Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

The Bergman property for semigroups

V. Maltcev, J. D. Mitchell, and N. Ruškuc

Abstract

In this article, we study the Bergman property for semigroups and the associated notions of cofinality and strong cofinality. A large part of the paper is devoted to determining when the Bergman property, and the values of the cofinality and strong cofinality, can be passed from semigroups to subsemigroups and vice versa.

Numerous examples, including many important semigroups from the literature, are given throughout the paper. For example, it is shown that the semigroup of all mappings on an infinite set has the Bergman property but that its finitary power semigroup does not; the symmetric inverse semigroup on an infinite set and its finitary power semigroup have the Bergman property; the Baer-Levi semigroup does not have the Bergman property.

1. *Introduction*

In this paper, we will consider the notion of Bergman's property for semigroups. This property has already been studied by several authors for groups, and we begin by discussing Bergman's property and related notions in this context.

Let G be a group. If U is a (group) generating set for G, then

$$G = \bigcup_{i=1}^{\infty} (U \cup U^{-1})^i$$

where $(U \cup U^{-1})^i = \{ u_1u_2 \cdots u_i : u_1, u_2, \ldots, u_i \in U \cup U^{-1} \}$. It is not always true that for a group G and a generating set U for G that

$$G = \bigcup_{i=1}^{j} (U \cup U^{-1})^i$$

for some $j \in \mathbb{N}$. For example, the free group $FG(X)$ on any set X does not satisfy this property. A group G is *group Cayley bounded* with respect to a subset U if there exists $n \in \mathbb{N}$ such that $G = V \cup \cdots \cup V^n$ where $V = U \cup U^{-1}$. In other words, the minimum distance between any two elements in the Cayley graph of G with respect to U is at most n. So, the free group $FG(X)$ is not group Cayley bounded with respect to X but is group Cayley bounded with respect to itself. More surprisingly, there are examples of non-finitely generated groups G that are group Cayley bounded with respect to every generating set. One of the first examples of such a group was provided by Bergman in [**3**] where it was shown that the symmetric group $\mathrm{Sym}(\Omega)$ is group Cayley bounded with respect to every generating set for all sets Ω. Consequently, a group is said to have the *group Bergman property* if it is Cayley bounded with respect to every generating set. Droste and Göbel [**6**] give sufficient conditions for a permutation group to have the group Bergman property. Examples of groups satisfying their conditions are: the symmetric groups, homeomorphism groups of Cantor's discontinuum $\mathfrak{C}$, the rationals $\mathbb{Q}$, and the irrationals $\mathbb{I}$. Other notable examples of groups satisfying the group Bergman property are: the infinite cartesian power of any finite perfect group, the full groups of measure-preserving and ergodic

---

2000 *Mathematics Subject Classification* 20M10 (primary), 20M20 (secondary).

transformations on the unit interval [**7**], ω_1-existentially closed groups [**5**], and the groups of measure-preserving homeomorphisms of the Cantor space or Lipschitz homeomorphisms of the Baire space, and certain closed oligomorphic subgroups of $\text{Sym}(\mathbb{N})$ [**16**].

A semigroup S is said to be *semigroup Cayley bounded* with respect to a generating set U if $S = U \cup U^2 \cup \cdots \cup U^n$ for some $n \in \mathbb{N}$. We will say that a semigroup S has the *semigroup Bergman property* if it is semigroup Cayley bounded with respect to every generating set.

Note that we must make separate definitions of these notions for semigroups because the definitions for groups involve inverses. The fact that the definitions of these two properties for semigroups and groups are not the same, accounts for the use of the word 'group' in the definitions above.

After making these definitions it is most natural to ask the following questions. Are there natural examples of semigroups that satisfy the semigroup Bergman property? In particular, do the semigroup theoretic analogues of the symmetric group satisfy the semigroup Bergman property? Groups are natural examples of semigroups, so how does the semigroup Bergman property compare with the group Bergman property? In this paper we attempt to answer these questions.

If a group satisfies the semigroup Bergman property, then it certainly satisfies the group Bergman property. It is not known if the converse is true or not. However, the majority of the groups that are known to satisfy the group Bergman property, such as those groups mentioned above, also satisfy the semigroup Bergman property; for more details see Corollary 2.5.

To answer the first of the questions above, let us introduce the *full transformation semigroup* of all self-maps of a set Ω, denoted by $\text{Self}(\Omega)$. Every semigroup can be embedded into a full transformation semigroup $\text{Self}(\Omega)$ for some set Ω. As such $\text{Self}(\Omega)$ plays an analogous role in semigroup theory as that played by $\text{Sym}(\Omega)$ in group theory. Other counterparts of $\text{Self}(\Omega)$ and $\text{Sym}(\Omega)$ are $\text{SymInv}(\Omega)$, $\text{Part}(\Omega)$, and $\text{Bin}(\Omega)$ the semigroups of all injective partial self-maps (the so-called *symmetric inverse semigroup*), partial self-maps, and binary relations, respectively, on Ω.

Most notable among the semigroups that we will show to satisfy the semigroup Bergman property are: $\text{Self}(\Omega)$, $\text{SymInv}(\Omega)$, $\text{Bin}(\Omega)$, $\text{Part}(\Omega)$, semigroups of continuous functions on the rationals $\mathbb{Q}$, irrationals $\mathbb{I}$, Cantor's discontinuum, and the finitary power semigroup of $\text{SymInv}(\Omega)$ (see Section 4). Equally notable for not satisfying the semigroup Bergman property are: the Baer-Levi semigroup on $\mathbb{N}$, the finitary power semigroups of $\text{Self}(\Omega)$, $\text{Bin}(\Omega)$, $\text{Part}(\Omega)$, and the semigroup of bounded self-maps of $\mathbb{Q}$ (see Section 5). The techniques used in resolving these specific examples are based on the more general results in Sections 2 and 3.

2. *Cofinality and Strong Cofinality*

We require the following notions analogous to those with the same names introduced by Macpherson and Neumann [**18**] and Droste and Göbel [**6**].

A sequence of sets $(U_i)_{i<\lambda}$, for some cardinal λ, such that $U_i \subseteq U_j$ for all $i \leq j < \lambda$ is called a *chain*. Let S be a non-finitely generated semigroup. Then the *cofinality* of S is the least cardinal λ such that there exists a chain of proper subsemigroups $(U_i)_{i<\lambda}$ of S where $S = \bigcup_{i<\lambda} U_i$. We follow the usual convention that λ is the collection of all ordinals less than λ. We will denote the cofinality of S by $\text{cf}(S)$ and refer to subsemigroups $(U_i)_{i<\text{cf}(S)}$ satisfying the above property as a *cofinal chain* for S. Obviously, the above definition of cofinality cannot be applied to finitely generated semigroups. The *strong cofinality* of S is the least cardinal λ such that there exists a chain of proper subsets $(U_i)_{i<\lambda}$ of S where for all $i < \lambda$ there exists $j < \lambda$ such that $U_iU_i \subseteq U_j$ and $S = \bigcup_{i<\lambda} U_i$. The strong cofinality of S is denoted by $\text{scf}(S)$ and a *strong cofinal chain* is defined analogously to a cofinal chain. It is clear that $\text{scf}(S) \leq \text{cf}(S)$.

The following technical lemma shows that the notions of cofinality and strong cofinality used here, when applied to a group, are equivalent to those used in [**3**], [**6**], and [**18**]. Lemma 2.1 and Corollary 2.5 follow by similar arguments as those given on page 435 and in the proofs of Theorems 5 and 6 in [**3**]. We include the proofs of these results for the sake of completeness.

LEMMA 2.1. *Let G be a non-finitely generated group. Then*

(i) $\mathrm{cf}(G)$ *is the least cardinal of a cofinal chain of subgroups for G;*

(ii) $\mathrm{scf}(G)$ *is the least cardinal λ of a strong cofinal chain $(U_i)_{i<\lambda}$ for G satisfying $U_i = U_i^{-1}$ for all $i < \lambda$.*

Proof. To prove Part (i), let λ be the least cardinal of a cofinal chain of subgroups for G and let $\kappa = \mathrm{cf}(G)$. By definition, $\mathrm{cf}(G) = \kappa \leq \lambda$. To prove that the converse inequality holds, note that there exists a chain of proper subsemigroups $(V_i)_{i<\kappa}$ of G where $G = \bigcup_{i<\kappa} V_i$. Hence

$$G = G^{-1} = \bigcup_{i<\kappa} V_i^{-1}$$

and so

$$G = G \cap G^{-1} = \bigcup_{i<\kappa} V_i \cap V_i^{-1}.$$

Although there may be $i < \kappa$ such that $V_i \cap V_i^{-1} = \emptyset$, after some point all the terms in $(V_i \cap V_i^{-1})_{i<\kappa}$ are nonempty. Thus we may assume without loss of generality that all the terms in $(V_i \cap V_i^{-1})_{i<\kappa}$ are nonempty. Hence $(V_i \cap V_i^{-1})_{i<\kappa}$ is a chain of proper subgroups of G and the proof is complete.

The proof of part (ii) is analogous and omitted. □

The following proposition relates cofinality, strong cofinality and the semigroup Bergman property. The proposition is analagous to [**6**, Proposition 2.2] and although the proof is similar we include it for completeness.

PROPOSITION 2.2. *Let S be a non-finitely generated semigroup. Then*

(i) $\mathrm{scf}(S) > \aleph_0$ *if and only if S has the semigroup Bergman property and* $\mathrm{cf}(S) > \aleph_0$*;*

(ii) *if* $\mathrm{scf}(S) > \aleph_0$, *then* $\mathrm{scf}(S) = \mathrm{cf}(S)$.

Proof. **Part (i).** ($\Rightarrow$) Since $\mathrm{cf}(S) \geq \mathrm{scf}(S)$ it follows immediately that $\mathrm{cf}(S) > \aleph_0$. Let U be any generating set for S and let $V_i = U \cup U^2 \cup \cdots \cup U^i$. Then $(V_i)_{i\in\mathbb{N}}$ is a chain of proper subsets of S such that $V_iV_i \subseteq V_{2i}$. Since U is a generating set for S, it also follows that $S = \bigcup_{i\in\mathbb{N}} V_i$. Hence, since $\mathrm{scf}(S) > \aleph_0$, there exists $j \in \mathbb{N}$ such that $S = V_j$ and so S is Cayley bounded with respect to U.

($\Leftarrow$) Again seeking a contradiction, assume that $\mathrm{scf}(S) = \aleph_0$ and $(U_i)_{i\in\mathbb{N}}$ is a strong cofinal chain for S. Then $S = \bigcup_{i\in\mathbb{N}} U_i$ and so certainly $S = \bigcup_{i\in\mathbb{N}} \langle U_i \rangle$. Since $\mathrm{cf}(S) > \aleph_0$ it follows that $\langle U_r \rangle = S$ for some $r \in \mathbb{N}$. Hence since S has the semigroup Bergman property $S = U_r \cup U_r^2 \cup \cdots \cup U_r^n$ for some n. But $(U_i)_{i\in\mathbb{N}}$ is a strong cofinal chain and so $U_r \cup U_r^2 \cup \cdots \cup U_r^n \subseteq U_j$ for some j. Thus $S \subseteq U_j$, a contradiction.

Part (ii). Let $\mathrm{scf}(S) = \kappa$ and let $(U_i)_{i<\kappa}$ be a strong cofinal chain for S. Without loss of generality assume that $U_iU_i \subseteq U_{i+1}$ for all $i < \kappa$. If I is the set of all limit ordinals less than κ, then for any $i \in I$, $V_i = \bigcup_{j<i} U_j$ is a proper subsemigroup of S. Thus

$$\mathrm{scf}(S) \leq \mathrm{cf}(S) \leq |I| \leq \kappa = \mathrm{scf}(S)$$

giving equality throughout. □

The following lemma will be used later in the paper as it gives a convenient way of proving that a semigroup has uncountable strong cofinality. The idea behind it is taken from [**3**] and [**17**]; we include a proof for completeness.

LEMMA 2.3. *Let S be a non-finitely generated semigroup. Then* $\mathrm{scf}(S) > \aleph_0$ *if and only if every function* $\Phi : S \to \mathbb{N}$ *satisfying*

$$(st)\Phi \leq (s)\Phi + (t)\Phi + k_\Phi, \tag{2.1}$$

for all $s, t \in S$ *and some constant* $k_\Phi \in \{0, 1, 2, \ldots\}$*, is bounded above.*

Proof. (⇒) Let $\Phi : S \to \mathbb{N}$ be any function satisfying (2.1) and let

$$U_n = \{\, s \in S \;:\; (s)\Phi \leq n \,\}.$$

Then $S = \bigcup_{n\in\mathbb{N}} U_n$ and $U_m U_n \subseteq U_{m+n+k_\Phi}$. Hence, since $\mathrm{scf}(S) > \aleph_0$, we have that $S = U_n$ for some n. Thus n is the required upper bound for Φ.

(⇐) By Proposition 2.2(i), it suffices to prove that $\mathrm{cf}(S) > \aleph_0$ and S has the semigroup Bergman property. Seeking a contradiction, assume that $\mathrm{cf}(S) = \aleph_0$. Then there exists a cofinal chain $(S_n)_{n\in\mathbb{N}}$ for S. Define $\Phi : S \to \mathbb{N}$ by

$$(s)\Phi = \min\{\, n \;:\; s \in S_n \,\}.$$

The function Φ satisfies (2.1) with $k_\Phi = 0$ but is unbounded above, a contradiction. Hence $\mathrm{cf}(S) > \aleph_0$.

Again in order to produce a contradiction, assume that there exists a generating set U for S such that S is not Cayley bounded with respect to U. As in the previous paragraph, define $\Phi : S \to \mathbb{N}$ by

$$(s)\Phi = \min\{\, n \;:\; s \in U^n \,\}.$$

Again, Φ satisfies (2.1) with $k_\Phi = 0$ but is unbounded above, a contradiction. Thus S satisfies the semigroup Bergman property and the proof is complete. □

The following notion and the subsequent lemma yield a convenient method for proving that a semigroup has uncountable strong cofinality. A semigroup S is called *strongly distorted* if there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of natural numbers and $N_S \in \mathbb{N}$ such that for all sequences $(s_n)_{n\in\mathbb{N}}$ of elements from S there exist $t_1, t_2, \ldots, t_{N_S} \in S$ such that each s_n can be written as a product of length at most a_n in the letters $t_1, \ldots, t_{N_S}$. The following lemma was suggested to us by Y. Cornulier and a similar result appears in Khelif [**17**, Theorem 6].

LEMMA 2.4. *If S is non-finitely generated and strongly distorted, then* $\mathrm{scf}(S) > \aleph_0$.

Proof. Let $\Phi : S \longrightarrow \mathbb{N}$ be any function satisfying (2.1) and seeking a contradiction assume that Φ is unbounded above. Let $\{a_n\}_{n\in\mathbb{N}}$ and $N_S \in \mathbb{N}$ be as given in the definition of a strongly distorted semigroup S and assume without loss of generality that $\{a_n\}_{n\in\mathbb{N}}$ is strictly increasing. Then there exist $s_1, s_2, \ldots \in S$ such that $(s_n)\Phi > a_n^2$ for all n. Since S is strongly distorted there exist $t_1, \ldots, t_{N_S} \in S$ such that each s_n can be written as a product of length at most a_n in the letters $t_1, \ldots, t_{N_S}$. But if $M = \max\{(t_1)\Phi, \ldots, (t_{N_S})\Phi\}$, then

$$(s_n)\Phi \leq a_n \cdot k_\Phi + a_n \cdot M < a_n^2$$

for all sufficiently large n, a contradiction. Thus Φ is bounded above and so, by Lemma 2.3, $\mathrm{scf}(S) > \aleph_0$. □

In light of Proposition 2.2 we observe that for a non-finitely generated semigroup S there are four possibilities:

(i) $\mathrm{cf}(S) = \mathrm{scf}(S) > \aleph_0$ and so S satisfies the semigroup Bergman property;
(ii) $\mathrm{cf}(S) > \aleph_0 = \mathrm{scf}(S)$ and so S does not satisfy the semigroup Bergman property;
(iii) $\mathrm{cf}(S) = \mathrm{scf}(S) = \aleph_0$ and S satisfies the semigroup Bergman property;
(iv) $\mathrm{cf}(S) = \mathrm{scf}(S) = \aleph_0$ and S does not satisfy the semigroup Bergman property.

Of course, the next question is: are there examples of semigroups that satisfy each of these four cases? Finding an example that satisfies case (iv) is routine. For example, the free semigroup on an infinite set X has countable cofinality and does not satisfy the semigroup Bergman property. The next corollary relates the group and semigroup Bergman properties, and consequently provides several examples of semigroups that satisfy case (i) above.

Corollary 2.5. *If a group G has $\mathrm{scf}(G) > \aleph_0$, then G satisfies both the group and semigroup Bergman properties.*

In particular, $\mathrm{Sym}(\Omega)$*, the homeomorphism groups of* $\mathfrak{C}$, $\mathbb{Q}$*, and* $\mathbb{I}$*, and the infinite cartesian power of any finite perfect group satisfy both the group and semigroup Bergman properties.*

Proof. Since $\mathrm{scf}(G) > \aleph_0$ it follows from Propostion 2.2(i) that G satisfies the semigroup Bergman property. Now, by Lemma 2.1 the least cardinal of a cofinal chain of subgroups for G is greater than $\aleph_0$. Hence by [**6**, Proposition 2.2] G satisfies the group Bergman property.

By Lemma 2.1 and Droste and Göbel [**6**] it follows that $\mathrm{scf}(G) > \aleph_0$ when G is any of the groups $\mathrm{Sym}(\Omega)$ or the homeomorphism groups of $\mathfrak{C}$, $\mathbb{Q}$, or $\mathbb{I}$. Again by Lemma 2.1 and Cornulier [**5**], the infinite cartesian power G of any finite perfect group satisfies $\mathrm{scf}(G) > \aleph_0$. □

The following example stems from [**6**] and provides a semigroup satisfying case (ii) above.

Example 2.6. Let $\mathrm{BSym}(\mathbb{Q})$ denote the group of all permutations $f \in \mathrm{Sym}(\mathbb{Q})$ where there exists $k \in \mathbb{N}$ such that $|x - (x)f| < k$ for all $x \in \mathbb{Q}$, called the *bounded permutation group* on $\mathbb{Q}$. Droste and Göbel [**6**] proved that the least cardinal of a cofinal chain of subgroups for $\mathrm{BSym}(\mathbb{Q})$ is uncountable but that $\mathrm{BSym}(\mathbb{Q})$ does not satisfy the group Bergman property. By [**6**, Proposition 2.2] and Lemma 2.1, $\mathrm{cf}(\mathrm{BSym}(\mathbb{Q})) > \aleph_0$ and $\mathrm{scf}(\mathrm{BSym}(\mathbb{Q})) = \aleph_0$. Thus by Proposition 2.2(i), $\mathrm{BSym}(\mathbb{Q})$ does not satisfy the semigroup Bergman property. So, $\mathrm{BSym}(\mathbb{Q})$ is an example of a (semi)group that satisfies case (ii) above.

It remains to find an example of semigroup satisfying case (iii). Khelif [**17**] provided an example of a group G where the least cardinal of a cofinal chain of subgroups for G is $\aleph_0$ and that satisfies the group Bergman property. Using the same reasoning as in Example 2.6 we deduce that Khelif's group satisfies (iii). However, Khelif's construction is somewhat too complicated to include here. Moreover it is straightforward to directly construct examples of semigroups, that are not groups, with countable cofinality and that satisfy the semigroup Bergman property.

The following examples are trivial but are included for the sake of completeness.

Example 2.7. A semigroup S of *left zeros* satisfies $xy = x$ for all $x, y \in S$. The unique generating set for such a semigroup S is S itself. Therefore every semigroup of left zeros has the semigroup Bergman property. If S is infinite, then S is not finitely generated. Hence if (the generating set) S is partitioned into $S_1, S_2, \ldots$, then $(\langle S_1, \ldots, S_i \rangle)_{i\in\mathbb{N}} = (S_1 \cup \cdots \cup S_i)_{i\in\mathbb{N}}$ is a cofinal chain for S. Hence $\mathrm{cf}(S) = \aleph_0$.

Example 2.8. A *rectangular band* R is the direct product $I \times \Lambda$ of arbitrary sets I and Λ with multiplication $(i, \lambda)(j, \mu) = (i, \mu)$. Every generating set for R must for all $i \in I$ and $\mu \in \Lambda$ contain elements of the form (i, λ) and (j, μ) for some $\lambda \in \Lambda$ and $j \in I$. Therefore if $R = \langle U \rangle$, then $R = U^2$ and R has the semigroup Bergman property. Moreover, if R is infinite, then, as in Example 2.7, $\mathrm{cf}(R) = \aleph_0$.

An element s of an arbitrary semigroup S is *indecomposable* if $s \neq xy$ for all $x, y \in S$. The indecomposable elements of S must be contained in every generating set. If S is Cayley bounded with respect to a generating set consisting of indecomposable elements, then S satisfies the semigroup Bergman property.

Example 2.9. Let S be the semigroup defined by the presentation

$$\langle A \mid abc = ab \ \ (a, b, c \in A) \rangle$$

for some infinite set of generators A. Then every element in A is indecomposable in S and $S = A \cup A^2$. Hence S has the semigroup Bergman property and $\mathrm{cf}(S) = \aleph_0$, as in Example 2.7.

Example 2.10. Let S be the set $\mathbb{N} \times \mathbb{N}$ with componentwise addition. Then the set

$$(\{1\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1\})$$

is a generating set for S consisting of indecomposable elements. Therefore S has the semigroup Bergman property and $\mathrm{cf}(S) = \aleph_0$, as in Example 2.7.

Example 5.7 is a further semigroup having uncountable cofinality and not having the semigroup Bergman property. However, this example relies on results from Section 4 and so cannot be included here.

3. *Subsemigroups, ideals, and homomorphic images*

In this section we give the main tools that will provide a method to find the cofinality and strong cofinality of the semigroup $\mathrm{Self}(\Omega)$ of all self-maps of any infinite set Ω, and several other fundamental semigroups.

Theorem 3.1. *Let S be a non-finitely generated semigroup that is Cayley bounded with respect to the union of a subsemigroup T and a finite set F. Then $\mathrm{cf}(T) \le \mathrm{cf}(S)$ and $\mathrm{scf}(T) \le \mathrm{scf}(S)$.*

Proof. We will prove the theorem for strong cofinality. The proof for cofinality follows by an analogous argument.

Let $\lambda = \mathrm{scf}(S)$ and $(S_i)_{i<\lambda}$ be a strong cofinal chain for S. Set $T_i = S_i \cap T$ for all $i < \lambda$. We will prove that $T_i \subsetneq T$ for all i. Assuming the contrary, there exists $i < \lambda$ such that $T_i = T$. Since S is Cayley bounded with respect to $T \cup F$, there exists $n \in \mathbb{N}$ such that $S = (T \cup F) \cup (T \cup F)^2 \cup \cdots \cup (T \cup F)^n$. The set F is finite and so there exists $j < \lambda$ such that $F \subseteq S_j$. Thus

$T \cup F = T_i \cup F$ is a subset of $S_{\max(i,j)}$. Since $(S_i)_{i<\lambda}$ is a cofinal chain, it follows that $S = S_m$ for some $m > \max(i, j)$, a contradiction. So, we have shown that for all $i < \lambda$, the set T_i is properly contained in T.

To conclude, $T_iT_i = (S_i \cap T)(S_i \cap T) \subseteq S_iS_i \cap T \subseteq S_k \cap T = T_k$, for some $k > i$. Therefore $\operatorname{scf}(T) \leq \lambda$. □

THEOREM 3.2. *Let T be a subsemigroup of a non-finitely generated semigroup S with $S \setminus T$ finite. Then $\operatorname{cf}(T) = \operatorname{cf}(S)$ and $\operatorname{scf}(T) = \operatorname{scf}(S)$.*

Furthermore, if T satisfies the semigroup Bergman property, then S does also.

Although Theorem 3.2 is similar to Theorem 3.1 it is somewhat harder to prove. The proof of Theorem 3.2 requires Lemma 2.3 and the following technical lemma.

LEMMA 3.3. *Let T be a subsemigroup of a non-finitely generated semigroup S with $S \setminus T$ finite and $T \cap \langle\, S \setminus T \,\rangle \neq \emptyset$. Then $T \cap \langle\, S \setminus T \,\rangle$ is finitely generated.*

Proof. It is shown in [**15**] that if U is a finitely generated semigroup and $V \leq U$ with $U \setminus V$ finite, then V is finitely generated also.

So, $T \cap \langle\, S \setminus T \,\rangle = \langle\, S \setminus T \,\rangle \setminus (S \setminus T) \leq \langle\, S \setminus T \,\rangle$. By assumption, $S \setminus T$ is finite and so $T \cap \langle\, S \setminus T \,\rangle$ has finite complement in $\langle\, S \setminus T \,\rangle$ and $\langle\, S \setminus T \,\rangle$ is finitely generated. Thus $T \cap \langle\, S \setminus T \,\rangle$ is finitely generated. □

Equipped with Lemmas 2.3 and 3.3 we can now give the proof of Theorem 3.2.

Proof of Theorem 3.2. Recall that T is a subsemigroup of a non-finitely generated semigroup S with $S \setminus T$ finite. Assume without loss of generality that S has an identity 1_S and that $1_S \in S \setminus T$. Note that T is not finitely generated, otherwise S would be finitely generated. The proof has three parts.

Part 1: cf(T)=cf(S).

The cofinality of T is at most the cofinality of S by Theorem 3.1; that is,

$$\operatorname{cf}(T) \leq \operatorname{cf}(S).$$

It remains to prove the opposite inequality: $\operatorname{cf}(T) \geq \operatorname{cf}(S)$. Let $\operatorname{cf}(T) = \lambda$ and let $(T_i)_{i<\lambda}$ be a cofinal chain for T. From this cofinal chain, we will construct a chain with length λ of proper subsemigroups of S whose union is S.

The first step is to give an alternate cofinal chain $(U_i)_{i<\lambda}$ for T that involves $S \setminus T$. Define

$$U_i = \{\, t \in T \;:\; (\forall x, y \in S \setminus T)\; (xty \in T_i \cup (S \setminus T)) \,\}.$$

To prove that $(U_i)_{i<\lambda}$ is a chain, let $i \leq j$ and let $t \in U_i$. Then $xty \in T_i \leq T_j$ whenever $xty \in T$, $x, y \in S \setminus T$. Thus U_i is contained in U_j and so $(U_i)_{i<\lambda}$ is a chain. Next we prove that the union of the sets U_i, $i < \lambda$, equals T. Let $t \in T$. Then there are only finitely many products xty in T where $x, y \in S \setminus T$. Hence there exists $i < \lambda$ such that all these products are in T_i. Hence $t \in U_i$ and so $\bigcup_{i<\lambda} U_i = T$.

It remains to prove that U_i is a proper subsemigroup of T for all $i < \lambda$. Let $i < \lambda$, $s, t \in U_i$, and $x, y \in S \setminus T$ such that $xsty \in T$. Of course such x and y exist since $1_S \in S \setminus T$. If either xs or $ty \in S \setminus T$, then $(xs)ty = xs(ty) \in T$ and so $xsty \in T_i$. On the other hand, if $xs, ty \in T$, then $xs1_S, 1_Sty \in T$ and so $xs1_S, 1_Sty \in T_i$. But T_i is a subsemigroup and so $xsty \in T_i$. Thus

$st \in U_i$ and U_i is a subsemigroup. If $x = y = 1_S$ and $t \in U_i$, then $xty \in T$ and so $t = xty \in T_i$. Hence U_i is contained in T_i and as such is a proper subsemigroup of T.

Now, let us construct a cofinal chain for S using the chain $(U_i)_{i<\lambda}$. Let S_i, $i < \lambda$, be the subsemigroup of S generated by U_i and $S \setminus T$; that is, $S_i = \langle\, U_i, S \setminus T\,\rangle$. Clearly, $(S_i)_{i<\lambda}$ is a chain and $\bigcup_{i<\lambda} S_i = S$. So, to prove that $(S_i)_{i<\lambda}$ is a cofinal chain for S it suffices to show that every S_i is properly contained in S. We will do this by showing that $S_i \cap T \leq T_i$ for all $N < i < \lambda$ for some N.

By Lemma 3.3, $T \cap \langle\, S \setminus T\,\rangle$ is finitely generated and so there exists $N < \lambda$ such that for all $i > N$ we have $T \cap \langle\, S \setminus T\,\rangle \subseteq U_i$. If $t \in S_i \cap T$ for some $i > N$, then there exist $w_1, w_2, \ldots, w_{k+1} \in \langle\, S \setminus T\,\rangle$ and $u_1, u_2, \ldots, u_k \in U_i$ such that

$$t = w_1 u_1 w_2 u_2 \cdots u_k w_{k+1}, \tag{3.1}$$

and $2k + 1$ is the least length of such a product. If $w_j \in \langle\, S \setminus T\,\rangle \cap T$, then, since $i > N$, $w_j \in U_i$ and the product (3.1) could be shortened. So, we conclude that $w_1, w_2, \ldots, w_{k+1} \in S \setminus T$. Consider the products $w_m u_m, w_n u_n w_{n+1} \in S$ where $1 \leq m, n \leq k$. If either product lies in $S \setminus T$, then again (3.1) could be shortened. Hence $w_m u_m, w_n u_n w_{n+1} \in T$, and by the definition of U_i, $w_m u_m, w_n u_n w_{n+1} \in T_i$. But T_i is a subsemigroup of T and so $t \in T_i$.

We conclude that $S_i \cap T \leq T_i$ and so if $S = S_i$ for some i, then $T = S \cap T = S_i \cap T \leq T_i < T$, a contradiction. Hence S_i is a proper subsemigroup of S. We have shown that $\mathrm{cf}(T) \geq \mathrm{cf}(S)$ and this part of the proof is concluded.

Part 2: scf(*T*)=scf(*S*).

If $\mathrm{scf}(S) = \aleph_0$, then by Theorem 3.1 we have $\aleph_0 \leq \mathrm{scf}(T) \leq \mathrm{scf}(S) = \aleph_0$, giving equality throughout. Assume that $\mathrm{scf}(S) > \aleph_0$. Then if $\mathrm{scf}(T) > \aleph_0$, we could deduce that $\mathrm{scf}(T) = \mathrm{cf}(T) = \mathrm{cf}(S) = \mathrm{scf}(S)$, by Proposition 2.2(ii) and the first part of the theorem. So, we are left with the task of proving that $\mathrm{scf}(T) > \aleph_0$.

Let $\Psi : T \to \mathbb{N}$ be any function satisfying

$$(st)\Psi \leq (s)\Psi + (t)\Psi + k_\Psi$$

for all $s, t \in T$ and for some constant $k_\Psi \in \{0, 1, 2, \ldots\}$. By Lemma 2.3, it suffices to prove that Ψ is bounded. We proceed in a similar fashion as in the proof of the previous part of the theorem. That is, we define $\Phi : T \to \mathbb{N}$ using Ψ and subsequently define $\Upsilon : S \to \mathbb{N}$ satisfying (2.1). Let $\Phi : T \to \mathbb{N}$ be defined by

$$(t)\Phi = \max\{\, (xty)\Psi \;:\; x, y \in S \setminus T,\ xty \in T\,\}.$$

Note that Φ is well-defined since the set $\{\, (xty)\Psi \;:\; x, y \in S \setminus T,\ xty \in T\,\}$ is non-empty and finite. To prove that Φ satisfies (2.1) let $s, t \in T$. Then

$$(st)\Phi = \max\{\, (x \cdot st \cdot y)\Psi \;:\; x, y \in S \setminus T,\ x \cdot st \cdot y \in T\,\}.$$

The set $\{\, (x \cdot st \cdot y)\Psi \;:\; x, y \in S \setminus T,\ x \cdot st \cdot y \in T\,\}$ is the union of the following three sets

$$A = \{\, (xs \cdot t \cdot y)\Psi \;:\; x, y \in S \setminus T,\ xs \cdot t \cdot y \in T, xs \in S \setminus T\,\},$$

$$B = \{\, (x \cdot s \cdot ty)\Psi \;:\; x, y \in S \setminus T,\ x \cdot s \cdot ty \in T, ty \in S \setminus T\,\},$$

$$C = \{\, (xs \cdot ty)\Psi \;:\; x, y \in S \setminus T,\ xs \cdot ty \in T, xs, ty \in T\,\}.$$

So,

$$\begin{aligned}(st)\Phi &\leq \max\{\, \max A,\ \max B,\ \max C\,\} \\ &\leq \max\{(t)\Phi, (s)\Phi, (s)\Phi + (t)\Phi + k_\Psi\} = (s)\Phi + (t)\Phi + k_\Psi,\end{aligned}$$

and Φ satisfies (2.1).

As the final step in the proof, define $\Upsilon : S \to \mathbb{N}$ by

$$(s)\Upsilon = \begin{cases} (s)\Phi & \text{if } s \in T \\ 1 & \text{if } s \in S \setminus T. \end{cases}$$

Note that $(t)\Upsilon = (t)\Phi \geq (t)\Psi$ for all $t \in T$. So, to prove that Ψ is bounded it suffices to prove that Υ satisfies (2.1). Let $s, t \in S$. Then there are four cases to consider.

Firstly, if $s, t \in T$, then Υ trivially satisfies (2.1) with constant k_Ψ since Φ does.

Secondly, let

$$M = \max\{\, (st)\Upsilon \ :\ s, t \in S \setminus T \,\}.$$

Then for all $s, t \in S \setminus T$ we have that if $st \in S \setminus T$, then $(st)\Upsilon = 1 < (s)\Upsilon + (t)\Upsilon + M$. On the other hand, if $st \in T$, then $(st)\Upsilon = (st)\Phi \leq M$. In either case,

$$(st)\Upsilon \leq (s)\Upsilon + (t)\Upsilon + M.$$

Thirdly, let $s \in S \setminus T$ and $t \in T$. If $st \in S \setminus T$, then $(st)\Upsilon = 1 \leq (s)\Upsilon + (t)\Upsilon$. Otherwise, $(st)\Upsilon = (st)\Phi = (x \cdot st \cdot y)\Psi$ for some $x, y \in S \setminus T$ with $x \cdot st \cdot y \in T$, from the definitions of Φ and Υ. Let

$$P = \{\, (us \cdot t \cdot v)\Psi \ :\ us, u, v \in S \setminus T,\ ustv \in T \,\}$$

$$Q = \{\, (us \cdot t \cdot v)\Psi \ :\ us \in T,\ u, v \in S \setminus T,\ ustv \in T \,\}.$$

Then $\max\{P\} \leq (t)\Phi = (t)\Upsilon$ from the definition and for all $(us \cdot t \cdot v)\Psi \in Q$

$$(us \cdot t \cdot v)\Psi \leq (us \cdot t)\Phi \leq (us)\Phi + (t)\Phi + k_\Psi \leq M + (t)\Phi + k_\Psi.$$

This implies that $\max\{Q\} \leq M + (t)\Phi + k_\Psi = M + (t)\Upsilon + k_\Psi$. Hence

$$(st)\Upsilon \leq \max\{\, P, Q \,\} \leq (t)\Upsilon + M + k_\Psi \leq (s)\Upsilon + (t)\Upsilon + M + k_\Psi.$$

Finally, if $s \in T$ and $t \in S \setminus T$, then $(st)\Upsilon \leq (s)\Upsilon + (t)\Upsilon + M + k_\Psi$ follows by symmetry.

Therefore Υ satisfies (2.1) with constant $M + k_\Psi$, and the proof of this part of the theorem is complete.

Part 3: if T satisfies the semigroup Bergman property, then S does also.

Let U be any generating set for S. We must prove that S is Cayley bounded with respect to U. Since $S \setminus T$ is finite, there exists $m \in \mathbb{N}$ such that $S \setminus T \subseteq U \cup U^2 \cup \cdots \cup U^m = V$. Obviously V generates S. By the Schreier Theorem for semigroups [**4**, Theorem 3.1] or [**15**], the set

$$X = \{\, xvy \ :\ x, y \in S \setminus T,\ v \in V,\ xv, xvy \in T \,\}$$

generates T. Clearly $X \subseteq V \cup V^2 \cup V^3$. But T satisfies the semigroup Bergman property and so $T = X \cup X^2 \cup \cdots \cup X^n$ for some $n \in \mathbb{N}$. Thus

$$S = (S \setminus T) \cup T = V \cup V^2 \cup \cdots \cup V^{3n} = U \cup U^2 \cup \cdots \cup U^{3mn},$$

as required. □

In light of Theorems 3.1 and 3.2, it is natural to ask: do the equalities $\mathrm{cf}(S) = \mathrm{cf}(T)$ and $\mathrm{scf}(S) = \mathrm{scf}(T)$ hold when S is a non-finitely generated semigroup that is Cayley bounded with respect to the union of a subsemigroup T and a finite set F? Perhaps the simplest case not covered by Theorem 3.2, is when $S = (T \cup F)^2$. We will show in Examples 5.4 and 5.5 that the conclusions of Theorem 3.2 no longer hold even for this simple case.

The other question we should ask is: if T is a subsemigroup of S such that $S \setminus T$ is finite and S has the semigroup Bergman property, then does T have the semigroup Bergman property too? Unfortunately, we do not know the answer to this question.

It was noted by Bergman in [**3**] that the group Bergman property is preserved by homomorphisms. However, as the following lemma demonstrates this is no longer true for the semigroup Bergman property.

LEMMA 3.4. *Let S be a semigroup. Then there exists a semigroup T such that S is a homomorphic image of T and T satisfies the semigroup Bergman property.*

Proof. The presentation

$$\langle\, A \mid a_s a_t = a_{st} \ \ (s,t \in S)\,\rangle,$$

derived from the Cayley table of S where $A = \{\, a_s \,:\, s \in S\,\}$, defines a semigroup isomorphic to S. Let T be the semigroup defined by the presentation

$$\langle\, A \mid a_s a_t a_u = a_{st} a_u \ \ (s,t,u \in S)\,\rangle.$$

The semigroup S satisfies the relations in the presentation for T. Thus S is a homomorphic image of T.

Now, the set A consists of indecomposable elements in T and so, by the comments preceding Example 2.9, every generating set for T contains A. But $A \cup A^2 = T$ and so T satisfies the semigroup Bergman property. □

Although not all homomorphisms preserve the semigroup Bergman property, one distinguished type does. A *Rees quotient* of a semigroup S by an ideal I is the quotient of S by the congruence with (at most) one non-singleton class $I \times I$, denoted S/I.

LEMMA 3.5. *Let S be a semigroup and I an ideal of S. Then*

(i) *if S has semigroup Bergman property, then so does the Rees quotient S/I;*
(ii) *if I and S/I have the semigroup Bergman property, then so does S.*

Proof. **Part (i).** Let $U = V \cup \{0\}$ be any generating set for S/I where $V \subseteq S \setminus I$. Since I is an ideal, $V \cup I$ generates S. But S satisfies the semigroup Bergman property and so $S = (V \cup I) \cup (V \cup I)^2 \cup \cdots \cup (V \cup I)^n$ for some n. Thus $S/I = (V \cup \{0\}) \cup (V \cup \{0\})^2 \cup \cdots \cup (V \cup \{0\})^n$, as required.

Part (ii). Let U be any generating set for S. Then $\langle\, U \setminus I, 0\,\rangle = S/I$ and so $S \setminus I \subseteq (U \setminus I) \cup (U \setminus I)^2 \cup \cdots \cup (U \setminus I)^n$ for some n.

Assume, without loss of generality, that $S \setminus I$ contains an identity for S. By [**4**, Theorem 3.1], the set

$$V = \{\, xuy \,:\, x, y \in S \setminus I,\ u \in U,\ xu, xuy \in I\,\}$$

generates I; Thus $I = V \cup V^2 \cup \cdots \cup V^m$ for some m.

To conclude, $V \subseteq (S \setminus I)U(S \setminus I) \subseteq U \cup U^2 \cup \cdots \cup U^{3n}$. It follows that

$$S = (S \setminus I) \cup I \subseteq U \cup U^2 \cup \cdots \cup U^{3mn},$$

as required. □

The converse of Lemma 3.5(i) obviously does not hold (if $I = S$, then S/I has the semigroup Bergman property). Example 5.6 shows that the converse of Lemma 3.5(ii) also does not hold.

4. *Positive Examples*

In this section we apply the results of the previous sections to prove that various standard semigroups satisfy the semigroup Bergman property. In Section 5, we provide some negative examples, that is, natural semigroups that do not satisfy the Bergman property.

Theorem 4.1. *Let Ω be an infinite set and let $S \in \{\mathrm{Self}(\Omega), \mathrm{SymInv}(\Omega), \mathrm{Part}(\Omega), \mathrm{Bin}(\Omega)\}$. Then $\mathrm{scf}(S) > |\Omega|$ and so S satisfies the semigroup Bergman property.*

Proof. By [**18**, Theorem 1.1] and Lemma 2.1, $\mathrm{cf}(\mathrm{Sym}(\Omega)) > |\Omega|$. Hence, by Proposition 2.2(i) and since $\mathrm{Sym}(\Omega)$ satisfies the Bergman property [**3**], $\mathrm{scf}(\mathrm{Sym}(\Omega)) > \aleph_0$. It follows, by Proposition 2.2(ii), that $\mathrm{scf}(\mathrm{Sym}(\Omega)) = \mathrm{cf}(\mathrm{Sym}(\Omega)) > |\Omega| \geq \aleph_0$.

It follows by [**14**, Proposition 1.7 and Theorem 4.5] and [**2**, Theorem 3.4] that there exist $f, g \in S$ such that $f\,\mathrm{Sym}(\Omega)g = S$. Hence, by Theorem 3.1, $\mathrm{scf}(S) \geq \mathrm{scf}(\mathrm{Sym}(\Omega)) > |\Omega| \geq \aleph_0$. In particular, by Proposition 2.2(i), S satisfies the semigroup Bergman property.

An alternative proof can be obtained using Lemma 2.4. It follows from [**24**], and [**12**, Proposition 4.2] that for all sequences $(f_n)_{n\in\mathbb{N}}$ of elements from S there exist $f, g \in S$ such that every f_n is a product of f and g wih length bounded by a linear function. Hence S is strongly distorted and so, by Lemma 2.4, $\mathrm{scf}(S) > \aleph_0$. □

Mesyan [**19**, Proposition 4] proved that $\mathrm{cf}(\mathrm{Self}(\Omega)) > \aleph_0$ using an elementary diagonal argument, and an alternative proof of Theorem 4.1 can be obtained using a similar argument. In Galvin [**11**] it was shown that the symmetric group on an infinite set is strongly distorted. Hence Bergman's original theorem follows immediately by Lemma 2.4.

The following theorem is an immediate consequence of the main theorems in [**1**] and [**20**] and Lemma 2.4.

Theorem 4.2. *Let S be one of the following semigroups: the linear functions of an infinite dimensional vector space, the endomorphism semigroup of the random graph, the continuous functions on the unit interval $[0, 1]$, the Lebesgue, or Borel measurable functions on $[0, 1]$, the order endomorphisms of $[0, 1]$, or the Lipschitz functions on $[0, 1]$. Then $\mathrm{scf}(S) > \aleph_0$ and S has the Bergman property.*

Next, following Cornulier [**5**, Theorem 3.1], we consider a further class of semigroups that satisfy the semigroup Bergman property. The notions of algebraically and existentially closed groups were introduced by Scott [**23**] in 1950 and an extensive analysis can be found in [**13**]. Neumann considered these notions for semigroups in [**21**]. Analogous notions have been considered in the more general context of model theory.

Let $\mathbb{S}$ be the class of all semigroups and let κ be an infinite cardinal. Then $S \in \mathbb{S}$ is *κ-algebraically closed in* $\mathbb{S}$ if every set E of equations with $|E| < \kappa$ and coefficients from S, which is solvable in some $T \in \mathbb{S}$ containing S, already has a solution in S. The analogous notions for groups and inverse semigroups can be obtained by replacing every occurrence of $\mathbb{S}$ in the preceding sentences with the class of all groups $\mathbb{G}$ or the class of all inverse semigroups $\mathbb{I}$. Recall that, semigroup S is *inverse* if for all $x \in S$ there exists a unique x^{-1} such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. Note that equations over $\mathbb{G}$ or $\mathbb{I}$ can include inverses of coefficients and variables.

THEOREM 4.3. *Let S be an ω_1-algebraically closed semigroup, inverse semigroup, or group where ω_1 denotes the first uncountable cardinal. Then $\mathrm{scf}(S) > \aleph_0$ and S has the semigroup Bergman property.*

Proof. We will prove that S is strongly distorted.

Let $f_1, f_2, \ldots \in S$ and assume without loss of generality that S is a subsemigroup of $T = \mathrm{Self}(\Omega)$, $\mathrm{SymInv}(\Omega)$, or $\mathrm{Sym}(\Omega)$, respectively, for some infinite set Ω. As in the proof of Theorem 4.1, by [**11**, Theorem 3.3], [**12**, Proposition 4.2], and [**24**], it follows that there exist $f, g \in T$ such that every f_n is a product of f and g with length bounded by a linear function. Since S is an ω_1-algebraically closed semigroup, it follows that there exist $f', g' \in S$ such that every f_n is a product of f' and g' with length bounded by a linear function. Hence S is strongly distorted and so by Lemma 2.4, $\mathrm{scf}(S) > \aleph_0$. □

THEOREM 4.4. *Let $\mathcal{C}_{\mathbb{Q}}, \mathcal{C}_{\mathbb{I}}$ and $\mathcal{C}_{\mathfrak{C}}$ denote the semigroups of all continuous functions on the rationals $\mathbb{Q}$, irrationals $\mathbb{I}$, and Cantor's discontinuum $\mathfrak{C}$, respectively, and let $S \in \{\mathcal{C}_{\mathbb{Q}}, \mathcal{C}_{\mathbb{I}}, \mathcal{C}_{\mathfrak{C}}\}$. Then $\mathrm{scf}(S) > \aleph_0$ and so S satisfies the semigroup Bergman property.*

In order to prove Theorem 4.4 we require the following straightforward lemma.

LEMMA 4.5. *Let $p \in \mathbb{R} \cup \{-\infty\}$ and $q \in \mathbb{R}$ with $p < q$. Then there exists an order preserving piecewise linear bijection from $\mathbb{Q}$ to $\mathbb{Q} \cap (p, q)$.*

Proof of Theorem 4.4. We will prove the theorem in the case that $S = \mathcal{C}_{\mathbb{Q}}$. The proofs in the other two cases are analogous.

Seeking a contradiction assume that $(U_i)_{i \in \mathbb{N} \cup \{0\}}$ is a strong cofinal chain for $\mathcal{C}_{\mathbb{Q}}$. Let $p \in \mathbb{R} \setminus \mathbb{Q}$ be arbitrary but fixed. Then define $\Sigma_0 = (-\infty, p) \cap \mathbb{Q}$ and for $n \geq 1$ define

$$\Sigma_n = (p + n - \frac{1}{2}, p + n) \cap \mathbb{Q}.$$

Let $\mathcal{C}_{\Sigma_n}$ denote the semigroup of continuous functions on Σ_n. Then we will prove that

$$U_n|_{\Sigma_n} = \{\, f \in \mathcal{C}_{\Sigma_n} \ : \ f = g|_{\Sigma_n},\ g \in U_n \,\} \neq \mathcal{C}_{\Sigma_n}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Assume otherwise, that is, there exists $n \in \mathbb{N} \cup \{0\}$ such that $U_n|_{\Sigma_n} = \mathcal{C}_{\Sigma_n}$. Then for some $n \geq 0$, by Lemma 4.5, there exists an order preserving continuous bijection $f : \mathbb{Q} \to \Sigma_n$. Since f is piecewise linear, f^{-1} is also an order preserving continuous bijection, and so f^{-1} can be extended to $g \in \mathcal{C}_{\mathbb{Q}}$. Thus $fU_ng = \mathcal{C}_{\mathbb{Q}}$ and so there exists $m \geq n$ such that $\mathcal{C}_{\mathbb{Q}} = U_m$, a contradiction.

Therefore for all $n \geq 0$ there exists $f_n \in \mathcal{C}_{\Sigma_n}$ such that $f_n \notin U_n|_{\Sigma_n}$. Let $f \in \mathcal{C}_{\mathbb{Q}}$ be any extension of the function defined by $x \mapsto xf_n$ for all $x \in \Sigma_n$ and for all n. Then $f \notin \bigcup_{n \in \mathbb{N} \cup \{0\}} U_n$, a contradiction. □

The *finitary power semigroup* of a semigroup S is the set of all finite subsets X and Y of S with multiplication $X \cdot Y = \{\, xy \ : \ x \in X \ \&\ y \in Y \,\}$. We will denote this semigroup by $\mathcal{P}(S)$.

The following theorem was initially motivated by the search for an example with the properties of the semigroup given in Example 5.5, as discussed after the proof of Theorem 3.2. Although very similar, of the four semigroups S appearing in Theorem 4.1, somewhat unexpectedly, only one has the property that $\mathcal{P}(S)$ has the semigroup Bergman property and the other three do not, see Theorem 5.2.

Theorem 4.6. *Let Ω be an infinite set. Then $\mathcal{P}(\mathrm{SymInv}(\Omega))$ satisfies the semigroup Bergman property.*

We will prove Theorem 4.6 in a series of lemmas. Although the next lemma is straightforward we state it explicitly because of its usefulness.

Lemma 4.7. *Let T be a subsemigroup of S and $\mathrm{scf}(T) > \aleph_0$. Then for any generating set U of S we have $T \subseteq U \cup U^2 \cup \cdots \cup U^n$ for some $n \in \mathbb{N}$.*

Proof. Let $V_i = U \cup U^2 \cup \cdots \cup U^i$. Since $S = \bigcup_{i\in\mathbb{N}} V_i$, we have that $T = \bigcup_{i\in\mathbb{N}} V_i \cap T$. It is clear that $V_i \subseteq V_{i+1}$ and that $V_i^2 \subseteq V_{2i}$. Hence $V_n \cap T = T$ for some n, from the assumption that $\mathrm{scf}(T) > \aleph_0$. Therefore $T \subseteq V_n$, as required. □

The following notion was first defined in [**8**]. Let S be a semigroup. Then a product $X_1X_2\cdots X_r$ in $\mathcal{P}(S)$ is said to be *without surplus elements* if for all i and for all $x \in X_i$

$$X_1X_2\cdots X_r \neq X_1\cdots X_{i-1}(X_i \setminus \{x\})X_{i+1}\cdots X_r.$$

Lemma 4.8. *Let $X \in \mathcal{P}(S)$ such that $X = Y_1Y_2\cdots Y_r$ for some $Y_1, Y_2, \ldots, Y_r \in \mathcal{P}(S)$. Then there exist $Z_1, Z_2, \ldots, Z_r \in \mathcal{P}(S)$ such that $Z_i \subseteq Y_i$, $|Z_i| \leq |X|$, and $X = Z_1Z_2\cdots Z_r$ is without surplus elements.*

Moreover, if $|Z_i| = |X|$ for some i, then $|Z_j| = 1$ for all $j \neq i$.

For a proof see [**8**, Lemma 3.1].

The following lemma is similar to Lemma 4.8 but is more specific to our considerations.

Lemma 4.9. *Let $X \in \mathcal{P}(\mathrm{Sym}(\Omega))$ such that $X = Y_1Y_2\cdots Y_r$ is without surplus elements for some $Y_1, Y_2, \ldots, Y_r \in \mathcal{P}(\mathrm{SymInv}(\Omega))$. Then there exist $Z_1, Z_2, \ldots, Z_r \in \mathcal{P}(\mathrm{Sym}(\Omega))$ with $|Z_i| = |Y_i|$ for all i and $X = Z_1Z_2\cdots Z_r$.*

Proof. Let $y_1 \in Y_1, y_2 \in Y_2, \ldots, y_r \in Y_r$. Then $y_1y_2\cdots y_r \in \mathrm{Sym}(\Omega)$. The sets $\Omega_1 = \Omega, \Omega_2, \ldots, \Omega_{r+1} = \Omega$ are defined by $\Omega_i = (\Omega)y_1y_2\cdots y_{i-1}$. From the definition of Ω_i, the restriction $y_i|_{\Omega_i}$ is a bijection from Ω_i to Ω_{i+1}.

Likewise, if $z_i \in Y_i$, then $z_i|_{\Omega_i}$ is a bijection from Ω_i to Ω_{i+1} also. Otherwise $y_1y_2\cdots y_{i-1}z_iy_{i+1}\cdots y_r \notin \mathrm{Sym}(\Omega)$, a contradiction. Hence

$$z_1z_2\cdots z_r = z_1|_{\Omega_1}z_2|_{\Omega_2}\cdots z_r|_{\Omega_r}.$$

Note that if $z_i \neq t_i \in Y_i$, then $z_i|_{\Omega_i} \neq t_i|_{\Omega_i}$ since $Y_1Y_2\cdots Y_r$ is without surplus elements.

So, if $g_i : \Omega \to \Omega_i$, $2 \leq i \leq r$, is a bijection and $g_1 = g_{r+1}$ is the identity, then

$$z_1z_2\cdots z_r = (g_1 \cdot z_1|_{\Omega_1} \cdot g_2^{-1})(g_2 \cdot z_2|_{\Omega_2} \cdot g_3^{-1})\cdots(g_r \cdot z_r|_{\Omega_r} \cdot g_{r+1}^{-1}).$$

Now, $g_i \cdot z_i|_{\Omega_i} \cdot g_{i+1}^{-1} \in \mathrm{Sym}(\Omega)$ for all i. So, let $Z_i = \{\, g_i \cdot z_i|_{\Omega_i} \cdot g_{i+1}^{-1} \;:\; z_i \in Y_i \,\}$. It remains to show that $|Z_i| = |Y_i|$ for all i. In fact, if $z_i \neq t_i \in Y_i$, then $z_i|_{\Omega_i} \neq t_i|_{\Omega_i}$ and so $g_i \cdot z_i|_{\Omega_i} \cdot g_{i+1}^{-1} \neq g_i \cdot t_i|_{\Omega_i} \cdot g_{i+1}^{-1}$. □

An element $X \in \mathcal{P}(\mathrm{Sym}(\Omega))$ is said to be *power indecomposable* if it cannot be given as a product of sets Y and Z where $|Y|, |Z| < |X|$. In [**9**, Lemma 2] it is shown that a set $X \in \mathcal{P}(\mathrm{Sym}(\Omega))$ is power indecomposable if and only if X satisfies

(i) $x \neq yz^{-1}t$ for all distinct $x, y, z, t \in X$;
(ii) $x \neq yz^{-1}y$ for all distinct $x, y, z \in X$.

Moreover, in [**9**, Lemma 3] it is proved that for all $n \in \mathbb{N}$ there exists a set satisfying conditions (i) and (ii) with size n.

Proof of Theorem 4.6. We will prove that $\mathcal{P}(\mathrm{SymInv}(\Omega)$ has the semigroup Bergman property. Let $\mathfrak{U}$ be any generating set for $\mathcal{P}(\mathrm{SymInv}(\Omega))$. We will start by showing that it suffices to prove that there exists $n \in \mathbb{N}$ such that

$$\mathcal{P}(\mathrm{Sym}(\Omega)) \subseteq \mathfrak{U} \cup \mathfrak{U}^2 \cup \cdots \cup \mathfrak{U}^n. \tag{4.1}$$

Of course, there exist subsemigroups of $\mathcal{P}(\mathrm{SymInv}(\Omega))$ isomorphic to $\mathrm{SymInv}(\Omega)$ and $\mathrm{Sym}(\Omega)$, i.e. those consisiting of singletons. For the sake of simplicity we will denote these subsemigroups by $\mathrm{SymInv}(\Omega)$ and $\mathrm{Sym}(\Omega)$, respectively.

If Ω is an infinite set, then a subset Σ is called a *moiety* if $|\Sigma| = |\Omega \setminus \Sigma| = |\Omega|$. Let Σ be a moiety in Ω and $f : \Omega \to \Sigma$ be bijective. Then, by [**14**, Theorem 4.5], it follows that $\{f\}\,\mathrm{Sym}(\Omega)\{f^{-1}\} = \mathrm{SymInv}(\Omega)$. So, we deduce that $\{f\}\mathcal{P}(\mathrm{Sym}(\Omega))\{f^{-1}\} = \mathcal{P}(\mathrm{SymInv}(\Omega))$. By Theorem 4.1, $\mathrm{scf}(\mathrm{SymInv}(\Omega)) > \aleph_0$, and so by Lemma 4.7 there exists $m \in \mathbb{N}$ such that

$$\mathrm{SymInv}(\Omega) \subseteq \mathfrak{U} \cup \mathfrak{U}^2 \cup \cdots \cup \mathfrak{U}^m. \tag{4.2}$$

In particular, $\{f\}, \{f^{-1}\} \in \mathfrak{U} \cup \mathfrak{U}^2 \cup \cdots \cup \mathfrak{U}^m$. Hence to prove that $\mathcal{P}(\mathrm{SymInv}(\Omega))$ is Cayley bounded with respect to $\mathfrak{U}$ it suffices to prove that (4.1) holds for some n.

Let $\mathfrak{V}$ denote the power indecomposable elements in $\mathcal{P}(\mathrm{Sym}(\Omega))$. We now prove that $\mathcal{P}(\mathrm{Sym}(\Omega)) \subseteq \{f\}\mathfrak{V}\{f^{-1}\}$. Let $X = \{x_1, x_2, \ldots, x_t\} \in \mathcal{P}(\mathrm{Sym}(\Omega))$ be arbitrary. Then there exist $y_1, y_2, \ldots, y_t \in \mathrm{Sym}(\Sigma)$ such that $x_i = f y_i f^{-1}$ for all i. As mentioned in the comments just before the proof, by [**9**, Lemma 3] there exists a set $\{z_1, z_2, \ldots, z_t\} \in \mathcal{P}(\mathrm{Sym}(\Omega \setminus \Sigma))$ that does satisfy conditions (i) and (ii). Let $v_i \in \mathrm{Sym}(\Omega)$ be defined by

$$(\alpha)v_i = \begin{cases} (\alpha)y_i & \alpha \in \Sigma \\ (\alpha)z_i & \alpha \in \Omega \setminus \Sigma. \end{cases}$$

Then $V = \{v_1, v_2, \ldots, v_t\}$ satisfies conditions (i) and (ii) and so $V \in \mathfrak{V}$. It follows that $X = fVf^{-1} \in \{f\}\mathfrak{V}\{f^{-1}\}$ and so $\mathcal{P}(\mathrm{Sym}(\Omega)) \subseteq \{f\}\mathfrak{V}\{f^{-1}\}$ as required.

Finally, we will prove that $\mathfrak{V} \subseteq \mathrm{SymInv}(\Omega)\mathfrak{U}\,\mathrm{SymInv}(\Omega)$. Let $V \in \mathfrak{V}$. Then there exist $U_1, U_2, \ldots, U_r \in \mathfrak{U}$ such that $V = U_1U_2\cdots U_r$ for some r. Then by Lemma 4.8 there exist $X_1, X_2, \ldots, X_r$ such that $X_i \subseteq U_i$, $|X_i| \leq |V|$ and $V = X_1X_2\cdots X_r$ is without surplus elements and if $|X_i| = |V|$ for some i, then $|X_j| = 1$ for all $j \neq i$. Hence by Lemma 4.9 there exist $Y_1, Y_2, \ldots, Y_r \in \mathcal{P}(\mathrm{Sym}(\Omega))$ such that $V = Y_1Y_2\cdots Y_r$ and $|Y_i| = |X_i|$ for all i. But $V \in \mathfrak{V}$ and so there exists i such that $|Y_i| = |V|$. Thus $|X_i| = |V|$ and $|X_j| = 1$ for all $j \neq i$. So,

$$V \subseteq X_1 \cdots X_{i-1}U_iX_{i+1}\cdots X_r \subseteq U_1U_2\ldots U_r = V.$$

Hence $V = X_1\cdots X_{i-1}U_iX_{i+1}\cdots X_r \in \mathrm{SymInv}(\Omega)\mathfrak{U}\,\mathrm{SymInv}(\Omega)$. Therefore

$$\mathcal{P}(\mathrm{Sym}(\Omega)) \subseteq \{f\}\mathfrak{V}\{f^{-1}\} \subseteq \mathrm{SymInv}(\Omega)\mathfrak{U}\,\mathrm{SymInv}(\Omega) \subseteq \mathfrak{U} \cup \mathfrak{U}^2 \cup \cdots \cup \mathfrak{U}^{2m+1}.$$

Thus (4.1) is satisfied with $n = 2m + 1$, as required. □

It is natural to ask if it is possible to construct new semigroups with the semigroup Bergman property from semigroups that are known to have the property. It is known [**5**] that the infinite cartesian power of infinitely many copies of a finite group G has the group Bergman property if and only if G is perfect. If G, in the previous sentence, is replaced with an infinite group, then no such necessary and sufficient conditions are known. In fact, very little is known even for specific examples of infinite groups, see [**5**]. The situation for semigroups is perhaps even worse. However, as our final positive example shows, the cartesian product of at most $|\Omega|$ copies of $\mathrm{Self}(\Omega)$ has the semigroup Bergman property.

THEOREM 4.10. *Let Ω be an infinite set, let $S \in \{\mathrm{Self}(\Omega), \mathrm{SymInv}(\Omega), \mathrm{Part}(\Omega), \mathrm{Bin}(\Omega)\}$, and let T denote the cartesian product $\Pi_{i\in I} S$ where I is an index set. Then*

(i) *if Ω is countable, then $\mathrm{scf}(T) > \aleph_0$ and so T satisfies the semigroup Bergman property;*
(ii) *if $|I| \leq |\Omega|$, then $\mathrm{scf}(T) > \aleph_0$ and so T satisfies the semigroup Bergman property.*

Proof. **Part (i).** Let S_i be a semigroup of transformations or binary relations isomorphic to S acting on a set Ω_i. Then we may assume without loss of generality that $T = \Pi_{i\in I} S_i$. Then, as in the proof of Theorem 4.1, for all $i \in I$ there exist $f_i, g_i \in S_i$ such that $f_i \,\mathrm{Sym}(\Omega_i) g_i = S_i$. Hence $(f_i)_{i\in I} \Pi_{i\in I} \mathrm{Sym}(\Omega_i)(g_i)_{i\in I} = T$ and so, by Theorem 3.1, $\mathrm{scf}(T) \geq \mathrm{scf}(\Pi_{i\in I} \mathrm{Sym}(\Omega_i))$. Finally, it was shown in [**6**, Lemma 3.5] that $\mathrm{scf}(\Pi_{i\in I} \mathrm{Sym}(\Omega_i)) > \aleph_0$, and so the proof is complete.

Part (ii). We will prove that $\mathrm{scf}(T) \geq \mathrm{scf}(S)$ by showing that T is Cayley bounded with respect to a finite set and a subsemigroup isomorphic to S. Let Ω_i with $|\Omega_i| = |\Omega|$, $i \in I$, partition Ω and let $f_i : \Omega \to \Omega_i$ be arbitrary bijections for all $i \in I$. If $g_i \in S$, then there exists $h_i : \Omega_i \to \Omega$ such that $g_i = f_i h_i$. So,

$$(g_i)_{i\in I} = (f_i h_i)_{i\in I} = (f_i)_{i\in I}(h)_{i\in I},$$

where $h \in S$ satisfies $(\alpha)h = (\alpha)h_i$ whenever $\alpha \in \Omega_i$. Thus T is the product of the fixed element $(f_i)_{i\in I}$ in T and the subsemigroup U consisting of all constant sequences of elements from S. That is, $T = (f_i)_{i\in I} U$. Hence, by Theorem 3.1, $\mathrm{scf}(T) \geq \mathrm{scf}(U)$. Now, $U \cong S$ and so $\mathrm{scf}(U) > \aleph_0$, as required. □

5. *Negative Examples*

In this section we apply the results of the previous sections to prove that various standard semigroups do not satisfy the semigroup Bergman property. If $f \in \mathrm{Self}(\Omega)$, then denote the image (or range) of f by $\mathrm{im}(f)$.

THEOREM 5.1. *Let $\mathcal{BL}(\mathbb{N})$ denote the so-called Baer-Levi semigroup of injective mappings f in $\mathrm{Self}(\mathbb{N})$ such that $\mathbb{N} \setminus \mathrm{im}(f)$ is infinite. Then $\mathrm{cf}(\mathcal{BL}(\mathbb{N})) = \aleph_0$ and $\mathcal{BL}(\mathbb{N})$ does not satisfy the semigroup Bergman property.*

Proof. Let $S_n = \{\, f \in \mathcal{BL}(\mathbb{N}) \,:\, \{1, 2, \ldots, n\} \not\subseteq \mathrm{im}(f) \,\}$. Then $(S_n)_{n\in\mathbb{N}}$ forms a cofinal chain for $\mathcal{BL}(\mathbb{N})$ and so $\mathrm{cf}(\mathcal{BL}(\mathbb{N})) = \aleph_0$.

It remains to prove that $\mathcal{BL}(\mathbb{N})$ does not satisfy the semigroup Bergman property. We start by making a simple observation that will be used many times in the rest of the proof. Let Σ, Γ be infinite subsets of $\mathbb{N}$ where $\mathbb{N} \setminus \Gamma$ is infinite. Then any injection $f : \Sigma \to \Gamma$ can be extended to an element of $\mathcal{BL}(\mathbb{N})$.

We will give a generating set U for $\mathcal{BL}(\mathbb{N})$ such that $\mathcal{BL}(\mathbb{N})$ is not Cayley bounded with respect to U. Let $p_1, p_2, \ldots$ denote the prime numbers. Then for every $n \in \mathbb{N}$ let $f_n \in \mathcal{BL}(\mathbb{N})$ such that

$$if_n = \begin{cases} i & i < n \text{ or } i = p_n^m \text{ for some } m > 1 \\ n & i = p_n. \end{cases}$$

Note that since $p_n \geq n+1$ for all n we may assume that $n+1 \notin \operatorname{im}(f_n)$. Define

$$U_n = \{ f \in \mathcal{BL}(\mathbb{N}) : n \notin \operatorname{im}(f) \text{ and } if = i \text{ when } i < n \}.$$

Then set

$$U = \bigcup_{n \in \mathbb{N}} U_n \cup \{f_1, f_2, \ldots\}.$$

The semigroup $\mathcal{BL}(\mathbb{N})$ can be given as the union of the sets

$$V_n = \{ f \in \mathcal{BL}(\mathbb{N}) : n \notin \operatorname{im}(f) \text{ and } \{1, 2, \ldots, n-1\} \subseteq \operatorname{im}(f) \},$$

where $V_1 = \{ f \in \mathcal{BL}(\mathbb{N}) : 1 \notin \operatorname{im}(f) \}$.

We will prove that U is a generating set for $\mathcal{BL}(\mathbb{N})$ by showing that $V_n \subseteq U^{2n-1}$ for all n using induction. The base case when $n = 1$ follows from the fact that $V_1 = U_1 \subseteq U$. Assume that $n \geq 1$. Then the inductive hypothesis states that $V_n \subseteq U^{2n-1}$. Let $f \in V_{n+1}$ and let $\operatorname{im}(f) \setminus \{1, 2, \ldots, n\} = \{x_1, x_2, \ldots\}$. Then define $g : \mathbb{N} \to \mathbb{N}$ by

$$ig = \begin{cases} p_n^{j+1} & if = x_j \\ p_n & if = n \\ if & if < n \end{cases}$$

and let $h \in \mathcal{BL}(\mathbb{N})$ be any mapping satisfying $n+1 \notin \operatorname{im}(h)$ and

$$ih = \begin{cases} i & i < n+1 \\ x_j & i = p_n^{j+1} \end{cases}.$$

Then $g \in V_n$ and $h \in U_{n+1}$. Moreover, $gf_nh = f$ and so $f \in V_nU^2 \subseteq U^{2n+1}$. Hence $V_{n+1} \subseteq U^{2n+1}$ and U is a generating set for $\mathcal{BL}(\mathbb{N})$.

It remains to prove that $\mathcal{BL}(\mathbb{N})$ is not Cayley bounded with respect to U. Let $n \in \mathbb{N}$ and $g_n \in \mathcal{BL}(\mathbb{N})$ be any element satisfying $(2^k)g_n = k$ for all $k \leq n$. We will prove that if

$$g_n = u_1 u_2 \cdots u_m$$

where $u_1, u_2, \ldots, u_m \in U$ and m is the least length of such a product, then $m \geq n$. It suffices to prove that the elements $f_1, f_2, \ldots, f_n$ occur in the product $u_1 u_2 \cdots u_m$.

To start, let $F_{(1,2,\ldots,r)}$ denote the pointwise stabilizer of $\{1, 2, \ldots, r\}$ in $\mathcal{BL}(\mathbb{N})$. That is, $f \in F_{(1,2,\ldots,r)}$ implies that $if = i$ for all $1 \leq i \leq r$. Note that

$$U \setminus \left[\bigcup_{i=1}^{r} U_i \cup \{f_1, f_2, \ldots, f_r\} \right] \subseteq F_{(1,2,\ldots,r)}$$

and that U_r is a left ideal in $F_{(1,2,\ldots,r-1)}$ (U_1 is a left ideal in $F_{(\emptyset)} = \mathcal{BL}(\mathbb{N})$).

The mapping g_n is not an element of $F_{(1)}$ since $2g_n = 1$ and g_n is injective. Hence there exists $j \in \{1, 2, \ldots, m\}$ such that $u_j \in \{f_1\} \cup U_1$. Assume that i_1 is the largest such number j. Now, either $u_{i_1} \in U_1$ or $u_{i_1} = f_1$. In the former, $u_1 \cdots u_{i_1} \in U_1$ and so $i_1 = 1$ since m is the least length of product as defined above. It follows that $u_2, u_3, \ldots, u_m \in F_{(1)}$ and so $1 \notin \operatorname{im}(g_n)$, a contradiction. Thus $u_{i_1} = f_1$.

So, $2 \notin \operatorname{im}(u_1 \cdots u_{i_1})$ and $u_{i_1+1}, \ldots, u_m \in U \setminus [\{f_1\} \cup U_1] \subseteq F_{(1)}$. If $u_{i_1+1}, \ldots, u_m \in U \setminus [\{f_1, f_2\} \cup U_1 \cup U_2] \subseteq F_{(1,2)}$, then $2 \notin \operatorname{im}(g_n)$, a contradiction. Hence there exists $j \in \{i_1 + 1, i_1 + 2, \ldots, m\}$ such that $u_j \in \{f_2\} \cup U_2$. Assume that i_2 is the largest such j. As above, either $u_{i_2} \in U_2$ or $u_{i_2} = f_2$. In the former, as before, $u_{i_1+1} \cdots u_{i_2} \in U_2$ and so $i_2 = i_1 + 1$. Hence $u_{i_1+2}, \ldots, u_m \in F_{(1,2)}$ and so $2 \notin \operatorname{im}(g_n)$, a contradiction. Thus $u_{i_2} = f_2$.

Repeating this process n times we deduce that $f_1, f_2, \ldots, f_n$ occur in the product $u_1 u_2 \cdots u_m$, as required. □

Theorem 5.2. *Let Ω be an infinite set and let $S \in \{\mathrm{Self}(\Omega), \mathrm{Part}(\Omega), \mathrm{Bin}(\Omega)\}$. Then $\mathcal{P}(S)$ does not satisfy the semigroup Bergman property.*

Proof. We will prove the theorem in the case that $S = \mathrm{Self}(\Omega)$. Let U denote the set of all finite subsets of $\mathrm{Self}(\Omega)$ with at most 2 elements. It was shown in [**10**, Proposition 5.7.3 and Example 5.7.4] and [**22**] that the set U generates $\mathcal{P}(\mathrm{Self}(\Omega))$. However for completeness we include a short proof of this fact and show that $\mathcal{P}(\mathrm{Self}(\Omega))$ is not Cayley bounded with respect to U.

Let $\{f_1, f_2, \ldots, f_n\} \in \mathcal{P}(\mathrm{Self}(\Omega))$ be arbitrary. Then using induction we show that $\{f_1, f_2, \ldots, f_n\} \in \langle\, U \,\rangle$. If $n = 1$ or 2, then by definition $\{f_1, f_2, \ldots, f_n\} \in \langle\, U \,\rangle$. Otherwise, if $n > 2$, the inductive hypothesis states that every $n-1$ element subset of $\mathrm{Self}(\Omega)$ lies in $\langle\, U \,\rangle$. Let $\Omega_1, \Omega_2, \ldots, \Omega_n$ be any disjoint subsets of Ω satisfying $|\Omega| = |\Omega_1| = |\Omega_2| = \cdots = |\Omega_n|$ and let $g_1 : \Omega \to \Omega_1, g_2 : \Omega \to \Omega_2, \ldots, g_n : \Omega \to \Omega_n$ be bijections. It suffices to prove that $\{g_1, g_2, \ldots, g_n\} \in \langle\, U \,\rangle$, since there exists $r \in \mathrm{Self}(\Omega)$ such that $\{g_1, g_2, \ldots, g_n\} \cdot \{r\} = \{f_1, f_2, \ldots, f_n\}$.

Let $\Sigma \subseteq \Omega$ be a moiety and let $f : \Omega \to \Sigma$ and $g : \Omega \to \Omega \setminus \Sigma$ be arbitrary bijections. Then there exist $\{h_1, h_2, \ldots, h_{n-1}\} \in \mathcal{P}(\mathrm{Self}(\Omega))$ such that $fh_i = g_i$ and $gh_i = g_{i+1}$ for all $1 \leq i \leq n-1$. Thus

$$\{f, g\} \cdot \{h_1, h_2, \ldots, h_{n-1}\} = \{g_1, g_2, \ldots, g_n\}.$$

The proof is concluded by observing that any 2^n-element subset in $\mathcal{P}(\mathrm{Self}(\Omega))$ is the product of at least n subsets in U.

The proofs in the remaining two cases follow by an analogous arguments. □

The following theorem and its proof are analogues of [**6**, Theorem 3.6]; however the proof is somewhat more straightforward in the case presented here.

Theorem 5.3. *Let $\mathrm{BSelf}(\mathbb{Q})$ denote the semigroup of $f \in \mathrm{Self}(\mathbb{Q})$ such that there exists $k \in \mathbb{N}$ such that $|x - xf| \leq k$ for all $x \in \mathbb{Q}$. Then $\mathrm{cf}(\mathrm{BSelf}(\mathbb{Q})) > \aleph_0$ and $\mathrm{BSelf}(\mathbb{Q})$ does not satisfy the semigroup Bergman property.*

Proof. Throughout the proof we will use the usual notation to denote rational intervals, i.e. $[a, b] = \{\, c \in \mathbb{Q} \,:\, a \leq c \leq b \,\}$ and likewise for (a, b), $[a, b)$, and $(a, b]$. We begin by showing that $\mathrm{BSelf}(\mathbb{Q})$ does not satisfy the Bergman property.

Let U be the set of all elements f in $\mathrm{BSelf}(\mathbb{Q})$ such that $|x - xf| \leq 1$ for all $x \in \mathbb{Q}$. We will prove that U is a generating set for $\mathrm{BSelf}(\mathbb{Q})$.

With this aim in mind, let $f \in \mathrm{BSelf}(\mathbb{Q})$ such that $|x - xf| \leq k$ for some k. We will find $g, h \in \mathrm{BSelf}(\mathbb{Q})$ such that $f = gh$, $|x - xg| \leq (2/3)k$, and $|x - xh| \leq (2/3)k$. The image of f is infinite and countable and so we can enumerate the elements of $\mathrm{im}(f)$ as $x_1, x_2, \ldots$. Obviously, $x_m f^{-1} \cap x_n f^{-1} = \emptyset$ if $m \neq n$, and $x_n f^{-1} \subseteq [x_n - k, x_n + k]$.

Choose $y_1 \in [x_1 - (2/3)k, x_1 - (1/3)k]$, $z_1 \in [x_1 + (1/3)k, x_1 + (2/3)k]$ and for $n > 1$ choose

$$y_n \in [x_n - (2/3)k, x_n - (1/3)k] \setminus \{y_1, z_1, y_2, z_2, \ldots, y_{n-1}, z_{n-1}\}$$

and

$$z_n \in [x_n + (1/3)k, x_n + (2/3)k] \setminus \{y_1, z_1, y_2, z_2, \ldots, y_{n-1}, z_{n-1}\}.$$

Using the chosen elements y_n and z_n define a function $g : \mathbb{Q} \to \{y_1, z_1, y_2, z_2 \ldots\}$ by

$$xg = \begin{cases} y_n & x \in x_n f^{-1} \cap [x_n - k, x_n] \\ z_n & x \in x_n f^{-1} \cap [x_n, x_n + k]. \end{cases}$$

Define $h : \mathbb{Q} \to \mathbb{Q}$ by

$$xh = \begin{cases} x_n & x \in \{y_n, z_n\} \\ x & x \notin \{y_1, z_1, y_2, z_2 \ldots\}. \end{cases}$$

Hence we have shown that if $f \in \mathrm{BSelf}(\mathbb{Q})$ such that $|x - xf| \leq k$, then there exist $g, h \in \mathrm{BSelf}(\mathbb{Q})$ such that $f = gh$, $|x - xg| \leq (2/3)k$, and $|x - xh| \leq (2/3)k$. We may repeat this process for g and h and subsequently their factors and their factors' factors and so on, until f is given as a product of elements of U. Therefore we have shown that the set U generates $\mathrm{BSelf}(\mathbb{Q})$. It is obvious that $\mathrm{BSelf}(\mathbb{Q})$ is not Cayley bounded with respect to U and so $\mathrm{BSelf}(\mathbb{Q})$ does not satisfy the semigroup Bergman property.

It remains to prove that $\mathrm{cf}(\mathrm{BSelf}(\mathbb{Q})) > \aleph_0$. Let

$$G = \{\, f \in \mathrm{BSelf}(\mathbb{Q}) \;:\; [4n, 4n+4)f \subseteq [4n, 4n+4) \text{ for all } n \in \mathbb{Z} \,\}$$

and

$$H = \{\, f \in \mathrm{BSelf}(\mathbb{Q}) \;:\; [4n+2, 4n+6)f \subseteq [4n+2, 4n+6) \text{ for all } n \in \mathbb{Z} \,\}.$$

It is straightforward to verify that

$$G \cong H \cong \prod_{i \in \mathbb{Z}} \mathrm{Self}([0,4)).$$

We will now prove that $U \subseteq GH$. Let $f \in U$ and $\mathrm{im}(f) = \{x_1, x_2, \ldots\}$. The proof follows a similar argument to that used to show that $\langle\, U \,\rangle = \mathrm{BSelf}(\mathbb{Q})$. Let $n \geq 1$. We will define elements $y_n, z_n \in \mathbb{Q}$, $n \in \mathbb{N}$, and functions

$$g_n : \{x_1, x_2, \ldots, x_n\}f^{-1} \to \{y_1, z_1, y_2, z_2, \ldots, y_n, z_n\}$$

that depend on x_n and extend g_{n-1} and h_{n-1}. There are three cases to consider.

If $x_n \in [4k, 4k+1)$, then $x_n f^{-1} \subseteq [4k-1, 4k+2)$ since $f \in U$. Elements of G take $[4k, 4k+2)$ to $[4k, 4k+4)$ and $[4k-1, 4k)$ to $[4k-4, 4k)$. Hence choose

$$y_n \in [4k, 4k+2) \setminus \{y_1, z_1, y_2, z_2, \ldots, y_{n-1}, z_{n-1}\}$$

and

$$z_n \in [4k-1, 4k) \setminus \{y_1, z_1, y_2, z_2, \ldots, y_{n-1}, z_{n-1}\}.$$

Define g_n by

$$xg_n = \begin{cases} xg_{n-1} & x \notin x_n f^{-1} \\ y_n & x \in x_n f^{-1} \cap [4k, 4k+2) \\ z_n & x \in x_n f^{-1} \cap [4k-1, 4k). \end{cases}$$

If $x_n \in [4k+i, 4k+i+1)$ where $i = 1$ or 2, then choose

$$y_n, z_n \in [4k+i, 4k+i+1) \setminus \{y_1, z_1, y_2, z_2, \ldots, y_{n-1}, z_{n-1}\}$$

and define g_n by

$$xg_n = \begin{cases} xg_{n-1} & x \notin x_n f^{-1} \\ y_n & x \in x_n f^{-1}. \end{cases}$$

If $x_n \in [4k+3, 4k+4)$, then $x_n f^{-1} \subseteq [4k+2, 4k+5)$. Choose

$$y_n \in [4k+4, 4k+5) \setminus \{y_1, z_1, y_2, z_2, \ldots, y_{n-1}, z_{n-1}\}$$

and

$$z_n \in [4k+2, 4k+4) \setminus \{y_1, z_1, y_2, z_2, \ldots, y_{n-1}, z_{n-1}\}.$$

Define g_n by

$$xg_n = \begin{cases} xg_{n-1} & x \notin x_n f^{-1} \\ y_n & x \in x_n f^{-1} \cap [4k+4, 4k+5) \\ z_n & x \in x_n f^{-1} \cap [4k+2, 4k+4). \end{cases}$$

Finally, define

$$h_n : \{y_1, z_1, y_2, z_2, \ldots, y_n, z_n\} \to \{x_1, x_2, \ldots, x_n\}$$

by

$$xh_n = x_n \text{ if } x \in \{y_n, z_n\}.$$

Repeating the previous procedure *ad infinitum* produces two functions $g : \mathbb{Q} \to \{y_1, z_1, y_2, z_2, \ldots\} \in G$ and $h : \mathbb{Q} \to \mathbb{Q} \in H$ where $f = gh$, as required.

If $(S_n)_{n\in\mathbb{N}}$ is a cofinal chain for $\mathrm{BSelf}(\mathbb{Q})$, then $(S_n \cap G)_{n\in\mathbb{N}}$ is a chain of subsemigroups whose union is G. We showed in Theorem 4.10 that $\mathrm{cf}(\prod_{i\in\mathbb{Z}} \mathrm{Self}([0,4))) > \aleph_0$. Thus there exists $M \in \mathbb{N}$ such that $G \subseteq S_M$. Likewise, there exists N such that $H \subseteq S_N$. Assume without loss of generality that $N > M$. We proved in the previous paragraph that $U \subseteq G.H \subseteq S_N$. But then $\mathrm{BSelf}(\mathbb{Q}) = \langle\, U \,\rangle = S_N$, a contradiction. Hence $\mathrm{cf}(\mathrm{BSelf}(\mathbb{Q})) > \aleph_0$ and the proof is complete. □

We stated in Section 3 that it is possible to find a non-finitely generated semigroup S with subsemigroup T and finite set F such that $S = (T \cup F)^2$, $\mathrm{cf}(S) > \mathrm{cf}(T)$, $\mathrm{scf}(S) > \mathrm{scf}(T)$, and S satisfies the semigroup Bergman property but T does not. Using Theorems 4.1 and 5.1 we can now state this example explicitly.

Example 5.4. Let $S = \mathrm{SymInv}(\mathbb{N})$ and $T = \mathcal{BL}(\mathbb{N})$. Obviously $T \leq S$. It is easy to verify that for any bijection f from a moiety X in $\mathbb{N}$ to $\mathbb{N}$ we have $Tf = S$. Thus if $F = \{f\}$, then $(T \cup F)^2 = S$. Moreover, we showed in Theorems 4.1 and 5.1 that

$$\mathrm{cf}(S) \geq \mathrm{scf}(S) > \aleph_0 = \mathrm{cf}(T) \geq \mathrm{scf}(T),$$

and so S satisfies the semigroup Bergman property and T does not, as required.

The following example shows that it is not true that if $T \leq S$, T satisfies the semigroup Bergman property and $(T \cup F)^2 = S$, then S satisfies the semigroup Bergman property.

Example 5.5. Let Ω be an infinite set, $S = \mathcal{P}(\mathrm{Part}(\Omega))$, and $T = \mathcal{P}(\mathrm{SymInv}(\Omega))$. Then partition Ω into moieties Ω_α indexed by $\alpha \in \Omega$ and let $f \in \mathrm{Part}(\Omega)$ be the unique function satisfying $(\Omega_\alpha)f = \alpha$. Then it is straightforward to verify that $\mathrm{SymInv}(\Omega).f = \mathrm{Part}(\Omega)$ and so $T.f = S$. However, in Theorems 4.6 and 5.2 we showed that $T = \mathcal{P}(\mathrm{SymInv}(\Omega))$ does satisfy the semigroup Bergman property but $S = \mathcal{P}(\mathrm{Part}(\Omega))$ does not.

Recall that Lemma 3.5(ii) states that if S is a semigroup, I an ideal of S, and I and S/I satisfy the semigroup Bergman property, then S does also. The next example shows that there exists a semigroup satisfying the semigroup Bergman property that contains an ideal that does not satisfy it. Thus proving that the converse of Lemma 3.5(ii) does not hold.

Example 5.6. The union of $\mathrm{Sym}(\mathbb{N})$ and $\mathcal{BL}(\mathbb{N})$ forms a semigroup S and $I = \mathcal{BL}(\mathbb{N})$ is an ideal in S. In fact, for all $f \in I$ we have that $f.\mathrm{Sym}(\mathbb{N}) = I$. Thus if $(S_n)_{n\in\mathbb{N}}$ is a strong cofinal chain for S, then since $\mathrm{scf}(\mathrm{Sym}(\mathbb{N})) > \aleph_0$ there exists $M \in \mathbb{N}$ such that $\mathrm{Sym}(\mathbb{N}) \subseteq S_M$.

But then there exists $f \in S_{M+1} \cap I$, and so $S \subseteq S_N$ for some N, a contradiction. Therefore $\mathrm{scf}(S) > \aleph_0$ and S has the semigroup Bergman property but by Theorem 5.1, I does not have the semigroup Bergman property.

The following examples have uncountable cofinality but do not satisfy the semigroup Bergman property.

Example 5.7. Let X be an infinite set and let S be the semi-direct product $X^* \rtimes \mathrm{Self}(X)$ where $\mathrm{Self}(X)$ acts on free semigroup X^* (with empty word $\emptyset$) by extending every mapping from X to X to an endomorphism of X^*. We will prove that $\mathrm{cf}(S) > \aleph_0$. Assume otherwise. Then there exists a cofinal chain $(S_n)_{n \in \mathbb{N}}$. Since $\{\emptyset\} \times \mathrm{Self}(X) \cong \mathrm{Self}(X)$ it follows that $\mathrm{cf}(\{\emptyset\} \times \mathrm{Self}(X)) > \aleph_0$. Hence we deduce that there exists $N \in \mathbb{N}$ such that $\{\emptyset\} \times \mathrm{Self}(X) \subseteq S_N$. Without loss of generality there exists $(x, 1_X) \in S_N$ for some $x \in X$. If $y \in X \setminus \{x\}$ and σ the transposition that swaps x and y, then

$$(y, 1_X) = (\emptyset, \sigma)(x, 1_X)(\emptyset, \sigma) \in S_N.$$

Thus $X^* \times \{1_X\} \subseteq S_N$ and so $S \subseteq S_N$, a contradiction.

It remains to prove that S does not satisfy the semigroup Bergman property. The set $U = \{ (x, \tau) \ : \ x \in X \cup \{\emptyset\}, \tau \in \mathrm{Self}(X) \}$ generates S and S is not Cayley bounded with respect to U.

References

1. J. Araújo, J. D. Mitchell, and N. Silva, 'On generating countable sets of endomorphisms', *Algebra Universalis* 50 (2003) 61–67.
2. J. Araújo and J. D. Mitchell, 'Relative ranks in the semigroup of endomorphisms of an independence algebra', *Monatsh. Math.* 151 (2007) 1–10.
3. G. M. Bergman, 'Generating infinite symmetric groups', *Bull. London Math. Soc.* 38 (2006) 429–440.
4. C. M. Campbell, E. F. Robertson, N. Ruškuc, and R. M. Thomas, 'Reidemeister-Schreier type rewriting for semigroups', *Semigroup Forum* 51 (1995) 47–62.
5. Y. de Cornulier, 'Strongly bounded groups and infinite powers of finite groups', *Comm. Algebra* 34 (2006) 2337–2345.
6. M. Droste and R. Göbel, 'Uncountable cofinalities of permutation groups', *J. London Math. Soc.* 71 (2005) 335–344.
7. M. Droste, W. C. Holland, and G. Ulbrich, 'On full groups of measure-preserving and ergodic transformations with uncountable cofinalities', *Bull. Lond. Math. Soc.* 40 (2008) 463–472.
8. P. Gallagher, 'On the finite and non-finite generation of finitary power semigroups', *Semigroup Forum* 71 (2005) 481–494.
9. P. Gallagher and N. Ruškuc, 'Finitary power semigroups of infinite groups are not finitely generated', *Bull. London Math. Soc.* 37 (2005) 386–390.
10. P. Gallagher, 'On finite generation and presentability of diagonal acts, finitary power semigroups and Schützenberger products', Ph.D. Thesis, St Andrews, 2005.
11. F. Galvin, 'Generating countable sets of permutations', *J. London Math. Soc.* 51 (1995) 230–242.
12. P. M. Higgins, J. M. Howie, J. D. Mitchell, and N. Ruškuc, 'Countable versus uncountable ranks in infinite semigroups of transformations and relations', *Proc. Edinb. Math. Soc.* 46 (2003) 531–544.
13. G. Higman and E. Scott, *Existentially closed groups*, London Mathematical Society Monographs, New Series, 3, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1988.
14. P. M. Higgins, J. M. Howie, J. D. Mitchell, and N. Ruškuc, 'Countable versus uncountable ranks in infinite semigroups of transformations and relations', *Proc. Edinb. Math. Soc.* 46 (2003) 531–544.
15. A. Jura, 'Determining ideals of a given finite index in a finitely presented semigroup', *Demonstratio Math.* 11 (1978) 813–827.
16. A. S. Kechris and C. Rosendal, 'Turbulence, amalgamation, and generic automorphisms of homogeneous structures', *Proc. Lond. Math. Soc.* 94 (2007) 302–350.
17. A. Khelif, 'À propos de la propriété de Bergman', *C. R. Math. Acad. Sci. Paris* 342 (2006) 377–380.
18. H. D. Macpherson and P. M. Neumann, 'Subgroups of infinite symmetric groups', *J. London Math. Soc.* 42 (1990) 64–84.
19. Z. Mesyan 'Generating self-map monoids of infinite sets', *Semigroup Forum* 75 (2007) 649–676.

20. J. D. Mitchell, Y. Péresse, and M. R. Quick, 'Generating sequences of functions', *Q. J. Math.* 58 (2007) 71–79.
21. B. H. Neumann, 'Algebraically closed semigroups', *1971 Studies in Pure Mathematics*, Academic Press, London, pp. 185–194.
22. E. F. Robertson, N. Ruškuc, and M. R. Thomson, 'On diagonal acts of monoids', *Bull. Austral. Math. Soc.* 63 (2001) 167–175.
23. W. R. Scott, 'Algebraically closed groups', *Proc. Amer. Math. Soc.* 2 (1951) 118–121.
24. W. Sierpiński, 'Sur les suites infinies de fonctions définies dans les ensembles quelconques', *Fund. Math.* 24 (1935) 209–212.

V. Maltcev, J. D. Mitchell, and N. Ruškuc
Mathematical Institute,
North Haugh,
St Andrews,
Fife,
KY16 9SS,
Scotland

vm55@st-and.ac.uk
jdm3@st-and.ac.uk
nr1@st-and.ac.uk