Generators and Relations for Subsemigroups via Boundaries in Cayley Graphs

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Abstract

Given a finitely generated semigroup S and subsemigroup T of S we define the notion of the boundary of T in S which, intuitively, describes the position of T inside the left and right Cayley graphs of S. We prove that if S is finitely generated and T has a finite boundary in S then T is finitely generated. We also prove that if S is finitely presented and T has a finite boundary in S then T is finitely presented. Several corollaries and examples are given.

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1 Introduction

Given a semigroup S and a subsemigroup T of S it is natural to consider which properties S and T have in common. In the case of groups, for example, it is known that a group shares many of its properties with its subgroups of finite index. In particular we have the Reidemeister–Schreier theorem which says that subgroups of finitely presented groups with finite index are finitely presented; see [18, Proposition 4.2]. The general study of subgroups of finitely presented groups continues to receive a lot of attention; see for example [2], [8] and [26]. An important problem in the development of a similar theory for arbitrary monoids has been the search for a suitable notion of index for subsemigroups. One approach is

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to define the index of T in S to be the cardinality of the set $S\backslash T$. This is normally known as the *Rees* index of T in S. In [16] and [17] Jura discussed the problem of finding all the ideals of a given Rees index in a finitely presented semigroup. In order to obtain this result he proved the Hilbert–Schreier theorem for semigroups i.e. that if S is a finitely generated semigroup and T is a subsemigroup of S with finite Rees index then T is finitely generated. This result was reproved in [22] where, in addition, it was also shown that subsemigroups of finitely presented semigroups with finite Rees index are themselves finitely presented. An important tool that was used in the proof of this result is the Reidemeister–Schreier rewriting theorem for semigroups introduced in [4].

In [28] and [29] the groups of units of finitely presented monoids were investigated. The author considered so called special monoids and proved that, for this class of monoid, from a finite presentation for the monoid one may obtain a finite presentation for the group of units (with the same number of defining relations). In [23] and [24] presentations for arbitrary subgroups of finitely presented monoids were considered. In particular in [23] an example was given of a finitely presented monoid whose group of units is not finitely presented. Presentations of ideals of finitely presented semigroups were considered in [6] and of those of arbitrary subsemigroups in [5].

In [14] automatic semigroups were investigated and it was shown that if T is a finite Rees index subsemigroup of S then S is automatic if and only if T is.

The theory of monoid presentations is closely linked to that of string-rewriting systems. An important problem in this area is to classify all monoids that may be presented by some finite complete string-rewriting system. Monoids that may be defined by such presentations have nice properties: for example they all have solvable word problem. On the other hand, in [25] Squier showed that not every monoid that has solvable word problem is presented by some finite complete string-rewriting system. In a subsequent paper Squier introduced the notion of finite derivation type, proving that a monoid has finite derivation type if it can be presented by a finite complete rewriting system. In [27] it was shown that if T is a finite Rees index subsemigroup of S and T has finite derivation type then so does S. In the same paper it was also shown that if T has finite Rees index in S then S can be presented by a finite complete rewriting system if T can. The converses of both of these results are still open problems.

In this paper we introduce a new notion of index for subsemigroups which is significantly weaker than Rees index but is still strong enough to force T to inherit certain properties from S. The general idea is that rather than forcing the entire complement $S \setminus T$ to be finite we need only restrict the number of points where T and $S \setminus T$ meet each other in the Cayley graphs to be finite.

Let S be a finitely generated semigroup with T a subsemigroup of S. Let A be a finite generating set of S. Let $\Gamma_r(A, S)$ and $\Gamma_l(A, S)$ denote the right and left Cayley graphs of S with respect to A. Thus the vertices of $\Gamma_r(A, S)$ are the elements of S and there is a directed edge from s to t, labelled with $a \in A$, if and only if $sa = t$. The left Cayley graph is defined analogously. We define the *right boundary edges of T in $\Gamma_r(A,S)$* to be those edges whose initial vertex is in $S \setminus T$ and terminal vertex is in T. The left boundary edges are defined in the same way but using the left Cayley graph. We define the *right boundary of T in S with respect to A* to be the set of terminal vertices of the right boundary edges of T in $\Gamma_r(A,S)$ together with the elements of A that belong to T. The *left boundary of T in S with respect to A* is defined to be the set of terminal vertices of the left boundary edges of T in $\Gamma_l(A,S)$ together with the elements of A that belong to T. We define the *(two-sided) boundary* of T in S to be the union of the left and right boundaries. We use $\mathcal{B}_l(A,T)$, $\mathcal{B}_r(A,T)$ and $\mathcal{B}(A,T)$ to denote the left, right and two-sided boundaries, respectively, of T in S with respect to A. Formally these sets are given by

$$\mathcal{B}_l(A,T) = AU^1 \cap T = \{au : u \in U^1, a \in A\} \cap T,$$

$$\mathcal{B}_r(A,T) = U^1 A \cap T = \{ua : u \in U^1, a \in A\} \cap T$$

and

$$\mathcal{B}(A,T) = \mathcal{B}_l(A,T) \cup \mathcal{B}_r(A,T)$$

where S^1 denotes S with an identity adjoined (even if it already has one), U denotes the complement $S \setminus T$ and U^1 denotes $S^1 \setminus T$. We say that T *has a finite boundary in S* if for some finite generating set A of S the boundary $\mathcal{B}(A,T)$ is finite.

Clearly the sets defined above depend on the choice of generating set A. However, the finiteness (or otherwise) of these sets is independent of the choice of generating set (see Proposition 2.1). Thus we may speak of T being a subsemigroup with finite (left, right or two-sided) boundary without reference to the generating set for S.

Our notion of boundary is consistent with the way the term is used in the theory of ends of groups and graphs; see [9] and [20]. Boundaries are also encountered in the definition of the Cheeger constant of a graph, which gives rise to one of the equivalent definitions of amenable group; see [11, Section 7]. Correspondingly, boundaries of subsets in semigroup Cayley graphs appear in the study of amenable semigroups, and semigroups satisfying various Følner-type conditions; see for instance [13, 21]. More generally, boundaries of subsets of vertices in directed graphs are used in the definition of the Cheeger-type constant for directed graphs introduced and investigated in [7].

Our main results show that the properties of finite generation and presentability are inherited by subsemigroups with finite boundary.

Theorem A. *If S is a finitely generated semigroup and T is a subsemigroup of S with finite boundary then T is finitely generated.*

Theorem B. *Let S be a semigroup and T be a subsemigroup of S. If S is finitely presented and T has a finite boundary in S then T is finitely presented.*

The paper is structured as follows. We begin by describing the basic properties of boundaries in §2. In §3 we show the connection between boundaries and generating sets of subsemigroups and in the process prove Theorem A. We consider semigroup presentations in §4 and we prove Theorem B. In §5 we give some illustrative examples, applications and corollaries of our main results. The question of finite presentability when only one of the right or left boundaries is finite is the subject of §6 and in §7 we consider the converse of Theorem B.

2 Properties of subsemigroups with finite boundaries

Let S be a semigroup generated by a finite set A. Let A^+ denote the set of all non-empty words over the alphabet A, and let A^* denote the set of all words over the alphabet A including the empty word ϵ. There is a natural homomorphism $\theta : A^+ \to S$ mapping each word in A^+ to its corresponding product of generators in S. Since A generates S the map θ is surjective. Associated with the map θ is a congruence η on the free semigroup A^+ given by $(w, v) \in \eta$ if and only if $w\theta = v\theta$. Then the quotient A^+/η is isomorphic to S under the natural map $w/\eta \mapsto w\theta$. Given some $w \in A^+$ we will, where there is no chance of confusion, often omit reference to the function θ or the relation η altogether and talk of w in S rather than $w\theta$ in S or $w/\eta \in A^+/\eta$.

Given a word $w \in A^+$ we will use $|w|$ to denote its length. Given $w, v \in A^+$ we write $w = v$ if they represent the same element of S (i.e. if $w\theta = v\theta$) and write $w \equiv v$ if they are identical as words in A^+. Furthermore, given $w \in A^+$ and $s \in S$ we write $w = s$ meaning that $w\theta = s$ in S. We write S^1 to denote the semigroup S with an identity adjoined (even if S already has an identity) and we extend the definition of θ so that $\theta : A^* \to S^1$ by setting $\epsilon\theta = 1$.

For two words $w, v \in A^*$ we say that w is a *prefix* (respectively suffix) of v if $v \equiv w\beta$ (respectively $v \equiv \beta w$) for some $\beta \in A^*$. In particular, the empty word is both a prefix and a suffix of every word from A^*. We say that w is a *subword* of v if $v \equiv \alpha w \beta$ where $\alpha, \beta \in A^*$ (in the literature w is also often called a *factor* of v). Also, for a subset Y of S^1 we define $\mathcal{L}(A, Y) = \{w \in A^* : w\theta \in Y\}$ and call this set the *language of* Y *in* A^*. Note that from the convention described in the previous paragraph it follows that, for $Y \subseteq S$, we have that $\mathcal{L}(A, Y)$ does not contain the empy word, and $\mathcal{L}(A, Y^1) = \mathcal{L}(A, Y) \cup \{\epsilon\}$.

We now show that whether the boundary is finite or not is independent of the choice of generating set.

Proposition 2.1. *Let S be a finitely generated semigroup, let T be a subsemigroup of S and let A and B be two finite generating sets for S. Then $\mathcal{B}_r(A, T)$ is finite if and only if $\mathcal{B}_r(B, T)$ is finite. Also, $\mathcal{B}_l(A, T)$ is finite if and only if $\mathcal{B}_l(B, T)$ is finite.*

Proof. We will prove the first statement only. The second may be proved using a dual argument. For each $b \in B$ let $\pi_A(b) \in A^+$ be some fixed decomposition of b into generators from A so that $b = \pi_A(b)$ in S. Let $m = \max\{|\pi_A(b)| : b \in B\}$ which exists since B is finite. We claim that

$$\mathcal{B}_r(B,T) \subseteq \bigcup_{i=1}^{m-1} \mathcal{B}_r(A,T)A^i$$

which is a finite set since both A and $\mathcal{B}_r(A,T)$ are finite. To verify our claim first let $t \in \mathcal{B}_r(B,T)$. By the definition of $\mathcal{B}_r(B,T)$ we can write $t = ub = u\pi_A(b) = ua_1 \dots a_k$ where $u \in (S \setminus T)^1$, $b \in B$, $a_i \in A$ for $1 \leq i \leq k$ and $k \leq m$. Let l be the smallest subscript such that $ua_1 \dots a_l$ belongs to T. It follows that $ua_1 \dots a_l \in \mathcal{B}_r(A,T)$ and we have

$$t = (ua_1 \dots a_l)(a_{l+1} \dots a_k) \in \mathcal{B}_r(A,T)A^{k-l} \subseteq \bigcup_{i=1}^{m-1} \mathcal{B}_r(A,T)A^i$$

since $k - l \leq m - 1$. $\square$

Finite boundaries arise in many natural situations. Some of the most obvious such situations are listed in the following proposition. The proof follows straight from the definition of the boundary.

Proposition 2.2. *Let S be a semigroup generated by a finite set A and let T be a subsemigroup of S. Then we have:*

(i) if T is finite then $\mathcal{B}(A,T)$ is finite;

(ii) if $S \setminus T$ is finite then $\mathcal{B}(A,T)$ is finite;

(iii) if $S \setminus T$ is a right (resp. left) ideal in S then $\mathcal{B}_r(A,T)$ (resp. $\mathcal{B}_l(A,T)$) is finite;

(iv) if $S \setminus T$ is an ideal in S then $\mathcal{B}(A,T)$ is finite.

We shall see further examples of subsemigroups with finite boundary in Section 5.

The *depth* of an element $s \in S$ is defined to be the minimal possible length of a product in A^+ that equals s in S, and is denoted by $d(s)$. In other words:

$$d(s) = \min\{|w| : w \in A^+, w = s \text{ in S}\}.$$

For a subset X of S we define the depth of X to be:

$$d(X) = \max\{d(x) : x \in X\}$$

when it exists and say that X has infinite depth otherwise. Also, given a word $w \in A^+$ we define the depth of w by $d(w) = d(w\theta)$, and we denote by $\overline{w}$ a fixed word such that $\overline{w} = w$ and $|\overline{w}| = d(w)$.

The next result gives a characterization of subsemigroups with finite boundary that does not refer to the generating set of S. The properties described in the proposition will be used frequently in later sections.

Proposition 2.3. *Let S be a finitely generated semigroup with T a subsemigroup of S and $U = S \setminus T$. If T has finite boundary in S then the following properties hold:*

(i) for every finite subset X of S the set $U^1X \cap T$ is finite;

(ii) for every finite subset X of S the set $XU^1 \cap T$ is finite;

(iii) the set $U^2 \cap T$ is finite.

Proof. Suppose that T has a finite boundary in S. We now show that each of the three conditions given in the proposition must hold.

(i) Let X be a finite subset of S. Define $B = X \cup A$ which is a finite generating set for S. Now we have:

$$U^1X \cap T \subseteq U^1B \cap T = \mathcal{B}_r(B, T)$$

where $\mathcal{B}_r(B, T)$ is finite by Proposition 2.1. Condition (ii) is proved using a dual argument.

(iii) Let $m = d(\mathcal{B}(A, T))$, the depth of the boundary of T in S, which is well defined since $\mathcal{B}(A, T)$ is finite. Define $Z = \{w \in \mathcal{L}(A, T) : |w| \leq 3m\}$, $Y = (Z\theta) \cap U^2$ and let

$$k = \max_{y \in Y} \min\{|v| : u, v \in U, uv = y\}$$

which must exist since Y is finite.

Claim 1. *For all $u, v \in \mathcal{L}(A, U)$ where $uv \in \mathcal{L}(A, T)$ there exist $u_1, v_1 \in \mathcal{L}(A, U)$ such that $|v_1| \leq k$ and $uv = u_1v_1$ in S.*

Proof. We prove the claim by induction on the length of the word $|uv|$. Let $u, v \in \mathcal{L}(A, U)$ where $uv \in \mathcal{L}(A, T)$. If $|uv| = 2$ then the result holds trivially. Now suppose that the result holds for all pairs $\gamma, \delta \in \mathcal{L}(A, U)$ where $\gamma\delta \in \mathcal{L}(A, T)$ and $|\gamma\delta| < |uv|$. We prove the result for uv by considering the following cases.

Case 1: *u has no prefix in $\mathcal{L}(A, T)$.* Since $\mathcal{L}(A, T)$ does not contain the empty word, this is equivalent to saying that u has no nontrivial prefix in $\mathcal{L}(A, T)$. In this case since $v \in \mathcal{L}(A, U)$ and $uv \in \mathcal{L}(A, T)$ we can write $uv \equiv u'\beta_1$ where u' is a (possibly empty) prefix of u, v is a suffix of β_1 and $\beta_1 \in \mathcal{L}(A, \mathcal{B}_l(A, T))$. We have $uv = u'\overline{\beta_1}$ where, since u has no prefix in $\mathcal{L}(A, T)$, $u' \in \mathcal{L}(A, U^1)$ and

$u'\overline{\beta_1} \in \mathcal{L}(A,T)$. We can, therefore, write $u'\overline{\beta_1} = \beta_2\gamma$ where $\beta_2 \in \mathcal{L}(A, \mathcal{B}_r(A,T))$, u' is a prefix of β_2 and γ is a suffix of $\overline{\beta_1}$. Therefore $uv = \beta_2\gamma = \overline{\beta_2}\gamma$ where

$$|\overline{\beta_2}\gamma| = |\overline{\beta_2}| + |\gamma| \leq |\overline{\beta_2}| + |\overline{\beta_1}| \leq 2m \leq 3m.$$

It follows that $d(uv) \leq 3m$ which implies that $(uv)\theta \in Y$ and, by the definition of k, we can write $uv = u_1v_1$ where $|v_1| \leq k$.

Case 2: *u has a prefix in $\mathcal{L}(A,T)$*. In this case, since u has a prefix in $\mathcal{L}(A,T)$, we write $uv \equiv \beta u'v = \overline{\beta}u'v$ where $\beta \in \mathcal{L}(A, \mathcal{B}_r(A,T))$ is nonempty and, since T is a subsemigroup of S, $u' \in \mathcal{L}(A,U)$. This case now splits into two subcases depending on whether or not $u'v \in \mathcal{L}(A,T)$.

Case 2.1: $u'v \notin \mathcal{L}(A,T)$. Since $\overline{\beta}u'v \in \mathcal{L}(A,T)$ we can write $\overline{\beta}u'v \equiv \gamma\beta_2$ where $\beta_2 \in \mathcal{L}(A, \mathcal{B}_l(A,T))$, γ is a prefix of $\overline{\beta}$ and $u'v$ is a suffix of β_2. Now we have $uv = \overline{\beta}u'v = \gamma\overline{\beta_2}$ which satisfies $|\gamma\overline{\beta_2}| \leq 2m \leq 3m$. It follows that $d(uv) \leq 3m$ which implies that $(uv)\theta \in Y$ and, by the definition of k, we can write $uv = u_1v_1$ where $|v_1| \leq k$.

Case 2.2: $u'v \in \mathcal{L}(A,T)$. In this case $u', v \in \mathcal{L}(A,U)$, $u'v \in \mathcal{L}(A,T)$ and $|u'v| < |uv|$, so we can apply induction writing $u'v = u_2v_2$ where $|v_2| \leq k$. Now we have $uv \equiv \beta u'v = \overline{\beta}u_2v_2$. If $\overline{\beta}u_2 \in \mathcal{L}(A,U)$ then we are done since $uv = (\overline{\beta}u_2)v_2$ where $\overline{\beta}u_2, v_2 \in \mathcal{L}(A,U)$ and $|v_2| \leq k$. On the other hand, if $\overline{\beta}u_2 \in \mathcal{L}(A,T)$ then since $u_2 \in \mathcal{L}(A,U)$ and $\overline{\beta}u_2 \in \mathcal{L}(A,T)$ we can write $\overline{\beta}u_2 \equiv \gamma\beta_1$ where $\beta_1 \in \mathcal{L}(A, \mathcal{B}_l(A,T))$, γ is a prefix of $\overline{\beta}$ and u_2 is a suffix of β_1. Now we have $uv = \beta u'v = \overline{\beta}u_2v_2 = \gamma\overline{\beta_1}v_2$. Since $v_2 \in \mathcal{L}(A,U)$ and $\gamma\overline{\beta_1}v_2 \in \mathcal{L}(A,T)$ we can write $\gamma\overline{\beta_1}v_2 \equiv \delta\beta_2$ where $\beta_2 \in \mathcal{L}(A, \mathcal{B}_l(A,T))$, δ is a prefix of $\gamma\overline{\beta_1}$ and v_2 is a suffix of β_2. Therefore $uv = \delta\overline{\beta_2}$ where

$$|\delta\overline{\beta_2}| \leq |\delta| + |\overline{\beta_2}| \leq |\gamma| + |\overline{\beta_1}| + |\overline{\beta_2}| \leq |\overline{\beta}| + |\overline{\beta_1}| + |\overline{\beta_2}| \leq 3m.$$

It follows that $d(uv) \leq 3m$ which implies that $(uv)\theta \in Y$ and, by the definition of k, we can write $uv = u_1v_1$ where $|v_1| \leq k$.

This completes the proof of the claim. □

Returning to the proof of Proposition 2.3, let W be the set of words of $\mathcal{L}(A,U)$ that have length no greater than k. This set is finite and as a consequence so is the set $W\theta$. It now follows from the claim that $U^2 \cap T \subseteq U(W\theta) \cap T$ which, by condition (i), is a finite set. □

Note that if (i) and (ii) hold, then T has finite boundary in S trivially (taking $X = A$), and it follows that the converse of Proposition 2.3 also holds.

3 Generating subsemigroups using boundaries

In this section we will prove the first of our main theorems:

Theorem A. *If S is a finitely generated semigroup and T is a subsemigroup of S with finite boundary then T is finitely generated.*

The right (or left) boundary of a subsemigroup T of a semigroup S may be used to construct a generating set for T.

Proposition 3.1. *Let S be a finitely generated semigroup, let A be a finite generating set for S, let T be a subsemigroup of S, and let $U = S \setminus T$. Then each of the sets*

$$X_\rho = \mathcal{B}_r(A,T)U^1 \cap T, \quad X_\lambda = U^1\mathcal{B}_l(A,T) \cap T$$

generates T.

Proof. Let $t \in T$ be arbitrary. Write $t = a_1 \dots a_k$ where $a_i \in A$ for $1 \leq i \leq k$. Let m be the smallest subscript such that $\beta_1 = a_1 \dots a_m$ belongs to T, let $\gamma_1 = a_{m+1} \dots a_k$ and note that $\beta_1 \in \mathcal{B}_r(A,T)$. If $\gamma_1 \in U$ or is empty then stop. Otherwise repeat the same process on the word $a_{m+1} \dots a_k$ writing it as $\beta_2\gamma_2$ where β_2 is the shortest prefix that belongs to $\mathcal{L}(A,T)$ so that $\beta_2 \in \mathcal{B}_r(A,T)$. Continuing in this way, in a finite number of steps, we can write $t = \beta_1 \dots \beta_{m-1}\beta_m\gamma_m$ where $\beta_1, \dots, \beta_m \in \mathcal{B}_r(A,T)$ and $\gamma_m \in U^1$. The elements $\beta_1, \dots, \beta_{m-1}, \beta_m\gamma_m$ all belong to X_ρ and, since t was arbitrary, it follows that X_ρ generates T. The fact that X_λ generates T follows from a dual argument. □

Proof of Theorem A. If $\mathcal{B}_l(A,T)$ and $\mathcal{B}_r(A,T)$ are finite then X_ρ is finite by Proposition 2.3(ii), and generates T by Proposition 3.1. □

Note that if only the right (or left) boundary is finite then T need not inherit the property of being finitely generated as the following example demonstrates.

Example 3.2. Let $F = A^+$, the free semigroup over the alphabet A, where $A = \{a,b\}$. Let R be the subsemigroup of all words that begin with the letter a. Then $\mathcal{B}_r(A,T) = \{a\}$ which is finite but R is not finitely generated since all the elements ab^i where $i \in \mathbb{N}$ must be included in any generating set.

4 Presentations

Preliminaries: definitions and notation

A semigroup presentation is a pair $\mathfrak{P} = \langle A|\mathfrak{R}\rangle$ where A is a an alphabet and $\mathfrak{R} \subseteq A^+ \times A^+$ is a set of pairs of words. An element (u,v) of $\mathfrak{R}$ is called a *relation* and is usually written $u = v$. We say that S is the *semigroup defined by the presentation* $\mathfrak{P}$ if $S \cong A^+/\eta$ where η is the smallest congruence on A^+ containing $\mathfrak{R}$. We may think of S as the largest semigroup generated by the set A which satisfies all the relations of $\mathfrak{R}$. We say that a semigroup S is finitely presented if it can be defined by $\langle A|\mathfrak{R}\rangle$ where A and $\mathfrak{R}$ are both finite. For example the free semigroup on the alphabet $\{a, b\}$ is given by the presentation $\langle a, b | \rangle$ and hence is finitely presented. At the other extreme, every finite semigroup is finitely presented, by including the entire multiplication table in the set of relations if necessary. Not every semigroup is finitely presented: consider the semigroup defined by the presentation $\langle a, b \mid ab^i a = aba,\ (i \in \mathbb{N})\rangle$ for example.

We say that the word $w \in A^+$ *represents the element* $s \in S$ if $s = w/\eta$. As in §2 given two words $w, v \in A^+$ we write $w = v$ if w and v represent the same element of S and write $w \equiv v$ if w and v are identical as words. Also, given an element $s \in S$ and a word $w \in A^+$ we write $w = s$ when $w/\eta = s$ in S.

We say that w is obtained from v by one application of a relation from $\mathfrak{R}$ if there exist $\alpha, \beta \in A^*$ and $(x = y) \in \mathfrak{R} \cup \mathfrak{R}^{-1}$ such that $w \equiv \alpha x \beta$ and $v \equiv \alpha y \beta$. We say that the relation $w = v$ is a consequence of the relations $\mathfrak{R}$ (or of the presentation $\mathfrak{P}$) if there is a finite sequence of words $(\alpha_1, \ldots, \alpha_m)$ such that $w \equiv \alpha_1$, $v \equiv \alpha_m$ and, for all k, α_{k+1} is obtained from α_k by one application of a relation from $\mathfrak{R}$. We now state a basic result that will be used frequently in the paper.

Proposition 4.1. *Let $\mathfrak{P} = \langle A|\mathfrak{R}\rangle$ be a semigroup presentation, let $S = A^+/\eta$ be the semigroup defined by it, and let $\alpha, \beta \in A^+$ be any two words. Then the relation $\alpha = \beta$ holds in S if and only if it is a consequence of $\mathfrak{P}$.*

The rest of this section will be devoted to proving the following result.

Theorem B. *Let S be a semigroup and T be a subsemigroup of S. If S is finitely presented and T has a finite boundary in S then T is finitely presented.*

Proof of Theorem B: bars and hats

Before proving Theorem B we give an overview of our method. We will use the Reidemeister–Schreier theorem for semigroups described in [4] to construct a presentation for T from a given presentation of S. This presentation will be of the form $\langle B|\mathcal{Q}\rangle$ where B is finite and $\mathcal{Q}$ is infinite. We then go on to prove that there exists a finite set of relations $\mathcal{D}$ that all hold in T with the property that every relation of $\mathcal{Q}$ is a consequence of the relations $\mathcal{D}$. It will follow that T is defined by $\langle B|\mathcal{D}\rangle$ where B and $\mathcal{D}$ are both finite.

Let S be the semigroup defined by the presentation $\mathfrak{P} = \langle A|\mathfrak{R}\rangle$ with A finite and let η be the smallest congruence on A^+ containing $\mathfrak{R}$. Let T be a subsemigroup of S with a finite boundary, and let $U = S \setminus T$.

Fix a transversal $\mathcal{R}$ of the η-classes of A^+ chosen so that every $w \in \mathcal{R}$ is a word of shortest length in its η-class. Recalling the bar notation from Section 2 we let $\overline{w} = (w/\eta) \cap \mathcal{R}$: the fixed shortest length word in $\mathcal{R}$ that equals w in S.

Define $\mathcal{PI}(A, T) \subseteq \mathcal{L}(A, T)$ to be the set of words w such that every prefix of w (with the exception of w itself) belongs to $\mathcal{L}(A, U^1)$. We call $\mathcal{PI}(A, T)/\eta$ the *strict right boundary of T in S*. Note that the strict right boundary of T in S is a subset of the right boundary of T. In Section 3 we found a generating set for T by multiplying the elements of the right boundary on the right by elements of U. In fact, if we just take the strict right boundary and multiply on the right by elements of U we obtain a generating set. This is the generating set with respect to which we will write a presentation for T. Let $\mathcal{SB}_r(A,T) = \mathcal{PI}(A,T)/\eta$ and define

$$X_\rho = \mathcal{SB}_r(A,T)U^1 \cap T$$

which is a generating set for T by exactly the same argument as in the proof of Lemma 3.1.

We will now partition the elements of U into classes depending on how they interact with T.

Definition 4.2. Choose a symbol $0 \notin S$ and for each $u \in U^1$ define $f_u, g_u : T \cup \{0\} \to T \cup \{0\}$ by:

$$xf_u = \begin{cases} xu & \text{if } x \in T, xu \in T \\ 0 & \text{otherwise,} \end{cases} \qquad xg_u = \begin{cases} ux & \text{if } x \in T, ux \in T \\ 0 & \text{otherwise.} \end{cases}$$

Then, given a subset X of T and given $u, v \in U^1$ we write $u \sim_X v$ if and only if $f_u\restriction_X = f_v\restriction_X$ and $g_u\restriction_X = g_v\restriction_X$. (Where $h\restriction_X$ denotes the restriction of the mapping h to the set X.)

The relation $\sim_X$ so defined is clearly an equivalence relation. Moreover, if X is finite then, by Proposition 2.3, $\sim_X$ has finitely many equivalence classes. Indeed, we have $f_u\restriction_X : X \to (XU^1 \cap T) \cup \{0\}$ and $g_u\restriction_X : X \to (U^1X \cap T) \cup \{0\}$ and, by Proposition 2.3, the ranges of both of these functions are finite. In particular, since the generating set X_ρ is finite, the subset U^1 has finitely many $\sim_{X_\rho}$-classes.

Given $w, u \in \mathcal{L}(A, U^1)$ we write $w \approx_X u$ if $w/\eta \sim_X u/\eta$. Again, this is an equivalence relation. Let $\Sigma \subseteq \mathcal{L}(A, U^1)$ be a set of smallest length word representatives of the $\approx_{X_\rho}$ classes. Clearly $\epsilon \in \Sigma$.

We define the operation *hat* ($w \mapsto \widehat{w}$) on the words of $\mathcal{L}(A, U^1)$ by $\{\widehat{w}\} = (w/\approx_{X_\rho}) \cap \Sigma$ and $\widehat{\epsilon} = \epsilon$. Note that any word in A^* may be barred but only words in $\mathcal{L}(A, U^1)$ may be hatted.

The following lemma summarizes several properties of the hat operation. Its proof is an immediate consequence of the above definitions and discussion.

Lemma 4.3. *The following properties hold:*

(i) For $u \in \mathcal{L}(A, U^1)$ we have $|\widehat{u}| \leq |u|$.

(ii) The set Σ is finite.

(iii) If $\gamma \in \mathcal{L}(A, X_\rho), u \in \mathcal{L}(A, U^1)$ and $u\gamma \in \mathcal{L}(A, T)$ then $u\gamma = \widehat{u}\gamma$.

(iv) If $\gamma \in \mathcal{L}(A, X_\rho), u \in \mathcal{L}(A, U^1)$ and $\gamma u \in \mathcal{L}(A, T)$ then $\gamma u = \gamma\widehat{u}$.

Proof (continued): representation and rewriting mappings

In this subsection we find a presentation for T. It will have infinitely many relations and the rest of the proof will be devoted to reducing this infinite set to a finite one.

In view of Lemma 4.3(iv), the generating set X_ρ can be represented by the following set of words in A^+:

$$\{\overline{v}u : v \in \mathcal{PI}(A,T), u \in \Sigma, vu \in \mathcal{L}(A,T)\}.$$

We construct a new alphabet B in one-one correspondence with these generating words:

$$B = \{b_{\overline{v},u} : v \in \mathcal{PI}(A,T), u \in \Sigma, vu \in \mathcal{L}(A,T)\}.$$

This set is finite since A and Σ are finite and T has a finite boundary in S.

Let $\psi : B^+ \to A^+$ be the unique homomorphism extending $b_{\overline{v},u} \mapsto \overline{v}u$ and, following [4], call this map the *representation mapping*. It has the property that for every $w \in B^+$, the words w and $w\psi \in A^+$ represent the same element of S (and, of course, of T).

Now define a map $\phi : \mathcal{L}(A,T) \to B^+$ as follows. For $w \in \mathcal{L}(A,T)$ write $w \equiv \alpha\beta$ where $\alpha \in A^+, \beta \in A^*$ and α is the shortest prefix of w belonging to $\mathcal{L}(A,T)$: so α is the unique prefix of w that belongs to $\mathcal{PI}(A,T)$. Then ϕ is defined inductively by:

$$w\phi = \begin{cases} b_{\overline{\alpha},\widehat{\beta}} & \text{if } \beta \notin \mathcal{L}(A,T) \\ b_{\overline{\alpha},\epsilon}(\beta\phi) & \text{if } \beta \in \mathcal{L}(A,T). \end{cases}$$

It is easy to see that for every $w \in \mathcal{L}(A,T)$ the relation $w\phi\psi = w$ holds in S. (But we usually have $w\phi\psi \not\equiv w$.) In the terminology of [4], the map ϕ is a *rewriting mapping*.

It now follows from [4, Theorem 2.1] that the semigroup T is defined by the presentation with generators B and relations:

$$b_{\overline{v},u} = (\overline{v}u)\phi \tag{4.1}$$
$$(w_1w_2)\phi = (w_1\phi)(w_2\phi) \tag{4.2}$$
$$(w_3xw_4)\phi = (w_3yw_4)\phi \tag{4.3}$$

where $v \in \mathcal{PI}(A,T)$, $u \in \Sigma$, $vu \in \mathcal{L}(A,T)$, $w_1, w_2 \in \mathcal{L}(A,T)$, $w_3, w_4 \in A^*$, $(x = y) \in \mathfrak{R}$, $w_3xw_4 \in \mathcal{L}(A,T)$.

The set of relations (4.1) is finite since B is finite. The remainder of the proof is concerned with proving that the relations (4.2) and (4.3) are all consequences of a fixed finite set of relations $\mathcal{D}$ that we define below.

Before we do that, we state a lemma which gives a canonical decomposition of words from $\mathcal{L}(A,T)$ that is compatible with the operation of ϕ. The proof is an immediate consequence of the definition of ϕ.

Lemma 4.4. *Let $w \in \mathcal{L}(A,T)$ be arbitrary. The word w can be written uniquely as*

$$w \equiv \alpha_1 \dots \alpha_{k-1}\alpha_k\alpha_{k+1}$$

where $k \geq 1$, $\alpha_1, \dots, \alpha_k \in \mathcal{PI}(A,T)$, $\alpha_{k+1} \in \mathcal{L}(A,U^1)$ and $\alpha_k\alpha_{k+1} \in \mathcal{L}(A,T)$. When applying the rewriting mapping we obtain:

$$w\phi \equiv (\alpha_1\phi)\dots(\alpha_{k-1}\phi)(\alpha_k\alpha_{k+1})\phi \equiv b_{\overline{\alpha_1},\epsilon}\dots b_{\overline{\alpha_{k-1}},\epsilon} b_{\overline{\alpha_k},\widehat{\alpha_{k+1}}}.$$

We call the words $\alpha_1, \dots, \alpha_{k-1}, \alpha_k\alpha_{k+1}$ the *principal factors* of w and when we write $w \in \mathcal{L}(A,T)$ as $\alpha_1 \dots \alpha_{k-1}\alpha_k\alpha_{k+1}$ we say that it has been *decomposed* into principal factors.

Proof (continued): a finite set $\mathcal{D}$ of relations

We use the fact that T has a finite boundary in S and that, by Proposition 2.3, $U^2 \cap T$ is finite to define the following four numbers:

(i) $M_{\mathcal{B}} = \max\{|\overline{\gamma\delta}| : \gamma \in \mathcal{PI}(A,T), \delta \in \mathcal{L}(A,U^1), \gamma\delta \in \mathcal{L}(A,T)\}$ (well defined by Proposition 3.1);

(ii) $M_\Sigma = \max\{|\sigma| : \sigma \in \Sigma\} = \max\{|\widehat{u}| : u \in \mathcal{L}(A,U^1)\}$ (by Lemma 4.3(ii));

(iii) $M_{UU} = \max\{|\overline{uv}| : u,v \in \mathcal{L}(A,U^1), uv \in \mathcal{L}(A,T)\}$ (by Proposition 2.3);

(iv) $M_{\mathfrak{R}} = \max\{|uv| : (u = v) \in \mathfrak{R}\}$ (well defined since $\mathfrak{R}$ is finite).

Let $\mathcal{D}$ be the set of all relations in the alphabet B which hold in T and have length that does not exceed

$$N = 4(\max\{M_{\mathcal{B}}, M_\Sigma, M_{UU}, M_{\mathfrak{R}}\} + 1).$$

In other words $\mathcal{D} = \{(u,v) \in B^+ \times B^+ : |uv| \leq N, u\psi = v\psi \text{ holds in } S\}$. The rest of this section will be spent proving the following theorem.

Theorem 4.5. *The presentation $\langle B|\mathcal{D}\rangle$ defines T.*

Proof (continued): three technical lemmas

We now present three key lemmas that are used to prove Theorem 4.5.

Lemma 4.6. *The relations $(uv)\phi = (\overline{uv})\phi$ where $u,v \in \mathcal{L}(A,U)$ and $uv \in \mathcal{L}(A,T)$ are consequences of $\mathcal{D}$.*

Proof. Note that since, by definition, all of the relations in $\mathcal{D}$ hold in S, once we have shown that the relations $(uv)\phi = (\overline{uv})\phi$ are consequences of $\mathcal{D}$ it will also follow that these relations all hold in S.

We proceed by induction on the length of the word uv. When $|uv| \leq 2$ the relation $(uv)\phi = (\overline{uv})\phi$ is in $\mathcal{D}$ since

$$|(uv)\phi(\overline{uv})\phi| = |(uv)\phi| + |(\overline{uv})\phi| \leq |uv| + |\overline{uv}| \leq 2 + 2 = 4.$$

Now let $u, v \in \mathcal{L}(A, U)$ where $uv \in \mathcal{L}(A, T)$ and suppose that the result holds for all $u_1, v_1 \in \mathcal{L}(A, U)$ with $u_1v_1 \in \mathcal{L}(A, T)$ and $|u_1v_1| < |uv|$. There are two cases to consider depending on whether or not u has a prefix in $\mathcal{L}(A, T)$.

Case 1: *u has a prefix in $\mathcal{L}(A, T)$.* Write $u \equiv u'u''$ where u' is the shortest such prefix. Since T is a subsemigroup, $u \in \mathcal{L}(A, U)$ and $u' \in \mathcal{L}(A, T)$ it follows that $u'' \notin \mathcal{L}(A, T)$. The case now splits into two subcases.

Case 1.1: $u''v \notin \mathcal{L}(A, T)$. In this case $(uv)\phi$ is a single letter and $(uv)\phi = (\overline{uv})\phi$ belongs to $\mathcal{D}$ since

$$|(uv)\phi(\overline{uv})\phi| = 1 + |(\overline{uv})\phi| \leq 1 + |\overline{uv}| \leq 1 + M_{UU}.$$

Case 1.2: $u''v \in \mathcal{L}(A, T)$. In this case since $u'', v \in \mathcal{L}(A, U)$, $u''v \in \mathcal{L}(A, T)$ and $|u''v| < |uv|$ we can apply induction giving

$$\begin{array}{rcll} (uv)\phi & \equiv & (u')\phi(u''v)\phi & \text{(by definition of } \phi) \\ & = & (u')\phi(\overline{u''v})\phi & \text{(induction)} \\ & = & (\overline{uv})\phi & \text{(in } \mathcal{D}). \end{array}$$

In the last step the relation $(u')\phi(\overline{u''v})\phi = (\overline{uv})\phi$ is in $\mathcal{D}$ since

$$|(u')\phi(\overline{u''v})\phi\ (\overline{uv})\phi| = 1 + |(\overline{u''v})\phi| + |(\overline{uv})\phi| \leq 1 + |\overline{u''v}| + |\overline{uv}| \leq 1 + 2M_{UU}.$$

Case 2: *u has no prefix in $\mathcal{L}(A, T)$.* First decompose $uv \equiv u\beta_1 \dots \beta_b\beta_{b+1}$ where the principal factors are $u\beta_1, \beta_2, \dots, \beta_{b-1}, \beta_b\beta_{b+1}$. We follow the convention that β_1 always exists and β_{b+1} may be the empty word. This case now splits into two subcases.

Case 2.1: $b = 1$. In this case $(uv)\phi$ is a single letter and $(uv)\phi = (\overline{uv})\phi$ is in $\mathcal{D}$ since it has length $|(uv)\phi(\overline{uv})\phi| \leq 1 + |\overline{uv}| \leq 1 + M_{UU}$.

Case 2.2: $b \geq 2$. First note that since $\beta_b\beta_{b+1} \in \mathcal{L}(A, T)$ and $v \notin \mathcal{L}(A, T)$ it follows that $\beta_1 \dots \beta_{b-1} \notin \mathcal{L}(A, T)$. Then we have:

$$\begin{array}{rcll} (uv)\phi & \equiv & (u\beta_1 \dots \beta_{b-1}\beta_b\beta_{b+1})\phi & \\ & \equiv & (u\beta_1 \dots \beta_{b-1})\phi(\beta_b\beta_{b+1})\phi & \text{(by Lemma 4.4)} \\ & = & (\overline{u\beta_1 \dots \beta_{b-1}})\phi(\beta_b\beta_{b+1})\phi & \text{(induction)} \\ & = & (\overline{uv})\phi & \text{(in } \mathcal{D}). \end{array}$$

In the last step the relation $(\overline{u\beta_1 \dots \beta_{b-1}})\phi(\beta_b\beta_{b+1})\phi = (\overline{uv})\phi$ is in $\mathcal{D}$ since

$$\begin{array}{rcl} |(\overline{u\beta_1 \dots \beta_{b-1}})\phi(\beta_b\beta_{b+1})\phi(\overline{uv})\phi| & = & |(\overline{u\beta_1 \dots \beta_{b-1}})\phi| + |(\beta_b\beta_{b+1})\phi| + |(\overline{uv})\phi| \\ & \leq & |\overline{u\beta_1 \dots \beta_{b-1}}| + 1 + |\overline{uv}| \leq 1 + 2M_{UU} \end{array}$$

as required.

$\square$

Lemma 4.7. *The relations $(\beta\gamma\delta)\phi = (\widehat{\beta}\,\overline{\gamma\delta})\phi$ where $\beta \in \mathcal{L}(A, U^1), \gamma \in \mathcal{PI}(A, T), \delta \in \mathcal{L}(A, U^1), \gamma\delta \in \mathcal{L}(A, T)$ and $\beta\gamma\delta \in \mathcal{L}(A, T)$ are consequences of $\mathcal{D}$.*

Proof. Note that since, by definition, all of the relations in $\mathcal{D}$ hold in S, once we have shown that the relations $(\beta\gamma\delta)\phi = (\widehat{\beta}\,\overline{\gamma\delta})\phi$ are consequences of $\mathcal{D}$ it will also follow that these relations all hold in S.

We proceed by induction on the length of the word $|\beta\gamma\delta|$. When $|\beta\gamma\delta| \leq 3$ the relation $(\beta\gamma\delta)\phi = (\widehat{\beta}\,\overline{\gamma\delta})\phi$ is in $\mathcal{D}$ since

$$|(\beta\gamma\delta)\phi(\widehat{\beta}\,\overline{\gamma\delta})\phi| \leq |\beta\gamma\delta| + |\widehat{\beta}| + |\overline{\gamma\delta}| \leq 3 + M_\Sigma + M_\mathcal{B}.$$

Now let $\beta \in \mathcal{L}(A, U^1), \gamma \in \mathcal{PI}(A, T), \delta \in \mathcal{L}(A, U^1)$ be such that $\gamma\delta \in \mathcal{L}(A, T)$ and $\beta\gamma\delta \in \mathcal{L}(A, T)$, and suppose that the result holds for all $\beta_1, \gamma_1, \delta_1$ satisfying the analogous conditions with $|\beta_1\gamma_1\delta_1| < |\beta\gamma\delta|$. First observe that if β is empty then the relation becomes $(\gamma\delta)\phi = (\overline{\gamma\delta})\phi$ which has length $|(\gamma\delta)\phi(\overline{\gamma\delta})\phi| \leq 1 + M_\mathcal{B}$ and so belongs to $\mathcal{D}$. When β is not empty there are two cases to consider depending on whether or not β has a prefix in $\mathcal{L}(A, T)$.

Case 1: *β has a prefix in $\mathcal{L}(A, T)$.* Let β' be the shortest such prefix and write $\beta \equiv \beta'\beta''$. Note that since $\beta \in \mathcal{L}(A, U^1)$, $\beta' \in \mathcal{L}(A, T)$ and T is a subsemigroup of S it follows that $\beta'' \in \mathcal{L}(A, U^1)$. This case now splits into two subcases depending on whether or not $\beta''\gamma\delta \in \mathcal{L}(A, T)$.

Case 1.1: $\beta''\gamma\delta \notin \mathcal{L}(A, T)$. In this case $(\beta\gamma\delta)\phi$ is a single letter and $(\beta\gamma\delta)\phi = (\widehat{\beta}\,\overline{\gamma\delta})\phi$ is in $\mathcal{D}$ since

$$|(\beta\gamma\delta)\phi(\widehat{\beta}\,\overline{\gamma\delta})\phi| = 1 + |(\widehat{\beta}\,\overline{\gamma\delta})\phi| \leq 1 + |\widehat{\beta}\,\overline{\gamma\delta}| = 1 + |\widehat{\beta}| + |\overline{\gamma\delta}| \leq 1 + M_\Sigma + M_\mathcal{B}.$$

Case 1.2: $\beta''\gamma\delta \in \mathcal{L}(A, T)$. In this case we have:

$$\begin{array}{rcll}
(\beta\gamma\delta)\phi & \equiv & (\beta'\beta''\gamma\delta)\phi & \\
& \equiv & (\beta')\phi(\beta''\gamma\delta)\phi & \text{(by definition of } \phi) \\
& = & (\beta')\phi(\widehat{\beta''\overline{\gamma\delta}})\phi & \text{(induction)} \\
& = & (\widehat{\beta}\,\overline{\gamma\delta})\phi & (\text{in } \mathcal{D}).
\end{array}$$

In the last step the relation $(\beta')\phi(\widehat{\beta''\overline{\gamma\delta}})\phi = (\widehat{\beta}\,\overline{\gamma\delta})\phi$ is in $\mathcal{D}$ since

$$\begin{array}{rcl}
|(\beta')\phi(\widehat{\beta''\overline{\gamma\delta}})\phi(\widehat{\beta}\,\overline{\gamma\delta})\phi| & = & 1 + |(\widehat{\beta''\overline{\gamma\delta}})\phi| + |(\widehat{\beta}\,\overline{\gamma\delta})\phi| \leq 1 + |\widehat{\beta''\overline{\gamma\delta}}| + |\widehat{\beta}\,\overline{\gamma\delta}| \\
& = & 1 + |\widehat{\beta''}| + |\overline{\gamma\delta}| + |\widehat{\beta}| + |\overline{\gamma\delta}| \leq 1 + 2M_\Sigma + 2M_\mathcal{B}.
\end{array}$$

Case 2: *β has no prefix in $\mathcal{L}(A, T)$.* In this case we decompose:

$$\beta\gamma\delta \equiv \beta\gamma_1 \ldots \gamma_{c+1}\delta_1 \ldots \delta_d\delta_{d+1},$$

where $\gamma \equiv \gamma_1 \ldots \gamma_{c+1}$, $\delta \equiv \delta_1 \ldots \delta_{d+1}$ and the principal factors of $\beta\gamma\delta$ are

$$\beta\gamma_1, \gamma_2, \ldots, \gamma_c, \gamma_{c+1}\delta_1, \delta_2, \ldots, \delta_{d-1}, \delta_d\delta_{d+1}.$$

A few words of explanation are in order here. As usual, we think of the principal factors as being obtained by reading the word $\beta\gamma\delta$ from left to right, and writing successive prefixes that belong to $\mathcal{PI}(A,T)$, as long as the remaining suffix is in $\mathcal{L}(A,T)$. Thus, $\beta\gamma_1$ is the first such prefix, provided it is also a prefix of $\beta\gamma$. If the first such prefix is longer than $\beta\gamma$ we take $c = 0$ and $\gamma_{c+1} \equiv \gamma_1 \equiv \gamma$. Also, γ_c is the last of these prefixes which ends inside γ, and γ_{c+1} is the rest of γ. Of course, it may happen that γ_c ends at the last letter of γ, in which case we take $\gamma_{c+1} \equiv \epsilon$. Furthermore, in this case, $\gamma\delta$ is the final principal factor since $\delta \notin \mathcal{L}(A,T)$ and so we take $d = 0$ and $\delta_{d+1} \equiv \delta_1 \equiv \delta$.

This case now splits into two subcases.

Case 2.1: $d \geq 2$. In this case, by Lemma 4.4, we have

$$(\beta\gamma\delta)\phi \equiv (\beta\gamma\delta_1 \ldots \delta_{d-1})\phi(\delta_d\delta_{d+1})\phi.$$

This subcase now splits into two further subcases depending on whether or not $\gamma\delta_1 \ldots \delta_{d-1} \in \mathcal{L}(A,T)$.

Case 2.1.1: $\gamma\delta_1 \ldots \delta_{d-1} \notin \mathcal{L}(A,T)$. Then since $\beta \notin \mathcal{L}(A,T)$ we can apply the previous lemma to give:

$$\begin{array}{rcll} (\beta\gamma\delta)\phi & \equiv & (\beta\gamma\delta_1 \ldots \delta_{d-1})\phi(\delta_d\delta_{d+1})\phi & \\ & = & (\overline{\beta\gamma\delta_1 \ldots \delta_{d-1}})\phi(\delta_d\delta_{d+1})\phi & \text{(by Lemma 4.6)} \\ & = & (\widehat{\beta}\,\overline{\gamma\delta})\phi & \text{(in } \mathcal{D}\text{).} \end{array}$$

In the last step the relation $(\overline{\beta\gamma\delta_1 \ldots \delta_{d-1}})\phi(\delta_d\delta_{d+1})\phi = (\widehat{\beta}\,\overline{\gamma\delta})\phi$ is in $\mathcal{D}$ since

$$\begin{array}{rcl} |(\overline{\beta\gamma\delta_1 \ldots \delta_{d-1}})\phi(\delta_d\delta_{d+1})\phi(\widehat{\beta}\,\overline{\gamma\delta})\phi| & = & |(\overline{\beta\gamma\delta_1 \ldots \delta_{d-1}})\phi| + 1 + |(\widehat{\beta}\,\overline{\gamma\delta})\phi| \\ & \leq & |\overline{\beta\gamma\delta_1 \ldots \delta_{d-1}}| + 1 + |\widehat{\beta}| + |\overline{\gamma\delta}| \\ & \leq & M_{UU} + 1 + M_\Sigma + M_\mathcal{B}. \end{array}$$

Case 2.1.2: $\gamma\delta_1 \ldots \delta_{d-1} \in \mathcal{L}(A,T)$. Then, since $\delta_d\delta_{d+1} \in \mathcal{L}(A,T)$, $\delta \in \mathcal{L}(A,U^1)$ and T is a subsemigroup of S, we have $\delta_1 \ldots \delta_{d-1} \notin \mathcal{L}(A,T)$ and so $\gamma\delta_1 \ldots \delta_{d-1} \in \mathcal{L}(A,X_\rho)$ and we can apply induction giving:

$$\begin{array}{rcll} (\beta\gamma\delta)\phi & \equiv & (\beta\gamma\delta_1 \ldots \delta_{d-1})\phi(\delta_d\delta_{d+1})\phi & \text{(by Lemma 4.4)} \\ & = & (\widehat{\beta}\,\overline{\gamma\delta_1 \ldots \delta_{d-1}})\phi(\delta_d\delta_{d+1})\phi & \text{(induction)} \\ & = & (\widehat{\beta}\,\overline{\gamma\delta})\phi & \text{(in } \mathcal{D}\text{).} \end{array}$$

In the last step the relation $(\widehat{\beta}\,\overline{\gamma\delta_1 \ldots \delta_{d-1}})\phi(\delta_d\delta_{d+1})\phi = (\widehat{\beta}\,\overline{\gamma\delta})\phi$ is in $\mathcal{D}$ since

$$\begin{array}{rcl} |(\widehat{\beta}\,\overline{\gamma\delta_1 \ldots \delta_{d-1}})\phi(\delta_d\delta_{d+1})\phi(\widehat{\beta}\,\overline{\gamma\delta})\phi| & = & |(\widehat{\beta}\,\overline{\gamma\delta_1 \ldots \delta_{d-1}})\phi| + 1 + |(\widehat{\beta}\,\overline{\gamma\delta})\phi| \\ & \leq & |\widehat{\beta}| + |\overline{\gamma\delta_1 \ldots \delta_{d-1}}| + 1 + |\widehat{\beta}| + |\overline{\gamma\delta}| \\ & \leq & 2M_\Sigma + 2M_\mathcal{B} + 1. \end{array}$$

Case 2.2: $d \in \{0,1\}$. This subcase splits into two further subcases depending on the value of c.

Case 2.2.1: $c \geq 2$. Since $\gamma \in \mathcal{PI}(A,T)$ no strict prefix of γ belongs to $\mathcal{L}(A,T)$. In particular we have $\gamma_1 \dots \gamma_{c-1} \in \mathcal{L}(A,U^1)$. Now we have:

$$\begin{array}{rcll} (\beta\gamma\delta)\phi & \equiv & (\beta\gamma_1 \dots \gamma_{c-1}\gamma_c\gamma_{c+1}\delta)\phi & \\ & \equiv & (\beta\gamma_1 \dots \gamma_{c-1})\phi(\gamma_c\gamma_{c+1}\delta)\phi & \text{(by Lemma 4.4 and since } c \geq 2) \\ & = & (\overline{\beta\gamma_1 \dots \gamma_{c-1}})\phi(\gamma_c\gamma_{c+1}\delta)\phi & \text{(by Lemma 4.6)} \\ & = & (\widehat{\beta}\ \overline{\gamma\delta})\phi & \text{(in } \mathcal{D}). \end{array}$$

In the last step the relation $(\overline{\beta\gamma_1 \dots \gamma_{c-1}})\phi(\gamma_c\gamma_{c+1}\delta)\phi = (\widehat{\beta}\ \overline{\gamma\delta})\phi$ belongs to $\mathcal{D}$ since

$$\begin{array}{rcl} |(\overline{\beta\gamma_1 \dots \gamma_{c-1}})\phi(\gamma_c\gamma_{c+1}\delta)\phi(\widehat{\beta}\ \overline{\gamma\delta})\phi| & = & |(\overline{\beta\gamma_1 \dots \gamma_{c-1}})\phi| + |(\gamma_c\gamma_{c+1}\delta)\phi| + |(\widehat{\beta}\ \overline{\gamma\delta})\phi| \\ & \leq & |\overline{\beta\gamma_1 \dots \gamma_{c-1}}| + 2 + |\widehat{\beta}| + |\overline{\gamma\delta}| \\ & \leq & M_{UU} + 2 + M_\Sigma + M_\mathcal{B}. \end{array}$$

Case 2.2.2: $c \in \{0,1\}$. In this case $c \in \{0,1\}$ and $d \in \{0,1\}$ and β has no prefix in $\mathcal{L}(A,T)$. It follows that $|(\beta\gamma\delta)\phi| \leq 2$ and so $(\beta\gamma\delta)\phi = (\widehat{\beta}\ \overline{\gamma\delta})\phi$ is in $\mathcal{D}$ since

$$|(\beta\gamma\delta)\phi(\widehat{\beta}\ \overline{\gamma\delta})\phi| \leq |(\beta\gamma\delta)\phi| + |(\widehat{\beta}\ \overline{\gamma\delta})\phi| \leq 2 + |\widehat{\beta}| + |\overline{\gamma\delta}| = 2 + M_\Sigma + M_\mathcal{B}$$

as required. □

Lemma 4.8. *The relations $(\alpha\beta\gamma\delta)\phi = (\alpha\beta)\phi(\gamma\delta)\phi$ where $\alpha, \gamma \in \mathcal{PI}(A,T)$, $\beta, \delta \in \mathcal{L}(A,U^1), \alpha\beta \in \mathcal{L}(A,T)$ and $\gamma\delta \in \mathcal{L}(A,T)$ are consequences of $\mathcal{D}$.*

Proof. First note that if β is the empty word then, by definition of ϕ, we have:

$$(\alpha\beta\gamma\delta)\phi \equiv (\alpha\gamma\delta)\phi \equiv \alpha\phi(\gamma\delta)\phi \equiv (\alpha\beta)\phi(\gamma\delta)\phi.$$

Now suppose that β is non-empty. There are two cases to consider depending on whether or not $\beta\gamma\delta \in \mathcal{L}(A,T)$.

Case 1: $\beta\gamma\delta \notin \mathcal{L}(A,T)$. In this case $(\alpha\beta\gamma\delta)\phi$ is a single letter and the relation $(\alpha\beta\gamma\delta)\phi = (\alpha\beta)\phi(\gamma\delta)\phi$ is in $\mathcal{D}$ since

$$|(\alpha\beta\gamma\delta)\phi(\alpha\beta)\phi(\gamma\delta)\phi| = |(\alpha\beta\gamma\delta)\phi| + |(\alpha\beta)\phi| + |(\gamma\delta)\phi| \leq 3.$$

Case 2: $\beta\gamma\delta \in \mathcal{L}(A,T)$. In this case we have:

$$\begin{array}{rcll} (\alpha\beta\gamma\delta)\phi & \equiv & \alpha\phi(\beta\gamma\delta)\phi & \\ & = & \alpha\phi(\widehat{\beta}\ \overline{\gamma\delta})\phi & \text{(by Lemma 4.7)} \\ & = & (\alpha\beta)\phi(\gamma\delta)\phi & \text{(in } \mathcal{D}). \end{array}$$

In the last step the relation $\alpha\phi(\widehat{\beta}\,\overline{\gamma\delta})\phi = (\alpha\beta)\phi(\gamma\delta)\phi$ is in $\mathcal{D}$ since

$$\begin{aligned}
|\alpha\phi(\widehat{\beta}\,\overline{\gamma\delta})\phi(\alpha\beta)\phi(\gamma\delta)\phi| &\leq |\alpha\phi| + |(\widehat{\beta}\,\overline{\gamma\delta})\phi| + |(\alpha\beta)\phi| + |(\gamma\delta)\phi| \\
&\leq 1 + |\widehat{\beta}| + |\overline{\gamma\delta}| + 1 + 1 \\
&\leq 3 + |\widehat{\beta}| + |\overline{\gamma\delta}| \leq 3 + M_\Sigma + M_{\mathcal{B}}
\end{aligned}$$

as required. □

Completing the proof of Theorem B

We now complete the proof by proving that the relations (4.2) and (4.3) are all consequences of our fixed finite set $\mathcal{D}$ of relations.

Lemma 4.9. *The relations $(w_1w_2)\phi = (w_1)\phi(w_2)\phi$ where $w_1, w_2 \in \mathcal{L}(A,T)$ are consequences of $\mathcal{D}$.*

Proof. We proceed by induction on the length of the word w_1w_2. When $|w_1w_2| \leq 2$ the relation $(w_1w_2)\phi = (w_1)\phi(w_2)\phi$ is in $\mathcal{D}$ since:

$$|(w_1w_2)\phi(w_1)\phi(w_2)\phi| = |(w_1w_2)\phi| + |w_1\phi| + |w_2\phi| \leq 2 + 1 + 1 = 4.$$

Let $w_1, w_2 \in \mathcal{L}(A,T)$ and suppose that the result holds for all $w_1', w_2' \in \mathcal{L}(A,T)$ such that $|w_1'w_2'| < |w_1w_2|$. Decompose the word w_2:

$$w_2 \equiv \beta_1 \ldots \beta_b\beta_{b+1}$$

where the principal factors are $\beta_1, \ldots, \beta_{b-1}, \beta_b\beta_{b+1}$. Now consider the following prefixes of the word w_1w_2:

$$\xi_0 \equiv w_1, \quad \xi_i \equiv w_1\beta_1 \ldots \beta_i \quad (1 \leq i \leq b-1).$$

These words all belong to $\mathcal{L}(A,T)$ since they are products of elements of $\mathcal{L}(A,T)$. There are two cases to consider:

Case 1: $|\xi_k\phi| = 1$ *for all* $0 \leq k \leq b-1$. In this case we can repeatedly apply Lemma 4.8 to get:

$$\begin{aligned}
(w_1\phi)(w_2\phi) &\equiv (w_1)\phi(\beta_1\beta_2\ldots\beta_{b-1}\beta_b\beta_{b+1})\phi \\
&\equiv (w_1)\phi(\beta_1)\phi(\beta_2)\phi\ldots(\beta_{b-1})\phi(\beta_b\beta_{b+1})\phi \\
&= (w_1\beta_1)\phi(\beta_2)\phi\ldots(\beta_{b-1})\phi(\beta_b\beta_{b+1})\phi \\
&= (w_1\beta_1\beta_2)\phi\ldots(\beta_{b-1})\phi(\beta_b\beta_{b+1})\phi \\
&= \ldots \\
&= (w_1\beta_1\beta_2\ldots\beta_{b-1})\phi(\beta_b\beta_{b+1})\phi \\
&= (w_1\beta_1\beta_2\ldots\beta_{b-1}\beta_b\beta_{b+1})\phi \\
&\equiv (w_1w_2)\phi.
\end{aligned}$$

Case 2: $|\xi_k\phi| > 1$ *for some* $0 \le k \le b-1$. Let k be the smallest number such that $|\xi_k\phi| > 1$. Decompose ξ_k into principal factors:

$$\xi_k \equiv \gamma_1 \dots \gamma_c \gamma_{c+1}$$

where, since $|\xi_k\phi| > 1$, we know that $c \ge 2$. Proceeding as in Case 1 we first obtain

$$(w_1\phi)(w_2\phi) = (w_1\beta_1\beta_2 \dots \beta_k)\phi(\beta_{k+1} \dots \beta_b\beta_{b+1})\phi.$$

This time we continue as follows:

$$\begin{array}{lll} & (w_1\beta_1\beta_2 \dots \beta_k)\phi(\beta_{k+1} \dots \beta_b\beta_{b+1})\phi & \\ \equiv & (\gamma_1 \dots \gamma_c\gamma_{c+1})\phi(\beta_{k+1} \dots \beta_b\beta_{b+1})\phi & \\ \equiv & (\gamma_1)\phi(\gamma_2 \dots \gamma_{c+1})\phi(\beta_{k+1} \dots \beta_b\beta_{b+1})\phi & \text{(by Lemma 4.4 and since } c \ge 2) \\ = & (\gamma_1)\phi(\gamma_2 \dots \gamma_{c+1}\beta_{k+1} \dots \beta_b\beta_{b+1})\phi & \text{(induction)} \\ \equiv & (\gamma_1\gamma_2 \dots \gamma_{c+1}\beta_{k+1} \dots \beta_b\beta_{b+1})\phi & \text{(by Lemma 4.4)} \\ \equiv & (w_1w_2)\phi & \end{array}$$

as required. □

Lemma 4.10. *The relations* $(w_3xw_4)\phi = (w_3yw_4)\phi$ *where* $w_3, w_4 \in A^*, (x = y) \in \mathfrak{R}$ *and* $w_3xw_4 \in \mathcal{L}(A,T)$ *are consequences of* $\mathcal{D}$.

Proof. We proceed by induction on the combined length of w_3xw_4 and w_3yw_4. When $|w_3xw_4w_3yw_4| = 2$ the words w_3 and w_4 are empty and the relation $(w_3xw_4)\phi = (w_3yw_4)\phi$ belongs to $\mathcal{D}$ since

$$|(w_3xw_4)\phi(w_3yw_4)\phi| = |(w_3xw_4)\phi| + |(w_3yw_4)\phi| \le |w_3xw_4| + |w_3yw_4| = 2.$$

Let $w_3, w_4 \in A^*, (x = y) \in \mathfrak{R}$ and $w_3xw_4 \in \mathcal{L}(A,T)$ and suppose that the result holds for all w_3', w_4' and $(x' = y')$ satisfying the analogous conditions where $|w_3'x'w_4'w_3'y'w_4'| < |w_3xw_4w_3yw_4|$. There are two cases to consider depending on whether or not w_3 has a prefix that belongs to $\mathcal{L}(A,T)$.

Case 1: w_3 *has a prefix that belongs to* $\mathcal{L}(A,T)$. Then write $w_3 \equiv w_3'w_3''$ where w_3' is the shortest such prefix. This case now splits into two subcases depending on whether or not $w_3''xw_4 \in \mathcal{L}(A,T)$.

Case 1.1: $w_3''xw_4 \notin \mathcal{L}(A,T)$. Then we also have $w_3''yw_4 \notin \mathcal{L}(A,T)$ and so $(w_3xw_4)\phi$ and $(w_3yw_4)\phi$ are both single letters and the relation $(w_3xw_4)\phi = (w_3yw_4)\phi$ is trivial.

Case 1.2: $w_3''xw_4 \in \mathcal{L}(A,T)$. Then we have:

$$\begin{array}{rcll} (w_3xw_4)\phi & \equiv & (w_3'w_3''xw_4)\phi & \\ & \equiv & (w_3')\phi(w_3''xw_4)\phi & \text{(by Lemma 4.4)} \\ & = & (w_3')\phi(w_3''yw_4)\phi & \text{(induction)} \\ & \equiv & (w_3yw_4)\phi. & \end{array}$$

Case 2: *w_3 has no prefix that belongs to $\mathcal{L}(A,T)$.* In this case we decompose our words into principal factors:

$$w_3xw_4 \equiv w_3\beta_1\ldots\beta_{b+1}\gamma_1\ldots\gamma_{c+1}, \quad w_3yw_4 \equiv w_3\beta'_1\ldots\beta'_{b'+1}\gamma'_1\ldots\gamma'_{c'+1}$$

where $x \equiv \beta_1\ldots\beta_{b+1}$, $y \equiv \beta'_1\ldots\beta'_{b'+1}$ and the principal factors of w_3xw_4 are $w_3\beta_1$, β_2, ..., β_b, $\beta_{b+1}\gamma_1$, γ_2, ..., γ_{c-1} and $\gamma_c\gamma_{c+1}$, and those of w_3yw_4 are $w_3\beta'_1$, β'_2, ..., $\beta'_{b'}$, $\beta'_{b'+1}\gamma'_1$, γ'_2, ..., $\gamma'_{c'-1}$ and $\gamma'_{c'}\gamma'_{c'+1}$. There are two cases to consider depending on the values of c and of c'.

Case 2.1: *$c \geq 2$ or $c' \geq 2$.* If $c \geq 2$ we have:

$$\begin{aligned}
(w_3xw_4)\phi &\equiv (w_3x\gamma_1\ldots\gamma_c\gamma_{c+1})\phi \\
&\equiv (w_3x\gamma_1\ldots\gamma_{c-1})\phi(\gamma_c\gamma_{c+1})\phi && \text{(by Lemma 4.4)} \\
&= (w_3y\gamma_1\ldots\gamma_{c-1})\phi(\gamma_c\gamma_{c+1})\phi && \text{(induction)} \\
&= (w_3y\gamma_1\ldots\gamma_{c-1}\gamma_c\gamma_{c+1})\phi && \text{(by Lemma 4.9)} \\
&\equiv (w_3yw_4)\phi.
\end{aligned}$$

The case $c' \geq 2$ is dealt with analogously.

Case 2.2: *$c, c' \in \{0,1\}$.* In this case first note that $|(w_3xw_4)\phi| = b + c \leq M_{\mathfrak{R}} + 1$. Likewise $|(w_3yw_4)\phi| = b' + c' \leq M_{\mathfrak{R}} + 1$ and we conclude that $|(w_3xw_4)\phi(w_3yw_4)\phi| \leq 2M_{\mathfrak{R}}+2$ and therefore the relation $(w_3xw_4)\phi = (w_3yw_4)\phi$ belongs to $\mathcal{D}$. $\square$

5 Applications

In this section we shall give some examples, corollaries and illustrative applications of Theorems A and B.

As already observed in Proposition 2.2, two situations where the boundary of a subsemigroup T of a semigroup S is obviously finite are (a) when $|S\setminus T|$ is finite, or (b) when $S \setminus T$ forms an ideal of S. In particular this means that the main result of [22], which states that finite presentability is inherited by subsemigroups with finite Rees index, is an immediate corollary of Theorem B. Recall that by the Rees index of a subsemigroup T in a semigroup S we simply mean the cardinality of the complement $S \setminus T$. So, as a corollary of Theorem B we have:

Corollary 5.1. *[22, Theorem 1.3] Let S be a semigroup and let T be a subsemigroup of S. If S is finitely presented and $S \setminus T$ is finite then T is finitely presented.*

We note that the proof of this fact given in [22] is approximately the same length as the proof of the more general Theorem B given above. The proof of Theorem B is also less complicated; in particular, it avoids resorting to nested inductions.

This is an appropriate place for us to point out that in the process of writing this paper it was discovered that there is a slight problem in the proof of [22, Theorem 1.3] given in [22]. Specifically, at the end of what is called 'Stage 2 of the proof' three cases, Cases 4–6, are considered. It turns out that for each of these cases, using the notation of [22], the step that claims a relation from $\mathcal{D}$ is being applied is only valid when the words γ_i appearing in the expressions represent elements in the complement of the subsemigroup. As a result of this, each of these cases splits into further subcases that need to be dealt with, and are not handled in [22]. However, as it turns out, these cases can be patched up without too much difficulty. Since the fix is straightforward, and since Theorem B generalises [22, Theorem 1.3] in any case, we shall not go to the trouble of working through the details here of a direct fix of the proof of [22, Theorem 1.3], leaving it as an entertaining exercise for the interested reader.

Applications of our main theorems go way beyond merely providing a common generalisation for the finite Rees index case and the ideal complement case. For instance, subsemigroups of groups can have finite boundaries in their parent groups (consider the very easy example of the natural numbers inside the infinite cyclic group for instance) while it is easy to see that an infinite group does not have either any proper ideals, or any proper subsemigroups of finite Rees index. Of course, many other such examples exist. Let us now present one such example in detail showing that subsemigroups with finite boundary can be simultaneously very far away from having finite Rees index, and from having an ideal complement.

Let S be the monoid defined by the presentation

$$\langle\, a, b \mid ba = a,\ a^4b^4 = a^4b \,\rangle.$$

It is an easy exercise to check that this is a finite complete rewriting system, and hence a set of normal forms for the elements of the semigroup is given by the set of all words in $\{a, b\}^*$ not containing either of the left hand sides ba or a^4b^4 as a subword (see [3] for background on string rewriting systems). Let

$$T = \{a^i b^j : 0 \leq j \leq 3, i \geq 2\} \leq S.$$

The elements of this subsemigroup are indicated in Figures 1 and 2 where the left and right Cayley graphs of the semigroup S are illustrated. It is clear that T has a finite boundary in S, the complement $S \setminus T$ of T is infinite, and $S \setminus T$ is not an ideal. What is more, $S \setminus T$ is in a sense far from being an ideal: for every ideal I of S the symmetric difference $(S \setminus T)\Delta I$ is infinite. This example generalises in a natural way to a family of semigroups

$$S_{k,l,m} = \langle\, a, b \mid ba = a,\ a^k b^l = a^k b^m \,\rangle$$

where $k, l, m \in \mathbb{N}$ with $l > m \geq 1$, in which we can similarly find subsemigroups with finite boundaries that do not simply have finite Rees index or ideal complement.

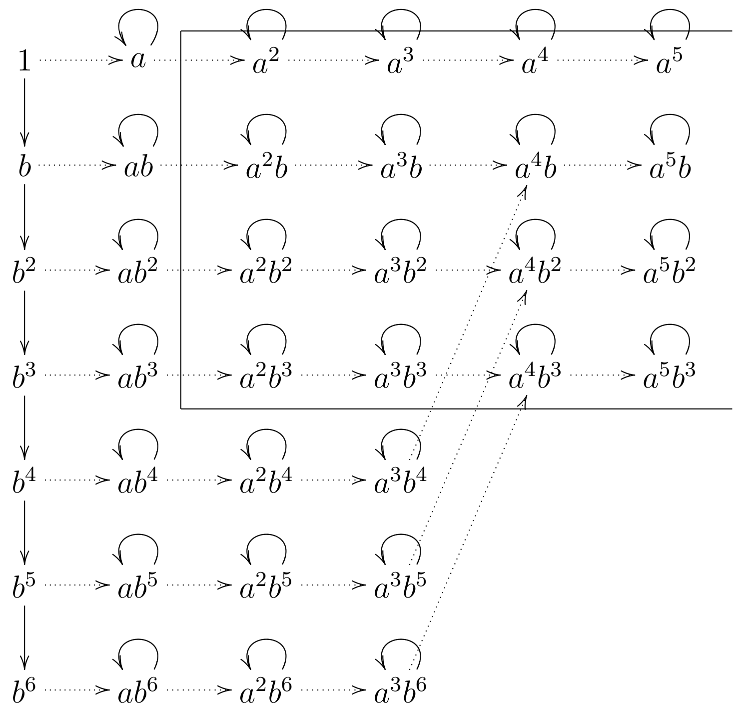

Figure 1: A partial view of the left Cayley graph of S where $\rightarrow$ corresponds to multiplication by b and $\dashrightarrow$ corresponds to multiplication by a. The elements of the subsemigroup T are those in the rectangle.

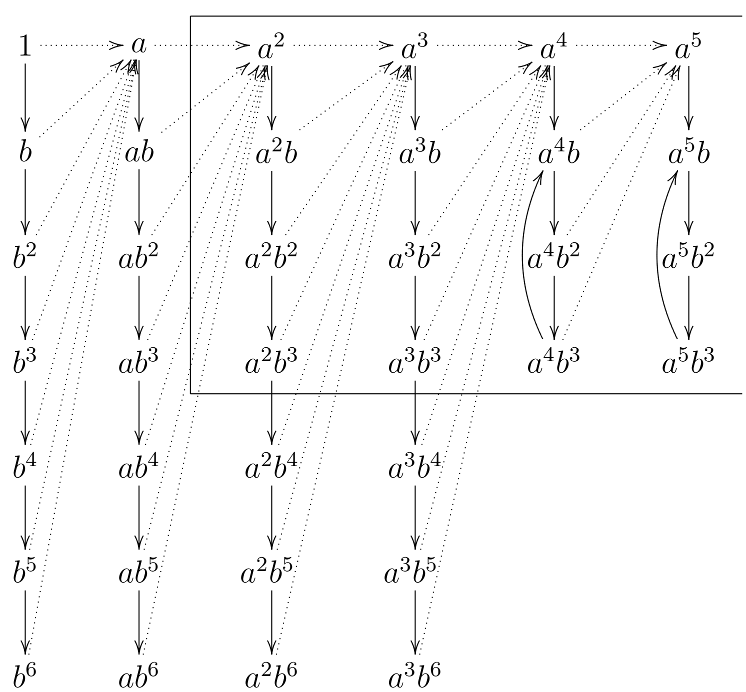

Figure 2: The right Cayley graph of S where $\rightarrow$ corresponds to b and $\dashrightarrow$ to a.

We now move on to give some general results that can be derived from Theorems A and B.

Semigroups with tree-like Cayley graphs

Since Theorems A and B are geometric in nature, one could justifiably hope for applications with a geometric flavour. In this subsection we provide one such application.

Geometrically, free groups are precisely those groups that have a tree as a Cayley graph. So a natural next step is to investigate groups that are 'tree-like'. This way of thinking has been very fruitful in the study of finitely generated infinite groups, giving rise to numerous interesting areas such as the study of Gromov hyperbolic groups, and the theory of ends of groups and Stallings theorem; see [12] for more background on these topics from geometric group theory.

Analogously, the Cayley graph of a free semigroup is a directed tree, and so it is not unreasonable to investigate finitely generated semigroups and monoids whose Cayley graphs are 'directed tree-like'. Of course there are many possible ways that one can try and capture the notion of tree-like in a definition. Here we shall consider one possible such definition which, in particular, will include all the standard examples that one would regard as 'obviously' having tree-like Cayley graphs, such as free semigroups, free products of finite semigroups, the bicyclic monoid etc.

It is well known that, in stark contrast to the situation in group theory, finitely generated subsemigroups of free semigroups need not be finitely presented (see [19, page 7]). But what if we instead try restricting our attention to ideals? For instance, [5, Theorem 3.5] asserts that if an ideal I of a free semigroup F is finitely generated (as a subsemigroup), then it has finite Rees index, and hence it is finitely presented. In what follows we prove a much more general theorem, and show in the process that boundaries, rather than Rees index, provide the appropriate framework for these considerations. Note that in general it is not true that ideals of finitely presented semigroups that are finitely generated as subsemigroups are necessarily finitely presented; see [6, Theorem 3.1]. So the tree-like restriction we are about to introduce will be playing a crucial role.

Let S be a semigroup generated by a finite set A, and as usual let $\Gamma_r(A, S)$ and $\Gamma_l(A, S)$ be the right and left Cayley graphs of S. By a *walk* p in $\Gamma_r(A, S)$ we simply mean a sequence of (not necessarily distinct) vertices $(v_0, v_1, \ldots v_n)$ such that $v_i \to v_{i+1}$ for all i. A *path* is a walk where all the vertices in the walk are distinct. We call v_0 the initial vertex of the walk and v_n the terminal vertex of the walk, and we denote these vertices by ιp and τp, respectively. The *length* of the walk p is n. Given to vertices s and t in the Cayley graph $\Gamma_r(A, S)$ we use $d_A(s, t)$ to denote the shortest length of a path from s to t in $\Gamma_r(A, S)$ if such a path exists, and set $d_A(s, t) = \infty$ otherwise. We say that the walks p and q are *parallel* if $\iota p = \iota q$ and $\tau p = \tau q$, and we say that a pair (p, q) of parallel walks is

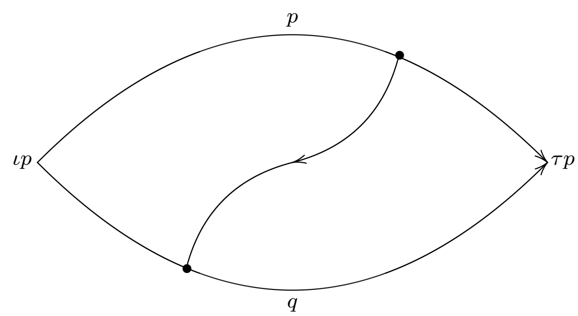

Figure 3: The splitting parallel paths property.

disjoint if they have no other vertices in common.

If p is a path in a Cayley graph $\Gamma_r(A, S)$, and v is a vertex in the path p, we say that v is an *interior point* of p if v is neither the initial nor terminal vertex of the path p.

Definition 5.2. Let S be a semigroup generated by a finite set A. We say that the right Cayley graph $\Gamma_r(A, S)$ has *splitting parallel paths* (SPP) if there is a constant $K > 0$ such that for every (ordered) pair (p, q) of parallel paths in $\Gamma_r(A, S)$, if $d_A(\iota p, \tau p) > K$ then there is a path in $\Gamma_r(A, S)$ from an interior point of p to an interior point of q (we say that (p, q) *splits*). A dual definition applies to the left Cayley graph $\Gamma_l(A, S)$.

Intuitively this condition says that any sufficiently large directed 2-cell in the Cayley graph can be split into two directed 2-cells (although these new directed 2-cells need not be smaller than the original). This is illustrated in Figure 5. Note that, in particular, any pair of paths (p, q) that is not disjoint automatically splits, by taking an interior point in the intersection of the two paths, along with the empty path from that vertex to itself. Thus if there is a bound on the size (i.e. distance between endpoints) of pairs of disjoint parallel paths in the Cayley graph, then it will have SPP. It is in this sense that SPP generalises the property of being a directed tree, since in a directed tree there is at most one directed path between any pair of vertices, and so there are *no* pairs of disjoint parallel paths and thus the Cayley graph automatically has SPP.

Even though the definition of SPP refers to a particular generating set it turns out this is not needed.

Lemma 5.3. *Let S be a finitely generated semigroup and let A and B be two finite generating sets for S. Then $\Gamma_r(A, S)$ (resp. $\Gamma_l(A, S)$) has splitting parallel paths if and only if $\Gamma_r(B, S)$ (resp. $\Gamma_l(B, S)$) has splitting parallel paths.*

Definition 5.4. We say that a finitely generated semigroup S has splitting parallel paths if, with respect to some (and hence any) finite generating set, the left and right Cayley graphs of S both have splitting parallel paths.

The proof of Lemma 5.3 is straightforward, and is relegated to the end of this subsection so that we can first state the result we want to present about such semigroups.

The class of semigroups with SPP includes: finite semigroups, all groups, free semigroups, the bicyclic monoid, the semigroups $S_{k,l,m}$ defined above, any semigroup defined by $\langle\ A, B \mid \alpha\beta = \gamma\delta\ \rangle$ where $\alpha \in A^+$, $\beta \in B^+$, $\gamma \in A^*$, $\delta \in B^*$, and $A \cap B = \emptyset$. In contrast, it is easy to see that the free commutative semigroup $\langle\ a, b \mid ab = ba\ \rangle$ does *not* have splitting parallel paths (nor would we expect it to, since its Cayley graph is not at all tree-like). Further, the semigroup free product of two semigroups with SPP again has SPP, and the monoid free product of two monoids with SPP again has SPP. All these assertions are straightforward to prove. Less trivially, generalising Lemma 5.3, the SPP property is a quasi-isometry invariant of finitely generated semigroups (in the sense of quasi-isometry between semigroups considered in [10]). Thus any semigroup that looks sufficiently tree-like when 'viewed from far away' will have SPP.

Applying Theorem B we can now characterise finite generation and presentability for ideals of SPP monoids in terms of boundaries in Cayley graphs.

Theorem 5.5. *Let S be a finitely presented monoid with splitting parallel paths, and let I be an ideal of S. Then the following are equivalent:*

(i) I is finitely generated (as a subsemigroup of S);

(ii) I is finitely presented;

(iii) I has finite boundary in S.

Proof. Clearly (ii) implies (i). That (iii) implies (ii) follows from Theorem B. This just leaves the task of proving that (iii) follows from (i), so let us suppose that I is finitely generated. Let $Y \subseteq I$ be a finite generating set for I and let $X \subseteq S \setminus I$ be a finite set such that $A = X \cup Y$ generates S. Such choices are possible since I and S are both finitely generated. If $1 \in I$ then $I = S$ since I is an ideal and the result clearly holds, so we may suppose that $1 \in S \setminus I$. Now $\Gamma_r(A, S)$ satisfies SPP with some constant $K > 0$ say. Let $b \in \mathcal{B}_r(A, I)$. Since $b \in I = \langle Y \rangle$, writing b as a word over Y, we see that there is a path p in $\Gamma_r(A, S)$ from 1 to b satisfying $\{p\} \cap (S \setminus I) = \{1\}$ (where $\{p\}$ denotes the set of vertices in the walk p). On the other hand, since $b \in \mathcal{B}_r(A, S)$ we can write $b = ua$ for some $u \in S \setminus I$ and $a \in A$. Let q' be a path in $\Gamma_r(A, S)$ from 1 to u. Since I is an ideal and $u \notin I$ it follows that $\{q'\} \cap I = \varnothing$. Let q be the path $q = (q', b)$. Now (p, q) is a pair of parallel paths from 1 to b. Since I is an ideal, and the interior points of q are in $S \setminus I$ while the interior points of p are all in I, it follows that there is no path in $\Gamma_r(A, S)$ from an interior point of p to an interior point of q. Therefore $d_A(1, b) \leq K$. Since S is finitely generated, and b was an arbitrary element of the right boundary, this shows that $|\mathcal{B}_r(A,S)|$ is finite. A dual argument shows $|\mathcal{B}_l(A,S)|$ is finite, completing the proof. $\square$

Note that part (iii) of Theorem 5.5 cannot be replaced by the statement 'I has finite Rees index in S'. Indeed, let S be the monoid defined by the presentation $S = \langle\, a, b \mid ba = a,\ a^4b^4 = a^4b \,\rangle$, which we discussed above, and which has SPP. The ideal $I = \{a^ib^j \mid i \geq 4, 0 \leq j \leq 3\}$ of S is finitely generated and presented, since it has finite boundary, but it clearly does not have finite Rees index in S. Therefore the right way to prove Theorem 5.5 is via the notion of boundaries and by applying Theorem B.

Let us end this subsection by verifying Lemma 5.3: the invariance of SPP under change of generators.

Definition 5.6. Let S be a semigroup generated by a finite set A. We say that the right Cayley graph $\Gamma_r(A,S)$ has *splitting parallel walks* (SPW) if there is a constant $K > 0$ such that for every (ordered) parallel pair of walks (p,q), where $p = (p_i)_{0 \leq i \leq n} = (p_0, p_1, \ldots, p_n)$ and $q = (q_i)_{0 \leq i \leq m} = (q_0, q_1, \ldots, q_m)$, if $d_A(\iota p, \tau p) \geq K$ then there exist some p_k ($k \notin \{0,n\}$) and q_l ($l \notin \{0,m\}$) such that there is a walk from p_k to q_l in $\Gamma_r(A,S)$ (i.e. such that $q_l \in p_kS^1$). A dual definition applies to the left Cayley graph $\Gamma_l(A,S)$.

Note that in the above definition the vertex p_0 might possibly be visited multiple times in the course of the walk p, and that the definition of SPW includes the possibility that the p_k that is asserted to exist actually satisfies $p_k = p_0$.

For the rest of this subsection S will denote a semigroup generated by a finite set A.

Lemma 5.7. *The Cayley graph $\Gamma_r(A,S)$ (resp. $\Gamma_l(A,S)$) has SPP if and only if $\Gamma_r(A,S)$ (resp. $\Gamma_l(A,S)$) has SPW.*

Proof. One implication is obvious, since every path in $\Gamma_r(A,S)$ is a walk. Now consider the converse. Suppose that $\Gamma_r(A,S)$ has SPP with constant $K > 0$. We claim that $\Gamma_r(A,S)$ has SPW with the same constant $K > 0$. Let (p,q) be a pair of parallel walks in $\Gamma_r(A,S)$ with $d_A(\iota p, \tau p) > K$. Then each of the walks p and q can be cut down (by removing circuits one at a time) to obtain a pair of paths p' and q' from ιp to τp where $\{p'\} \subseteq \{p\}$ (where $\{p\}$ is the set of vertices in the path p) and $\{q'\} \subseteq \{q\}$. Then (p', q') splits by assumption and since $\{p'\} \subseteq \{p\}$ and $\{q'\} \subseteq \{q\}$ this also constitutes a splitting for (p,q). $\square$

Roughly speaking, the next lemma tells us that by increasing the constant K we can force the crossing path in the splitting to be between vertices that are not close to the endvertices.

Lemma 5.8. *Suppose that $\Gamma_r(A,S)$ has SPW with constant $K > 0$, and let $N > 0$ be any integer. Then for any pair of parallel walks (p,q) in $\Gamma_r(A,S)$, with*

$p = (p_i)_{0\le i\le n}$ and $q = (q_i)_{0\le i\le m}$, if $d_A(\iota p, \tau p) > K + 2N$ then there exits p_k with $N < k < n$ and q_l with $0 < l < m - N$ such that there is a walk in $\Gamma_r(A, S)$ from p_k to q_l (i.e. $q_l \in p_k S^1$).

Proof. Let $L = K + 2N$. Let (p, q) be a pair of parallel walks in $\Gamma_r(A, S)$, with $p = (p_i)_{0\le i\le n}$ and $q = (q_i)_{0\le i\le m}$, and $d_A(\iota p, \tau p) > L$. This implies $n > N$ and $m > N$. Now $\Gamma_r(A, S)$ has SPW with constant K and so in particular this implies that there is a walk π_1 from p_1 to q_{m-1}. (Note that we could have $p_1 = p_0$ here, and similarly could have $q_{m-1} = q_m$). By the definition of L, and the triangle inequality, we see that $d(p_1, p_n) > K$ and so we may apply SPW to the pair of walks

$$((p_1, p_2, \ldots, p_n), (\pi_1, q_m))$$

to obtain a walk π_2 from p_2 to q_{m-1}. Continuing in this way after $N + 1$ steps we obtain a walk π_{N+1} from p_{N+1} to q_{m-1}. At each step we use the definition of L, together with the triangle inequality, to conclude that $d(p_i, p_n) > K$ (for $0 \le i \le N + 1$) and hence SPW may be applied.

Next we go through the whole process again, beginning with

$$((p_1, p_2, \ldots, p_N, \pi_{N+1}), (q_0, q_1, \ldots, q_{m-1})),$$

and finding walks from p_1 to q_{m-2}, p_2 to q_{m-2} and so on until we obtain a walk from p_{N+1} to q_{m-2}. At each step we make use of the fact that $d(p_i, q_{m-2})$ (for $0 \le i \le N+1$) is greater than K, which follows from the definition of L together with the triangle inequality. In this way we eventually obtain (after $(N + 1)^2$ steps) a walk from p_{N+1} to q_{m-N-1} , proving the lemma. □

Proof of Lemma 5.3. Let $S = \langle A\rangle$ where A is finite, and let $b \in S \setminus A$. We claim that $\Gamma_r(A, S)$ has SPP if and only if $\Gamma_r(A \cup \{b\}, S)$ has SPP. By Lemma 5.7 this is equivalent to proving that $\Gamma_r(A, S)$ has SPW if and only if $\Gamma_r(A \cup \{b\}, S)$ has SPW, so we shall prove this instead. This clearly suffices to prove the lemma, since once proved it will imply that $\Gamma_r(A, S)$ has SPP if and only if $\Gamma_r(A \cup B, S)$ has SPP if and only if $\Gamma_r(B, S)$ has SPP.

Write $b = w \in A^+$ and set $N = |w| \ge 1$. Now $\Gamma_r(A, S)$ is naturally a subdigraph of $\Gamma_r(A \cup \{b\}, S)$. Let $s, t \in S$ be arbitrary and such that $t \in sS^1$. Then clearly we have

$$d_{A\cup\{b\}}(s, t) \le d_A(s, t) \le N d_{A\cup\{b\}}(s, t),$$

since any path from s to t in $\Gamma_r(A \cup \{b\}, S)$ can be transformed into a path in $\Gamma_r(A, S)$ from s to t by replacing every occurrence of $x \to xb$ by a path from x to xb in $\Gamma(A, S)$ labelled by the word $w \in A^+$.

($\Leftarrow$) Suppose $\Gamma_r(A \cup \{b\}, S)$ has SPW with constant $K > 0$. We claim that $\Gamma_r(A, S)$ has SPW with constant $L = NK$. Indeed, let (p, q) be a pair of parallel walks in $\Gamma_r(A, S)$ with $d_A(\iota p, \tau p) > L$. Then

$$d_{A\cup\{b\}}(\iota p, \tau p) > \frac{1}{N} d_A(\iota p, \tau p) > \frac{L}{N} = K.$$

Therefore (p, q) splits when viewed as a pair of parallel walks in $\Gamma_r(A \cup \{b\}, S)$. This clearly immediately implies that (p, q) splits in $\Gamma_r(A, S)$ as well.

($\Rightarrow$) Suppose that $\Gamma_r(A, S)$ has SPW with constant $L > 0$. Recall that $N = |w|$ where $w \in A^+$ is a fixed word representing b. We claim that $\Gamma_r(A \cup \{b\}, S)$ has SPW with constant $K = L + 2N$. Let (p, q) be a pair of parallel walks in $\Gamma_r(A \cup \{b\}, S)$, where $p = (p_i)_{0 \leq i \leq n}$ and $q = (q_i)_{o \leq i \leq m}$, such that $d_{A \cup \{b\}}(\iota p, \tau p) > K$. Replacing each occurrence of $x \to xb$ in p and q by a path from x to xb labelled by $w \in A^+$ we obtain a pair of walks $p' = (p'_i)_{0 \leq i \leq n'}$ and $q' = (q'_i)_{0 \leq i \leq m'}$ in $\Gamma_r(A, S)$ from ιp to τp. Now $d_A(\iota p, \tau p) \geq d_{A \cup \{b\}}(\iota p, \tau p) > K$ and by Lemma 5.8, it follows that $q'_{m'-N-1} \in p'_{N+1} S^1$. But then since $|w| = N$ this implies that $q_{m-1} \in p_1 S^1$, as required. $\square$

As mentioned above, with just a little extra work one can adapt the above arguments to show that SPP is a quasi-isometry invariant of finitely generated semigroups, in the sense of [10].

Unions of semigroups and partial actions

Let S be a semigroup that can be decomposed as a disjoint union of two subsemigroups T and V. Then the right multiplicative action of S on itself induces a right action $(t, v) \mapsto t \cdot v$ of the semigroup V on the set $T \cup \{0\}$ (where 0 is a new symbol not in S) where

$$t \cdot v = \begin{cases} tv & \text{if } t \in T \text{ and } tv \in T \\ 0 & \text{otherwise.} \end{cases}$$

By the *orbit* of $t \in T$ under this action we simply mean the set $t \cdot V^1$. Dually V acts on $T \cup \{0\}$ on the left in a natural way.

Corollary 5.9. *Let $S = T \cup V$, a disjoint union, where T and V are subsemigroups of S. If S is finitely generated (resp. presented), and the natural right and left multiplicative actions of V on $T \cup \{0\}$ have finite orbits, then T is finitely generated (resp. presented).*

Proof. Since all the orbits are finite and V is a subsemigroup of S it follows that properties (i) and (ii) of Proposition 2.3 both hold. But this implies, taking $X = A$, that T has finite boundary in S and is therefore finitely generated (resp. presented) by Theorem A (resp. Theorem B). $\square$

In general if $S = T \cup V$, a disjoint union of two subsemigroups, is finitely presented it need not be the case that either T or V is finitely presented; see the comment that immediately follows Corollary 7.2 below. We shall return to the subject of unions of semigroups again below in Section 7.

Corollary 5.9 applies, for example, in the following situation. Let S be a finitely generated semigroup and suppose that there is a homomorphism $\theta : S \to$

$\mathbb{Z}$ from S into the infinite cyclic group $\mathbb{Z}$, such that $S^{\leq} = \theta^{-1}(\mathbb{Z}^{\leq}) \neq \varnothing$ and $S^{>} = \theta^{-1}(\mathbb{Z}^{>}) \neq \varnothing$ where $\mathbb{Z}^{\leq} = \{0, -1, -2, \ldots\}$ and $\mathbb{Z}^{>} = \{1, 2, \ldots\}$. Then $S = S^{\leq} \cup S^{>}$ a disjoint union of subsemigroups. If the homomorphism θ has finite fibres (that is, $\theta^{-1}(z)$ is finite for every $z \in \mathbb{Z}$) then the hypotheses of Corollary 5.9 will be satisfied. Thus if S is finitely presented then so are both $S^{\leq}$ and $S^{>}$. In fact, combining with Theorem 7.1 below, we have that S is finitely presented if and only if $S^{\leq}$ and $S^{>}$ both are.

Virtual ideals

The symmetric difference between two sets can be considered a measure of how 'far apart' they are. Let us say that a subset X (not necessarily a subsemigroup) of a semigroup S is a *virtual ideal* if there exists an ideal I of S such that the symmetric difference $X \Delta I$ is finite.

Corollary 5.10. *Let S be a semigroup and let T be a subsemigroup of S such that $S \setminus T$ is a virtual ideal. If S is finitely generated (resp. presented) then T is finitely generated (resp. presented).*

Proof. Let $U = S \setminus T$. Let I be an ideal of S such that $U \Delta I$ is finite. So $I = Z \cup X$, a disjoint union, where X is a finite subset of T, and $Z = U \setminus Y$ for some finite subset Y of U. With this notation, for any finite subset F of S, since $ZF \cap T \subseteq I \cap T = X$, we have

$$U^1 F \cap T = (Z \cup Y)^1 F \cap T \subseteq (X \cup YF \cup F) \cap T$$

which is finite since all of X, Y and F are. Dually we see that $FU^1 \cap T$ is finite. It follows, taking $F = A$, that T has finite boundary in S, and the corollary follows by applying Theorems A and B. $\square$

Subsemigroups of subsemigroups

In this subsection we shall give an example of a chain of semigroups $S \geq T \geq V$ where T has finite boundary in S, and V has finite boundary in T, but V does *not* have finite boundary in S. Thus finite presentability is inherited by V from S by applying Theorem B twice, going via T. This shows that Theorems A and B may sometimes even be applied in situations when the subsemigroup under consideration does not have a finite boundary in its containing semigroup.

Example 5.11. Let S be the semigroup with underlying set

$$S = [\mathbb{N}^0 \times \mathbb{N}^0 \times \mathbb{N}^0 \setminus \{(0,0,0)\}] \cup \{0\}$$

and multiplication that we describe below.

Define $F : S \setminus \{0\} \to \{1, 2, 3\}$ where, for $\alpha \in S \setminus \{0\}$, $F(\alpha)$ is the position of the first non-zero entry of the triple α (e.g. $F(0,1,1) = 2$). Also define

$L : S \setminus \{0\} \to \{1,2,3\}$ where $L(\alpha)$ is the position of the last non-zero entry of α (e.g. $L(0,1,1) = 3$). Now multiplication in S is given by:

$$(x_1,y_1,z_1)(x_2,y_2,z_2) = \begin{cases} (x_1+x_2, y_1+y_2, z_1+z_2) & \\ \qquad\qquad\qquad \text{if } L(x_1,y_1,z_1) \le F(x_2,y_2,z_2) & \\ 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise} & \end{cases}$$

and

$$(x,y,z)0 = 0(x,y,z) = 0 \cdot 0 = 0.$$

So for example $(1,2,3)(4,5,6) = 0$ since $L(1,2,3) = 3 > 1 = F(4,5,6)$. On the other hand $(1,2,0)(0,5,6) = (1,7,6)$ since $L(1,2,0) = 2 \le 2 = F(0,5,6)$. It is routine to check that the multiplication is associative.

The semigroup S is finitely generated by the set $A = \{(1,0,0),(0,1,0),(0,0,1)\}$ since for $(x,y,z) \in S \setminus \{0\}$ we can write

$$(x,y,z) = (1,0,0)^x(0,1,0)^y(0,0,1)^z$$

and we generate 0 with $(0,0,1)(1,0,0) = 0$. In fact, S is defined by the presentation

$$\langle a,b,c,0 \mid ba = 0, cb = 0, ca = 0, 0^2 = a0 = 0a = b0 = 0b = c0 = 0c = 0\rangle$$

where a, b and c correspond to the generators $(1,0,0),(0,1,0)$ and $(0,0,1)$, respectively. Let $B = \{(1,0,0),(0,1,0),(0,1,1)\}$, $C = \{(1,0,0),(1,1,1)\}$, $T = \langle B\rangle$ and $K = \langle C\rangle$. Clearly $K \le T \le S$. We begin with a straightforward observation concerning the elements of these subsemigroups:

$$T = \{(x,y,0) : x \ge 0, y \ge 0,\ x \text{ and } y \text{ not both zero}\}$$
$$\cup\{(x,y,1) : x \ge 0, y \ge 1\} \cup \{0\}$$

and

$$K = \{(x,0,0) : x \ge 1\} \cup \{(x,1,1) : x \ge 1\} \cup \{0\}.$$

Claim 1. *The boundary of K in S is infinite.*

Proof. We will show that $\mathcal{B}(A,K)$ in S is infinite. For all $x \ge 1$ we have $(x,1,0) \notin K$. However $(x,1,0)(0,0,1) = (x,1,1) \in K$ where $(0,0,1) \in A$. Thus $\{(x,1,1) : x \ge 1\}$ is an infinite subset of the boundary of K in S. □

Claim 2. *The boundary of K in T is finite.*

Proof. We will show that $\mathcal{B}(B,K)$ in T is finite. Let $Q_1 = \{(x,y,0) : y \ge 1\}$, $Q_2 = \{(x,y,1) : y \ge 2\}$, and note that $T \setminus K = Q_1 \cup Q_2 \cup \{(0,1,1)\}$. Next note that every $(x,y,z) \in K$ has $y \le 1$ so that the intersection of each of the sets

$$Q_2B, \quad BQ_2, \quad \{(0,1,0),(0,1,1)\}Q_1, \quad Q_1\{(0,1,0),(0,1,1)\},$$

$$\{(0,1,1)\}\{(0,1,0),(0,1,1)\}, \quad \{(0,1,0),(0,1,1)\}\{(0,1,0)\}$$

with K is either empty or equal to $\{0\}$. Also

$$Q_1\{(1,0,0)\} = \{0\} = \{(0,1,1)\}\{(1,0,0)\}$$

by the definition of multiplication. Hence the right boundary is equal to $\{(1,0,0),0\}$.

For the left boundary we have

$$\begin{aligned} B(T \setminus K)^1 \cap K &= (B \cup (1,0,0)Q_1 \cup \{(1,0,0),(0,1,0)\}\{(0,1,1)\} \cup \{0\}) \cap K \\ &= (B \cup \{(x,y,0) : x, y \geq 1\} \cup \{(1,1,1),(0,2,1)\} \cup \{0\}) \cap K \\ &= \{(1,0,0),(1,1,1),0\}. \end{aligned}$$

We conclude that the boundary of K in T is equal to $\{(1,0,0),(1,1,1),0\}$. $\square$

Claim 3. *The semigroup T has a finite boundary in S.*

Proof. We will show that $\mathcal{B}(A,T)$ in S is finite. Let $P_1 = \{(x,0,1) : x \geq 0\}$, $P_2 = \{(x,y,z) : z \geq 2\}$ noting that $S \setminus T = P_1 \cup P_2$. Note that every $(x,y,z) \in T$ has $z \leq 1$ and so the intersections of each of the sets

$$(0,0,1)P_1, \quad P_1(0,0,1), \quad P_2A, \quad AP_2$$

with T is either empty or is equal to $\{0\}$. In addition $P_1(1,0,0) = P_1(0,1,0) = \{0\}$ and so the right boundary is equal to $\{0,(1,0,0),(0,1,0)\}$. For the left boundary we have

$$\begin{aligned} A(S \setminus T)^1 \cap T &= (A \cup (1,0,0)P_1 \cup (0,1,0)P_1) \cap T \\ &= (A \cup \{(x,0,1) : x \geq 1\} \cup \{(0,1,1),0\}) \cap T \\ &= \{(0,1,1),(1,0,0),(0,1,0),0\}. \end{aligned}$$

We conclude that the boundary of T in S is equal to $\{(1,0,0),(0,1,0),(0,1,1),0\}$. $\square$

6 One sided boundaries

In §3 we saw that subsemigroups of finitely presented semigroups with only the right (or left) boundary finite need not be finitely generated, never mind finitely presented. This still leaves us with the question of whether finitely generated subsemigroups of finitely presented semigroups, with only a finite right (or left) boundary are always finitely presented. We now answer this question in the negative.

Let M be a monoid and let θ be an endomorphism of M. The Bruck–Reilly extension of M with respect to θ is the semigroup of triples $\mathbb{N}^0 \times M \times \mathbb{N}^0$ with multiplication defined by:

$$(m,a,n)(p,b,q) = (m-n+t, (a\theta^{t-n})(b\theta^{t-p}), q-p+t)$$

where $t = \max(n,p)$. Bruck–Reilly extensions are an important class of infinite simple semigroups. (For more details on Bruck–Reilly extensions see [15].)

Proposition 6.1. *Suppose that the Bruck–Reilly extension $S = BR(M,\theta)$ of a monoid M is finitely generated and consider the subsemigroup $T = \{(0,a,n) : a \in M, n \in \mathbb{N}^0\}$. Then the right boundary of T in S is finite, while the left boundary is infinite.*

Proof. Let $U = \mathrm{BR}(M,\theta) \setminus T$. Since

$$m - n + t = m - n + \max(n,p) \geq m - n + n = m$$

it follows that U is a right ideal in $\mathrm{BR}(M,\theta)$ and thus, by Proposition 2.2, the right boundary of T in $\mathrm{BR}(M,\theta)$ is finite.

Let X be a finite generating set for $\mathrm{BR}(M,\theta)$. Since, by Proposition 2.1, the finiteness or otherwise of the left boundary is independent of the choice of generating set, we may assume without loss of generality that $(0, 1_M, 0) \in X$. For $n \in \mathbb{N}$ we have:

$$(0, 1_M, 1)(1, s, n) = (0, s, n);$$

note that here $(1,s,n) \in U$ and $(0,s,n) \in T$. Therefore, the left boundary of T in S is infinite (and equal to the whole of T). $\square$

Example 6.2. Let M be a non-finitely presented monoid which has a finitely presented Bruck–Reilly extension $S = \mathrm{BR}(M,\theta)$. One possible choice for M is the group defined by the presentation:

$$\langle a, b, c, d \mid a^{2^i} b^{2^i} = c^{2^i} d^{2^i} \ (i \in \mathbb{N}^0) \rangle$$

where $\theta : M \to M$ extends the map $x\theta = x^2$ for $x \in \{a,b,c,d\}$. This example is taken from [23, Proposition 3.3] where it was shown that M is finitely generated but not finitely presented and $\mathrm{BR}(M,\theta)$ is finitely presented.

Let T be as in the proposition and let $N = \{(0,a,0) : a \in M\}$. Clearly $N \cong M$, $N \subseteq T$ and $T \setminus N$ is an ideal of T. Hence, by Corollary 5.10, T is not finitely presented, although, it is finitely generated: any finite generating set for N together with the element $(0, 1_M, 1)$ is a generating set for T. It now follows from Proposition 6.1 that T has a finite right boundary.

7 The converse: unions of semigroups

When defining the boundary $\mathcal{B}(A,T)$ it is essential to assume that S is finitely generated. Therefore the converse of Theorem A is not a sensible thing to consider. The converse of Theorem B may be formulated as follows. Let S be a semigroup generated by a finite set A and let T be a subsemigroup of S. If T is finitely presented and $\mathcal{B}(A,T)$ is finite then is S necessarily finitely presented? It is not hard to see that the answer to this question is no in general. For example, if S is any non-finitely presented semigroup that has a finite subsemigroup T then T is finitely presented and has a finite boundary in S.

One interesting situation where the converse does hold is when the complement of T happens to be a subsemigroup of S, i.e. when S is a disjoint union of two subsemigroups. In general we can prove the following result when S is a disjoint union of finitely many subsemigroups.

Theorem 7.1. *Let $S = \bigcup_{i \in I} S_i$, a disjoint union, where I is finite and each S_i is a subsemigroup of S. If each S_i is finitely presented and has a finite right boundary in S then S itself is finitely presented.*

Proof. For each $i \in I$ let S_i be defined by the presentation $\langle A_i | R_i \rangle$. We will write a presentation for S of the form $\langle B | \bigcup_{i \in I} R_i, R \rangle$ where $B = \bigcup_{i \in I} A_i$ and R is a finite set of relations holding in S that we describe below.

Let $i, j \in I$ with $i \neq j$. Consider the set of words $w \in A_i^+$ such that there exists some $a \in B$ with $wa \in \mathcal{L}(B, S_j)$. Denote this set of words by $W_r(i,j) \subseteq A_i^+$. Note that the elements that the words $W_r(i,j)$ represent may constitute an infinite subset of S_i. Let $w \in W_r(i,j)$ and $a \in B$ with $wa \in \mathcal{L}(B, S_j)$. Amongst all the words $w_1 \in A_i^+$ with the property that $wa = w_1 a$ in S let $\pi_r(w,a)$ be such a word of shortest length. So we have $wa = \pi_r(w,a)a$ in S. Now define:

$$d_r(i,j) = \max\{|\pi_r(w,a)| : w \in A_i^+, a \in B, wa \in \mathcal{L}(B, S_j)\}$$

provided $W_r(i,j)$ is non-empty; when $W_r(i,j)$ is empty we define $d_r(i,j) = 0$. The number $d_r(i,j)$ is well defined since B is finite and the boundary of S_j in S is finite. Now define

$$f = \max\{d_r(i,j) : i, j \in I, i \neq j\}$$

which is well defined since I is finite. For every word $w \in B^+$ let $\widetilde{w} \in A_1^+ \cup \ldots \cup A_{|I|}^+$ be a fixed word such that $w = \widetilde{w}$ holds in S. Now let:

$$R = \{(w = \widetilde{w}) : w \in B^+, |w| \leq f + 2\}.$$

Note that the relations $wa = \pi_r(w,a)a$ with $|w| \leq f+1$ are consequences of $\bigcup_{i \in I} R_i \cup R$.

Claim 1. *For every $w \in A_i^+$ and every $a \in B$ there exists some $u \in A_1^+ \cup \ldots \cup A_{|I|}^+$ such that $wa = u$ is a consequence of the relations R.*

Proof. We prove the claim by induction on the length of the word w. If $|w| \leq f+1$ then the relation $wa = \widetilde{wa}$ belongs to R and we are done. Now let $w \in A_i^+$ with $|w| > f+1$ and suppose that the result holds for all $v \in A_i^+$ such that $|v| < |w|$. Write $wa \equiv w'w''a$ where $|w''| = f+1$. There are two cases to consider:

Case 1: $w''a \in S_i$. Then the relation $w''a = \widetilde{w''a}$ belongs to R. Now $w' \in A_i^+$ and $\widetilde{w''a} \in A_i^+$ and so we can deduce

$$wa = w'(w''a) = w'(\widetilde{w''a}) \in A_i^+$$

as required.

Case 2: $w''a \in S_j$ where $j \neq i$. Then the relation $w''a = \pi_r(w'', a)a$ is a consequence of $\bigcup_{i\in I} R_i \cup R$ where $|\pi_r(w'', a)| = f < f + 1 = |w''|$ and so $|w'\pi_r(w'', a)| < |w'w''|$. Therefore we may apply induction to deduce:

$$wa = w'w''a = w'\pi_r(w'', a)a = u \in A_1^+ \cup \ldots \cup A_{|I|}^+$$

as required. $\square$

Claim 2. *For every $w \in B^+$ there exists $u \in A_1^+ \cup \ldots \cup A_{|I|}^+$ such that $w = u$ is a consequence of the relations R.*

Proof. We prove the claim by induction on the length of the word w. When $|w| \leq f + 2$ the relation $w = \widetilde{w}$ belongs to R and we are done. Now let $w \in B^+$ with $|w| > f + 2$ and suppose that the result holds for all $v \in B^+$ with $|v| < |w|$. Write $w \equiv w'a$ where $a \in B$ is the last letter of w. By induction we can deduce $w' = u$ where $u \in A_1^+ \cup \ldots \cup A_{|I|}^+$. By Claim 1 we can deduce $ua = v$ where $v \in A_1^+ \cup \ldots \cup A_{|I|}^+$. It follows that we can deduce:

$$w \equiv w'a = ua = v \in A_1^+ \cup \ldots \cup A_{|I|}^+$$

as required. $\square$

Let $w, v \in B^+$ such that $w = v$ holds in S and $w, v \in \mathcal{L}(B, S_i)$, say. Then there exist $w', v' \in A_i^+$ such that the relations $w = w'$ and $v = v'$ are consequences of R. Furthermore, the relation $w' = v'$ is a consequence of the relations R_i. Therefore, using the relations $\bigcup_{i\in I} R_i \cup R$ we may deduce $w = w' = v' = v$ and, since w and v were arbitrary, S is defined by the presentation $\langle B | \bigcup_{i\in I} R_i, R\rangle$. $\square$

There is an obvious dual result where the left boundaries are all finite. Now if we combine Theorem 7.1 with Theorem B we obtain:

Corollary 7.2. *Let S be a finitely generated semigroup which can be decomposed into a finite disjoint union $S = \bigcup_{i\in I} S_i$ of subsemigroups with finite boundaries. Then S is finitely presented if and only if all the S_i are finitely presented.*

Note that the converse of Theorem 7.1 does not hold in general. For example let $A = \{a, b\}$, $T_1 = \{aw : w \in A^*\}$ and $T_2 = \{bw : w \in A^*\}$. Then $S = A^+$ is the disjoint union of T_1 and T_2, both T_1 and T_2 have finite right boundaries in S but neither of them is finitely presented (since they are not even finitely generated).

Without the restriction that the boundaries are finite Corollary 7.2 no longer holds. For example, in [1, Example 3.4] an example is given of a non-finitely presented semigroup S that is a disjoint union of two finitely presented semigroups.

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