Automatic presentations for semigroups

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Abstract

This paper applies the concept of FA-presentable structures to semigroups. We give a complete classification of the finitely generated FA-presentable cancellative semigroups: namely, a finitely generated cancellative semigroup is FA-presentable if and only if it is a subsemigroup of a virtually abelian group. We prove that all finitely generated commutative semigroups are FA-presentable. We give a complete list of FA-presentable one-relation semigroups and compare the classes of FA-presentable semigroups and automatic semigroups.

Key words: Automatic presentation, FA-presentable, cancellative semigroup, virtually abelian group
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1 Introduction

Automatic presentations were introduced by Khoussainov and Nerode [23] to fulfill a need to extend finite model theory to infinite structures in such a way that interesting decision problems remain soluble; the present paper applies this concept to semigroups. We give definitions and examples, survey some previously published results, and establish some new ones, most importantly

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a complete characterization of the finitely generated cancellative semigroups admitting automatic presentations.

Recall that a *structure* $\mathcal{A}$ is a tuple $(A, R_1, \ldots, R_n)$ where:

- A is a set called the *domain* of $\mathcal{A}$;
- for each i with $1 \leqslant i \leqslant n$, there is an integer $r_i \geqslant 1$ such that R_i is a subset of A^{r_i}; r_i is called the *arity* of R_i.

An obvious instance of a structure is a relational database. However, there are many other natural examples; for instance, a semigroup is a structure $(S, \circ)$, where $\circ$ has arity 3, and a group is a structure $(G, \circ, e, {}^{-1})$, where $\circ$ has arity 3, e has arity 1, and ${}^{-1}$ has arity 2.

Informally, an automatic presentation for the structure $(A, R_1, \ldots, R_n)$ consists of a regular language of abstract representatives for the elements of A such that the relations R_i are all recognizable by synchronous finite automata; see Definition 2.3. A structure that admits an automatic presentation is said to be FA-presentable.

One important field of research has been the attempt to classify FA-presentable structures with specific classes of structures. As any finite structure is FA-presentable, we are really only interested in infinite structures here. In some cases this means that we have no real examples (for example, any FA-presentable integral domain is finite [24]). Essentially the only cases where we have a complete classification are those of:

- Boolean algebras [24];
- ordinals [9];
- finitely generated groups [28].

(For a number of partial results for FA-presentable groups, see [27]; for some necessary conditions for trees and linear orders to be FA-presentable, see [25].)

As far as groups are concerned, we also have the notion of an 'automatic group' in the sense of [13]. This has been generalized to semigroups (as in [7,29,21]). The considerable success of the theory of automatic groups was another motivation to have a general notion of FA-presentable structures; see also [30,32]. We note that a structure admitting an automatic presentation is often called an 'automatic structure'; although we will avoid that term, the reader should be aware of the terminological clash with the different notion of an automatic structure for a group or semigroup in the sense of [13,7].

In this paper we will be particularly concerned with FA-presentable semigroups. When one moves from groups to semigroups, it appears that the problem becomes significantly more difficult. For example, if one has an undirected graph Γ with vertices V and edges E, then we have a semigroup with elements $S = V \cup \{e, 0\}$, where we have the following products:

$$uv = \begin{cases} e & \text{if } u, v \in V \text{ and } \{u, v\} \in E; \\ 0 & \text{if } u, v \in V \text{ and } \{u, v\} \notin E; \end{cases}$$
$$ue = eu = u0 = 0u = 0 \text{ for } u \in V \cup \{e, 0\}.$$

Moreover, if we form the semigroup S from the graph Γ in this way, then S is FA-presentable if and only if Γ is FA-presentable. It is known [24] that the isomorphism problem for FA-presentable graphs is Σ_1^1-complete (and hence undecidable); hence the isomorphism problem for FA-presentable semigroups is also Σ_1^1-complete.

Given this, it seems sensible to restrict oneself to some naturally occurring classes of semigroups. Given the classification of the FA-presentable finitely generated groups referred to above, a natural class to consider is that of the FA-presentable finitely generated cancellative semigroups. In this paper we give a complete classification of these structures: a finitely generated cancellative semigroup is FA-presentable if and only if it embeds into a virtually abelian group (Theorem 10.1).

We remark that there are many examples of *non-cancellative* finitely generated FA-presentable semigroups. It is easy to see that adjoining a zero to a semigroup always preserves FA-presentability and destroys cancellativity. All finite semigroups, whether cancellative or not, are FA-presentable. Another example is the bicyclic monoid; see Example 3.2.

In Section 6, we prove that all finitely generated commutative semigroups are FA-presentable (Theorem 6.1). We also classify the FA-presentable one-relation semigroups (Proposition 9.1).

Finally, in Section 11, we consider the relationship between the classes of FA-presentable semigroups and automatic semigroups.

2 Automatic presentations

A semigroup is a set equipped with an associative binary operation $\circ$, although the operation symbol is often suppressed, so that $s \circ t$ is denoted st. We recall the idea of a "convolution mapping" which we will need throughout this paper:

Definition 2.1 *Let L be a regular language over a finite alphabet A. Define, for $n \in \mathbb{N}$,*

$$L^n = \{(w_1, \ldots, w_n) : w_i \in L \textit{ for } i = 1, \ldots, n\}.$$

Let \$ be a new symbol not in A. The mapping conv : $(A^*)^n \to ((A \cup \{\$\})^n)^*$ *is defined as follows. Suppose*

$$w_1 = w_{1,1}w_{1,2}\cdots w_{1,m_1},\ w_2 = w_{2,1}w_{2,2}\cdots w_{2,m_2},\ \ldots,\ w_n = w_{n,1}w_{n,2}\cdots w_{n,m_n},$$

where $w_{i,j} \in A$*. Then* $\mathrm{conv}(w_1, \ldots, w_n)$ *is defined to be*

$$(w_{1,1}, w_{2,1}, \ldots, w_{n,1})(w_{1,2}, w_{2,2}, \ldots, w_{n,2})\cdots(w_{1,m}, w_{2,m}, \ldots, w_{n,m}),$$

where $m = \max\{m_i : i = 1, \ldots, n\}$ *and with* $w_{i,j} = \$$ *whenever* $j > m_i$*.*

Observe that the map conv sends an n-tuple of words to a word of n-tuples. We then have:

Definition 2.2 *Let A be a finite alphabet, and let $R \subseteq (A^*)^n$ be a relation on A^*. Then R is said to be* regular *if*

$$\{\mathrm{conv}(w_1, \ldots, w_n) : (w_1, \ldots, w_n) \in R\}$$

is a regular language over $(A \cup \{\$\})^n$*.*

Having done this, we can now define the concept of an 'automatic presentation' for a structure:

Definition 2.3 *Let $\mathcal{S} = (S, R_1, \ldots, R_n)$ be a relational structure. Let L be a regular language over a finite alphabet A, and let $\phi : L \to S$ be a surjective mapping. Then (L, ϕ) is an* automatic presentation *for $\mathcal{S}$ if:*

(1) the relation $L_= = \{(w_1, w_2) \in L^2 : \phi(w_1) = \phi(w_2)\}$ is regular, and
(2) for each relation R_i of arity r_i, the relation

$$L_{R_i} = \{(w_1, w_2, \ldots, w_{r_i}) \in L^{r_i} : (\phi(w_1), \ldots, \phi(w_{r_i})) \in R_i\}$$

is regular.

A structure with an automatic presentation is said to be FA-presentable*.*

As noted in Section 1, a semigroup can be viewed as a relational structure in which the binary operation $\circ$ becomes a ternary relation. The following definition simply restates the preceding one in the special case where the structure is a semigroup:

Definition 2.4 *Let S be a semigroup. Let L be a regular language over a finite alphabet A, and let $\phi : L \to S$ be a surjective mapping. Then (L, ϕ) is an* automatic presentation *for S if the relations*

$$L_= = \{(w_1, w_2) \in L^2 : \phi(w_1) = \phi(w_2)\},$$
$$L_\circ = \{(w_1, w_2, w_3) \in L^3 : \phi(w_1)\phi(w_2) = \phi(w_3) \text{ in } S\}$$

are both regular.

3 Examples

In this section, we give some examples of FA-presentable semigroups. We first exhibit a well-known example:

Example 3.1 *The natural numbers under addition are FA-presentable: let* $L = \{0,1\}^*\{1\} \cup \{0\}$ *and define* $\phi : L \to \mathbb{N}$ *by letting* $\phi(w)$ *be the natural number expressed by* w *in reverse binary notation. The equality relation* $L_=$ *is the diagonal relation* $\{(w,w) : w \in L\}$*, for every natural number has a unique representative of* L*. A finite automaton can recognize the relation* $L_+ = \{(u,v,w) : \phi(u) + \phi(v) = \phi(w)\}$ *because it can add* u *to* v *digit by digit and compare the result with* w*, storing the carry in its internal state. So* (L,ϕ) *is an automatic presentation for* $(\mathbb{N},+)$*.*

Example 3.2 *The bicyclic monoid* B*, which is presented by* $\langle b,c \mid bc = 1\rangle$*, is FA-presentable. Notice that every element of the bicyclic monoid has a normal form* c^ib^j *and that*

$$c^ib^j \circ c^kb^l = \begin{cases} c^ib^{l+(j-k)} & \text{if } j \geqslant k \\ c^{i+(k-j)}b^l & \text{if } j < k \end{cases}.$$

Retain the language L *and the mapping* ϕ *from the previous example. Let* $K = \{\mathrm{conv}(x,y) : x,y \in L\}$*, where* $\psi : K \to B$ *is given by*

$$\mathrm{conv}(x,y) \mapsto c^{\phi(x)}b^{\phi(y)}.$$

Then (K,ψ) *is an automatic presentation for* B*: the equality relation* $K_=$ *is the diagonal relation, and the multiplication relation* $K_\circ$ *is easily seen to be automatic, since addition of natural numbers (in reverse binary notation) can be carried out by an automaton, as can subtraction and comparison.*

4 Basic results

The following notions and proposition will be useful in what follows:

Definition 4.1 *Let* (L,ϕ) *be an automatic presentation for a structure. Then* (L,ϕ) *is a* binary *automatic presentation if the language* L *is over a two-letter alphabet; it is an* injective *automatic presentation if the mapping* ϕ *is injective (so that every element of the structure has exactly one representative in* L*).*

Proposition 4.2 ([23, Corollary 4.3] & [2, Lemma 3.3]) *Any structure that admits an automatic presentation admits an injective binary automatic presentation.*

An *interpretation* of one structure inside another is, loosely speaking, a copy of the former inside the latter. The following definition is restricted to an interpretation of one semigroup inside another.

Definition 4.3 *Let S and T be semigroups. Let $n \in \mathbb{N}$. An* (n-dimensional) interpretation *of T in S consists of the following:*

- *a first-order formula $\psi(x_1, \ldots, x_n)$, called the* domain formula*, which specifies those n-tuples of elements of S used in the interpretation;*
- *a surjective map $f : \psi(S^n) \to T$, called the* co-ordinate map *(where $\psi(S^n)$ denotes the set of n-tuples of elements of S satisfying the formula ψ);*
- *a first-order formula $\theta_=(x_1, \ldots, x_n; y_1, \ldots, y_n)$ that is satisfied by*

$$(a_1, \ldots, a_n; b_1, \ldots, b_n)$$

  in the semigroup S if and only if $f(a_1, \ldots, a_n) = f(b_1, \ldots, b_n)$ in the semigroup T;
- *a first-order formula $\theta_\circ(x_1, \ldots, x_n; y_1, \ldots, y_n; z_1, \ldots, z_n)$ that is satisfied by*

$$(a_1, \ldots, a_n; b_1, \ldots, b_n; c_1, \ldots, c_n)$$

  in the semigroup S if and only if $f(a_1, \ldots, a_n)f(b_1, \ldots, b_n) = f(c_1, \ldots, c_n)$ in the semigroup T.

The following result, although here stated only for semigroups, is true for structures generally:

Proposition 4.4 ([2, Proposition 3.13]) *Let S and T be semigroups. If S has an automatic presentation and there is an interpretation of T in S, then T has an automatic presentation.*

The fact that a tuple of elements $(a_1, \ldots, a_n)$ of a structure S satisfies a first-order formula $\theta(x_1, \ldots, x_n)$ is denoted $S \models \theta(a_1, \ldots, a_n)$. We then have:

Proposition 4.5 ([23]) *Let $\mathcal{S}$ be a structure with an automatic presentation. For every first-order formula $\theta(x_1, \ldots, x_n)$ over the structure there is an automaton which accepts $\mathrm{conv}(w_1, \ldots, w_n)$ if and only if $S \models \theta(\phi(w_1), \ldots, \phi(w_n))$. Moreover, there is an algorithm which effectively constructs such an automaton from such a formula.*

(Proposition 4.4 is actually a consequence of Proposition 4.5.)

As a consequence of Proposition 4.5, FA-presentable structures have decidable first-order theories. In the context of semigroups, this means that any first-order definable property or relation of semigroups is decidable. For example, Green's relations and cancellativity are both decidable for FA-presentable semigroups. This contrasts the situation for automatic semigroups, where Green's relation $\mathcal{R}$ [22] and cancellativity [5] are undecidable.

5 Finitely generated FA-presentable groups

As mentioned in Section 1, a classification of the finitely generated groups with an automatic presentation was given in [28]. For convenience, we state the result here (along with some extra details from [28] that we will need later). Recall that a group G is said to be *virtually abelian* if it has an abelian subgroup A of finite index. If G is finitely generated, then the subgroup A is finitely generated as well. Using the fact that any finitely generated abelian group is the direct sum of finitely many cyclic groups, we may assume that A is of the form $\mathbb{Z}^n$ for some $n \geqslant 0$.

Theorem 5.1 ([28]) *A finitely generated group admits an automatic presentation if and only if it is virtually abelian. In particular, a group G with a subgroup $\mathbb{Z}^n$ of index ℓ admits an automatic presentation (L, ϕ), where L is the language of words*

$$g_i \mathrm{conv}(\varepsilon_1 z_1, \ldots, \varepsilon_n z_n),$$

where $\varepsilon_i \in \{+, -\}$, z_i is a natural number in reverse binary notation, $g_1, \ldots, g_\ell$ are representatives of the cosets of $\mathbb{Z}^n$ in G, with $\phi : L \to G$ being defined in the natural way:

$$\phi(g_i \mathrm{conv}(\varepsilon_1 z_1, \ldots, \varepsilon_n z_n)) = g_i(\varepsilon_1 z_1, \ldots, \varepsilon_n z_n).$$

6 Commutative semigroups

Commutative semigroups often have pleasant properties with regard to finite 'descriptions'. For example, Rédei's Theorem shows that all finitely generated commuative semigroups are finitely presented [31], and finitely generated commutative monoids are presented by finite confluent Noetherian rewriting systems [10]. The following result is thus perhaps unsurprising:

Theorem 6.1 *Every finitely generated commutative semigroup admits an automatic presentation.*

To prove this result, we will need the following lemma:

Lemma 6.2 ([2, Corollary 3.14]) *The class of FA-presentable structures is closed under forming quotients by first-order definable congruences, under forming finitary direct products, and under passing to first-order definable substructures. Moreover, in each case an automatic presentation is effectively constructable.*

We now proceed with the proof of Theorem 6.1:

PROOF. Finitely generated free commutative semigroups are isomorphic to $(\mathbb{N},+)^n - \{(0,\ldots,0)\}$ for some n and are thus FA-presentable, since the class of FA-presentable structure is closed under direct products, the exclusion of a single element gives a first-order definable substructure.

Every commutative semigroup is a quotient of a free commutative semigroup and, by [33], the corresponding congruence is first-order definable; the result then follows from Lemma 6.2.

We observe that not all countable commutative semigroups are FA-presentable: for example, any monoid which contains $(\mathbb{N},\times)$ is not FA-presentable [24, Theorem 3.6].

7 Growth

In the proof of Theorem 5.1 above, given in [28], one essential ingredient was the notion of growth. Before defining the growth of a semigroup, we first establish notation for and state a basic property of lengths of the words representing the elements of the domain of the structure.

Definition 7.1 *Let S be a semigroup with an injective automatic presentation (L,ϕ). For any $s \in S$, denote by $l(s)$ the length of the unique word in L representing s.*

Proposition 7.2 ([2, Proposition 5.1]) *Let S be a semigroup with an injective automatic presentation; then there is a constant $N \in \mathbb{N}$ such that, for all $s,t \in S$,*

$$l(st) \leqslant \max\{l(s),l(t)\} + N.$$

We now turn to the concept of *growth*:

Definition 7.3 *Let S be a semigroup generated by a finite set X. Define $\delta(s)$*

to be the length of the shortest product of elements of X that equals s, i.e.

$$\delta(s) = \min\{n \in \mathbb{N} : s = x_1 \cdots x_n \textit{ for some } x_i \in X\}.$$

The growth function $\gamma : \mathbb{N} \to \mathbb{N}$ *of S is given by*

$$\gamma(n) = |\{s \in S : \delta(s) \leqslant n\}|.$$

If the function γ is bounded above by a polynomial function (that is, if there exists a polynomial function β and some $N \in \mathbb{N}$ such that $\beta(n) > \gamma(n)$ for $n > N$), then S is said to have polynomial growth.

Note that whether a semigroup has polynomial growth or not is independent of the choice of finite generating set [17]. We now have the following result:

Theorem 7.4 *Any finitely generated subsemigroup of a semigroup admitting an automatic presentation has polynomial growth.*

Before embarking on a proof of this result, we pause to emphasize that polynomial growth is dependent on the structures in question being semigroups: general algebras admitting automatic presentations are only guaranteed to have at most exponential growth [23, Lemma 4.5].

PROOF. Let S be a semigroup, finitely generated by X, that admits an automatic presentation. By Proposition 4.2, assume without loss of generality that this automatic presentation is injective and binary. This proof follows that in [28], which dealt with groups. The main ingredient is provided by the following lemma:

Lemma 7.5 *Let $R = \max\{l(a) : a \in X\}$. There is a constant N such that, for all $m \in \mathbb{N}$,*

$$\max\{l(a_1 \cdots a_m) : a_i \in X\} \leqslant R + \lceil \log_2 m \rceil N. \tag{1}$$

PROOF. Let N be the constant of Proposition 7.2. We proceed by induction on m.

For $m = 1$, the inequality (1) holds, since

$$\max\{l(a_1) : a_1 \in X\} = R = R + \lceil \log_2 1 \rceil N.$$

Now assume that (1) is true for $1 \leqslant m \leqslant k$. The cases of k being odd or even must be considered separately:

(1) Suppose k is odd, with $k = 2r - 1$. Then, by Proposition 7.2:

$$\begin{aligned}
&\max\{l(a_1 \cdots a_{k+1}) : a_i \in X\} \\
&= \max\{l(a_1 \cdots a_{2r}) : a_i \in X\} \\
&\leqslant \max\{l(a_1 \cdots a_r),\ l(a_{r+1} \cdots a_{2r}) : a_i \in X\} + N \\
&\leqslant \max\{R + \lceil \log_2 r \rceil N,\ R + \lceil \log_2 r \rceil N\} + N \\
&= R + \lceil \log_2 r \rceil N + N \\
&= R + (\lceil \log_2 r + 1 \rceil) N \\
&= R + \lceil \log_2 2r \rceil N \\
&= R + \lceil \log_2 (k+1) \rceil N,
\end{aligned}$$

as required.

(2) Suppose k is even, say $k = 2r$. Then:

$$\begin{aligned}
&\max\{l(a_1 \cdots a_{k+1}) : a_i \in X\} \\
&= \max\{l(a_1 \cdots a_{2r+1}) : a_i \in X\} \\
&\leqslant \max\{l(a_1 \cdots a_r),\ l(a_{r+1} \cdots a_{2r+1}) : a_i \in X\} + N \\
&\leqslant \max\{R + \lceil \log_2 r \rceil N,\ R + \lceil \log_2 (r+1) \rceil N\} + N \\
&= R + \lceil \log_2 (r+1) \rceil N + N.
\end{aligned}$$

At this point, two subcases are required, depending on whether r is a power of two:

(a) Suppose that r is not a power of 2. Since the function $\lceil \log_2 y \rceil$ on the set $\{y \in \mathbb{N} : y > 0\}$ takes the same value on y and $y + 1$ except when y is a power of 2, $\lceil \log_2 (r+1) \rceil = \lceil \log_2 r \rceil$. Therefore, by the reasoning in part (1),

$$R + \lceil \log_2 (r+1) \rceil N + N = R + \lceil \log_2 r \rceil N + N = R + \lceil \log_2 (k+1) \rceil N,$$

as required.

(b) Suppose that $r = 2^x$, where $x \in \mathbb{N}$. Observe that

$$\lceil \log_2 (k+1) \rceil = \lceil \log_2 (2r+1) \rceil = \lceil \log_2 (2^{x+1} + 1) \rceil = x + 2.$$

Consequently,

$$\begin{aligned}
&R + \lceil \log_2 (r+1) \rceil N + N \\
&= R + \lceil \log_2 (r+1) + 1 \rceil N \\
&= R + \lceil \log_2 (r+1) + \log_2 2 \rceil N \\
&= R + \lceil \log_2 2(r+1) \rceil N \\
&= R + \lceil \log_2 2(2^x + 1) \rceil N \\
&= R + \lceil \log_2 (2^{x+1} + 2) \rceil N \\
&= R + (x+2) N \\
&= R + \lceil \log_2 (k+1) \rceil N,
\end{aligned}$$

as required.

We now return to the proof of Theorem 7.4. By Lemma 7.5, the number of possible words in L for elements of the form $x_1 \cdots x_m$, where $x_i \in X$, is no greater than

$$2^{(R+\lceil \log_2 m \rceil N)+1} = 2^{R+1}(2^{\lceil \log_2 m \rceil})^N \leqslant 2^{R+1}(2^{1+\log_2 m})^N = km^N,$$

where $k = 2^{R+1}2^N$ is a constant. So there are at most km^N elements $s \in S$ with $\delta(s) = m$. Consequently,

$$\gamma(n) = |\{s \in S : \delta(s) \leqslant n\}| \leqslant k \cdot 1^N + k \cdot 2^N + \ldots + k \cdot n^N \leqslant kn^{N+1}.$$

So S has polynomial growth. This establishes Theorem 7.4.

8 Maximum Group Homomorphic Image

Given the classification of FA-presentable finitely generated groups (see Theorem 5.1 above), it makes sense to investigate (finitely generated) groups related to semigroups. The *maximum group homomorphic image* of a semigroup S, if it exists, is the largest group G such that there is a surjective homomorphism from S onto G, in the sense that there is a homomorphism from this group G onto any group H that is a homomorphic image of S. The congruence associated to this homomorphic image is called the *minimum group congruence*. (For futher background information, see [20, Section 5.3].)

Definition 8.1 *Let S be a semigroup. A subset K of S is:*

- unitary *if for all $s \in S$ and $k \in K$, we have $(sk \in K \vee ks \in K) \implies s \in K$;*
- dense *if for all $s \in S$ there exists $x, y \in S$ such that $sx \in K$ and $ys \in K$;*
- reflexive *if for all $a, b \in S$, we have $ab \in K \implies ba \in K$.*

The subsemigroup generated by K is denoted $\langle K \rangle$.

Definition 8.2 *Let S be a semigroup, with E its set of idempotents. Then S is:*

- *regular if for every $s \in S$ there exists $s' \in S$ such that $ss's = s$;*
- *π-regular if for every $s \in S$, there exists $n \in N$ and $s' \in S$ such that $s^n s' s^n = s^n$;*
- *strongly π-inverse if it is π-regular and E is commutative;*
- *a unitary dense E-semigroup if E is a subsemigroup, and E is unitary and dense;*

- *a strongly $\langle E\rangle$-unitary dense monoid if it is a monoid and $\langle E\rangle$ is reflexive, unitary and dense.*

Using a variety of results from the literature, we obtain the following result:

Proposition 8.3 *If S is FA-presentable and either*

- *a regular semigroup,*
- *a strongly π-inverse semigroup,*
- *a unitary dense E-semigroup, or*
- *a strongly $\langle E\rangle$ unitary dense monoid,*

then the maximum group homomorphic image of S exists and is FA-presentable.

PROOF. For each of the given species of semigroup, the minimum group congruence exists and is first-order definable [14,35]. So the maximum group homomorphic image of any such semigroup will be FA-presentable by Lemma 6.2.

Corollary 8.4 *Let S be the free inverse monoid on the set A; then, S has an automatic presentation if and only if $|A| = 1$.*

PROOF. The monoid S is regular and its maximum group homomorphic image is the free group on A. Thus if S is FA-presentable, then so is the free group with on A, whence $|A| = 1$ since otherwise it contains a free subsemigroup on two generators, which does not have polynomial growth, which would contradict Theorem 7.4.

Conversely, if $|A| = 1$, then free inverse monoid on A is isomorphic to the semigroup formed by the set

$$\{(r, s, t) \in \mathbb{Z}^3 : r \geqslant 0, s \geqslant 0, -s \leqslant t \leqslant r\}$$

under the operation

$$(r, s, t)(r', s', t') = (\max\{r, r' + t\}, \max\{s, s' - t\}, s + s');$$

see [20, p.219]. Since a finite automaton can add, subtract, and compare integers in reverse binary notation, it is clear that this semigroup is FA-presentable.

9 One-relation semigroups

In this section, we characterize those one-relation semigroup presentations that define FA-presentable semigroups.

Proposition 9.1 *A semigroup S with one defining relation has an automatic presentation if and only if either S is monogenic, or S is generated by two elements, say a and b, and the defining relation is one of:*

$$a = b^k; \quad ab = ba; \quad ab = b^k; \quad ba = b^k; \quad ab = aba;$$
$$ba = aba; \quad ab = bab^2; \quad ba = b^2ab; \quad a = bab; \quad a^2 = b^2.$$

PROOF. Vazhenin [34] proved that these semigroups are precisely the one-relation semigroups with decidable first-order theory. The proof involves an interpretation of each of these semigroups in $(\mathbb{N}, +)^k$ for some $k \in N$. The semigroup $(\mathbb{N}, +)^k$ is FA-presentable by Theorem 6.1; thus each of these semigroups is FA-presentable by Proposition 4.4.

10 Characterization of FA-presentable cancellative semigroups

The present section is dedicated to proving the following characterization theorem:

Theorem 10.1 *A finitely generated cancellative semigroup is FA-presentable if and only if it embeds into a virtually abelian group.*

Recall that a semigroup S has a group of left (respectively, right) quotients G if S embeds into G and every element of G is of the form $t^{-1}s$ (respectively, st^{-1}) for $s, t \in S$. If a semigroup S has a group of left (respectively, right) quotients, then this group is unique up to isomorphism. For further information on groups of left and right quotients, see [8, Section 1.10].

The following result, due to Grigorchuk, generalizes the result of Gromov [18] that a finitely generated group of polynomial growth is virtually nilpotent (i.e. it has a nilpotent subgroup of finite index):

Theorem 10.2 ([16]) *A finitely generated cancellative semigroup has polynomial growth if and only if it has a virtually nilpotent group of left quotients.*

We then have the following immediate consequence of Theorems 10.2 and 7.4:

Corollary 10.3 *Let S be a finitely generated cancellative semigroup that admits an automatic presentation. Then the group of left quotients of S exists and is virtually nilpotent.*

Note that the groups of left and right quotients of subsemigroups of virtually nilpotent groups coincide (see [26] or [4, Sections 5.2–5.3]). We now have:

Proposition 10.4 *Let S be a finitely generated cancellative semigroup that admits an automatic presentation. Then the [necessarily virtually nilpotent] group of left (and right) quotients of S admits an automatic presentation.*

PROOF. Let G be the group of left (and right) quotients of S. The strategy is to show that G has a 2-dimensional interpretation in S.

- The domain formula is tautological: $\phi(x_1, x_2) := x_1 = x_1$. Thus all pairs of elements of S are used.
- The co-ordinate map is $f(x_1, x_2) = x_1^{-1}x_2$. Since G is the group of left quotients of S, the mapping f is surjective as required.
- The formula $\theta_=$ is given by

$$\theta_=(x_1, x_2; y_1, y_2) := (\exists a, b)(x_1 a = x_2 b \wedge y_1 a = y_2 b),$$

  since

$$\begin{aligned} & f(x_1, x_2) = f(y_1, y_2) \\ \iff & (\exists a, b)(f(x_1, x_2) = ab^{-1} \wedge f(y_1, y_2) = ab^{-1}) \\ \iff & (\exists a, b)(x_1^{-1}x_2 = ab^{-1} \wedge y_1^{-1}y_2 = ab^{-1}) \\ \iff & (\exists a, b)(x_1 a = x_2 b \wedge y_1 a = y_2 b). \end{aligned}$$

- The formula $\theta_\circ$ is given by

$$\begin{aligned} & \theta_\circ(x_1, x_2; y_1, y_2; z_1, z_2) := \\ & \qquad (\exists a, b, c, d)(cx_1 a = dy_2 b \wedge cx_2 = dy_1 \wedge z_2 b = z_1 a), \end{aligned}$$

  since

$$\begin{aligned} & f(x_1, x_2)f(y_1, y_2) = f(z_1, z_2) \\ \iff & (\exists a, b)(f(x_1, x_2)f(y_1, y_2) = ab^{-1} \wedge f(z_1, z_2) = ab^{-1}) \\ \iff & (\exists a, b)(x_1^{-1}x_2 y_1^{-1} y_2 = ab^{-1} \wedge z_1^{-1}z_2 = ab^{-1}) \\ \iff & (\exists a, b, c, d)(c^{-1}d = x_2 y_1^{-1} \wedge x_1^{-1}c^{-1}dy_2 = ab^{-1} \wedge z_1^{-1}z_2 = ab^{-1}) \\ \iff & (\exists a, b, c, d)(cx_2 = dy_1 \wedge dy_2 b = cx_1 a \wedge z_2 b = z_1 a). \qquad \square \end{aligned}$$

We are now in a position to prove one direction of Theorem 10.1:

Proposition 10.5 *A finitely generated cancellative semigroup admitting an automatic presentation embeds into a finitely generated virtually abelian group.*

PROOF. Let S be a finitely generated cancellative semigroup with an automatic presentation. By Proposition 10.4, its group of left quotients G has an automatic presentation. Since S is finitely generated, G is also. Theorem 5.1 then shows that G is virtually abelian.

The other direction is provided by:

Proposition 10.6 *Every finitely generated subsemigroup of a virtually abelian group admits an automatic presentation.*

PROOF. Let G be a virtually abelian group. Let $\mathbb{Z}^n$ be a finite-index abelian subgroup of G. By replacing $\mathbb{Z}^n$ by its core (the maximal normal subgroup of G contained in $\mathbb{Z}^n$) if necessary, we may assume that $\mathbb{Z}^n$ is normal in G. Let k be the index of $\mathbb{Z}^n$ in G. Let A be a finite alphabet representing a subset of G, and let S be the semigroup generated by this subset. Throughout this proof, denote by $\overline{w}$ the element of S represented by the word w over an alphabet representing a generating set. This notational distinction is necessary to avoid confusion when there are several representatives for the same element.

Let $B = \{a \in A : \overline{a} \in \mathbb{Z}^n\}$ and let $C = A - B$. So B consists of all letters in A representing elements of the abelian subgroup $\mathbb{Z}^n$ and C consists of letters representing elements of other cosets of $\mathbb{Z}^n$.

Introduce a new alphabet D representing the set

$$\{\overline{w} : w \in C^{\leqslant k}, \overline{w} \in \mathbb{Z}^n\},$$

where $C^{\leqslant k}$ denotes the set of words over C of length at most k. Notice that since the set $C^{\leqslant k}$ is finite, so is D. Furthermore, the semigroup S is generated by $\overline{B} \cup \overline{C} \cup \overline{D}$. We next observe the following lemma:

Lemma 10.7 *Every element of the semigroup S is represented by a word over $B \cup C \cup D$ that contains at most $k^2 - 1$ letters from C.*

PROOF. Let $s \in S$, and let $w \in (B \cup C \cup D)^+$ with $\overline{w} = s$. Then w is of the form

$$u_0 c_1 u_1 c_2 \cdots u_{m-1} c_m u_m, \tag{2}$$

where each u_i lies in $(B \cup D)^*$ and each c_i in C. The aim is to show that such a word w can be transformed into one that still represents $s \in S$ but contains at most $k^2 - 1$ letters from C

First stage. For any word w of the form (2) and for $i = 0, \ldots, m-1$, let $\psi_w(i)$ be maximal such that $\overline{c_{i+1}u_{i+1}\cdots c_m u_m}$ and $\overline{c_{\psi_w(i)+1}u_{\psi_w(i)+1}\cdots c_m u_m}$ lie in the same coset of $\mathbb{Z}^n$ in G. It is clear that $\psi_w(i)$ is always defined and is not less than i. Notice that since there are k distinct cosets of $\mathbb{Z}^n$ in G, $\psi_w(i)$ can take at most k distinct values as i ranges from 0 to $m-1$. Furthermore, for each i, $\overline{c_{i+1}u_{i+1}\cdots c_{\psi_w(i)}u_{\psi_w(i)}}$ lies in $\mathbb{Z}^n$ and so commutes with $\overline{u_i}$.

Define a mapping $\beta' : (B \cup C \cup D)^+ \to (B \cup C \cup D)^+$ as follows: for w of the form (2), $\beta'(w)$ is defined to be

$$u_0 c_1 u_1 c_2 \cdots c_i c_{i+1} u_{i+1} \cdots c_{\psi_w(i)} u_{\psi_w(i)} u_i c_{\psi_w(i)+1} \cdots u_{m-1} c_m u_m,$$

where i is minimal with $\psi_w(i) \neq i$, and $\beta'(w) = w$ if $\psi_w(i) = i$ for all i. By the remark at the end of the last paragraph, $\overline{w} = \overline{\beta'(w)}$.

The mapping $\beta : (B \cup C \cup D)^+ \to (B \cup C \cup D)^+$ is defined by $\beta(w) = (\beta')^p(w)$, where p is minimal with $(\beta')^p(w) = (\beta')^{p+1}(w)$. Again, $\overline{w} = \overline{\beta(w)}$.

So $\beta(w)$ is the word obtained from w by shifting each u_i rightwards to one of at most k distinct positions between the various letters c_j. Thus $\beta(w)$ has the form (2) with at most k of the words u_i being non-empty.

Second stage. Define a mapping $\gamma' : (B \cup C \cup D)^+ \to (B \cup C \cup D)^+$ as follows: if $w \in (B \cup C \cup D)^+$ has a subword $v \in C^{\leqslant k}$ with $\overline{v} \in \mathbb{Z}^n$, then choose the leftmost, shortest such subword and replace it with the letter of D representing the same element of S. (Such a letter exists by the definition of D.)

The mapping $\gamma : (B \cup C \cup D)^+ \to (B \cup C \cup D)^+$ is defined by $\gamma(w) = (\gamma')^p(w)$, where p is minimal with $(\gamma')^p(w) = (\gamma')^{p+1}(w)$. Since each application of γ' that results in a different word decreases the number of letters from C present, such a p must exist. Observe that $\overline{w} = \overline{\gamma(w)}$ and that $\gamma(w)$ cannot contain a subword of k letters from C, for such a string must contain a subword representing an element of $\mathbb{Z}^n$.

Third stage. The final mapping $\delta : (B \cup C \cup D)^+ \to (B \cup C \cup D)^+$ is given by $\delta(w) = (\gamma\beta)^p(w)$, where p is minimal with $(\gamma\beta)^p(w) = (\gamma\beta)^{p+1}(w)$. Observe that $\overline{w} = \overline{\delta(w)}$. Now, $\delta(w)$ is of the form (2) with at most k words u_i being nonempty and does not contain k consecutive letters from C. So separated by the k nonempty words u_i are strings of at most $k-1$ letters from C. So the total number of letters from C in $\delta(w)$ is at most $(k-1) \times (k+1) = k^2 - 1$.

We now return to the proof of Proposition 10.6. Choose a set of representatives $g_1, \ldots, g_k$ for the cosets of $\mathbb{Z}^n$ in G. Suppose $B \cup D = \{b_1, \ldots, b_q\}$.

For $c_1, \ldots, c_m \in C$ with $0 \leqslant m \leqslant k^2 - 1$, define

$$P_{c_1 \cdots c_m} = \{u_0 c_1 u_1 c_2 \cdots u_{m-1} c_m u_m : u_i = b_1^{\alpha_{i,1}} \cdots b_q^{\alpha_{i,q}}, \alpha_{i,j} \in \mathbb{N} \cup \{0\}\}.$$

By Lemma 10.7 and the fact that the elements $\overline{b_j}$ commute, every element of S is represented by an element in at least one of the sets $P_{c_1 \cdots c_m}$. That is,

$$S = \bigcup_{\substack{c_1, \ldots, c_m \in C \\ 0 \leqslant m \leqslant k^2 - 1}} \overline{P_{c_1 \cdots c_m}}. \tag{3}$$

By Theorem 5.1, the virtually abelian group G has an automatic presentation (L, ϕ), where L is the language of words

$$g_h \mathrm{conv}(\varepsilon_1 z_1, \ldots, \varepsilon_n z_n), \tag{4}$$

where $\varepsilon_i \in \{+, -\}$ and z_i is a natural number in reverse binary notation. (In L, the coset representative g_h functions simply as a *symbol*.) The aim is now to show that the subset of L representing elements of S is regular. To do so, it suffices to show that the set of words in L representing elements of $\overline{P_{c_1 \cdots c_m}}$ is regular, since (3) is a finite union.

To this end, fix $c_1, \ldots, c_m$ and write P for $P_{c_1 \cdots c_m}$. Let $z_{i,j} \in \mathbb{Z}^n$ be such that $\overline{b_j} \overline{c_{i+1} \cdots c_m} = \overline{c_{i+1} \cdots c_m} z_{i,j}$. Let $u_0 c_1 u_1 \cdots c_m u_m \in P$ with $u_i = b_1^{\alpha_{i,1}} \cdots b_q^{\alpha_{i,q}}$. Then

$$\overline{u_0 c_1 u_1 \cdots c_m u_m} = \overline{c_1 \cdots c_m} \prod_{i=0}^{m} \prod_{j=1}^{q} z_{i,j}^{\alpha_{i,j}},$$

or, switching to additive notation and supposing $\overline{c_1 \cdots c_m} = g_h(z_1', \ldots, z_n')$ and $z_{i,j} = (z_{i,j,1}, \ldots, z_{i,j,n})$ for all i, j:

$$\overline{u_0 c_1 u_1 \cdots c_m u_m} = g_h(z_1', \ldots, z_n') \sum_{i=0}^{m} \sum_{j=1}^{q} \alpha_{i,j}(z_{i,j,1}, \ldots, z_{i,j,n}).$$

Therefore define $\theta(z_1, \ldots, z_n)$ to be

$$\begin{aligned}
(\exists \alpha_{0,1}, \ldots, \alpha_{m,q})\Big((\alpha_{0,1} \geqslant 0) \wedge \ldots \wedge (\alpha_{m,q} \geqslant 0)& \\
\wedge \Big(z_1 = z_1' + \sum_{i=0}^{m} \sum_{j=1}^{m} \alpha_{i,j} z_{i,j,1}\Big)& \\
\wedge \Big(z_2 = z_2' + \sum_{i=0}^{m} \sum_{j=1}^{m} \alpha_{i,j} z_{i,j,2}\Big)& \\
\vdots \qquad& \\
\wedge \Big(z_n = z_n' + \sum_{i=0}^{m} \sum_{j=1}^{m} \alpha_{i,j} z_{i,j,n}\Big)\Big),&
\end{aligned}$$

where $\alpha_{i,j} z_{i,j,k}$ is understood to be shorthand for

$$\underbrace{\alpha_{i,j} + \ldots + \alpha_{i,j}}_{z_{i,j,k} \text{ times}}.$$

By a special case of Theorem 5.1, the structure $(\mathbb{Z}, +)$ admits an automatic presentation (M, ψ), where M is the set of words ϵz, where $\epsilon \in \{+, -\}$ and z is in reverse binary notation. Furthermore, it is clear that, in this presentation, the relation $\geqslant$ is regular. That is, (M, ψ) is an automatic presentation for $(\mathbb{Z}, +, \geqslant)$.

The set of words in L representing elements of $\overline{P}$ is then

$$\{g_h \mathrm{conv}(z_1, \ldots, z_n) : (\mathbb{Z}, +, \geqslant) \models \theta(\psi(z_1), \ldots, \psi(z_n))\}.$$

(Recall that g_h is the representative of the coset in which $\overline{c_1 \cdots c_m}$ lies.) By Proposition 4.5, this set is a regular subset of L.

Union together the [finitely many] regular subsets of L obtained for the various $c_1, \ldots, c_m$ to see that the set L_S consisting of those words in L representing elements of S is regular. So S admits the automatic presentation $(L_S, \phi|_{L_S})$.

Propositions 10.6 and 10.5 together yield Theorem 10.1.

11 FA-presentability, automaticity, and Cayley graphs

We recall the definition of an automatic semigroup; see [7] for further background information:

Definition 11.1 *A semigroup S is* automatic *if there exists a finite generating set A for S and a regular language L over A such that every element of S is represented by at least one element of L, and, for all $a \in A \cup \{\varepsilon\}$, the relation*

$$L_a = \{(u, v) : ua = v \text{ in } S\}$$

is regular.

If S is an automatic semigroup, then the Cayley graph of S (viewed as a labelled graph) is FA-presentable: the language L (as in the definition of 'automatic') is a regular language of representatives for the vertices of the Cayley graph (the elements of S), and the adjacency relations (the relations L_a for $a \in A$) and the equality relation (the relation L_ε) are all regular.

The converse of this does not hold: let H be the discrete Heisenberg group — that is, the multiplicative group of matrices of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \text{ where } x, y, z \in \mathbb{Z}.$$

The Cayley graph of H is FA-presentable, but H is not automatic [3, p. 651].

Observe that whether the Cayley graph of a semigroup is FA-presentable is not dependent on the choice of generating set:

Proposition 11.2 *Let S be a semigroup and suppose the Cayley graph of S with respect to some finite generating set X is FA-presentable. Let Y be any finite generating set for S. Then the Cayley graph of S with respect to Y is also FA-presentable.*

PROOF. Let (L, ϕ) be an automatic presentation for the Cayley graph of S with respect to X. Let $y \in Y$. Since X generates S, there exists a word $w = w_1 \cdots w_k$ with $w_i \in X$ with $y = w$ in S. Then the adjacency relation L_y is given by

$$\begin{aligned} L_y &= L_{w_1} \circ L_{w_2} \circ \cdots \circ L_{w_k} \\ &= \{(u, v) : (\exists t_1, \dots t_{k-1}) \\ &\qquad ((u, t_1) \in L_{w_1} \wedge (t_2, t_3) \in L_{w_2} \wedge \dots \wedge (t_{k-1}, v) \in L_{w_k})\}. \end{aligned}$$

So the relations L_y are first-order definable and thus regular. So (L, ϕ) is also an automatic presentation for the Cayley graph of S with respect to Y.

Let $\mathcal{S}$, $\mathcal{C}$, and $\mathcal{G}$ be respectively the classes of finitely generated semigroups, finitely generated cancellative semigroups, and finitely generated groups. Let $\mathcal{F}$ be the class of FA-presentable semigroups, $\mathcal{A}$ the class of automatic semigroups, and $\mathcal{T}$ the class of semigroups whose Cayley graphs are FA-presentable.

With this notation, the discussion above can be summarized by the following result:

Proposition 11.3 $\mathcal{S} \cap \mathcal{A} \subsetneq \mathcal{S} \cap \mathcal{T}$ *and* $\mathcal{C} \cap \mathcal{A} \subsetneq \mathcal{C} \cap \mathcal{T}$.

Within the class of finitely generated groups $\mathcal{G}$, we can say more:

Proposition 11.4 $\mathcal{G} \cap \mathcal{F} \subsetneq \mathcal{G} \cap \mathcal{A} \subsetneq \mathcal{G} \cap \mathcal{T}$.

PROOF. By Theorem 5.1, the finitely generated FA-presentable groups are precisely the virtually abelian groups, which are known to be automatic [13, Section 4.1]. Free groups are automatic but not FA-presentable. This establishes the first proper inclusion. For the second proper inclusion, recall that every automatic group has an FA-presentable Cayley graph, but that the non-automatic group H defined above has an FA-presentable Cayley graph.

However, this does not generalize to semigroups:

Proposition 11.5 *The classes $\mathcal{C} \cap \mathcal{F}$ and $\mathcal{C} \cap \mathcal{A}$ are incomparable; thus the classes $\mathcal{S} \cap \mathcal{F}$ and $\mathcal{S} \cap \mathcal{A}$ are also.*

PROOF. The non-inclusion of $\mathcal{C}\cap\mathcal{A}$ in $\mathcal{C}\cap\mathcal{F}$ follows from the non-inclusion of $\mathcal{G}\cap\mathcal{A}$ in $\mathcal{G}\cap\mathcal{F}$. The first author has previously exhibited an example of a non-automatic finitely generated subsemigroup S of a virtually abelian group [6]. This semigroup S must admit an automatic presentation by Theorem 10.1. This establishes the non-inclusion of $\mathcal{C} \cap \mathcal{A}$ in $\mathcal{C} \cap \mathcal{F}$.

12 Unary automatic presentations & Word problems

An automatic presentation (L, ϕ) is *unary* if the language L consists of words over a one-letter alphabet. This section considers unary automatic presentations and connections to word problems for semigroups and groups.

Theorem 12.1 *Let S be a cancellative semigroup that admits a unary automatic presentation. Then S is finite.*

PROOF. The proof of [2, Theorem 7.19], which asserts that groups admitting unary automatic presentations are finite, holds in the more general setting of cancellative semigroups.

However, there do exist infinite non-cancellative semigroups admitting unary automatic presentations: for example, a countable semigroup of right zeros $Z = \{z_i : i \in \mathbb{N}\}$ (with $z_i z_j = z_j$ for all $i, j \in \mathbb{N}$) admits the automatic presentation (L, ϕ), where $L = a^*$ and $\phi : L \to Z$ is defined by $a^k \mapsto z_k$. The multiplication relation is then

$$\{(a^i, a^j, a^j) : i, j \in \mathbb{N}\},$$

which is clearly regular. Note that Z is left-cancellative, so even one-sided cancellative unary FA-presentable semigroups can be infinite.

Recall that the word problem of a group G with respect to a [semigroup] generating set X is the set of words over X that are equal to 1_G. The word problem for a group is said to be *one-counter* if it is accepted by a one-counter automaton. (See [1] for background information on one-counter automata.) The word problem for a semigroup S with respect to a generating set X, as defined by Duncan and Gilman [11], is the set $\{u\#v^r : u, v \in X^*, u = v \text{ in } S\}$, where v^r denotes the reverse of the word v and # is a new symbol not in X.

Blumensath [2, Proposition 7.22] proved that the Cayley graph of a finitely generated group G is virtually cyclic if and only if the Cayley graph of G admits a unary automatic presentation. This, together with Herbst's [19] result that finitely generated groups with one-counter word problem are precisely the virtually cyclic groups, yields the following corollary:

Corollary 12.2 *A finitely generated group has a one-counter word problem if and only if its Cayley graph has a unary automatic presentation.*

Finitely generated virtually abelian groups — which are precisely the finitely generated FA-presentable groups — are characterized by having word problems recognizable by blind one-counter automata [12, Theorem 1]. This fact, together with the preceding corollary, suggests the following question:

Problem 12.3 *Are finitely generated FA-presentable semigroups classifiable by having word problems recognizable by some 'natural' class of automata?*

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