, since $|B_n| = n+1$ for all $n \in \mathbb{N}$ and since $\phi$ is injective, we know that $i \leq k$. As $\phi^{-1} : \mathscr{C}_Y \to \mathscr{C}_X$ is also an isomorphism, we can repeat the argument above to show that $B_k\,\phi^{-1} \subseteq B_l$ for some $l \in \mathbb{N}$ with $k \leq l$. Then since $B_i \subseteq B_k\,\phi^{-1}$, it follows that $B_i \subseteq B_l$. But this implies that $i = l$ and hence $k = i$. Thus $B_i\,\phi \subseteq B_i$ and since $\phi$ is bijective it follows that $B_i\phi = B_i$ as required. $\square$

**Theorem 8.18.** *Let $X$ and $Y$ be any two enumerations of $\mathbb{Q}$ and suppose that $\phi : \mathscr{C}_X \to \mathscr{C}_Y$ is an isomorphism. Then $\phi = \mathbf{1}_C$.*

*Proof.* By Lemma 8.17 we know that $B_i\phi = B_i$ for all $i \in \mathbb{N}$. Now since both the substructure of $\mathscr{C}_X$ induced by $B_i$ and the substructure of $\mathscr{C}_Y$ induced by $B_i$ are well orders of size $i+1$, we must have that $\phi|_{B_i} = \mathbf{1}_{B_i}$ for all $i \in \mathbb{N}$. It now follows that $\phi = \mathbf{1}_C$. $\square$

**Corollary 8.19.** *Let $X$ be an enumeration of $\mathbb{Q}$. Then* $\mathrm{Aut}(\mathscr{C}_X) = \mathbf{1}_C$.

*Proof.* This follows immediately from Theorem 8.18 when we let $X = Y$. $\square$

**Theorem 8.20.** *Let $X$ and $Y$ be any two enumerations of $\mathbb{Q}$. Then $\mathscr{C}_X \cong \mathscr{C}_Y$ if and only if the map $f : \mathbb{Q} \to \mathbb{Q}$ defined by $x_n f = y_n$ for all $n \in \mathbb{N}$ defines an automorphism of $(\mathbb{Q}, \leq)$.*

*Proof.* Let $X = (x_n : n \in \mathbb{N})$ and let $Y = (y_m : m \in \mathbb{N})$. Suppose that $\mathscr{C}_X \cong \mathscr{C}_Y$. Then there exists an isomorphism $\phi : \mathscr{C}_X \to \mathscr{C}_Y$. By Theorem 8.18, we know that $a_{ij}\phi = a_{ij}$ for all $i, j \in \mathbb{N}$, $j \leq i$. Thus since $\phi$ is an isomorphism we can conclude that $a_{ij} \leq_X a_{kl}$ if and only if $a_{ij} \leq_Y a_{kl}$. Now consider the map $f : \mathbb{Q} \to \mathbb{Q}$ defined by $x_n f = y_n$ for all $n \in \mathbb{N}$. Then clearly $f$ is a bijective function. Furthermore, by construction of $\mathscr{C}_X$, $x_i \leq x_k$ if and only if $a_{i0} \leq_X a_{k0}$ and hence (by our previous observation) if and only if $a_{i0} \leq_Y a_{k0}$. But by construction of $\mathscr{C}_Y$, $a_{i0} \leq_Y a_{k0}$ if and only if $y_i \leq y_k$. Thus since $x_i f = y_i$ and $x_k f = y_k$ we can conclude that $x_i \leq x_k$ if and only if $x_i f \leq x_k f$. Hence $f$ defines an automorphism of $(\mathbb{Q}, \leq)$.

Now suppose instead that the map $f : \mathbb{Q} \to \mathbb{Q}$ defined by $x_n f = y_n$ for all $n \in \mathbb{N}$ defines an automorphism of $(\mathbb{Q}, \leq)$. Let $\phi = \mathbf{1}_C$. Then clearly $\phi$ defines a bijective function $C \to C$. Now suppose that $a_{ij} \leq_X a_{kl}$. Then $x_i \leq x_k$ and hence since $f$ is an isomorphism, $y_i \leq y_k$. But this means that $a_{ij} \leq_Y a_{kl}$ and hence $a_{ij}\phi \leq_Y a_{kl}\phi$. A similar argument shows that $a_{ij}\phi \leq_Y a_{kl}\phi$ implies that $a_{ij} \leq_X a_{kl}$. Thus $\phi$ defines an automorphism $\mathscr{C}_X \to \mathscr{C}_Y$ and the result is complete. □

We will now show that there exist $2^{\aleph_0}$ enumerations of the rational numbers which give rise to non-isomorphic total orders which have trivial automorphism group.

**Lemma 8.21.** *Let $X = (x_n : n \in \mathbb{N})$ be an enumeration of $\mathbb{Q}$ and let $\pi \in S_{\mathbb{N}}$. Let $\phi : \mathbb{Q} \to \mathbb{Q}$ be the function defined by $x_n \phi = x_{n\pi}$ for all $n \in \mathbb{N}$. If the disjoint cycle notation of $\pi$ contains a finite cycle, then $\phi \notin \operatorname{Aut}(\mathbb{Q})$.*

*Proof.* Seeking a contradiction, suppose that $\phi \in \operatorname{Aut}(\mathbb{Q})$. Let $(n_1\, n_2 \ldots n_k)$ be a finite cycle in the disjoint cycle notation of $\pi$. Without loss of generality we can assume that $x_{n_1} < x_{n_i}$ for all $1 < i \leq k$. Now since $\phi$ is an automorphism we have that $x_{n_1}\phi < x_{n_k}\phi$, but this says that $x_{n_2} < x_{n_1}$, a contradiction. Thus $\phi \notin \operatorname{Aut}(\mathbb{Q})$ as required. □

**Theorem 8.22.** *There exist a set $P$ of $2^{\aleph_0}$ enumerations of $\mathbb{Q}$ such that if $X, Y \in P$ and $X \not\equiv Y$ then $\mathscr{C}_X \not\cong \mathscr{C}_Y$.*

*Proof.* Let $Z = (z_n : n \in \mathbb{N})$ be any enumeration of $\mathbb{Q}$ and for $i \in \mathbb{N}$, let $\pi_i = (2i\ 2i+1) \in S_{\mathbb{N}}$. Now for $\Sigma \subseteq \mathbb{N}$ define,

$$\pi_\Sigma = \prod_{\sigma \in \Sigma} \pi_\sigma.$$

Then for each $\Sigma \subseteq \mathbb{N}$, $\pi_\Sigma \in S_{\mathbb{N}}$ and $\pi_\Sigma \pi_\Sigma = \mathbf{1}_{S_{\mathbb{N}}}$. Furthermore if we also have $\Psi \subseteq \mathbb{N}$, then $\pi_\Sigma \pi_\Psi = \pi_{\Sigma \ominus \Psi}$, where $\Sigma \ominus \Psi = (\Sigma \cup \Psi) \setminus (\Sigma \cap \Psi)$, the symmetric difference of $\Sigma$ and $\Psi$. Thus $\pi_\Sigma \pi_\Psi$ is a product of finite and disjoint cycles. Now for $\Sigma \subseteq \mathbb{N}$ we let,

$$Y_\Sigma = (x_n : n \in \mathbb{N}) \text{ where } x_n = z_{n\pi_\Sigma} \text{ for all } n \in \mathbb{N}.$$

Then clearly $Y_\Sigma$ is an enumeration of $\mathbb{Q}$ for all $\Sigma \subseteq \mathbb{N}$. Furthermore, if $\Psi \subseteq \mathbb{N}$ and $\Sigma \neq \Psi$ then $Y_\Sigma \not\equiv Y_\Psi$. Additionally, by Lemma 8.21 it follows that the map $\phi_{\pi_\Sigma \pi_\Psi} : \mathbb{Q} \to \mathbb{Q}$ defined by $z_n \phi_{\pi_\Sigma \pi_\Psi} = z_{n\pi_\Sigma \pi_\Psi}$ is such that $\phi_{\pi_\Sigma \pi_\Psi} \notin \operatorname{Aut}(\mathbb{Q})$. If we let

$$Y_\Sigma = (x_n : n \in \mathbb{N}) \text{ and } Y_\Psi = (y_n : n \in \mathbb{N}),$$

then for all $n \in \mathbb{N}$,

$$x_n \phi_{\pi_\Sigma \pi_\Psi} = z_{n\pi_\Sigma \pi_\Sigma \pi_\Psi} = z_{n\pi_\Psi} = y_n.$$

Thus by Theorem 8.20, $\mathscr{C}_{Y_\Sigma} \not\cong \mathscr{C}_{Y_\Psi}$ for $\Sigma \neq \Psi$. Letting $P = \{Y_\Sigma : \Sigma \subseteq \mathbb{N}\}$ completes the proof. $\square$

We will now show that for each enumeration $X$ of $\mathbb{Q}$, we can find an embedding $f : \mathscr{C}_X \to \mathbb{Q}$ such that $\operatorname{im} f$ is a retract of $\mathbb{Q}$. First we require the following two lemmas.

**Lemma 8.23.** *Let $n \in \mathbb{N}$ and let $\{i_1, \ldots, i_n\} \subseteq \mathbb{N}$. Suppose that*

$$f : \langle B_{i_1} \cup \cdots \cup B_{i_n} \rangle \to \mathbb{Q}$$

*is an embedding of $\langle B_{i_1} \cup \cdots \cup B_{i_n} \rangle \subset \mathscr{C}_X$ into $\mathbb{Q}$. Now let $i_{n+1} \in \mathbb{N} \setminus \{i_1, \ldots, i_n\}$. Then $f$ can be extended to an embedding,*

$$\tilde{f} : \langle B_{i_1} \cup \cdots \cup B_{i_n} \cup B_{i_{n+1}} \rangle \to \mathbb{Q}.$$

*Proof.* By the construction of $\mathscr{C}_X$, one of the following three statements hold.

(i) There exists $k, l \in \{i_1, \ldots, i_n\}$ such that $B_k <_X B_{i_{n+1}} <_X B_l$ and such that for any $p, q \in \{i_1, \ldots, i_n\}$, if $B_k \leq_X B_p <_X B_{i_{n+1}} <_X B_q \leq_X B_l$ then $k = p$ and $l = q$. Roughly speaking, $B_k$ is the maximum $B_{i_j}$ less than $B_{i_{n+1}}$ and $B_l$ is the minimum $B_{i_j}$ greater than $B_{i_{n+1}}$ for $j = 1, \ldots, n$.

(ii) Only $k$ exists as above.

(iii) Only $l$ exists as above.

So suppose that we are in case (i). Since $\mathbb{Q}$ is dense we can find $q_j \in \mathbb{Q}$, $j = 0, \dots, i_{n+1}$ such that $a_{kk}f < q_0 < q_1 < \cdots < q_{i_{n+1}} < a_{l0}f$. Now define $\tilde{f} : B_{i_1} \cup \cdots \cup B_{i_{n+1}} \to \mathbb{Q}$ by,

$$c\tilde{f} = \begin{cases} cf & \text{if } c \in B_{i_m},\ m = 1, \dots, n, \\ q_j & \text{if } c = a_{i_{n+1}j} \text{ for } j = 0, \dots, i_{n+1}. \end{cases}$$

Then clearly $\tilde{f}$ is a injective map $B_{i_1} \cup \cdots \cup B_{i_{n+1}} \to \mathbb{Q}$. Now if $b, c \in B_{i_1} \cup \cdots \cup B_{i_n}$, then clearly $b \leq_X c$ if and only if $b\tilde{f} \leq c\tilde{f}$ since $f$ was an embedding. Also, by construction of $\mathscr{C}_X$, if $b, c \in B_{i_{n+1}}$ then $b \leq_X c$ if and only if $b = a_{i_{n+1}j}$, $c = a_{i_{n+1}k}$ and $j \leq k$. But by choice, $q_j \leq q_k$ if and only if $j \leq k$. Thus it follows that $b \leq c$ if and only if

$$b\tilde{f} = q_j \leq q_k = c\tilde{f}.$$

Now suppose that $b \in B_{i_{n+1}}$ and $c \in B_{i_1} \cup \cdots \cup B_{i_n}$. Then $c \in B_{i_j}$ for some $j \in \{1, \dots, n\}$ such that $B_{i_{n+1}} <_X B_l \leq_X B_{i_j}$. Hence $b \leq_X c$ if and only if $a_{l0} \leq c$. Thus it now follows that $b \leq_X c$ if and only if

$$b\tilde{f} \leq q_{i_{n+1}} < a_{l0}f \leq cf = c\tilde{f}.$$

A dual argument shows that if $b \in B_{i_1} \cup \cdots \cup B_{i_n}$ and $c \in B_{i_{n+1}}$ then $b \leq_X c$ if and only if,

$$b\tilde{f} = bf \leq a_{kk}f < q_0 \leq c\tilde{f}.$$

Thus $\tilde{f}$ is indeed an embedding of $\langle B_{i_1} \cup \cdots \cup B_{i_{n+1}} \rangle$ into $\mathbb{Q}$ as required. A similar argument (using the fact that $\mathbb{Q}$ is without endpoints) shows that if we are in cases (ii) or (iii), then we can again extend $f$ to an embedding $\tilde{f} : \langle B_{i_1} \cup \cdots \cup B_{i_n} \cup B_{i_{n+1}} \rangle \to \mathbb{Q}$. □

**Lemma 8.24.** *Let $n \in \mathbb{N}$ and let $\{i_1, \dots, i_n\} \subseteq \mathbb{N}$. Suppose that,*

$$f : \langle B_{i_1} \cup \cdots \cup B_{i_n} \rangle \to \mathbb{Q}$$

*is an embedding of $\langle B_{i_1} \cup \cdots \cup B_{i_n} \rangle \subset \mathscr{C}_X$ into $\mathbb{Q}$. Now let $q \notin \operatorname{im} f$ and suppose that for all $j \in \{i_1, \dots, i_n\}$ there exists no $k \in \mathbb{N}$, $k < j$ such that $a_{jk}f < q < a_{j(k+1)}f$. Then there exists $i_{n+1} \in \mathbb{N} \setminus \{i_1, \dots, i_n\}$ and an extension*

$$\tilde{f} : \langle B_{i_1} \cup \cdots \cup B_{i_n} \cup B_{i_{n+1}} \rangle \to \mathbb{Q}$$

*of $f$ such that $\tilde{f}$ is an embedding and such that $q \in \operatorname{im} \tilde{f}$.*

*Proof.* By assumption, for all $j \in \{i_1, \dots, i_n\}$ there exists no $k \in \mathbb{N}$, $k < j$ such that $a_{jk}f < q < a_{j(k+1)}f$. Thus one of the following three cases must hold.

(i) There exists $k, l \in \{i_1, \ldots, i_n\}$ such that $B_k f < q < B_l f$ and such that for any $p, q \in \{i_1, \ldots, i_n\}$, if $B_k f \leq B_p f < q < B_q f \leq B_l f$ then $k = p, l = q$. Roughly speaking, $B_k f$ is the maximum $B_{i_j} f$ less than $q$ and $B_l f$ is the minimum $B_{i_j} f$ greater than $q$ for $j = 1, \ldots, n$.

(ii) Only $k$ exists as above.

(iii) Only $l$ exists as above.

So let us suppose that we are in case (i). Since $\mathbb{Q}$ is dense we can find $x_{i_{n+1}} \in \mathbb{Q}$ such that $x_k < x_{i_{n+1}} < x_l$ and hence $B_k <_X B_{i_{n+1}} <_X B_l$. Furthermore, we can find $q_j \in \mathbb{Q}$, $j = 0, \ldots, i_{n+1}$ such that,

$$q < q_1 < \cdots < q_{i_{n+1}} < a_{l0} f.$$

Now define $\tilde{f} : B_{i_1} \cup \cdots \cup B_{i_{n+1}} \to \mathbb{Q}$ by,

$$c\tilde{f} = \begin{cases} cf & \text{if } c \in B_{i_m},\ m = 1, \ldots, n \\ q & \text{if } c = a_{i_{n+1}0} \\ q_j & \text{if } c = a_{i_{n+1}j} \text{ for } j = 1, \ldots, i_{n+1}. \end{cases}$$

Then clearly, $\tilde{f}$ is an injective map. We will show that it also defines an embedding. Since by assumption $f$ was an embedding, if $b, c \in B_{i_1} \cup \cdots \cup B_{i_n}$ then clearly $b \leq_X c$ if and only if $b\tilde{f} \leq c\tilde{f}$. So suppose that $b \in B_{i_{n+1}}$ and $c \in B_{i_1} \cup \cdots \cup B_{i_n}$, then by definition of $\tilde{f}$, $b \leq_X c$ if and only if

$$b\tilde{f} \leq q_{i_{n+1}} < a_{l0} f \leq cf = c\tilde{f}.$$

Similarly if $b \in B_{i_1} \cup \cdots \cup B_{i_n}$ and $c \in B_{i_{n+1}}$ then $b \leq_X c$ if and only if,

$$b\tilde{f} = bf \leq a_{kk} f < q \leq c\tilde{f}.$$

Finally if $b, c \in B_{i_{n+1}}$ then $b \leq_X c$ if and only if $b = a_{i_{n+1}j}$ and $c = a_{i_{n+1}m}$ for some $1 \leq j \leq m \leq i_{n+1}$. But since $a_{i_{n+1}0}\tilde{f} = q < q_m = a_{i_{n+1}m}\tilde{f}$ for all $1 < m \leq i_{n+1}$, and since $a_{i_{n+1}j}\tilde{f} = q_j < q_m = a_{i_{n+1}m}\tilde{f}$ for all $1 < j \leq m$, it follows that if $b, c \in B_{i_{n+1}}$ then $b \leq_X c$ if and only if $b\tilde{f} \leq c\tilde{f}$. Hence $\tilde{f}$ is an embedding $\langle B_{i_1} \cup \cdots \cup B_{i_{n+1}} \rangle \to \mathbb{Q}$ and $q \in \operatorname{im} \tilde{f}$. A similar argument (using the fact that $\mathbb{Q}$ is without endpoints) shows that if we are in cases (ii) or (iii), then we can again extend $f$ to an embedding $\tilde{f} : \langle B_{i_1} \cup \cdots \cup B_{i_n} \cup B_{i_{n+1}} \rangle \to \mathbb{Q}$ with $q \in \operatorname{im} f$. $\square$

**Theorem 8.25.** *There exists an embedding $g : \mathscr{C}_X \to \mathbb{Q}$ such that* $\operatorname{im} g$ *is a retract of* $\mathbb{Q}$.

*Proof.* First, recall that $\mathscr{C}_X = (C, \leq)$, where $C = \bigcup_{i \in \mathbb{N}} B_n$. Enumerate $\mathbb{Q}$ as $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$. We will construct an embedding $g : C \to \mathbb{Q}$ inductively as follows. Define the map $f_0 : \{a_{00}\} \to \{q_0\}$ by $a_{00}f_0 = q_0$. Then clearly $f_0$ is an embedding $\langle a_{00} \rangle \to \langle q_0 \rangle$. Now suppose that for $n \in \mathbb{N}$, $f_n$ has been defined and is an embedding $\langle B_0 \cup B_{i_1} \cup \cdots \cup B_{i_n} \rangle \to \mathbb{Q}$ for some $i_j \in \mathbb{N}$, $j = 1, \ldots, n$. If $n$ is even let

$$a = \min\{i \in \mathbb{N} : B_i \nsubseteq \operatorname{dom} f_n\}$$

and by use of Lemma 8.23, extend $f_n$ to an embedding $f_{n+1}$ such that $B_a \in \operatorname{dom} f_n$. If on the other hand $n$ is odd, let

$$\begin{aligned} b = \min\{l \in \mathbb{N} : q_l \notin \operatorname{im} f_n \text{ such that for all } j \in \{i_1, \ldots, i_m\} \text{ there} \\ \text{exists no } k < j \text{ such that } a_{jk}f < q_l < a_{j(k+1)}f\}. \end{aligned}$$

Then by use of Lemma 8.24, extend $f_n$ to an embedding $f_{n+1}$ such that $B_{i_{n+1}} \in \operatorname{dom} f_{n+1}$ for some $i_{n+1} \notin \{0, i_1, \ldots, i_n\}$ and such that $q_b \in \operatorname{im} f_{n+1}$. Now let

$$g = \bigcup_{n=0}^{\infty} f_n.$$

Then $g$ is a well defined injective function since each $f_{n+1}$ was injective and was an extension of $f_n$. By alternately going back and forth we have ensured that $g$ is defined on every element of $\mathscr{C}_X$ and that if $q \notin \operatorname{im} g$ then there exist $i, j \in \mathbb{N}$ such that $a_{ij}g < q < a_{i(j+1)}g$. Furthermore, since each $f_n$ was an embedding $g$ is an embedding of $\mathscr{C}_X \to \mathbb{Q}$.

We will now show that $\operatorname{im} g$ is a retract of $\mathbb{Q}$. For $i, j \in \mathbb{N}$, $j < i$, let $T_{ij} = (a_{ij}g, a_{i(j+1)}g)$. Then clearly $T_{ij}$ is a non-closed interval in $\mathbb{Q}$ for all $i, j \in \mathbb{N}$ and since $g$ is an embedding, $T_{ij} \neq T_{kl}$ whenever $(i, j) \neq (k, l)$. Furthermore, since there exists no $c \in C$ such that $a_{ij} <_X c <_X a_{i(j+1)}$ and since $g$ is an embedding, it follows that there exists no $c \in C$ such that $a_{ij}g < cg < a_{i(j+1)}g$. Thus $T_{ij} \cap \operatorname{im} g = \emptyset$ for all $i, j \in \mathbb{N}$, $j < i$. Thus we can deduce that,

$$\operatorname{im} g \subseteq \mathbb{Q} \setminus \bigcup_{\substack{i,j \in \mathbb{N} \\ j < i}} T_{ij}.$$

Moreover, by our previous observations, if $q \notin \operatorname{im} g$ there must exist $i, j \in \mathbb{N}$ such that $a_{ij}g < q < a_{i(j+1)}g$. In other words $q \in T_{ij}$ and hence have shown that

$$\operatorname{im} g = \mathbb{Q} \setminus \bigcup_{\substack{i,j \in \mathbb{N} \\ j < i}} T_{ij}.$$

We claim that

$$\bigcup_{\substack{i,j\in\mathbb{N}\\ j<i}} T_{ij},$$

satisfies conditions (i), (ii) and (iii) of Theorem 8.2. We have already observed that $T_{ij}$ is a non-closed interval in $\mathbb{Q}$ for all $i, j \in \mathbb{N}$ and thus statement (i) of Theorem 8.2 holds. Now suppose that $T_{ij} \neq T_{kl}$. If $i \neq k$ then we can suppose without loss of generality that $x_i < x_k$. Hence by definition of $\mathscr{C}_X$, $a_{i(j+1)} < a_{kl}$ and thus since $g$ is an embedding $a_{i(j+1)}g < a_{kl}g$ and $T_{ij} < T_{kl}$. On the other hand, if $i = k$ but $j \neq l$ we can assume without loss of generality that $j < l$. Then since $a_{ij} < a_{in}$ for all $j < n$, it follows that $a_{i(j+1)} \leq a_{il} = a_{kl}$. Thus since $g$ is an embedding we can conclude that $a_{i(j+1)}g \leq a_{kl}g$ and hence that $T_{ij} < T_{kl}$. In either case we have shown that $T_{ij} \cap T_{kl} = \emptyset$ whenever $T_{ij} \neq T_{kl}$ and thus statement (ii) of Theorem 8.2 holds. Finally if $T_{ij} < T_{kl}$ then we observed above that $a_{i(j+1)} \leq a_{kl}$ and hence it follows that $a_{i(j+1)}g \leq a_{kl}g$. Thus since $a_{i(j+1)}g \in \operatorname{im} g$ and $T_{ij} < a_{i_(j+1)}g \leq a_{kl}g < T_{kl}$, statement (iii) of Theorem 8.2 holds. Thus since

$$\operatorname{im} g = \mathbb{Q} \setminus \bigcup_{\substack{i,j\in\mathbb{N}\\ j<i}} T_{ij}$$

and conditions (i), (ii) and (iii) of Theorem 8.2 are satisfied, it follows that $\operatorname{im} g$ is a retract of $\mathbb{Q}$. $\square$

**Theorem 8.26.** *There exist $2^{\aleph_0}$ idempotents $f \in \operatorname{End}(\mathbb{Q})$ such that $H_f \cong \mathbf{1}$.*

*Proof.* By Theorem 8.22, there exists a set $P$ of $2^{\aleph_0}$ enumerations of $\mathbb{Q}$ such that if $X, Y \in P$ and $X \not\equiv Y$ then $\mathscr{C}_X \ncong \mathscr{C}_Y$. Now by Theorem 8.25, for each $X \in P$ there exists an embedding $g_X : \mathscr{C}_X \to \mathbb{Q}$ such that $\operatorname{im} g_X$ is a retract. Hence by Theorem 8.2, for each $X \in P$ there exists $f_X \in E(\operatorname{End}(\mathbb{Q}))$ such that $\operatorname{im} f_X = \operatorname{im} g_X$. Since $\operatorname{im} f_X \cong \mathscr{C}_X$ for all $X \in P$ and since $\mathscr{C}_X \ncong \mathscr{C}_Y$ for all $Y \in P$ with $X \not\equiv Y$, we can deduce that the idempotents $f_X$ are all distinct. Now by Theorem 2.7 it follows that,

$$H_{f_X} \cong \operatorname{Aut}(\operatorname{im} f_X) = \operatorname{Aut}(\operatorname{im} g) \cong \operatorname{Aut}(\mathscr{C}_X).$$

But by Corollary 8.19, $\operatorname{Aut}(\mathscr{C}_X) = \mathbf{1}$ for all $X \in P$. Thus $H_{f_X} \cong \mathbf{1}$ for all $X \in P$ and the result is complete. $\square$

## 8.4 Group $\mathscr{H}$-classes of End($\mathbb{Q}$)

In this section, we will show that if $\Omega$ is a countable total order and there exists an embedding $f : \Omega \to \mathbb{Q}$ such that $\operatorname{im} f$ is a retract of $\mathbb{Q}$, then there exist $2^{\aleph_0}$ group $\mathscr{H}$-classes $H$ of End($\mathbb{Q}$) such that $H \cong \mathrm{Aut}(\Omega)$. First, we will see that if we are presented with a total order $\Omega$, we can construct $2^{\aleph_0}$ total orders with the same automorphism group as $\Omega$.

**Definition 8.27.** Let $\Omega = (V_\Omega, \leq_\Omega)$ and $\Lambda = (V_\Lambda, \leq_\Lambda)$ be total orders. When $V_\Omega \cap V_\Lambda = \emptyset$ we can construct a new total order, $\Omega + \Lambda = (V_{\Omega+\Lambda}, \leq_{\Omega+\Lambda})$ from $\Omega$ and $\Lambda$ as follows. We define $V_{\Omega+\Lambda} = V_\Omega \cup V_\Lambda$ and say that $u \leq_{\Omega+\Lambda} v$ if and only if either,

$u, v \in V_\Omega$ and $u \leq_\Omega v$,

$u, v \in V_\Lambda$ and $u \leq_\Lambda v$ or,

$u \in V_\Omega$ and $v \in V_\Lambda$.

Thus in $\Omega + \Lambda$, $V_\Omega <_{\Omega+\Lambda} V_\Lambda$. To avoid cumbersome notation we will denote $\leq_{\Omega+\Lambda}$ by $\preceq$ from now on.

Notice that if $\Omega = (V_\Omega, \leq_\Omega)$ and $\Lambda = (V_\Lambda, \leq_\Lambda)$ are total orders and $V_\Omega \cap V_\Lambda \neq \emptyset$ then we can consider the total orders $\Omega' = (V_{\Omega'}, \leq_{\Omega'})$ and $\Lambda' = (V_{\Lambda'}, \leq_{\Lambda'})$ defined by setting

$$\begin{aligned} V_{\Omega'} =& \{(u,1) : u \in V_\Omega\}, \\ V_{\Lambda'} =& \{(v,2) : v \in V_\Lambda\}, \end{aligned}$$

and where $(t,1) \leq_{\Omega'} (u,1)$ if and only if $t \leq_\Omega u$ and $(v,2) \leq_{\Lambda'} (w,2)$ if and only if $v \leq_\Lambda w$. Then it is easy to see that $\Omega \cong \Omega'$, $\Lambda \cong \Lambda'$ and $V_{\Omega'} \cap V_{\Lambda'} = \emptyset$. For this reason we will abuse notation in this chapter and often write $\Omega + \Lambda$ even when we have not asserted that $V_\Omega \cap V_\Lambda = \emptyset$.

Let $\bar{\mathbb{R}}_1 = (\mathbb{R}_1 \cup \{-\infty_1, \infty_1\}, \leq)$ and $\bar{\mathbb{R}}_2 = (\mathbb{R}_2 \cup \{-\infty_2, \infty_2\}, \leq)$ be two disjoint copies of the affinely extended real numbers with the natural ordering. Let $\mathbb{Q}_1$ and $\mathbb{Q}_2$ be copies of the total order $\mathbb{Q}$, thought of as substructures of $\bar{\mathbb{R}}_1$ and $\bar{\mathbb{R}}_2$ respectively. Then the following lemma is easy to prove.

**Lemma 8.28.** *Let $\mathbb{Q}_1$ and $\mathbb{Q}_2$ be copies of $\mathbb{Q}$. Then $\mathbb{Q}_1 + \mathbb{Q}_2 \cong \mathbb{Q}$.*

*Proof.* We observed that $\mathbb{Q}$ is the unique countable dense total order without endpoints. Thus it suffices to observe that $\mathbb{Q}_1 + \mathbb{Q}_2$ is countable, dense and without endpoints. □

Clearly, $\mathbb{Q}_1 + \mathbb{Q}_2$ is a relational substructure of $\bar{\mathbb{R}}_1 + \bar{\mathbb{R}}_2$. Thus, similar to the case with $\mathbb{Q}$, we can define an interval in $\mathbb{Q}_1 + \mathbb{Q}_2$ with real or infinite endpoints as follows. For $p, q \in \mathbb{R}_1 \cup \mathbb{R}_2 \cup \{-\infty_1, -\infty_2, \infty_1, \infty_2\}$, $p \leq q$ we define

$$\begin{aligned}
[p, q] &= \{x \in \mathbb{Q}_1 + \mathbb{Q}_2 : p \leq x \leq q\}, \\
(p, q] &= \{x \in \mathbb{Q}_1 + \mathbb{Q}_2 : p < x \leq q\}, \\
[p, q) &= \{x \in \mathbb{Q}_1 + \mathbb{Q}_2 : p \leq x < q\}, \\
(p, q) &= \{x \in \mathbb{Q}_1 + \mathbb{Q}_2 : p < x < q\}.
\end{aligned}$$

It should be clear that every interval in $\mathbb{Q}_1 + \mathbb{Q}_2$ can be written in at least one of the above forms. By a non-closed interval in $\mathbb{Q}_1 + \mathbb{Q}_2$ we will mean an interval $U$ in $\mathbb{Q}_1 + \mathbb{Q}_2$ such that $U$ cannot be written in the form $U = [p, q]$ for some $p, q \in \bar{\mathbb{R}}_1 \cup \bar{\mathbb{R}}_2 \cup \{-\infty_1, -\infty_2, \infty_1, \infty_2\}$. For example, if $p \in \mathbb{Q}_1 + \mathbb{Q}_2$ then the intervals $(p, q]$ and $(p, q)$ are non-closed, having the rational open endpoint $p$. However, if $p, q \notin \mathbb{Q}_1 + \mathbb{Q}_2$ then $[p, q] = [p, q) = (p, q] = (p, q)$ and hence each of these intervals is not a non-closed interval.

**Lemma 8.29.** *Let $\Omega = (V_\Omega, \leq_\Omega)$ be a countable total order. Suppose that there exists an embedding $h : \Omega \to \mathbb{Q}_1 + \mathbb{Q}_2$ such that $\operatorname{im} h = \mathbb{Q}_1 + \mathbb{Q}_2$ or $\operatorname{im} h = (\mathbb{Q}_1 + \mathbb{Q}_2) \setminus U$, where $U = \bigcup_{i \in I} V_i$ satisfies the following properties.*

- *(a) For each $i \in I$, $V_i$ is a non-closed interval in $\mathbb{Q}_1 + \mathbb{Q}_2$.*
- *(b) For $i \neq j$, $V_i \cap V_j = \emptyset$.*
- *(c) If $V_i < V_j$ then there exists $x \in \operatorname{im} h$ such that $V_i < x < V_j$.*

*Then there exists an embedding $\hat{h} : \Omega \to \mathbb{Q}$ such that $\operatorname{im} \hat{h}$ is a retract of $\mathbb{Q}$.*

*Proof.* By Lemma 8.28, there exists an isomorphism of total orders $\phi : \mathbb{Q}_1 + \mathbb{Q}_2 \to \mathbb{Q}$. Let $\hat{h} = h\phi$. Then $\hat{h}$ is an embedding of the total order $\Omega$ into $\mathbb{Q}$. By Theorem 8.2, it suffices to show that either $\operatorname{im} \hat{h} = \mathbb{Q}$ or $\operatorname{im} \hat{h} = \mathbb{Q} \setminus S$ where $S = \bigcup_{j \in J} T_j$ satisfies conditions (i), (ii) and (iii) of the theorem. If $\operatorname{im} h = \mathbb{Q}_1 + \mathbb{Q}_2$, then $\operatorname{im} \hat{h} = \mathbb{Q}$ and hence $\operatorname{im} \hat{h}$ is a retract of $\mathbb{Q}$ as required. So suppose instead that $\operatorname{im} h = (\mathbb{Q}_1 + \mathbb{Q}_2) \setminus U$, where $U = \bigcup_{i \in I} V_i$ satisfies properties $(a)$, $(b)$ and $(c)$ above. Then clearly $\operatorname{im} \hat{h} = \mathbb{Q} \setminus U\phi$. So let $T_i = V_i\phi$ for all $i \in I$ and let $S = \bigcup_{i \in I} T_i$. Then since $\phi$ is an isomorphism of total orders, $T_i$ is an interval in $\mathbb{Q}$ for all $i \in I$. We will now show that $T_i$ is non-closed for all $i \in I$. By property $(b)$, $V_i$ is a non-closed interval in $\mathbb{Q}_1 + \mathbb{Q}_2$ for all $i \in I$. Thus it is not hard to see that $V_i$ must have one of the following forms (for otherwise written in the form $V_i = [p, q]$ for some $p, q \in \bar{\mathbb{R}}_1 \cup \bar{\mathbb{R}}_2 \cup \{-\infty_1, -\infty_2, \infty_1, \infty_2\}$).

(1) $[x, y)$ where $x, y \in \mathbb{Q}_1 \cup \mathbb{Q}_2$.

(2) $(x, y]$ where $x, y \in \mathbb{Q}_1 \cup \mathbb{Q}_2$.

(3) $(x, y)$ where $x \in \mathbb{Q}_1 \cup \mathbb{Q}_2$, $y \notin \mathbb{Q}_1 \cup \mathbb{Q}_2$.

(4) $(x, y)$ where $y \in \mathbb{Q}_1 \cup \mathbb{Q}_2$, $x \notin \mathbb{Q}_1 \cup \mathbb{Q}_2$.

(5) $(x, y)$ where $x, y \in \mathbb{Q}_1 \cup \mathbb{Q}_2$.

Suppose first that $V_i = [x, y)$ for $x, y \in \mathbb{Q}_1 \cup \mathbb{Q}_2$, as in case (1). Then it should be easy to see that since $\phi$ is an isomorphism $T_i = [x, y)\phi = [x\phi, y\phi)$. Hence since $x\phi, y\phi \in \mathbb{Q}$, $T_i$ is a non-closed interval in $\mathbb{Q}$ as desired. A similar argument shows that if $V_i$ is of the form in case (2) and (5) then $T_i$ is a non-closed interval in $\mathbb{Q}$. If $V_i = (x, y)$ for $x \in \mathbb{Q}_1 \cup \mathbb{Q}_2$ as in case (3), then $x\phi$ is a rational open (left) endpoint for $T_i$ and so $T_i$ is a non-closed interval in $\mathbb{Q}$. A dual argument for case (4) now completes the proof that $T_i$ is a non-closed interval in $\mathbb{Q}$ in all cases.

Furthermore we can easily show that if $T_i \neq T_j$, then $T_i \cap T_j = \emptyset$ as follows. For if $x \in T_i \cap T_j$, then $x \in V_i\phi \cap V_j\phi$. But then $x\phi^{-1} \in V_i \cap V_j$, a contradiction to property $(b)$. Additionally, if $T_i < T_j$ then $V_i\phi < V_j\phi$ and since $\phi$ is an isomorphism it follows that $V_i < V_j$. Hence by property $(c)$ there exists $x \in \operatorname{im} h$ such that $V_i < x < V_j$. Thus $T_i < x\phi < T_j$ and $x\phi \in \operatorname{im} h\phi = \operatorname{im} \hat{h}$. We have hence shown that $\operatorname{im} \hat{h} = \mathbb{Q} \setminus S$ satisfies conditions (i), (ii) and (iii) of Theorem 8.2, and hence $\operatorname{im} \hat{h}$ is a retract of $\mathbb{Q}$. $\square$

**Lemma 8.30.** *Let $\Omega$ and $\Lambda$ be total orders and let $f : \Omega \to \mathbb{Q}$ and $g : \Lambda \to \mathbb{Q}$ be embeddings. Suppose that $\operatorname{im} f$ and $\operatorname{im} g$ are retracts. Then there exists an embedding $h : \Omega + \Lambda \to \mathbb{Q}$ such that $\operatorname{im} h$ is a retract of $\mathbb{Q}$.*

*Proof.* Let $\Omega = (V_\Omega, \leq)$ and let $\Omega = (V_\Omega, \leq)$. Let $\mathbb{Q}_1$ and $\mathbb{Q}_2$ be copies of the total order $\mathbb{Q}$. We can assume without loss of generality that $f : \Omega \to \mathbb{Q}_1$ and $g : \Lambda \to \mathbb{Q}_2$. By Lemma 8.29 it suffices to show that $\Omega + \Lambda$ can be embedded into $\mathbb{Q}_1 + \mathbb{Q}_2$ *via* an embedding $h$ such that either $\operatorname{im} h = \mathbb{Q}_1 + \mathbb{Q}_2$ or $\operatorname{im} h = (\mathbb{Q}_1 + \mathbb{Q}_2) \setminus S$ where $S$ is a union of non-closed intervals in $\mathbb{Q}_1 + \mathbb{Q}_2$ satisfying properties $(b)$ and $(c)$ of Theorem 8.29. Define $h : \Omega + \Lambda \to \mathbb{Q}_1 + \mathbb{Q}_2$ by,

$$vh = \begin{cases} vf & \text{if } v \in V_\Omega \\ vg & \text{if } v \in V_\Lambda. \end{cases}$$

Then since $f$ and $g$ are embeddings, $h$ is an embedding of $\Omega + \Lambda$ into $\mathbb{Q}_1 + \mathbb{Q}_2$. Furthermore, $\operatorname{im} h = \operatorname{im} g \cup \operatorname{im} f$. By assumption, $\operatorname{im} f$ and $\operatorname{im} g$ are retracts and so satisfy Theorem 8.2. If $\operatorname{im} f = \mathbb{Q}_1$ and $\operatorname{im} g = \mathbb{Q}_2$ then $\operatorname{im} h = \mathbb{Q}_1 + \mathbb{Q}_2$ and we are finished. On the other hand, suppose that $\operatorname{im} f = \mathbb{Q}_1$ and $\operatorname{im} g = \mathbb{Q}_2 \setminus \bigcup_{j \in J} U_j$, where the $U_j$ satisfy conditions (i)–(iii) of Theorem 8.2. Then $\operatorname{im} h = (\mathbb{Q}_1 + \mathbb{Q}_2) \setminus \bigcup_{j \in J} U_j$. Since by assumption $U_j$ is a non-closed interval in $\mathbb{Q}_2$ for all $j \in J$, $U_j$ is a non-closed interval in $\mathbb{Q}_1 + \mathbb{Q}_2$ for all $j \in J$. Furthermore, it is easy to see since the $U_j$ satisfy conditions $(ii) - (iii)$ of Theorem 8.2, properties $(b)$ and $(c)$ of Theorem 8.29 are satisfied. Thus letting $S = \bigcup_{j \in J} U_j$ we are finished. Similarly suppose that $\operatorname{im} g = \mathbb{Q}_2$ and $\operatorname{im} f = \mathbb{Q}_1 \setminus \bigcup_{i \in I} T_i$, where the $T_i$ satisfy conditions (i)–(iii) of Theorem 8.2. Then a similar argument shows that $\operatorname{im} h = (\mathbb{Q}_1 + \mathbb{Q}_2) \setminus \bigcup_{i \in I} T_i$ where $T_i$ is a non-closed interval in $\mathbb{Q}_1 + \mathbb{Q}_2$ for all $i \in I$ and where conditions $(b)$ and $(c)$ of Theorem 8.29 are satisfied. Thus by setting $S = \bigcup_{i \in I} T_i$ we are again finished.

Finally suppose that $\operatorname{im} f = \mathbb{Q}_1 \setminus \bigcup_{i \in I} T_i$ and $\operatorname{im} g = \mathbb{Q}_2 \setminus \bigcup_{j \in J} U_j$, where the $T_i$ and $U_j$ satisfy conditions (i)–(iii) of Theorem 8.2. Then

$$\operatorname{im} h = \mathbb{Q} \setminus \left( \left( \bigcup_{i \in I} T_i \right) \cup \left( \bigcup_{j \in J} U_j \right) \right).$$

Let

$$S = \left( \bigcup_{i \in I} T_i \right) \cup \left( \bigcup_{j \in J} U_j \right).$$

We first show that $S$ is a union of non-closed intervals in $\mathbb{Q}_1 + \mathbb{Q}_2$. Seeking a contradiction, suppose that there exists a closed interval $[q, r] \subseteq S$ for some $q, r \in \bar{\mathbb{R}}_1 \cup \bar{\mathbb{R}}_1 \cup \{-\infty_1, -\infty_2, \infty_1, \infty_2\}$. By assumption $T_i$ and $U_j$ are non-closed intervals in $\mathbb{Q}_1$ and $\mathbb{Q}_2$ respectively for all $i \in I$ and for all $j \in J$. Thus it follows that there must exist $i \in I$ and $j \in J$ such that $T_i \cup U_j$ is a closed interval. It is not hard to see that since $\operatorname{im} f \prec \operatorname{im} g$ this is possible only if $T_i = [q, \infty)$ and $U_j = (-\infty, r]$ for some $q \in \bar{\mathbb{R}}_1 \cup \{-\infty_1\}$ and some $r \in \bar{\mathbb{R}}_2 \cup \{\infty_2\}$. But then $T_i = [q, \infty]$ and $U_j = [-\infty, r]$, a contradiction to $T_i$ and $U_j$ being non-closed. Thus it follows that $S$ is a union of non-closed intervals in $\mathbb{Q}_1 + \mathbb{Q}_2$.

To see that $S$ satisfies condition $(b)$ of Theorem 8.29, we recall that since $\operatorname{im} f \subseteq \mathbb{Q}_1$ and $\operatorname{im} g \subseteq \mathbb{Q}_2$, $\operatorname{im} f \prec \operatorname{im} g$. Thus $T_i \cap U_j = \emptyset$ for all $i \in I$ and for all $j \in J$. Furthermore, since $\operatorname{im} f$ and $\operatorname{im} g$ are retracts it follows that by condition (ii) of Theorem 8.2 that $T_i \cap T_k = \emptyset$ for all $i, k \in I$ such that $i \neq k$ and similarly that $U_j \cap U_l = \emptyset$ for all $j, l \in J$ such that $j \neq l$. Thus $S$ satisfies condition $(b)$ of Theorem 8.29.

To finish we verify that $S$ satisfies condition $(c)$ of Theorem 8.29. Since $\operatorname{im} f$ is a retract it follows that for all $i, k \in I$ such that $T_i < T_k$ there exists $x \in \operatorname{im} f \subseteq \operatorname{im} h$ such that $T_i < x < T_k$. Similarly since $\operatorname{im} g$ is a retract it follows that for all $j, l \in I$ such that $U_j < U_l$ there exists $y \in \operatorname{im} f \subseteq \operatorname{im} h$ such that $U_j < y < U_l$. It remains to show that if $i \in I$, $j \in J$ and $T_i < U_j$, then there exists $z \in \operatorname{im} h$ such that $T_i < z < U_j$. If there exists $x \in \operatorname{im} f$ such that $T_i < x$, then $T_i < x < U_j$ and we are done. Similarly if there exists $y \in \operatorname{im} g$ such that $y < U_j$, then $T_i < y < U_j$ and we are finished. So suppose that for all $x \in \operatorname{im} f$ and for all $y \in \operatorname{im} g$, $x \leq T_i$ and $U_j \leq y$. Then it must be the case that $T_i = (q, \infty)$ and $U_j = (-\infty, r)$ for some $q \in \mathbb{Q}_1$ and some $r \in \mathbb{Q}_2$. Hence $T_i \cup U_j = (q, r)$ and $T_i \cup U_j$ is actually one of the non-closed interval in $S$. Moreover if $T_k < T_i$, then since $\operatorname{im} f$ is a retract there exists $x_k \in \operatorname{im} f \subseteq \operatorname{im} h$ such that $T_k < x_k < T_i \cup U_j$ and similarly if $U_j < U_l$ there exists $y_l \in \operatorname{im} g \subseteq \operatorname{im} h$ such that $T_i \cup U_j < y_l < U_l$ . Thus it follows that condition $(c)$ of Theorem 8.29 is satisfied. In any case, we have shown that $\operatorname{im} h = \mathbb{Q} \setminus S$ satisfies conditions $(a)$, $(b)$ and $(c)$ of Theorem 8.2 and thus $\operatorname{im} h$ is a retract of $\mathbb{Q}$. $\square$

**Lemma 8.31.** *Let $\Omega = (V_\Omega, \leq_\Omega)$ be a total order. Let $X$ be an enumeration of $\mathbb{Q}$ and let $\mathscr{C}_X$ be the total order defined in Definition 8.14. Then* $\operatorname{Aut}(\Omega + \mathscr{C}_X) \cong \operatorname{Aut}(\Omega)$.

*Proof.* First recall that $\mathscr{C}_X = (C, \leq_X)$. We will first show that if $g \in \operatorname{Aut}(\Omega + \mathscr{C}_X)$ and $Cg \subseteq C$, then in fact $Cg = C$. Consider $B_i \subseteq C$ for some $i \in \mathbb{N}$. We start by showing that $B_i g \subseteq B_k$ for some $k \in \mathbb{N}$. The method is essentially the same as the proof of Lemma 8.17. Seeking a contradiction, suppose that $B_i g \not\subseteq B_j$ for all $j \in \mathbb{N}$. Then

$$m = \max\{n \in \mathbb{N} : a_{in} g \in B_j \, , \, a_{i(n+1)} g \notin B_j, \text{ for some } j \in \mathbb{N}\}$$

exists and $m < i$. Since $a_{im} <_X a_{i(m+1)}$ it follows that $a_{im} \prec a_{i(m+1)}$. Thus $a_{i(m+1)} g \in C$ and thus $a_{i(m+1)} g \in B_k$ for some $k \in \mathbb{N}$ such that $x_j < x_k$. Now, by Lemma 8.15 there exists $c \in C$ such that $a_{im} g <_X c <_X a_{i(m+1)} g$ and hence $a_{im} g \prec c \prec a_{i(m+1)} g$. Since $g$ is an automorphism we now deduce that $a_{im} \prec cg^{-1} \prec a_{i(m+1)}$. But then $cg^{-1} \in C$ and $a_{im} <_X cg^{-1} <_X a_{i(m+1)}$, a contradiction to Lemma 8.15. We can hence conclude that $B_i \phi \subseteq B_k$ for some $k \in \mathbb{N}$ as required. Furthermore, since $|B_n| = n + 1$ for all $n \in \mathbb{N}$ and since $f$ is injective, we know that $i \leq k$.

Now suppose that $i \neq k$ so that $i < k$. Then there exists some $a_{kl} \in B_k$

such that $a_{kl} \neq ag$ for any $a \in B_i$. Let,

$$s = \max\{n < l : n \in \mathbb{N},\ a_{kn} = bg \text{ for some } b \in B_i\}, \text{ and}$$
$$t = \min\{m > l : m \in \mathbb{N},\ a_{km} = cg \text{ for some } c \in B_i\}.$$

Since $B_i\, g \subset B_k$, at least one of $s$ and $t$ exist. So suppose that $s$ exists. Then $s + 1 \leq l$, $a_{ks}g^{-1} \in B_i$ and $a_{k(s+1)}g^{-1} \notin B_i$. Since $g$ is an automorphism $a_{ks}g^{-1} \prec a_{k(s+1)}g^{-1}$ and so $a_{k(s+1)}g^{-1} \in C$. Thus there must exist some $l \in \mathbb{N}$ such that $x_i < x_l$ and $a_{k(s+1)}g^{-1} \in B_l$. However, by Lemma 8.15 there exists $c \in C$ such that $a_{ks}g^{-1} <_X c <_X a_{k(s+1)}g^{-1}$ and hence $a_{ks}g^{-1} \prec c \prec a_{k(s+1)}g^{-1}$. Since $g$ is an automorphism we can thus conclude that $a_{ks} \prec cg \prec a_{j(s+1)}$. But then $a_{ks} <_X cg <_X a_{k(s+1)}$ which is clearly a contradiction to Lemma 8.15.

So suppose instead that $t$ exists. Then $l \leq t - 1$, $a_{k(t-1)} \notin B_i$ and $a_{kt} \in B_i$. Also, since $g$ is an automorphism $a_{k(t-1)}g^{-1} \prec a_{kt}g^{-1}$. Suppose that $a_{k(t-1)}g^{-1} = v$ for some $v \in V_\Omega$. Then $v \prec C$. By Lemma 8.15, there exists infinitely many elements $c \in C$ such that $c <_X a_{kt}g^{-1}$ and hence $v \prec c \prec a_{kt}g^{-1}$. Thus, since $g$ is an automorphism $vg \prec cg \prec a_{kt}$. But then $a_{k(t-1)} \prec cg \prec a_{kt}$ and hence $a_{k(t-1)} <_X cg <_X a_{kt}$, a contradiction to Lemma 8.15. Thus we conclude that $a_{k(t-1)}g^{-1} \in C$. Thus there must exist some $l \in \mathbb{N}$ such that $x_l < x_i$ and $a_{k(t-1)}g^{-1} \in B_l$. However, by Lemma 8.15 there exists $c \in C$ such that $a_{k(t-1)}g^{-1} <_X c <_X a_{kt}g$ and hence $a_{k(t-1)}g^{-1} \prec c \prec a_{kt)}g^{-1}$. Since $g$ is an automorphism we can thus conclude that $a_{k(t-1)} \prec cg \prec a_{kt}$. But then $a_{k(t-1)} <_X cg <_X a_{kt}$ which is clearly a contradiction to Lemma 8.15. In any case we have shown that the assumption that $i < k$ leads us to a contradiction. Hence we can now conclude that $i = k$ and hence $B_i g \subseteq B_i$. Thus since $g$ is bijective $B_i g = B_i$ and it now follows that $Cg = C$ as claimed.

We will now show that if $f \in \mathrm{Aut}(\Omega + C_X)$, then $Cf = C$ and $V_\Omega f = V_\Omega$. If $Cf \subseteq C$, then by the observations above $Cf = C$ and hence since $f$ is bijective it follows that $V_\Omega f = V_\Omega$. Suppose on the other hand that there exists $c \in C$ such that $cf = v$ for some $v \in V_\Omega$. Consider then inverse automorphism $f^{-1}$. Then $vf^{-1} = c$. Furthermore, since $f^{-1}$ is an automorphism, if $u \in V_{\Omega+\mathscr{C}_X}$ and $v \prec u$ then $c = vf^{-1} \prec uf^{-1}$. In particular, since $v \prec C$, $c \prec Cf^{-1}$ and hence $Cf^{-1} \subseteq C$. Thus by a further application of the observation above, $Cf^{-1} = C$ and hence $Cf = C$. Now since $f$ is bijective we can again conclude that $V_\Omega f = V_\Omega$. In ether case, we have shown that $Cf = C$ and $V_\Omega f = V_\Omega$. Moreover, since $f$ is an automorphism and since we have just shown that $Cf = C$, $f|_C$ must be an automorphism on the relational substructure of $\Omega + \mathscr{C}_X$ induced by $C$. In other words $f|_C \in \mathrm{Aut}(\mathscr{C}_x)$. But by Lemma 8.18 this implies that $f|_C = \mathbf{1}_C$.

Now define a map $\phi : \mathrm{Aut}(\Omega + \mathscr{C}_X) \to \mathrm{Aut}(\Omega)$ by $f\phi = f|_{V_\Omega}$ for all $f \in \mathrm{Aut}(\Omega + \mathscr{C}_X)$. Then by our previous observations $\phi$ is a well defined function. Moreover, $\phi$ defines a group homomorphism since

$$(fg)\phi = (fg)|_\Omega = f|_\Omega \cdot g|_\Omega = f\phi \cdot g\phi.$$

To see that $\phi$ is injective suppose that $f, g \in \mathrm{Aut}(\Omega + \mathscr{C}_X)$ are such that $f\phi = g\phi$. Then $f|_\Omega = g|_\Omega$ and since we know that $f|_C = g|_C = \mathbf{1}_C$ we can conclude that $f = g$. To show that it is surjective we note that if $h \in \mathrm{Aut}(\Omega)$ then the map $\hat{h} : \Omega + \mathscr{C}_X \to \Omega + \mathscr{C}_X$ defined by,

$$v\hat{h} = \begin{cases} v & \text{if } v \in C \\ vh & \text{if } v \in V_\Omega. \end{cases}$$

is an automorphism of $\Omega + \mathscr{C}_X$ and $\hat{h}\phi = h$. Thus $\phi$ defines a group isomorphism $\mathrm{Aut}(\Omega + \mathscr{C}_X) \to \mathrm{Aut}(\Omega)$ and hence $\mathrm{Aut}(\Omega + \mathscr{C}_X) \cong \mathrm{Aut}(\Omega)$. □

As a direct consequence of the above lemmas, if we have a total order $\Omega$ and an embedding $f : \Omega \to \mathbb{Q}$ such that $\operatorname{im} f$ is a retract of $\mathbb{Q}$, then we can gain some insight into the number of group $\mathscr{H}$-classes which are isomorphic to the automorphism group of $\Omega$.

**Theorem 8.32.** *Let $\Omega$ be a total order. If there exists an embedding $f : \Omega \to \mathbb{Q}$ such that $\operatorname{im} f$ is a retract of $\mathbb{Q}$, then there exist $2^{\aleph_0}$ group $\mathscr{H}$-classes $H$ of $\mathrm{End}(\mathbb{Q})$ such that $H \cong \mathrm{Aut}(\Omega)$.*

*Proof.* By Theorem 8.22, there exists a set $P$ of $2^{\aleph_0}$ enumerations of $\mathbb{Q}$ such that if $X, Y \in P$ and $X \not\equiv Y$ then $\mathscr{C}_X \not\cong \mathscr{C}_Y$. Now by Lemma 8.30, for each $X \in P$ there exists an embedding $g_X : \Omega + \mathscr{C}_X \to \mathbb{Q}$ such that $\operatorname{im} g_X$ is a retract. Hence by Theorem 8.2, for each $X \in P$ there exists $f_X \in E(\mathrm{End}(\mathbb{Q}))$ such that $\operatorname{im} f_X = \operatorname{im} g_X$. Since $\operatorname{im} f_X \cong \mathscr{C}_X$ for all $X \in P$ and since $\mathscr{C}_X \not\cong \mathscr{C}_Y$ for all $Y \in P$ with $X \neq Y$, we can deduce that the idempotents $f_X$ are all distinct. Now by Theorem 2.7 it follows that,

$$H_{f_X} \cong \mathrm{Aut}(\operatorname{im} f_X) = \mathrm{Aut}(\operatorname{im} g) \cong \mathrm{Aut}(\Omega + \mathscr{C}_X).$$

But by Lemma 8.31, $\mathrm{Aut}(\Omega + \mathscr{C}_X) \cong \mathrm{Aut}(\Omega)$, for all $X \in P$. Thus $H_{f_X} \cong \mathrm{Aut}(\Omega)$ for all $X \in P$ and the result is complete. □

We now know that if we can embed a total order $\Omega$ into $\mathbb{Q}$ such that the image of $\Omega$ in $\mathbb{Q}$ is a retract, then the automorphism group of $\Omega$ is isomorphic to $2^{\aleph_0}$ maximal subgroups of $\mathrm{End}(\mathbb{Q})$.

## 8.5 Regular $\mathscr{D}$-classes of End($\mathbb{Q}$)

What we can deduce about the $\mathscr{D}$-classes of End($\mathbb{Q}$) now follows in this section.

**Theorem 8.33.** *There exist $2^{\aleph_0}$ regular $\mathscr{D}$-classes of* End($\mathbb{Q}$) *for which any two group $\mathscr{H}$-classes are isomorphic.*

*Proof.* By the proof of Theorem 8.26 there exists a set $P$ of size $2^{\aleph_0}$ and idempotents $f_X$ such that $H_{f_X} \cong \mathbf{1}$ for all $X \in P$ and such that $\operatorname{im} f_X \not\cong \operatorname{im} f_Y$ for all $X, Y \in P$, $X \not\equiv Y$. Thus by Theorem 2.10, the $f_X$ lie in distinct $\mathscr{D}$-classes but $H_{f_X} \cong \mathbf{1}$ for all $X \in P$. Now since any two group $\mathscr{H}$-classes which are contained in the same $\mathscr{D}$-class are isomorphic, the result follows. $\square$

We cannot yet say whether there exist $2^{\aleph_0}$ $\mathscr{D}$-classes for which any two $\mathscr{H}$-classes from different $\mathscr{D}$-classes are not isomorphic. To do so we would need to assert that there are uncountably many groups which can be realised as the automorphism group of a retract of $\mathbb{Q}$. This is, as yet, undetermined.

**Theorem 8.34.** *Let $f \in E(\text{End}(\mathbb{Q}))$. If $\mathbb{Q} \setminus \operatorname{im} f$ contains an open interval with rational endpoints then $D_f$ contains $2^{\aleph_0}$ group $\mathscr{H}$-classes.*

*Proof.* Since $\mathbb{Q} \setminus \operatorname{im} f$ contains an open interval it follows from Theorem 8.2 that there exist $2^{\aleph_0}$ distinct idempotents $g$ such that $\operatorname{im} g = \operatorname{im} f$. Furthermore, it follows from Theorem 2.10 that each of these idempotents $g$ lies in $D_f$. Thus each $H_g$ is contained in $D_f$ and so there exist $2^{\aleph_0}$ group $\mathscr{H}$-classes in $D_f$ as required. $\square$

In view of Theorem 8.34 it is now easy to give an example of a $\mathscr{D}$-classes of End($\mathbb{Q}$) which contains $2^{\aleph_0}$ group $\mathscr{H}$-classes.

**Example 8.35.** Let $S = (-\infty, -1] \cup [1, \infty)$. Then $\mathbb{Q} \setminus S = (-1, 1)$ and hence by Theorem 8.2, there exists an idempotent $f \in \text{End}(\mathbb{Q})$ such that $\operatorname{im} f = \langle S \rangle$. Then by Theorem 8.34 above, $D_f$ contains $2^{\aleph_0}$ group $\mathscr{H}$-classes.

On the other hand, we can show that there exists a $\mathscr{D}$-class of End($\mathbb{Q}$) which contains countably many group $\mathscr{H}$-classes.

**Example 8.36.** Consider any element $q \in \mathbb{Q}$. Then $\langle \{q\} \rangle = (\{q\}, (q, q))$ is trivially a total order. Furthermore, since $\mathbb{Q} \setminus \{q\} = (-\infty, q) \cup (q, \infty)$ it follows by Theorem 8.2 that there exists an idempotent $f \in \text{End}(\mathbb{Q})$ such that $\operatorname{im} f = \langle \{q\} \rangle$. Furthermore, it is not hard to see that the only such idempotent is the map $g_q$ where $xg_q = q$ for all $x \in \mathbb{Q}$. Now for any other idempotent $h \in E(\mathrm{End}(\mathbb{Q}))$, we know from Theorem 2.10, that $h \in D_f$ if and only if $\mathrm{im}\, h \cong \mathrm{im}\, f = (\{q\}, (q, q))$. Clearly this is only possible when $\mathrm{im}\, h = \langle\{p\}\rangle$ and $h = g_p$ for some $p \in \mathbb{Q}$. Since $\mathbb{Q}$ is countable there exist only countably many distinct elements $p \in \mathbb{Q}$ and thus there exist exactly $\aleph_0$ idempotents $g_p$ with $\mathrm{im}\, g_p \cong \mathrm{im}\, f$. Hence $D_f$ contains exactly $\aleph_0$ group $\mathscr{H}$-classes (namely $H_{g_p}$ for each $p \in \mathbb{Q}$) as claimed.

# Chapter 9

# Countable Groups which are the Automorphism Group of a Total Order

In Chapter 3 we observed that if $\Gamma$ is a countable graph, then $\mathrm{End}(R)$ contains a maximal subgroup isomorphic to $\mathrm{Aut}(\Gamma)$. Thus by use of Frucht's Theorem, we were able to show that every countable group is contained as a maximal subgroup $\mathrm{End}(R)$. Analogous results were also obtained for $\mathrm{End}(D)$ and $\mathrm{End}(B)$ in Chapters 4 and 7, respectively. In this chapter we will show that this analogy breaks down in the setting of total orders. We will show that if $\Omega$ is a total order and $\mathrm{Aut}(\Omega)$ is countable, then $\mathrm{Aut}(\Omega) \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}$. In particular, this means that any countable maximal subgroup of $\mathrm{End}(\mathbb{Q})$ must be of this form. The proof will require the introduction of many technical lemmas and so we provide a short overview of the chapter as follows.

Throughout we fix a total order $\Omega = (V_\Omega, \leq)$. In Section 9.1, we define what is meant by a *orbital* $U \subseteq V_\Omega$ of an automorphism of $\Omega$ and develop the necessary theory for use in this chapter. We will show *via* Sections 9.2–9.4 that if $\mathrm{Aut}(\Omega)$ is countable, then $\mathrm{Aut}(\langle U \rangle)$ is either cyclic or there exists an order $\leq$ on $\mathrm{Aut}(\Omega)$ such that $(\mathrm{Aut}(\langle U \rangle), \leq)$ is dense. In the latter case, there is a close connection between the $\lambda$-coloured rationals and $\langle U \rangle$. Accordingly we use Section 9.5 to introduce the definition of the $\lambda$-coloured rationals and show that the automorphism group of this structure has cardinality $2^{\aleph_0}$. We bring all the ideas together in the final sections, Section 9.6 and 9.7, and conclude in Theorem 9.51 that, when $\mathrm{Aut}(\Omega)$ is countable, $\mathrm{Aut}(\langle U \rangle)$ is cyclic and that $\mathrm{Aut}(\Omega) \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}$.

## 9.1 Fundamentals of Orbitals

We begin with a definition.

**Definition 9.1.** Let $\Omega = (V_\Omega, \leq)$ be a total order and fix $f \in \mathrm{Aut}(\Omega)$. For $x \in V_\Omega$ we define $U_x \subseteq V_\Omega$ to be the set,

$$U_x = \{y \in V_\Omega : xf^m \leq y \leq xf^n \text{ for some } m, n \in \mathbb{Z}\},$$

and call $U_x$ an *orbital* of $f$. Clearly, $x \in U_x$ for all $x \in V_\Omega$.

Orbitals will be pivotal to the proof of the major results in this chapter. Accordingly, we will use this subsection to briefly develop the theory of orbitals and provide some key lemmas for use later. For the rest of this subsection we will let $\Omega = (V_\Omega, \leq)$ be a total order and fix $f \in \mathrm{Aut}(\Omega)$ with orbital $U_x$.

**Lemma 9.2.** *For $x \in V_\Omega$, either:*

(i) *$u < uf$ for all $u \in U_x$ and $U_x$ is infinite,*

(ii) *$uf < u$ for all $u \in U_x$ and $U_x$ is infinite, or*

(iii) *$xf = x$ and $U_x = \{x\}$.*

*In case (i) we say that $U_x$ is a positive orbital and in case (ii) we say that $U_x$ is a negative orbital.*

*Proof.* If $x \in V_\Omega$ is such that $xf = x$, then clearly $xf^n = x$ for all $n \in \mathbb{Z}$. Thus we can immediately conclude that $U_x = \{x\}$. So suppose that $xf \neq x$. Then either $x < xf$ or $xf < x$. Suppose that $x < xf$ and thus, since $f$ is an automorphism, that $xf^n < xf^{n+1}$ for all $n \in \mathbb{Z}$. Clearly, since $xf^n \in U_x$ for all $n \in \mathbb{N}$ and since $xf^n \neq xf^m$ for all $m \in \mathbb{Z}$ with $m \neq n$, it follows that $U_x$ is infinite. Now, seeking a contradiction, suppose that there exists $u \in U_x$ such that $uf < u$ and hence that $uf^{i+1} < u$ for all $i \in \mathbb{N}$. Since $u \in U_x$ there exists $m, n \in \mathbb{N}$ such that $xf^m \leq u \leq xf^n$ and since $x < xf$ it follows that $m < n$. Now let $p = n - m$. Then $xf^m \leq u$ but $uf^p < u \leq xf^n = xf^{m+p}$. This is clearly a contradiction since $f$ is order preserving. Hence we can conclude that $u < uf$ for all $u \in U_x$. A similar argument starting with $xf < x$ shows that in this case $uf < u$ for all $u \in U_x$ and that $U_x$ is infinite. $\square$

**Corollary 9.3.** *If $f \in \mathrm{Aut}(\Omega) \setminus \{\mathbf{1}\}$, then $f$ has an infinite orbital.*

*Proof.* Suppose that $f \in \mathrm{Aut}(\Omega)$ and that all orbitals of $f$ are finite. Then by Lemma 9.2 all orbitals are singletons and $xf = x$ for all $x \in V_\Omega$. Hence $f = \mathbf{1}$ and the result now follows. $\square$

**Lemma 9.4.** *Let $x \in V_\Omega$ and suppose that $U_x$ is an infinite orbital. Then $U_x$ is an interval in $\Omega$.*

*Proof.* If $u, v \in U_x$ then there exists $i, j, m, n \in \mathbb{Z}$ such that $xf^m \leq u \leq xf^n$ and $xf^i \leq v \leq xf^j$. Hence if $w \in V_\Omega$ is such that $u \leq w \leq v$ then,

$$xf^m \leq u < w < v \leq xf^j.$$

Hence we can immediately conclude that $w \in U_x$ and hence that $U_x$ is an interval in $\Omega$. $\square$

**Lemma 9.5.** *Suppose that $U_x$ is an infinite orbital. Then $\langle U_x \rangle$ is without endpoints.*

*Proof.* Seeking a contradiction, suppose that there exists $u \in U_x$ such that $u \leq v$ for all $v \in V$. Since $u \in U_x$ there exists $m, n \in \mathbb{Z}$ such that $xf^m \leq u \leq xf^n$. Hence we can conclude that $xf^m = u$. If $U_x$ is positive then $xf^{m-1} < xf^m = u$, a contradiction. If $U_x$ is negative then $xf^{m+1} < xf^m = u$, another contradiction. Hence it must be the case that such a $u$ does not exist. A dual argument to dismiss the existence of an element $w \in U_x$ such that $v \leq w$ for all $v \in V$ completes the proof. $\square$

**Lemma 9.6.** *Let $x, y \in V_\Omega$. If $y \in U_x$, then $U_x = U_y$. Thus for any two elements $y, z \in U_x$, there exists $m, n \in \mathbb{Z}$ such that $yf^m \leq z \leq yf^n$.*

*Proof.* If $y \in U_x$ then there exists $m, n \in \mathbb{Z}$ such that $xf^m \leq y \leq xf^n$. Now suppose that $u \in U_y$. Then there exists $i, j \in \mathbb{Z}$ such that $yf^i \leq u \leq yf^j$. Hence,

$$xf^{m+i} \leq yf^i \leq u \leq yf^j \leq xf^{n+j}$$

and $u \in U_x$. Now suppose that $v \in U_x$. Then there exists $k, l \in \mathbb{Z}$ such that $xf^k \leq v \leq xf^l$. Hence,

$$yf^{k-n} \leq xf^k \leq v \leq xf^l \leq yf^{l-m},$$

and $v \in U_y$. Thus we have shown that $U_x \subseteq U_y$ and $U_y \subseteq U_x$ and hence $U_x = U_y$ as required. $\square$

**Lemma 9.7.** *Let $x \in V_\Omega$. Then $U_x = U_{xf} = U_x f$.*

*Proof.* The first equality is immediate from Lemma 9.6 since clearly $xf \in U_x$. Now suppose that $u \in U_x$, then there exists $m, n \in \mathbb{Z}$ such that $xf^m \leq u \leq xf^n$. Hence $xf^{m-1} \leq uf^{-1} \leq xf^{n-1}$ so that $uf^{-1} \in U_x$. In other words $u \in U_x f$. On the other hand suppose that $v \in U_x f$. Then $v = uf$ for some $u \in U_x$. Thus there exists $i, j \in \mathbb{Z}$ such that $xf^i \leq u \leq xf^j$. It now follows that $xf^{i+1} \leq v \leq xf^{j+1}$ and hence $v \in U_x$. Thus $U_x = U_x f$ as required. $\square$

**Corollary 9.8.** *Let $x \in V_\Omega$ Then $f|_{U_x} \in \mathrm{Aut}(\langle U_x \rangle)$.*

*Proof.* By Lemma 9.7, $U_x f = U_x$. Now since $f$ is an automorphism of $\Omega$ it follows that $f|_{U_x}$ is an automorphism of $\langle U_x \rangle$ and the result is complete. $\square$

Define an equivalence relation $\sim$ on $V_\Omega$ by $x \sim y$ if and only if $U_x = U_y$. Let $I \subseteq V_\Omega$ be a transversal of the set of equivalence classes of $V_\Omega$ under $\sim$.

**Lemma 9.9.** *$\{U_x : x \in I\}$ is a partition of $V_\Omega$.*

*Proof.* Let $x \in V_\Omega$. Then $x \in U_x$ and hence $x \in U_y$ for some $y \in I$ with $U_x = U_y$. Hence $\bigcup_{x \in I} U_x = V_\Omega$. Now let $x, y \in I$ and suppose that $x \neq y$. Suppose that $U_x \cap U_y \neq \emptyset$. Then there exists $u \in V_\Omega$ such that $u \in U_x$ and $u \in U_y$. But by Lemma 9.6 it follows that $U_u = U_y$ and $U_u = U_x$. Hence $U_x = U_y$. But this is a contradiction to $I$ being a transversal. Thus we can conclude that $U_x \cap U_y = \emptyset$ for all $x, y \in I$ and $\{U_x : x \in I\}$ is a partition of $V_\Omega$. $\square$

As a point of interest, Lemmas 9.4 and 9.9 allow the formation of a natural order on the set of orbitals $\{U_x : x \in I\}$. For if $x \neq y$, then by Lemma 9.9, $U_x \cap U_y = \emptyset$. Then, since $U_x$ and $U_y$ are intervals by Lemma 9.4, for $x \neq y$ we can define $U_x \prec U_y$ if and only if $U_x < U_y$. If we set $U_x \preceq U_x$ for all $x \in I$, then it is not hard to see that $(\{U_x : x \in I\}, \preceq)$ is a total order.

## 9.2 Orbital Constraints when Aut(Ω) is Countable

For the rest of the chapter, we now assume that $\mathrm{Aut}(\Omega)$ is countable (although we often restate this fact for clarity). The next few results will show that if $\Omega = (V_\Omega, \leq)$ is a total order, then the assumption that $\mathrm{Aut}(\Omega)$ is countable places strong conditions on the orbitals of an automorphism $f \in \mathrm{Aut}(\Omega)$. First we will require the following lemma.

**Lemma 9.10.** *Let $\Omega$ be a total order and let $\{S_i : i \in \mathbb{N}\}$ be a set of non-empty disjoint intervals in $\Omega$. Let $g_i \in \mathrm{Aut}(S_i)$ for all $i \in \mathbb{N}$. Define a map $f : V_\Omega \to V_\Omega$ by*

$$xf = \begin{cases} xg_i & \text{if } v \in S_i, \\ x & \text{otherwise.} \end{cases}$$

*Then $f \in \mathrm{Aut}(\Omega)$.*

*Proof.* First we note that $f$ is a well defined function since by assumption the intervals $S_i$ are mutually disjoint. Since $g_i \in \mathrm{Aut}(S_i)$, we know that $g_i$ is a bijection on $S_i$ and that $S_i g_i = S_i$ for all $i \in \mathbb{N}$. Thus by construction of $f$ it clearly follows that $f$ is a bijective function $V_\Omega \to V_\Omega$.

Now suppose that $x, y \in V_\Omega$ and that $x \leq y$. If $x, y \in S_i$, then since $g_i$ is an automorphism of $S_i$ it follows that $xf = xg_i \leq yg_i = yf$. Similarly, if $x \in S_i$ and $y \in S_j$ for some $i, j \in \mathbb{N}$ with $i \neq j$, then since $S_i$ and $S_j$ are disjoint intervals, $S_i \leq S_j$. Since $g_i$ and $g_j$ are automorphisms of $S_i$ and $S_j$ respectively, it then follows that $xf = xg_i \leq yg_j = yf$. If $x, y \in V_\Omega \setminus \bigcup_{i\in I} S_i$ then $xf = x \leq y = yf$. So suppose that $x \in S_j$ for some $j \in I$ and that $y \in V_\Omega \setminus \bigcup_{i\in I} S_i$. Seeking a contradiction, suppose that $yf \leq xf$. Then $y \leq xg_j$ and hence $x \leq y \leq xg_j$. But this means that $y \in S_j$, a contradiction. Hence we can conclude that $xf \leq yf$. A similar argument when $x \in V_\Omega \setminus \bigcup_{i\in I} S_i$ and $y \in S_j$ completes the proof that $x \leq y$ implies that $xf \leq yf$.

Conversely, suppose that $x, y \in V_\Omega$ and that $xf \leq yf$. If $xf \in S_i$ and $yf \in S_j$ for some $i, j \in \mathbb{N}$, then $S_i \leq S_j$. Since $g_i$ and $g_j$ are automorphisms of $S_i$ and $S_j$ respectively, we can then deduce that $x \leq y$. If $xf, yf \in V_\Omega \setminus \bigcup_{i\in I} S_i$ then it follows immediately that $x = xf \leq yf = y$. So suppose that $xf \in S_j$ for some $j \in I$ and that $yf \in V_\Omega \setminus \bigcup_{i\in I} S_i$. Then $yf = y$ and $xf = xg_j$. Furthermore since $g_j$ is an automorphism of $S_j$ it follows that $x \in S_j$. If $y \leq x$, then $xg_j = xf < yf < x$. But then $y \in S_j$, a contradiction, and hence we conclude that $x \leq y$. A similar argument when $xf \in V_\Omega \setminus \bigcup_{i\in I} S_i$ and $yf \in S_j$ completes the proof that $f$ is an automorphism of $\Omega$. $\square$

Let $\Omega = (V_\Omega, \leq)$ be a total order. For $u, v \in V_\Omega$ with $u < v$, we define $(u, v) = \{w \in V_\Omega : u < w < v\}$ and $(u, v] = \{w \in V_\Omega : u < w \leq v\}$. Then $(u, v)$ and $(u, v]$ are intervals in $\Omega$.

**Theorem 9.11.** *Let $\Omega = (V_\Omega, \leq)$ be a total order and suppose that $\mathrm{Aut}(\Omega)$ is countable. Let $f \in \mathrm{Aut}(\Omega)$ and suppose that $U_x$ is an infinite orbital of $f$. Let $u, v \in U_x$ with $u < v$. If $(u, v) \neq \emptyset$, then $\mathrm{Aut}(\langle (u, v)\rangle) = \mathbf{1}$.*

*Proof.* First assume without loss of generality, that $U_x$ is a positive orbital of $f$ (otherwise replace $f$ by $f^{-1}$). Note that since $u \in U_x = U_u$, there exists $m \in \mathbb{Z}$ such that $v < uf^m$. Then $(uf^{km}, vf^{km}) < (uf^{(k+1)m}, vf^{(k+1)m})$ for all $k \in \mathbb{N}$. Furthermore since $f \in \mathrm{Aut}(\Omega)$ it follows that $\langle (u, v)\rangle \cong \langle (uf^{km}, vf^{km})\rangle$ and hence $\mathrm{Aut}(\langle (u, v)\rangle) \cong \mathrm{Aut}(\langle (uf^{km}, vf^{km})\rangle)$ for all $k \in \mathbb{N}$. Define a map

$$\phi : \prod_{k\in\mathbb{N}} \mathrm{Aut}(\langle (uf^{km}, vf^{km})\rangle) \to \mathrm{Aut}(\Omega),$$

by defining, for $g \in \prod_{k\in\mathbb{N}} \mathrm{Aut}(\langle (uf^{km}, vf^{km})\rangle)$ and for $w \in V_\Omega$,

$$(w)g\phi = \begin{cases} (w)((k)g) & \text{if } w \in (uf^{km}, vf^{km}), \\ w & \text{otherwise .} \end{cases}$$

By Lemma 9.10, $g\phi$ is a well defined map. It should also be easy to see that $\phi$ is an injective group homomorphism so that $\phi$ defines an embedding of groups. Now if $\mathrm{Aut}(\langle (u,v)\rangle) \neq \mathbf{1}$, then $|\mathrm{Aut}(\langle (uf^{km}, vf^{km})\rangle)| \geq 2$ for all $k \in \mathbb{N}$. Hence,

$$\left| \prod_{k\in\mathbb{N}} \mathrm{Aut}(\langle (uf^{km}, vf^{km})\rangle) \right| \geq 2^{\aleph_0}.$$

But since $\mathrm{Aut}(\Omega)$ is countable this is clearly impossible. Hence it follows that $\mathrm{Aut}(\langle (u,v)\rangle) = \mathbf{1}$ and the result is complete. $\square$

Theorem 9.11 shows us that the assumption that $\mathrm{Aut}(\Omega)$ is countable places a strong condition on the internal structure of an orbital $U$ of $f \in \mathrm{Aut}(\Omega)$. Using the next few lemmas, we will now show that the same assumption also places a restriction on the number of infinite orbitals an automorphism $f \in \mathrm{Aut}(\Omega)$ can have.

**Lemma 9.12.** *Let $\Omega$ be a total order and let $f \in \mathrm{Aut}(\Omega)$. Suppose that $\{U_x : x \in I\}$ is a partition of $V_\Omega$ into orbitals of $f$. For each $x \in I$ define a map $f_x : V_\Omega \to V_\Omega$ by*

$$vf_x = \begin{cases} vf & \textit{if } v \in U_x, \\ v & \textit{otherwise.} \end{cases}$$

*Then for all $x \in I$, $f_x \in \mathrm{Aut}(\Omega)$. Clearly if $U_x = \{x\}$, then $f_x = \mathbf{1}_{V_\Omega}$.*

*Proof.* Note first that by Corollary 9.8, $f|_{U_x} \in \mathrm{Aut}(\langle U_x\rangle)$. Then by applying Lemma 9.10 it follows immediately that $f_x \in \mathrm{Aut}(\Omega)$. $\square$

**Lemma 9.13.** *Let $\Omega$ be a total order and let $f \in \mathrm{Aut}(\Omega)$. Suppose that $\{U_x : x \in I\}$ is a partition of $V_\Omega$ into orbitals of $f$. Let $x, y \in I$. Then $f_x f_y = f_y f_x$.*

*Proof.* If $x = y$ there is nothing to do. So suppose that $x \neq y$. Then since $\{U_x : x \in I\}$ is a partition of $V_\Omega$ into orbitals of $f$ it should be easy to see that

$$vf_xf_y = vf_yf_x = \begin{cases} vf_x & \text{if } v \in U_x, \\ vf_y & \text{if } v \in U_y, \\ v & \text{otherwise.} \end{cases}$$

Thus for all $x, y \in I$, $f_xf_y = f_yf_x$ as required. $\square$

**Lemma 9.14.** *Let $\Omega$ be a total order and let $f \in \mathrm{Aut}(\Omega)$. Suppose that $\{U_x : x \in I\}$ is a partition of $V_\Omega$ into the orbitals of $f$. Let $J = \{x \in I : U_x \text{ is infinite}\}$. Then there exists an isomorphism of groups*

$$\langle f_x : U_x \text{ is infinite}\rangle \cong \mathbb{Z}^J.$$

*Proof.* Notice first that by Lemma 9.13, $f_x f_y = f_y f_x$ for all $x, y \in J$. Thus, since $f_x$ has infinite order for all $x \in J$, we can write every element of $\langle f_x : U_x \text{ is infinite}\rangle$ as a unique product $\prod_{x\in J} f_x^{i_x}$ for some $i_x \in \mathbb{Z}$. Now define a map $\phi : \langle f_x : U_x \text{ is infinite}\rangle \to \mathbb{Z}^J$ by setting,

$$(y)\left(\prod_{x\in J} f_x^{i_x}\right)\phi = i_y,$$

for all $y \in J$. Then $\phi$ is a group homomorphism since if $g, h \in G$ then,

$$g = \prod_{x\in J} f_x^{i_x} \text{ and } h = \prod_{x\in J} f_x^{j_x},$$

for some $i_x, j_x \in \mathbb{Z}$ and,

$$\begin{aligned}(y)(gh)\phi &= (y)\left(\prod_{x\in J} f_x^{i_x+j_x}\right)\phi \\ &= i_y + j_y \\ &= (y)\left(\prod_{x\in J} f_x^{i_x}\right)\phi + (y)\left(\prod_{x\in J} f_x^{j_x}\right)\phi.\end{aligned}$$

Furthermore $\phi$ is injective since $g\phi = h\phi$ implies $i_x = j_x$ for all $x \in J$ and hence $g = h$. Also $\phi$ is surjective since if $p \in \mathbb{Z}^J$ and $(x)p = i_x$ for $x \in J$, $i_x \in \mathbb{Z}$ say, then clearly

$$\left(\prod_{x\in J} f_x^{i_x}\right)\phi = p.$$

Thus $\phi$ is a bijective group homomorphism and hence an isomorphism. □

**Theorem 9.15.** *Let $\Omega$ be a total order and let $f \in \mathrm{Aut}(\Omega)$. Suppose that $\{U_x : x \in I\}$ is a partition of $V_\Omega$ into orbitals of $f$. If $\mathrm{Aut}(\Omega)$ is countable then the set $J = \{x \in I : U_x \text{ is infinite}\}$ is finite.*

*Proof.* By Lemma 9.14, if there exist infinitely many infinite orbitals then there exists an isomorphism $\phi : \langle f_x : U_x \text{ is infinite}\rangle \to \mathbb{Z}^J$. Since the group $\langle f_x : U_x \text{ is infinite}\rangle$ is contained as a subgroup of $\mathrm{Aut}(\Omega)$ and since $\mathbb{Z}^J$ is uncountable when $J$ is infinite, it follows that the set $J = \{x \in I : U_x \text{ is infinite}\}$ is finite. □

From Theorem 9.15 we can thus conclude that if $\mathrm{Aut}(\Omega)$ is countable then every automorphism of $\Omega$ has finitely many distinct infinite orbitals.

## 9.3 Orbitals of Distinct Automorphisms

So far we have shown that if $\Omega$ is a total order and $\mathrm{Aut}(\Omega)$ is countable, then we are able to deduce strong results on the orbitals of a single automorphism $f \in \mathrm{Aut}(\Omega)$. In this subsection we will show that the assumption that $\mathrm{Aut}(\Omega)$ is countable also allows us to deduce information on the relationship between the orbitals of distinct automorphisms from $\mathrm{Aut}(\Omega)$. The main result is stated below and proof of this theorem will take up the remainder of this subsection.

**Theorem 9.16.** *Let $\Omega$ be a total order. Suppose that $\mathrm{Aut}(\Omega)$ is countable and let $f, g \in \mathrm{Aut}(\Omega)$. Then for all infinite orbitals $U$ of $f$ and for all infinite orbitals $T$ of $g$, either $U \cap T = \emptyset$ or $U = T$.*

### The Proof of Theorem 9.16

Suppose that $\mathrm{Aut}(\Omega)$ is countable and let $f, g \in \mathrm{Aut}(\Omega)$. Let $U$ be an infinite orbital of $f$ and let $T$ be an infinite orbital of $g$. If $U \cap T = \emptyset$ then there is nothing to do. So suppose that $U \cap T \neq \emptyset$ but that $U \neq T$. Then without loss of generality one of the following cases must hold (otherwise swap the labels on $U$ and $T$).

(A) $U \subset T$ and there exists $s, t \in T$ such that $s < U < t$.

(B) $U \subset T$ and there exists $s \in T$ such that $s < U$ but no $t \in T$ such that $U < t$.

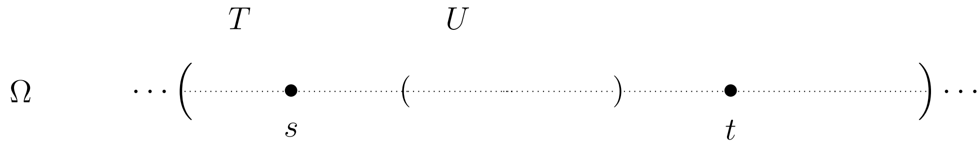
(C) $U \subset T$ and there exists $t \in T$ such that $U < t$ but no $s \in T$ such that $s < U$.

(D) $U \not\subset T$ and there exists $u \in U$, $t \in T$ such that $u < U \cap T < t$.

See Figures 9.1 through 9.4 for a pictorial representation of each case. We will show case by case that each of these scenarios leads us to a contradiction and hence to the conclusion that $U = T$.

We will further split the proof into two cases. First, case I, where we assume that $f$ and $g$ commute, and second, case II when $f$ and $g$ do not commute. The following lemma will be of importance in both cases.

Figure 9.1: The Orbital Intersection in case (A)




Large brackets denote the interval $T$, small brackets the interval $U$.

Figure 9.2: The Orbital Intersection in case (B)


**Lemma 9.17.** *Let $\Omega$ be a total order. Let $f, g \in \mathrm{Aut}(\Omega)$ and let $U$ be an infinite orbital of $f$. Suppose that $i \in \mathbb{Z}$. Then $(U)g^{-i}$ is an infinite orbital of $g^i f g^{-i}$.*

*Proof.* First observe that since $g$ is an automorphism of $\Omega$ and since $U$ is infinite, $(U)g$ is an infinite set. Now let $x \in U$. Then,

$$U = \{y \in V_\Omega : xf^m \leq y \leq xf^n \text{ for some } m, n \in \mathbb{Z}\}.$$

Hence,

$$\begin{aligned}
(U)g^{-i} &= \{yg^{-i} : y \in V_\Omega,\, xf^m \leq y \leq xf^n \text{ for some } m, n \in \mathbb{Z}\} \\
&= \{z \in V_\Omega : xf^m \leq zg^i \leq xf^n \text{ for some } m, n \in \mathbb{Z}\} \\
&= \{z \in V_\Omega : xf^m g^{-i} \leq z \leq xf^n g^{-i} \text{ for some } m, n \in \mathbb{Z}\} \\
&= \{z \in V_\Omega : (x)g^{-i}g^i f^m g^{-i} \leq z \leq (x)g^{-i}g^i f^n g^{-i} \text{ for some } m, n \in \mathbb{Z}\} \\
&= \{z \in V_\Omega : (xg^{-i})(g^i f g^{-i})^m \leq z \leq (xg^{-i})(g^i f g^{-i})^n \text{ for some } m, n \in \mathbb{Z}\}
\end{aligned}$$

Thus $(U)g^{-i}$ is the infinite orbital of $xg^{-i}$ under $g^i f g^{-i}$. □

## I. The automorphisms $f$ and $g$ commute.

First we prove the following key lemma.

**Lemma 9.18.** *Let $\Omega$ be a total order. Let $f, g \in \mathrm{Aut}(\Omega)$ and let $U$ be an infinite orbital of $f$. If $f$ and $g$ commute then $(U)g = U$.*

Figure 9.3: The Orbital Intersection in case (C)

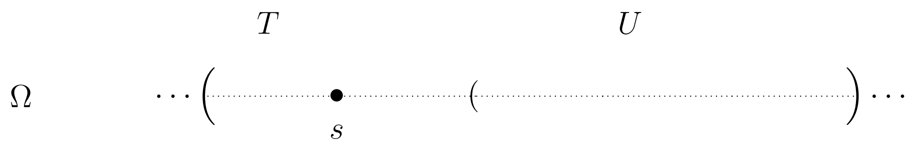



Figure 9.4: The Orbital Intersection in case (D)


*Proof.* First note that since $U$ is an infinite orbital of $f$, $f \neq \mathbf{1}$. If $g = \mathbf{1}$, then clearly $(U)g = U$ and there is nothing to do. So suppose that $g \neq \mathbf{1}$. By Lemma 9.15, $f$ and $g$ have only finitely many infinite orbitals. So let $\{U_1, \ldots, U_n\}$ be the infinite orbitals of $f$ and let $\{T_1, \ldots T_m\}$ be the infinite orbitals of $g$ for some $m, n \in \mathbb{N}$. Now for $k = 1, \ldots n$, let $x_k \in U_k$. By Lemma 9.17, $(U_k)g$ is the infinite orbital of $x_k g$ under $g^{-1}fg$. But since $f$ and $g$ commute this says that $(U_k)g$ is the infinite orbital of $x_k g$ under $f$. Since $f$ has only the infinite orbitals $\{U_1, \ldots, U_n\}$ and since $g$ is an automorphism, we can conclude that $g$ must permute the orbitals of $f$. But since there are only finitely many orbitals of $f$, it must be the case that $(U_k)g = U_k$ for all $k = 1, \ldots, n$. For suppose without loss of generality that $U_i < U_j$ for all $i < j \leq n$. Let $m = \max\{k \in \{1, \ldots n\} : (U_k)g \neq U_k\}$. If $m$ exists, then $(U_m)g = U_k$ for some $k < m$ and there exists some $j < m$ such that $(U_j)g = U_m$. But this contradicts $g$ being order preserving since $U_j < U_m$ but $(U_m)g = U_k < U_m = (U_j)g$. Hence $m$ cannot exist and we deduce that $(U_k)g = U_k$ for all $k = 1, \ldots n$. The result now follows. $\square$

**Case (A)**

In this case we have assumed that $U \subset T$ and there exists $s, t \in T$ such that $s < U < t$. Then $(s,t) \subseteq T$ and hence, by Lemma 9.11, $\mathrm{Aut}(\langle (s,t) \rangle) = \mathbf{1}$. But by Lemma 9.10, if $h \in \mathrm{Aut}(\langle U \rangle)$, then the map $\tilde{h} : (s,t) \to (s,t)$ defined by,

$$v\tilde{h} = \begin{cases} vh & \text{if } v \in U, \\ v & \text{otherwise,} \end{cases}$$

is an automorphism of $\langle (s,t) \rangle$. Thus we can conclude that $\mathrm{Aut}(\langle U \rangle) = \mathbf{1}$. But this is clearly a contradiction since by Corollary 9.8, $f|_U \in \mathrm{Aut}(\langle U \rangle)$ and $f|_U \neq \mathbf{1}$ since $U$ was an infinite orbital of $f$. Thus we have quickly ruled out this scenario for the orbitals $U$ and $T$.

**Case (B)**

In case $(B)$ we suppose that, $U \subset T$ and there exists $t \in T$ such that $U < t$. Let $x \in U$. Then since $x \in T$ and since $T$ is an infinite orbital of $g$, there exists $m \in \mathbb{Z}$ such that $t < xg^m$. But By Lemma 9.18, $xg^m \in U$ and hence $xg^m < t$. A contradiction. Hence this case also ruled out.

**Case (C)**

In case $(C)$ we suppose that, $U \subset T$ and there exists $s \in T$ such that $s < U$. Let $x \in U$. Then since $x \in T$ and since $T$ is an infinite orbital of $g$, there exists $n \in \mathbb{Z}$ such that $xg^n < s$. But by Lemma 9.18, $xg^n \in U$ and hence $s < xg^n$. A contradiction. Hence this case cannot occur.

**Case(D)**

In this case we have assumed that $U \not\subset T$ and $T \not\subset U$ and there exists $u \in U$, $t \in T$ such that $u < U \cap T < t$. Let $x \in U \cap T$. Then since $x \in T$ and since $T$ is an infinite orbital of $g$, there exists $m \in \mathbb{Z}$ such that $t < xg^m$. But by Lemma 9.18, $xg^m \in U$ and hence $xg^m < t$. A contradiction. Hence this case cannot occur either.

We have thus shown that for all four cases, the assumption that $f$ and $g$ commute leads us to a contradiction. Thus we can now conclude that if $f$ and $g$ commute and $U \cap T \neq \emptyset$ then $U = T$.

## II. The automorphisms $f$ and $g$ do not commute.

Suppose on the other hand that $f$ and $g$ do not commute. Then, $f, g \in \mathrm{Aut}(\Omega) \setminus \mathbf{1}$, $f \neq g$ and $f^{-1} \neq g$. For the proof that follows, we can assume without loss of generality that $U$ and $T$ are positive orbitals, since otherwise we simply consider $f^{-1}$ or $g^{-1}$. Similarly, we can assume that $U$ and $T$ are the only infinite orbitals of $f$ and $g$, since otherwise we can consider the maps $f_x$ and $g_y$ from Lemma 9.12, for some $x \in U$ and some $y \in T$.

**Case (A)**

A contradiction is obtained exactly as in case I.(A).

**Case (B)**

In case $(B)$ we suppose that, $U \subset T$ and there exists $t \in T$ such that $U < t$. In this case we will consider the automorphisms $g^i f g^{-i}$ for $i \in \mathbb{N}$.

**Lemma 9.19.** *For $i \in \mathbb{Z}$, let $\theta_i = g^i f g^{-i}$. Then $\theta_i$ has one infinite orbital which is equal to $(U)g^{-i}$.*

*Proof.* By Lemma 9.17, $(U)g^{-i}$ is an infinite orbital of $\theta_i$. Now suppose that $u \in V_\Omega$ and $u \notin (U)g^{-i}$. Then $ug^i \notin U$ and since $U$ was the only infinite orbital of $f$ by assumption, it follows that $ug^i f = ug^i$. Then $ug^i f g^{-i} = ug^i g^{-i} = u$ for all $u \notin (U)g^{-i}$. Thus $(U)g^{-i}$ is the only infinite orbital of $\theta_i$. □

Using the automorphisms $\theta_i$, we will show that we can produce $2^{\aleph_0}$ automorphisms of $\Omega$, and hence provide the contradiction we require. Let $\Sigma \subseteq \mathbb{N}$ be an infinite set where $\Sigma = \{\sigma_n : n \in \mathbb{N}\}$ and $\sigma_n < \sigma_{n+1}$ for all $n \in \mathbb{N}$. Define the function

$$h_\Sigma = \lim_{n\to\infty} \theta_{\sigma_n} \ldots \theta_{\sigma_0}.$$

First we observe that this function is well defined. For let $u \in V_\Omega$. If $u \notin T$, then $ug = u$ so that in particular $ug \notin U$. If $u \in T$ then, since $T$ is a positive orbital and by Lemma 9.6, there exists some $k \in \mathbb{N}$ such that $t < ug^k$. But since $U < t$ it follows that $ug^k \notin U$. So for $u \in V_\Omega$, let

$$m(u) = \min\{n \in \mathbb{N} : ug^{\sigma_n} \notin U\}.$$

Then by our previous observations $m(u)$ exists for all $u \in V_\Omega$. Moreover, for all $i \in \mathbb{N}$ such that $i \geq m(u)$, $u \notin Ug^{-i}$, and hence by Lemma 9.19, $u\theta_{\sigma_i} = u$. Thus for each $u \in V_\Omega$,

$$(u)h_\Sigma = (u) \lim_{n\to\infty} \theta_{\sigma_n} \cdots \theta_{\sigma_0} = (u)\theta_{\sigma_{m(u)}} \cdots \theta_{\sigma_0}.$$

Hence the function $h_\Sigma$ is well defined at every point $u \in V_\Omega$.

Furthermore since each $\theta_{\sigma_n}$ is an automorphism, it follows that $h_\Sigma$ is an automorphism of $\Omega$. For if $u, v \in V_\Omega$, let $l = \max\{m(u), m(v)\}$. Then if $u \leq v$ it follows that,

$$(u)h_\Sigma = (u)\theta_{\sigma_l} \cdots \theta_{i_0} \leq (v)\theta_{\sigma_l} \cdots \theta_{\sigma_0} = (v)h_\Sigma.$$

Conversely if $(u)h_\Sigma \leq (v)h_\Sigma$ then,

$$(u)\theta_{\sigma_l}\cdots\theta_{\sigma_0} = (u)h_\Sigma \leq (v)h_\Sigma = (u)\theta_{\sigma_l}\cdots\theta_{\sigma_0},$$

and we can deduce that $u \leq v$.

Now let $\Psi \subset \mathbb{N}$ where $\Psi = \{\psi_n : n \in \mathbb{N}\}$ and $\psi_n < \psi_{n+1}$ for all $n \in \mathbb{N}$. We will show that if $\Sigma \neq \Psi$ then $h_\Sigma \neq h_\Psi$. If $\Sigma \neq \Psi$, then $r = \min\{n \in \mathbb{N} : n \in (\Sigma \cup \Psi) \setminus (\Sigma \cap \Psi)\}$ exists. Without loss of generality assume that $r \in \Sigma$ so that $r = \sigma_p$ for some $p \in \mathbb{N}$. Pick $u \in V_\Omega$ such that $ug^{\sigma_p} \in U$ and $ug^{\sigma_p+1} \notin U$. To see that $u$ exists, let $v \in Ug^{-\sigma_p}$. If $vg^{\sigma_p+1} \notin U$, the we are done. Otherwise $vg^{\sigma_p+1} \in U$ and so $\sigma_p+1 < \sigma_{m(u)}$. Now let $w = vg^{\sigma_{m(u)}-\sigma_p-1}$, then $wg^{\sigma_p} = wg^{\sigma_{m(u)}-1} \in U$ and $wg^{\sigma_p+1} = wg^{\sigma_{m(u)}} \notin U$, as required. Now if we apply the maps $h_\Sigma$ and $h_\Psi$ we find that,

$$\begin{aligned} wh_\Sigma &= w \lim_{n\to\infty} \theta_{\sigma_n}\cdots\theta_{\sigma_0} = w\theta_{\sigma_p}\theta_{\sigma_{p-1}}\cdots\theta_{\sigma_0} \text{ and,} \\ wh_\Psi &= w \lim_{n\to\infty} \theta_{\psi_n}\cdots\theta_{\psi_0} = w\theta_{\sigma_{p-1}}\cdots\theta_{\sigma_0}, \end{aligned}$$

since $\psi_k = \sigma_k$ for all $k < p$ and since $\psi_k \geq \sigma_p$ for all $k \geq p$. Suppose that $wh_\Sigma = wh_\Psi$. Then since the $\theta_{\sigma_n}$ are automorphisms it follows that $w\theta_{\sigma_p} = w$. In other words $wg^{\sigma_p}fg^{-\sigma_p} = w$ and hence $wg^{\sigma_p}f = wg^{\sigma_p}$. But this brings us to the conclusion that $wg^{\sigma_p} \notin U$, which is clearly a contradiction. Hence $wh_\Sigma \neq wh_\Psi$ and so $h_\Sigma \neq h_\Psi$.

Thus since the set $\{h_\Sigma : \Sigma \subseteq \mathbb{N}, \Sigma \text{ infinite}\}$ has size $2^{\aleph_0}$, we have constructed $2^{\aleph_0}$ distinct automorphisms of $\Omega$. Since $\mathrm{Aut}(\Omega)$ was countable by assumption, we can conclude that this configuration for the orbitals of $f$ and $g$ is impossible.

### Case (C)

In case $(C)$ we suppose that, $U \subset T$ and there exists $s \in T$ such that $s < U$. This setting is effectively dual to case $(B)$. As a result we can rule out case $(C)$ by making analogous arguments for the automorphisms $\phi_i = g^{-i}f^{-1}g^i$ which have orbital $(U)g^i$.

### Case (D)

In this case we have assumed that $U \not\subset T$ and $T \not\subset U$ and there exists $u \in U$, $t \in T$ such that $u < U \cap T < t$. Consider $[g, f] = g^{-1}f^{-1}gf$. We claim that $[g, f]$ has an infinite orbital $S$, and that it lies in $T$.

**Lemma 9.20.** $[g, f]$ *has an infinite orbital* $S$ *which is a subset of* $T$ *and there exists* $r \in T$ *such that* $S < r$.

*Proof.* First we note that since $[f, g] \neq \mathbf{1}$, it follows that there exists $v \in V_\Omega$ such that $v[f, g] \neq v$. Thus by Lemma 9.2, the orbital $S_v$ of $[f, g]$ is infinite. Now suppose that $v \in V_\Omega$ and that $v < U \cap T$. Then $v \notin T$ and so $vg^{-1}f^{-1}gf = vf^{-1}gf$. But since $vf^{-1} < v < U$ it now follows that $(vf^{-1})gf = vf^{-1}f$ and hence $vf^{-1}gf = v$. Thus $v[g, f] = v$ for all $v \in V_\Omega$ such that $v < U \cap T$. Similarly if $w \in V_\Omega$ and $tg \leq w$, then $U < t \leq wg^{-1}$ and hence $wg^{-1}f^{-1}gf = wg^{-1}gf = wf = w$. Thus since $tg \in T$, any infinite orbital $S$ of $[f, g]$ must lie in $T$ and $S < tg$. □

Thus the orbitals $S$ of $[f, g]$ and $T$ of $g$, fall into case $(B)$ previously discussed. We proved that this would lead us to contradict $\mathrm{Aut}(\Omega)$ being countable and so we can now discard case $(D)$ as a possible configuration.

We have thus shown that for all four cases, the assumption that $f$ and $g$ do not commute, leads us to a contradiction. Thus we can now conclude that if $f$ and $g$ do not commute and $U \cap T \neq \emptyset$ then $U = T$ and the proof of Theorem 9.16 is complete. In due course we shall in fact observe that if $\mathrm{Aut}(\Omega)$ is countable then $f$ and $g$ always commute. However the proof of Theorem 9.16 in both cases is a necessary step towards this result.

## 9.4 The Automorphism Group of an Orbital

The work from the previous sections will aid us in determining the automorphism group of an orbital $U$ when $\mathrm{Aut}(\Omega)$ is a countable group. We will show that $\mathrm{Aut}(\langle U \rangle)$, when equipped with a particular total order, is an Archimedean group. This together with the results on Archimedean groups introduced in Chapter 2, will help us to show that $\mathrm{Aut}(\langle U \rangle) \cong \mathbb{Z}$. First we have the following useful lemmas.

**Lemma 9.21.** *Let* $\Omega$ *be a total order. Suppose that* $f \in \mathrm{Aut}(\Omega)$ *and that* $U$ *is an infinite orbital of* $f$. *If* $\mathrm{Aut}(\Omega)$ *is countable, then* $\mathrm{Aut}(\langle U \rangle)$ *is countable.*

*Proof.* Suppose that $\phi \in \mathrm{Aut}(\langle U \rangle)$. Then by Lemma 9.10, the map $\hat{\phi} : V_\Omega \to V_\Omega$ defined by

$$v\hat{\phi} = \begin{cases} v\phi & \text{if } v \in U, \\ v & \text{otherwise,} \end{cases}$$

is an automorphism of $\Omega$. Furthermore if $\phi, \theta \in \mathrm{Aut}(\langle U \rangle)$ and $\phi \neq \theta$, then $\hat{\phi}|_U \neq \hat{\theta}|_U$ and so $\hat{\phi} \neq \hat{\theta}$. Thus if $\mathrm{Aut}(\Omega)$ is countable, it follows that $\mathrm{Aut}(\langle U \rangle)$ must also be countable. □

**Lemma 9.22.** *Suppose that $f \in \mathrm{Aut}(\Omega)$ and that $U$ is an infinite orbital of $f$. Let $\phi \in \mathrm{Aut}(\langle U\rangle) \setminus \mathbf{1}$. If $\mathrm{Aut}(\Omega)$ is countable then the map $\hat{\phi} : V_\Omega \to V_\Omega$ defined by*

$$v\hat{\phi} = \begin{cases} v\phi & \textit{if } v \in U \\ v & \textit{otherwise,} \end{cases}$$

*is an automorphism of $\Omega$ and has infinite orbital $U$. Thus either $y < y\phi$ or $y\phi < y$ for all $y \in U$.*

*Proof.* It follows immediately by Lemma 9.10 that $\hat{\phi} \in \mathrm{Aut}(\Omega)$. Furthermore, since $\phi \neq \mathbf{1}_U$ it follows that $\hat{\phi} \neq \mathbf{1}_{V_\Omega}$ and so by Corollary 9.3, $\hat{\phi}$ has an infinite orbital. But since $v\hat{\phi} = v$ for all $v \in V_\Omega \setminus U$, any infinite orbital $\hat{\phi}$ must be contained in $U$. Thus $\hat{\phi}$ has an infinite orbital, $T$ say, contained in $U$. Now, by Theorem 9.16 we can conclude that $T = U$. Thus $U$ is an infinite orbital of $\hat{\phi}$ and so by Lemma 9.2, either $y < y\hat{\phi}$ or $y\hat{\phi} < y$ for all $y \in U$. In other words $y < y\phi$ or $y\phi < y$ for all $y \in U$. □

**Lemma 9.23.** *Suppose that $f \in \mathrm{Aut}(\Omega)$ and that $U$ is an infinite orbital of $f$. Let $y \in U$. If $\mathrm{Aut}(\Omega)$ is countable then,*

$$\mathrm{stab}_{\mathrm{Aut}(\langle U\rangle)}(y) = \{\phi \in \mathrm{Aut}(\langle U\rangle) : y\phi = y\} = \mathbf{1}_U.$$

*Proof.* Seeking a contradiction, suppose that there exists $\phi \in \mathrm{Aut}(\langle U\rangle)$, where $\phi \neq \mathbf{1}_U$ and $y\phi = y$. The map $\hat{\phi} : V_\Omega \to V_\Omega$ defined in Lemma 9.22 by

$$v\hat{\phi} = \begin{cases} v\phi & \text{if } v \in U \\ v & \text{otherwise,} \end{cases}$$

is an automorphism of $\Omega$ with infinite orbital $U$. Thus $y\hat{\phi} \neq y$ for all $y \in U$. But this says that $y\phi \neq y$, a contradiction. Thus it follows that $\phi = \mathbf{1}_U$ and the proof is complete. □

**Corollary 9.24.** *Suppose that $f \in \mathrm{Aut}(\Omega)$ and that $U$ is an infinite orbital of $f$. Let $x \in U$. If $\mathrm{Aut}(\Omega)$ is countable then the map $\xi : \mathrm{Aut}(\langle U\rangle) \to U$ defined by $(\phi)\xi = x\phi$ for all $\phi \in \mathrm{Aut}(\langle U\rangle)$ is an injective map.*

*Proof.* Suppose that $\phi, \theta \in \mathrm{Aut}(\langle U\rangle)$ are such that $(\phi)\xi = (\theta)\xi$. Then $x\phi = x\theta$ and so $x\phi\theta^{-1} = x$. But since by Lemma 9.23, $\mathrm{stab}_{\mathrm{Aut}(\langle U\rangle)}(x) = \mathbf{1}_U$, it follows that $\phi\theta = \mathbf{1}_U$ and hence $\phi = \theta$. Thus $\xi$ is injective as claimed. □

The map $\xi$ from Corollary 9.24, allows us to induce an order on $\mathrm{Aut}(\langle U\rangle)$ in the following way. Fix $x \in U$. This element $x \in U$ will now be fixed until otherwise stated. For $\phi, \theta \in \mathrm{Aut}(\langle U\rangle)$, we define $\phi \leq \theta$ if and only if $x\phi \leq x\theta$.

**Lemma 9.25.** *Let $\Omega$ be a total order. Suppose that $f \in \mathrm{Aut}(\Omega)$ and that $U$ is an orbital of $f$. If $\mathrm{Aut}(\Omega)$ is countable then $(\mathrm{Aut}(\langle U \rangle), \cdot, \leq)$ is an Archimedean group.*

*Proof.* We begin by checking that $(\mathrm{Aut}(\langle U \rangle), \leq)$ is a total order. It should be clear that $\leq$ is reflexive since $\Omega$ is a total order and hence $x\phi \leq x\phi$ for all $\phi \in \mathrm{Aut}(\langle U \rangle)$. It is similarly easy to show that $\leq$ is transitive since for all $\phi, \theta, \psi \in \mathrm{Aut}(\langle U \rangle)$, $x\phi \leq x\theta \leq x\psi$ implies that $x\phi \leq x\psi$. To check symmetry let $\phi, \theta \in \mathrm{Aut}(\langle U \rangle)$ and suppose that $\phi \leq \theta$ and $\theta \leq \phi$. Then $x\phi \leq x\theta$ and $x\theta \leq x\phi$. Since $\Omega$ was a total order it thus follows that $x\phi = x\theta$. Consequently, by Corollary 9.24 we can now deduce that $\phi = \theta$. Finally, totality of $(\mathrm{Aut}(\langle U \rangle), \leq)$ follows from totality of $(\langle U \rangle, \leq)$.

We now check that $\leq$ is translation invariant. So let $\phi, \theta, \psi \in \mathrm{Aut}(\langle U \rangle)$ and suppose that $\phi \leq \theta$. Then $x\phi \leq x\theta$ and since $\psi$ is an automorphism $x\phi\psi \leq x\theta\psi$. Thus it immediately follows that $\phi\psi \leq \theta\psi$. Furthermore, it follows that $x \leq x\theta\phi^{-1}$. Thus by Lemma 9.22, $y \leq y\theta\phi^{-1}$ for all $y \in U$. Hence $x\psi \leq x\psi\theta\phi^{-1}$ since $\psi$ is an automorphism, $x\psi\phi \leq x\psi\theta$. Thus $\psi\phi \leq \psi\theta$ and hence $\leq$ is indeed translation invariant. Thus $\mathrm{Aut}(\langle U \rangle)$ equipped with $\leq$ is a totally ordered group. Finally, suppose that $\theta, \phi \in \mathrm{Aut}(\langle U \rangle)$ are positive elements and that $\phi < \theta$. Let $\hat{\phi} : V_\Omega \to V_\Omega$ be the map defined in Lemma 9.22 by,

$$v\hat{\phi} = \begin{cases} v\phi & \text{if } v \in U, \\ v & \text{otherwise.} \end{cases}$$

Then Lemma 9.22, $\hat{\phi} \in \mathrm{Aut}(\Omega)$ and $U$ is an infinite positive orbital of $\hat{\phi}$. Thus there exists $i, j \in \mathbb{N}$ such that $x\hat{\phi}^i \leq x\theta < x\hat{\phi}^j$. But this says that $x\theta < x\phi^j$, and hence $\theta < \phi^j$. We have now completed all steps to show that $(\mathrm{Aut}(\langle U \rangle), \cdot, \leq)$ is an Archimedean group. $\square$

By applying Lemma 9.25 together with Theorem 2.2 we can deduce that if $\mathrm{Aut}(\Omega)$ is countable and $f \in \mathrm{Aut}(\Omega)$, then for an infinite orbital $U$ of $f$, either $\mathrm{Aut}(\langle U \rangle) \cong \mathbb{Z}$ or $(\mathrm{Aut}(\langle U \rangle), \leq)$ is dense. We will now show that the assumption that $(\mathrm{Aut}(\langle U \rangle), \leq)$ is dense leads to a contradiction, and thus to the conclusion that $\mathrm{Aut}(\langle U \rangle) \cong \mathbb{Z}$.

To do this we will consider the structure of the induced total order $\langle U \rangle$, when $(\mathrm{Aut}(\langle U \rangle), \leq)$ is dense. We first we examine the action of $\mathrm{Aut}(\langle U \rangle)$ on the elements of $U$. For $y \in U$, define

$$\mathrm{orb}_{\mathrm{Aut}(\langle U \rangle)}(y) = \{y\phi : \phi \in \mathrm{Aut}(\langle U \rangle)\}.$$

That is, $\mathrm{orb}_{\mathrm{Aut}(\langle U\rangle)}(y)$ is the orbit of $y$ under the natural action of $\mathrm{Aut}(\langle U\rangle)$. Then, $U = \bigcup_{y\in U} \mathrm{orb}_{\mathrm{Aut}(\langle U\rangle)}(y)$. When $(\mathrm{Aut}(\langle U\rangle), \leq)$ is dense, the following lemmas hold.

**Lemma 9.26.** *Suppose that* $(\mathrm{Aut}(\langle U\rangle), \leq)$ *is dense. Let* $y \in U$. *Then,* $\langle \mathrm{orb}_{\mathrm{Aut}(\langle U\rangle)}(y)\rangle \cong (\mathbb{Q}, \leq)$.

*Proof.* First we recall from Corollary 9.24 that the map $\xi : \mathrm{Aut}(\langle U\rangle) \to U$ defined by $(\phi)\xi = x\phi$ for all $\phi \in \mathrm{Aut}(\langle U\rangle)$ is an injective map. Furthermore, for $\phi, \theta \in \mathrm{Aut}(\langle U\rangle)$, $\phi \leq \theta$ if and only if $x\phi \leq x\theta$. That is $\phi \leq \theta$ if and only if $(\phi)\xi \leq (\theta)\xi$ and hence $\xi$ is an embedding of $(\mathrm{Aut}(\langle U\rangle), \leq)$ into $\langle U\rangle$. Furthermore $\mathrm{im}\,\xi = \mathrm{orb}_{\mathrm{Aut}(\langle U\rangle)}(x)$. Thus $\xi$ defines an isomorphism between $(\mathrm{Aut}(\langle U\rangle), \leq)$ and $\mathrm{orb}_{\mathrm{Aut}(\langle U\rangle)}(x)$. We will show that $(\mathrm{Aut}(\langle U\rangle), \leq)$ $\cong (\mathbb{Q}, \leq)$. It suffices to show that $(\mathrm{Aut}(\langle U\rangle), \leq)$ is countable, dense and without endpoints. By assumption $(\mathrm{Aut}(\langle U\rangle), \leq)$ is countable and dense. Now suppose that $\phi \in \mathrm{Aut}(\langle U\rangle)$. Since $U$ is an infinite orbital of $f$, it follows by Corollary 9.8 that $f|_U \in \mathrm{Aut}(\langle U\rangle)$. Furthermore, since $U$ is an infinite orbital of $f$, there exists $m, n \in \mathbb{Z}$ such that $xf^m \leq x\phi \leq xf^n$. Hence $(f|_U)^m \leq \phi \leq (f|_U)^n$ and so $(\mathrm{Aut}(\langle U\rangle), \leq)$ is without endpoints as required. □

**Lemma 9.27.** *Suppose that* $(\mathrm{Aut}(\langle U\rangle), \leq)$ *is dense. Let* $y, z \in U$. *Then for* $\phi, \theta \in \mathrm{Aut}(\langle U\rangle)$ *such that* $\phi < \theta$, *there exists* $\psi \in \mathrm{Aut}(\langle U\rangle)$ *such that* $y\phi < z\psi < y\theta$.

*Proof.* If $y \in \mathrm{orb}_{\mathrm{Aut}(\langle U\rangle)}(z)$, then since $\mathrm{orb}_{\mathrm{Aut}(\langle U\rangle)}(z) \cong \mathbb{Q}$ by Lemma 9.26 the result follows immediately. So suppose that $y \notin \mathrm{orb}_{\mathrm{Aut}(\langle U\rangle)}(z)$ and so $\mathrm{orb}_{\mathrm{Aut}(\langle U\rangle)}(y) \neq \mathrm{orb}_{\mathrm{Aut}(\langle U\rangle)}(z)$. Note that if there exists $\psi \in \mathrm{Aut}(\langle U\rangle)$ such that $y < z\psi < y\theta\phi^{-1}$ then $y\phi < z\psi\phi < y\theta$. Thus it suffices to show that for all positive $\theta \in \mathrm{Aut}(\langle U\rangle)$, there exists $\psi \in \mathrm{Aut}(\langle U\rangle)$ such that $y < z\psi < y\theta$.

We split the proof into two cases. First the case that $z < y$ and second the case where $y < z$. So suppose that $z < y$. Then $z\theta < y\theta$. If $y < z\theta < y\theta$ we are done by setting $\psi = \theta$. Otherwise $z\theta < y$ and hence $z\theta^2 < y\theta$. If $y < z\theta^2 < y\theta$, then we are again finished by setting $\psi = \theta^2$ otherwise $z\theta^2 < y$ and hence $z\theta^3 < y\theta$. Continuing this argument we find that either there exists $m \in \mathbb{N}$ such that $y < z\theta^m < y\theta$ or $z\theta^n < y$ for all $n \in \mathbb{N}$. We will show that the latter case cannot happen. Consider the map $\hat{\phi} : V_\Omega \to V_\Omega$ defined in Lemma 9.22 by

$$v\hat{\theta} = \begin{cases} v\phi & \text{if } v \in U \\ v & \text{otherwise .} \end{cases}$$

Then by Lemma 9.22 $\hat{\theta}$ is an automorphism of $\Omega$ with infinite positive orbital $U$. Consequently, there exists $i \in \mathbb{N}$ such that $y < z\hat{\theta}^i$, or in other words $y < z\theta^i$. Thus we can conclude that there exists $m \in \mathbb{N}$ such that $y < z\theta^m < y\theta$ and we can take $\psi = \theta^m$.

Now suppose on the other hand that $y < z$. If $y < z < y\theta$, then we are finished by setting $h = 1$. So suppose that $y < y\theta < z$. Then $y < z\theta^{-1}$. If $y < z\theta^{-1} < y\theta$, then take $\psi = \theta^{-1}$. Otherwise $y\theta < z\theta^{-1}$ and $y < z\theta^{-2}$. Continuing this argument as before we find that either there exists $m \in \mathbb{N}$ such that $y < z\theta^{-m} < y\theta$, or $y\theta < z\theta^{-n}$ for all $n \in \mathbb{N}$. In the latter case this means that $y\theta^n < z$ for all $n \in \mathbb{N}$. But above we saw that the map $\hat{\theta}$ is an automorphism of $\Omega$ with infinite positive orbital $U$. Hence there exists $j \in \mathbb{N}$ such that $y < z\hat{\theta}^j$, or in other words $y < z\theta^j$. Thus we conclude that there must exist some $m \in \mathbb{N}$ such that $y < z\theta^{-m} < y\theta$. Taking $\psi = \theta^{-m}$ completes the proof. □

In other words, Lemma 9.27 says that for any $y, z \in U$ and for any two elements $u, v \in \mathrm{orb}_{\mathrm{Aut}(\langle U \rangle)}(y)$ such that $u < v$ there exists $w \in \mathrm{orb}_{\mathrm{Aut}(\langle U \rangle)}(z)$ such that $u < w < v$.

Now consider the set,

$$X_0 = \{S \subseteq U : S \text{ is an interval in } U, S\phi \cap S = \emptyset \text{ for all } \phi \in \mathrm{Aut}(\langle U \rangle) \setminus \mathbf{1}\}.$$

Since $\{y\}f \neq \{y\}$ for all $y \in U$, it follows that $X_0$ is not empty. If we order the elements of $S$ by inclusion, *i.e.* we let $S \leq T$ if and only if $S \subseteq T$, then it is easy to see that $(X_0, \leq)$ is a partially ordered set. Now let $S_0 \leq S_1 \leq \ldots$ be a chain in $(X_0, \leq)$ and consider $\bigcup_{n=1}^{\infty} S_n$. Clearly $S_n \subseteq \bigcup_{n=1}^{\infty} S_n$ for all $n \in \mathbb{N}$. Moreover, $\bigcup_{n=1}^{\infty} S_n$ is a interval in $(X_0, \leq)$. For if $r, t \in \bigcup_{n=1}^{\infty} S_n$, then there exists some $k \in \mathbb{N}$ such that $r, t \in S_k$. Thus since $S_k$ is an interval in $U$, if $s \in U$ and $r < s < t$, it follows that $s \in S_n$ and hence $s \in \bigcup_{n=1}^{\infty} S_n$. Additionally, if $\phi \in \mathrm{Aut}(\langle U \rangle) \setminus \mathbf{1}$, then we can show that $(\bigcup_{n=1}^{\infty} S_n)\phi \cap (\bigcup_{n=1}^{\infty} S_n) = \emptyset$, in the following way. Seeking a contradiction, suppose that $(\bigcup_{n=1}^{\infty} S_n)\phi \cap (\bigcup_{n=1}^{\infty} S_n) \neq \emptyset$. Then there exists some $s \in \bigcup_{n=1}^{\infty} S_n$ such that $s \in S_j$ and $s \in S_k\phi$, for some $j, k \in \mathbb{N}$. Suppose without loss of generality that $j < k$ (otherwise consider $\phi^{-1}$). Then by assumption $S_j \subseteq S_k$. But this says that $s \in S_k\phi \cap S_k$, a contradiction to the definition of $S_k$. Hence $(\bigcup_{n=1}^{\infty} S_n)\phi \cap (\bigcup_{n=1}^{\infty} S_n) = \emptyset$ as required.

It now follows that every chain $S_0 \leq S_1 \leq \ldots$ in $X_0$ has $\bigcup_{i=1}^{\infty} S_i$ as an upper bound. Thus by Zorn's Lemma $X_0$ contains a maximal element, $M_0$

say. Define

$$T_0 = U \setminus \bigcup_{y \in M_0} \mathrm{orb}_{\mathrm{Aut}(\langle U \rangle)}(y) = U \setminus \bigcup_{m=0}^{\infty} M_0 \psi_m,$$

where we let $\mathrm{Aut}(\langle U \rangle) = \{\psi_m : m \in \mathbb{N}\}$.

Now for a countable ordinal $\alpha$ suppose that for all $i < \alpha$ we have constructed sets $X_i$ and $M_i$ such that $M_i$ is a maximal element of $X_i$ and such that

$$T_\alpha := U \setminus \bigcup_{i<\alpha} \bigcup_{m=0}^{\infty} M_i \psi_m \neq \emptyset.$$

Let

$$X_\alpha = \{S \subseteq T_\alpha : S \text{ is an interval in } U, S\phi \cap S = \emptyset \text{ for all } \phi \in \mathrm{Aut}(\langle U \rangle) \setminus \mathbf{1}\}.$$

Then, once again, by ordering the sets in $X_\alpha$ by inclusion we find that Zorn's Lemma guarantees the existence of a maximal element $M_\alpha$. Since $U$ is a countable set, there must exist a countable ordinal $\lambda$ such that,

$$T_\lambda := U \setminus \bigcup_{i<\lambda} \bigcup_{m=0}^{\infty} M_i \psi_m = \emptyset,$$

and thus

$$U = \bigcup_{i<\lambda} \bigcup_{m=0}^{\infty} M_i \psi_m.$$

Observe that since $\phi_m \in \mathrm{Aut}(\langle U \rangle)$ for all $m \in \mathbb{N}$, $\langle M_i \phi_m \rangle \cong \langle M_i \rangle$ for all $i < \lambda$ and for all $m \in \mathbb{N}$.

**Lemma 9.28.** *Let $i < \lambda$ and let $\phi \in \mathrm{Aut}(\langle U \rangle)$. Then $M_i\phi$ is an interval in $U$. Furthermore if $j < \lambda$ then $M_i\phi \cap M_j\theta = \emptyset$ for all $\theta \in \mathrm{Aut}(\langle U \rangle)$ such that $\theta \neq \phi$.*

*Proof.* Since $M_i$ is an interval in $U$ and since $\phi \in \mathrm{Aut}(\langle U \rangle)$ it follows immediately that $M_i\phi$ is an interval in $U$ for all $\phi \in \mathrm{Aut}(\langle U \rangle)$. Furthermore, by definition of $M_i$, $M_i\theta\phi^{-1} \cap M_i = \emptyset$ for all $\phi, \theta \in \mathrm{Aut}(\langle U \rangle)$ such that $\phi \neq \theta$. Thus, $M_i\theta \cap M_i\phi = \emptyset$ for all $\phi, \theta \in \mathrm{Aut}(\langle U \rangle)$. Now suppose that $i \neq j$. Then we can suppose without loss of generality that $i < j$. By construction, $M_j$ is an interval in $U \setminus \bigcup_{k<j} \bigcup_{m=0}^{\infty} M_k \psi_m$, where we set $\mathrm{Aut}(\langle U \rangle) = \{\psi_m : m \in \mathbb{N}\}$. Thus $M_i\phi\theta^{-1} \cap M_j \neq \emptyset$, and hence $M_i\phi \cap M_j\theta = \emptyset$ as required. □

We will examine $U = \bigcup_{i<\lambda} \bigcup_{m=0}^{\infty} M_i\psi_m$ to show that if $(\mathrm{Aut}(\langle U \rangle), \leq)$ is dense, then $\mathrm{Aut}(\langle U \rangle)$ is uncountable.

**Lemma 9.29.** *Suppose that* $(\mathrm{Aut}(\langle U \rangle), \leq)$ *is dense. Let* $\phi \in \mathrm{Aut}(\langle U \rangle)$ *and let* $i < \lambda$. *Then there exists* $\theta, \psi \in \mathrm{Aut}(\langle U \rangle)$ *such that* $M_i\theta < M_i\phi < M_i\psi$.

*Proof.* Let $y \in M_i$. Then since $\langle \mathrm{orb}_{\mathrm{Aut}(\langle U \rangle)}(y) \rangle \cong (\mathbb{Q}, \leq)$ by Lemma 9.26, and is thus without endpoints, there exists $\theta, \psi \in \mathrm{Aut}(\langle U \rangle)$ such that $y\theta < y\phi < y\psi$. By Lemma 9.28, $M_i\theta$ and $M_i\phi$ are disjoint intervals in $U$ and hence we can conclude that $M_i\theta < M_i\phi$. Similarly $M_i\phi < M_i\psi$ and the result follows. □

**Lemma 9.30.** *Suppose that* $(\mathrm{Aut}(\langle U \rangle), \leq)$ *is dense. Let* $\phi, \theta \in \mathrm{Aut}(\langle U \rangle)$ *and let* $i < \lambda$. *Then there exists* $\psi \in \mathrm{Aut}(\langle U \rangle)$ *such that* $M_i\phi < M_i\psi < M_i\theta$.

*Proof.* Let $y \in M_i$. Then since $\langle \mathrm{orb}_{\mathrm{Aut}(\langle U \rangle)}(y) \rangle \cong (\mathbb{Q}, \leq)$, and is thus dense, there exists $\psi \in \mathrm{Aut}(\langle U \rangle)$ such that $y\phi < y\psi < y\theta$. But by Lemma 9.28 $M_i\phi$, $M_i\theta$ and $M_i\psi$ are disjoint intervals in $U$ and so it follows that $M_i\phi < M_i\psi < M_i\theta$. □

**Corollary 9.31.** *Suppose that* $(\mathrm{Aut}(\langle U \rangle), \leq)$ *is dense. Then,*

$$(\{M_i\psi_m : i < \lambda, m \in \mathbb{N}\}, \leq) \cong (\mathbb{Q}, \leq).$$

*Proof.* Since $\mathbb{Q}$ is the unique countable dense total order without endpoints, we need only show that $(\{M_i\psi_m : i < \lambda, m \in \mathbb{N}\}, \leq)$ is countable, dense and without endpoints. First notice that $\{M_i\psi_m : i < \lambda, m \in \mathbb{N}\}$ is countable since $U$ is countable. Furthermore, by Lemma 9.30, $\{M_i\psi_m : i < \lambda, m \in \mathbb{N}\}$ is dense and by Lemma 9.29, $\{M_i\psi_m : i < \lambda, m \in \mathbb{N}\}$ is without endpoints. The result now follows immediately. □

**Lemma 9.32.** *Suppose that* $(\mathrm{Aut}(\langle U \rangle), \leq)$ *is dense. Let* $\phi, \theta \in \mathrm{Aut}(\langle U \rangle)$ *and let* $i, j < \lambda$. *If* $M_i\phi < M_j\theta$ *then for all* $k < \lambda$ *there exists* $\psi_k \in \mathrm{Aut}(\langle U \rangle)$ *such that* $M_i\phi < M_k\psi_k < M_j\theta$.

*Proof.* Suppose that $M_i\phi < M_j\theta$. If $i = j$, let $y \in M_i$. Then by Lemma 9.27, for all $k < \lambda$ there exists $z_k \in M_k$ and $\psi_k \in \mathrm{Aut}(\langle U \rangle)$ such that $y\phi < z_k\psi_k < y\theta$. By Lemma 9.28 we know that $M_i\phi$, $M_i\theta$ and $M_k\psi_k$ $(k < \lambda)$ are disjoint intervals in $U$. Thus it now follows that $M_i\phi < M_k\psi_k < M_i\theta$ for all $k < \lambda$.

Suppose on the other hand that $i \neq j$. If there exists $l < \lambda$ and $\psi, \psi' \in \mathrm{Aut}(\langle U \rangle)$ such that $M_i\phi \leq M_l\psi < M_l\psi' \leq M_j\theta$, then we are done by applying the above argument to $M_l\psi$ and $M_l\psi'$. So suppose not. Then for each $l < \lambda$, $l \neq i, j$, there exists at most one $\psi_l \in \mathrm{Aut}(\langle U \rangle)$ such that $M_i\phi < M_l\psi_l < M_j\theta$. So let $m < \lambda$ and let $\{M_{k_n} : n < m\}$ be the maximal subset of $\{M_k : k < \lambda, k \neq i, j\}$ such that for all $k_n$, there exists $\psi_{k_n}$ such that $M_i\phi < M_{k_n}\psi_{k_n} < M_j\theta$. Then since $U$ is an interval and $U = \bigcup_{i<\lambda} \bigcup_{\psi \in \mathrm{Aut}(\langle U \rangle)} M_i\psi$ it follows that

$$N = \left( \bigcup_{n<m} M_{k_n}\psi_{k_n} \right) \cup M_i\phi \cup M_j\theta$$

is an interval in $U$. Furthermore by definition, $(M_i\phi)\pi \cap M_i\phi = \emptyset$, $(M_j\theta)\pi \cap M_j\theta = \emptyset$ and $(M_{k_n}\psi_{k_n})\pi \cap M_{k_n}\psi = \emptyset$ for all $n < m$ and for all $\pi \in \mathrm{Aut}(\langle U \rangle)$, $\pi \neq \mathbf{1}_U$. Thus by applying Lemma 9.28, we can conclude that $(M_i\phi)\pi \cap N = \emptyset$, $(M_j\theta)\pi \cap N = \emptyset$ and $(M_{k_n}\psi_{k_n})\phi \cap N = \emptyset$ for all $n < m$ and for all $\pi \in \mathrm{Aut}(\langle U \rangle)$, $\pi \neq \mathbf{1}_U$. Thus $N\pi \cap N = \emptyset$ for all $\pi \in \mathrm{Aut}(\langle U \rangle)$, $\pi \neq \mathbf{1}_U$. But then since $M_i, M_j\theta\pi^{-1} \subseteq N\pi^{-1}$ it follows that $M_i, \subset N\pi^{-1}$ and $N\pi^{-1} \cap N = \emptyset$ for all $\pi \in \mathrm{Aut}(\langle U \rangle)$, $\pi \neq \mathbf{1}_U$. This contradicts maximality of $M_i$. Thus we can conclude that there must exist $l < \lambda$ and $\psi, \psi' \in \mathrm{Aut}(\langle U \rangle)$ such that $M_i\phi \leq M_l\psi < M_l\psi' \leq M_j\theta$ and the result follows. □

**Corollary 9.33.** *Suppose that* $(\mathrm{Aut}(\langle U \rangle), \leq)$ *is dense. Let* $\phi \in \mathrm{Aut}(\langle U \rangle)$ *and let* $i < \lambda$. *Then for all* $k < \lambda$ *there exists* $\theta_k, \pi_k \in \mathrm{Aut}(\langle U \rangle)$ *such that* $M_k\theta_k < M_i\phi < M_k\pi_k$.

*Proof.* By Lemma 9.29, there exists $\phi', \phi'' \in \mathrm{Aut}(\langle U \rangle)$ such that $M_i\phi' < M_i\phi < M_i\phi''$. Then by Lemma 9.32, for all $k < \lambda$ there exists $\theta_k, \pi_k \in \mathrm{Aut}(\langle U \rangle)$ such that $M_i\phi' < M_k\theta_k < M_i\phi$ and $M_i\phi < M_k\pi_k < M_i\phi''$, and the result follows. □

## 9.5 Interlude on Coloured Total Orders

Before we can continue on our way towards the proof of Theorem 9.51, we must now introduce a new type of relational structure – a $\lambda$-coloured total order. In particular, we will introduce the $\lambda$-coloured rationals. In the next section we will show that the $\lambda$-coloured rationals are closely connected to the orbital $U$ of an automorphism $f \in \mathrm{Aut}(U)$.

**Definition 9.34.** Let $\lambda$ be a countable ordinal. Then an $\lambda$-coloured total order is a relational structure $\Gamma = (V_\Gamma, \leq_\Gamma, (C_i)_{i<\lambda})$, where the following conditions are satisfied.

(i) $(V_\Gamma, \leq_\Gamma)$ is a total order,

(ii) $V_\Gamma = \bigcup_{i<\lambda} U_i$ with $U_i \cap U_j = \emptyset$ for $i \neq j$,

(iii) $C_i = U_i \times U_i$ for all $i < \lambda$.

The elements $u \in U_i$, for $i < \lambda$, are said to have colour $i$.

It is not hard to see that if $\Omega = (V_\Omega, \leq)$ is a total order, $\lambda$ is a countable ordinal and $V_\Omega = \bigcup_{i<\lambda} U_i$ is any partition of $V_\Omega$, then the relational structure $\Gamma_\Omega = (V_\Omega, \leq, (C_i)_{i<\lambda})$ formed by setting $C_i = U_i \times U_i$ for all $i < \lambda$, is an $\lambda$-coloured total order.

**Lemma 9.35.** *For a countable ordinal $\lambda$, let $\Gamma = (V_\Gamma, \leq, (C_i)_{i<\lambda})$ be a $\lambda$-coloured total order and let $f : V_\Gamma \to V_\Gamma$ be a function. Then $f \in \mathrm{Aut}(\Gamma)$ if and only if $f$ defines an automorphism of the total order $(V_\Gamma, \leq)$ and $U_i f = U_i$ for all $i < \lambda$.*

*Proof.* Suppose first that $f \in \mathrm{Aut}(\Gamma)$. Then $f$ is a bijective function $V_\Gamma \to V_\Gamma$ and it holds that $u \leq v$ if and only if $uf \leq vf$. Thus $f$ defines an automorphism of $(V_\Gamma, \leq)$. Now for $i < \lambda$, let $u \in U_i$. Then $(u, u) \in C_i$ and hence $(uf, uf) \in C_i$. In other words, $uf \in U_i$ and thus $U_i f = U_i$ as required.

Now suppose that $f$ defines an automorphism of $(V_\Gamma, \leq)$ and $U_i f = U_i$ for all $i < \lambda$. Then $f$ is a bijective function $V_\Gamma \to V_\Gamma$ and it holds that $u \leq v$ if and only if $uf \leq vf$. Furthermore since $U_i f = U_i$ for all $i < \lambda$, it follows that $(t, u) \in C_i$ if and only if $(tf, uf) \in C_i$. Thus $f$ defines an automorphism of $\Gamma$ and the result is complete. $\square$

Lemma 9.35 tells us that any automorphism of a $\lambda$-coloured total order, must map elements of colour $i$ to elements of colour $i$. An important consequence of Lemma 9.35 is the following.

**Corollary 9.36.** *Let $\Gamma = (V_\Gamma, \leq, (C_i)_{i<\lambda})$ be an $\lambda$-coloured total order. Then there exists an embedding of $\mathrm{Aut}(\Gamma)$ into $\mathrm{Aut}((V_\Gamma, \leq))$.*

*Proof.* Define an embedding $\phi : \mathrm{Aut}(\Gamma) \to \mathrm{Aut}((V_\Gamma, \leq))$ by $f\phi = f$. By Lemma 9.35 $\phi$ is well defined. It is straightforward to see that $\phi$ is an injective group homomorphism and so the result follows immediately. $\square$

It is not hard to show that the class of finite $\lambda$-coloured total orders, for any countable ordinal $\lambda$, has the hereditary, joint embedding and amalgamation properties. Consequently, the class of finite $\lambda$-coloured total orders has a Fraïssé limit.

**Definition 9.37.** Let $\lambda$ be a countable ordinal. We define $\mathbb{Q}_\lambda = (V_{\mathbb{Q}_\lambda}, \leq, (C_i)_{i<\lambda})$ to be a $\lambda$-coloured total order with $(V_{\mathbb{Q}_\lambda}, \leq) \cong \mathbb{Q}$, and such that for all $v, w \in V_{\mathbb{Q}_\lambda}$, $v < w$, and for all $i < \lambda$, there exists $u \in U_i$ such that $v < u < w$. The $\lambda$-coloured total order $\mathbb{Q}_\lambda$ is known as the $\lambda$-coloured rationals.

It can be shown that, for countable ordinals $\lambda$ and $\alpha$ such that $|\lambda| = |\alpha|$, every finite $\alpha$-coloured total order can be embedded into $\mathbb{Q}_\lambda$. Therefore is the unique homogeneous Fraïssé limit of the class of $\alpha$-coloured total orders where $|\lambda| = |\alpha|$ and therefore $\mathbb{Q}_\lambda$ exists. In the next theorem, we will show that the automorphism group of $\mathbb{Q}_\lambda$ has cardinality $2^{\aleph_0}$. In order to prove this, we will need make the following definition.

**Definition 9.38.** Let $\Lambda_n = (V_{\Lambda_n}, \leq_{\Lambda_n}, (D_{i,n})_{i<\lambda})$ be a $\lambda$-coloured total order for all $n \in \mathbb{Z}$, where $D_{i,n} = U_{i,n} \times U_{i,n}$ for all $i < \lambda$. Suppose that the sets $V_{\Lambda_n}$ are mutually disjoint for all $n \in \mathbb{Z}$. We define

$$\bigoplus_{n\in\mathbb{Z}} \Lambda_n = (V_{\bigoplus_{n\in\mathbb{Z}}\Lambda_n}, \preceq, (E_i)_{i<\lambda})$$

to be the $\lambda$-coloured total order formed from the $\Lambda_n$ by setting $V_{\bigoplus_{n\in\mathbb{Z}}\Lambda_n} = \bigcup_{n\in\mathbb{Z}} V_{\Lambda_n}$, $E_i = \left(\bigcup_{n\in\mathbb{Z}} U_{i,n}\right) \times \left(\bigcup_{n\in\mathbb{Z}} U_{i,n}\right)$ and where for $u, v \in$, $u \prec v$ if and only if either,

$u, v \in V_{\Lambda_n}$ for some $n \in \mathbb{Z}$ and $u \leq_{\Lambda_n} v$ or,

$u \in V_{\Lambda_m}$, $v \in V_{\Lambda_n}$ and $m < n$.

Notice that by definition $V_{\Lambda_m} \prec V_{\Lambda_n}$ for all $m \leq n$

**Theorem 9.39.** *For all countable ordinals $\lambda$, $\mathrm{Aut}(\mathbb{Q}_\lambda)$ has cardinality $2^{\aleph_0}$.*

*Proof.* For $n \in \mathbb{Z}$, let $\mathbb{Q}_{\lambda,n}$ be a copy of the $\lambda$-coloured rationals so that $\mathbb{Q}_{\lambda,n} = (V_{\mathbb{Q}_{\lambda,n}}, \leq, (C_{i,n})_{i<\lambda})$. First we claim that

$$\bigoplus_{n\in\mathbb{Z}} \mathbb{Q}_{\lambda,n} = \left(\bigcup_{n\in\mathbb{Z}} V_{\mathbb{Q}_{\lambda,n}}, \preceq, (E_i)_{i<\lambda}\right) \cong \mathbb{Q}_\lambda.$$

Since $\mathbb{Q}_\lambda$ is the unique homogeneous Fraïssé limit of the class of finite $\lambda$-coloured orders it suffices to show that $\bigoplus_{n\in\mathbb{Z}} \mathbb{Q}_{\lambda,n}$ satisfies the properties described in Definition 9.37. In other words, we need only show that

$$\left(\bigcup_{n\in\mathbb{Z}} V_{\mathbb{Q}_{\lambda,n}}, \preceq\right) \cong \mathbb{Q}$$

and show that for all $v, w \in \bigcup_{n\in\mathbb{Z}} V_{\mathbb{Q}_{\lambda,n}}$ with $v \prec w$, and for all $i < \lambda$, there exists $u \in E_i$ such that $v \prec u \prec w$. It should be clear that since $(V_{\mathbb{Q}_{\lambda,n}}, \leq) \cong \mathbb{Q}$ for all $n \in \mathbb{Z}$,

$$\left(\bigcup_{n\in\mathbb{Z}} V_{\mathbb{Q}_{\lambda,n}}, \preceq\right) \cong \bigoplus_{n\in\mathbb{Z}} \mathbb{Q}.$$

It can easily be shown that $\bigoplus_{n\in\mathbb{Z}}\mathbb{Q}$ is dense and without endpoints. Thus since $\mathbb{Q}$ is the unique total order which is both dense and without endpoints, it follows that $(\bigcup_{n\in\mathbb{Z}} V_{\mathbb{Q}_{\lambda,n}}, \preceq) \cong \mathbb{Q}$ as required. Now suppose that $v, w \in \bigcup_{n\in\mathbb{Z}} V_{\mathbb{Q}_{\lambda,n}}$, $v \prec w$ and $i < \lambda$. If $v, w \in V_{\mathbb{Q}_{\lambda,n}}$ for some $n \in \mathbb{Z}$, then by definition of $\mathbb{Q}_{\lambda,n}$, there exists $u \in C_{i,n}$ such that $v < u < w$. Thus $u \in E_i$ and $u \prec v \prec w$. Now suppose instead that $v \in V_{\mathbb{Q}_{\lambda,m}}$, $w \in V_{\mathbb{Q}_{\lambda,n}}$ for some $m, n \in \mathbb{Z}$, $m \neq n$. Since $(V_{\mathbb{Q}_{\lambda,m}}, \leq) \cong \mathbb{Q}$ it follows that there exists $x \in V_{\mathbb{Q}_{\lambda,m}}$ such that $v < x$ and hence $v \prec x$. Furthermore, since $V_{\mathbb{Q}_{\lambda,m}} \prec V_{\mathbb{Q}_{\lambda,n}}$ it follows that $x \prec w$. Now by definition of $\mathbb{Q}_{\lambda,n}$, there exists $u \in C_{i,n}$ such that $v < u < x$. Thus $u \in E_i$ and $v \prec u \prec w$ as required. In either case we have shown that for all $v, w \in \bigcup_{n\in\mathbb{Z}} V_{\mathbb{Q}_{\lambda,n}}$, $v \prec w$, and for all $i < \lambda$, there exists $u \in E_i$ such that $v \prec u \prec w$. Thus

$$\bigoplus_{n\in\mathbb{Z}} \mathbb{Q}_{\lambda,n} = \left(\bigcup_{n\in\mathbb{Z}} V_{\mathbb{Q}_{\lambda,n}}, \preceq, (E_i)_{i<\lambda}\right) \cong \mathbb{Q}_\lambda,$$

as claimed. We will show that $\mathrm{Aut}(\bigoplus_{n\in\mathbb{Z}} \mathbb{Q}_{\lambda,n})$ has cardinality $2^{\aleph_0}$ and the result will then follow.

Now consider a sequence of automorphisms $\mathbf{f} = (f_n)_{n\in\mathbb{Z}}$ such that $f_n \in \mathrm{Aut}(\mathbb{Q}_{\lambda,n})$ for all $n \in \mathbb{Z}$. Define a map $\hat{\mathbf{f}} : \bigoplus_{n\in\mathbb{Z}} \mathbb{Q}_{\lambda,n} \to \bigoplus_{n\in\mathbb{Z}} \mathbb{Q}_{\lambda,n}$ by $v\hat{\mathbf{f}} = vf_n$ where $v \in V_{\mathbb{Q}_{\lambda,n}}$. Since $f_n \in \mathrm{Aut}(\mathbb{Q}_{\lambda,n})$ for all $n \in \mathbb{Z}$, it follows that for all $n \in \mathbb{Z}$, $\hat{\mathbf{f}}$ maps the set $V_{\mathbb{Q}_{\lambda,n}}$ back to itself. It should thus be easy to see that $\hat{\mathbf{f}}$ is a well defined automorphism of $\bigoplus_{n\in\mathbb{Z}} \mathbb{Q}_{\lambda,n}$. Furthermore, if $\mathbf{g} = (g_n)_{n\in\mathbb{Z}}$ is another sequence of automorphisms such that such that $g_n \in \mathrm{Aut}(\mathbb{Q}_{\lambda,n})$ for all $n \in \mathbb{Z}$, and there exists $m \in \mathbb{Z}$ with $f_m \neq g_m$, then clearly $\hat{\mathbf{f}} \neq \hat{\mathbf{g}}$. Now since $\mathbb{Q}_{\lambda,n}$ is a copy of the $\lambda$-coloured rationals, it is homogeneous for all $n \in \mathbb{Z}$. In particular this means that $|\mathrm{Aut}(\mathbb{Q}_{\lambda,n})| \geq 2$ for all $n \in \mathbb{Z}$. Thus there exist $2^{\aleph_0}$ distinct sequences of automorphisms $\mathbf{f} = (f_n)_{n\in\mathbb{Z}}$ such that $f_n \in \mathrm{Aut}(\mathbb{Q}_{\lambda,n})$ for all $n \in \mathbb{Z}$. Consequently, $\{\hat{\mathbf{f}} : \mathbf{f} = (f_n)_{n\in\mathbb{Z}}, f_n \in \mathrm{Aut}(\mathbb{Q}_{\lambda,n}) \text{ for all } n \in \mathbb{Z}\}$ is a set of size $2^{\aleph_0}$ contained in $\mathrm{Aut}(\bigoplus_{n\in\mathbb{Z}} \mathbb{Q}_{\lambda,n})$ and since $\bigoplus_{n\in\mathbb{Z}} \mathbb{Q}_{\lambda,n} \cong \mathbb{Q}_\lambda$ the result now follows. □

**Definition 9.40.** Let $\lambda$ be a countable ordinal and for each $i < \lambda$, let $\Omega_i$ be a countable total order. By $\mathbb{Q}_\lambda((\Omega_i)_{i<\lambda})$ we will mean the relational structure formed from $\mathbb{Q}_\lambda$, where each $u \in U_i$ is replaced by a copy of $\Omega_i$. More formally, suppose that for $i < \lambda$, $U_i = \{u_{im} : m \in \mathbb{N}\}$ and let $\Omega_i = (\{z_{ir} : r \in \mathbb{N}\}, \leq)$ (replacing the natural numbers with a finite set if $\Omega_i$ is finite). For each $i \in \mathbb{N}$, form a copy of $\Omega_i$ as follows. Let $X_{im} = \{x_{imr} : r \in \mathbb{N}\}$ and set $x_{imr} \leq x_{ims}$

if and only if $z_{ir} \leq z_{is}$. Now let $X_{im} \leq X_{jn}$ if and only if $u_{im} \leq u_{jn}$. Then

$$\mathbb{Q}_\lambda((\Omega_i)_{i<\lambda}) = \left( \bigcup_{m=0}^{\infty} \bigcup_{i<\lambda} X_{im} \, , \leq, (D_i)_{i<\lambda} \right),$$

where

$$D_i = \left( \bigcup_{m=0}^{\infty} X_{im} \right) \times \left( \bigcup_{m=0}^{\infty} X_{im} \right).$$

**Lemma 9.41.** *Let $\Omega_i$, $i < \lambda$, be countable total orders. Then $\mathbb{Q}_\lambda((\Omega_i)_{i<\lambda})$ is a countable $\lambda$-coloured total order.*

*Proof.* First we note that since $\Omega_i$ is countable for all $i < \lambda$ and since $\mathbb{Q}_\lambda$ is countable, $\mathbb{Q}_\lambda((\Omega_i)_{i<\lambda})$ is a countable relational structure. We will now check that $\left(\bigcup_{m=0}^{\infty} \bigcup_{i<\lambda} X_{im} \, , \leq\right)$, is a total order. It should be clear that $\leq$ is reflexive since $\Omega_i$ is a total order and hence for all $i < \lambda$ and for all $r \in \mathbb{N}$, $z_{ir} \leq z_{ir}$ and hence $x_{imr} \leq x_{imr}$. To check symmetry suppose that $x_{imr} \leq x_{jns}$ and $x_{jns} \leq x_{imr}$. Then $u_{im} \leq u_{jn}$ and $u_{jn} \leq u_{im}$. Since $\mathbb{Q}_\lambda$ is a total order it follows that $u_{im} = u_{jn}$ and hence $i = j$ and $m = n$. It now follows that $z_{ir} \leq z_{is} = z_{js}$ and $z_{js} = z_{is} \leq z_{ir}$. But since $\Omega_i$ is a total order, this implies that $r = s$. Thus $x_{imr} = x_{jms}$ and symmetry is satisfied. To see that $\leq$ is transitive, suppose that $x_{imr} \leq x_{jns}$ and $x_{jns} \leq x_{kpt}$. Then $u_{im} \leq u_{jn} \leq u_{kp}$ and hence $u_{im} \leq u_{kp}$. If $i \neq j$ or $m \neq n$ then $X_{im} < X_{jn} \leq X_{kp}$ and hence $x_{imr} < x_{kpt}$. Similarly if $j \neq k$ or $m \neq p$ then $X_{im} \leq X_{jn} < X_{kp}$ and hence $x_{imr} < x_{kpt}$. So suppose that $i = j = k$ and $m = n = p$. Then it must be the case that $z_{ir} \leq z_{is} \leq z_{it}$ and since $\Omega_i$ is a total order it follows that $z_{ir} \leq z_{it}$. Hence $x_{imr} \leq x_{imt} = x_{kpt}$ and transitivity is satisfied. Finally, totality follows from the totality of the orders on $\mathbb{Q}_\lambda$ and $\Omega_i$ for all $i < \lambda$. To finish the proof we observe that if $i, j < \lambda$ and $i \neq j$, then $(\bigcup_{m=0}^{\infty} X_{im}) \cap (\bigcup_{m=0}^{\infty} X_{jm}) = \emptyset$, since $u_{im} \neq u_{jn}$ if $i \neq j$. □

**Lemma 9.42.** *Let $i, j < \lambda$ and let $m, n \in \mathbb{N}$. If $X_{im} < X_{jn}$ then for all $k < \lambda$ there exists $p_k \in \mathbb{N}$ such that $X_{im} < X_{kp_k} < X_{jn}$.*

*Proof.* If $X_{im} < X_{jn}$ then $u_{im} < u_{jn}$ and so by definition of $\mathbb{Q}_\lambda$, for all $k < \lambda$ there exists $p_k \in \mathbb{N}$ such that $u_{im} < u_{kp_k} < u_{jn}$. Then $X_{im} < X_{kp_k} < X_{jn}$ as required. □

**Lemma 9.43.** *Let $i < \lambda$ and let $m \in \mathbb{N}$. Then for all $k < \lambda$ there exists $p_k, q_k \in \mathbb{N}$ such that $X_{kp_k} < X_{im} < X_{kq_k}$.*

*Proof.* By definition of $\mathbb{Q}_\lambda$, there exists $j < \lambda$ and $n \in N$ such that $u_{im} < u_{jn}$. Thus $X_{im} < X_{jn}$ and so by Lemma 9.42 for all $k < \lambda$ there exists $q_k \in \mathbb{N}$ such that $X_{im} < X_{kq_k} < X_{jn}$. A dual argument shows the existence of $p_k \in \mathbb{N}$ such that $X_{kp_k} < X_{im}$ for all $k < \lambda$. □

**Lemma 9.44.** *Let $\lambda$ be a countable ordinal and let $\Omega_i$, $i < \lambda$, be countable total orders. Then there exists an embedding* $\mathrm{Aut}(\mathbb{Q}_\lambda) \to \mathrm{Aut}(\mathbb{Q}_\lambda((\Omega_i)_{i<\lambda}))$.

*Proof.* Let $f \in \mathrm{Aut}(\mathbb{Q}_\lambda)$. Then by Lemma 9.35, $U_i f = U_i$ for all $i < \lambda$. Let $\phi : \mathrm{Aut}(\mathbb{Q}_\lambda) \to \mathrm{Aut}(\mathbb{Q}_\lambda((\Omega_i)_{i<\lambda}))$ be the map defined on $f \in \mathrm{Aut}(\mathbb{Q}_\lambda)$ by setting $(x_{imr})f\phi = x_{inr}$, where $u_{im}f = u_{in}$. We claim that $f\phi \in \mathrm{Aut}(\mathbb{Q}_\lambda((\Omega_i)_{i<\lambda}))$. To see that $f\phi$ is injective first note that by definition of $f\phi$, $(x_{imr})f\phi = (x_{jps})f\phi$ implies that $i = j$ and $r = s$. Suppose that $(x_{imr})f\phi = x_{inr} = (x_{ipr})f\phi$. Then $u_{im}f = u_{ip}f = u_{in}$ and since $f$ was injective we can deduce that $m = p$. Thus $x_{imr} = x_{jpr}$ and it follows that $f\phi$ is injective. Furthermore, $f\phi$ is surjective. For consider any element $x_{inr}$. Since $f$ is an automorphism of $\mathbb{Q}_n$, there exists $u_{im}$ such that $u_{im}f = u_{in}$. Then $(x_{imr})f\phi = x_{inr}$.

We must now check that $f\phi$ is an automorphism of the $\lambda$-coloured total order $\mathbb{Q}_\lambda((\Omega_i)_{i<\lambda})$. So suppose that $x_{imr} \le x_{jps}$ and that $x_{imr}f\phi = x_{inr}$ and $x_{jps}f\phi = x_{jqs}$. We seek to show that $x_{inr} \le x_{jqs}$. Since $x_{imr} \le x_{jps}$ we know that $u_{im} \le u_{jp}$ and since $f$ is an automorphism it follows that $u_{in} \le u_{jq}$. Thus $X_{in} \le X_{jq}$. If $i \ne j$ or $n \ne q$ then $X_{in} < X_{jq}$ and hence $x_{inr} < x_{jqs}$. If $i = j$ and $n = q$ then $u_{im}f = u_{in} = u_{jp} = u_{jp}f$ and hence since $f$ is injective we can conclude that $m = p$. But since $x_{imr} \le x_{jps} = x_{ims}$ it must be the case that $z_{ir} \le z_{is}$ and hence $x_{inr} \le x_{ins} = x_{jqs}$ as required. Now suppose instead that $x_{inr} \le x_{jqs}$, where $x_{imr}f\phi = x_{inr}$ and $x_{jps}f\phi = x_{jqs}$. We will show that $x_{imr} \le x_{jps}$. Since $x_{inr} \le x_{jqs}$, we know that $u_{in} \le u_{jq}$, and since $f$ is an automorphism it follows that $u_{im} \le u_{jp}$. If $i \ne j$ or $m \ne p$ then $u_{im} < u_{jp}$. Thus $X_{im} < X_{jp}$ and we can deduce that $x_{imr} < x_{jps}$. If on the other hand $i = j$ and $m = p$, then $u_{in}f = u_{ip}f$ and hence we can conclude that $n = p$. Since $x_{inr} \le x_{jps} = x_{ins}$ it must be the case that $z_{ir} \le z_{is} = z_{js}$ and hence it now follows that $x_{imr} \le x_{ims} = x_{jps}$ as required. We must also show that $(x_{imr}, x_{ins}) \in D_i$ if and only if $(x_{imr}f\phi, x_{ins}f\phi) \in D_i$, where for $i < \lambda$,

$$D_i = \left(\bigcup_{m=0}^{\infty} X_{im}\right) \times \left(\bigcup_{m=0}^{\infty} X_{im}\right).$$

This should be clear since by definition $x_{imr} \in X_{im}$ if and only if $x_{imr}f\phi \in X_{ip}$ for some $p$ and similarly $x_{ins} \in X_{in}$ if and only if $x_{ins}f\phi \in X_{iq}$ for some $q$.

To finish the proof, we show that $\phi$ is an injective group homomorphism. For suppose that $f, g \in \mathrm{Aut}(\mathbb{Q}_n)$ and that $u_{im}f = u_{in}$ and $u_{in}g = u_{ip}$. Then,

$$(x_{imr})(f \cdot g)\phi = x_{ipr} = (x_{inr})g\phi = (x_{imr})(f\phi) \cdot (g\phi).$$

Thus it easily follows that $\phi$ is a group homomorphism. It is injective since if $f\phi = g\phi$, then $x_{imr}f\phi = x_{imr}g\phi$ for all $i < \lambda$ and for all $m, r \in \mathbb{N}$. Thus $u_{im}f = u_{im}g$ for all $i < \lambda$ and for all $m \in \mathbb{N}$. In other words $f = g$ and we can conclude that $\phi$ is an injective group homomorphism. $\square$

## 9.6 The Automorphism Group of an Orbital and the $\lambda$-coloured Rationals

We now return to the orbital $U$ of an automorphism $f \in \mathrm{Aut}(\Omega)$ and consider the induced total order $\langle U \rangle$. Recall from page 142 that

$$U = \bigcup_{m=0}^{\infty} \bigcup_{i<\lambda} M_i \psi_m,$$

where $\mathrm{Aut}(\langle U \rangle) = \{\psi_m, m \in \mathbb{N}\}$. Moreover, by Lemma 9.28,

$$\left( \bigcup_{m=0}^{\infty} M_i \psi_m \right) \cap \left( \bigcup_{m=0}^{\infty} M_j \psi_m \right) = \emptyset$$

for all $i \neq j$. Thus, the relational structure $(U, \leq, (E_i)_{i<\lambda})$, formed by setting

$$E_i = \left( \bigcup_{m=0}^{\infty} M_i \psi_m \right) \times \left( \bigcup_{m=0}^{\infty} M_i \psi_m \right),$$

is a $\lambda$-coloured total order (to see this recall Definition 9.34). Since $U$ is a countable union of countable sets, we can write $U = \bigcup_{l \in \mathbb{N}} M_{j_l} \psi_{j_l}$ where for $l \in \mathbb{N}$, $j_l = i$ for some $i < \lambda$ and $\psi_{j_l} \in \mathrm{Aut}(\langle U \rangle)$. It will be convenient for us to write $U$ in this way for the next few lemmas.

Since $U$ is countable, for each $i < \lambda$ we can write, $M_i = \{z_{ir} : r \in \mathbb{N}\}$ (replacing the natural numbers with a finite set if $M_i$ is finite). Let $X_{in} = \{x_{inr} : r \in \mathbb{N}\}$ where $x_{inr} \leq x_{ins}$ if and only if $z_{ir} \leq z_{is}$. Now recall from Definition 9.34, that

$$\mathbb{Q}_\lambda(\langle M_i \rangle_{i<\lambda}) = \left( \bigcup_{i<\lambda} \bigcup_{n=0}^{\infty} X_{in}\,, \leq, (D_i)_{i<\lambda} \right)$$

is the $\lambda$-coloured total order formed from the $\lambda$-coloured rationals,

$$\mathbb{Q}_\lambda = \left( \bigcup_{i<\lambda} \{u_{im} : m \in \mathbb{N}\}, \leq, (C_i)_{i<\lambda} \right),$$

by setting $X_{in} \leq X_{js}$ if and only if $u_{in} \leq u_{js}$ and where $D_i = \left(\bigcup_{n=0}^{\infty} X_{in}\right) \times \left(\bigcup_{n=0}^{\infty} X_{in}\right)$. It will be convenient to write

$$\bigcup_{i<\lambda} \bigcup_{n=0}^{\infty} X_{in} = \bigcup_{k\in\mathbb{N}} X_{i_k p_k},$$

where for $k \in \mathbb{N}$, $i_k < \lambda$ and $p_k \in \mathbb{N}$. Note that this is possible since $\lambda$ is a countable ordinal.

We will show that if $(\mathrm{Aut}(\langle U\rangle), \leq)$ is dense, then $(U, \leq, (E_i)_{i<\lambda})$ is isomorphic to $\mathbb{Q}_\lambda(\langle M_i\rangle_{i<\lambda})$, the $\lambda$-coloured total order formed from the $\lambda$-coloured rationals by replacing each element of colour $i < \lambda$ by a copy of the total order $\langle M_i\rangle$.

**Lemma 9.45.** *Let $S \subseteq \mathbb{N}$ be finite. Suppose that*

$$\varphi : \left\langle \bigcup_{l\in S} M_{j_l}\psi_{j_l} \right\rangle \to \mathbb{Q}_\lambda(\langle X_i\rangle_{i<\lambda})$$

*is an embedding of $\lambda$-coloured total orders such that for all $r \in \mathbb{N}$ and for all $l \in S$, $((z_{j_l r})\psi_{j_l})\varphi = x_{i_{k_l} p_{k_l} r}$ for some $k_l \in \mathbb{N}$ such that $i_{k_l} = j_l$. Suppose that $t \notin S$. If $(\mathrm{Aut}(\langle U\rangle), \leq)$ is dense then there exists an extension,*

$$\tilde{\varphi} : \left\langle \bigcup_{l\in S\cup\{t\}} M_{j_l}\psi_{j_l} \right\rangle \to \mathbb{Q}_\lambda(\langle X_i\rangle_{i<\lambda}),$$

*of $\varphi$ such that $\tilde{\varphi}$ is an embedding and such that for all $r \in \mathbb{N}$, $((z_{j_t r})\psi_{j_t})\varphi = x_{i_{k_t} p_{k_t} r}$ for some $k_t \in \mathbb{N}$ such that $i_{k_t} = j_t$.*

*Proof.* First note that by assumption, for each $l \in S$, $(M_{j_l}\psi_{jl})\varphi = X_{i_{k_l} p_l}$ for some $k_l \in \mathbb{N}$. So let $A = \{l \in S : M_{j_l}\psi_{i_l} < M_{j_t}\psi_{j_t}\}$ and let $B = \{l \in S : M_{j_t}\psi_{j_t} < M_{j_l}\psi_{i_l}\}$. Then since $S$ is finite, both $A$ and $B$ are finite. If $a \in A$ and let $b \in B$, then since $M_{j_a}\psi_{i_a} < M_{j_b}\psi_{j_b}$ and $\varphi$ is an automorphism, it follows that $X_{i_{k_a} p_{k_a}} < X_{i_{k_b} p_{k_a}}$. Suppose that both $A$ and $B$ are non-empty. Then by Lemma 9.42 there exists $k_t \in \mathbb{N}$ such that $i_{k_t} = j_t$ and such that $X_{i_{k_a} p_{k_a}} < X_{i_{k_t} p_{k_t}} < X_{i_{k_b} p_{k_a}}$ for all $a \in A$ and for all $b \in B$. If on the other hand $A = \emptyset$ but $B$ is non-empty, then by Lemma 9.43 there exists $k_t \in \mathbb{N}$ such that $i_{k_t} = j_t$ and such that $X_{i_{k_t} p_{k_t}} < X_{i_{k_b} p_{k_a}}$ for all $b \in B$. Similarly if $B = \emptyset$ and $A$ is non-empty, then by Lemma 9.43 there exists $k_t \in \mathbb{N}$ such that $i_{k_t} = j_t$ and such that $X_{i_{k_a} p_{k_a}} < X_{i_{k_t} p_{k_t}}$ for all $a \in A$. In any case define $\tilde{\varphi} : \langle \bigcup_{l\in S\cup\{t\}} M_{j_l}\psi_{j_l}\rangle \to \mathbb{Q}_\lambda(\langle X_i\rangle_{i<\lambda})$ by

$$(z_{j_l r}\psi_{j_l})\tilde{\varphi} = \begin{cases} (z_{j_l r}\psi_{j_l})\varphi & \text{if } l \in S, \\ x_{i_{k_t} p_{k_t} r} & \text{if } l = t. \end{cases}$$

Then clearly $\tilde{\varphi}$ is an injective function since $\varphi$ is an injective function and since by choice $X_{i_{k_t}p_{k_t}} \neq X_{i_{k_l}p_{k_l}}$ for all $l \in S$.

Since $\varphi$ was an embedding $\langle \bigcup_{l\in S} M_{j_l}\psi_{j_l}\rangle \to \mathbb{Q}_\lambda(\langle X_i\rangle_{i<\lambda})$, it follows that if $n, l \in S$ and $r, s \in \mathbb{N}$, then $z_{j_l r}\psi_{j_l} \leq z_{i_n s}\psi_{i_n}$ if and only if $(z_{j_l r}\psi_{j_l})\tilde{\varphi} \leq (z_{i_n s}\psi_{i_n})\tilde{\varphi}$. Furthermore, $z_{j_l r}\psi_{j_l} < z_{j_t s}\psi_{j_t}$ if and only if $l \in A$. Hence $z_{j_l r}\psi_{j_l} < z_{j_t s}\psi_{j_t}$ if and only if $X_{i_{k_l}p_{k_l}} < X_{i_{k_t}p_{k_t}}$. Thus it follows that $z_{j_l r}\psi_{j_l} \leq z_{j_t s}\psi_{j_t}$ if and only if $x_{i_{k_l}p_{k_l}r} \leq x_{i_{k_t}p_{k_t}s}$, that is, if and only if $(z_{j_l r}\psi_{j_l})\tilde{\varphi} \leq (z_{j_t s}\psi_{j_t})\tilde{\varphi}$. A similar argument shows that $z_{j_t r}\psi_{j_t} \leq z_{j_l s}\psi_{j_l}$ if and only if $(z_{j_t r}\psi_{j_t})\tilde{\varphi} \leq (z_{j_l s}\psi_{j_l})\tilde{\varphi}$. Finally, since $\psi_{j_l}$ is an automorphism for all $l \in S \cup \{k\}$, it follows that $z_{j_l r}\psi_{j_l} \leq z_{j_l s}\psi_{j_l}$ if and only if $z_{j_l r} \leq z_{j_l s}$. But by construction $x_{j_l p_{k_l} r} \leq x_{j_l p_{k_l} s}$ if and only if $z_{j_l r} \leq z_{j_l s}$. Thus since $j_l = i_{k_l}$, it follows that $x_{j_{k_l}p_{k_l}r} \leq x_{j_{k_l}p_{k_l}s}$ if and only if $z_{j_l r} \leq z_{j_l s}$. Hence we can conclude that $z_{j_l r}\psi_{j_l} \leq z_{j_l s}\psi_{j_l}$ if and only if $(z_{j_l r}\psi_{j_l})\tilde{\varphi} \leq (z_{j_l s}\psi_{j_l})\tilde{\varphi}$.

Furthermore, for all $l \in S \cup \{t\}$, $((z_{j_l r})\psi_{j_l})\varphi = x_{i_{k_l}p_{k_l}r}$ for some $k_l \in \mathbb{N}$ such that $i_{k_l} = j_l$. Hence it follows that $(u, v) \in (\bigcup_{m=0}^{\infty} M_i\psi_m) \times (\bigcup_{m=0}^{\infty} M_i\psi_m)$ if and only if $(u\varphi, v\varphi) \in \left(\bigcup_{n=0}^{\infty} X_{in}\right) \times \left(\bigcup_{n=0}^{\infty} X_{in}\right)$. Thus $\tilde{\varphi}$ is an embedding of $\langle \bigcup_{l\in S\cup\{k\}} M_{j_l}\psi_{j_l}\rangle$ into $Q_\lambda(\langle M_i\rangle_{i<\lambda})$ as required. $\square$

**Lemma 9.46.** *Let $S \subseteq \mathbb{N}$ be finite. Suppose that*

$$\varphi : \left\langle \bigcup_{l\in S} M_{j_l}\psi_{j_l} \right\rangle \to \mathbb{Q}_\lambda(\langle X_i\rangle_{i<\lambda})$$

*is an embedding of $\lambda$-coloured total orders such that for all $r \in \mathbb{N}$ and for all $l \in S$, $((z_{j_l r})\psi_{j_l})\varphi = x_{i_{k_l}p_{k_l}r}$ for some $p_l \in \mathbb{N}$. Suppose that $X_{i_n p_n} \nsubseteq \operatorname{im}\varphi$. If $(\operatorname{Aut}(\langle U\rangle), \leq)$ is dense then there exists $t \in \mathbb{N}$ such that $j_t = i_n$ and an extension*

$$\tilde{\varphi} : \left\langle \bigcup_{l\in S\cup\{t\}} M_{j_l}\psi_{j_l} \right\rangle \to \mathbb{Q}_\lambda(\langle X_i\rangle_{i<\lambda}),$$

*of $\varphi$ such that $\tilde{\varphi}$ is an embedding and such for all for all $r \in \mathbb{N}$, $((z_{j_t r})\psi_{j_t})\varphi = x_{i_n p_n r} = x_{j_t p_n r}$.*

*Proof.* By assumption, for each $l \in S$, $(M_{j_l}\psi_{j_l})\phi = X_{i_{k_l}p_{k_l}}$ for some $p_l \in \mathbb{N}$. Now let $A = \{l \in S : X_{i_{k_l}p_{k_l}} < X_{i_n p_n}\}$ and let $B = \{l \in S : X_{i_n p_n} < X_{i_{k_l}p_{k_l}}\}$. If $a \in A$ and $b \in B$, then since $X_{j_{k_a}p_{k_a}} < X_{j_{k_b}p_{k_b}}$ and $\varphi$ is an automorphism, it follows that $M_{j_a}\psi_{j_a} < M_{j_b}\psi_{j_b}$. Furthermore, both $A$ and $B$ are finite sets. Thus if $A, B \neq \emptyset$ then by an application of Lemma 9.32, there exists $t \in \mathbb{N}$ such that $j_t = i_n$ and such that $M_{j_a}\psi_{j_a} < M_{j_t}\psi_{j_t} < M_{j_b}\psi_{j_b}$ for all $a \in A$ and for all $b \in B$. If on the other hand $A = \emptyset$ or $B = \emptyset$ then by an application of Corollary 9.33, there exists $t \in \mathbb{N}$ such that $j_t = i_n$ and such that $M_{j_t}\psi_{j_t} < M_{j_b}\psi_{j_b}$ for all $a \in A$ or $M_{j_t}\psi_{j_t} < M_{j_b}\psi_{j_b}$ for all $b \in B$, respectively. In any case define $\tilde{\varphi} : \langle \bigcup_{l \in S \cup \{t\}} M_{j_l}\psi_{j_l}\rangle \to \mathbb{Q}_\lambda(\langle X_i\rangle_{i<\lambda})$ by

$$(z_{j_l r}\psi_{j_l})\tilde{\phi} = \begin{cases} (z_{j_l r}\psi_{t_l})\phi & \text{if } l \in S, \\ x_{j_t p_n r} & \text{if } l = t. \end{cases}$$

Then $\tilde{\varphi}$ is a well defined function since $M_{j_l}\psi_{j_l} \neq M_{j_n}\psi_{j_n}$ for all $l \in S$. Furthermore, $\tilde{\varphi}$ is an injective function since $\varphi$ was injective and since by assumption $X_{i_n p_n} \nsubseteq \operatorname{im}\phi$. It remains to show that $\varphi$ is an embedding.

Since $\varphi$ was a embedding $\langle \bigcup_{l \in S} M_{j_l}\psi_{j_l}\rangle \to \mathbb{Q}_\lambda(\langle X_i\rangle_{i<\lambda})$, it follows that if $n, l \in S$ then $z_{j_l r}\psi_{j_l} \leq z_{i_n s}\psi_{i_n}$ if and only if $(z_{j_l r}\psi_{j_l})\tilde{\varphi} \leq (z_{i_n s}\psi_{i_n})\tilde{\varphi}$. Furthermore, $x_{i_{k_l} p_{k_l} r} < x_{i_n p_n s}$ if and only if $l \in A$. Thus $x_{i_{k_l} p_{k_l} r} < x_{i_n p_n s}$ if and only if $M_{j_l}\psi_{j_l} < M_{j_t}\psi_{j_t}$ and hence if and only if $z_{j_l r}\psi_{j_l} < z_{j_t s}\psi_{j_t}$. In other words $z_{j_l r}\psi_{j_l}\varphi < z_{j_t s}\psi_{j_t}\varphi$ if and only if $z_{j_l r}\psi_{j_l} < z_{j_t s}\psi_{j_t}$. A similar argument shows that $z_{j_t s}\psi_{j_t}\varphi < z_{j_n r}\psi_{j_n}\varphi$ if and only if $z_{j_t s}\psi_{j_t} < z_{j_n r}\psi_{j_n}$. Finally, since $\psi_{j_l}$ is an automorphism for all $l \in S \cup \{t\}$, it follows that for all $l \in S \cup \{t\}$, $z_{j_l r}\psi_{j_l} \leq z_{j_l s}\psi_{j_l}$ if and only if $z_{j_l r} \leq z_{j_l s}$. But by construction of $X_{j_l p_{k_l}}$, $z_{j_l r} \leq z_{j_l s}$ if and only if $x_{j_l p_{k_l} r} \leq x_{j_l p_{k_l} s}$. Thus since $j_l = i_{k_l}$, it follows that $z_{j_l r} \leq z_{j_l s}$ if and only if $x_{j_{k_l} p_{k_l} r} \leq x_{j_{k_l} p_{k_l} s}$. Hence we can conclude that $z_{j_l r}\psi_{j_l} \leq z_{j_l s}\psi_{j_l}$ if and only if $(z_{j_l r}\psi_{j_l})\tilde{\varphi} \leq (z_{j_l s}\psi_{j_l})\tilde{\varphi}$.

Furthermore, for all $l \in S \cup \{t\}$, $((z_{j_l r})\psi_{j_l})\varphi = x_{i_{k_l} p_{k_l} r}$ for some $k_l \in \mathbb{N}$ such that $i_{k_l} = j_l$. Hence it follows that $(u, v) \in (\bigcup_{m=0}^{\infty} M_i\psi_m) \times (\bigcup_{m=0}^{\infty} M_i\psi_m)$ if and only if $(u\varphi, v\varphi) \in (\bigcup_{n=0}^{\infty} X_{in}) \times (\bigcup_{n=0}^{\infty} X_{in})$. Thus $\tilde{\varphi}$ is an embedding of $\langle \bigcup_{l \in S \cup \{t\}} M_{j_l}\psi_{j_l}\rangle$ into $Q_\lambda(\langle M_i\rangle_{i<\lambda})$ as required. $\square$

**Theorem 9.47.** *If* $(\operatorname{Aut}(\langle U\rangle), \leq)$ *is dense then there exists an isomorphism*

$$\phi : (U, \leq, (E_i)_{i<\lambda}) \to \mathbb{Q}_\lambda(\langle M_i\rangle_{i<\lambda}),$$

*where* $(U, \leq, (E_i)_{i<\lambda})$ *is the* $\lambda$*-coloured total order defined on page 150.*

*Proof.* We will define the isomorphism $\phi : U \to V_{\mathbb{Q}_\lambda(\langle M_i\rangle_{i<\lambda})}$ as follows. Let $m = \min\{n \in \mathbb{N} : i_n = j_0\}$. Define $f_0 : M_{j_0}\psi_{j_0} \to X_{i_m 0}$ by $(z_{j_0 r}\psi_{j_0})\phi = x_{i_m 0 r}$ for all $r \in \mathbb{N}$. Then clearly $f_0$ is an injective map. Moreover, since $\psi_{j_0}$ is an automorphism $z_{j_0 r}\psi_{j_0} \leq z_{j_0 s}\psi_{j_0}$ if and only if $z_{j_0 r} \leq z_{j_0 s}$ and hence if and only if $x_{i_m 0 r} \leq x_{i_m 0 s}$. Also since $i_m = j_0$ it follows that $(C_{j_0} \cap ((M_{j_0}\psi_{j_0}) \times (M_{j_0}\psi_{j_0})))\,\phi \subseteq D_{j_0}$ Thus $f_0$ is an embedding $\langle M_{j_0}\psi_{j_0}\rangle \to \mathbb{Q}_\lambda(\langle M_i\rangle_{i<\lambda})$ and $(M_{j_0}\psi_{j_0})\phi = X_{i_m 0}$.

Now let $n \in \mathbb{N}$ and let $S \subseteq \mathbb{N}$ such that $|S| = n+1$. Suppose that $f_n$ has been defined such that $f_n : \left\langle \bigcup_{l \in S} M_{j_l}\psi_{j_l} \right\rangle \to \mathbb{Q}_\lambda(\langle X_i \rangle_{i<\lambda})$ is an embedding and such that for all $l \in \mathbb{N}$, $(M_{j_l}\psi_{j_l})f_n = X_{i_{k_l}p_{k_l}}$ for some $k_l \in \mathbb{N}$ where $i_{k_l} = j_l$. If $n$ is even let $m = \min\{k \in \mathbb{N} : X_{i_k p_k} \not\subseteq \operatorname{im} f_n\}$. By Lemma 9.46, $f_n$ can be extended to an embedding $f_{n+1}$ such that $X_{i_m p_m} \subseteq \operatorname{im} f_n$ and such that $X_{i_m p_m} = (M_{j_s}\phi_{j_s})f_{n+1}$ for some $j_s \in \mathbb{N}$. If $n$ is odd let $m = \min\{k \in N : M_{j_k}\psi_{j_k} \not\subseteq \operatorname{dom} f_n\}$. Then by Lemma 9.45 $f_n$ can be extended to an embedding $f_{n+1}$ such that $M_{j_m}\psi_{j_m} \in \operatorname{dom} f_{n+1}$ and such that $(M_{j_m}\psi_{j_m})f_{n+1} = X_{i_{k_m}p_{k_m}}$ for some $k_m \in \mathbb{N}$. Let

$$g = \bigcup_{n=0}^{\infty} f_n.$$

Then since each $f_{n+1}$ is an extension of $f_n$ it follows that $g$ is a well defined function. By alternately going back and forth we have ensured that $g$ is defined on every member of $U = \bigcup_{l \in \mathbb{N}} M_{j_l}\psi_{j_l}$ and that every member of $V_{\mathbb{Q}_\lambda(\langle M_i \rangle_{i<\lambda})} = \bigcup_{k \in \mathbb{N}} X_{i_k p_k}$ is in the image of $g$. In fact, since $f_n$ was an embedding at each stage and since $g$ is surjective, it follows that $g$ is an isomorphism of $\lambda$-coloured total orders. $\square$

The isomorphism from Theorem 9.47, will now allow us to produce a contradiction to $\mathrm{Aut}(\Omega)$ being a countable group.

**Theorem 9.48.** *Let* $\mathrm{Aut}(\Omega)$ *be countable. Suppose that* $f \in \mathrm{Aut}(\Omega)$ *and that* $U$ *is an infinite orbital of* $f$*. Then* $\mathrm{Aut}(\langle U \rangle) \cong \mathbb{Z}$.

*Proof.* By Lemma 9.25 $(\mathrm{Aut}(\langle U \rangle), \cdot, \leq)$ is an Archimedean group. Thus by Theorem 2.2, either $\mathrm{Aut}(\langle U \rangle) \cong \mathbb{Z}$ or $(\mathrm{Aut}(\langle U \rangle), \leq)$ is dense. But by Theorem 9.47, if $\mathrm{Aut}(\langle U \rangle)$ is dense, then $\mathrm{Aut}((U, \leq, (E_i)_{i<\lambda})) \cong \mathrm{Aut}(\mathbb{Q}_\lambda(\langle M_i \rangle_{i<\lambda})$, and hence by Lemma 9.44 and Corollary 9.36, there exists an an injective function from $\mathrm{Aut}(\mathbb{Q}_\lambda)$ into $\mathrm{Aut}(\langle U \rangle)$. But by Theorem 9.39, $\mathrm{Aut}(\mathbb{Q}_\lambda)$ has cardinality $2^{\aleph_0}$, a contradiction to Lemma 9.21 which states that $\mathrm{Aut}(\langle U \rangle))$ is countable. We thus conclude that $\mathrm{Aut}(\langle U \rangle) \cong \mathbb{Z}$ and the result follows. $\square$

## 9.7 Countable Groups which are the Automorphism Group of a Total Order

In this last section, we will finally show that if $\mathrm{Aut}(\Omega)$ is countable then it is isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$. We first prove the following theorem.

**Theorem 9.49.** *Let* $\Omega$ *be a total order and let* $\mathrm{Aut}(\Omega)$ *be countable. Then* $\mathrm{Aut}(\Omega)$ *is a countable abelian group.*

*Proof.* Let $f, g \in \mathrm{Aut}(\Omega)$. By Theorem 9.16, we know that for all infinite orbitals $U$ of $f$ and $T$ of $g$, either $T \cap U = \emptyset$ or $T = U$. Now, by Theorem 9.48 we also know that $\mathrm{Aut}(\langle U \rangle) \cong \mathbb{Z}$ for each $U$. Thus since $f|_U$ and $g|_U$ are both automorphisms of $U$ by Corollary 9.8, we find that for any $u \in U$, $ufg = ugf$. Thus $f$ and $g$ commute on any mutual orbital. Clearly if $v \in V_\Omega$ is contained in no infinite orbital of $f$ nor $g$ then $vfg = vg = vgf$. Furthermore, if $v$ is contained in an infinite orbital of $f$ but not of $g$, then $vgf = vf = vfg$ and similarly if $u$ is contained in an infinite orbital of $g$ but not of $f$ then $vfg = vg = vgf$. It now it follows that $f$ and $g$ commute at all points $u \in V_\Omega$ and hence $\mathrm{Aut}(\Omega)$ is a countable abelian group. □

**Lemma 9.50.** *Let $\Omega$ be a total order and let*

$$\mathcal{O} = \{U : U \textit{ is an infinite orbital of } f \textit{ for some } f \in \mathrm{Aut}(\Omega)\}.$$

*If* $\mathrm{Aut}(\Omega)$ *is countable then* $|\mathcal{O}| = n$ *for some* $n \in \mathbb{N}$.

*Proof.* For each $U \in \mathcal{O}$, let $g_U \in \mathrm{Aut}(\Omega)$ be chosen such that $U$ is an infinite orbital of $g_U$. Define a map $f : V_\Omega \to V_\Omega$ by,

$$vf = \begin{cases} vg_U & \text{if } v \in U \text{ for some } U \in \mathcal{O}, \\ v & \text{otherwise.} \end{cases}$$

By Theorem, 9.16, $U \cap T = \emptyset$ for all distinct orbitals $U$ and $T$ and so by Lemma 9.10, $f \in \mathrm{Aut}(\Omega)$. Thus $f$ is an automorphism of $\Omega$ whose infinite orbitals are exactly those in $\mathcal{O}$. Now by Lemma 9.15, $f$ has at most a finite number of distinct infinite orbitals. Thus $\mathcal{O}$ must be finite as required. □

We now have the main result.

**Theorem 9.51.** *Let $\Omega$ be a countable total order such that* $\mathrm{Aut}(\Omega)$ *is countable. Then* $\mathrm{Aut}(\Omega) \cong \mathbb{Z}^n$ *for some* $n \in \mathbb{N}$.

*Proof.* Again let

$$\mathcal{O} = \{U : U \text{ is an infinite orbital of } f \text{ for some } f \in \mathrm{Aut}(\Omega)\},$$

and let $|\mathcal{O}| = n$. By Lemma 9.50, $n \in \mathbb{N}$ and so let $\mathcal{O} = \{U_1, \ldots, U_n\}$. Since Theorem 9.48 tells us that $\mathrm{Aut}(\langle U_k \rangle) \cong \mathbb{Z}$ for all $k = 1, \ldots n$. So let $h_k \in \mathrm{Aut}(\langle U_k \rangle)$ be chosen such that $\langle h_k \rangle \cong \mathbb{Z}$. By Lemma 9.22, we can extend each $h_k$ to an automorphism $g_k \in \mathrm{Aut}(\Omega)$ such that $g_k|_{V_\Omega \setminus U_k} = \mathbf{1}|_{V_\Omega \setminus U_k}$. Now define a map $\phi : \mathbb{Z}^n \to \mathrm{Aut}(\Omega)$ by $(i_1, \ldots, i_n)\phi = g_1^{i_1} \cdots g_n^{i_n}$. Then $\phi$ is well defined and since we know from Theorem 9.16 that $\mathrm{Aut}(\Omega)$ is an abelian group, it follows that,

$$\begin{aligned}((i_1,\ldots,i_n)+(j_1,\ldots,j_n))\phi =&(i_1+j_1,\ldots,i_n+j_n)\phi,\\ =&g_1^{i_1+j_1}\cdots g_n^{i_n+j_n},\\ =&g_1^{i_1}\cdots g_n^{i_n}g_1^{j_1}\cdots g_n^{j_n},\\ =&(i_1,\ldots,i_n)\phi+(j_1,\ldots,j_n)\phi.\end{aligned}$$

Thus $\phi$ is a group homomorphism. To see that it is injective suppose that $(i_1,\ldots,i_n),(j_1,\ldots,j_n)\in\mathbb{Z}^n$ are such that $(i_1,\ldots,i_n)\phi=(j_1,\ldots,j_n)\phi$. Then $g_1^{i_1}\cdots g_n^{i_n}=g_1^{j_1}\cdots g_n^{j_n}$. Clearly since each $g_k$ is such that $g_k|_{V_\Omega\setminus U_k}=\mathbf{1}|_{V_\Omega\setminus U_k}$ and since $U_k\cap U_l=\emptyset$ for all $k\neq l$, it follows that $g_k^{i_k}=g_k^{j_k}$ for all $k=1,\ldots,n$. Hence $i_k=j_k$ for all $k=1,\ldots,n$ and thus $(i_1,\ldots,i_n)=(j_1,\ldots,j_n)$. Now suppose that $f\in\mathrm{Aut}(\Omega)$. Then by Corollary 9.8, $f|_{U_k}\in\mathrm{Aut}(\langle U_k\rangle)$ for all $k=1,\ldots,n$ and $f|_{V_\Omega\setminus\cup_{k=1}^n U_k}=\mathbf{1}$. Hence $f|_{U_k}=g_k^{i_k}$ for some $i_k\in\mathbb{Z}$ and $f=g_1^{i_1}\cdots g_n^{i_n}$. Then $(i_1,\ldots,i_n)\phi=f$ and so $\phi$ is surjective. Since we have now shown that $\phi$ is a bijective group homomorphism, the result is complete. □

**Corollary 9.52.** *Suppose $H$ is a countable group $\mathscr{H}$-class of* $\mathrm{End}(\mathbb{Q})$. *Then $H\cong\mathbb{Z}^n$ for some $n\in\mathbb{N}$.*

*Proof.* Let $f\in\mathrm{End}(\mathbb{Q})$ be the idempotent contained in $H$. Then, by Theorem 2.7, $H\cong\mathrm{Aut}(\mathrm{im}\,f)$. Thus if $H$ is countable, $\mathrm{Aut}(\mathrm{im}\,f)$ countable. Hence by Theorem 9.51, $\mathrm{Aut}(\mathrm{im}\,f)\cong\mathbb{Z}^n$ for some $n\in\mathbb{N}$ and the result now follows. □

# Chapter 10

# Questions and Open Problems

In this final chapter we will discuss some questions and possible directions for further research which arise from the results presented in this thesis.

Most obviously, this thesis has dealt with only a handful of the most common Fraïssé limits. Given time, we could also ask the same type of questions about the maximal subgroups, regular $\mathscr{D}$-classes and $\mathscr{J}$-classes of other Fraïssé limits. For example one could consider:

> *The random poset* – the Fraïssé limit of the class of all finite partial orders.
>
> *The random n-independence free directed graph* – the Fraïssé limit of the class of directed graphs with no independent sets of size $n$.
>
> *The ordered Urysohn space* – the Fraïssé limit of the class of finite ordered metric spaces with rational distances.

Focussing on the Fraïssé limits that are covered in this thesis, there are many additional questions we could ask about Green's relations on $\text{End}(\Omega)$ where $\Omega = R, D, T, B, G_n, \mathbb{Q}$. For example, we produced many results about the regular $\mathscr{D}$-classes in each setting, but we might naturally ask the following questions.

**Question 10.1.** How many non-regular $\mathscr{D}$-classes of $\text{End}(\Omega)$ are there? What sizes are they?

**Question 10.2.** Can we gain any information on the number of $\mathscr{H}$-classes contained in non-regular $\mathscr{D}$-classes of $\text{End}(\Omega)$?

The primary focus on this thesis was on group $\mathscr{H}$-classes. However might also want to investigate the $\mathscr{H}$-classes which do not contain an idempotent and are therefore not groups. In a natural way we can associate a group to such an $\mathscr{H}$-class as follows.

Let $H$ be a $\mathscr{H}$-class of a semigroup $S$ and let $T_H = \{s \in S^1 : Hs \subseteq H\}$. Then for each $s \in T_H$ we can define a function $f_s : H \to H$, where $hf_s = hs$ for all $h \in H$. It is not hard to see that the set $\{f_s : s \in T_H\}$ forms a group under composition of mappings. In fact this group is known as the *Schutzenberger group* of the $\mathscr{H}$-class $H$. It can be shown that if $K$ is a maximal group $\mathscr{H}$-class then the Schutzenberger group of $K$ is isomorphic to $K$ itself. If we let $S = \mathrm{End}(\Omega)$ for some relational structure $\Omega$ then it can be shown that for any $\mathscr{D}$-class $D$, all Schutzenberger groups associated to an $\mathscr{H}$-class of $D$ are isomorphic and are isomorphic to the group $\mathscr{H}$-classes in $D$ (see [Mag75, Theorem 3.1]). Furthermore, if $K$ is any $\mathscr{H}$-class of $\mathrm{End}(\Omega)$ and $k \in K$, then the Schutzenberger groups associated to $K$ is isomorphic to a subgroup of $\mathrm{Aut}(\mathrm{im}\, k)$, [Mag75, Theorem 3.2]. We might now ask the following question.

**Question 10.3.** Which groups arise as Schutzenberger groups associated to $\mathscr{H}$-classes from non regular $\mathscr{D}$-classes of $\mathrm{End}(\Omega)$?

In Chapter 6, we briefly discussed triangle free graphs which have property $\star$ (recall Definition 6.9). We were able to classify the finite triangle free graphs with property $\star$ which have exactly two maximal independent sets. As a result we provided a complete description of the groups which occur as the automorphism group of such finite triangle free graphs. However, as already mentioned in that chapter, the following is still an open problem.

**Question 10.4.** Which groups can occur as the automorphism group of a finite triangle-free graph with property $\star$ which has three or more maximal independent sets?

Similarly, we also showed in Chapter 6 that the automorphism group of a countably infinite triangle-free graph with property $\star$ which has finitely many vertices of infinite degree has cardinality $2^{\aleph_0}$. A natural open problem which the arose was the following.

**Question 10.5.** What is the cardinality of the automorphism group of a countably infinite triangle-free graph with property $\star$ which has infinitely many vertices of infinite degree?

In Chapter 8 we were able to show that if a total order $\Omega$ can be embedded into $\mathbb{Q}$ *via* an embedding $f$ such that $\mathrm{im}\, f$ was a retract of $\mathbb{Q}$, then $\mathrm{Aut}(\Omega)$ was isomorphic to $2^{\aleph_0}$ maximal subgroups of $\mathrm{End}(\mathbb{Q})$. The following still remains an open question.

**Question 10.6.** Exactly which total orders $\Omega$ can be embedded into $\mathbb{Q}$ *via* an embedding $f$ such that $\operatorname{im} f$ is a retract of $\mathbb{Q}$? Can we find an example of a total which cannot be embedded into $\mathbb{Q}$ *via* an embedding $f$ such that $\operatorname{im} f$ is a retract of $\mathbb{Q}$?

# Bibliography


[AG07] Geir Agnarsson and Raymond Greenlaw, *Graph theory: modelling, applications and algorithms*, Pearson, 2007.

[BD00] Anthony Bonato and Dejan Delić, *The monoid of the random graph*, Semigroup Forum **61** (2000), 138–148.

[BDD10] Anthony Bonato, Dejan Delić, and Igor Dolinka, *All countable monoids embed into the monoid of the infinite random graph*, Discrete Mathematics **310** (2010), 373–375.

[Cam97] Peter Cameron, *The random graph*, The Mathematics of Paul Erdős II, Algorithms and Combinatorics, vol. 14, pp. 333–351, Springer-Verlag, 1997.

[Cam01] ______, *Oligomorphic permutation groups*, London Mathematical Lecure Note Series 152, Cambridge University Press, 2001.

[Cam08] ______, *Sets, logic and categories*, Springer Undergraduate Mathematics Series, Springer, London, 2008.

[CP61] Alfred H. Clifford and Gordon B. Preston, *The algebraic theory of semigroups (Vol. 1)*, American Mathematical Society, 1961.

[Dar97] Michael R. Darnel, *Theory of lattice ordered groups*, Marcel Dekker Inc., 1997.

[DD04] Dejan Delić and Igor Dolinka, *The endomorphism monoid of the random graph has uncountably many ideals*, Semigroup Forum **69** (2004), 75–79.

[Dol07] Igor Dolinka, *The endomorphism monoid of the random poset contains all countable semigroups*, Algebra Universalis **56** (2007), 469–474.

[Dol12] ———, *A characterization of retracts in certain Fraïssé limits*, Mathematical Logic Quarterly **58** (2012), 46–54.

[EKR76] Paul Erdős, Daniel J. Kleitman, and Bruce L. Rothschild, *Asymptotic enumeration of $K_n$-free graphs*, International Colloquium on Combinatorial Theory, Atti dei Convegni Lincei, vol. 2, pp. 19–27, 1976.

[ER63] Paul Erdős and Alfréd Rényi, *Asymmetric graphs*, Acta Mathematica Academiae Scientiarum Hungaricae **14** (1963), 295–315.

[Fra53] Roland Fraïssé, *Sur certains relations qui généralisent l'ordre des nombres rationnels*, Comptes Rendus de l'Académie des Sciences Paris **237** (1953), 540–542.

[Fru39] Robert Frucht, *Herstellung von graphen mit vorgegebener abstrakter gruppe*, Compositio Mathematica **6** (1939), 239–250.

[Goo86] Ken R. Goodearl, *Partially ordered abelian groups with interpolation*, Mathematical Surveys and Monographs 20, American Mathematical Society, 1986.

[Gro59] Johannes de Groot, *Groups represented by homeomorphism groups*, Mathematische Annalen **138** (1959), 80–120.

[Hen71] C. Ward Henson, *A family of countable homogeneous graphs*, Pacific Journal of Mathematics **38** (1971), no. 1, 69–83.

[Hod97] Wilfrid Hodges, *A shorter model theory*, Cambridge University Press, 1997.

[How95] John Howie, *Fundamentals of semigroup theory*, London Mathematical Society Monographs, Clarendon Press, 1995.

[Mag75] K. D. Magill, *A survey of semigroups of continuous selfmaps*, Semigroup Forum **11** (1975), no. 1, 189–282.

[MS74] K. D. Magill and S. Subbiah, *Green's relations for regular elements of semigroups of endomorphisms*, Canadian Journal of Mathematics **26** (1974), 1484–1497.

[MT11] Dugald Macpherson and Katrin Tent, *Simplicity of some automorphism groups*, Journal of Algebra **342** (2011), 40–52.

[Mud10] Nebojša Mudrinski, *Notes on endomorphisms of Henson graphs and their complements*, Ars Combinatoria **96** (2010), 173–183.

[PV10] F. Petrov and A. Vershik, *Uncountable graphs and invariant measures on the set of universal countable graphs*, Random Structures and Algorithms **37** (2010), no. 3, 389–406.

[Rad64] Richard Rado, *Universal graphs and universal functions*, Acta Arithmetica **9** (1964), 331–340.

[Ros99] Kenneth H. Rosen, *Handbook of discrete and combinatorial mathematics*, second ed., CRC Press, Florida, 1999.

[RS09] John Rhodes and Benjamin Steinberg, *The q-theory of finite semigroups*, Springer Monographs in Mathematics, Springer, New York, 2009.

[Sab60] Gert Sabidussi, *Graphs with given infinite group*, Monatshefte fur Mathematik **64** (1960), 64–67.

[Tru85] John Truss, *The group of the countable universal graph*, Mathematical Proceedings of the Cambridge Philosophical Society **98** (1985), 213–245.