PROCEEDINGS A

rspa.royalsocietypublishing.org

Research

Check for updates

Cite this article: Bagarello F, Gargano F, Spagnolo S, Triolo S. 2017 Coordinate representation for non-Hermitian position and momentum operators. *Proc. R. Soc. A* **473**: 20170434.
http://dx.doi.org/10.1098/rspa.2017.0434

Received: 20 June 2017
Accepted: 11 August 2017

Subject Areas:
applied mathematics, mathematical physics, quantum physics

Keywords:
position operators, generalized eigenvectors, quasi*-algebras

Author for correspondence:
F. Bagarello
e-mail: fabio.bagarello@unipa.it

Coordinate representation for non-Hermitian position and momentum operators

F. Bagarello[1,2,3], F. Gargano[1], S. Spagnolo[1] and S. Triolo[1]

[1]DEIM—Dipartimento di Energia, Ingegneria dell'Informazione e Modelli Matematici, Scuola Politecnica, Università di Palermo, 90128 Palermo, Italy
[2]INFN, Sezione di Napoli, Naples, Italy
[3]Department of Mathematics and Applied Mathematics, University of Cape Town, Cape Town, South Africa

FB, 0000-0002-4454-091X

In this paper, we undertake an analysis of the eigenstates of two non-self-adjoint operators $\hat{q}$ and $\hat{p}$ similar, in a suitable sense, to the self-adjoint position and momentum operators $\hat{q}_0$ and $\hat{p}_0$ usually adopted in ordinary quantum mechanics. In particular, we discuss conditions for these eigenstates to be *biorthogonal distributions*, and we discuss a few of their properties. We illustrate our results with two examples, one in which the similarity map between the self-adjoint and the non-self-adjoint is bounded, with bounded inverse, and the other in which this is not true. We also briefly propose an alternative strategy to deal with $\hat{q}$ and $\hat{p}$, based on the so-called *quasi *-algebras*.

1. Introduction

Quantum mechanics driven by non-self-adjoint Hamiltonians with real eigenvalues has been investigated by several authors for some years, both from a physical point of view [1–4] and with a more mathematical perspective (see [5–15] and the recent volume [16]). Quite often interest is focused on the analysis of the eigenvalues and the eigenstates of some Hamiltonian operator H which, contrary to what is usually assumed in (ordinary) quantum mechanics, is different from $H^\dagger$. In many cases, however, the eigenvalues of such an H are still real,

THE ROYAL SOCIETY PUBLISHING

at least for a certain range of parameters of the model, the *unbroken phase*, which differs from the *broken phase* since, in this case, some of the eigenvalues of H can be complex [3,17].

What has not been discussed in the literature so far, to the best of our knowledge, is the role of non-self-adjointness in the main standard ingredients of H, i.e. in the position and in the momentum operators. In fact, in many physical systems the self-adjoint Hamiltonian H_0 is just some suitable function of the operators $\hat{q}_0 = \hat{q}_0^\dagger$ and $\hat{p}_0 = \hat{p}_0^\dagger$, satisfying $[\hat{q}_0, \hat{p}_0] = \mathrm{i}\mathbb{1}$, where $\mathbb{1}$ is the identity operator. For instance, for the harmonic oscillator, we have $H_0 = \frac{1}{2}(\hat{p}_0^2 + \hat{q}_0^2)$. Therefore, it is surely interesting, and natural, to also look at the functional properties of both $\hat{q}_0$ and $\hat{p}_0$. And, in fact, these have been studied over the years by several authors. In particular, we refer to [18] for a rather interesting and clear review on this topic. Recently, Bagarello [19] began the analysis of some aspects of two operators, $\hat{q}$ and $\hat{p}$, which behave as a deformed version of $\hat{q}_0$ and $\hat{p}_0$, while retaining some of their essential aspects. This was done in connection with bicoherent states, in order to prove their completeness under suitable conditions. Here, rather than bicoherent states, we investigate the possibility of defining, in some mathematically rigorous manner, the eigenstates of $\hat{q}$ and $\hat{p}$, and we discuss some of their properties. We believe that this analysis can be relevant in the foregoing discussion on the coexistence of two (or more) non-self-adjoint observables; see [20,21] for instance.

The paper is organized as follows. In §2, we shall analyse in a rigorous way the family of eigenstates of the deformed position and momentum operators $\hat{q}$ and $\hat{p}$ by adopting a distributional approach which essentially extends the one in [18]. We shall also briefly propose an alternative approach based on the so-called *quasi *-algebras* [22,23]. Section 3 contains two different examples. In the first one, the operators $\hat{q}_0$ and $\hat{p}_0$ are related to $\hat{q}$ and $\hat{p}$ by a bounded operator T with bounded inverse. In the second example T is still bounded, but T^{-1} is not. Of course, there are other possibilities, and some of these can be found in the literature [24]. For this reason, we believe that having a general setting where all these possibilities fit can be important and useful. Our conclusions are given in §4.

2. Well-behaved sets of distributions

Let $\hat{q}_0$ and $\hat{p}_0$ be the self-adjoint position and momentum operators defined as follows:

$$\hat{q}_0\varphi(x) = x\varphi(x) \quad \text{and} \quad \hat{p}_0\varphi(x) = -\mathrm{i}\varphi'(x),$$

for all $\varphi(x) \in \mathcal{S}(\mathbb{R})$, the Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}$. Note that $\mathcal{S}(\mathbb{R})$ is not the maximal domain of these operators. In fact, these sets are

$$D_{\max}(\hat{q}_0) = \{f(x) \in \mathcal{L}^2(\mathbb{R}) : xf(x) \in \mathcal{L}^2(\mathbb{R})\} \quad \text{and} \quad D_{\max}(\hat{p}_0) = \{f(x) \in \mathcal{L}^2(\mathbb{R}) : f'(x) \in \mathcal{L}^2(\mathbb{R})\}.$$

Of course, $\mathcal{S}(\mathbb{R}) \subset D_{\max}(\hat{q}_0) \cap D_{\max}(\hat{p}_0)$, and is dense in $\mathcal{L}^2(\mathbb{R})$. In the rest of the paper, also in view of its role in quantum mechanics, we will often use $\mathcal{S}(\mathbb{R})$ as a suitable domain for the operators we introduce in our analysis, even when it lacks maximality. This is useful also because $\mathcal{S}(\mathbb{R})$ is stable under the action of $\hat{q}_0$ and $\hat{p}_0$, while $D_{\max}(\hat{q}_0)$ and $D_{\max}(\hat{p}_0)$ are not. In particular, $\mathcal{S}(\mathbb{R}) \subseteq D^\infty(\hat{q}_0) \cap D^\infty(\hat{p}_0)$, where $D^\infty(X) = \cap_{k\geq 0} D(X^k)$ is the domain of all the powers of the operator X. It is well known that neither $\hat{q}_0$ nor $\hat{p}_0$ admit square integrable eigenvectors, and that their spectra coincide with $\mathbb{R}$,

$$\hat{q}_0\xi_{x_0}(x) = x_0\xi_{x_0}(x) \quad \text{and} \quad \hat{p}_0\theta_{p_0}(x) = p_0\theta_{p_0}(x), \tag{2.1}$$

where x_0 and p_0 are real numbers, and

$$\xi_{x_0}(x) = \delta(x - x_0) \quad \text{and} \quad \theta_{p_0}(x) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{\mathrm{i}p_0x}. \tag{2.2}$$

Of course, $\xi_{x_0}(x), \theta_{p_0}(x) \in \mathcal{S}'(\mathbb{R})$, the set of tempered distributions (i.e. the continuous linear functionals on $\mathcal{S}(\mathbb{R})$). Two well-known properties of the generalized eigenvectors $\{\xi_{x_0}(x), x_0 \in \mathbb{R}\}$

are the following equalities:

$$\langle \xi_{x_0}, \xi_{y_0} \rangle = \delta(x_0 - y_0), \quad \int_{\mathbb{R}} \mathrm{d}x_0 |\xi_{x_0}\rangle\langle \xi_{x_0}| = \mathbb{1}. \tag{2.3}$$

Here $\langle \xi_{x_0}, \xi_{y_0} \rangle$ can be seen as an extension of the scalar product in $\mathcal{L}^2(\mathbb{R})$ to two elements in $\mathcal{S}'(\mathbb{R})$, which results in another element in $\mathcal{S}'(\mathbb{R})$. The way in which this extension is constructed is as follows: given two distributions F and G in $\mathcal{S}'(\mathbb{R})$, we define $\langle F, G\rangle = (\bar{F} * \tilde{G})(0)$,[1] where $\tilde{G}(x) = G(-x)$ [25]. In other words, we use the convolution between distributions to extend the scalar product to $\mathcal{S}'(\mathbb{R})$. The convolution $\bar{F} * \tilde{G}$ is further defined as follows:

$$(\bar{F} * \tilde{G}, \varphi) = \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{F(x)}\, \tilde{G}(y) \varphi(x+y)\, \mathrm{d}x\, \mathrm{d}y = \langle F, G * \varphi\rangle, \tag{2.4}$$

$\forall \varphi(x) \in \mathcal{S}(\mathbb{R})$. It is well known [25] that the convolution between two distributions does not always exist. However, sufficient conditions on the support of the distributions are considered in the literature which ensure the existence of $\bar{F} * \tilde{G}$, and therefore of $\langle F, G\rangle$, and these conditions are satisfied in our case: if we take $F = \xi_{x_0}$ and $G = \xi_{y_0}$, simple computations show that

$$(\bar{\xi}_{x_0} * \tilde{\xi}_{y_0}, \varphi) = \langle \xi_{x_0}, \xi_{y_0} * \varphi \rangle = \int_{\mathbb{R}} \xi_{x_0}(x) \varphi(x - y_0)\, \mathrm{d}x = \varphi(x_0 - y_0) = (\xi_{t_0}, \varphi),$$

where $t_0 = x_0 - y_0$, for all $\varphi(x) \in \mathcal{S}(\mathbb{R})$. Hence $(\bar{\xi}_{x_0} * \tilde{\xi}_{y_0})(x) = \xi_{t_0}(x)$, and therefore

$$\langle \xi_{x_0}, \xi_{y_0} \rangle = (\bar{\xi}_{x_0} * \tilde{\xi}_{y_0})(0) = \xi_{t_0}(0) = \delta(x_0 - y_0),$$

which is the first equality in (2.3). The second equality follows from the fact that, for all $\varphi(x) \in \mathcal{S}(\mathbb{R})$, $\varphi(x_0) = \langle \xi_{x_0}, \varphi\rangle$. Then we have

$$\varphi(x) = \int_{\mathbb{R}} \delta(x - x_0) \varphi(x_0)\, \mathrm{d}x_0 = \int_{\mathbb{R}} \xi_{x_0}(x) \langle \xi_{x_0}, \varphi\rangle\, \mathrm{d}x_0,$$

as we had to prove. Incidentally, we see that the resolution of the identity in (2.3) has to be intended (at least) on $\mathcal{S}(\mathbb{R})$.

Let us now consider an operator T, not necessarily bounded, with domain $D(T)$ larger than (or equal to) $\mathcal{S}(\mathbb{R})$.

Definition 2.1. T is $\mathcal{S}(\mathbb{R})$-stable if T is invertible and if T, T^{-1}, $T^\dagger$ and $(T^{-1})^\dagger = (T^\dagger)^{-1}$ all map $\mathcal{S}(\mathbb{R})$ in $\mathcal{S}(\mathbb{R})$. Moreover, an $\mathcal{S}(\mathbb{R})$-stable operator T is called fully $\mathcal{S}(\mathbb{R})$-stable if $T^\dagger$ and T^{-1} map $\mathcal{S}(\mathbb{R})$ into itself continuously.

Examples of these operators will be given later in the paper: essentially, a fully $\mathcal{S}(\mathbb{R})$-stable operator T leaves $\mathcal{S}(\mathbb{R})$ stable, together with its adjoint, its inverse and the inverse of its adjoint, and is such that, if $\{\varphi_n(x)\}$ is a sequence in $\mathcal{S}(\mathbb{R})$ which converges to $\varphi(x)$ in the $\tau_{\mathcal{S}}$ topology, both $\{T^{-1}\varphi_n(x)\}$ and $\{T^\dagger\varphi_n(x)\}$ converge in the same topology. Each $\mathcal{S}(\mathbb{R})$-stable operator can be used to construct a non-self-adjoint version of $\hat{q}_0$ and $\hat{p}_0$, as we will show now.

Let T be an $\mathcal{S}(\mathbb{R})$-stable operator. Then, two operators $\hat{q}$ and $\hat{p}$ can be defined as follows:

$$\hat{q}\varphi = T\hat{q}_0 T^{-1}\varphi \quad \text{and} \quad \hat{p}\varphi = T\hat{p}_0 T^{-1}\varphi, \tag{2.5}$$

for all $\varphi(x) \in \mathcal{S}(\mathbb{R})$. Of course, $\hat{q}$ and $\hat{p}$ map $\mathcal{S}(\mathbb{R})$ in $\mathcal{S}(\mathbb{R})$, so that they are, in particular, densely defined. As for $\hat{q}_0$ and $\hat{p}_0$, we are not interested here in $D_{\max}(\hat{q})$ and $D_{\max}(\hat{p})$, even if these sets could be larger than $\mathcal{S}(\mathbb{R})$. It is also possible to check that their adjoints satisfy the following:

$$\hat{q}^\dagger\varphi = (T^{-1})^\dagger \hat{q}_0 T^\dagger\varphi \quad \text{and} \quad \hat{p}^\dagger\varphi = (T^{-1})^\dagger \hat{p}_0 T^\dagger\varphi, \tag{2.6}$$

for all $\varphi(x) \in \mathcal{S}(\mathbb{R})$. Hence, also $\hat{q}^\dagger, \hat{p}^\dagger$ leave $\mathcal{S}(\mathbb{R})$ stable.

Remark 2.2. A *physical way* to understand these results is by noticing that $\hat{q}_0$, $\hat{q}$ and $\hat{q}^\dagger$ are similar or, in other words, that they satisfy suitable intertwining relations. When this happens for

[1]This is because of the following equality: $\langle \varphi, \psi\rangle = (\bar{\varphi} * \tilde{\psi})(0)$, which holds for each $\varphi(x), \psi(x) \in \mathcal{S}(\mathbb{R})$. Here $\tilde{\psi}(x) = \psi(-x)$.

(bounded) operators with discrete spectra, this implies that their eigenvectors are related by the intertwining operators (T, in our case). Of course, the operators $\hat{p}_0$, $\hat{p}$ and $\hat{p}^\dagger$ are also similar.

Now, we want to construct the eigenvectors of $\hat{q}$ and $\hat{q}^\dagger$, as well as those of $\hat{p}$ and $\hat{p}^\dagger$, and to check that they produce a family of *well-behaved* states, in the sense of Bagarello [19]. We recall that, if $x \in \mathbb{R}$ labels a tempered distribution $\eta_x \in \mathcal{S}'(\mathbb{R})$, and if $\mathcal{F}_\eta$ is the set of all these distributions, $\mathcal{F}_\eta = \{\eta_x,\ x \in \mathbb{R}\}$, then we have the following definition.

Definition 2.3. $\mathcal{F}_\eta$ is called $\hat{q}$-*well-behaved* (or, simply, *well-behaved*) if:

1. each η_x is a generalized eigenstate of $\hat{q}$: $\hat{q}\eta_x = x\eta_x$, for all $x \in \mathbb{R}$;
2. a second family of generalized vectors $\mathcal{F}^\eta = \{\eta^x \in \mathcal{S}'(\mathbb{R}), x \in \mathbb{R}\}$ exists such that $\langle \eta_x, \eta^y\rangle = \delta(x-y)$ and $\int_{\mathbb{R}} \mathrm{d}x |\eta_x\rangle\langle\eta^x| = \int_{\mathbb{R}} \mathrm{d}x |\eta^x\rangle\langle\eta_x| = \mathbb{1}$, at least on $\mathcal{S}(\mathbb{R})$.

To do that, we first need to extend T and $(T^{-1})^\dagger$ to all of $\mathcal{S}'(\mathbb{R})$ even if, in principle, it would be sufficient to extend them to $\xi_{x_0}(x)$. This would be enough to define the eigenstates of $\hat{q}$ and $\hat{q}^\dagger$. But, since we also want to construct eigenstates of $\hat{p}$ and $\hat{p}^\dagger$, we prefer to take a more general approach.

First of all, it is clear that T and $(T^{-1})^\dagger$ can be extended to $\mathcal{S}'(\mathbb{R})$ by duality. In fact, we define

$$\langle TF, \varphi\rangle = \langle F, T^\dagger\varphi\rangle \quad \text{and} \quad \langle (T^{-1})^\dagger F, \varphi\rangle = \langle F, T^{-1}\varphi\rangle, \tag{2.7}$$

for all $F \in \mathcal{S}'(\mathbb{R})$ and $\varphi \in \mathcal{S}(\mathbb{R})$. Here $\langle .,.\rangle$ is the form which puts in duality $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$, which extends the standard scalar product in $\mathcal{L}^2(\mathbb{R})$.[2] It is obvious that the formulae in (2.7) make sense, since $\mathcal{S}(\mathbb{R})$ is stable under the action of $T^\dagger$ and T^{-1}. It is also clear that TF and $(T^{-1})F$ define linear functionals on $\mathcal{S}(\mathbb{R})$. To conclude that they are both tempered distributions, we still have to check that they are both $\tau_{\mathcal{S}}$-continuous, i.e. that if $\{\varphi_n(x) \in \mathcal{S}(\mathbb{R})\}$ is a sequence $\tau_{\mathcal{S}}$-convergent to $\varphi(x) \in \mathcal{S}(\mathbb{R})$, then $\langle TF, \varphi_n\rangle \to \langle TF, \varphi\rangle$, and $\langle (T^{-1})^\dagger F, \varphi_n\rangle \to \langle (T^{-1})^\dagger F, \varphi\rangle$, for all $F \in \mathcal{S}'(\mathbb{R})$. This is certainly true if T is fully $\mathcal{S}(\mathbb{R})$-stable since, for instance, we have

$$\langle TF, \varphi_n\rangle = \langle F, T^\dagger\varphi_n\rangle \to \langle F, T^\dagger\varphi\rangle = \langle TF, \varphi\rangle,$$

due to the fact that, if $\varphi_n(x)$ converges in the topology τ_S, $(T^\dagger\varphi_n)(x)$ converges as well, in the same topology. Then the following corollary holds.

Corollary 2.4. *Let T be a fully $\mathcal{S}(\mathbb{R})$-stable operator. Then*

$$\eta_{x_0}(x) = (T\xi_{x_0})(x) \quad \text{and} \quad \eta^{x_0}(x) = ((T^{-1})^\dagger\xi_{x_0})(x) \tag{2.8}$$

are tempered distributions. Moreover, $\eta_{x_0}(x) \in D(\hat{q})$ and $\eta^{x_0}(x) \in D(\hat{q}^\dagger)$, and we have:

$$(\hat{q}\eta_{x_0})(x) = x_0\eta_{x_0}(x) \quad \text{and} \quad (\hat{q}^\dagger\eta^{x_0})(x) = x_0\eta^{x_0}(x). \tag{2.9}$$

Remark 2.5. We recall that, in general, the domain of a given operator X acting on $\mathcal{H}$, $D(X)$, is the set of the following subset of $\mathcal{H}$: $D(X) = \{f \in \mathcal{H} : Xf \in \mathcal{H}\}$. Here we are slightly extending this notion, while keeping the same notation. In fact, $\eta_{x_0}(x) \in D(\hat{q})$ would not be compatible with definition 2.3, since $\eta_{x_0}(x)$ does not belong to $\mathcal{L}^2(\mathbb{R})$. For this reason, $D(\hat{q})$ and $D(\hat{q}^\dagger)$ should be understood as *generalized domains*: $D(\hat{q}) = \{F \in \mathcal{S}'(\mathbb{R}) : \hat{q}F \in \mathcal{S}'(\mathbb{R})\}$, $D(\hat{q}^\dagger) = \{F \in \mathcal{S}'(\mathbb{R}) : \hat{q}^\dagger F \in \mathcal{S}'(\mathbb{R})\}$ and so on.

In the same way, out of $(\hat{p}_0, \theta_{p_0}(x))$ in (2.1) and (2.2), we can deduce the eigenstates of the operators $\hat{p}$ and $\hat{p}^\dagger$ introduced in (2.5) and (2.6), as in (2.8),

$$\mu_{p_0}(x) = (T\theta_{p_0})(x) \quad \text{and} \quad \mu^{p_0}(x) = ((T^{-1})^\dagger\theta_{p_0})(x). \tag{2.10}$$

They are both tempered distributions belonging, respectively, to $D(\hat{p})$ and $D(\hat{p}^\dagger)$ (in the sense of remark 2.5) and they are both generalized eigenstates of $\hat{p}$ and $\hat{p}^\dagger$, respectively:

$$(\hat{p}\mu_{p_0})(x) = p_0\mu_{p_0}(x) \quad \text{and} \quad (\hat{p}^\dagger\mu^{p_0})(x) = p_0\mu^{p_0}(x). \tag{2.11}$$

[2]This form can be defined as before, via convolution of distributions.

Let us now introduce the following sets of tempered distributions: $\mathcal{F}_\eta=\{\eta_{x_0}(x), x_0\in\mathbb{R}\}$, $\mathcal{F}^\eta=\{\eta^{x_0}(x), x_0\in\mathbb{R}\}$, $\mathcal{F}_\mu=\{\mu_{p_0}(x), p_0\in\mathbb{R}\}$ and $\mathcal{F}^\mu=\{\mu^{p_0}(x), p_0\in\mathbb{R}\}$.

For our purposes, it is convenient now (but not strictly necessary) to assume some extra properties for $(T^{-1})^\dagger$. In particular, we assume that

$$((T^{-1})^\dagger\xi_{y_0})*\varphi=(T^{-1})^\dagger(\xi_{y_0}*\varphi), \tag{2.12}$$

for all $\varphi(x)\in\mathcal{S}(\mathbb{R})$. Here $*$ is the convolution. Note that both sides of this equality are well defined. In fact, we have already seen that $\eta^{y_0}=(T^{-1})^\dagger\xi_{y_0}$ is in $\mathcal{S}'(\mathbb{R})$, and, therefore, it admits a convolution with $\varphi(x)$, since this function belongs to $\mathcal{S}(\mathbb{R})$. The result is, in general, a C^∞ function increasing not faster than some polynomial. As for the right-hand side of (2.12), we see that $(\xi_{y_0}*\varphi)(x)=\int_{\mathbb{R}}\xi_{y_0}(s)\varphi(x-s)\,\mathrm{d}s=\int_{\mathbb{R}}\delta(s-y_0)\varphi(x-s)\,\mathrm{d}s=\varphi(x-y_0)$, which belongs to $\mathcal{S}(\mathbb{R})$. Hence, we can act on this function with $(T^{-1})^\dagger$. However, it is not granted *a priori* that both sides of (2.12) coincide. However, we observe that (2.12) is very similar to some well-known properties of convolutions of distributions, such as the property that $(F*G)'=F'*G$, for all distributions F and G for which their convolution exists. We refer to [25] for many details on distribution theory, and for more situations in which (2.12) is again satisfied.

Now we have the following proposition.

Proposition 2.6. *Under assumption* (2.12) *the set* $\mathcal{F}_\eta$ *is well behaved on* $\mathcal{S}(\mathbb{R})$.

Proof. Let us take $\varphi(x),\psi(x)\in\mathcal{S}(\mathbb{R})$. Then we have, using the stability of $\mathcal{S}(\mathbb{R})$ under the action of both $T^\dagger$ and T^{-1}, as well as the resolution of the identity in (2.3), valid for all functions in $\mathcal{S}(\mathbb{R})$,

$$\begin{aligned}\int_{\mathbb{R}}\mathrm{d}x_0\langle\varphi,\eta_{x_0}\rangle\langle\eta^{x_0},\psi\rangle&=\int_{\mathbb{R}}\mathrm{d}x_0\langle\varphi,T\xi_{x_0}\rangle\langle(T^{-1})^\dagger\xi_{x_0},\psi\rangle\\&=\int_{\mathbb{R}}\mathrm{d}x_0\langle T^\dagger\varphi,\xi_{x_0}\rangle\langle\xi_{x_0},T^{-1}\psi\rangle=\langle T^\dagger\varphi,T^{-1}\psi\rangle=\langle\varphi,\psi\rangle.\end{aligned}$$

In a similar way, we get

$$\int_{\mathbb{R}}\mathrm{d}x_0\langle\varphi,\eta^{x_0}\rangle\langle\eta_{x_0},\psi\rangle=\int_{\mathbb{R}}\mathrm{d}x_0\langle\varphi,(T^{-1})^\dagger\xi_{x_0}\rangle\langle T\xi_{x_0},\psi\rangle=\langle\varphi,\psi\rangle.$$

To prove that $\langle\eta_{x_0},\eta^{y_0}\rangle=\delta(x_0-y_0)$ we first recall that (see (2.4)) $(\bar{\eta}_{x_0}*\tilde{\eta}^{y_0},\varphi)=\langle\eta_{x_0},\eta^{y_0}*\varphi\rangle$, for all $\varphi(x)\in\mathcal{S}(\mathbb{R})$. But, since $(\eta^{y_0}*\varphi)(x)=\varphi(x-y_0)$ is in $\mathcal{S}(\mathbb{R})$, using (2.12),

$$\langle\eta_{x_0},\eta^{y_0}*\varphi\rangle=\langle T\xi_{x_0},\eta^{y_0}*\varphi\rangle=\langle\xi_{x_0},T^\dagger(\eta^{y_0}*\varphi)\rangle=\langle\xi_{x_0},(\xi_{y_0}*\varphi)\rangle=(\bar{\xi}_{x_0}*\tilde{\xi}_{y_0},\varphi).$$

Hence $(\bar{\eta}_{x_0}*\tilde{\eta}^{y_0},\varphi)(x)=(\bar{\xi}_{x_0}*\tilde{\xi}_{y_0},\varphi)(x)$, and, therefore, we have

$$\langle\eta_{x_0},\eta^{y_0}\rangle=(\bar{\eta}_{x_0}*\tilde{\eta}_{y_0},\varphi)(0)=(\bar{\xi}_{x_0}*\tilde{\xi}_{y_0},\varphi)(0)=\langle\xi_{x_0},\xi_{y_0}\rangle=\delta(x_0-y_0),$$

which is what we needed to prove. ■

Remark 2.7. (1) A similar proof can be repeated to prove that $\mathcal{F}_\mu$ is also $\hat{p}$-well-behaved.

(2) It might be interesting to observe that the equality $\langle\eta_{x_0},\eta^{y_0}\rangle=\delta(x_0-y_0)$ can be deduced following a very different approach from the one adopted in proposition 2.6, i.e. by making use of the so-called quasi-bases [24]. To construct the quasi-bases we consider an o.n. basis for $\mathcal{L}^2(\mathbb{R})$, say $\mathcal{F}=\{e_n(x)\in\mathcal{S}(\mathbb{R})\}$. To be concrete, we can think of $\mathcal{F}_e$ as the set of eigenstates of the harmonic oscillator, since these are all in $\mathcal{S}(\mathbb{R})$, and form an o.n. basis of $\mathcal{L}^2(\mathbb{R})$. Then we introduce $\mathcal{F}_\varphi=\{\varphi_n(x)=Te_n(x)\}$ and $\mathcal{F}_\Psi=\{\Psi_n(x)=(T^{-1})^\dagger e_n(x)\}$. Assuming that T is $\mathcal{S}(\mathbb{R})$-stable, then the functions $\varphi_n(x)$ and $\Psi_n(x)$ are all in $\mathcal{S}(\mathbb{R})$. Also, $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are biorthogonal, $\langle\varphi_n,\Psi_m\rangle=\delta_{n,m}$, and they are $\mathcal{S}(\mathbb{R})$-quasi-bases [24]. Indeed, it is easy to check that, for all $\gamma(x),\eta(x)\in\mathcal{S}(\mathbb{R})$, we have

$$\langle\gamma,\eta\rangle=\sum_n\langle\gamma,\varphi_n\rangle\langle\Psi_n,\eta\rangle=\sum_n\langle\gamma,\Psi_n\rangle\langle\varphi_n,\eta\rangle. \tag{2.13}$$

Our working assumption here is that this same equality can be extended outside $\mathcal{S}(\mathbb{R})$, to all distributions ξ_{x_0}, $x_0 \in \mathbb{R}$. Indeed, assuming for instance that

$$\delta(x_0 - y_0) = \langle \xi_{x_0}, \xi_{y_0} \rangle = \sum_n \langle \xi_{x_0}, \varphi_n \rangle \langle \Psi_n, \xi_{y_0} \rangle, \tag{2.14}$$

it is easy to conclude, again, that

$$\langle \eta_{x_0}, \eta^{y_0} \rangle = \langle \eta^{y_0}, \eta_{x_0} \rangle = \delta(x_0 - y_0). \tag{2.15}$$

In fact, we have

$$\begin{aligned}\langle \eta^{y_0}, \eta_{x_0} \rangle &= \sum_n \langle (T^{-1})^\dagger \xi_{y_0}, \varphi_n \rangle \langle \Psi_n, T^{-1} \xi_{y_0} \rangle = \sum_n \langle \xi_{x_0}, T^{-1} \varphi_n \rangle \langle T^\dagger \Psi_n, \xi_{y_0} \rangle \\ &= \sum_n \langle \xi_{x_0}, e_n \rangle \langle e_n, \xi_{y_0} \rangle = \sum_n e_n(x_0) \overline{e_n(y_0)} = \delta(x_0 - y_0).\end{aligned}$$

This approach is particularly interesting as it is heavily connected with the general settings proposed in recent years for deformed canonical commutation and anti-commutation relations (see [9–11] for recent applications), and since it makes no use of equality (2.12), which is not always satisfied even in simple cases, as in the first example discussed in §3a.

Following [19], we can also introduce two operators, S_η and S^η, in the following generalized domains:

$$D(S_\eta) = \left\{ F \in \mathcal{S}'(\mathbb{R}) : \int_{\mathbb{R}} \mathrm{d}x \langle \eta_x, F \rangle \eta_x \in \mathcal{S}'(\mathbb{R}) \right\},$$

$$D(S^\eta) = \left\{ G \in \mathcal{S}'(\mathbb{R}) : \int_{\mathbb{R}} \mathrm{d}x \langle \eta^x, G \rangle \eta^x \in \mathcal{S}'(\mathbb{R}) \right\}$$

and

$$S_\eta F = \int_{\mathbb{R}} \mathrm{d}x \langle \eta_x, F \rangle \eta_x, \quad S^\eta G = \int_{\mathbb{R}} \mathrm{d}x \langle \eta^x, G \rangle \eta^x, \tag{2.16}$$

for all $F \in D(S_\eta)$ and $G \in D(S^\eta)$. In particular, it is clear that $\eta^y \in D(S_\eta)$ and that $\eta_y \in D(S^\eta)$, for all $y \in \mathbb{R}$. In particular, $S_\eta \eta^y = \eta_y$, while $S^\eta \eta_y = \eta^y$. Furthermore, if T is bounded, then for all $\varphi(x) \in \mathcal{S}(\mathbb{R})$ we get $S_\eta \varphi = TT^\dagger \varphi$. Also, if T^{-1} is bounded, then $S^\eta f = (T^{-1})^\dagger T^{-1} \varphi$. Of course, when they are both bounded, we see that S_η and S^η are the inverse of each other. More results on S_η and S^η are discussed in [19], where some connections of these operators to the so-called kq-representation (see, for example, [26]) are also considered. Here we just want to note that similar operators are somehow used in the literature to define new scalar products in the Hilbert space; see [24] and references therein.

(a) A brief algebraic view of $\hat{q}$ and $\hat{p}$

In what we have done so far, we have used techniques borrowed from functional analysis and distribution theory to deal with $\hat{q}$, $\hat{p}$ and their adjoints. Now, we briefly suggest a possible alternative approach to deal with these operators, based on certain algebras of unbounded operators. We refer to [22] for a mathematically oriented monograph, and to [23] for a more physically focused review.

If $\mathcal{D}$ is a dense subspace of a (separable) Hilbert space $\mathcal{H}$ we denote by $\mathcal{L}^\dagger(\mathcal{D})$ the set of all the operators which leave, together with their adjoints, $\mathcal{D}$ invariant. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a *-algebra with respect to the usual operations. In particular, the adjoint in $\mathcal{L}^\dagger(\mathcal{D})$ is just the restriction of the usual adjoint to $\mathcal{D}$. It is worth remarking that $\mathcal{L}^\dagger(\mathcal{D})$ contains suitable unbounded operators, and this is, in fact, its main raison d'etre [22,23]. This can be easily understood since, in many concrete applications, $\mathcal{D}$ is taken to be the domain of all the powers of some suitable unbounded, densely defined, self-adjoint operator N on $\mathcal{H}$: $\mathcal{D} = D^\infty(N) = \cap_{k \geq 0} D(N^k)$, which, due to the assumptions on N, is automatically dense in $\mathcal{H}$. In particular, if $N = p^2 + x^2$, where $p = -\mathrm{i}(\mathrm{d}/\mathrm{d}x)$, it is known that $\mathcal{D} = \mathcal{S}(\mathbb{R})$, and that the topology $\tau_{\mathcal{S}}$ is equivalent to the one defined by the seminorms

$f \to \|N^n f\|$, $n \geq 0$, [27]. Then we get the following *rigged Hilbert space*:

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R});$$

see also [28]. From now on we identify $\mathcal{D}$ with $\mathcal{S}(\mathbb{R})$. Therefore, the set of the distributions $\mathcal{S}'(\mathbb{R})$ is just the dual of $\mathcal{D}$.

Definition 2.8. An invertible operator T in the Hilbert space $\mathcal{H}$, such that $T, T^{-1} \in \mathcal{L}^\dagger(\mathcal{D})$ is called *admissible* if there exists an o.n. basis $\mathcal{F}_e = \{e_n(x) \in \mathcal{S}(\mathbb{R})\}$ for $\mathcal{L}^2(\mathbb{R})$, such that $\mathcal{F}_\varphi = \{\varphi_n(x) = Te_n(x)\}$ and $\mathcal{F}_\Psi = \{\Psi_n(x) = (T^{-1})^\dagger e_n(x)\}$ are $\mathcal{D}'$ quasi-bases, in the following sense: for every $F, G \in \mathcal{D}'$

$$\langle F, G\rangle = \sum_n \langle F, \varphi_n\rangle\langle \Psi_n, G\rangle = \sum_n \langle F, \Psi_n\rangle\langle \varphi_n, G\rangle. \tag{2.17}$$

Remark 2.9. Because of the properties of $\mathcal{L}^\dagger(\mathcal{D})$ [22], an element $A \in \mathcal{L}^\dagger(\mathcal{D})$ is automatically *fully admissible*. By this we mean that, for all sequences $\varphi_n(x) \in \mathcal{S}(\mathbb{R})$ $\tau_{\mathcal{S}}$-converging to $\varphi(x)$, the sequences $(T^\dagger\varphi_n)(x)$ and $(T^{-1}\varphi_n)(x)$ both converge in the same topology. It is evident how the concept of full admissibility can be seen as an *algebraic counterpart* of the full stability we have introduced before. In fact, what we are doing in this short section is just adopting a different language to deduce the same results.

Now let $A \in \mathcal{L}^\dagger(\mathcal{D})$ be admissible, and therefore fully admissible. Then A can be extended, by duality, to a continuous operator $A_{\text{ex}} : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$. In fact, to keep the notation simple and so no confusion can arise, in the following we identify A and A_{ex}. We have: $\langle AF, \varphi\rangle = \langle F, A^\dagger\varphi\rangle$, $\forall F \in \mathcal{S}'(\mathbb{R}), \varphi \in \mathcal{S}(\mathbb{R})$. This operator is still linear and it is also continuous: in fact, due to definition 2.8, if $\varphi_n(x) \to \varphi(x)$ in the topology $\tau_{\mathcal{S}}$, then $A^\dagger\varphi_n \to A^\dagger\varphi$ in the same topology. Hence $\langle AF, \varphi_n\rangle = \langle F, A^\dagger\varphi_n\rangle \to \langle F, A^\dagger\varphi\rangle = \langle AF, \varphi\rangle$, for all $\varphi(x) \in \mathcal{S}(\mathbb{R})$ and $F \in \mathcal{S}'(\mathbb{R})$. In particular, then, $\eta_{x_0} = T\xi_{x_0}$ and $\eta^{x_0} = (T^{-1})^\dagger\xi_{x_0}$ are both in $\mathcal{S}'(\mathbb{R})$, and the following hold:

$$\langle \eta_{x_0}, \varphi\rangle = (T^\dagger\varphi)(x_0) \quad \text{and} \quad \langle \eta^{x_0}, \varphi\rangle = (T^{-1}\varphi)(x_0),$$

for all $\varphi(x) \in \mathcal{S}(\mathbb{R})$. Moreover, again using definition 2.8 and the fact that $\hat{q}_0 \in \mathcal{L}^\dagger(\mathcal{D})$, it is clear that $\hat{q} = T\hat{q}_0T^{-1}$ is also in $\mathcal{L}^\dagger(\mathcal{D})$. Then we have the following proposition.

Proposition 2.10. *If $T \in \mathcal{L}^\dagger(\mathcal{D})$ is admissible, the set $\mathcal{F}_\eta = \{\eta_{x_0}, x_0 \in \mathbb{R}\}$ is well behaved.*

Proof.

1. For every $x_0 \in \mathbb{R}, \varphi(x) \in \mathcal{S}(\mathbb{R})$,

$$\langle \hat{q}\eta_{x_0}, \varphi\rangle := \langle (T\hat{q}_0T^{-1})(T\xi_{x_0}), \varphi\rangle = \langle T\hat{q}_0\xi_{x_0}, \varphi\rangle = \langle x_0T\xi_{x_0}, \varphi\rangle = \langle x_0\eta_{x_0}, \varphi\rangle.$$

   Hence $\hat{q}\eta_{x_0} = x_0\eta_{x_0}$.
2.

$$\langle \eta^{y_0}, \eta_{x_0}\rangle = \sum_n \langle (T^{-1})^\dagger\xi_{y_0}, \varphi_n\rangle\langle \Psi_n, T^{-1}\xi_{y_0}\rangle = \sum_n \langle \xi_{x_0}, T^{-1}\varphi_n\rangle\langle T^\dagger\Psi_n, \xi_{y_0}\rangle$$
$$= \sum_n \langle \xi_{x_0}, e_n\rangle\langle e_n, \xi_{y_0}\rangle = \sum_n e_n(x_0)\overline{e_n(y_0)} = \delta(x_0 - y_0).$$

3. For every $x_0, y_0 \in \mathbb{R}$, and for all $\varphi(x), \psi(x) \in \mathcal{S}(\mathbb{R})$, using the resolution of the identity in (2.3), valid for all functions in $\mathcal{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} \mathrm{d}x_0 \langle \varphi, \eta_{x_0}\rangle\langle \eta^{x_0}, \psi\rangle = \int_{\mathbb{R}} \mathrm{d}x_0 \langle \varphi, T\xi_{x_0}\rangle\langle (T^{-1})^\dagger\xi_{x_0}, \psi\rangle = \int_{\mathbb{R}} \mathrm{d}x_0 \langle T^\dagger\varphi, \xi_{x_0}\rangle\langle \xi_{x_0}, T^{-1}\psi\rangle$$
$$= \langle T^\dagger\varphi, T^{-1}\psi\rangle = \langle \varphi, \psi\rangle.$$

■

Of course, a similar procedure can be repeated for $\hat{p} = T\hat{p}_0T^{-1}$.

The conclusion of this analysis is that we could use the algebraic settings provided by $\mathcal{L}^\dagger(\mathcal{D})$ rather than the one adopted previously by simply replacing the notion of stability with that of admissibility. A deeper analysis of this alternative approach is postponed to a future paper.

3. Examples

This section is devoted to the discussion of two examples of our general results. In particular, we will first analyse a situation in which both T and T^{-1} are bounded, and then a different situation in which T is bounded but T^{-1} is not.

(a) First example

For every $u, v \in \mathcal{S}(\mathbb{R})$ such that $\langle u, v\rangle = 1$, we define the operator $P_{u,v}f := \langle u, f\rangle v$. Note that u and v cannot have different parity, since in this case they would be automatically orthogonal. Then, either they have the same parity or their parity is not defined. Assume that α, β are complex numbers such that $\alpha + \beta + \alpha\beta = 0$. Then, if $\alpha \neq -1$, $\beta = -\alpha/(1+\alpha)$, and the new operator

$$T = \mathbb{1} + \alpha P_{u,v}$$

is invertible, with $T^{-1} = \mathbb{1} + \beta P_{u,v}$. Unless $u = v$ and $\alpha \in \mathbb{R}$, T is not Hermitian, nor unitary, and we have $T^\dagger = \mathbb{1} + \bar{\alpha} P_{v,u} \neq T^{-1}$. Then $(T^{-1})^\dagger = \mathbb{1} + \bar{\beta} P_{v,u} = (T^\dagger)^{-1}$.

Recalling that $u, v \in \mathcal{S}(\mathbb{R})$, T turns out to be fully $\mathcal{S}(\mathbb{R})$-stable. In fact, first of all it is evident that $T, T^{-1}, T^\dagger, (T^{-1})^\dagger$ all map $\mathcal{S}(\mathbb{R})$ in $\mathcal{S}(\mathbb{R})$. Moreover, if $\{\varphi_n \in \mathcal{S}(\mathbb{R})\}$ is a sequence $\tau_{\mathcal{S}}$-convergent to $\varphi \in \mathcal{S}(\mathbb{R})$ then, for each $F \in \mathcal{S}'(\mathbb{R})$,

$$\begin{aligned}\langle F, T^\dagger \varphi_n\rangle &= \langle F, \varphi_n + \bar{\alpha}\langle v, \varphi_n\rangle u\rangle = \langle F, \varphi_n\rangle + \bar{\alpha}\langle v, \varphi_n\rangle\langle F, u\rangle \longrightarrow \langle F, \varphi\rangle + \bar{\alpha}\langle v, \varphi\rangle\langle F, u\rangle \\ &= \langle F, \varphi + \bar{\alpha}\langle v, \varphi\rangle u\rangle = \langle F, T^\dagger \varphi\rangle.\end{aligned}$$

Similarly, $\langle F, T^{-1}\varphi_n\rangle \to \langle F, T^{-1}\varphi\rangle$, and therefore both $T^\dagger$ and T^{-1} map $\mathcal{S}(\mathbb{R})$ into itself with continuity. As a consequence of the full $\mathcal{S}(\mathbb{R})$-stability of T, corollary 2.4 implies that, for each $x_0, x \in \mathbb{R}$, $\eta_{x_0}(x), \eta^{x_0}(x) \in \mathcal{S}'(\mathbb{R})$, and, in particular, that $\eta_{x_0}(x) \in D(\hat{q})$ and $\eta^{x_0}(x) \in D(\hat{q}^\dagger)$. The following expressions for $\eta_{x_0}(x), \eta^{x_0}(x)$ follow from (2.8):

$$\eta_{x_0}(x) = (T\xi_{x_0})(x) = \xi_{x_0}(x) + \alpha\langle u, \xi_{x_0}\rangle v(x) = \xi_{x_0}(x) + \alpha v(x)\overline{u(x_0)} \tag{3.1}$$

and

$$\eta^{x_0}(x) = ((T^{-1})^\dagger \xi_{x_0})(x) = \xi_{x_0}(x) + \overline{\beta v(x_0)} u(x). \tag{3.2}$$

We now prove that the sets $\mathcal{F}_\eta = \{\eta_{x_0} \in \mathcal{S}'(\mathbb{R}), x_0 \in \mathbb{R}\}$ and $\mathcal{F}^\eta = \{\eta^{x_0} \in \mathcal{S}'(\mathbb{R}), x_0 \in \mathbb{R}\}$ form two families of well-behaved states in the sense of definition 2.3.

In fact, we obtain the following:

1. From corollary 2.4, (2.9), it follows that $\hat{q}\eta_{x_0}(x) = x_0\eta_{x_0}(x)$. It is instructive to show how this result also follows from a direct computation. Using (2.5) and (3.1), we get

$$\begin{aligned}\hat{q}\eta_{x_0}(x) &= (\mathbb{1} + \alpha P_{u,v})\hat{q}_0(\mathbb{1} + \beta P_{u,v})(\xi_{x_0}(x) + \alpha v(x)\overline{u(x_0)}) \\ &= (\mathbb{1} + \alpha P_{u,v})\hat{q}_0(\xi_{x_0}(x) + (\alpha + \beta + \alpha\beta)\overline{u(x_0)} v(x)) \\ &= (\mathbb{1} + \alpha P_{u,v})(x_0\xi_{x_0}(x)) = x_0 T\xi_{x_0}(x) = x_0\eta_{x_0}(x).\end{aligned}$$

2. $\forall \varphi, \psi \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{d}x_0 \langle \varphi, \eta^{x_0}\rangle\langle \eta_{x_0}, \psi\rangle &= \int_{\mathbb{R}} \mathrm{d}x_0 \langle T^\dagger\varphi, \xi_{x_0}\rangle\langle \xi_{x_0}, T^{-1}\psi\rangle \\ &= \int_{\mathbb{R}} \mathrm{d}x_0 (\bar{\varphi}(x_0) + \alpha\overline{\langle v, \varphi\rangle}\bar{u}(x_0))(\psi(x_0) + \beta\langle u, \psi\rangle v(x_0)) \\ &= \langle \varphi, \psi\rangle + (\alpha + \beta + \alpha\beta)\langle u, \psi\rangle\langle \varphi, v\rangle = \langle \varphi, \psi\rangle.\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}} dx_0 \langle \varphi, \eta_{x_0} \rangle \langle \eta^{x_0}, \psi \rangle = \langle \varphi, \psi \rangle.$$

3. $\forall x_0, y_0 \in \mathbb{R}$, using (3.1) and (3.2) and the constraint $\alpha + \beta + \alpha\beta = 0$, we have

$$\langle \eta_{x_0}, \eta^{y_0} \rangle = \langle \xi_{x_0}, \xi_{y_0} \rangle + \alpha \overline{u(y_0)} v(x_0) + \beta \overline{u(y_0)} v(x_0) + \alpha\beta \overline{u(y_0)} v(x_0) \langle u, v \rangle = \delta(x_0 - y_0).$$

We also note that the condition $\langle \eta_{x_0}, \eta^{y_0} \rangle = \delta(x_0 - y_0)$ is ensured by (2.14), which can be checked to hold. In fact, let $\mathcal{F} = \{e_n(x) \in \mathcal{L}^2(\mathbb{R})\}$ be an o.n. basis for $\mathcal{L}^2(\mathbb{R})$, and suppose also that the $e_n(x)$ belongs to $\mathcal{S}(\mathbb{R})$. Then we construct the sets $\mathcal{F}_\varphi = \{\varphi_n(x) = Te_n(x)\}$ and $\mathcal{F}_\Psi = \{\Psi_n(x) = (T^{-1})^\dagger e_n(x)\}$, where

$$\varphi_n(x) = e_n(x) + \alpha \langle u, e_n \rangle v(x) \quad \text{and} \quad \Psi_n(x) = e_n(x) + \bar{\beta} \langle v, e_n \rangle u(x).$$

The functions $\varphi_n(x)$ and $\Psi_n(x)$ are all in $\mathcal{S}(\mathbb{R})$, and it is easy to show that they form a biorthonormal family, $\langle \varphi_n, \Psi_m \rangle = \delta_{n,m}$. Actually, since both T and T^{-1} are bounded, they form two biorthonormal Riesz bases. Using the expansion $u(x_0) = \sum_n \langle e_n, u \rangle e_n(x_0)$, true in particular for all $u(x) \in \mathcal{S}(\mathbb{R}), x_0 \in \mathbb{R}$, we obtain

$$\begin{aligned}
\sum_n \langle \xi_{x_0}, \Psi_n \rangle \langle \varphi_n, \xi_{y_0} \rangle &= \sum_n \langle \xi_{x_0}, e_n + \bar{\beta} \langle v, e_n \rangle u \rangle \langle e_n + \alpha \langle u, e_n \rangle v, \xi_{y_0} \rangle \\
&= \sum_n [e_n(x_0) + \bar{\beta} \langle v, e_n \rangle u(x_0)] \left[\overline{e_n(y_0)} + \overline{\alpha \langle u, e_n \rangle} v(x_0) \right] \\
&= \left(\sum_n e_n(x_0) \overline{e_n(y_0)} \right) + (\alpha + \beta + \alpha\beta) u(x_0) \overline{v(y_0)} = \left(\sum_n e_n(x_0) \overline{e_n(y_0)} \right) \\
&= \delta(x_0 - y_0) = \langle \xi_{x_0}, \xi_{y_0} \rangle.
\end{aligned}$$

Conditions (1–3) above ensure that $\mathcal{F}_\eta$ is well behaved. We could further check that, for every $\varphi \in \mathcal{S}(\mathbb{R})$, the following relations hold:

$$(\hat{q}\varphi)(x) = x\varphi(x) + (\alpha \langle u, x\varphi \rangle + \beta \langle u, \varphi \rangle x + \alpha\beta \langle u, \varphi \rangle \langle u, xv \rangle) v(x),$$

$$(\hat{p}\varphi)(x) = -\mathrm{i} \frac{d\varphi(x)}{dx} - \mathrm{i} \left(\beta \langle u, \varphi \rangle \frac{dv(x)}{dx} + \alpha \left\langle u, \frac{d\varphi}{dx} \right\rangle v(x) + \alpha\beta \langle u, \varphi \rangle \left\langle u, \frac{dv(x)}{dx} \right\rangle \right) v(x),$$

which give the explicit action of $\hat{q}$ and $\hat{p}$ on functions of $\mathcal{S}(\mathbb{R})$. In fact, these can be seen as particular cases of the more general situation: let Θ_0 be a self-adjoint operator, mapping $\mathcal{S}(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$ (for instance $\hat{q}_0$ or $\hat{p}_0$), and let T be as before. Then the operator $\Theta = T\Theta_0 T^{-1}$ works on $\mathcal{S}(\mathbb{R})$ as follows:

$$(\Theta\varphi)(x) = (\Theta_0\varphi)(x) + (\delta\varphi)(x),$$

where

$$(\delta\varphi)(x) = \beta \langle u, \varphi \rangle (\Theta_0 v)(x) + \alpha [\langle \Theta_0 u, \varphi \rangle + \beta \langle \Theta_0 u, v \rangle \langle u, \varphi \rangle] v(x).$$

It is interesting to see that, when Θ_0 coincides with $\hat{q}_0$ or with $\hat{p}_0$, if $v(x)$ has definite parity, $(\delta\varphi)(x)$ is necessarily not zero. This is easy to see. Suppose this is not so, i.e. that $(\delta\varphi)(x) = 0$ for all $x \in \mathbb{R}$. Hence

$$\beta \langle u, \varphi \rangle (\Theta_0 v)(x) = -\alpha [\langle \Theta_0 u, \varphi \rangle + \beta \langle \Theta_0 u, v \rangle \langle u, \varphi \rangle] v(x),$$

which is impossible since the two sides of this equation would have different parities, both if $\Theta_0 = \hat{q}_0$ and if $\Theta_0 = \hat{p}_0$. Hence our map T is non-trivial: it really changes the action of $\hat{q}_0$ and $\hat{p}_0$, while maintaining the commutation rules between the deformed operators: $[\hat{q}_0, \hat{p}_0] = [\hat{q}, \hat{p}] = \mathrm{i}\mathbb{1}$ (in the sense of unbounded operators).

From a more physical side, we see that $\hat{q}$ and $\hat{p}$ differ from their self-adjoint counterparts for an additive term which, in the first case, is a linear combination of $v(x)$ and $xv(x)$, and, in the second case, is a linear combination of $v(x)$ and $v'(x)$, with coefficients depending on the function on which these operators are applied. In analogy with the models discussed in the recent literature on *position-dependent mass* (see, for instance, [29,30] and references therein), we can call our deformed operators $\hat{q}$ and $\hat{p}$ *coordinate-dependent position and momentum operators*. These operators, when suitably used in the construction of quadratic Hamiltonians of the harmonic oscillator type, give rise to completely solvable models (see, for instance, [11] for the analysis of this kind of model), even in the presence of this explicit dependence on x.

(b) Second example

Let T^{-1} be the following unbounded operator:

$$T^{-1} := \mathbb{1} - \mathrm{i}(\hat{p}_0)^2.$$

First of all, it is clear that T^{-1} and its adjoint map $\mathcal{S}(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$. To see that T is $\mathcal{S}(\mathbb{R})$-stable, we also have to check that T and $T^\dagger$ do the same. First, we need to compute T, which can be found by introducing the Green function for T^{-1}: $(T^{-1}G)(x) = \delta(x)$. Then, standard computations give

$$G(x) = \frac{\mathrm{i}}{\sqrt{2}(1+\mathrm{i})}\,\mathrm{e}^{-|x|\left(\sqrt{2}/2\right)(1+\mathrm{i})},$$

so that the actions of T and $T^\dagger$ on $\varphi(x) \in \mathcal{S}(\mathbb{R})$ are given by

$$T(\varphi(x)) = \frac{\mathrm{i}}{\sqrt{2}(1+\mathrm{i})}\int_{\mathbb{R}} \varphi(x-s)\,\mathrm{e}^{-|s|\left(\sqrt{2}/2\right)(1+\mathrm{i})}\,\mathrm{d}s \tag{3.3}$$

and

$$T^\dagger(\varphi(x)) = \frac{-\mathrm{i}}{\sqrt{2}(1-\mathrm{i})}\int_{\mathbb{R}} \varphi(x-s)\,\mathrm{e}^{-|s|\left(\sqrt{2}/2\right)(1-\mathrm{i})}\,\mathrm{d}s. \tag{3.4}$$

We want to check that T is fully $\mathcal{S}(\mathbb{R})$-stable. We have already observed that T^{-1} and $(T^{-1})^\dagger$ both map $\mathcal{S}(\mathbb{R})$ into itself. Less trivial is to check that T also does the same. To see this, we will now check that $x^l(\mathrm{d}^k/\mathrm{d}x^k)T(\varphi(x))$ is well defined and goes to zero when $|x|$ diverges, for all k and $l \geq 0$.

First we can see that, for all $k \geq 0$,

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}x^k}T(\varphi(x)) &= \frac{\mathrm{i}}{\sqrt{2}(1+\mathrm{i})}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\int_{\mathbb{R}} \varphi(x-s)\,\mathrm{e}^{-|s|\left(\sqrt{2}/2\right)(1+\mathrm{i})}\,\mathrm{d}s \\ &= \frac{\mathrm{i}}{\sqrt{2}(1+\mathrm{i})}\int_{\mathbb{R}} \varphi^{(k)}(x-s)\,\mathrm{e}^{-|s|\left(\sqrt{2}/2\right)(1+\mathrm{i})}\,\mathrm{d}s.\end{aligned} \tag{3.5}$$

This can be proved easily since the function $g(s,x) := \varphi(x-s)\,\mathrm{e}^{-|s|\left(\sqrt{2}/2\right)(1+\mathrm{i})}$ satisfies the conditions that ensure the possibility of exchanging integrals and derivatives. In fact, $|\partial^k g(s,x)/\partial x^k| \leq M_k\,\mathrm{e}^{-|s|/\sqrt{2}}$, for all x, where $M_k = \sup_{x\in\mathbb{R}} |\varphi^{(k)}(x)|$, which is finite since $\varphi(x) \in \mathcal{S}(\mathbb{R})$. Then, in particular, $T(\varphi(x))$ is a C^∞ function. Of course, from (3.5) we also deduce that

$$x^l\frac{\mathrm{d}^k}{\mathrm{d}x^k}T(\varphi(x)) = \frac{\mathrm{i}}{\sqrt{2}(1+\mathrm{i})}\int_{\mathbb{R}} x^l\varphi^{(k)}(x-s)\,\mathrm{e}^{-|s|\left(\sqrt{2}/2\right)(1+\mathrm{i})}\,\mathrm{d}s,$$

for all $k, l \geq 0$. Finally, since

$$\lim_{|x|,\infty} x^l\frac{\mathrm{d}^k}{\mathrm{d}x^k}T(\varphi(x)) = \frac{\mathrm{i}}{\sqrt{2}(1+\mathrm{i})}\int_{\mathbb{R}} \lim_{|x|,\infty} x^l\varphi^{(k)}(x-s)\,\mathrm{e}^{-|s|\left(\sqrt{2}/2\right)(1+\mathrm{i})}\,\mathrm{d}s \tag{3.6}$$

and since $\lim_{|x|,\infty} x^l\varphi^{(k)}(x-s) = 0$ a.e. in s, we conclude that $T(\varphi(x))$ belongs to $\mathcal{S}(\mathbb{R})$. The equality in (3.6) follows again from the possibility of exchanging the limit and the integral, which is true because

$$|x^l\varphi^{(k)}(x-s)\,\mathrm{e}^{-|s|\left(\sqrt{2}/2\right)(1+\mathrm{i})}| \leq M_{l,k}\,\mathrm{e}^{-|s|/\sqrt{2}},$$

where $M_{l,k} = \sup_{x,s\in\mathbb{R}} |x^l\varphi^{(k)}(x-s)|$, which is finite for all $l, k \geq 0$, since $\varphi(x) \in \mathcal{S}(\mathbb{R})$.

Of course, the same holds true for $T^\dagger$; see (3.4). Hence, $T(\varphi(x)) \in \mathcal{S}(\mathbb{R})$.

To prove that T is fully $\mathcal{S}(\mathbb{R})$-stable, it remains to prove that for any sequence $\varphi_n(x) \in \mathcal{S}(\mathbb{R})$ $\tau_{\mathcal{S}}$-convergent to $\varphi(x) \in \mathcal{S}(\mathbb{R})$, then $(T^{-1}\varphi_n)(x)$ and $(T^\dagger\varphi_n)(x)$ are $\tau_{\mathcal{S}}$-convergent to $(T^{-1}\varphi)(x)$ and to $(T^\dagger\varphi)(x)$, respectively. It is clear that this condition is indeed true for T^{-1}. Regarding the convergence of $(T^\dagger\varphi_n)(x)$, using (3.4) we have

$$\begin{aligned}
&\lim_{n\to+\infty} x^l \frac{\mathrm{d}^k}{\mathrm{d}x^k}[(T^\dagger\varphi_n)(x) - (T^\dagger\varphi)(x)]\\
&\quad = \frac{-\mathrm{i}}{\sqrt{2}(1-\mathrm{i})} \lim_{n\to+\infty} \int_{\mathbb{R}} \mathrm{e}^{-|s|(\sqrt{2}/2)(1+\mathrm{i})} x^l \frac{\mathrm{d}^k}{\mathrm{d}x^k}[\varphi_n(x-s) - \varphi(x-s)]\,\mathrm{d}s = 0,
\end{aligned}$$

due to the Lebesgue dominated convergence theorem. Hence T is $\mathcal{S}(\mathbb{R})$-fully stable.

We are now ready to see how the results in §2 look like in the present case. First of all, by corollary 2.4, $\eta_{x_0} \in \mathcal{S}'(\mathbb{R}), \eta^{x_0} \in \mathcal{S}'(\mathbb{R})$, $\forall x_0 \in \mathbb{R}$, and that, for each $x \in \mathbb{R}$, $\eta_{x_0}(x) \in D(\hat{q})$ and $\eta^{x_0}(x) \in D(\hat{q}^\dagger)$. Their explicit expressions are

$$\begin{aligned}
\eta_{x_0}(x) = (T\xi_{x_0})(x) &= \frac{\mathrm{i}}{\sqrt{2}(1+\mathrm{i})} \int_{\mathbb{R}} \mathrm{e}^{-|s|(\sqrt{2}/2)(1+\mathrm{i})} \delta(x - x_0 - s)\,\mathrm{d}s\\
&= \frac{\mathrm{i}}{\sqrt{2}(1+\mathrm{i})}\, \mathrm{e}^{-|x-x_0|(\sqrt{2}/2)(1+\mathrm{i})}
\end{aligned} \tag{3.7}$$

and

$$\eta^{x_0}(x) = ((T^{-1})^\dagger\xi_{x_0})(x) = \xi_{x_0}(x) - \mathrm{i}\xi''_{x_0}(x). \tag{3.8}$$

We can then prove that $\mathcal{F}_\eta = \{\eta_{x_0} \in \mathcal{S}'(\mathbb{R}), x_0 \in \mathbb{R}\}$ is well behaved.

In fact, we first observe that, because of the fully $\mathcal{S}(\mathbb{R})$-stable condition, corollary 2.4 implies that $\hat{q}\eta_{x_0}(x) = x_0\eta_{x_0}(x)$.

Remark 3.1. It is instructive to verify that η_{x_0} is an eigenfunction for $\hat{q}$ by a direct computation,

$$\begin{aligned}
\hat{q}\eta_{x_0}(x) = T\hat{q}_0T^{-1}\eta_{x_0}(x) &= T\hat{q}_0T^{-1}\frac{\mathrm{i}}{\sqrt{2}(1+\mathrm{i})}\,\mathrm{e}^{-|x-x_0|(\sqrt{2}/2)(1+\mathrm{i})}\\
&= T\hat{q}_0\Bigg[\frac{\mathrm{i}x_0}{\sqrt{2}(1+\mathrm{i})}\,\mathrm{e}^{-|x-x_0|(\sqrt{2}/2)(1+\mathrm{i})} + \delta(x-x_0)\,\mathrm{e}^{-|x-x_0|(\sqrt{2}/2)(1+\mathrm{i})}\\
&\qquad - \frac{(1+\mathrm{i})x_0}{2\sqrt{2}}\,\mathrm{e}^{-|x-x_0|(\sqrt{2}/2)(1+\mathrm{i})}\Bigg]\\
&= x_0\frac{\mathrm{i}}{\sqrt{2}(1+\mathrm{i})}\,\mathrm{e}^{-|x-x_0|(\sqrt{2}/2)(1+\mathrm{i})} = x_0\eta_{x_0}(x).
\end{aligned}$$

Now let us take $x_0, y_0 \in \mathbb{R}$. Then

$$\begin{aligned}
\langle\eta_{x_0}, \eta^{y_0}\rangle &= \left\langle\frac{\mathrm{i}}{\sqrt{2}(1+\mathrm{i})}\,\mathrm{e}^{-|x-x_0|(\sqrt{2}/2)(1+\mathrm{i})}, \xi_{y_0}(x) - \mathrm{i}\xi''{}_{y_0}(x)\right\rangle\\
&= \left\langle\frac{\mathrm{i}}{\sqrt{2}(1+\mathrm{i})}\,\mathrm{e}^{-|x-x_0|(\sqrt{2}/2)(1+i)}, \xi_{y_0}(x)\right\rangle + \left\langle\frac{\mathrm{i}}{\sqrt{2}(1+\mathrm{i})}\,\mathrm{e}^{-|x-x_0|(\sqrt{2}/2)(1+\mathrm{i})}, -\mathrm{i}\xi''{}_{y_0}(x)\right\rangle\\
&= \left\langle\frac{\mathrm{i}}{\sqrt{2}(1+\mathrm{i})}\,\mathrm{e}^{-|x-x_0|(\sqrt{2}/2)(1+\mathrm{i})}, \xi_{y_0}(x)\right\rangle\\
&\quad + \left\langle\delta(x-x_0)\,\mathrm{e}^{-|x-x_0|(\sqrt{2}/2)(1+\mathrm{i})} - \frac{(1+\mathrm{i})}{2\sqrt{2}}\,\mathrm{e}^{-|x-x_0|(\sqrt{2}/2)(1+\mathrm{i})}, \xi_{y_0}(x)\right\rangle = \langle\xi_{x_0}, \xi_{y_0}\rangle = \delta(x_0 - y_0),
\end{aligned}$$

using the distributional derivative. Moreover, $\forall\varphi(x), \psi(x) \in \mathcal{S}(\mathbb{R})$, we can check that

$$\int_{\mathbb{R}} \mathrm{d}x_0 \langle\varphi, \eta^{x_0}\rangle\langle\eta_{x_0}, \psi\rangle = \int_{\mathbb{R}} \mathrm{d}x_0 \langle\varphi, \eta_{x_0}\rangle\langle\eta^{x_0}, \psi\rangle = \langle\varphi, \psi\rangle.$$

Hence $\mathcal{F}_\eta$ is well behaved, as claimed above.

Moreover, it is easy to check that, for every $\varphi(x) \in \mathcal{S}(\mathbb{R})$,

$$\hat{q}\varphi(x) = T\hat{q}_0T^{-1}\varphi(x) = \frac{\mathrm{i}}{\sqrt{2}(1+\mathrm{i})}\int_{\mathbb{R}} (x\varphi''(x-s) + \mathrm{i}x\varphi(x-s))\,\mathrm{e}^{-|x-s|\left(\sqrt{2}/2\right)(1+\mathrm{i})}\,\mathrm{d}s$$

and

$$\hat{p}\varphi(x) = T\hat{p}_0T^{-1}\varphi(x) = \frac{\mathrm{i}}{\sqrt{2}(1+\mathrm{i})}\int_{\mathbb{R}} (-\mathrm{i}\varphi'''(x-s) + \varphi'(x-s))\,\mathrm{e}^{-|x-s|\left(\sqrt{2}/2\right)(1+\mathrm{i})}\,\mathrm{d}s,$$

which give the explicit expressions of $\hat{q}$ and $\hat{p}$ on functions of $\mathcal{S}(\mathbb{R})$.

A possible framework where the example of this section could be useful is that of quantum field theory. In fact, the one-particle Feyman propagator [31], $D(p_0) = \mathrm{i}[\hat{p}_0^2 + \mathrm{i}(\epsilon + \mathrm{i}m^2)\mathbb{1}]^{-1} = \mathrm{i}[\hat{p}_0^2 + \mathrm{i}z\mathbb{1}]^{-1}$, where ϵ is a constant and m is the particle mass, is equal to operator T introduced in this section when $z = 1$. This suggests the possibility of studying and rigorously analysing some mathematical techniques used to circumvent the constraints imposed by the standard formulation of the quantum field theory based on the use of Hermitian operators.

4. Conclusion

We have seen how two non-self-adjoint *position* and *momentum* operators, $\hat{q}$ and $\hat{p}$, can be analysed when they are related to the self-adjoint ones, $\hat{q}_0$ and $\hat{p}_0$, by some suitable similarity map T. In particular, we have shown that biorthogonal eigenvectors can be found for $\hat{q}$ and $\hat{p}$, and also for $\hat{q}^\dagger$ and $\hat{p}^\dagger$, which are distributions in $\mathcal{S}'(\mathbb{R})$, and which are well behaved in the sense of [19]. We have also discussed in detail two examples of a different nature, where in particular one can see the explicit form of these eigenvectors. An alternative algebraic setting has also been proposed.

We plan to continue this analysis in the near future, in particular in connection with bicoherent states, extending what was originally done in [19]. We also plan to work more on the physical side of this paper, looking for concrete applications in which the mathematical framework discussed here can be of some utility, for instance in the analysis of time-dependent models.

Data accessibility. This work does not have any experimental data.

Authors' contributions. F.B. carried out the mathematical part of the paper, with the help of F.G., S.S. and S.T. F.G., S.S. and S.T. carried out the concrete applications, with the help of F.B. All authors gave final approval for publication.

Competing interests. We have no competing interests.

Funding. This work was partly supported by Palermo University. F.B. and F.G. also acknowledge partial support from G.N.F.M. of the INdAM. S.T. acknowledges partial support from G.N.A.M.P.A. of the INdAM.

Acknowledgements. F.B. thanks the *Distinguished Visitor Program* of the Faculty of Science of the University of Cape Town, 2017.

References

1. Bender CM, Boettcher S. 1998 Real spectra in non-Hermitian Hamiltonians having PT-symmetry. *Phys. Rev. Lett.* **80**, 5243. (doi:10.1103/PhysRevLett.80.5243)
2. Bender CM, Jones HF. 2008 Interactions of Hermitian and non-Hermitian Hamiltonians. *J. Phys. A* **41**, 244006. (doi:10.1088/1751-8113/41/24/244006)
3. Bender CM. 2005 Introduction to *PT*-symmetric quantum theory. *Contemp. Phys.* **46**, 277–292. (doi:10.1080/00107500072632)
4. Li J-Q, Li Q, Miao Y-G. 2012 Investigation of PT-symmetric Hamiltonian systems from an alternative point of view. *Commun. Theor. Phys.* **58**, 497. (doi:10.1088/0253-6102/58/4/08)
5. Krejcirik D, Siegl P. 2012 On the metric operator for the imaginary cubic oscillator. *Phys. Rev. D* **86**, 121702(R). (doi:10.1103/PhysRevD.86.121702)
6. Siegl P. 2011 PT-symmetric square well-perturbations and the existence of metric operator. *Int. J. Theor. Phys.* **50**, 991–996. (doi:10.1007/s10773-010-0593-x)
7. Davies EB, Kuijlaars BJ. 2004 Spectral asymptotics of the non-self-adjoint harmonic oscillator. *J. Lond. Math. Soc.* **70**, 420–426. (doi:10.1112/S0024610704005381)

8. Davies EB. 1999 Pseudospectra, the harmonic oscillator and complex resonances. *Proc. R. Soc. Lond. A* **455**, 585–599. (doi:10.1098/rspa.1999.0325)
9. Bebiano N, da Providencia J. 2016 Mathematical aspects of quantum systems with a pseudo-Hermitian Hamiltonian. *Braz. J. Phys.* **46**, 152–156. (doi:10.1007/s13538-015-0390-3)
10. Bebiano N, da Providencia J. In press. Implications of losing Hermiticity in quantum mechanics. *Linear Algebr. Appl.* (doi:10.1016/j.laa.2017.01.033)
11. Bagarello F, Gargano F, Volpe D. 2015 $\mathcal{D}$-deformed harmonic oscillator. *Int. J. Theor. Phys.* **54**, 4110–4123. (doi:10.1007/s10773-014-2487-9)
12. Bagarello F. 2015 Some results on the dynamics and transition probabilities for non self-adjoint hamiltonians. *Ann. Phys.* **356**, 171–184. (doi:10.1016/j.aop.2015.02.034)
13. Bagarello F. 2015 Transition probabilities for non self-adjoint Hamiltonians in infinite dimensional Hilbert spaces. *Ann. Phys.* **362**, 424–435. (doi:10.1016/j.aop.2015.08.010)
14. Houri T, Sakamoto M, Tatsumi. K. 2017 Spectral intertwining relations in exactly solvable quantum mechanical systems. (http://arxiv.org/abs/1701.04307)
15. Bagarello F. 2016 Intertwining operators for non self-adjoint Hamiltonians and bicoherent states. *J. Math. Phys.* **57**, 103501. (doi:10.1063/1.4964128)
16. Bagarello F, Gazeau JP, Szafraniec FH, Znojil M (eds). 2015 *Non-selfadjoint operators in quantum physics: mathematical aspects*. New York, NY: Wiley.
17. Bagarello F, Gargano F. 2014 Model pseudofermionic systems: connections with exceptional points. *Phys. Rev. A* **89**, 032113. (doi:10.1103/PhysRevA.89.032113)
18. Gieres F. 2000 Mathematical surprises and Dirac's formalism in quantum mechanics. *Rep. Prog. Phys.* **63**, 1893. (doi:10.1088/0034-4885/63/12/201)
19. Bagarello F. 2017 kq-representation for pseudo-bosons, and completeness of bi-coherent states. *J. Math. Anal. Appl.* **450**, 631–643. (doi:10.1016/j.jmaa.2017.01.048)
20. Znojil M, Semoradova I, Ruzicka F, Moulla H, Leghrib I. 2017 Problem of the coexistence of several non-Hermitian observables in PT-symmetric quantum mechanics. *Phys. Rev. A* **95**, 042122. (doi:10.1103/PhysRevA.95.042122)
21. Bagchi B, Fring A. 2009 Minimal length in quantum mechanics and non-Hermitian Hamiltonian systems. *Phys. Lett. A* **373**, 4307–4310. (doi:10.1016/j.physleta.2009.09.054)
22. Antoine J-P, Inoue A, Trapani C. 2002 *Partial *-algebras and their operator realizations*. Dordrecht, The Netherlands: Kluwer.
23. Bagarello F. 2007 Algebras of unbounded operators, physical applications: a survey. *Rev. Math. Phys.* **19**, 231–272. (doi:10.1142/S0129055X07002961)
24. Bagarello F. 2015 Deformed canonical (anti-)commutation relations and non hermitian Hamiltonians. In *Non-selfadjoint operators in quantum physics: mathematical aspects* (eds F Bagarello, JP Gazeau, FH Szafraniec, M Znojil), pp. 121–187. New York, NY: Wiley.
25. Vladimirov VS. 1981 *Le distribuzioni nella fisica matematica*. Moscow, Russia: Edizioni MIR.
26. Zak J. 1968 Dynamics of electrons in solids in external fields. *Phys. Rev.* **168**, 686. (doi:10.1103/PhysRev.168.686)
27. Antoine J-P, Trapani. C. 2009 *Partial inner product spaces: theory and applications*. Berlin, Germany: Springer.
28. Bogolubov NN, Logunov AA, Todorov. IT. 1975 *Introduction to axiomatic quantum field theory*. New York, NY: W. A. Benjamin.
29. Amir N, Iqbal S. 2016 Algebraic solutions of shape-invariant position-dependent effective mass systems. *J. Math. Phys.* **57**, 062105. (doi:10.1063/1.4954283)
30. Cruz y Cruz S. 2013 Factorization method and the position-dependent mass problem. In *Geometric methods in physics* (eds P Kielanowski, S Ali, A Odzijewicz, M Schlichenmaier, T Voronov), pp. 229–237. Trends in Mathematics. Basel, Switzerland: Birkhäuser.
31. Maggiore M. 2009 *A modern introduction to quantum field theory*. Oxford, UK: Oxford University Press.