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ON HOLOMORPHICALLY PROJECTIVE MAPPINGS OF PARABOLIC KÄHLER MANIFOLDS

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Abstract. In this paper we study fundamental equations of holomorphically projective mappings of parabolic Kähler spaces (which are generalized classical, pseudo- and hyperbolic Kähler spaces) with respect to the smoothness class of metrics. We show that holomorphically projective mappings preserve the smoothness class of metrics.

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1. INTRODUCTION

First we study the general dependence of holomorphically-projective mappings of parabolic Kähler manifolds in dependence on the smoothness class of the metric. We present well known facts, which were proved by M. Shiha, J. Mikeš et al, see [2, 3, 14, 17–21].

I. Hinterleitner [5] has solved the analogically problems for classical, pseudo- and hyperbolic Kähler manifolds. In this paper were clarified results which were proved by Bácsó, Domashev, Kurbatova, Mikeš, Prvanović, Otsuki, Tashiro, see [4, 10–13, 15, 16, 23, 25]. In these results no details about the smoothness class of the metric were stressed. They were formulated "for sufficiently smooth" geometric objects. This result was inspired in [6, 7] of geodesic mappings.

2. PARABOLIC KÄHLER MANIFOLDS

In the following definition we introduce generalizations of Kähler manifolds [8], see [9, 11, 13]. A basis on this definition see the monography by V.V. Vishnevskii, A.P. Shirokov and V.V. Shurigin [24].

Definition 1. An n-dimensional (pseudo-) Riemannian manifold (M, g) is called an *m-parabolic Kähler manifold* $K_n^{o(m)}$, if beside the metric tensor g, a tensor field

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F of a rank $m \geq 2$ of type $(1,1)$ is given on the manifold M_n, called a *structure* F, such that the following conditions hold:

$$F^2 = 0, \quad g(X, FX) = 0, \quad \nabla F = 0, \tag{2.1}$$

where X is an arbitrary vector of TM_n, and ∇ denotes the covariant derivative in $K_n^{o(m)}$.

We remind, that Kähler spaces, were characterized by conditions $F^2 = -Id$, $g(X, FX) = 0$, $\nabla F = 0$, were first considered by P.A. Shirokov, see [22]. Independently they were studied by E. Kähler [8]. Hyperbolic Kähler space (also *para Kähler space*, see D.V. Alekseevsky [1]) characterized $F^2 = Id$, $g(X, FX) = 0$, $\nabla F = 0$, were considered P.K. Rashevskij, see [9].

3. HOLOMORPHICALLY-PROJECTIVE MAPPING THEORY FOR $K_n^{o(m)} \to \bar{K}_n^{o(\bar{m})}$ OF CLASS C^1

Assume the parabolic Kähler manifolds $K_n^{o(m)} = (M, g, F)$ and $\bar{K}_n^{o(\bar{m})} = (\bar{M}, \bar{g}, \bar{F})$ with metrics g and $\bar{g}$, structures F and $\bar{F}$, Levi-Civita connections ∇ and $\bar{\nabla}$, respectively. Here $\bar{K}_n$, $\bar{K}_n \in C^1$, i.e. $g, \bar{g} \in C^1$ which means that their components g_{ij}, $\bar{g}_{ij} \in C^1$. Likewise, as in [17, 18] we introduce the following notations, this is an analogy by [15], see [11, p. 240].

Definition 2. A curve ℓ in K_n which is given by the equation $\ell = \ell(t)$, $\lambda = d\ell/dt$ $(\neq 0), t \in I$, where t is a parameter is called *analytical planar*, if under the parallel translation along the curve, the tangent vector λ belongs to the two-dimensional distribution $D = Span\{\lambda, F\lambda\}$ generated by λ and its conjugate $F\lambda$, that is, it satisfies

$$\nabla_t \lambda = a(t)\lambda + b(\lambda) F\lambda,$$

where $a(t)$ and $b(t)$ are some functions of the parameter t. Particularly, in the case $b(t) = 0$, an analytical planar curve is a geodesic.

Definition 3. A diffeomorphism $f\colon K_n^{o(m)} \to \bar{K}_n^{o(\bar{m})}$ is called a *holomorphically-projective mapping* of $K_n^{o(m)}$ onto $\bar{K}_n^{o(\bar{m})}$ if f maps any analytical planar curve in $K_n^{o(m)}$ onto an analytical planar curve in $\bar{K}_n^{o(\bar{m})}$.

Assume a holomorphically-projective mapping $f\colon K_n^{o(m)} \to \bar{K}_n^{o(\bar{m})}$. Since f is a diffeomorphism, we can suppose local coordinate charts on M or $\bar{M}$, respectively, such that locally, $f : K_n^{o(m)} \to \bar{K}_n^{o(\bar{m})}$ maps points onto points with the same coordinates, and $\bar{M} = M$. A manifold $K_n^{o(m)}$ admits a holomorphically-projective mapping onto $\bar{K}_n^{o(\bar{m})}$ if and only if the following equations [17]:

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X + \varphi(FX)FY + \varphi(FY)FX \tag{3.1}$$

hold for any tangent fields X, Y and where ψ is a gradient-like form and $\psi(X) = \varphi(FX)$. If $\psi \equiv 0$ than f is *affine* or *trivially holomorphically-projective* . Moreover, structures F and $\bar{F}$ are preserved, i.e. $\bar{F} = F$, and $\bar{m} = m$. This fact implies from the theory of F-planar mappings, see [11, pp. 219-220]. In local form:

$$\bar{\Gamma}_{ij}^h = \Gamma_{ij}^h + \psi_i \delta_j^h + \psi_j \delta_i^h + \varphi_i F_j^h + \varphi_j F_i^h, \quad \psi_i = \varphi_j F_i^j,$$

where Γ_{ij}^h and $\bar{\Gamma}_{ij}^h$ are the Christoffel symbols of K_n and $\bar{K}_n$, ψ_i, F_i^h are components of ψ, F and δ_i^h is the Kronecker delta,

$$\psi_i = \frac{\partial \Psi}{\partial x^i}, \quad \Psi = \frac{1}{2(n+2)} \ln \left| \frac{\det \bar{g}}{\det g} \right|.$$

Here and in the following we will use the conjugation operation of indices in the way

$$A_{\dots \bar{i} \dots} = A_{\dots k \dots} F_i^k.$$

Equations (3.1) are equivalent to the following equations

$$\nabla_Z \bar{g}(X,Y) = 2\psi(Z)\bar{g}(X,Y) + \psi(X)\bar{g}(Y,Z) + \psi(Y)\bar{g}(X,Z) + \varphi(FX)\bar{g}(FY,Z) + \varphi(FY)\bar{g}(FX,Z). \tag{3.2}$$

In local form:

$$\bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi \bar{g}_{ik} + \varphi_i \bar{g}_{\bar{j}k} + \varphi_j \bar{g}_{\bar{i}k},$$

where "," denotes the covariant derivative on $K_n^{o(m)}$. M. Shiha and J. Mikeš [18] proved that equations (3.1) and (3.2) are equivalent to

$$\nabla_Z a(X,Y) = \lambda(X) g(Y,Z) + \lambda(Y) g(X,Z) + \theta(X) g(FY,Z) + \theta(Y) g(FX,Z). \tag{3.3}$$

In local form:

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} + \theta_i g_{\bar{j}k} + \theta_j g_{\bar{i}k},$$

where

$$\text{(a) } a_{ij} = \mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}, \quad \text{(b) } \lambda_i = -\mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\beta i} \psi_\alpha, \quad \text{(c) } \theta_i = -\mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\beta \bar{i}} \varphi_\alpha. \tag{3.4}$$

From (3.3) follows that λ_i is gradient-like vector and it holds

$$\lambda_i = \partial_i \Lambda, \qquad \Lambda = 1/4 a_{\alpha\beta} g^{\alpha\beta}. \tag{3.5}$$

On the other hand [11]:

$$\bar{g}_{ij} = \mathrm{e}^{2\Psi} \tilde{g}_{ij}, \quad \Psi = \frac{1}{2} \ln \left| \frac{\det \tilde{g}}{\det g} \right|, \quad \|\tilde{g}_{ij}\| = \|g^{i\alpha} g^{j\beta} a_{\alpha\beta}\|^{-1}. \tag{3.6}$$

The above formulas are the criterion for holomorphically-projective mappings $K_n^{o(m)} \to \bar{K}_n^{o(m)}$, globally as well as locally.

Theorem 1. *A diffeomorphism $f : K_n^{o(m)} \to \bar{K}_n^{o(\bar{m})}$ is a holomorphically-projective mapping if and only if there exist a solution of the following linear Cauchy-like system*

$$
\begin{aligned}
a)\ \ & a_{ij,k} = \lambda_{(i} g_{j)k} + \theta_{(i} F_{j)k};\\
b)\ \ & \theta_{i,j} = \tau F_{ij} + a_{\alpha\beta} M^{\alpha\beta}_{1|ij};\\
c)\ \ & \tau_{,i} = \theta_\alpha M^\alpha_{2|i} + a_{\alpha\beta} M^{\alpha\beta}_{3|i}
\end{aligned}
\tag{3.7}
$$

on unknown tensor a_{ij} ($a_{ij} = a_{ji}$, $a_{\bar{i}j} + a_{i\bar{j}} = 0$, $\det a_{ij} \neq 0$), a vector λ_i, and a function τ. Here $M^{\alpha\beta}_{1|ij}$, $M^\alpha_{2|i}$, $M^{\alpha\beta}_{3|i}$ are tensors determined from metric and structure tensors g_{ij} and F_i^h of the space $K_n^{o(m)}$.

Remark 1. This theorem was proved with assuming that $K_n^{o(m)}$ and $\bar{K}_n^{o(m)}$ belong to C^3 class. We will prove, that the Theorem 1 valides too if $K_n^{o(m)} \in C^3$ and $\bar{K}_n^{o(\bar{m})} \in C^2$.

The system (3.7) has at most one solution for the initial values in a point x_0 : $a_{ij}(x_0)$, $\lambda_i(x_0)$ and $\tau(x_0)$. Hence, the general solution of this system depends on no more than $(n+2)(n+1)/2 - m(n-m+1)$ essential parameters.

The integrability of conditions (3.7) and their differential prolongations are linear algebraic equations on the components of the unknown tensors a_{ij}, λ_{ij} and τ with coefficients from $K_n^{o(m)}$.

4. Holomorphically-projective mapping of parabolic Kähler space of class C^2

The direct substitution of (3.1) implies that Riemannian tensors of spaces $K_n^{o(m)}$ and $\bar{K}_n^{o(m)}$, which are holomorphically-projectively corresponding of m-parabolic Kähler space are connected by the following relations:

$$\bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik} + F^h_k \varphi_{ij} - F^h_j \varphi_{ik} + F^h_i \varphi_{[kj]}, \tag{4.1}$$

where $\bar{R}^h_{ijk}$, R^h_{ijk} are Riemannian tensors of $K_n^{o(m)}$ and $\bar{K}_n^{o(m)}$,

$$\varphi_{ik} \equiv \varphi_{i,j} - \psi_i \varphi_j - \varphi_i \psi_j; \quad \psi_{ij} \equiv \varphi_{\bar{i}j}. \tag{4.2}$$

Tensor $\bar{\varphi}_{ij}$ has this form:

$$\psi_{ij} = \psi_{i,j} - \psi_i \psi_j. \tag{4.3}$$

As ψ_i is a gradient, that this tensor is symmetrical, that is

$$\psi_{ij} = \psi_{ji}. \tag{4.4}$$

Contracting (4.1) with respect to indices h and k, we obtain connection between Ricci tensors

$$\bar{R}_{ij} = R_{ij} + n\psi_{ij}, \tag{4.5}$$

where R_{ij} and $\bar{R}_{ij}$ are Ricci tensors of $K_n^{o(m)}$ and $\bar{K}_n^{o(m)}$.

Next, we will proceed similarly as in work of M. Shiha [17, 18]. Let we consider the integrability condition of equation (3.3):

$$a_{\alpha(i}R^{\alpha}_{j)kl} = g_{k(i}\lambda_{j),l} - g_{l(}\lambda_{j),k} - F_{k(i}\theta_{j),l} + F_{l(i}\theta_{j),k}. \tag{4.6}$$

Contracting (4.6) with $F^k_{k'}$ and also with $F^l_{l'}$, we obtain two expressions. After removing primes, we sum them up. Since $K_n^{o(m)}$ it holds $R^h_{i\alpha k} + R^h_{ij\alpha}F^{\alpha}_k = 0$, we get $F_{k(i}\Lambda_{j)l} - F_{l(i}\Lambda_{j)k} = 0$, where $\Lambda_{ij} \equiv \lambda_{i,j} + \theta_{i,\alpha}F^{\alpha}_j$. It follows that $\Lambda_{ij} = \Lambda F_{ij}$, i.e. $\lambda_{i,j} + \theta_{i,\alpha}F^{\alpha}_j = \Lambda F_{ij}$ where Λ is a function. Contracting (4.6) with g^{jk}, we obtain

$$n\lambda_{i,j} = \mu g_{ij} + \nu F_{ij} - a_{\alpha i}R^{\alpha}_j - a_{\alpha\beta}R^{\alpha..\beta}_{.ij.}. \tag{4.7}$$

where R^h_{ijk} and R_{ij} are Riemann and Ricci tensors, respectively, the operation of lifting and lowering indices are induced by the metric tensor, and μ, ν are certain functions. After symmetrizing (4.7) we get

$$n\lambda_{i,j} = \mu g_{ij} - \frac{1}{2}a_{\alpha(i}R^{\alpha}_{j)} - a_{\alpha\beta}R^{\alpha..\beta}_{.ij.}. \tag{4.8}$$

Substituting (4.8) to (4.6) we obtain

$$a_{\alpha\beta}M^{\alpha\beta}_{jkl} = F_{li}\theta_{j,k} + F_{lj}\theta_{i,k} - F_{ki}\theta_{j,l} + F_{kj}\theta_{i,l}. \tag{4.9}$$

From this implies that M are tensors determined by g_{ij} and F^h_i on $K_n^{o(m)}$. More precisely

$$M^{\alpha\beta}_{ijkl} \equiv \delta^{\alpha}_{(i}R^{\beta}_{j)kl} + M^{\alpha\beta}_{4|k(i}g_{j)l} - M^{\alpha\beta}_{4|l(i}g_{j)k}; \qquad nM^{\alpha\beta}_{4|li} \equiv \frac{1}{2}\delta^{\alpha}_{(i}R^{\beta}_{j)} - R^{\alpha..\beta}_{.ij.},$$

where δ^h_i is the Kronecker symbol. Let ε^j and ν^k be vectors such that $\varepsilon^j\nu^k F_{jk} = 1$. Denote $M_i \equiv \varepsilon^{\alpha}F_{\alpha i}$. Contracting (4.9) with $\varepsilon^i\varepsilon^j\nu^k$ we get

$$\varepsilon^{\alpha}\theta_{\alpha,l} = \tau M_l + a_{\alpha\beta}M^{\alpha\beta}_{5|l}, \tag{4.10}$$

where $\tau \equiv \lambda_{\alpha,\beta}\varepsilon^{\alpha}\nu^{\beta}$. Contracting (4.9) with $\varepsilon^j\nu^k$ and using (4.10) we have the following formula

$$\theta_{i,j} = \tau F_{ij} + \Lambda_i M_j + a_{\alpha\beta}M^{\alpha\beta}_{6|ij}, \tag{4.11}$$

where λ_i is a vector. Substituting (4.11) into (4.9) we have

$$\Lambda_i(M_k F_{lj} - M_l F_{kj}) + \Lambda_j(M_k F_{li} - M_l F_{ki}) = a_{\alpha\beta}M^{\alpha\beta}_{7|ijkl}. \tag{4.12}$$

So we proved the following lemma:

Lemma 1. *Let $K_n^{o(m)}$ and $\bar{K}_n^{o(\bar{m})}$ belong to class C^2. If $K_n^{o(m)}$ admit a holomorphically-projective mapping onto $\bar{K}_n^{o(\bar{m})}$ than formulae (3.7a) and (3.7b) hold.*

5. Holomorphically-projective mapping $K_n^{o(m)} \to \bar{K}_n^{o(\bar{m})}$ for $K_n^{o(m)} \in C^r$ $(r \geq 3)$ and $\bar{K}_n^{o(\bar{m})} \in C^2$

Theorem 2. *Let $K_n^{o(m)} \in C^r$ $(r \geq 3)$ and $\bar{K}_n^{o(\bar{m})} \in C^2$. If $K_n^{o(m)}$ admits holomorphically-projective mapping onto $\bar{K}_n^{o(\bar{m})}$ then $\bar{K}_n^{o(\bar{m})} \in C^r$.*

Proof of Theorem 2 is based on the proof of Theorem 3.

Theorem 3. *Let $K_n^{o(m)} \in C^3$ and $\bar{K}_n^{o(\bar{m})} \in C^2$. If $K_n^{o(m)}$ admits holomorphically-projective mapping onto $\bar{K}_n^{o(\bar{m})}$ then $\bar{K}_n^{o(\bar{m})} \in C^3$.*

For first, we prove that metric g and structure F have the same differentiation.

Lemma 2. *If $K_n^{o(m)} = (M, g, F) \in C^r$, i.e. $g(x) \in C^r$, then $F(x) \in C^r$, for $r \in \mathbb{N}$ and $r = \infty, \omega$.*

Proof. Let $K_n^{o(m)} \in C^r$, i.e. the components of metric $g_{ij}(x) \in C^r$ in a coordinate chart x. It is a priori valid, that $F_i^h \in C^1$. The formula $\nabla F = 0$ can be written $\partial_k F_i^h = F_\alpha^h \Gamma_{ik}^\alpha - F_i^\alpha \Gamma_{\alpha k}^h$, where $\Gamma_{ijk} = 1/2(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$, $\partial_k = \partial/\partial x^k$, and $\Gamma_{ij}^h = g^{hk}\Gamma_{ijk}$ are the Christoffel symbols of the first and second kind, respectively. It holds, that Γ_{ijk} and $\Gamma_{ij}^h \in C^{r-1}$. From this equation immediately follows $F_i^h(x) \in C^r$, i.e. $F \in C^r$. □

For proving Theorem 3, we need the lemmae:

Lemma 3 ([9]). *Let $\lambda^h(x) \in C^1$ be a vector field, $\Phi_i^h = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right)$ is special form with $I = \delta_b^a$, where $1 \leq a, b \leq r$, $r \geq 2$. If $\partial_i \lambda^h - \rho \Phi_i^h \in C^1$ then $\lambda^h \in C^2$ and $\rho \in C^1$.*

Lemma 4. *Let $\lambda^h(x) \in C^1$ be a vector field, $F_i^h(x) \in C^2$ is a tensor field of* rank $F \geq 2$. *If $\partial_i \lambda^h - \rho F_i^h \in C^1$ then $\lambda^h \in C^2$ and $\rho \in C^1$.*

Proof. Let
$$\partial_i \lambda^h - \rho F_i^h(x) = f_i^h(x) \in C^1.$$
Because rank $F_i^h \geq r$ and $F_i^h(x) \in C^2$, then exist a regular tensor field $\Omega_i^h(x) \in C^2$ that $F_i^h \Omega_j^i = \Phi_i^h$

We put $v^h = \lambda^\alpha \Omega_\alpha^h$. Then we have
$$\partial_i v^h - \rho \Phi_\alpha^h = \partial_i(\lambda^\alpha \Omega_\alpha^h) - \rho \Phi_\alpha^h = f_i^\alpha \Omega_\alpha^h + \lambda^\alpha \partial_i \Omega_\alpha^h.$$
Because $f_i^\alpha \Omega_\alpha^h + \lambda^\alpha \partial_i \Omega_\alpha^h \in C^1$, from Lemma 3 it implies $v^h \in C^2$ and $\rho \in C^1$. Since $\rho \in C^1$, then $\lambda^h(x) \in C^2$. □

It follows prooving of Theorem 3:

Proof. Equations (3.7b) can be written in the following form:

$$\partial_j \theta^h = \tau F_j^h - \theta^\alpha \Gamma_{\alpha j}^h + a_{\alpha\beta} \underset{1|j}{M}^{\alpha\beta h},$$

i.e. $\partial_j \theta^h - \tau F_j^h = f_j^h$, where $f_j^h = -\theta^\alpha \Gamma_{\alpha j}^h + a_{\alpha\beta} \underset{1|j}{M}^{\alpha\beta h}$. Because $f_j^i \in C^1$ and $F_i^h \in C^3$ then from Lemma 4 follows $\theta^i \in C^2$ and $\tau \in C^1$. Using (3.7a) and (3.7b) we obtain $a_{ij}(x) \in C^3$ and $\Psi \in C^3$. Finally, from (3.6) we have $\bar{g}_{ij}(x) \in C^3$. □

In to this moment, we proved formulae (3.7a) and (3.7b). For complete proof of Theorem 1, we have to prove the latest formula (3.7c).

Proof. Proof of Theorem 3 allows us differentiation of equation (3.7b). Application to this, we have:

$$\theta_{i,jk} = \tau_{,k} F_{ij} + a_{\alpha\beta,k} \underset{1|ij}{M}^{\alpha\beta} + a_{\alpha\beta} \underset{1|ij,k}{M}^{\alpha\beta}. \tag{5.1}$$

After alternation with respect to indices j and k and using Ricci identity, we have

$$\tau_{,j} F_{ik} - \tau_{,k} F_{ij} = a_{\alpha\beta} \underset{8|ijk}{M}^{\alpha\beta} + \theta_\alpha \underset{9|ijk}{M}^{\alpha\beta}. \tag{5.2}$$

Contracting (5.2) with $\varepsilon^i \varepsilon^j \nu^k$, we obtain $\tau_{,a}\varepsilon^\alpha = a_{\alpha\beta} \underset{9|}{M}^{\alpha\beta} + \lambda_\alpha \underset{10|}{M}^{\alpha\beta}$. Finally, contracting (5.2) with $\varepsilon^j \nu^i$, we have (3.7c) and Theorem 1 is proved. □

At the end, we prove Theorem 2.

Proof. If $K_n^{o(m)} \in C^r$ $(r \geq 3)$ and $\bar{K}_n^{o(\bar{m})} \in C^2$, then by Theorem 3, $\bar{K}_n^{o(\bar{m})} \in C^3$ and formulas (3.7) hold. Because the system of equations (3.7) is closed, we can differentiate equation (3.1) $(r-1)$ times. So we can convince ourselves that $a_{ij} \in C^r$, and also $\bar{g}_{ij} \in C^r$ $(\equiv \bar{K}_n^{o(\bar{m})} \in C^r)$. □

REFERENCES

[1] D. V. Alekseevsky, "Pseudo-Kähler and para-Kähler symmetric spaces." in *Handbook of pseudo-Riemannian geometry and supersymmetry*. Zürich: EMS, 2010, pp. 703–729, doi: 10.4171/079-1/21.

[2] H. Chudá, M. Chodorová, and M. Shiha, "On composition of conformal and holomorphically projective mappings between conformally Kählerian spaces," *J. Appl. Math. Bratislava*, vol. 5, no. 3, pp. 91–96, 2012.

[3] H. Chudá and M. Shiha, "Conformal holomorphically projective mappings satisfying a certain initial condition." *Miskolc Math. Notes*, vol. 14, no. 2, pp. 569–574, 2013.

[4] V. Domashev and J. Mikeš, "Theory of holomorphically projective mappings of Kählerian spaces." *Math. Notes*, vol. 23, pp. 160–163, 1978, doi: 10.1007/BF01153160.

[5] I. Hinterleitner, "On holomorphically projective mappings of e-Kähler manifolds." *Arch. Math., Brno*, vol. 48, no. 5, pp. 333–338, 2012, doi: 10.5817/am2012-5-333.

[6] I. Hinterleitner and J. Mikeš, "Geodesic mappings of (pseudo-)Riemannian manifolds preserve class of differentiability." *Miskolc Math. Notes*, vol. 14, no. 2, pp. 575–582, 2013.

[7] I. Hinterleitner and J. Mikeš, "Geodesic mappings and Einstein spaces." in *Geometric methods in physics. XXX workshop*. Basel: Birkhäuser, 2013, pp. 331–335, doi: 10.1007/978-3-0348-0448-6_28.
[8] E. Kähler, "Über eine bemerkenswerte Hermitesche Metrik," *Sem. Hamburg. Univ.*, vol. 9, pp. 173–786, 1933, doi: 10.1007/bf02940642.
[9] J. Mikeš and all, *Differential geometry of special mappings*. Olomouc: Palacký University, Faculty of Science, 2015.
[10] J. Mikeš, S. Bácsó, and V. Berezovski, "Geodesic mappings of weakly Berwald spaces and Berwald spaces onto Riemannian spaces." *Int. J. Pure Appl. Math.*, vol. 45, no. 3, pp. 413–418, 2008.
[11] J. Mikeš, A. Vanžurová, and I. Hinterleitner, *Geodesic mappings and some generalizations.* Olomouc: Palacký University, Faculty of Science, 2009.
[12] Mikeš, J., "On holomorphically projective mappings of Kählerian spaces. (Russian)," *Ukr. Geom. Sb.*, vol. 23, pp. 90–98, 1980.
[13] Mikeš, J., "Holomorphically projective mappings and their generalizations," *J. Math. Sci., New York*, vol. 89, pp. 1334–1353, 1998, doi: 10.1007/bf02414875.
[14] Mikeš, J. and Shiha M. and Vanžurová A., "Invariant objects by holomorhpically projective mappings of parabolically Kähler spaces." *J. Appl. Math. 2:1*, pp. 135–141, 2009.
[15] T. Ōtsuki and Y. Tashiro, "On curves in Kählerian spaces," *Math. J. Okayama Univ.*, vol. 4, pp. 57–78, 1954.
[16] M. Prvanović, "Holomorphically projective transformations in a locally product space," *Math. Balk.*, vol. 1, pp. 195–213, 1971.
[17] M. Shiha, "Geodesic and holomorphically projective mappings of parabolically kählerian spaces," Ph.D. dissertation, Moscow Ped. Inst., 1992.
[18] M. Shiha, "On the theory of holomorphically-projective mappings of parabolically-Kählerian spaces," *Math. Publ. 1, Silesian Univ. Opava*, pp. 157–160, 1993.
[19] M. Shiha, L. Juklová, and J. Mikeš, "Holomorphically projective mappings onto Riemannian tangent-product spaces," *J. Appl. Math. Bratislava, 5:3*, pp. 259–266, 2012.
[20] M. Shiha and J. Mikeš, "On holomorphically projective flat parabolically-Kählerian spaces." in *Proc. Conf. on Contemp. Geom. and Related Topics*. Belgrade: Univ. Belgrade, Faculty of Mathematics, 2006, pp. 467–474.
[21] M. Shikha and J. Mikeš, "On equidistant, parabolically Kählerian spaces," *Tr. Geom. Semin.*, vol. 22, pp. 97–107, 1994.
[22] P. A. Shirokov, "Selected investigations on geometry," *Kazan' Univ. Press*, p. 432, 1966.
[23] S.-I. Tachibana and S. Ishihara, "On infinitesimal holomorphically projective transformations in Kählerian manifolds." *Tohoku Math. J. (2)*, vol. 12, pp. 77–101, 1960, doi: 10.2748/tmj/1178244489.
[24] V. Vishnevskij, A. Shirokov, and V. Shurygin, "Spaces over algebras," Kazan': Izd. Kazansk. Univ. 263 p., 1985.
[25] K. Yano, "Differential geometry on complex and almost complex spaces," Oxford-London-New York-Paris-Frankfurt: Pergamon Press. XII, 323 p. (1965)., 1965.

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