Substitution-closed pattern classes

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Abstract

The substitution closure of a pattern class is the class of all permutations obtained by repeated substitution. The principal pattern classes (those defined by a single restriction) whose substitution closure can be defined by a finite number of restrictions are classified by listing them as a set of explicit families.

1 Introduction

A pattern class of permutations is a set of permutations closed under taking subpermutations. We recall that a permutation α is a *subpermutation* of a permutation β if β contains a subsequence which is isomorphic to α (its terms are ordered relatively the same as those of α). It is convenient to have a phrase for the opposite condition: β *avoids* α if α is not a subpermutation of β. Similarly, it is convenient to refer to permutations β which have α as a subpermutation as *extensions* of α.

A very common way of specifying a pattern class X is to give it as the set of permutations that avoid a given set B, in which case we write $X = \mathrm{Av}(B)$. Indeed, every pattern class X may be defined in this way and if the avoided set B is taken to be minimal it is called the *basis* of X. Not every pattern class has a finite basis and distinguishing the finitely based classes from the infinitely based classes is one of the most common pattern class questions.

As a classical example of these notions we have the class S of all stack-sortable permutations. This class may be defined as $S = \mathrm{Av}(231)$, an example of a

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pattern class whose basis is a single permutation.

In this paper we consider pattern classes with an additional property – being closed under substitution, defined as follows. Suppose that $\sigma, \alpha_1, \ldots, \alpha_n$ are permutations where $n = |\sigma|$. Then $\sigma[\alpha_1, \ldots, \alpha_n]$ denotes the permutation in which the ith term of σ is substituted by α_i. In other words $\sigma[\alpha_1, \ldots, \alpha_n]$ consists of n consecutive subsequences isomorphic to $\alpha_1, \ldots, \alpha_n$ whose relative order is the same as the relative order of the terms of σ. For example

$$231[21, 321, 12] = 4376512$$

A pattern class X is said to be *substitution-closed* or *wreath-closed* if, whenever $\sigma, \alpha_1, \ldots, \alpha_n \in X$ with $n = |\sigma|$, we have $\sigma[\alpha_1, \ldots, \alpha_n] \in X$. It is evident that the intersection of substitution-closed pattern classes is also substitution-closed and therefore every pattern class is contained in a smallest substitution-closed class called its *substitution closure*. The substitution closure of a pattern class X can equivalently be defined as the set of all permutations obtained from X by repeated substitution operations. For example, the substitution closure of $S = \mathrm{Av}(231)$ is the class $\mathrm{Av}(2413, 3142)$ of all separable permutations.

Substitution-closed classes were first defined in [1] where it was shown that their structure can be defined in terms of simple permutations. We recall the basic concepts. An *interval* of a permutation θ is a (non-empty) subsequence of consecutive terms of θ whose values form a set of consecutive integers. Thus, in 2645173 the subsequence 645 is an interval. Every permutation has itself and singletons as intervals; other intervals are said to be *proper*. A permutation with no proper intervals is said to be *simple*. The connection with substitution-closed classes is provided by the following result from [1]:

Proposition 1.1 *A pattern class is substitution-closed if and only if its basis consists of simple permutations.*

Simple permutations and substitution-closed classes have been much studied. It was shown in [1] that the enumeration of a substitution-closed class is determined by the enumeration of its simple permutations; and if the substitution-closed class has only finitely many simple permutations then it and every subclass has an algebraic generating function. Very recently, simple permutations have been used to answer a wide range of questions about pattern classes [4, 5, 6, 7]. The number of simple permutations of length n is approximately $n!/e^2$ and more exact asymptotics are given in [2].

This paper is about the substitution closure of pattern classes of the form $\mathrm{Av}(\psi)$ and particularly about whether these substitution closures are finitely based. If the substitution closure of $\mathrm{Av}(\psi)$ is finitely based we shall say that ψ has *finite type*, otherwise that ψ has *infinite type*. If two permutations α and β are equivalent up to symmetry, then α and β have the same type. As we shall see, simple permutations are the key to determining the type of ψ.

An extension ξ of a permutation ψ is a *minimal simple extension* of ψ if

1. ξ is simple, and
2. among all simple extensions of ψ, ξ is minimal under the subpermutation order.

In other words, if any set of terms of ξ is removed the resulting permutation either avoids ψ or is not simple. The following result is crucial for the rest of the paper.

Proposition 1.2 *The basis of the substitution closure of $Av(\psi)$ is the set of minimal simple extensions of ψ.*

Proof: Let $X = \mathrm{Av}(\psi)$ and let Y be its substitution closure. Since permutations in Y arise from permutations in X by iterated substitution, X and Y contain the same set of simple permutations. Therefore, the set of minimal simple permutations not in X is the same as the set of minimal simple permutations not in Y. But then Proposition 1.1 and the fact that permutations not in X are precisely the extensions of ψ completes the proof. ■

We now recall a few other definitions that will be used frequently in this paper. A permutation is *plus-decomposable* if it can be expressed as $12[\alpha_1, \alpha_2]$ (which we also write as $\alpha_1 \oplus \alpha_2$) and *minus-decomposable* if it can be expressed as $21[\alpha_1, \alpha_2]$ (written also as $\alpha_1 \ominus \alpha_2$). A *decomposable* permutation is one that is either plus-decomposable or minus-decomposable. Naturally, permutations that are not (respectively) plus-decomposable, minus-decomposable, or decomposable are called plus-indecomposable, minus-indecomposable, or indecomposable. These notions feature in the following result from [1].

Proposition 1.3 *Let ψ be any permutation. Then there is a unique simple permutation σ together with permutations $\alpha_1, \alpha_2, \ldots, \alpha_n$ (which correspond to intervals of ψ) such that*

$$\psi = \sigma[\alpha_1, \alpha_2, \ldots, \alpha_n]$$

If $\sigma \neq 12, 21$, then $\alpha_1, \alpha_2, \ldots, \alpha_n$ are also uniquely determined by ψ. If $\sigma = 12$ or 21, then α_1, α_2 are unique so long as we require that α_1 is plus-indecomposable or minus-indecomposable respectively.

The simple permutation σ appearing in this result is called the *skeleton* of ψ.

Our main objective is to explicitly determine the permutations of finite type. Similar notions have recently been studied in graph theory [3, 8, 9, 10, 12, 13, 14, 15]. For graphs there are natural analogues of the notions of 'basis', 'interval' and 'simple' and the substitution closure has been a tool in graph theory since Lovász's work [11] on perfect graphs. In the last decade much work has been done on determining the graphs of finite type and this recently culminated in a complete characterisation [10]. There is a connection between permutations and graphs in that every permutation determines a graph with the set of permuted points as vertices and the inversions as edges; then, in the terminology of [8, 9, 10], intervals correspond to modules and simple permutations to prime graphs. However two different permutations (such as 4123 and 2341) can give rise to the same graph (a star graph on 4 vertices in this case, sometimes called a 3-claw) and so the connection is quite weak. In particular the finite type property is not preserved (for example, 4123 and 2341 both have infinite type whereas the 3-claw has finite type [3]).

Our paper is organised as follows. The next section sets up some terminology, reviews some basic results from [5], and refines some others. Then, in Section 3, we discuss ways in which permutations can be extended to simple permutations. Section 4 gives conditions in terms of the decomposition in Proposition 1.3 to guarantee that a permutation has finite type and Section 5 treats permutations of infinite type. Taken together these results give an explicit description of all permutations of finite type and make it readily decidable to find the type of any given permutation. In Section 6 we give first a table of the low degree finite type permutations ψ together with the basis of the substitution closure of $\mathrm{Av}(\psi)$; then we give tables that summarise our characterisation according to the different types of skeleton. Finally we make some remarks about the substitution closure of pattern classes with more than one basis element.

2 Permutation diagrams and pin sequences

We shall often find it convenient to describe a permutation $\sigma = (s_1, s_2, \ldots, s_n)$ by its *diagram* – the set of points (i, s_i) plotted in the (x, y)-plane. In such plots $\mathrm{Rect}(A)$ denotes the axes-parallel rectangular region that bounds the points of a subset A of σ. A point that lies outside $\mathrm{Rect}(A)$ but between either its first and last point or between its greatest and smallest point is said to *cut* A. For example, if $\sigma = 317296485$ and A is the subset of image points $\{1, 7, 6, 4\}$ then σ together with $\mathrm{Rect}(A)$ and its cut points (shown as stars) are displayed in Figure 1.

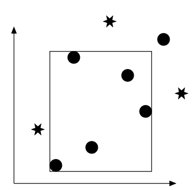

Figure 1: The permutation 317296485 and $\mathrm{Rect}(1, 7, 6, 4)$ with cut points

Notice that the rectangle of an interval is a square and no points outside the square cut it either by position or by value.

If p_1, p_2 are two points of a permutation then a *proper pin sequence* from $\{p_1, p_2\}$ is a sequence of points $p_1, p_2, p_3, \ldots$ such that, for each $i \geq 3$,

1. p_{i+1} lies outside $\mathrm{Rect}(p_1, p_2, \ldots, p_i)$,
2. p_{i+1} cuts $\mathrm{Rect}(p_1, p_2, \ldots, p_i)$ either to the left (L), right (R), below (B) or above (A) it,
3. p_{i+1} separates p_i from $\mathrm{Rect}(p_1, p_2, \ldots, p_{i-1})$ by lying in position or in value between p_i and $\mathrm{Rect}(p_1, p_2, \ldots, p_{i-1})$.

The letters L, R, B, A of condition 2 may be used to describe the points $p_3, p_4, \ldots$ and occasionally we shall describe proper pin sequences by words in these letters. Notice that, in these words, letters from $\{L, R\}$ alternate with letters from $\{A, B\}$. Proper pin sequences were first defined in [5]. There conditions 1, 2 were the defining properties of a pin sequence and the additional condition 3 defined a proper pin sequence; however, in this paper, all our pin sequences will be proper and so we shall simply refer to them as pin sequences. They are usually drawn with horizontal and vertical guide lines displaying property 3 (see, for example, Figure 2). Two properties of pin sequences that we shall require from [5] (Proposition 3.2 and Lemma 3.6) are:

Proposition 2.1 *If $P = p_1, p_2, \ldots, p_m$ with $m \geq 5$ is a pin sequence in a permutation σ then the only subsets of P that can be proper intervals are $\{p_1, p_m\}$, $\{p_2, p_m\}$, $\{p_1, p_3, \ldots, p_m\}$, and $\{p_2, p_3, \ldots, p_m\}$.*

Proposition 2.2 *If p_1, p_2 are points of a simple permutation σ, with p_1 not the last point of σ, then there is a pin sequence $P = p_1, p_2, \ldots, p_m$ whose final point is the last point of σ (a* right-reaching *pin sequence). Analogous statements hold for left-, top- and bottom-reaching pin sequences.*

In addition we shall need the following further properties of pin sequences:

Lemma 2.3 *Let α be any permutation except for $1, 12, 21, 132, 213, 231, 312$. Then there are arbitrarily long pin sequences not containing α as a subsequence.*

Proof: Consider the pin sequences in Figure 2. The left hand pin sequence P_1 avoids both 321 and 3412 so its reverse avoids 123 and 2143. So, if α contains any of $123, 321, 3412, 2143$, then P_1 or its reverse will avoid α. On the other hand, if α avoids all of $123, 321, 3412, 2143$ then α is easily seen to be one of the permutations in the statement of the lemma or it is 2413 or 3142. However, the right hand pin sequence avoids 2413, and its reverse avoids 3142. ■

Lemma 2.4 *Let $P = p_1, p_2, \ldots$ be any pin sequence of length 5 or more and let $p_a, p_{a+1}, p_{a+2}, p_{a+3}, p_{a+4}$ be five consecutive points of P. Let θ be any of $12, 21, 132, 213, 231, 312$. Then P has a subset Q which is a pin sequence in its own right with the following properties:*

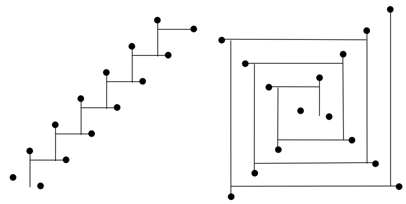

Figure 2: Two pin sequences; the corresponding words are $(AR)^6$ and $(ALBR)^3A$

1. $Q \subseteq \{p_a, p_{a+1}, \ldots\}$,
2. *Regarded as a pin sequence Q ends with the points $p_{a+5}, p_{a+6}, \ldots$,*
3. $Q \cap \{p_a, p_{a+1}, p_{a+2}, p_{a+3}, p_{a+4}\}$ *contains a subset θ' isomorphic to θ,*
4. *the unique two points of θ' that form an interval of θ' are consecutive points of the pin sequence Q.*

Proof: We consider the case $\theta = 231$; the other permutations are similar. As described above we represent each p_i by a directional letter L, R, A, B according to its placement with respect to $\mathrm{Rect}(p_1, \ldots, p_{i-1})$. Then the 5 point sequence $p_a, p_{a+1}, p_{a+2}, p_{a+3}, p_{a+4}$ can be described by its initial points p_a, p_{a+1} followed by three directional letters. These three directional letters either begin or end with AL, BL, LA, or RA, or they are one of ARB, BRB, LBR or RBR. The pin sequence fragments corresponding to these are shown in Figure 3. In each diagram the arrows indicate the order of points in P, the points drawn as squares display the pattern 231, the dotted line shows the two possible directions in which the pin sequence P and Q can continue, and the initial points of Q are explicitly labelled. In every case we have a pin sequence Q with the desired properties. ■

3 Hook points and unique embeddings

We have seen in Proposition 1.2 that the basis for the substitution closure of a pattern class $\mathrm{Av}(\psi)$ is closely related to the simple extensions of ψ. A helpful technique for constructing such extensions is to add hook points to a permutation. Suppose that we want to construct a simple extension of some permutation ψ. We consider its minimal non-singleton intervals and for each of them we define a new point outside $\mathrm{Rect}(\psi)$ that cuts the interval either by position or by value; such a point is called a *hook point* of the interval. Hook points that cut an interval to its left or right are called H-hook points while those that cut it from above or below are called V-hook points. Usually we shall

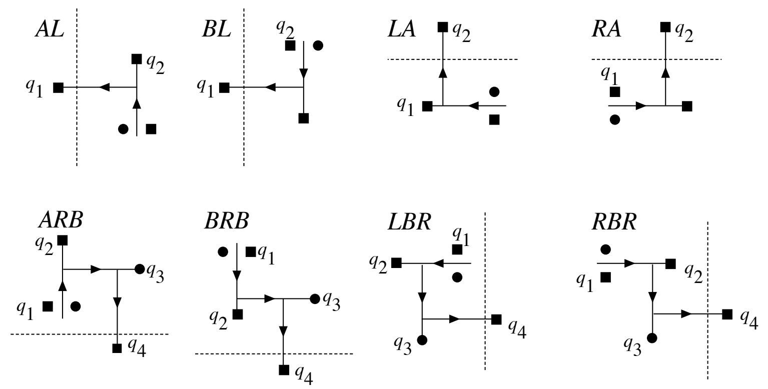

Figure 3: Pin sequence fragments containing 231

take the hook points in each direction out of ψ to be monotonically increasing or decreasing (see Figure 4). By a judicious choice of the directions we can usually ensure that the new permutation will be simple. In diagrams of permutations we shall, for clarity, connect each H-hook point (respectively, V-hook point) to a horizontal (respectively, vertical) line pointing to the interval it cuts.

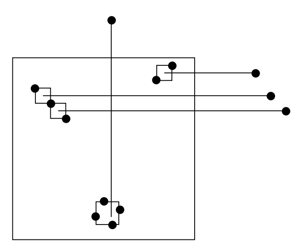

Figure 4: 4 minimal intervals of a permutation extended by 4 hook points

Ultimately, however, we are interested in *minimal* simple extensions ξ of ψ and to verify the minimality property it is useful to have an additional property: that ξ contains a *unique* subpermutation isomorphic to ψ. In Proposition 3.1 we show that such extensions exist while Lemma 3.2 prepares the way for constructing infinitely many such extensions (when this is possible – see Section 5).

Proposition 3.1 *Let α be any permutation. Then it has a minimal simple extension χ that contains no other copy of α. Furthermore*

1. *For every point $h \in \chi - \alpha$ there is an interval of α that is cut only by h,*
2. *If $|\alpha| > 2$, χ may be written as a juxtaposition $\lambda\mu$ where neither λ or μ is empty and where every interval of α is contained within λ or μ.*

Proof: The proof divides into two main cases depending on whether the skeleton of α is of length 2 or of length at least 4. We begin with the former case and, by symmetry, we may assume that α is minus-decomposable.

Figure 5 illustrates simple extensions χ of such a minus-decomposable permutation α. The centre diagram shows the case that α is decreasing while the right hand diagram shows the case that α has two increasing components. In these diagrams the points of α itself are depicted by squares whilst the hook points are depicted by circles. In these two cases we leave the (elementary) check to the reader that the displayed permutations are minimal simple extensions of α containing no other subpermutation isomorphic to α, and that the auxiliary conditions hold. All remaining cases, when α is minus-decomposable, are handled by the left hand diagram of Figure 5. In that diagram we have $\alpha = \theta_1 \ominus \cdots \ominus \theta_n$ where the blocks $\theta_1, \ldots, \theta_n$ are of two types: each block is either decreasing or it is minus indecomposable; and no two decreasing blocks are consecutive. The minimal intervals of these blocks are cut by hook points as shown. The H-hook points are in a monotone order different from the order of the last two points of α and the V-hook points are in a monotone order different from the order of the top two points of α.

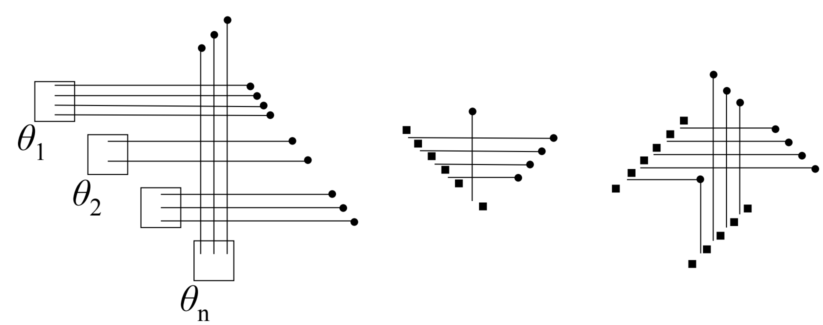

Figure 5: Simple extensions of a minus-decomposable permutation α

The permutation χ is simple because the hook points have cut all the intervals within blocks, there are no intervals across two blocks, and no intervals containing hook points exist. Furthermore, as every point of $\chi - \alpha$ is a hook point cutting a unique interval the first auxiliary condition holds. The second auxiliary condition holds because we may take $\lambda = \theta_1 \cdots \theta_{n-1}$. The next task is to show that χ contains no other copy of α.

Suppose that χ contains another subpermutation α' isomorphic to α. Then α' can contain only one H-hook point; for if it contained two, the final two points of α' would be H-hook points, and would be ordered incorrectly. Similarly, α' can contain at most one V-hook point.

If α' contains an H-hook point h but not a V-hook point then h is its final point. Consider the sequence $\beta = \alpha h = b_1 b_2 \cdots b_p$ which has the property that, for some b_i (the unique point not in α'), $\beta - b_i = \alpha'$ is isomorphic to $\beta - b_p = \alpha$. In particular, the rank of b_p (the number of terms less than or equal to b_p) within $\beta - b_i$ is equal to the rank of b_{p-1} within $\beta - b_p$; let this common rank be called r. Therefore the rank of b_p within β itself is either r or $r+1$ and the same is true for the rank of b_{p-1} within β. These ranks are obviously not equal but nor can they differ by one since $b_p = h$ is a hook point of a block of α different from the one that contains the final point b_{p-1} of α.

If α' contains a V-hook point v but not an H-hook point then v is its largest point, and therefore lies in the first block θ'_1 of α'. On the other hand, v together with the points of α to the right of v number at most k where $k = |\theta_n|$. These must all lie in the last block θ'_n of α' and so v is simultaneously in θ'_1 and θ'_n. Hence α and α' have one block only. This single block is either minus indecomposable (a contradiction) or is decreasing which is a case that we left to the reader (centre diagram of Figure 5).

Suppose now that α' contains both an H-hook point h and a V-hook point v. The points of α' between v and h by position all lie (except for v, h themselves) within θ_n and so including v and h, there are at most $|\theta_n| + 1$ of these points since the first point of θ_n precedes v. Hence all of these points except possibly v must lie in θ'_n, since they are rightmost points of α'. If $v \in \theta'_n$, then θ'_n is the top block of α' and so α and α' have one block only; then we can argue as in the previous paragraph. The only alternative is that v is not contained in θ'_n and is therefore forced by position to lie in θ'_{n-1}. This would mean that α has two blocks only. Moreover θ'_1 would end with v, its maximum, and θ'_2 would end with h, also its maximum. Furthermore $\theta'_1 \subset \theta_1 v$ and $\theta_1 v - \theta'_1$ contains only the smallest element t_1 of θ_1 ($t_1 \notin \theta'_1$ since $t_1 < h$). Thus, as θ_1 and θ'_1 are isomorphic, they must be increasing; likewise θ_2 is increasing. This is the other case that we left to the reader (right-hand diagram of Figure 5).

There remains the case that $\alpha = \sigma[\theta_1, \ldots, \theta_n]$ where the skeleton σ has size 4 or more, in which case (because it is simple) it does not end in its largest element. Consider the extension χ of α shown in Figure 6. In this figure the intervals θ_i of α are shown as squares, and the minimal intervals of the θ_i are cut by hook points as shown. We use H-hook points for all intervals except the last, and V-hook points for the last. The ordering of the hook points is monotone in each case; the H-hook point monotone order is the opposite of the ordering of the final two points of α and the V-hook point monotone order is opposite to the ordering of the top two points of α. This is illustrated in the first diagram of Figure 6 where it is implicitly assumed that the final two points of α, and its top two points, are decreasing. There is one exception to the direction in which the V-hook points are positioned: when θ_{n-1} is the top interval of α. In that case, we will instead take the V-hook points to cut θ_n from below, with their monotone order being different from the order in which the *bottom* two points of α are ordered. This is illustrated in the second diagram of Figure 6.

It is easy to see that the resulting permutation χ is simple. Moreover the auxiliary conditions hold. For the first one, notice that every point of $\chi - \alpha$ is a hook point and the interval it cuts is cut by no other point of χ. For the second condition, inspection of the diagrams in Figure 6 shows that the sequences λ and μ may be chosen in many ways. Therefore only the unique embedding of α in χ needs to be verified. Suppose χ contained another copy $\alpha' = \sigma[\theta'_1, \theta'_2, \ldots, \theta'_n]$ of α. As in the first case, α' can contain at most one H-hook point and at most one V-hook point.

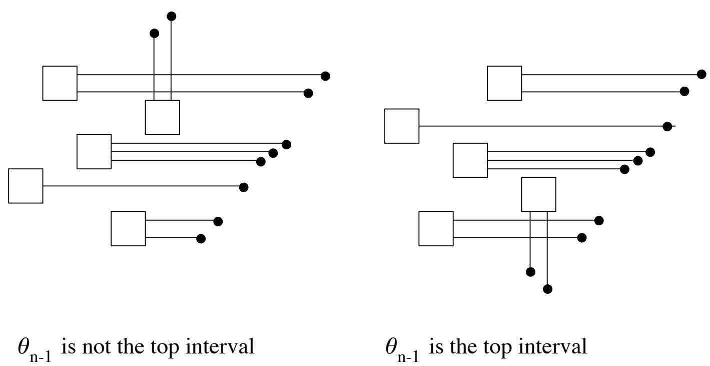

Figure 6: Simple extension of $\sigma[\theta_1, \ldots, \theta_n]$

Consider first the case that α' contains both a V-hook point v and an H-hook point h. By exactly the same argument as in the first case we have $v \in \theta'_n$ or $v \in \theta'_{n-1}$. But, if $v \in \theta'_n$, then θ'_n is an interval of α' that contains both its rightmost point (namely h) and v which is either its top or bottom point; hence θ'_n is the rightmost interval of α' and also either its topmost or bottommost interval; but this contradicts that σ is simple. On the other hand if $v \in \theta'_{n-1}$ then θ'_{n-1} is either the highest or lowest interval of α' and the V-hook points were chosen to extend downwards or upwards respectively in these two cases, giving again a contrdiction.

If α' contains an H-hook point h but not a V-hook point then we can derive a contradiction by exactly the same argument as in the case that the skeleton of α is of length 2. The case that α' contains a V-hook point but not an H-hook point is treated analogously.

In all cases the constructions clearly produce minimal simple extensions of α because any proper subpermutation β that extended α would fail to contain some hook point and one of the intervals of α would then be an interval of β.

■

Lemma 3.2 *Let α be the permutation 123, or 321, or any permutation of length at least 4. Then there are infinitely many permutations $\hat{\alpha}$ which can be expressed as a union $\alpha^* \cup P$, where:*

1. *α^* is a fixed minimal simple extension of α and all points of $\alpha^* - \alpha$ cut α,*
2. *P is a pin sequence,*
3. *Only the first point of P cuts α^* and, except when $\alpha = 2413$ or 3142, all others lie above α^*,*
4. *The final point of P is an A-pin point and hence the largest point (by value) of $\hat{\alpha}$,*
5. *$\hat{\alpha}$ is simple,*
6. *$\hat{\alpha}$ contains a unique copy of α, and*
7. *If $\alpha \preceq \beta \preceq \hat{\alpha}$ and $\beta \neq \hat{\alpha}$ then β has an interval M such that $|M| > 1$ and such that M does not contain the final point of P.*

Proof: Suppose first that $\alpha = a_1 \cdots a_m$ is neither of the permutations 2413 or 3142. By applying the reverse symmetry if necessary we may assume that $a_1 > a_m$. Then we define a minimal simple extension α^* of α having the properties guaranteed by Proposition 3.1. Next we define $\hat{\alpha}$ by the diagram in Figure 7 using a pin sequence that ascends to the right. In this diagram the first pin point is between the sequences λ and μ (where $\alpha^* = \lambda\mu$ as in Proposition 3.1).

Property 1 follows from Proposition 3.1 and properties 3, 4 and 5 follow by inspection. Property 6, the fact that α is uniquely embedded in $\hat{\alpha}$, follows because it is embedded once only in α^*, not at all in P (because of the conditions on α), and not with points in both α^* and P because $a_1 > a_m$.

Finally we check property 7. Suppose that $\alpha \preceq \beta \preceq \hat{\alpha}$ and $\beta \neq \hat{\alpha}$. Let $t \in \hat{\alpha} - \beta$. If $t \in \alpha^*$ then there is a minimal interval M of α that is cut by no point of α^* except for t, nor by any point of P (from Proposition 3.1). Therefore M is also an interval of β and M does not contain the final point of P confirming the property in this case. We may therefore assume that $\alpha^* \preceq \beta$, and we may take t to be the first point of P not in β. Then, by inspection, α^* together with all points of P that precede t forms an interval with the desired properties.

In the cases that $\alpha = 2413$ and 3142, we let $\alpha^* = \alpha$ and we define $\hat{\alpha}$ as in Figure 8. The required properties all follow by inspection. ■

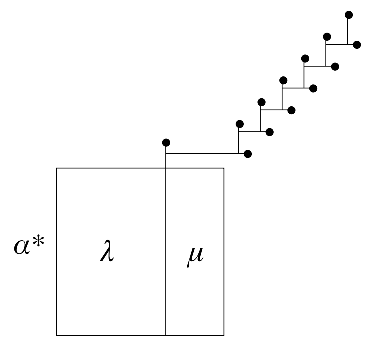

Figure 7: The permutation $\hat{\alpha}$ when $\alpha \neq 2413, 3142$ and $a_1 > a_m$

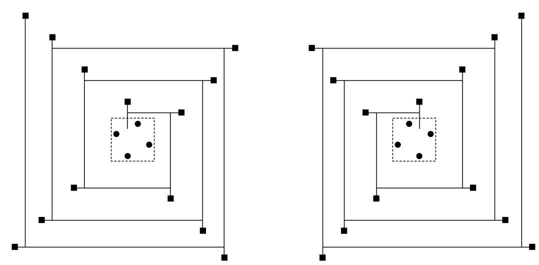

Figure 8: The permutations $\hat{\alpha}$ for $\alpha = 3142$ and $\alpha = 2413$

4 Finite types

In this section we describe permutations of finite type. We will divide these permutations into three families: 'generic' indecomposable permutations, 'generic' decomposable permutations, and the 'sporadic' family of spiral permutations defined in Subsection 4.3 below.

4.1 Indecomposable permutations of finite type

Theorem 4.1 *Suppose that the permutation ψ has the form $\sigma[\alpha_1, \ldots, \alpha_n]$ where σ is the skeleton of ψ and $|\sigma| = n \geq 4$. If every interval α_i lies in the set $\{1, 12, 21, 132, 213, 231, 312\}$ then ψ has finite type.*

Proof: We shall prove that every minimal simple extension of ψ has length at most $7n$. Let ξ be an arbitrary minimal simple extension of ψ.

There may be several embeddings of ψ in ξ and, in due course, we shall choose one to minimise a certain parameter. For now let $\psi = \sigma[\alpha_1, \alpha_2, \ldots, \alpha_n]$ be any of these embeddings (where, with the usual abuse of notation, we use α_i to mean the corresponding subsequence of ξ also).

If $|\alpha_i| \geq 2$ let p_{i1}, p_{i2} be two points that form an interval of α_i. Consider a pin sequence $P_i = p_{i1}, p_{i2}, p_{i3}, \ldots$ in ξ whose first two points are p_{i1} and p_{i2}. Since ξ is simple we may, by Proposition 2.2, take P_i to be either left-reaching or right-reaching. If $i \neq n$ we take P_i to be right-reaching, and for $i = n$ left-reaching.

Let π_{ij} denote the subpermutation of ξ formed by $\alpha_i, p_{i3}, \ldots, p_{ij}$ (provided that all these points exist). Then, by definition, $\pi_{i2} = \alpha_i$ always exists and $\{\pi_{i2} \mid 1 \leq i \leq n\}$ is a collection of subpermutations of ξ which do not overlap either by position or by value and the relative order of these subpermutations is the simple permutation σ. We can therefore choose $v_1, v_2, \ldots, v_n$ such that $\{\pi_{i,v_i} \mid 1 \leq i \leq n\}$ is a collection of subpermutations of ξ which do not overlap either by position or by value (and therefore whose relative order is the simple permutation σ) such that $\sum v_i$ is maximal.

Consider any one of the subpermutations π_{i,v_i} for which $|\alpha_i| \geq 2$. Note first that p_{i,v_i} is not the last point on the pin sequence P_i. This follows for $i < n$ because P_i is right-reaching and so its final point is inside α_n or separated from α_i by α_n; and it follows for $i = n$ because P_n is left reaching and so its final point is inside α_1 or separated from α_n by all of $\alpha_1, \ldots, \alpha_{n-1}$. So there is a point $f_i = p_{i,v_i+1}$ and, by the maximal choice of $\sum v_i$, there must be a point in some π_{j,v_j} with $j \neq i$ that separates f_i from π_{i,v_i}.

Put $F = \{f_i : |\alpha_i| \geq 2\}$. Furthermore let $\chi_i = \pi_{i,v_i}$ if $|\alpha_i| \geq 2$ and $\chi_i = \alpha_i$ if $|\alpha_i| = 1$. Then, clearly, the subpermutation β of ξ defined by

$$\beta = F \cup \bigcup_{i=1}^{n} \chi_i$$

is an extension of ψ.

The above series of definitions can be made starting from *any* embedding of ψ. We now choose and fix an embedding for which the length of the resulting permutation β is as small as possible. Our aim is to prove

1. $|\beta| \leq 7n$, and
2. β is simple.

To prove the first assertion suppose that $v_i > 5$ for some i. Then we may apply Lemma 2.4 to the 5 points $p_{i,v_i-4}, p_{i,v_i-3}, \ldots, p_{i,v_i}$ of the pin sequence $p_{i1}, \ldots, p_{i,v_i+1}$. The lemma shows that there is a copy α_i' of α_i contained in the sequence $p_{i,v_i-4}, p_{i,v_i-3}, \ldots, p_{i,v_i}$ and a pin sequence beginning from the unique interval of size 2 in α_i' that ends at p_{i,v_i+1}. Thus, replacing α_i by α_i' would give another embedding of ψ with a shorter pin sequence associated with α_i' and this would contradict the minimality of $|\beta|$. Hence $v_i \leq 5$ for all i. This implies that $|\chi_i| \leq 6$ and so

$$|\beta| \leq \sum_i |\chi_i| + |F| \leq 6n + n = 7n$$

For the second statement let M be any non-trivial interval of β.

If $M \subseteq F$ then we could replace all the points in M by a single point; this point would have the separating properties previously enjoyed by the points of M, and this would contradict the minimality of $|\beta|$.

We next suppose that M intersects two distinct χ_i. The permutation defined by the relative order of those χ_i that do intersect M is an interval of the simple permutation σ; therefore M intersects every χ_i. But again, by the simplicity of σ, every χ_i lies between (by value or position) two other χ_k, and so every χ_i is contained in M. Now, since f_i separates p_{i,v_i} from all p_{ij} with $j < v_i$, we have $f_i \in M$ and hence $M = \beta$.

From now on we may suppose that M intersects one χ_t only. Recall that χ_t consists of a single point if $|\alpha_t| = 1$ but otherwise consists of α_t together with a sequence of pin points. If $|\alpha_t| = 2$ then, in fact, $\chi_t = \{p_{t1}, \ldots, p_{t,v_t}\}$. Furthermore if $|\alpha_t| = 3$ then χ_t consists of $p_{t1}, \ldots, p_{t,v_t}$ together with another point a_t, the point of α_t not in its interval of size 2 (it is however possible that a_t is among $p_{t1}, \ldots, p_{t,v_t}$).

Suppose that $f_t \in M$. Then, by construction, there would be a point of some $\pi_{j,v_j} = \chi_j$ with $j \neq t$ that separates f_t from χ_t. Since M is an interval, this point would belong to M. Now M would intersect the distinct χ_t and χ_j which is a contradiction.

If $|\alpha_t| \geq 2$ and M contains two or more points from $p_{t1}, \ldots, p_{t,v_t}$ then, by Proposition 2.1, M must contain p_{t,v_t} and some prior point, hence M contains f_t which we have just excluded.

If $|\alpha_t| = 3$ and M contains a_t and a point $p_{tj} \neq a_t$ then, by removing a_t and considering p_{tj} in its stead, we could obtain an embedding of ψ with smaller $|\beta|$ which is impossible.

This proves that M intersects χ_t in exactly one point z and so M must also contain a point of F. We have seen that M cannot contain f_t and so it must contain some f_s with $s \neq t$. Since f_s separates p_{s,v_s} from $\alpha_s, p_{s3}, \ldots, p_{s,v_s-1}$, z also must separate p_{s,v_s} from $\alpha_s, p_{s3}, \ldots, p_{s,v_s-1}$ and this contradicts that χ_s does not overlap χ_t.

Therefore no such M can exist and β is simple.

But ξ is a minimal simple extension of ψ and $\psi \preceq \beta \preceq \xi$. Therefore $\beta = \xi$ from which we conclude that $|\xi| \leq 7n$. ∎

4.2 Decomposable permutations of finite type

Theorem 4.2 *If ψ is of any of the following permutations:*

1. $\alpha_1 \oplus \alpha_2$ *where* $\alpha_1, \alpha_2 \in \{1, 21, 231, 312\}$,
2. $21 \oplus 1 \oplus 21$, $1 \oplus 21 \oplus 1$, *or any subpermutation of these, or*
3. $2413 \oplus 1$, $1 \oplus 2413$, $3142 \oplus 1$, *or* $1 \oplus 3142$,

then ψ has finite type.

Proof: The first two parts are similar and we will only prove the first. The third part is a special case of Theorem 4.5 which is proved in the next subsection. Therefore, suppose $\psi = \alpha_1 \oplus \alpha_2$ where $\alpha_1, \alpha_2 \in \{1, 21, 231, 312\}$. Further suppose that ξ is a minimal simple extension of ψ.

We construct a subpermutation β of ξ exactly as in the proof of Theorem 4.1. Using the notation of that proof we have $\beta = F \cup \chi_1 \cup \chi_2$ with $|F| \leq 2$ and $|\chi_i| \leq 6$. Thus $|\beta| \leq 14$. The result will therefore follow if we can prove that β is simple for then $\psi \preceq \beta \preceq \xi$ implies that $\beta = \xi$ and hence demonstrates that $|\xi| \leq 14$.

To this end let M be a non-trivial interval of β. We can now argue that $M = \beta$ exactly as in the proof of Theorem 4.1 with one exception: we have to give a different argument in the case that M intersects both χ_1 and χ_2 (since there is no further interval of ψ lying between α_1 and α_2). So suppose that M contains a point $u \in \chi_1$ and a point $v \in \chi_2$.

We shall prove that M contains a point of α_1. Obviously this is true if $u \in \alpha_1$ so we assume that $u \in \chi_1 \setminus \alpha_1$ and therefore (by the construction) u is a point on a pin sequence that starts in α_1. Then u is not both the rightmost and greatest point of χ_1, since a pin sequence has no such point. Suppose u is not the rightmost point of χ_1 (the case that u is not the greatest point is similar). Then the rightmost point w of χ_1 lies between u and v by position and so belongs to M. The intersection of M with the pin sequence is then a non-trivial interval of the pin sequence and Proposition 2.1 shows that it must contain one of the two initial points which both lie in α_1.

Because of the form of α_1 and the fact that M contains the point v which is larger and to the right of α_1 we have $\alpha_1 \subseteq M$. Another appeal to Proposition 2.1 now demonstrates that M contains χ_1.

Similarly M also contains χ_2 and therefore, as in Theorem 4.1, $M = \beta$ and so β is simple. ■

4.3 Spiral permutations

A *spiral permutation* ψ is made up of a central pattern 3142 which forms an interval and a non-empty pin sequence $p_2, \ldots p_k$ that winds clockwise around this interval as shown in Figure 9. Also a (*dual*) *spiral permutation* is made up of a central interval 2413 and a counter-clockwise pin sequence as shown in Figure 10.

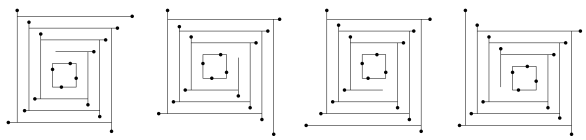

Figure 9: The four spiral permutations of length 17 with central interval 3142

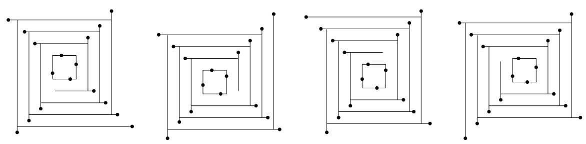

Figure 10: The four dual spiral permutations of length 17 with central interval 2413

Lemma 4.3 *Suppose a permutation ζ is contained in a spiral permutation ψ, contains an interval isomorphic to* 3142 *[or* 2413*], but no longer proper intervals. Then ζ is itself a spiral permutation.*

Proof: Without loss we will assume that the central interval of ψ is 3142. From Figure 9, it is clear that ψ contains only one subsequence θ isomorphic to 3142 and so θ is contained in ζ.

By way of contradiction suppose that the sequence of pin points of ψ contained in ζ is not contiguous. Let p_i be the first pin point contained in ζ and let p_{j+1} be the first pin point after p_i not contained in ζ. Now θ and $p_i, p_{i+1}, \ldots, p_j$ form a single interval in ζ. This contradicts the assumption that the largest interval of ζ is 3142. Hence ζ consists of θ and a contiguous sequence of pin points from the clockwise pin sequence. Therefore ζ is a spiral permutation. ■

Lemma 4.4 *A spiral permutation ψ where $|\psi| \neq 6, 7, 9$ has only one proper interval, that is, the central 3142 [or 2413] pattern. If $|\psi| = 9$, then ψ has exactly two proper intervals where the second of these intervals is made up of the first and last pin points. If $|\psi| = 6, 7$, then ψ has a plus or minus decomposition with one interval of length 1 and another interval of length $|\psi| - 1$.*

Proof: The claims for the cases when $5 \leq |\psi| \leq 9$ are clear upon inspection of diagrams shown in Figure 11.

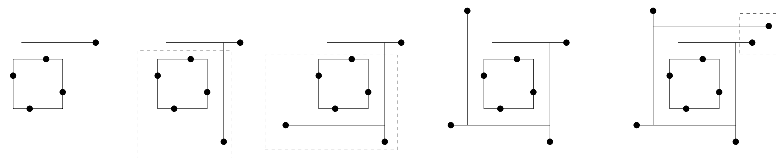

Figure 11: The spiral permutation ψ and its intervals when $5 \leq |\psi| \leq 9$

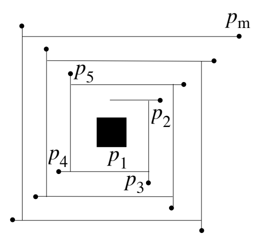

Figure 12: A spiral permutation with central interval 3142

Consider ψ with $|\psi| = m+3 > 9$ as the pin sequence P shown in Figure 12 where the centre solid square p_1 represents the pattern 3142, and the remaining points are labeled $p_2, \ldots, p_m$. Then by Proposition 2.1, if P has a proper interval I, then it must be one of $\{p_1, p_m\}$, $\{p_2, p_m\}$, $\{p_1, p_3, \ldots, p_m\}$, or $\{p_2, p_3, \ldots, p_m\}$. If $I = \{p_2, p_m\}$ or $I = \{p_2, p_3, \ldots, p_m\}$, then $p_1 \notin I$, and so p_m must be in the same quadrant (with respect to the central interval) as p_2. But since $m > 6$, this quadrant also contains $p_{m-4} \neq p_2$, and so $p_{m-4} \in I$. It then follows that $p_{m-3} \in I$, and this point lies in a different quadrant, which in turn implies $p_1 \in I$, a contradiction. A similar argument shows that $I = \{p_1, p_m\}$ and $I = \{p_1, p_3, \ldots, p_m\}$ are impossible. ∎

Theorem 4.5 *Any spiral permutation ψ where $|\psi| \neq 6, 7$ has finite type.*

Proof: Suppose without loss that the central interval C of ψ is isomorphic to 3142. Let $|\psi| = n$ and let $q_1, \ldots, q_{n-4}$ be the remaining points of ψ which spiral around C clockwise. For now, in addition to $n \neq 6, 7$, let us also suppose $n \neq 9$, so that, by Lemma 4.4, C is the only proper interval of ψ.

Suppose that ξ is a minimal simple extension of ψ. Let p_1, p_2 be two points of C; by Proposition 2.2 there must be right-reaching and left-reaching pin sequences that begin from p_1, p_2 and we choose a pin sequence $P = \{p_1, p_2, \ldots, p_m\}$ whose final point is on the same side (left or right) of C as q_1. Then q_1 separates the final point of P from C by position unless q_1 actually is the final point of P; in this latter case $q_1 = p_m$ is a right pin point and it separates C from the penultimate point of P by value. In any case there is a point of P that is separated from C by some q_i. Hence there is a first such point of P. So there exists some $v \geq 3$ such that $p_1, p_2, \ldots, p_v$ are not separated from C by any q_i but p_{v+1} is separated from C by some q_i.

The sequence of points $p_3, \ldots, p_{v+1}$ can be defined for any embedding ψ' of ψ in ξ and any pair of points p_1, p_2 in the central interval of ψ'. We shall choose the embedding of ψ and the pair of points p_1, p_2 for which the value of v is as small as possible.

We observe that $C \cup \{p_1, \ldots, p_v\}$ contains no copy of 3142 other than C itself. For if there were another copy C' it would contain p_t for some $2 < t \leq v$ which we can take as large as possible. Then $C' \cup \{q_1, \ldots, q_{n-4}\}$ would be a subpermutation isomorphic to ψ. Furthermore $p_{t+1}, \ldots, p_{v+1}, \ldots, p_m$ would be a pin sequence extending from it such that $p_t, \ldots, p_v$ is not separated from C' by any q_i but p_{v+1} is separated from C' by some q_i. This would contradict the minimal property of the embedding of ψ.

The point p_3 lies outside $\mathrm{Rect}(C)$. If not, the minimality property is contradicted since we could instead take p_3 to be the first point of P outside $\mathrm{Rect}(C)$ and p_1, p_2 to be points of C separated by p_3: it cuts $\mathrm{Rect}(C)$ but lies either below, left, above or right of $\mathrm{Rect}(C)$. Using the fact that $C \cup \{p_1, \ldots, p_v\}$ contains no other copy of 3142 a case inspection shows that p_4 lies (respectively), left of, above, right of or below $\mathrm{Rect}(C)$ and does not cut $\mathrm{Rect}(C)$. Indeed, by analogous inspection, all subsequent p_i lie outside $\mathrm{Rect}(C)$ and they spiral around $\mathrm{Rect}(C)$ in a clockwise fashion.

It is now easily seen that $C \cup \{p_1, \ldots, p_v\}$ is simple (except possibly if $v = 3$) and that $C \cup \{p_1, \ldots, p_v\} \setminus \{p_3\}$ is a spiral permutation of length $v+1$. However ψ, a spiral permutation of length n, is involved in every spiral permutation of length $n+3$ or more. Therefore $v + 1 < n + 3$ for otherwise $C \cup \{p_1, \ldots, p_v\}$ would be a simple permutation that contained a subpermutation isomorphic to ψ and this would contradict the minimality of ξ.

Let $\alpha = \psi \cup \{p_3, p_4, \ldots, p_{v+1}\}$. From the above we have

$$|\alpha| \leq |\psi| + v - 1 < n + n + 2 - 1 = 2n + 1$$

Now we will show that α is a simple extension of ψ and thus $\alpha = \xi$. To do this we have to consider an arbitrary interval M of α with $|M| > 1$ and prove that $M = \alpha$. Then from $\psi \preceq \alpha \preceq \xi$ we can deduce that $\alpha = \xi$.

Define the subpermutations $\beta = C \cup \{p_1, \ldots, p_v, p_{v+1}\}$ and $\gamma = q_1, \ldots, q_{n-4}$ of α; then $\alpha = \beta \cup \gamma$. Note that β is obtained by adding a further point (necessarily outside $\mathrm{Rect}(C)$) to $C \cup \{p_1, \ldots, p_v\}$ and is easily seen to be simple. In fact, because $n - 4 \neq 2, 3, 5$, we have, by inspection, that γ is also simple.

Suppose first that M contains at least two points of β. Then, as β is simple and $M \cap \beta$ is an interval of β, we have $\beta \subseteq M$. However, there is some point of γ that separates $p_{v+1} \in \beta$ from $C \subseteq \beta$ and so M contains that point of γ. If $|\gamma| = 1$ (i.e. $n = 5$) then certainly $M = \beta \cup \gamma = \alpha$. On the other hand if $|\gamma| > 1$ then, since $n \neq 6, 7$, we have $|\gamma| \geq 4$. However then, for every point q_i of γ, there is some other point of γ that separates it from C (namely q_{i+1} if $i < n-4$ and q_{i-3} if $i = n-4$). As at least one point of γ belongs to M so therefore does every point and hence $M = \alpha$.

Suppose next that M contains at least two points from γ (and so in particular γ has at least two points). Since γ is simple and $M \cap \gamma$ is an interval of γ, we have $\gamma \subseteq M$. As any two consecutive points of γ are separated by C, it follows that M contains C and therefore M contains β. So again, $M = \alpha$.

The remaining case is that $M \cap \beta$ and $M \cap \gamma$ are non-empty intervals of simple permutations β and γ respectively and therefore are singletons $\{b\}$ and $\{c\}$. The point c does not cut C and, without loss, we shall assume it lies below and to the right of C. If $b \in C$ then a further point (the least or rightmost point) of C would lie between b and c and so would lie in M which is impossible. It follows that no point of C lies by position or by value between b and c, and so b also lies below and to the right of C. In particular $b = p_i$ for some $i > 3$. Then p_{i-1}, the previous point on the clockwise sequence $p_3, p_4, \ldots$, is above and to the right of b. Since $i - 1 \leq v$, the point p_{i-1} lies between b and c and therefore lies in M, a contradiction.

Since ξ is bounded, we have shown that ψ has finite type except when $|\psi| \neq 6, 7, 9$. A similar argument that uses ideas from the proof of Theorem 4.1 can be used to show that ψ also has finite type when $|\psi| = 9$. ∎

In the cases when $|\psi| = 6, 7$, the permutation ψ is shown to have infinite type in Proposition 5.4 below based on the decomposition mentioned in Lemma 4.4. Also when $|\psi| = 5$, Theorem 4.5 is equivalent under symmetry to the third case of Theorem 4.2.

5 Infinite types

To prove that a permutation ψ has infinite type we have to display infinitely many minimal simple extensions of ψ. Our approach to this is to find simple extensions ξ of ψ which contain a unique copy of ψ and then to show that every proper subpermutation of ξ that contains this unique copy of ψ has a proper interval. It then follows that ξ contains no proper subpermutation that is a simple extension of ψ and hence that ξ is minimal.

Our methods for carrying out this programme depend on the skeleton of ψ. We treat the case that the skeleton is 12 (or 21, which is equivalent by symmetry) somewhat differently from the case of larger skeletons.

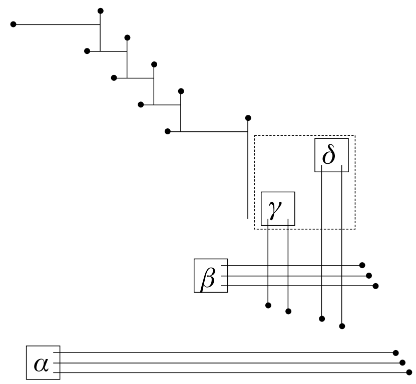

Figure 13: Minimal simple permutations containing $\alpha \oplus \beta \oplus \gamma \oplus \delta$

5.1 The decomposable infinite types

Theorem 5.1 *Suppose ψ is plus-decomposable and is not of the form in Theorem 4.2. Then ψ has infinite type.*

Proof: This statement will follow from Propositions 5.2, 5.3, and 5.4. ■

Proposition 5.2 *For all $\alpha, \beta, \gamma, \delta$, $\alpha \oplus \beta \oplus \gamma \oplus \delta$ has infinite type.*

Proof: In the permutations exemplified by Figure 13 the minimal intervals of α and β are cut by hook points on the right and these hook points are ordered in monotonic decreasing order; the minimal intervals of $\gamma \oplus \delta$ are cut by hook points which lie above α and below β in value and which are also in monotonic decreasing order. The pin sequence in the top left of the figure is of arbitrary length; its left-most point is the first element of the permutation and its other points lie between α and β. In such permutations, $\alpha \oplus \beta \oplus \gamma \oplus \delta$ has a unique embedding. This follows because there are no other subpermutations which are the plus sum of four permutations of lengths $|\alpha|$,$|\beta|$, $|\gamma|$ and $|\delta|$. Furthermore such a permutation is clearly simple. Finally, this simple extension of $\alpha \oplus \beta \oplus \gamma \oplus \delta$ is minimal. Indeed, consider any subpermutation ξ that contains $\alpha \oplus \beta \oplus \gamma \oplus \delta$. If there is a hook point that ξ does not contain then, by construction, one of the intervals of α, β, $\gamma \oplus \delta$ will be an interval of ξ. On the other hand if one of the pin points p is not contained in ξ then $\beta \oplus \gamma \oplus \delta$ together with their hook points and pin points preceding p will form an interval of ξ. ■

Proposition 5.3 *Suppose that α, β, γ are plus-indecomposable permutations. Then $\alpha \oplus \beta \oplus \gamma$ has infinite type unless we have*

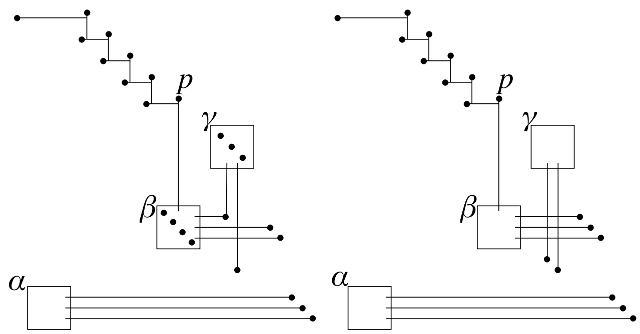

Figure 14: Minimal simple permutations containing $\alpha \oplus \beta \oplus \gamma$

1. *$|\alpha| \leq 2$, $|\gamma| \leq 2$ and $\beta = 1$, or*
2. *$\alpha = \gamma = 1$ and $|\beta| \leq 2$.*

Proof: Suppose that $\alpha \oplus \beta \oplus \gamma$ is not one of the permutations listed in the statement of the proposition. By taking an appropriate symmetry we may suppose that $|\alpha| \leq |\gamma|$.

Consider the extensions of $\alpha \oplus \beta \oplus \gamma$ exemplified by Figure 14; the left hand diagram is the case that β, γ are decreasing permutations of lengths $r, s \geq 2$ and the right hand diagram represents all the other cases. For both diagrams the hook points are chosen so that they cut the minimal intervals of α, β, and γ and are ordered monotonically as shown. The pin sequence shown is of arbitrary length. The point p is shown as lying among the positions of β; however, in the left diagram, should β have only one point then we will instead place p before or among the positions of γ in such a way that it does not lie immediately before or after the largest point of γ (this is possible since $|\gamma| > 2$).

By inspection, such permutations are simple. Suppose, for the moment, that they have a unique copy of $\alpha\oplus\beta\oplus\gamma$. Then they are all minimal simple extensions of $\alpha \oplus \beta \oplus \gamma$, because any proper subpermutation that includes all the points of $\alpha \oplus \beta \oplus \gamma$ either must omit a hook point thereby creating an interval in one of α, β, γ or must omit a pin point which creates an interval containing $\beta \oplus \gamma$.

So to complete the proof we only have to verify that there is no other copy of $\alpha \oplus \beta \oplus \gamma$ within the permutations of Figure 14. To this end suppose that $\alpha' \oplus \beta' \oplus \gamma'$ is another copy. In either case however it is easily seen that $\alpha' = \alpha$ (since α' clearly contains none of the points of the pin sequence, therefore ends within β or after β, and then the subsequent points cannot contain $\beta' \oplus \gamma'$).

Put $\beta = \lambda \ominus \delta_r$ and $\gamma = \mu \ominus \delta_s$ (with similar definitions for β', γ'), where δ_r, δ_s denote decreasing permutations of length r, s and r, s are taken as large as possible so that λ, μ do not end with their minimum.

We now show that λ is empty. This is certainly true if $|\beta| \leq 2$. But if $|\beta| \geq 3$ then β' can consist only of points of β and hook points of γ. In particular it follows that $\lambda' = \lambda$. If λ' is non-empty (so we have the right hand diagram of Figure 14) it must contain the top point of β and therefore we must have $\gamma' = \gamma$ and then $\beta = \beta'$ which is a contradiction. Thus $\beta = \delta_r$.

Next we show that μ is empty. Assume not so that we have the right hand diagram of Figure 14. If $\beta' = \beta$ then we must have $\gamma' = \gamma$, a contradiction. Therefore β' contains at least one of the hook points of γ. However γ' contains only points of γ and hook points of β and consequently $\mu' = \mu$; but that means that γ' contains points that precede points of β', a contradiction. Thus $\gamma = \delta_s$.

We now show that $\beta \neq 1$. If not then consider the right hand diagram again. If $\beta' = \beta$ then $\gamma' = \gamma$ which is impossible. If β' is a hook point of γ then we cannot embed γ' (as β has no hook points). The only other possibility is that β' embeds as a pin point. But then γ' also embeds as a pin point and so $|\gamma| \leq 2$ and therefore also $|\alpha| \leq 2$ which is the first exception listed in the theorem.

Now we show that $\gamma \neq 1$. If this were false then, in the right hand diagram and noting that there are no hook points of γ, we would either have $\beta' = \beta$ and therefore $\gamma' = \gamma$, a contradiction, or β' consisting of pin points; in the latter case $\beta \preceq 21$ and $\gamma = 1$, from which also $\alpha = 1$ and this is the second exception.

At this stage we have completed the proof unless $\beta = \delta_r$, $\gamma = \delta_s$ with $r, s \geq 2$. But here it is easily seen in the left hand diagram that no other copy of $\alpha \oplus \beta \oplus \gamma$ is possible. ■

Proposition 5.4 *Suppose that α and β are plus-indecomposable. Then $\alpha \oplus \beta$ has infinite type unless*

1. *One of α, β is in $\{2413, 3142\}$ and the other is trivial, or*
2. *α and β are amongst $\{1, 21, 312, 231\}$.*

Proof: Since the permutation 12 has finite type we may assume that not both $\alpha = 1$ and $\beta = 1$ so assume that $|\alpha| > 1$. Consider three cases.

A. α is a permutation of the type shown in Figure 15 with $|\alpha| \geq 4$; in particular α is minus-indecomposable, although we allow it to begin without the first point F and/or end without the last point E.

B. α is minus-indecomposable but not of the form in A.

C. α is minus-decomposable.

In case A, consider the family of permutations exemplified by Figure 16.

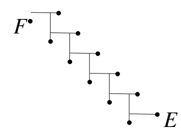

Figure 15: The type A permutations α

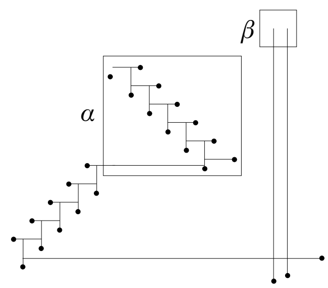

Figure 16: In case A, minimal simple permutations containing $\alpha \oplus \beta, |\alpha| > 4$

These permutations are all simple. Furthermore, so long as $|\alpha| > 4$, $\alpha \oplus \beta$ embeds in them uniquely. Now we argue as usual that every proper subpermutation that contains $\alpha \oplus \beta$ contains a proper interval; thus, when $|\alpha| > 4$, the permutations of Figure 16 are minimal simple extensions of $\alpha \oplus \beta$. However, if $|\alpha| = 4$ we need a rather different family. By symmetry we may assume that $\alpha = 2413$. Consider the permutations typified by Figure 17.

If $\beta \neq 1$ then, using its plus-indecomposability, these permutations have unique embeddings of $2413 \oplus \beta$. They are also clearly simple and, if any points other than points of the embedded $2413 \oplus \beta$ are removed, we produce a proper interval. So $2413 \oplus \beta$ has infinite type. On the other hand if $\beta = 1$ we get the first exception of the proposition.

For case B we begin by dividing into two subcases according to whether the last two symbols of α are in increasing order or not. We shall give full details for the former subcase and we shall appeal to the family of permutations exemplified by Figure 18. In this family we have a pin sequence emerging from the right of α whose first point is a value that lies between the smallest and second smallest values of α. The other subcase, where the last two points of α are decreasing, is very similar and uses a pin sequence that emerges from the bottom of α in between the final two positions of α but the essential arguments are the same.

In Figure 18 the hook points above and to the left of α represent points that extend α to a simple permutation (as in Proposition 3.1 and a suitable symmetry of Figure 6) that contains a unique copy of α. The permutations in Figure 18

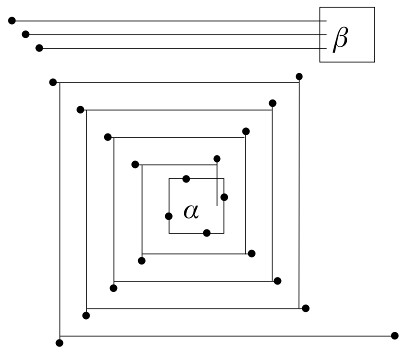

Figure 17: In case A, minimal simple permutations containing $2413 \oplus \beta$

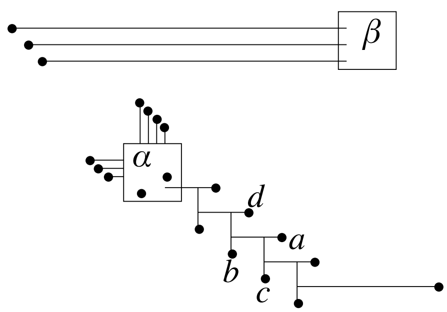

Figure 18: A type B permutation $\alpha \oplus \beta$ and a minimal simple permutation containing it

are simple and if we can prove that they contain one copy only of $\alpha \oplus \beta$ we shall, as usual, be able to conclude that they are minimal simple extensions of $\alpha \oplus \beta$. This will follow if we can show that the set of points consisting of the displayed copy of α together with the pin points that follow it does not contain another copy of α.

So now suppose that, within the displayed set α and the subsequent pin points, there is a second copy (say, α') of α. Consider the final point of α'. This point must be a "R" (right) pin point (for, if it is a "B" (below) pin point, α' would not be minus-indecomposable); so we may take it to be a point a illustrated as in Figure 18. Now, again by the minus-indecomposable property, we know that point b must also lie in α'. For the same reason d also lies in α'. But now also we can conclude that point c lies in α' because the final two points of α (and so the final two points of α') are increasing. Now we continue to use the minus

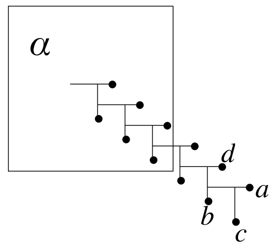

Figure 19: A type B permutation α together with α'

indecomposability of α' and deduce that all the pin points between a and α lie in α'. In particular, α terminates in a sequence of points that is isomorphic to this sequence of pin points; and, even more particularly, the penultimate point of α is its minimal point.

Thus α and α' are related as shown in Figure 19. Again the minus-indecomposable property proves that the entirety of points shown there belong to α' and we can extend backwards further in α. It follows that α is of type A, a contradiction.

For case C, let $\alpha = \theta \ominus \phi$. First assume that $|\theta| > 1$ and consider the permutations of Figure 20. Here the hook points cut the minimal non-trivial intervals as usual and the first pin point is positioned between the first and last points of θ. Note too that there might be an interval that intersects both θ and ϕ; this should be cut by an additional point in the same family as the hook points of ϕ. Again we have to confirm that there is a unique embedding of $\alpha \oplus \beta$. However another copy of α could only occur in the pin sequence extending from θ and the only minus-decomposable subsequences that it has are isomorphic to 21, 312 and 231. So, except for these cases, $\alpha \oplus \beta$ has infinite type.

We can argue in a similar way if $|\phi| > 1$ by first taking inverses (essentially reversing the roles of θ and ϕ). Thus we have shown that $\alpha \oplus \beta$ has infinite type unless α is one of 21, 312 and 231. However, when α has one of these forms we can argue by symmetry that, unless β is also of this form, $\alpha \oplus \beta$ has infinite type. This gives the second exception of the proposition. ∎

5.2 Indecomposable infinite types

In this subsection we shall be considering non-spiral permutations

$$\psi = \sigma[\alpha_1, \alpha_2, \ldots, \alpha_n]$$

where σ is simple, $n \geq 4$, and at least one of the intervals α_i is isomorphic to 123, or isomorphic to 321, or has length at least four. Our aim is to prove that such permutations have infinite type.

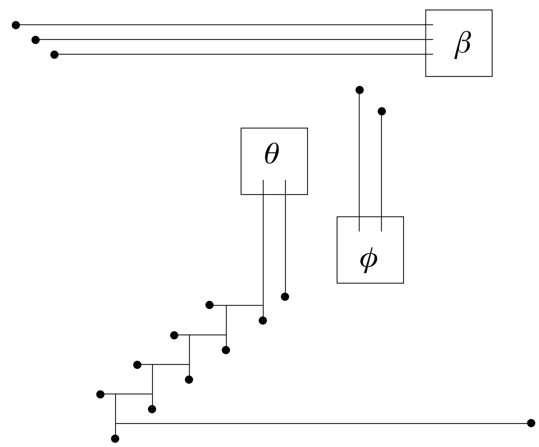

Figure 20: In case C, minimal simple permutations containing $\alpha \oplus \beta$

We pick a maximal length interval α_s of ψ. If there are no intervals of size 4 or more we choose α_s to be any 123 or 321 interval.

Since we can replace ψ by its reverse, we may assume $s < n$. Similarly, if necessary, we can replace ψ by its complement so that $\alpha_s > \alpha_n$.

To prove that ψ has infinite type we shall construct an infinite family of minimal simple extensions of ψ. As before, the minimality of such simple extensions will be proved using the fact that they contain a unique copy of ψ. We let ξ denote a typical permutation in this family.

Construction of ξ

We first apply Lemma 3.2 to α_s. This shows that there exists a minimal simple extension α_s^* of α_s where every point of $\alpha_s^* - \alpha_s$ cuts α_s, an arbitrarily long pin sequence $P = \{p_1, \ldots, p_m\}$ out of α_s^* and a permutation $\hat{\alpha}_s = \alpha_s^* \cup P$. To recapitulate their properties as stated in Lemma 3.2:

- $\hat{\alpha}_s$ is simple and has a unique subpermutation isomorphic to α_s (properties 5 and 6),
- the last pin point of P is the largest point of $\hat{\alpha}_s$ (property 4), and
- if $\alpha_s \neq 3142$ or 2413, then all of the pin points that do not cut α_s^* lie above α_s^* (property 3),
- If $\alpha_s \preceq \beta \preceq \hat{\alpha}_s$ and $\beta \neq \hat{\alpha}_s$ then β has an interval M such that $|M| > 1$ and such that M does not contain the final point of P (property 7).

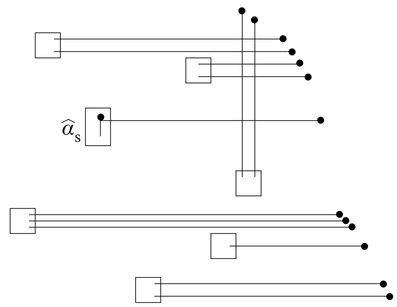

Figure 21: The permutation ξ

In broad terms the permutation ξ is constructed in two stages. First we form

$$\sigma[\alpha_1, \ldots, \alpha_{s-1}, \hat{\alpha}_s, \alpha_{s+1}, \ldots, \alpha_n]$$

and then we extend it to a simple permutation by inserting hook points. The hook points either cut intervals by value (H-hook points) or by position (V-hook points). The hook point cutting $\hat{\alpha}_s$ cuts it just below its highest point.

In more detail, we form the permutation shown in Figure 21. There is one exception to this: when α_{n-1} is the top interval of ψ. In that case, we will instead take the V-hook points to drop downwards to the bottom of ξ. The reason for this variation will emerge in Lemma 5.8. In both cases the monotone order in which the H-hook points and the V-hook points is chosen is as in Proposition 3.1: different from the order in which the final two points of ψ and the top (or, in the exceptional case, the bottom) two points of ψ are ordered.

For the remainder of this section, we will often refer to *blocks* of ξ, denoted $\hat{\alpha}_1, \ldots, \hat{\alpha}_n$ where

$$\hat{\alpha}_i = \begin{cases} \alpha_i & \text{if } i \neq s \\ \hat{\alpha}_s & \text{if } i = s \end{cases}$$

That is, the blocks of ξ will be the same as the corresponding intervals of ψ, except in the case of $\hat{\alpha}_s$. This block $\hat{\alpha}_s$ will be called the *special block*.

It follows from Proposition 3.1 (and its proof) that we have:

Lemma 5.5 *ξ is simple.*

In the following seven lemmas, we go about proving that the copy of ψ that has been designed to appear as a subpermutation of ξ is unique. To do this, we assume the contrary and suppose that ξ has another subpermutation ψ' isomorphic to ψ. Then ψ' has intervals represented by the subsequences $\alpha'_1, \ldots, \alpha'_n$ of ξ, which are isomorphic to $\alpha_1, \ldots, \alpha_n$ and $\psi' = \sigma[\alpha'_1, \ldots, \alpha'_n]$. By analyzing the possible ways in which $\alpha'_1, \ldots, \alpha'_n$ can intersect with $\hat{\alpha}_1, \ldots, \hat{\alpha}_n$, we shall eventually prove that ψ' does not exist.

Lemma 5.6 *No block of ξ intersects non-trivially all the intervals of ψ'.*

Proof: Suppose that R is a block of ξ that has non-trivial intersection with every interval of ψ'. We shall first show that these intersecting intervals are actually contained entirely within R. We will then show that this leads to a contradiction.

Let B be an interval of ψ' not wholly contained in R. Since every interval of ψ' intersects R non-trivially, B is an extremal (leftmost, rightmost, topmost, or bottommost) interval of ψ'. All points of $B \setminus R$ must cut R, otherwise B would be extremal in two directions and the intervals of ψ' would not have the simple pattern σ. Since R is a block of ξ the points of $B \setminus R$ are therefore hook points of R and so we have one of the three possibilities:

1. all points of $B \setminus R$ lie above R,
2. all points of $B \setminus R$ lie below R, or
3. all points of $B \setminus R$ lie to the right of R,

depending on which side of R its hook points lie. Therefore B is unique subject to not being wholly contained in R since R has hook points in one direction only and B is an interval of ψ'.

Suppose that R is not the special block; say $R = \alpha_i$. This implies that $B = \alpha'_i$, for otherwise we would have $\alpha'_i \subseteq \alpha_i$ and thus $\alpha'_i = \alpha_i$ leaving no points available for the other intervals of ψ'. Hence, R is extremal as an interval of ψ in the same direction that B is extremal as an interval of ψ'. However, the construction of ξ ensured that hook points of any extremal interval of ψ did not extend in the same direction in which the interval was extremal. Therefore we have a contradiction.

Thus $R = \hat{\alpha}_s$. For the same reasons as given at the end of the previous paragraph, $B \neq \alpha'_s$. Hence $\alpha'_s \subseteq R$ and by the uniqueness of the embedding of α_s in $\hat{\alpha}_s$ (given in the construction of ξ), we know $\alpha_s = \alpha'_s$.

However, since the hook point of $\hat{\alpha}_s$ is in B, this requires that $B = \alpha'_n$ is the rightmost interval of ψ'. By our construction, this hook point lies above α_s and thus $\alpha'_s < \alpha'_n$ which contradicts one of our initial assumptions about ψ.

Therefore if R is to have non-trivial intersection with all of the intervals of ψ', R must contain all of the intervals of ψ'. It is then clear that $R = \hat{\alpha}_s$, since otherwise R would not have enough points. In particular $\alpha'_s = \alpha_s$ and therefore none of the other intervals of ψ' cut α_s. However $\alpha^*_s - \alpha_s$ consists solely of points that cut α_s and so all the other intervals of ψ' are contained in the pin sequence. But, except when $\alpha_s = 2413$ or 3142, the pin points all lie above α^*_s

and (apart from the first) lie either all to the left or all to the right of α'_s. Thus all the other intervals of ψ' lie above and to one side of α'_s. This contradicts the simplicity of σ.

Finally, if we have $\alpha_s = 3142$ or 2413 (so that no other interval of ψ has length more than 4) Lemma 4.3 proves that ψ' and therefore ψ itself is a spiral permutation, contrary to the initial assumption of this subsection.

Hence, there is no block R having non-trivial intersection with all of the intervals of ψ'. ∎

Lemma 5.7 *Suppose that each of the blocks $\hat{\alpha}_r$, $\hat{\alpha}_{r+1}$, $\ldots$, $\hat{\alpha}_{r+t}$ of ξ intersects non-trivially at most one interval of ψ'. Additionally suppose the intervals $\alpha'_u, \alpha'_{u+1}, \ldots, \alpha'_{u+t}$ of ψ' are contained in the union of the blocks $\hat{\alpha}_r, \hat{\alpha}_{r+1}, \ldots, \hat{\alpha}_{r+t}$. Then α'_{u+j} is contained in $\hat{\alpha}_{r+j}$ for all $0 \leq j \leq t$. Also, if $r = u$, then $\alpha'_{r+j} = \alpha_{r+j}$ for all $0 \leq j \leq t$.*

Proof: Define $0 \leq a_0, a_1, \ldots, a_t \leq t$ such that $\hat{\alpha}_{r+a_j}$ is the leftmost block that intersects α'_{u+j} non-trivially. Since each block of ξ intersects at most one interval of ψ', we know that $\hat{\alpha}_{r+a_j}$ does not intersect α'_{u+j+1}. Also, due to their positions, none of $\hat{\alpha}_r, \ldots, \hat{\alpha}_{r+a_j-1}$ intersect α'_{u+j} non-trivially. Hence $0 \leq a_0 < a_1 < \ldots < a_t \leq t$ and so $a_j = j$ for all $0 \leq j \leq t$. Therefore α'_{u+j} is contained in $\hat{\alpha}_{r+j}$ for all $0 \leq j \leq t$.

Suppose now that $r = u$ and $0 \leq j \leq t$. If $r + j \neq s$ then $\alpha_{r+j} = \alpha'_{r+j}$ since these intervals are isomorphic. This is also true in the case $r + j = s$ since, by the construction of ξ, $\hat{\alpha}_s$ contains a unique copy of α_s. ∎

Lemma 5.8 *The permutation ψ' contains no V-hook points. It contains exactly one H-hook point h and $h \in \alpha'_n$.*

Proof: Because of the choice of the monotone direction in which the H-hook points were positioned in the construction, there can be at most one H-hook point h in ψ'. Also if ψ' contains h, then it will be the rightmost point of ψ' and thus h must be contained in α'_n.

If there were a V-hook point in ψ', let v be the rightmost such. Then $v \in \alpha'_{n-1}$ or $v \in \alpha'_n$ since there are at most $|\alpha_n|$ points of ψ' to the right of v. However, $v \notin \alpha'_{n-1}$ because α_{n-1} would be extremal in value in the same direction as the V-hook points extend which is not consistent with the construction of ξ. Also, $v \notin \alpha'_n$ because if $v \in \alpha'_n$, then α_n would be maximal in position and value contradicting the simplicity of σ. Hence there cannot be a V-hook point in ψ'.

Finally, if ψ' contained no hook points at all it would be contained in the union of the blocks of ξ. In addition each $\hat{\alpha}_i$ would intersect at most one interval of ψ'. To see this consider the set T of intervals of ψ' which intersect $\hat{\alpha}_i$. Every other interval of ψ' that cuts T would have to cut $\hat{\alpha}_i$ but only hook points cut $\hat{\alpha}_i$ and, by assumption, there are none such in ψ'. Therefore T defines an interval in the simple pattern formed by the intervals of ψ'. By Lemma 5.6 $|T| = 1$. Now we can apply Lemma 5.7 (with $r = u = 1$ and $t = n - 1$) and deduce that $\alpha'_i = \alpha_i$ for all i. But then $\psi = \psi'$, a contradiction. ∎

Notation: From now on we shall let h denote the unique H-hook point contained in ψ' and $\hat{\alpha}_i$ the block that it cuts.

Lemma 5.9 *Every block except possibly $\hat{\alpha}_i$ intersects at most one interval of ψ'. If the block $\hat{\alpha}_i$ intersects more than one interval of ψ' then two of these intervals are separated in value by h.*

Proof: Suppose that there is some block $\hat{\alpha}_j$ that intersects more than one interval of ψ'. Then the set of such intersecting intervals has size strictly between 1 and n (by Lemma 5.6). However T is not an interval in the pattern σ defined by all the intervals of ψ' which has no non-trivial intervals. Hence there is an interval α'_k of ψ' that separates the set T and so α'_k cuts $\hat{\alpha}_j$. As $\hat{\alpha}_j$ is cut only by hook points and the only hook point that belongs to ψ' is h it follows that $h \in \alpha'_k$ and that h cuts $\hat{\alpha}_i$. Hence $j = i$ proving the first statement. The second statement follows also since h is an H-hook point and so cuts $\hat{\alpha}_i$ by value. ∎

Lemma 5.10 *For all $1 \le j \le i$, the interval α'_j begins no earlier than $\hat{\alpha}_j$. Similarly for all $n > j \ge i$, the interval α'_j ends no later than $\hat{\alpha}_{j+1}$. In particular, α'_i is contained in $\hat{\alpha}_i \cup \hat{\alpha}_{i+1}$.*

Proof: Let us prove the first assertion by induction on j. If $j = 1$, then certainly α'_1 cannot begin before $\hat{\alpha}_1$. Now suppose α'_{j-1} does not begin before $\hat{\alpha}_{j-1}$ for some $1 \le j - 1 < i$. By way of contradiction, suppose α'_j begins in the block $\hat{\alpha}_t$ where $t < j$. Then because $t < j \le i$, we know from Lemma 5.9 that $\hat{\alpha}_t$ cannot have non-trivial intersection with any other interval of ψ'. In particular, $\hat{\alpha}_t$ cannot intersect α'_{j-1}. Since α'_{j-1} must begin before $\hat{\alpha}_j$, there must exist a $u < t$ such that α'_{j-1} begins in $\hat{\alpha}_u$. However, $u < j - 1$ which is a contradiction. Hence if $1 \le j \le i$, then α'_j begins no earlier than $\hat{\alpha}_j$.

The argument is similar for the second assertion. ∎

Lemma 5.11 *α'_i is contained in $\hat{\alpha}_i$.*

Proof: Suppose by way of contradiction that α'_i is not contained in $\hat{\alpha}_i$. Then by Lemma 5.10, α'_i would have non-trivial intersection with $\hat{\alpha}_{i+1}$. Consequently, α'_{i+1} cannot intersect $\hat{\alpha}_{i+1}$ by Lemma 5.9. It follows that all of the intervals $\alpha'_{i+1}, \ldots, \alpha'_{n-1}$ are contained in $\hat{\alpha}_{i+2} \cdots \hat{\alpha}_n$. In addition, by Lemma 5.9, each of $\hat{\alpha}_{i+2}, \ldots, \hat{\alpha}_n$ intersects at most one interval of ψ'. Therefore Lemma 5.7 can be applied and it proves that $\alpha'_j \subseteq \hat{\alpha}_{j+1}$ for all $i + 1 \le j < n$. This leaves h as the only point available for α'_n, and so $|\alpha_n| = |\alpha'_n| = 1$. But then $\alpha'_{n-1} \subseteq \hat{\alpha}_n = \alpha_n$ implies $|\alpha_{n-1}| = |\alpha'_{n-1}| = 1$. Continuing in this way we conclude that $|\alpha_j| = |\hat{\alpha}_j| = 1$ and $\alpha'_j = \hat{\alpha}_{j+1}$ for all $i < j < n$.

Since $\hat{\alpha}_i$ has a hook point, $|\alpha_i| = |\alpha'_i| \geq 2$. Because of this and since $|\hat{\alpha}_{i+1}| = 1$, we have $\hat{\alpha}_i \cap \alpha'_i \neq \emptyset$. Therefore it follows that both $\hat{\alpha}_i$ and $\hat{\alpha}_{i+1}$ have non-trivial intersection with α'_i. Since $\hat{\alpha}_i$ and $\hat{\alpha}_{i+1}$ do not form an interval in ψ, there must be another block B in ψ that separates them by value. Because some points of $\hat{\alpha}_i$ and $\hat{\alpha}_{i+1}$ combine to form a single interval in ψ', B does not intersect ψ'. Thus B must be to the left of $\hat{\alpha}_i$ because the blocks $\hat{\alpha}_{i+2}, \hat{\alpha}_{i+3}, \ldots, \hat{\alpha}_n$ of ξ are the intervals $\alpha'_{i+1}, \alpha'_{i+2}, \ldots, \alpha'_{n-1}$ of ψ' respectively.

We can now deduce that $\alpha'_{i-1} \subseteq \hat{\alpha}_i$. For if this were false each of $\alpha'_1, \ldots, \alpha'_{i-1}$ would have non-empty intersection with $\hat{\alpha}_1\hat{\alpha}_2 \cdots \hat{\alpha}_{i-1}$ and so there would be at least $i-1$ non-empty intersections $\hat{\alpha}_j \cap \alpha'_k$ with $1 \leq j, k \leq i-1$. However each such $\hat{\alpha}_j$ features in at most one such intersection (see Lemma 5.9) while $\hat{\alpha}_j = B$ features in none at all; so there are fewer than $i-1$ such intersections, a contradiction.

Suppose $\hat{\alpha}_i$ is not the special block. Since $\hat{\alpha}_{i+1}$ contributes only one point to α'_i, $|\hat{\alpha}_i| - 1 = |\alpha_i| - 1$ points of $\hat{\alpha}_i$ are in α'_i. This leaves only one point remaining in $\hat{\alpha}_i$ which must constitute α'_{i-1}. Thus $\hat{\alpha}_i$ has non-trivial intersection only with α'_{i-1} and α'_i and so $\alpha'_1 \cdots \alpha'_{i-2} \subseteq \hat{\alpha}_1 \cdots \hat{\alpha}_{i-1}$. As in the previous paragraph we count the non-empty intersections $\hat{\alpha}_j \cap \alpha'_k$ where $1 \leq j \leq i-1$ and $1 \leq k \leq i-2$ in two ways and find exactly $i-2$ such. This means that each of the $i-2$ blocks of ξ to the left of $\hat{\alpha}_i$ except for B must contain one of the intervals $\alpha'_1, \alpha'_2, \ldots, \alpha'_{i-2}$.

Suppose $\hat{\alpha}_i < \hat{\alpha}_{i+1}$. Then since $\alpha'_i \subset \hat{\alpha}_i \cup \hat{\alpha}_{i+1}$ and $\alpha'_{i-1} \subset \hat{\alpha}_i$, we must have $\alpha'_{i-1} < \alpha'_i$ and therefore $\hat{\alpha}_{i-1} < \hat{\alpha}_i$. Similarly, if $\hat{\alpha}_i > \hat{\alpha}_{i+1}$, we have $\hat{\alpha}_{i-1} > \hat{\alpha}_i$. Hence we have either $\hat{\alpha}_{i-1} < \hat{\alpha}_i < \hat{\alpha}_{i+1}$ or $\hat{\alpha}_{i-1} > \hat{\alpha}_i > \hat{\alpha}_{i+1}$. Thus $\hat{\alpha}_{i-1} \neq B$ since $\hat{\alpha}_{i-1}$ does not separate $\hat{\alpha}_i$ and $\hat{\alpha}_{i+1}$ by value. Assume without loss that $\hat{\alpha}_{i-1} < \hat{\alpha}_i$. Since $\hat{\alpha}_{i-1} \neq B$, we must have α'_{i-2} contained in $\hat{\alpha}_{i-1}$. Thus $\hat{\alpha}_{i-2} < \hat{\alpha}_i$ and thus $\hat{\alpha}_{i-2} \neq B$ either. Continuing this argument, one can see that $\hat{\alpha}_j \neq B$ for all $1 \leq j \leq i-1$. This is a contradiction and therefore $\hat{\alpha}_i$ is the special block, i.e. $i = s$.

The set of intervals that intersect $\hat{\alpha}_s$ has at least two members (for example, α'_{s-1} and α'_s). The second part of Lemma 5.9 tells us that h separates this set of intervals by value but, because h lies just below the top point of $\hat{\alpha}_s$ in value, this top point must belong to ψ'. If α'_s intersects $\hat{\alpha}_s$ in its top point then that will be the only point of intersection. In this case α'_s has only one further point (the single point in $\hat{\alpha}_{s+1}$) and so $|\alpha'_s| = 2$ but this contradicts the initial choice of α_s. Alternatively, if α'_s intersects $\hat{\alpha}_s$ in any other point, then $\alpha'_s < \alpha'_n$ which contradicts one of the initial assumptions about ξ. ∎

Lemma 5.12 *ψ' does not exist and so ψ embeds uniquely in ξ.*

Proof: Suppose first that $i = s$. By assumption h cuts $\hat{\alpha}_s$ and so, by Lemma 5.11, α'_s is contained in $\hat{\alpha}_s$. Therefore, by the construction of ξ, $\alpha'_s = \alpha_s$. Again from the construction h is above α_s and, as $h \in \alpha'_n$, $\alpha'_n > \alpha'_s$ and this contradicts an initial assumption about ξ.

Suppose next that $i \neq s$. Then, by Lemma 5.11, α'_i is contained in $\hat{\alpha}_i$ and so $\hat{\alpha}_i = \alpha'_i$. However $h \in \alpha'_n$ and h cuts $\hat{\alpha}_i$. This is also a contradiction since α'_i is cut by no point of another interval of ψ'. ■

Lemma 5.13 *ξ is a minimal simple extension of ψ.*

Proof: We have to prove that any proper subpermutation β of ξ that properly contains (the unique copy of) ψ is not simple. So consider such a subpermutation. If any hook point of ξ is not contained in β then, by design, β has a minimal non-trivial interval. On the other hand, if any point of $\hat{\alpha}_s - \alpha_s$ is not contained in β then $\beta \cap \hat{\alpha}_s$ is a proper subpermutation of $\hat{\alpha}_s$. One of the properties of the construction of ξ is that such a subpermutation would have an interval M with $|M| > 1$ and such that M does not contain the final point of P. Since M is not cut by any hook point it would also be an interval of β. ■

Since the pin sequences in our construction of ξ can be arbitrarily long we have proved

Theorem 5.14 *If $\psi = \sigma[\alpha_1, \ldots, \alpha_n]$ is a non-spiral permutation where σ is simple, $n \geq 4$, and at least one of the intervals α_i is isomorphic to 123, or isomorphic to 321, or has length at least four, then ψ has infinite type.*

6 Concluding remarks

In closing we give, in Table 1, the bases of the substitution closures of the classes of permutations $\mathrm{Av}(\psi)$ (up to symmetry) where ψ has finite type and $|\psi| = 3, 4$. These bases have been found by exhaustive search, testing simple permutations of increasing degree up to the bounds implicit in the proofs of Section 4. A summary of the results on type is given in Tables 2, 3, 4, 5, 6 for every permutation (up to symmetry).

This paper has been entirely about pattern classes with a single basis element. The same question can be asked about $\mathrm{Av}(B)$ for any finite set B: when is its substitution closure finitely based? We feel that this is a hard question but some progress is possible. It is easy to extend Proposition 1.2 to

Proposition 6.1 *The basis of the substitution closure of $Av(B)$ is the set of permutations ξ such that*

1. *ξ is simple and involves at least one permutation of B.*
2. *The proper simple subpermutations of ξ involve no permutations of B.*

In other words every basis element of the substitution closure of $Av(B)$ is a minimal simple extension of one of the permutations of B.

Table 1: Finite bases for substitution closures of small permutation classes

ψ	Basis of the substitution closure of $Av(\psi)$
231	2413,3142
123	24153, 25314, 31524, 41352, 246135, 415263
3142	3142
3412	35142, 42513, 351624, 415263, 246135
4132	41352, 35142, 263514, 531642, 264153, 526413, 362514
4231	463152, 364152, 264153, 536142, 531642, 531462 462513, 362514, 263514, 526413, 524613, 524163 526314, 426315, 513642, 362415, 461352, 416352
4312	463152, 364152, 264153, 536142, 531642, 531462 462513, 362514, 263514, 526413, 524613, 524163 361524, 264135, 514263, 531624

Table 2: Classification of $\psi = (12)[\alpha_1, \alpha_2]$

α_1	α_2	Type	Justification
$\{1, 21, 231, 312\}$	$\{1, 21, 231, 312\}$	finite	Theorem 4.2
$\{2413, 3142\}$	1	finite	Theorems 4.2, 4.5
1	$\{2413, 3142\}$	finite	Theorems 4.2, 4.5
All other combinations		infinite	Proposition 5.4

Table 3: Classification of $\psi = (123)[\alpha_1, \alpha_2, \alpha_3]$

α_1	α_2	α_3	Type	Justification
{1,21}	1	{1,21}	finite	Theorem 4.2
1	21	1	finite	Theorem 4.2
All other combinations			infinite	Proposition 5.3

Table 4: Classification of $\psi = (12 \cdots n)[\alpha_1, \ldots, \alpha_n]$, $n \geq 4$

α_i	Type	Justification
α_i is any permutation	infinite	Proposition 5.2

Table 5: Classification of Spiral Permutations ψ

k	Type	Justification
5	finite	Theorem 4.5
6,7	infinite	Proposition 5.4
8,9,10,...	finite	Theorem 4.5

Therefore we have

Theorem 6.2 *If every permutation in the finite set of permutations B has finite type then the substitution closure of $Av(B)$ is finitely based.*

Table 6: Classification of Non-spiral Permutations $\psi = \sigma[\alpha_1, \ldots, \alpha_n], n \geq 4$

α_i	Type	Justification
All $\alpha_i \in \{1, 12, 21, 132, 231, 213, 312\}$	finite	Theorem 4.1
Some $\alpha_i \notin \{1, 12, 21, 132, 231, 213, 312\}$	infinite	Theorem 5.14

On the other hand the permutations 1234 and 4321 each have infinite type yet the substitution closure of Av(1234, 4321) (which is a finite set) is finitely based by results of [1]. Even more disconcertingly the permutation 2134 is of finite type yet the substitution closure of Av(2134, 1234) is infinitely based as demonstrated by the family of basis permutations shown in Figure 22.

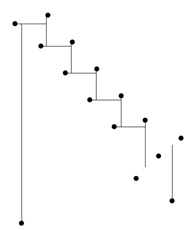

Figure 22: Basis permutations of the substitution closure of Av(2143, 1234)

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