

PEIFFER PRODUCT AND PEIFFER COMMUTATOR FOR INTERNAL PRE-CROSSED MODULES

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Abstract

In this work we introduce the notions of Peiffer product and Peiffer commutator of internal pre-crossed modules over a fixed object B , extending the corresponding classical notions to any semi-abelian category \mathcal{C} . We prove that, under mild additional assumptions on \mathcal{C} , crossed modules are characterized as those pre-crossed modules X whose Peiffer commutator $\langle X, X \rangle$ is trivial. Furthermore we provide suitable conditions on \mathcal{C} (fulfilled by a large class of algebraic varieties, including among others groups, associative algebras, Lie and Leibniz algebras) under which the Peiffer product realizes the coproduct in the category of crossed modules over B .

1. Introduction

The relevance of commutator theory in the study of internal structures in Mal'tsev categories has been clear since the pioneering works of Janelidze, Carboni, Pedicchio and Pirovano [27, 14, 37]. In particular, they showed that a reflexive graph

$$A \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{c} \\ \xrightarrow{c} \\ \xleftarrow{d} \end{array} B$$

underlies a category structure (which in this context is actually a groupoid) if and only if the kernel pairs of d and c centralize each other, i.e., their Pedicchio–Smith commutator is trivial. Moreover, under suitable conditions on the base category, the quotient over such a commutator realizes the reflection of reflexive graphs onto groupoids (see [37]).

In the case of the category \mathbf{Gp} of groups (where the so-called (SH) condition holds, see, e.g., [34]) this reflection can be obtained equivalently by means of the usual commutator subgroup $[\mathrm{Ker}(d), \mathrm{Ker}(c)]$ of the kernels of d and c .

On the other hand, groupoids in \mathbf{Gp} can be equivalently described as crossed modules (see [13, 30]). The latter were introduced by Whitehead in [42] while working on second relative homotopy groups (see also [41, 38], and [39] for an extensive recollection of results concerning crossed modules and their applications and generalizations).

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The crossed module representation of groupoids cited above is the result of a normalization process, that, applied to a reflexive graph, produces what was introduced in [12] under the name of *pre-crossed module*. More precisely, a pre-crossed module (X, δ) over a group B is a homomorphism $\delta: X \rightarrow B$ together with a (left) action of B on X such that, for b in B and x in X :

$$\delta({}^b x) = b\delta(x)b^{-1}.$$

A pre-crossed module becomes a crossed module if the following *Peiffer identity* (which appeared first in [41]) holds for any x, x' in X :

$$\delta x x' = x x' x^{-1}.$$

Looking at the normalized version of the above cited reflection, the role of the usual commutator is played by the so-called *Peiffer commutator*. In fact, for X and Y pre-crossed submodules of a given pre-crossed module A , following the work [18] by Conduché and Ellis, one can form the subgroup of A generated by the Peiffer words

$$x y x^{-1} (\delta x y)^{-1}, \quad y x y^{-1} (\delta y x)^{-1},$$

with x in X and y in Y . This subgroup is known as the Peiffer commutator of X and Y , and it is denoted by $\langle X, Y \rangle$. Let us observe that the Peiffer commutator generalizes the classical commutator for groups, which is recovered when B is the trivial group (a calculus of Peiffer commutators and some applications to homology can be found in [18, 3]). The reflection of a pre-crossed module X is then obtained by quotienting out the Peiffer subgroup $\langle X, X \rangle$.

The aim of this work is to develop a structural approach to the study of Peiffer commutators in the case of internal pre-crossed and crossed modules in a semi-abelian category \mathcal{C} . Semi-abelian categories were introduced by Janelidze, Márki and Tholen in [29] as a convenient setting where to develop non-abelian categorical algebra. In such a context, the notions of internal action, semidirect product and internal (pre)crossed module are available thanks to the crucial works [6, 9, 28].

In order to introduce the internal Peiffer commutator as in Definition 3.8, given two pre-crossed modules X and Y over an object B , we first define their Peiffer product $X \bowtie Y$, which is the internal version of the one introduced by Gilbert and Higgins for the group case [22]. In Proposition 3.11 we show that $X \bowtie Y$ plays for the Peiffer commutator the same role as $X \times Y$ plays for the Huq commutator [26], in the sense that it is universal among those objects where the images of X and Y have trivial Peiffer commutator. Notice that, when X and Y are trivial pre-crossed modules, then $X \bowtie Y \cong X \times Y$. In Theorem 3.18 we prove that, in any semi-abelian category satisfying the condition (SH), the reflection of pre-crossed modules onto crossed modules can be computed by means of the quotient over the Peiffer commutator. Since crossed modules form a Birkhoff subcategory $\mathbf{XMod}(\mathcal{C})$ of the category of pre-crossed modules (see [23]), one can consider the relative commutator introduced by Everaert and Van der Linden in [21] associated with $\mathbf{XMod}(\mathcal{C})$. In the case of groups, this relative notion, likewise the one introduced here, captures the classical Peiffer commutator, as shown by Everaert in [19] (see also [20]). In fact, Theorem 3.18 says that the internal Peiffer commutator $\langle X, X \rangle$ coincides with the relative one in any

semi-abelian category satisfying the condition (SH). It would be interesting to compare the two notions also in the general case, when any two pre-crossed submodules are considered.

Our investigation reveals that the generalization of the Peiffer commutator and Peiffer product to the semi-abelian context is far from being a mere translation of the corresponding notions for groups, since in general the necessary constructions have to be performed in the category $\text{PMod}_B(\mathcal{C})$ rather than in \mathcal{C} .

This observation naturally leads to the task of detecting assumptions under which these new notions can be described directly in the base category, as it happens for groups. The property of \mathcal{C} being *algebraically coherent* [17] turns out to be crucial in this sense. In this context, Proposition 4.3 and Proposition 4.5 show that both Peiffer product and Peiffer commutator can be obtained by using just coproducts in \mathcal{C} , avoiding the problem of computing coproducts in $\text{PMod}_B(\mathcal{C})$.

A further assumption is taken in Section 5 in order to prove that the Peiffer product yields the coproduct in the category of internal crossed B -modules, generalizing a result by Brown for crossed modules of groups in [10] (see also Chapter 4 of [11]). The problem of studying the coproduct of crossed modules was faced also for Lie algebras and associative algebras, respectively, in [16] and [40]. Our Theorem 5.2 generalizes all these results, since it applies, in particular, to any *category of interest* in the sense of Orzech [36], including, other than groups, associative algebras and Lie algebras, also Poisson algebras and Leibniz algebras among others.

The paper is organized as follows. In Section 2, we describe the category of internal pre-crossed modules over a fixed object B and its equivalence with internal reflexive graphs over B . Moreover, we provide the description of some relevant limits and colimits in $\text{PMod}_B(\mathcal{C})$ in terms of universal constructions in the base category. Peiffer product and Peiffer commutator of internal pre-crossed B -modules are introduced in Section 3. In Section 4, we reformulate, in the algebraically coherent context, the definitions given in Section 3 and we provide an explicit calculation of the Peiffer commutator in the cases of rings and Leibniz algebras. The last section is devoted to the proof of the fact that, under suitable assumptions, the Peiffer product is the coproduct in $\text{XMod}_B(\mathcal{C})$.

2. The category $\text{PMod}_B(\mathcal{C})$

From now on, \mathcal{C} will be a semi-abelian category [29]. Semi-abelian categories are a convenient setting for working with *internal actions*. Here we briefly recall their definition from [6].

For a fixed object B in \mathcal{C} , let us consider

$$\text{Pt}_B(\mathcal{C}) = 1_B \downarrow (\mathcal{C} \downarrow B),$$

i.e., the category of points of $\mathcal{C} \downarrow B$. The kernel functor $\text{Ker}_B : \text{Pt}_B(\mathcal{C}) \rightarrow \mathcal{C}$ is monadic and the corresponding monad, denoted by $B\flat(-)$, is defined, for any object X of \mathcal{C} , by the kernel diagram

$$B\flat X \xrightarrow{\kappa_{B,X}} B + X \xrightarrow{[1,0]} B.$$

The $B\flat(-)$ -algebras are called *internal B -actions* [6] and the category of such algebras

is denoted by $\mathbf{Act}(B, -)$. For an action $\xi: B \triangleright X \rightarrow X$, the semidirect product $X \rtimes B$ of X with B is introduced in [9] as the split epimorphism corresponding to ξ via the canonical comparison equivalence $\Xi: \mathbf{Pt}_B(\mathcal{C}) \rightarrow \mathbf{Act}(B, -)$. It can be computed explicitly (see [32]) via the coequalizer

$$B \triangleright X \begin{array}{c} \xrightarrow{\kappa_{B,X}} \\ \xrightarrow[\iota_X \cdot \xi]{} \end{array} B + X \xrightarrow{q_\xi} X \rtimes B.$$

Canonical examples of internal actions follow:

- the *trivial action* of A on X is given by the composite

$$\tau: B \triangleright X \xrightarrow{\kappa_{B,X}} B + X \xrightarrow{[0,1]} X;$$

- the *conjugation action* of X is given by the composite

$$\chi: X \triangleright X \xrightarrow{\kappa_{X,X}} X + X \xrightarrow{[1,1]} X;$$

- for a kernel $K \rightarrow X$, the *conjugation action* of X restricts to an action

$$\chi|: X \triangleright K \longrightarrow K;$$

- for any action $\xi: B \triangleright X \rightarrow X$ and any morphism $f: A \rightarrow B$, the composite

$$f^*(\xi): A \triangleright X \xrightarrow{f \triangleright 1_X} B \triangleright X \xrightarrow{\xi} X$$

defines an action, called the *pullback action* of ξ along f (indeed, the above composition amounts to a pullback via the canonical comparison Ξ).

An *internal pre-crossed module* $(\delta: X \rightarrow B, \xi)$ in \mathcal{C} is introduced in [28] as an arrow $\delta: X \rightarrow B$, together with an internal action ξ of B on X , making the following diagram commute:

$$\begin{array}{ccc} B \triangleright X & \xrightarrow{1 \triangleright \delta} & B \triangleright B \\ \xi \downarrow & & \downarrow \chi \\ X & \xrightarrow{\delta} & B \end{array}$$

A morphism between two pre-crossed modules $(\delta_X: X \rightarrow B, \xi_X)$ and $(\delta_Y: Y \rightarrow B, \xi_Y)$ with the same codomain B is given by a morphism f in the slice category $\mathcal{C} \downarrow B$ which is equivariant with respect to the B -actions, i.e., it is a morphism of $B \triangleright(-)$ -algebras:

$$\begin{array}{ccc} B \triangleright X & \xrightarrow{1 \triangleright f} & B \triangleright Y \\ \xi_X \downarrow & & \downarrow \xi_Y \\ X & \xrightarrow{f} & Y \end{array}$$

The purpose of the following sections is to study some basic constructions in the category $\mathbf{PXMod}_B(\mathcal{C})$ of pre-crossed modules in \mathcal{C} with codomain a fixed object B .

We widely use the equivalence between internal pre-crossed modules and internal reflexive graphs in \mathcal{C} (see [28]), which associates with every reflexive graph

$$X_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} B$$

its normalization $\delta: X \xrightarrow{c \cdot \ker(d)} B$, equipped with the conjugation action of B on X computed in X_1 (i.e., the pullback along e of the conjugation action of X_1 on its normal subobject X). Conversely, every pre-crossed module $(\delta: X \rightarrow B, \xi)$ yields a reflexive graph

$$X \rtimes B \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{i} \\ \xrightarrow{[\delta, 1]} \end{array} B,$$

where p and i are the canonical projection and inclusion, respectively, and $[\delta, 1]$ is the unique arrow making the following diagram commute:

$$\begin{array}{ccccc} X & \xrightarrow{j} & X \rtimes B & \xleftarrow{i} & B \\ & \searrow \delta & \downarrow [\delta, 1] & \swarrow 1 & \\ & & B & & \end{array}$$

Thanks to the equivalence mentioned above, every categorical construction in the sequentiable (in the sense of Bourn [7], see [20]) category $\mathbf{RG}_B(\mathcal{C})$ of internal reflexive graphs in \mathcal{C} with a given object of objects B has a counterpart in $\mathbf{PMod}_B(\mathcal{C})$.

We are going to describe how the main constructions we need for the present paper can be performed directly in the category $\mathbf{PMod}_B(\mathcal{C})$ by means of constructions in the base category \mathcal{C} .

2.1. The (regular epi, mono) factorization

Let f_1 be an arrow in $\mathbf{RG}_B(\mathcal{C})$. Its (regular epi, mono) factorization can be obtained by means of the (regular epi, mono) factorization (q_1, m_1) of f_1 as an arrow in \mathcal{C} :

$$\begin{array}{ccccc} X_1 & \xrightarrow{q_1} & \bullet & \xrightarrow{m_1} & Y_1 \\ \begin{array}{c} \uparrow \downarrow \\ c \uparrow \downarrow \\ e \uparrow \downarrow \\ d \end{array} & & \begin{array}{c} \uparrow \downarrow \\ c' m_1 \uparrow \downarrow \\ q_1 \uparrow \downarrow \\ e \end{array} & & \begin{array}{c} \uparrow \downarrow \\ c' \uparrow \downarrow \\ e' \uparrow \downarrow \\ d' \end{array} \\ B & \xrightarrow{1} & B & \xrightarrow{1} & B \end{array}$$

Indeed, m_1 is obviously a monomorphism in $\mathbf{RG}_B(\mathcal{C})$ and q_1 is the coequalizer in $\mathbf{RG}_B(\mathcal{C})$ of the following pair of morphisms:

$$\begin{array}{ccc} R & \begin{array}{c} \xrightarrow{r_1} \\ \xrightarrow{r_2} \end{array} & X_1 \\ \begin{array}{c} \uparrow \downarrow \\ cr_1 \uparrow \downarrow \\ \langle e, e \rangle \uparrow \downarrow \\ dr_1 \end{array} & & \begin{array}{c} \uparrow \downarrow \\ c \uparrow \downarrow \\ e \uparrow \downarrow \\ d \end{array} \\ B & \xrightarrow{1} & B \end{array}$$

where (R, r_1, r_2) is the kernel pair of q_1 .

By equivalence, we have a (regular epi, mono) factorization in $\text{PMod}_B(\mathcal{C})$ by taking the restrictions (q, m) to the kernels. Moreover, q is a regular epimorphism in \mathcal{C} being a pullback of q_1 , while m is obviously a monomorphism. Thus the following result holds:

Proposition 2.1. *Every morphism in $\text{PMod}_B(\mathcal{C})$ has a (regular epi, mono) factorization which can be obtained by means of the (regular epi, mono) factorization of the corresponding arrow in \mathcal{C} :*

$$\begin{array}{ccc}
 X & \xrightarrow{q} & \bullet & \xrightarrow{m} & Y \\
 & \searrow \delta_X & \downarrow \delta_Y m & \swarrow \delta_Y & \\
 & & B & &
 \end{array}$$

We state here the following lemma, which will be used later on.

Lemma 2.2. *Let X, Y and Z be pre-crossed B -modules in \mathcal{C} , and $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ arrows in \mathcal{C} such that the composite $g \cdot f$ is a pre-crossed module morphism. Then*

1. *if g is a monomorphism in $\text{PMod}_B(\mathcal{C})$, f is also a pre-crossed module morphism;*
2. *if f is a regular epimorphism in $\text{PMod}_B(\mathcal{C})$, g is also a pre-crossed module morphism.*

Proof. It is easy to prove that in both cases f and g are morphisms in the slice category $\mathcal{C} \downarrow B$. It remains to prove that they are equivariant. Let us consider the following diagram, where the outer square commutes since the composite $g \cdot f$ is a pre-crossed module morphism

$$\begin{array}{ccccc}
 BbX & \xrightarrow{1bf} & BbY & \xrightarrow{1bg} & BbZ \\
 \xi_X \downarrow & & \xi_Y \downarrow & & \downarrow \xi_Z \\
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z
 \end{array}$$

When g is a monomorphism of pre-crossed modules, the right hand square commutes and, by cancellation, the same holds for the one on the left. When f is a regular epimorphism of pre-crossed modules, the left hand square commutes and, moreover, $1bf$ is also a regular epimorphism since these are preserved by the functor $Bb-$ (see, e.g., [33]). Again, by cancellation, the right hand square commutes as desired. \square

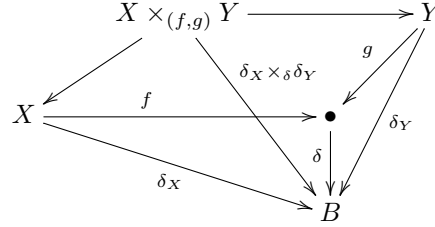
2.2. Limits

It is well known that the category $\text{RG}_B(\mathcal{C})$ has pullbacks that are computed in the following way. Given a cospan in $\text{RG}_B(\mathcal{C})$:

$$\begin{array}{ccccc}
 X_1 & \xrightarrow{f_1} & \bullet & \xleftarrow{g_1} & Y_1 \\
 \begin{array}{c} \uparrow c \\ \downarrow d \\ \uparrow e \\ \downarrow c' \end{array} & & \begin{array}{c} \uparrow \\ \downarrow \\ \uparrow \\ \downarrow \end{array} & & \begin{array}{c} \uparrow \\ \downarrow \\ \uparrow \\ \downarrow \end{array} \\
 B & \xrightarrow{1} & B & \xrightarrow{1} & B
 \end{array}$$

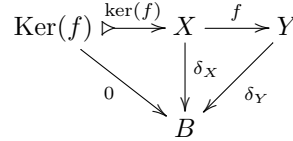
it is easy to see that the pullback $X_1 \times_{(f_1, g_1)} Y_1$ is endowed with a reflexive graph structure over B . Since the kernel functor $\text{Ker}_B: \text{Pt}_B(\mathcal{C}) \rightarrow \mathcal{C}$ preserves limits, the restriction of this pullback to the kernels of the domain projections is again a pullback, yielding the following result.

Proposition 2.3. *Every cospan in $\text{PXMod}_B(\mathcal{C})$ has a pullback which can be obtained by means of the pullback of the corresponding arrows in \mathcal{C} :*



Applying the last proposition to the particular case of a pullback along the initial map, we can describe explicitly how a kernel is computed in $\text{PXMod}_B(\mathcal{C})$.

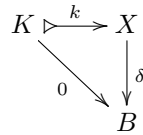
Proposition 2.4. *A kernel diagram in $\text{PXMod}_B(\mathcal{C})$ is a diagram of the following type:*



where the action of B on $\text{Ker}(f)$ is the restriction of the one on X .

The following result essentially depends on the characterization of kernels in the category $\text{RG}_B(\mathcal{C})$, which is a special case of Proposition 6.2.1 in [4]:

Proposition 2.5. *The kernels in the category $\text{PXMod}_B(\mathcal{C})$ are precisely the arrows of the form*



where k is a kernel in \mathcal{C} and the action of B on X passes to the quotient X/K .

In a semi-abelian category \mathcal{C} which is strongly protomodular the above condition on the B -action comes for free, so it suffices to ask that k is a kernel. In fact, in the semi-abelian context, the condition “every action which restricts to a kernel passes to the quotient” is equivalent to strong protomodularity (see [35] for details).

2.3. Colimits

While limit constructions are quite easy in $\text{PXMod}_B(\mathcal{C})$, colimits can be rather complicated to describe, unless we assume some additional conditions on the base category.

It is well known that a pushout in $\text{RG}_B(\mathcal{C})$ is constructed by means of level-wise pushouts in \mathcal{C} . In general, it may not be easy to compute pushouts directly in $\text{PMod}_B(\mathcal{C})$. An exception is given by the cokernel of a kernel. Following Bourn [7], in the sequentiable context, we call cokernel of $k: X \rightarrow Y$ the pushout of k along the unique arrow $X \rightarrow 0$ (if it exists, as in the case of kernels). Thanks to Proposition 2.5, it is easy to prove the following result.

Proposition 2.6. *The cokernel of a kernel k in $\text{PMod}_B(\mathcal{C})$ is a diagram of the following type (the action of B on $\text{Coker}(k)$ is induced by the one on X):*

$$\begin{array}{ccccc}
 K & \xrightarrow{k} & X & \xrightarrow{\text{coker}(k)} & \text{Coker}(k) \\
 & \searrow 0 & \downarrow \delta & \swarrow \delta' & \\
 & & B & &
 \end{array}$$

Another manageable colimit is the pushout of two regular epimorphisms.

Proposition 2.7. *The pushout of two regular epimorphisms in $\text{PMod}_B(\mathcal{C})$ can be obtained by means of the pushout of the corresponding arrows in \mathcal{C} :*

$$\begin{array}{ccccc}
 \bullet & \xrightarrow{g} & Y & & \\
 & \searrow f & \downarrow \delta_Y & \searrow & \\
 & & X & \xrightarrow{\quad} & X +_{(f,g)} Y \\
 & \searrow \delta & \downarrow \delta_X & \swarrow \delta_X + \delta_Y & \\
 & & B & &
 \end{array}$$

Proof. Thanks to Lemma 1.1 in [8], both in \mathcal{C} and in $\text{PMod}_B(\mathcal{C})$, a commutative square of regular epimorphisms is a pushout if and only if it induces a regular epimorphic restriction to the kernels. Since the kernels in $\text{PMod}_B(\mathcal{C})$ are computed as in \mathcal{C} , the pushout in $\text{PMod}_B(\mathcal{C})$ coincides with the pushout of the corresponding arrows in \mathcal{C} . \square

$\text{PMod}_B(\mathcal{C})$ being exact and sequentiable, the following proposition can be proved likewise the analogue one in the semi-abelian context (see Corollary 4.3.15 in [4]).

Proposition 2.8. *In $\text{PMod}_B(\mathcal{C})$, the join of two kernels is the kernel of the diagonal of the pushout of the corresponding cokernels.*

Corollary 2.9. *The join of two kernels in $\text{PMod}_B(\mathcal{C})$ can be obtained by means of the join of the corresponding arrows in \mathcal{C} .*

Let us consider now the coproduct construction. The coproduct of two internal reflexive graphs (X_1, d, c, e) and (Y_1, d', c', e') in $\text{RG}_B(\mathcal{C})$ is given by the pushout of

the corresponding initial maps:

But the induced square on the kernels of the domain projections is not in general a pushout in \mathcal{C} , so that $X +_{\text{PX}} Y$ needs not coincide with the coproduct $X + Y$ in \mathcal{C} (see Proposition 6.2 in [24]). It does happen when the base category is (LACC) (i.e., locally algebraically cartesian closed, see [24]), which implies that the kernel functor

$$\text{Ker}_B: \text{Pt}_B(\mathcal{C}) \rightarrow \mathcal{C}$$

preserves colimits. However, while groups and Lie algebras are examples of (LACC) categories, many other important algebraic varieties are not (e.g., the category of rings, as shown in [24]). In Section 4, we will use the weaker condition (CS) requiring the canonical arrow

$$\sigma: X + Y \rightarrow X +_{\text{PX}} Y$$

to be a regular epimorphism. In a semi-abelian context, this condition is equivalent to *algebraic coherence* in the sense of [17] (see Proposition 4.1 below).

Remark 2.10. Notice that in any case the comparison arrow σ is cancellable on the right with respect to morphisms in $\text{PXMod}_B(\mathcal{C})$, i.e., if f and g in $\text{PXMod}_B(\mathcal{C})$ are such that $f \cdot \sigma = g \cdot \sigma$, then $f = g$.

3. Peiffer product and Peiffer commutator

3.1. Definitions and properties

Lemma 3.1. *Let $(\delta: X \rightarrow B, \xi)$ be a pre-crossed module and $f: A \rightarrow B$ a morphism in \mathcal{C} . The pullback of δ along f is endowed with a pre-crossed module structure.*

Proof. This is a pre-crossed module version of a standard fact about internal reflexive graphs and fully faithful morphisms between them (see [1] for the crossed module case). We just recall here the construction of the induced action: it is the (unique)

arrow $\bar{\xi}$ making the diagram below commute:

$$\begin{array}{ccccc}
 \text{Ab}(A \times_{(f,\delta)} X) & \xrightarrow{fb\bar{f}} & BbX & & \\
 \downarrow \text{1b}\bar{\delta} & \searrow \bar{\xi} & \downarrow \xi & & \\
 & & A \times_{(f,\delta)} X & \xrightarrow{\bar{f}} & X \\
 & & \downarrow \bar{\delta} & & \downarrow \delta \\
 \text{Ab}A & \xrightarrow{\chi} & A & \xrightarrow{f} & B
 \end{array}
 \quad \square$$

Let $(\delta_X: X \rightarrow B, \xi_X)$ and $(\delta_Y: Y \rightarrow B, \xi_Y)$ be pre-crossed modules in \mathcal{C} . Then X and Y act on each other by means of the following actions:

$$\begin{aligned}
 \delta_Y^* \xi_X: YbX &\xrightarrow{\delta_Y b1} BbX \xrightarrow{\xi_X} X, \\
 \delta_X^* \xi_Y: XbY &\xrightarrow{\delta_X b1} BbY \xrightarrow{\xi_Y} Y.
 \end{aligned}$$

Proposition 3.2. *There exists a (unique) arrow $[\delta_X, \delta_Y]: X \rtimes Y \rightarrow B$, making the following diagram commute*

$$\begin{array}{ccccc}
 X & \xrightarrow{j_X} & X \rtimes Y & \xleftarrow{i_Y} & Y \\
 & \searrow \delta_X & \downarrow [\delta_X, \delta_Y] & \swarrow \delta_Y & \\
 & & B & &
 \end{array}$$

and it is endowed with a pre-crossed module structure, such that j_X and i_Y are morphisms in $\text{PMod}_B(\mathcal{C})$.

Notice that, as a particular case, when $(Y, \delta_Y) = (B, 1_B)$, we recover the arrow $[\delta_X, 1]: X \rtimes B \rightarrow B$ introduced in Section 2.

Proof. First of all we have to show that $[\delta_X, \delta_Y]$ exists. This is true (and the arrow is unique) if and only if the following diagram commutes:

$$\begin{array}{ccc}
 YbX & \xrightarrow{\kappa_{Y,X}} & X + Y \\
 \delta_Y^* \xi_X \downarrow & & \downarrow [\delta_X, \delta_Y] \\
 X & \xrightarrow{\delta_X} & B
 \end{array}$$

The latter depends on the fact that δ_X is a pre-crossed module.

Now we prove that there is an action of B on $X \rtimes Y$. By definition of $\delta_Y^* \xi_X$, we can form the following pullback:

$$\begin{array}{ccc}
 X \rtimes Y & \xrightarrow{p_Y} & Y \\
 \text{1} \times \delta_Y \downarrow & & \downarrow \delta_Y \\
 X \rtimes B & \xrightarrow{p_B} & B
 \end{array}
 \quad (1)$$

By Lemma 3.1, the arrow $1 \rtimes \delta_Y$ is a pre-crossed module, with the corresponding action $\overline{\xi_Y}$. Then B acts on $X \rtimes Y$ with the action $i_B^* \overline{\xi_Y}$ defined by the following composition:

$$\begin{array}{ccc} B\flat(X \rtimes Y) & \xrightarrow{i_B \flat 1} & (X \rtimes B)\flat(X \rtimes Y) \\ & \searrow i_B^* \overline{\xi_Y} & \downarrow \overline{\xi_Y} \\ & & X \rtimes Y \end{array}$$

Finally, we prove that $([\delta_X, \delta_Y], i_B^* \overline{\xi_Y})$ is a pre-crossed module. It suffices to show that the following diagram commutes:

$$\begin{array}{ccccc} B\flat(X \rtimes Y) & \xrightarrow{1\flat[\delta_X, \delta_Y]} & & & B\flat B \\ \downarrow i_B \flat 1 & & (a) & & \uparrow [\delta_X, 1]\flat[\delta_X, 1] \\ (X \rtimes B)\flat(X \rtimes Y) & \xrightarrow{1\flat(1 \rtimes \delta_Y)} & (X \rtimes B)\flat(X \rtimes B) & & \downarrow \chi \\ \downarrow \overline{\xi_Y} & & \downarrow \chi & & \downarrow \chi \\ X \rtimes Y & \xrightarrow{1 \rtimes \delta_Y} & (X \rtimes B) & \xrightarrow{[\delta_X, 1]} & B \\ & & \uparrow [1 \rtimes \delta_Y] & & \uparrow [\delta_X, \delta_Y] \end{array}$$

The commutativity of the bottom triangle follows from the uniqueness of $[\delta_X, \delta_Y]$, and consequently (a) commutes by functoriality of \flat . (c) commutes because every morphism is equivariant with respect to the conjugation actions of its domain and codomain; the commutativity of (b) depends on the fact that $1 \rtimes \delta_Y$ is a pre-crossed module.

In order to prove that $j_X: X \rightarrow X \rtimes Y$ is a morphism in $\text{PXMod}_B(\mathcal{C})$, we have to show that the following diagram commutes:

$$\begin{array}{ccc} B\flat X & \xrightarrow{\xi_X} & X \\ \downarrow 1\flat j_X & & \downarrow j_X \\ B\flat(X \rtimes Y) & \xrightarrow{i_B \flat 1} & (X \rtimes B)\flat(X \rtimes Y) \xrightarrow{\overline{\xi_Y}} X \rtimes Y \end{array}$$

It suffices to compose with the jointly monic projections $(p_Y, 1_X \rtimes \delta_Y)$ of the pull-back (1). Indeed, $p_Y \cdot \overline{\xi_Y} = \xi_Y \cdot (p_B \flat p_Y)$ by definition of $\overline{\xi_Y}$, and then

$$p_Y \cdot \overline{\xi_Y} \cdot (i_B \flat 1) \cdot (1\flat j_X) = \xi_Y \cdot (p_B \flat p_Y) \cdot (i_B \flat j_X) = \xi_Y \cdot (1\flat 0) = 0 = p_Y \cdot j_X \cdot \xi_X.$$

On the other hand, by the commutativity of the square (b) above, $(1 \rtimes \delta_Y) \cdot \overline{\xi_Y} = \chi \cdot (1\flat(1 \rtimes \delta_Y))$, and then

$$\begin{aligned} (1 \rtimes \delta_Y) \cdot \overline{\xi_Y} \cdot (i_B \flat 1) \cdot (1\flat j_X) &= \chi \cdot (1\flat(1 \rtimes \delta_Y)) \cdot (i_B \flat j_X) = \\ &= \chi \cdot (i_B \flat j_X) = j_X \cdot \xi_X = (1 \rtimes \delta_Y) \cdot j_X \cdot \xi_X, \end{aligned}$$

where the last but one equality holds by definition of $X \rtimes B$ (in other words, the

conjugation action of B as subobject of $X \rtimes B$ on the normal subobject X coincides with the action ξ_X defining the semidirect product).

To prove that $i_Y : Y \rightarrow X \rtimes Y$ is also a morphism in $\mathbf{PMod}_B(\mathcal{C})$, we have to show that the following diagram commutes:

$$\begin{array}{ccc} BbY & \xrightarrow{\xi_Y} & Y \\ \text{\scriptsize } 1bi_Y \downarrow & & \downarrow \text{\scriptsize } i_Y \\ Bb(X \rtimes Y) & \xrightarrow{i_B b1} (X \rtimes B)b(X \rtimes Y) \xrightarrow{\xi_Y} & X \rtimes Y \end{array}$$

As before, it suffices to compose with the projections of the pullback (1). Indeed,

$$p_Y \cdot \overline{\xi_Y} \cdot (i_B b1) \cdot (1bi_Y) = \xi_Y \cdot (p_B b p_Y) \cdot (i_B b i_Y) = \xi_Y \cdot (1b1) = \xi_Y = p_Y \cdot i_Y \cdot \xi_Y,$$

and

$$\begin{aligned} (1 \rtimes \delta_Y) \cdot \overline{\xi_Y} \cdot (i_B b1) \cdot (1bi_Y) &= \chi \cdot (1b(1 \rtimes \delta_Y)) \cdot (i_B b i_Y) = \chi \cdot (i_B b(i_B \delta_Y)) = \\ &= \chi \cdot (i_B b i_B) \cdot (1b \delta_Y) = i_B \cdot \chi \cdot (1b \delta_Y) = i_B \cdot \delta_Y \cdot \xi_Y = (1 \rtimes \delta_Y) \cdot i_Y \cdot \xi_Y, \end{aligned}$$

where in the last line we use the fact that δ_Y is a pre-crossed module. \square

As a consequence, we get a morphism $[j_X, i_Y]_{\text{px}} : X +_{\text{px}} Y \rightarrow X \rtimes Y$ of pre-crossed B -modules. We denote by $X \boxtimes Y$ the kernel of such an arrow in \mathcal{C} . As seen in Section 2.2, $0 : X \boxtimes Y \rightarrow B$ is the kernel in $\mathbf{PMod}_B(\mathcal{C})$ of $[j_X, i_Y]_{\text{px}}$. In a symmetric way, we obtain $X \boxtimes Y$ as the kernel of $[i_X, j_Y]_{\text{px}} : X +_{\text{px}} Y \rightarrow Y \rtimes X$.

Definition 3.3. We denote by $X \boxtimes Y := (X \boxtimes Y) \vee_{\text{px}} (X \boxtimes Y)$ the domain of the pre-crossed module which is the join in $\mathbf{PMod}_B(\mathcal{C})$ of these two normal subobjects of $X +_{\text{px}} Y$. By Proposition 2.8, this join is the kernel of the diagonal of the following pushout in $\mathbf{PMod}_B(\mathcal{C})$:

$$\begin{array}{ccc} X +_{\text{px}} Y & \xrightarrow{[j_X, i_Y]_{\text{px}}} & X \rtimes Y \\ [i_X, j_Y]_{\text{px}} \downarrow & & \downarrow \\ Y \rtimes X & \longrightarrow & X \boxtimes Y \end{array}$$

where $X \boxtimes Y$ is what we call the (internal) *Peiffer product* of δ_X and δ_Y , with $\delta_{X \boxtimes Y}$ the corresponding pre-crossed module over B (see [22] for the original definition of Peiffer product of groups).

In the following, we will denote l_X and l_Y the canonical morphisms from X and Y , respectively, to $X \boxtimes Y$, obtained by composition of the canonical injections in $X +_{\text{px}} Y$ with the regular epimorphism $X +_{\text{px}} Y \rightarrow X \boxtimes Y$. It follows that (l_X, l_Y) is a jointly strongly epimorphic pair in $\mathbf{PMod}_B(\mathcal{C})$.

Remark 3.4. Given two trivial pre-crossed B -modules associated with X and Y (i.e., $\delta = 0$ with B acting trivially), we have $X \rtimes Y \cong Y \rtimes X \cong X \times Y$ and consequently $X \boxtimes Y \cong X \times Y$.

Lemma 3.5. *Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be regular epimorphisms of pre-crossed B -modules. Then the following is a pushout in \mathcal{C} :*

$$\begin{array}{ccc} X + Y & \xrightarrow{[j,i]} & X \rtimes Y \\ f+g \downarrow & & \downarrow f \rtimes g \\ X' + Y' & \xrightarrow{[j,i]} & X' \rtimes Y' \end{array}$$

Proof. Let us consider the following commutative cube:

$$\begin{array}{ccccc} & & Y \bowtie X & \xrightarrow{\delta_{Y,X}^* \xi_X} & X \\ & \swarrow \kappa_{Y,X} & \downarrow & \searrow j & \downarrow f \\ X + Y & \xrightarrow{[j,i]} & X \rtimes Y & & X' \\ & \downarrow gb f & \downarrow f \rtimes g & & \downarrow j \\ & & Y' \bowtie X' & \xrightarrow{\delta_{Y',X'}^* \xi_{X'}} & X' \\ f+g \downarrow & \swarrow \kappa_{Y',X'} & \downarrow & \searrow j & \\ X' + Y' & \xrightarrow{[j,i]} & X' \rtimes Y' & & \end{array}$$

The upper and lower squares are pushouts by definition of semidirect product. gbf being a regular epimorphism (since $-b-$ preserves them) and $\delta_{Y',X'}^* \xi_{X'}$ a split epimorphism, the square on the rear and hence its composite with the lower square are pushouts. By cancellation, it follows that the square on the front is a pushout. \square

Lemma 3.6. *Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be regular epimorphisms of pre-crossed B -modules. Then the following is a pushout in $\text{PXMod}_B(\mathcal{C})$:*

$$\begin{array}{ccc} X +_{\text{PX}} Y & \xrightarrow{[j,i]_{\text{PX}}} & X \rtimes Y \\ f +_{\text{PX}} g \downarrow & & \downarrow f \rtimes g \\ X' +_{\text{PX}} Y' & \xrightarrow{[j,i]_{\text{PX}}} & X' \rtimes Y' \end{array}$$

Proof. Let us consider the diagram of solid arrows below, where u and v are morphisms in $\text{PXMod}_B(\mathcal{C})$ such that $u \cdot (f +_{\text{PX}} g) = v \cdot [j, i]_{\text{PX}}$:

$$\begin{array}{ccccc} X + Y & \xrightarrow{\sigma} & X +_{\text{PX}} Y & \xrightarrow{[j,i]_{\text{PX}}} & X \rtimes Y \\ f+g \downarrow & & (a) \downarrow f +_{\text{PX}} g & & (b) \downarrow f \rtimes g \\ X' + Y' & \xrightarrow{\sigma} & X' +_{\text{PX}} Y' & \xrightarrow{[j,i]_{\text{PX}}} & X' \rtimes Y' \\ & & & & \downarrow v \\ & & & & \bullet \\ & & & & \uparrow u \\ & & & & \bullet \\ & & & & \uparrow t \\ & & & & \bullet \end{array}$$

By Lemma 3.5, the rectangle (a) + (b) is a pushout in \mathcal{C} , hence there exists a unique t in \mathcal{C} such that $t \cdot (f \rtimes g) = v$ and $t \cdot [j, i]_{\text{PX}} \cdot \sigma = u \cdot \sigma$. Moreover, t is a morphism in

$\text{PXMod}_B(\mathcal{C})$ by Lemma 2.2, since v and the regular epimorphism $(f \times g)$ are. Then the equality $t \cdot [j, i]_{\text{PX}} = u$ follows by applying Remark 2.10. \square

Proposition 3.7. *Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be regular epimorphisms of pre-crossed B -modules. Then the induced arrows*

$$\begin{aligned} f \bowtie g: X \bowtie Y &\rightarrow X' \bowtie Y', \\ f \boxtimes g: X \boxtimes Y &\rightarrow X' \boxtimes Y' \end{aligned}$$

are also regular epimorphisms.

Proof. The arrow $f \bowtie g$ is obviously a regular epimorphism being the last part of a composite of regular epimorphisms. By Lemma 3.6 and a trivial argument on composition and cancellation of pushouts the following is a pushout in $\text{PXMod}_B(\mathcal{C})$:

$$\begin{array}{ccc} X +_{\text{PX}} Y & \xrightarrow{[l_X, l_Y]_{\text{PX}}} & X \bowtie Y \\ f +_{\text{PX}} g \downarrow & & \downarrow f \bowtie g \\ X' +_{\text{PX}} Y' & \xrightarrow{[l_{X'}, l_{Y'}]_{\text{PX}}} & X' \bowtie Y' \end{array}$$

As a consequence, the restriction to the kernels of the horizontal arrows, i.e., the morphism $f \boxtimes g$, is a regular epimorphism. Notice that all the pushouts above are also pushouts in \mathcal{C} as observed in Section 2.3. \square

Definition 3.8. Let X and Y be subobjects of A in $\text{PXMod}_B(\mathcal{C})$:

$$\begin{array}{ccccc} X & \xrightarrow{m} & A & \xleftarrow{n} & Y \\ & \searrow \delta_X & \downarrow \delta & \swarrow \delta_Y & \\ & & B & & \end{array} \quad (2)$$

The Peiffer commutator $\langle X, Y \rangle$ is given by the regular image of $X \boxtimes Y$ through the arrow $[m, n]_{\text{PX}}: X +_{\text{PX}} Y \rightarrow A$:

$$\begin{array}{ccc} X \boxtimes Y & \longrightarrow & \langle X, Y \rangle \\ \downarrow & & \downarrow \\ X +_{\text{PX}} Y & \xrightarrow{[m, n]_{\text{PX}}} & A \end{array} \quad (3)$$

Remark 3.9. Since the (regular epi, mono) factorization in $\text{PXMod}_B(\mathcal{C})$ is the same as in \mathcal{C} , $\langle X, Y \rangle$ turns out to be a subobject of A in $\text{PXMod}_B(\mathcal{C})$. Its corresponding arrow on B is $0: \langle X, Y \rangle \rightarrow B$, since it is the image of a kernel. Hence, it is possible to compute the cokernel in $\text{PXMod}_B(\mathcal{C})$ of the inclusion of $\langle X, Y \rangle$ in A . Moreover, when $[m, n]_{\text{PX}}$ is a regular epimorphism, then $\langle X, Y \rangle$ is the kernel of its cokernel in $\text{PXMod}_B(\mathcal{C})$.

Remark 3.10. $X \boxtimes Y$ is the Peiffer commutator of X and Y in $X +_{\text{PX}} Y$.

From the definition of Peiffer commutator and being the Peiffer product a cokernel, the next result follows.

Proposition 3.11. *Let X and Y be subobjects of A in $\text{PXMod}_B(\mathcal{C})$ as in diagram (2). The following are equivalent:*

1. $\langle X, Y \rangle = 0$;
2. *there exists a (necessarily unique) morphism φ making the following diagram commute:*

$$\begin{array}{ccccc} X & \xrightarrow{l_X} & X \boxtimes Y & \xleftarrow{l_Y} & Y \\ & \searrow m & \downarrow \varphi & \swarrow n & \\ & & A & & \end{array}$$

Remark 3.12. In case of trivial pre-crossed modules associated with X and Y , as already observed $X \boxtimes Y \cong X \times Y$, $l_X = \langle 1, 0 \rangle$ and $l_Y = \langle 0, 1 \rangle$, and φ is nothing but the coproduct of f and g in the sense of [4]. As a consequence, in this case, the normal closure of the Peiffer commutator coincides with the Huq commutator.

In the category of groups, the Peiffer commutator of X and Y defined above coincides with the Peiffer commutator defined by Conduché and Ellis in [18]. Notice that in this case $X +_{\text{PX}} Y = X + Y$. This is not necessarily true in a general semi-abelian category, hence the computation of the Peiffer commutator may not be easy. As we will see in Section 4, when \mathcal{C} is algebraically coherent, the construction of the Peiffer commutator can be performed entirely in \mathcal{C} , avoiding the use of the coproduct in $\text{PXMod}_B(\mathcal{C})$.

Proposition 3.13. *The Peiffer commutator preserves the (regular epi, mono) factorization of pre-crossed B -module morphisms. Namely, given the following commutative diagram in $\text{PXMod}_B(\mathcal{C})$:*

$$\begin{array}{ccccc} X & \xrightarrow{m} & A & \xleftarrow{n} & Y \\ \downarrow f & & \downarrow h & & \downarrow g \\ X' & \xrightarrow{m'} & A' & \xleftarrow{n'} & Y' \end{array}$$

where m, n, m' and n' are monomorphisms, we have the following factorization of the induced arrow between the Peiffer commutators:

$$\langle f, g \rangle: \langle X, Y \rangle \twoheadrightarrow \langle f(X), g(Y) \rangle \twoheadrightarrow \langle X', Y' \rangle$$

Proof. By the properties of the factorization, it suffices to show that:

1. $\langle f, g \rangle$ is a regular epimorphism whenever f, g and h are;
2. $\langle f, g \rangle$ is a monomorphism whenever f, g and h are.

The second assertion is trivial, while the first one follows from Proposition 3.7, since the diagonal of the following commutative square is a regular epimorphism:

$$\begin{array}{ccc} X \boxtimes Y & \twoheadrightarrow & \langle X, Y \rangle \\ \downarrow f \boxtimes g & & \downarrow \langle f, g \rangle \\ X' \boxtimes Y' & \twoheadrightarrow & \langle X', Y' \rangle \end{array} \quad \square$$

Corollary 3.14. *The Peiffer commutator is monotone: if $X \leq X'$ and $Y \leq Y'$ are pre-crossed submodules of a given pre-crossed module A , then $\langle X, Y \rangle \leq \langle X', Y' \rangle$.*

Corollary 3.15. *If X and Y are pre-crossed submodules of a given pre-crossed module A and q is the cokernel of the inclusion of $\langle X, Y \rangle$ in A , then $\langle q(X), q(Y) \rangle = 0$.*

3.2. Reflection onto crossed modules

We are going to show how the Peiffer commutator may allow to describe directly the reflection

$$\text{PMod}_B(\mathcal{C}) \begin{array}{c} \xrightarrow{I} \\ \xleftarrow{\perp} \\ \xleftarrow{H} \end{array} \text{XMod}_B(\mathcal{C}),$$

where $\text{XMod}_B(\mathcal{C})$ stands for the subcategory of internal crossed modules of codomain B , introduced by Janelidze in [28]. We recall from [34] that, when the semi-abelian category \mathcal{C} satisfies the condition (SH), an internal pre-crossed module $(\partial: A \rightarrow B, \xi)$ is a crossed module if and only if the following diagram commutes (Peiffer condition):

$$\begin{array}{ccc} AbA & \xrightarrow{\partial b1} & BbA \\ x \downarrow & & \downarrow \xi \\ A & \xrightarrow{1} & A \end{array}$$

As a corollary of Proposition 3.11 we get the following lemma.

Lemma 3.16. *Given a crossed module $(\partial: A \rightarrow B, \xi)$, we have $\langle A, A \rangle = 0$. Moreover, if X and Y are pre-crossed submodules of A , then $\langle X, Y \rangle = 0$.*

Proof. Since ∂ is a crossed module, it satisfies the Peiffer condition, which is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} AbA & \longrightarrow & A + A \\ \partial^* \xi \downarrow & & \downarrow [1,1] \\ A & \xrightarrow{1} & A \end{array}$$

which, in turn, is equivalent to the existence of the (unique) arrow $[1, 1]$ making the following diagram commute:

$$\begin{array}{ccccc} A & \xrightarrow{j} & A \rtimes A & \xleftarrow{i} & A \\ & \searrow 1 & \downarrow [1,1] & \swarrow 1 & \\ & & A & & \end{array}$$

By Proposition 3.2, we know that j and i are pre-crossed B -module morphisms of codomain $[\partial, \partial]: A \rtimes A \rightarrow B$, then there exists in $\text{PMod}_B(\mathcal{C})$ the canonical arrow $[j, i]_{\text{PX}}: A +_{\text{PX}} A \rightarrow A \rtimes A$, which is a regular epimorphism since (j, i) is a jointly strongly epimorphic pair in \mathcal{C} and then in $\text{PMod}_B(\mathcal{C})$. Since both $[j, i]_{\text{PX}}$ and the composite $[1, 1][j, i]_{\text{PX}} = [1, 1]_{\text{PX}}$ are morphisms in $\text{PMod}_B(\mathcal{C})$ and $[j, i]_{\text{PX}}$ is a regular

epimorphism, by Lemma 2.2 $[1, 1]$ is also a morphism in $\text{PXMod}_B(\mathcal{C})$:

$$\begin{array}{ccc} A \rtimes A & \xrightarrow{[1,1]} & A \\ & \searrow [\partial, \partial] & \swarrow \partial \\ & & B \end{array}$$

Moreover, $[1, 1][j, i]_{\text{PX}} = [1, 1][i, j]_{\text{PX}} = [1, 1]_{\text{PX}}$. From the definition of $A \rtimes A$ it follows that there exists a (unique) arrow φ making the following diagram commute:

$$\begin{array}{ccccc} A +_{\text{PX}} A & \xrightarrow{[j, i]_{\text{PX}}} & A \rtimes A & & \\ \downarrow [i, j]_{\text{PX}} & & \downarrow & \searrow [1,1] & \\ A \rtimes A & \xrightarrow{\quad} & A \rtimes A & \xrightarrow{\quad} & A \\ & \searrow [1,1] & \searrow \varphi & & \end{array}$$

By Proposition 3.11 this means that $\langle A, A \rangle = 0$.

The second statement follows from Corollary 3.14. \square

In fact, when the condition (SH) holds (as in the case of strongly protomodular categories and, in particular, for algebraically coherent categories), the property $\langle A, A \rangle = 0$ characterizes crossed modules among pre-crossed modules.

Proposition 3.17. *In a semi-abelian category \mathcal{C} satisfying (SH), a pre-crossed module $(\partial: A \rightarrow B, \xi)$ is a crossed module if and only if $\langle A, A \rangle = 0$.*

Proof. The condition $\langle A, A \rangle = 0$ is necessary by Lemma 3.16.

Suppose now that $\langle A, A \rangle = 0$. Then, by Proposition 3.11, there exists a unique arrow φ making the following diagram commute:

$$\begin{array}{ccccc} A & \xrightarrow{l_1} & A \rtimes A & \xleftarrow{l_2} & A \\ & \searrow 1 & \downarrow \varphi & \swarrow 1 & \\ & & A & & \end{array}$$

By composition with the canonical arrow $A \rtimes A \rightarrow A \rtimes A$, we get the arrow $[1, 1]$ below:

$$\begin{array}{ccccc} A & \xrightarrow{j} & A \rtimes A & \xleftarrow{i} & A \\ & \searrow 1 & \downarrow [1,1] & \swarrow 1 & \\ & & A & & \end{array}$$

As observed in the proof of Lemma 3.16, the existence of $[1, 1]$ is equivalent to the Peiffer condition, which, under the (SH) condition, implies that $(\partial: A \rightarrow B, \xi)$ is a crossed module. \square

Furthermore the reflection $\text{PXMod}_B(\mathcal{C}) \rightarrow \text{XMod}_B(\mathcal{C})$ can be obtained performing the quotient on the Peiffer commutator. More precisely, the following theorem holds.

Theorem 3.18. *Let $(\delta: X \rightarrow B, \xi)$ be an internal pre-crossed module in a semi-abelian category \mathcal{C} satisfying (SH). Then $HI(\delta)$ is obtained by the following cokernel in $\text{PMod}_B(\mathcal{C})$:*

$$\begin{array}{ccccc}
 \langle X, X \rangle & \xrightarrow{\quad} & X & \xrightarrow{\eta} & \frac{X}{\langle X, X \rangle} \\
 & \searrow & \downarrow \delta & \swarrow HI(\delta) & \\
 & & B & &
 \end{array}$$

0

Proof. The regular epimorphism η arises also as the right vertical map in the following pushout in $\text{PMod}_B(\mathcal{C})$:

$$\begin{array}{ccc}
 X +_{\text{PX}} X & \xrightarrow{[1,1]_{\text{PX}}} & X \\
 \Sigma \downarrow & & \downarrow \eta \\
 X \bowtie X & \xrightarrow{q} & \frac{X}{\langle X, X \rangle}
 \end{array} \tag{4}$$

(paste this diagram with diagram (3)).

The fact that $HI(\delta)$ is a crossed module follows from Corollary 3.15 and Proposition 3.17.

On the other hand, given any morphism $f: X \rightarrow A$ in $\text{PMod}_B(\mathcal{C})$, where the codomain is a crossed module, then by Lemma 3.16 we have that $\langle f(X), f(X) \rangle = 0$ and by Proposition 3.11 this yields an arrow $\varphi: X \bowtie X \rightarrow A$ making the following diagram commute:

$$\begin{array}{ccccc}
 X & \xrightarrow{l_1} & X \bowtie X & \xleftarrow{l_2} & X \\
 & \searrow f & \downarrow \varphi & \swarrow f & \\
 & & A & &
 \end{array}$$

Hence $\varphi \cdot \Sigma = f \cdot [1, 1]_{\text{PX}}$ and the universal property of $HI(\delta)$ follows from the one of the pushout (4). □

4. The algebraically coherent case

We consider from now on the following condition on the base category \mathcal{C} :

(CS) For any $(\delta_X: X \rightarrow B)$ and $(\delta_Y: Y \rightarrow B)$ in $\text{PMod}_B(\mathcal{C})$, the comparison arrow

$$\sigma: X + Y \rightarrow X +_{\text{PX}} Y$$

is a regular epimorphism.

It turns out (as Proposition 4.1 shows) that, in the semi-abelian context, this condition is equivalent to *algebraic coherence*, independently introduced in [17]. A semi-abelian category is algebraically coherent when, for any B in \mathcal{C} , the kernel functor

$$\text{Ker}: \text{Pt}_B(\mathcal{C}) \rightarrow \mathcal{C}$$

preserves jointly strongly epimorphic pairs. This condition is fulfilled by a wide class

of algebraic varieties such as the categories of groups, rings, Lie and Leibniz algebras, Poisson algebras and in general any *category of interest* in the sense of Orzech [36], as shown in [17].

Proposition 4.1. *A semi-abelian category \mathcal{C} is algebraically coherent if and only if it satisfies the condition (CS).*

Proof. Given X and Y in $\text{PMod}_B(\mathcal{C})$, it suffices to consider the coproduct of the corresponding objects in $\text{RG}_B(\mathcal{C})$ and restrict the canonical injections to the kernels of the domain projections. If \mathcal{C} is algebraically coherent, these restrictions are jointly strongly epimorphic or, equivalently, σ is a regular epimorphism.

Conversely, it suffices to observe that $\text{Pt}_B(\mathcal{C})$ is isomorphic to the full subcategory of $\text{RG}_B(\mathcal{C})$ whose objects are those reflexive graphs where domain and codomain projections coincide, and this embedding preserves coproducts. Then, given a jointly strongly epimorphic pair (f_1, g_1) in $\text{Pt}_B(\mathcal{C})$, whose restriction to kernels is (f, g) :

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & A & \xleftarrow{g} & Y \\
 \downarrow & & \downarrow & & \downarrow \\
 X_1 & \xrightarrow{f_1} & A_1 & \xleftarrow{g_1} & Y_1 \\
 \uparrow s & \downarrow p & \uparrow \beta & \downarrow \alpha & \downarrow s' \\
 B & \xrightarrow{1} & B & \xrightarrow{1} & B
 \end{array}$$

just doubling the split epimorphisms, one gets a jointly strongly epimorphic pair in $\text{RG}_B(\mathcal{C})$, and, by equivalence, in $\text{PMod}_B(\mathcal{C})$. Hence $[f, g]_{\text{PX}}: X +_{\text{PX}} Y \rightarrow A$ is a regular epimorphism. If σ is a regular epimorphism, by composition $[f, g]: X + Y \rightarrow A$ is also a regular epimorphism, proving that the kernel functor $\text{Ker}: \text{Pt}_B(\mathcal{C}) \rightarrow \mathcal{C}$ preserves jointly strongly epimorphic pairs as desired. \square

We saw in Section 2.3 that the join of two kernels in $\text{PMod}_B(\mathcal{C})$ coincides with the join of the corresponding objects in \mathcal{C} . Under (CS), this property extends to arbitrary subobjects. In fact, we have more.

Proposition 4.2. *A semi-abelian category \mathcal{C} satisfies the condition (CS) if and only if the join of any pair of subobjects in $\text{PMod}_B(\mathcal{C})$ coincides with the join of the corresponding subobjects in \mathcal{C} :*

$$\begin{array}{ccccc}
 & X \vee Y & \xleftarrow{\quad} & Y & \\
 & \swarrow & & \searrow & \\
 X & \xrightarrow{\delta_X \vee \delta_Y} & A & \xrightarrow{n} & Y \\
 & \searrow m & \downarrow \delta & \swarrow \delta_Y & \\
 & & B & & \\
 & \swarrow \delta_X & & &
 \end{array}$$

Proof. The joins of X and Y computed in \mathcal{C} and in $\text{PMod}_B(\mathcal{C})$ are obtained by means of the (regular epi, mono) factorization of the arrows $[m, n]: X + Y \rightarrow A$ and $[m, n]_{\text{PX}}: X +_{\text{PX}} Y \rightarrow A$, respectively. If (CS) holds, then the two factorizations yield the same subobject of A .

Conversely, let us consider X and Y as subobjects of $X +_{\text{PX}} Y$ in $\text{PXMod}_B(\mathcal{C})$. If their join in \mathcal{C} coincides with $X +_{\text{PX}} Y$, then $\sigma: X + Y \rightarrow X +_{\text{PX}} Y$ is a regular epimorphism. \square

4.1. Construction of the Peiffer product and commutator under (CS)

The condition (CS) on \mathcal{C} says that $\sigma: X + Y \rightarrow X +_{\text{PX}} Y$ is a regular epimorphism. Hence, considering the following commutative diagram where the two rows are short exact sequences:

$$\begin{array}{ccccc} X \diamond Y & \twoheadrightarrow & X + Y & \xrightarrow{[j,i]} & X \rtimes Y \\ \downarrow & & \downarrow \sigma & & \downarrow 1 \\ X \boxtimes Y & \twoheadrightarrow & X +_{\text{PX}} Y & \xrightarrow{[j,i]_{\text{PX}}} & X \rtimes Y \end{array}$$

the leftmost vertical arrow is a regular epimorphism, since the left hand square is a pullback. In a symmetric way, we obtain a canonical regular epimorphism by replacing $X \rtimes Y$ with $Y \rtimes X$ in the diagram above:

$$X \diamond Y \twoheadrightarrow X \boxtimes Y.$$

Let us denote $X \diamond Y := (X \diamond Y) \vee (X \diamond Y)$ (computed in $X + Y$). By Corollary 2.9, it follows that also $X \boxtimes Y$ can be obtained as the join $(X \boxtimes Y) \vee (X \boxtimes Y)$ in \mathcal{C} . Then we have a regular epimorphism:

$$X \diamond Y \twoheadrightarrow X \boxtimes Y.$$

Proposition 4.3. *The Peiffer product $X \rtimes Y$, introduced in Definition 3.3 as the quotient $\frac{X +_{\text{PX}} Y}{X \boxtimes Y}$ in $\text{PXMod}_B(\mathcal{C})$, under (CS) can be obtained as a quotient $\frac{X + Y}{X \diamond Y}$ of the coproduct in \mathcal{C} .*

Proof. Consider the following diagram, where the inner square is constructed as a pushout in $\text{PXMod}_B(\mathcal{C})$:

$$\begin{array}{ccccc} X + Y & & & & \\ & \searrow^{[j,i]} & & & \\ & & X +_{\text{PX}} Y & \xrightarrow{[j,i]_{\text{PX}}} & X \rtimes Y \\ & \searrow^{\sigma} & \downarrow [i,j]_{\text{PX}} & & \downarrow \\ & & Y \rtimes X & \longrightarrow & X \rtimes Y \\ & \searrow_{[i,j]} & & & \end{array}$$

Since σ is a regular epimorphism and the inner square is also a pushout in \mathcal{C} , so is the outer square. \square

Remark 4.4. In this case, the Peiffer product $X \rtimes Y$ can also be computed as the colimit of the following diagram:

$$\begin{array}{ccccc} & & Y \bowtie X & & X \bowtie Y \\ & \swarrow \delta_Y^* \xi_X & & \swarrow \kappa_{Y,X} & \swarrow \kappa_{X,Y} \\ X & & & & X + Y & & & & Y \\ & & & & \searrow \delta_X^* \xi_Y & & & & \end{array}$$

where $\kappa_{Y,X}$ and $\kappa_{X,Y}$ represent the formal conjugates in $X + Y$ of Y on X and of X on Y . Indeed, in order to obtain this colimit, we can take first the pushout of the span on the left producing the semi-direct product $X \rtimes Y$ (see [28]), i.e., the universal quotient of $X + Y$ where the action $\delta_Y^* \xi_X$ becomes a conjugation. In a symmetric way, we obtain $Y \rtimes X$ and by a final pushout, we get the desired colimit, which coincides then with $X \rtimes Y$, thanks to Proposition 4.3. As a consequence, $X \rtimes Y$ is the universal quotient of $X + Y$ where both the actions $\delta_Y^* \xi_X$ and $\delta_X^* \xi_Y$ become conjugations.

Proposition 4.5. *Let X and Y be pre-crossed submodules of A as in diagram (2). Under (CS) the Peiffer commutator $\langle X, Y \rangle$ can be obtained as the regular image of $X \rtimes Y$ through the arrow $[m, n]: X + Y \rightarrow A$:*

$$\begin{array}{ccccc}
 X \rtimes Y & \longrightarrow & X \rtimes Y & \longrightarrow & \langle X, Y \rangle \\
 \downarrow & & \downarrow & & \downarrow \\
 X + Y & \xrightarrow{\sigma} & X +_{\text{PX}} Y & \xrightarrow{[m,n]_{\text{PX}}} & A \\
 & \searrow & \searrow & \searrow & \\
 & & & & A \\
 & & & & \uparrow \\
 & & & & [m,n]
 \end{array}$$

Remark 4.6. In the case $[m, n]$ is a regular epimorphism, then $\langle X, Y \rangle$ is normal in A if considered in \mathcal{C} . But since algebraic coherence implies strong protomodularity (see [17]), thanks to Proposition 2.5, it becomes normal also in $\text{PXMod}_B(\mathcal{C})$.

Remark 4.7. In case of trivial pre-crossed modules, $X \rtimes Y$ coincides with the canonical object $X \diamond Y$ of [33] and the Peiffer commutator $\langle X, Y \rangle$ coincides with the Higgins commutator $[X, Y]$ introduced in the case of Ω -groups in [25] and in a categorical setting in [33].

Remark 4.8. In the present setting, the reflection described in Section 3.2 of a pre-crossed module $(\delta: X \rightarrow B, \xi)$ can be computed by means of the following pushout in \mathcal{C} :

$$\begin{array}{ccc}
 X + X & \xrightarrow{[1,1]} & X \\
 \downarrow & & \downarrow \eta \\
 X \rtimes X & \xrightarrow{q} & \overline{X} \\
 & & \langle X, X \rangle
 \end{array}$$

4.2. Examples

As an application of the constructions described above, we provide here some examples of explicit calculation of the Peiffer commutator for algebraically coherent varieties. In this context, following the description given in Remark 4.4, we can interpret $\langle X, Y \rangle$ as the ideal generated in $X \vee Y$ by the Peiffer words, i.e., those elements of A whose vanishing makes the actions $\delta_Y^* \xi_X$ and $\delta_X^* \xi_Y$ become conjugations.

Example 4.9. Let us consider first the case of (not necessarily unitary) rings. Following the notation of diagram (2), let $(\delta: A \rightarrow B, \xi)$ be a pre-crossed module in the

category of rings. The action ξ is given by the assignment of two bilinear maps:

$$\begin{aligned} B \times A &\rightarrow A, & (b, a) &\mapsto b \cdot a, \\ A \times B &\rightarrow A, & (a, b) &\mapsto a \cdot b, \end{aligned}$$

satisfying the following identities (for all $a, a' \in A$ and $b, b' \in B$):

$$\begin{aligned} (bb') \cdot a &= b \cdot (b' \cdot a), & (b \cdot a)a' &= b \cdot (aa'), \\ (b \cdot a) \cdot b' &= b \cdot (a \cdot b'), & (a \cdot b)a' &= a(b \cdot a'), \\ (a \cdot b) \cdot b' &= a \cdot (bb'), & (aa') \cdot b &= a(a' \cdot b). \end{aligned}$$

The pre-crossed module condition says that:

$$\delta(b \cdot a) = b \cdot \delta(a) \quad \text{and} \quad \delta(a \cdot b) = \delta(a) \cdot b.$$

Given X and Y pre-crossed submodules of A as in diagram (2), their Peiffer commutator $\langle X, Y \rangle \leq A$ is the ideal of $X \vee Y$ generated by the following Peiffer words (for all $x \in X$ and $y \in Y$):

$$xy - \delta_X(x) \cdot y, \quad xy - x \cdot \delta_Y(y), \quad yx - \delta_Y(y) \cdot x, \quad yx - y \cdot \delta_X(x).$$

Example 4.10. Consider now the category of Leibniz algebras over a fixed field. As above, δ is a pre-crossed module. Here, the action ξ is a pair of bilinear maps:

$$\begin{aligned} B \times A &\rightarrow A, & (b, a) &\mapsto [b, a], \\ A \times B &\rightarrow A, & (a, b) &\mapsto [a, b], \end{aligned}$$

satisfying the following identities (for all $a, a' \in A$ and $b, b' \in B$):

$$\begin{aligned} [[a, a'], b] &= [[a, b], a'] + [a, [a', b]], & [[a, b], b'] &= [[a, b'], b] + [a, [b, b']], \\ [[a, b], a'] &= [[a, a'], b] + [a, [b, a']], & [[b, a], b'] &= [[b, b'], a] + [b, [a, b']], \\ [[b, a], a'] &= [[b, a'], a] + [b, [a, a']], & [[b, b'], a] &= [[b, a], b'] + [b, [b', a]], \end{aligned}$$

and the pre-crossed module condition says that:

$$\delta([b, a]) = [b, \delta(a)] \quad \text{and} \quad \delta([a, b]) = [\delta(a), b].$$

Given X and Y pre-crossed submodules of A as in diagram (2), their Peiffer commutator $\langle X, Y \rangle \leq A$ is the ideal of $X \vee Y$ generated by the following Peiffer words (for all $x \in X$ and $y \in Y$):

$$[x, y] - [\delta_X(x), y], \quad [x, y] - [x, \delta_Y(y)], \quad [y, x] - [\delta_Y(y), x], \quad [y, x] - [y, \delta_X(x)].$$

5. Coproduct of crossed modules

Brown showed in [10] that the Peiffer product of two crossed B -modules of groups represents their coproduct in $\mathbf{XMod}_B(\mathbf{Gp})$. We show here that an internal version of this result holds in any algebraically coherent semi-abelian category satisfying the following condition.

(UA) Given a jointly strongly epimorphic cospan $A \xrightarrow{f} B \xleftarrow{g} C$ in \mathcal{C} , then for any 4-tuple $(\xi_1, \xi_2, \xi_3, \xi_4)$ of actions on a fixed object X making the following

diagram commute:

$$\begin{array}{ccccc}
 \text{Ab}X & \xrightarrow{fb^1} & \text{Bb}X & \xleftarrow{gb^1} & \text{Cb}X \\
 & \searrow \xi_1 & \Downarrow \xi_3 & \Downarrow \xi_4 & \swarrow \xi_2 \\
 & & X & &
 \end{array} \tag{5}$$

we have $\xi_3 = \xi_4$.

Proposition 5.1. *Let \mathcal{C} be an action representative semi-abelian category (see [6, 5]) or a category of interest. Then \mathcal{C} satisfies the condition (UA) above.*

Proof. In an action representative semi-abelian category \mathcal{C} , for every X in \mathcal{C} , there exists an object $\text{Act}(X)$ (the ‘‘actor’’ of X), such that the actions of any object Y on X are in one-to-one correspondence with the morphisms $Y \rightarrow \text{Act}(X)$. Through this correspondence, naming ϕ_i the morphism associated with the action ξ_i in diagram (5) above, we get the following commutative diagram:

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xleftarrow{g} & C \\
 & \searrow \phi_1 & \Downarrow \phi_3 & \Downarrow \phi_4 & \swarrow \phi_2 \\
 & & \text{Act}(X) & &
 \end{array}$$

Hence condition (UA) follows from the fact that f and g are jointly (strongly) epimorphic.

A similar phenomenon occurs in any *category of interest* \mathcal{C} , that might not be action representative. Nevertheless, it is shown in [15] that, viewing \mathcal{C} as a subvariety of a variety \mathcal{C}_G of groups with operations, for every object X in \mathcal{C} there exists an object $\text{USGA}(X)$ in \mathcal{C}_G (the ‘‘universal strict general actor’’ of X), such that every action of an object Y in \mathcal{C} on X yields a morphism $Y \rightarrow \text{USGA}(X)$ in \mathcal{C}_G . Then, as above, we get a commutative diagram in \mathcal{C}_G :

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xleftarrow{g} & C \\
 & \searrow \phi_1 & \Downarrow \phi_3 & \Downarrow \phi_4 & \swarrow \phi_2 \\
 & & \text{USGA}(X) & &
 \end{array}$$

The pair (f, g) is jointly strongly epimorphic in \mathcal{C}_G since the same holds in the subvariety \mathcal{C} , and condition (UA) follows by cancellation. \square

Thanks to the previous proposition, the following theorem applies also to the cases of crossed modules of Lie algebras, Leibniz algebras and commutative algebras studied in [16, 2, 40].

Theorem 5.2. *Let \mathcal{C} be an algebraically coherent semi-abelian category satisfying the condition (UA), $(\partial_X: X \rightarrow B, \xi_X)$ and $(\partial_Y: Y \rightarrow B, \xi_Y)$ internal crossed modules in \mathcal{C} . Then $(\partial_{\bowtie}: X \bowtie Y \rightarrow B, \xi_{\bowtie})$, constructed as in Proposition 4.3, is the coproduct of ∂_X and ∂_Y in $\text{XMod}_B(\mathcal{C})$.*

Proof. By definition, $(\partial_{\bowtie}, \xi_{\bowtie})$ is a pre-crossed module. Since in the context we are considering the property (SH) holds (see [17]), it suffices to verify the Peiffer identity for $X \bowtie Y$:

$$\begin{array}{ccc} (X \bowtie Y)b(X \bowtie Y) & \xrightarrow{\partial_{\bowtie}b1} & Bb(X \bowtie Y) \\ & \searrow \chi & \swarrow \xi_{\bowtie} \\ & & X \bowtie Y \end{array}$$

In other words, we have to show that the following two actions are equal:

$$(X \bowtie Y)b(X \bowtie Y) \xrightarrow[\chi]{\partial_{\bowtie}^* \xi_{\bowtie}} X \bowtie Y.$$

To prove this, we can pre-compose with the canonical injections l_X and l_Y to obtain the following diagram:

$$\begin{array}{ccccc} Xb(X \bowtie Y) & \xrightarrow{l_X b1} & (X \bowtie Y)b(X \bowtie Y) & \xleftarrow{l_Y b1} & Yb(X \bowtie Y) \\ & \searrow \partial_X^* \xi_{\bowtie} & \chi \downarrow \partial_{\bowtie}^* \xi_{\bowtie} & \swarrow \partial_Y^* \xi_{\bowtie} & \\ & & X \bowtie Y & & \end{array} \quad (6)$$

As already observed in Section 3.1, the pair (l_X, l_Y) is jointly strongly epimorphic in $\text{PXMod}_B(\mathcal{C})$, and then in \mathcal{C} , thanks to condition (CS). Hence, by condition (UA), we only have to prove that diagram (6) is commutative.

Let us focus on the left hand side triangles. Again by algebraic coherence, the functor $Xb-$ preserves jointly strongly epimorphic pairs (see [17]), so the two triangles commute if and only if they commute when pre-composed with the jointly strongly epimorphic pair

$$XbX \xrightarrow{1bl_X} Xb(X \bowtie Y) \xleftarrow{1bl_Y} XbY.$$

This is true because, on one hand, ∂_X is a crossed module, then the two squares

$$\begin{array}{ccc} XbX & \xrightarrow{l_X b l_X} & (X \bowtie Y)b(X \bowtie Y) \\ \chi = \partial_X^* \xi_X \downarrow & & \chi \downarrow \partial_{\bowtie}^* \xi_{\bowtie} \\ X & \xrightarrow{l_X} & X \bowtie Y \end{array}$$

commute because morphisms in \mathcal{C} are always equivariant with respect to the conjugation actions and because l_X is a morphism of pre-crossed modules. On the other hand, composing with $1bl_Y$ we get the following diagram:

$$\begin{array}{ccccc} XbY & \xrightarrow{1bl_Y} & Xb(X \bowtie Y) & \xrightarrow{l_X b1} & (X \bowtie Y)b(X \bowtie Y) \\ \partial_X^* \xi_Y \downarrow & \searrow \kappa_{X,Y} & & \swarrow \partial_X^* \xi_{\bowtie} & \chi \downarrow \partial_{\bowtie}^* \xi_{\bowtie} \\ Y & \xrightarrow{\iota_Y} & X + Y & \xrightarrow{[l_X, l_Y]} & X \bowtie Y \end{array}$$

where both the outer rectangles are commutative, since $[l_X, l_Y]$ coequalizes the pair $(\kappa_{X,Y}, \iota_Y \cdot \partial_X^* \xi_Y)$, $X \bowtie Y$ being a quotient of $Y \bowtie X$.

In a symmetric way, one can prove that the right hand side triangles in (6) commute because ∂_Y is a crossed module and $X \rtimes Y$ is also a quotient of $X \rtimes Y$. \square

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