

SYMMETRIC SUBGROUPS IN MODULAR GROUP ALGEBRAS

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ABSTRACT. Let $V(KG)$ be a normalised unit group of the modular group algebra of a finite p -group G over the field K of p elements. We introduce a notion of symmetric subgroups in $V(KG)$ as subgroups invariant under the action of the classical involution of the group algebra KG . We study properties of symmetric subgroups and construct a counterexample to the conjecture by V. Bovdi, which states that $V(KG) = \langle G, S_* \rangle$, where S_* is a set of symmetric units of $V(KG)$.

1. INTRODUCTION

In this preprint¹ we investigate the structure of the unit group $U(KG)$ of the modular group algebra KG , where G is a finite p -group and K is a field of p elements. $U(KG)$ is a direct product of the unit group of the field K and the normalized unit group $V(KG)$ that consists of all elements of the form $1 + x$, where x belongs to $I(G)$, the latter being the augmentation ideal of the modular group algebra KG . Thus, the task of the investigation of $U(KG)$ is reduced to the investigation of the p -group $V(KG)$.

Study of units and their properties is one of the main research problems in group ring theory. Results obtained in this direction are also useful for the investigation of Lie properties of group rings, isomorphism problem and other open questions in this area (see, for example, [2]).

Let us consider the mapping φ from KG into itself which maps an element $x = \sum \lambda_g g$ to $\sum \lambda_g g^{-1}$. This mapping is an antiautomorphism of order two, and it is called the *classical involution* of the group algebra KG . In what follows, we will denote $\varphi(x)$ by x^* . An element $x \in KG$ will be called *symmetric*, if $x = x^*$. Note that if $x^* \neq x^{-1}$, then x^*x — nontrivial symmetric element. Indeed, $\varphi(x^*x) = (x^*x)^* = x^*x^{**} = x^*x$ since $x^{**} = x$. Classical involution and symmetric elements were studied by V. Bovdi, M. Dokuchaev, L. Kovacs, S. Sehgal in [3, 4, 5, 6, 8].

In [9] the generating system for $V(KG)$ was given, however that system is minimal only in the abelian case. In the general case, the problem of the determination of the minimal generating system of $V(KG)$ remains open. For groups of small order the minimal generating system can be determined with the help of computer after the construction of $V(KG)$ using the algorithm from [9] implemented in the LAGUNA package [10] for the computational algebra system GAP [11]. Obtaining new results about the minimal generating system would be useful for the improvement of algorithms for the computation of $V(KG)$.

1991 *Mathematics Subject Classification*. Primary 16S34, 20C05.

Key words and phrases. modular group algebra, classical involution, normalized unit group, symmetric unit, symmetric subgroup, computational algebra system GAP.

¹This preprint is a slightly updated translation of the paper [7], earlier written by the same authors in Russian.

In 1996 V. Bovdi suggested the following conjecture:

Conjecture 1. *Let G be a finite nonabelian p -group, $V(KG)$ be the normalized unit group of the modular group algebra KG , and $S_* = \{x \in V(KG) \mid x^* = x\}$ be the set of symmetric units. Is it true that*

$$(1) \quad V(KG) = \langle G, S_* \rangle?$$

In the present paper we give a counterexample to this conjecture and investigate further properties of the classical involution. We introduce the notions of symmetric subsets of the group algebra and symmetric subgroups of $V(KG)$, and investigate their properties. Then we give some obvious examples of symmetric subgroups and formulate the question about the existence of non-trivial symmetric subgroup. After this, we consider the normalized unit group of the modular group algebra of the quaternion group of order 8, and check for it the conjecture (1). Using the LAGUNA package [10], we discovered that in this case the condition (1) does not hold, because $\langle G, S_* \rangle$ has the order 64. In the present paper we give a purely theoretical proof of this fact. Besides this, it appears that $\langle G, S_* \rangle$ is the symmetric subgroup, as well as the set S_* , which in this case is also a subgroup. Thus, we obtain two examples of non-trivial symmetric subgroups.

It would be interesting to continue studies of properties of symmetric subgroups of $V(KG)$ and conditions of the existence of non-trivial symmetric subgroups in $V(KG)$.

2. SYMMETRIC SUBGROUPS

Let H be a subset of the group algebra KG . Then H will be called *symmetric*, if $H^* = H$, where $H^* = \{h^* \mid h \in H\}$. Similarly, a subgroup $H \subseteq V(KG)$ will be called a *symmetric subgroup*, if $H^* = H$.

Clearly, symmetric subgroups exist. It is easy to see that $\{1\}$, G and $V(KG)$ are trivial examples of symmetric subgroups. This naturally raises a question of the existence of modular group algebras that possesses non-trivial symmetric subgroups. Moreover, in [3] conditions were obtained under which the set of symmetric elements forms a group, that in this case will be symmetric. Therefore, it would be interesting to find an example of non-trivial symmetric subgroup that contains non-symmetric units, thus the restriction of the classical involution on this subgroup will be not an identity mapping.

Before we construct such example, we will state and prove some properties of symmetric subgroup.

Lemma 1. *If H is a subgroup of $V(KG)$, then H^* also is a subgroup of $V(KG)$.*

Proof. We will check that H^* is a subgroup of $V(KG)$. Since $1 \in H$, we have $1 = \varphi(1) \in H^*$. Then, if $a^*, b^* \in H^*$, so does $a^*b^* \in H^*$, since $a^*b^* = \varphi(ba)$, and $ba \in H$. Finally, if $a^* \in H^*$, then $(a^*)^{-1} = (\varphi(a))^{-1} = \varphi(a^{-1}) \in H^*$, and the lemma is proved.

Lemma 2. *Let H' and H be subgroups of $V(KG)$, and H be a symmetric subgroup. If H' is a conjugate of H with the help of an element $g \in V(KG)$ such that $(g^*)^{-1} = (g^{-1})^*$, then H'^* is also a conjugate of H with the help of element $(g^*)^{-1}$.*

Proof. Let us assume that $g \in V(KG)$ is such that $H' = g^{-1}Hg$, and $H = H^*$. Then $H'^* = (g^{-1}hg)^* = g^*h(g^{-1})^* = g^*H(g^{-1})^*$. If $(g^*)^{-1} = (g^{-1})^*$, what holds,

in particular, for all $g \in G$, then $H'^* = H^{(g^{-1})^*}$, and the lemma is proved. Note that H' is not necessary to be symmetric, and also that $H'^* = (H^{g^{-1}})^*$.

Lemma 3. *Let H_1 and H_2 be symmetric subgroups. Then H_1H_2 also is a symmetric subgroup.*

Proof. Let H_1 and H_2 be symmetric subgroups. Consider an element from H_1H_2 having the form $h = a_1b_1a_2b_2 \dots a_kb_k$, where $a_i \in H_1$ and $b_i \in H_2$. Then $h^* = u_kv_k \dots u_2v_2u_1v_1$, where $u_i = b_i^*$, $v_i = a_i^*$, also belongs to H_1H_2 , because H_1 and H_2 are symmetric. Thus, the lemma is proved.

3. INVESTIGATION OF THE CONJECTURE ON GENERATORS OF $V(KG)$

To verify the conjecture by V. Bovdi about generators of the normalized unit group we used the LAGUNA package [10] for the computational algebra system GAP [11]. We developed a program in GAP language using functions available in the LAGUNA package, that performed the following steps:

- a) get the set of all elements of the group G ;
- b) compute the normalized unit group $V(KG)$ (generated by group algebra elements) with the help of the LAGUNA package, and the group W that is a power-commutator presentation of $V(KG)$;
- c) construct the embedding f from G to the group algebra KG ;
- d) construct the isomorphism ψ from $V(KG)$ to the group W ;
- e) compute the set S of symmetric units of the group algebra KG ;
- f) using the isomorphism ψ , find images of the set of symmetric units S and $f(G)$ in the group W ;
- g) compute the subgroup $\langle \psi(f(G)), \psi(S_*) \rangle \subseteq W$ and verify the condition (1).

As a result, we find out that the condition (1) does not hold for the quaternion group of order eight.

4. COMPUTATION OF THE SYMMETRIC SUBGROUP $\langle G, S_* \rangle$

In this section we will give a theoretical explanation why the quaternion group Q_8 provides a counterexample to the conjecture by V. Bovdi. The group Q_8 is given by the following presentation:

$$Q_8 = \langle a, b \mid a^4 = b^4 = 1, a^2 = b^2, b^{-1}ab = a^3 \rangle,$$

and consists of the following elements:

$$Q_8 = \{a^i b^j, 0 \leq i \leq 4, 0 \leq j \leq 2\}.$$

Every symmetric element $x \in S_*$ has the form

$$(2) \quad x = \alpha_0 + \alpha_1 a^2 + \gamma_x,$$

where

$$(3) \quad \gamma_x = \alpha_2(a + a^3) + \alpha_3(b + a^2b) + \alpha_4(ab + a^3b).$$

Since x is a unit from $V(KQ_8)$, we have that $\alpha_0 + \alpha_1 = 1$ (i.e. one and only one of coefficients α_0, α_1 is equal to one). From this follows that $|S_*| = 16$.

Lemma 4. *Let $S_* = \{s \in V(KQ_8) \mid s^* = s\}$ be the set of symmetric units. Then $S_* \subset Z(KQ_8)$, where $Z(KQ_8)$ is the center of the group algebra KQ_8 .*

Proof. The group Q_8 has the following decomposition on the conjugacy classes of elements:

$$Q_8 = \{1\} \cup \{a^2\} \cup \{a, a^3\} \cup \{b, a^2b\} \cup \{ab, a^3b\}.$$

Since class sums are central elements (see [1]), the lemma follows immediately from (2) and (3).

Lemma 5. *The set S_* is a subgroup of the normalized unit group $V(KQ_8)$.*

Proof. The lemma follows from the result from [3] stating that the set of symmetric units is a subgroup if and only if they all commute, in combination with the lemma 4.

It also follows from the lemma 5 that S_* is a non-trivial symmetric subgroup of $V(KQ_8)$. However, all its elements are symmetric, so we did not have an example of a symmetric subgroup which contain non-symmetric units yet.

Lemma 6. $\forall x \in S_* \quad x^2 = 1.$

Proof. By (2), we have that

$$x^2 = \alpha_0^2 + \alpha_1 a^4 + \gamma_x^2 = \alpha_0^2 + \alpha_1^2 = 1,$$

since the characteristic of the field K is two, elements $1, a^4$ and γ_x are in the center $Z(KQ_8)$, and $\gamma_x^2 = 0$.

Now we are ready to prove the main statement, that gives a counterexample to the conjecture by V. Bovdi.

Theorem 1. *Let Q_8 be the quaternion group of order eight, $V(KQ_8)$ be the normalized unit group of its modular group algebra over the field of two elements, and $S_* = \{x \in V(KQ_8) | x^* = x\}$ be the set of symmetric units. Then*

$$|H| = |\langle Q_8, S_* \rangle| = 64.$$

Proof. Since $S_* \subset Z(KQ_8)$, then each element from H can be written as $x = gs$, where $g \in Q_8$ and $s \in S_*$. Note that $Q_8 \cap S_* = \{1, a^2\}$ that coincides with $Z(Q_8)$. Thus, $\langle Q_8, S_* \rangle$ is a central product of groups Q_8 and S_* . From this follows that $|H| = \frac{|S_*| \cdot |Q_8|}{|Z(Q_8)|} = \frac{16 \cdot 8}{2} = 64$, and the theorem is proved.

Thus, we proved that the order of a subgroup generated by the group Q_8 and the set of symmetric elements S_* is equal to 64, so it does not coincide with the normalized unit group $V(KQ_8)$ of order 128. So, $H \neq V(KQ_8)$, and the quaternion group of order eight gives a counterexample to the conjecture by V. Bovdi. Besides this, $H = \langle Q_8, S_* \rangle$ is a non-trivial symmetric subgroup of the group $V(KQ_8)$, containing both symmetric and non-symmetric units.

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